The Gamma Function and Related Functions

2.1 *Introduction*

In the eighteenth century, L. Euler (1707–1783) concerned himself with the problem of interpolating between the numbers

$$n! = \int_0^\infty e^{-t} t^n \, dt, \qquad n = 0, 1, 2, \ldots$$

with nonintegral values of n. This problem led Euler in 1729 to the now famous *gamma function*, a generalization of the factorial function that gives meaning to $x!$ when x is any positive number. His result can be extended to certain negative numbers and even to complex numbers. The notation $\Gamma(x)$ that is now widely accepted for the gamma function is not due to Euler, however, but was introduced in 1809 by A. Legendre (1752–1833), who was also responsible for the *duplication formula* for the gamma function. Nearly 150 years after Euler's discovery of it, the theory concerning the gamma function was greatly expanded by means of the theory of entire functions developed by K. Weierstrass (1815–1897).

Because it is a generalization of $n!$, the gamma function has been examined over the years as a means of generalizing certain functions, operations, etc., that are commonly defined in terms of factorials. In addition to these applications, the gamma function is useful in the evaluation of many nonelementary integrals; the same is true of the related beta function. often called the Eulerian integral of the first kind. In 1771, forty-three years after discovering the gamma function, Euler discovered that the beta function is actually a particular combination of gamma functions.

The logarithmic derivative of the gamma function leads to the *digamma function*. Further differentiation of the digamma function produces the family of *polygamma functions*, all of which are also related to the *zeta function* of G. Riemann (1826–1866).

2.2 *Gamma Function*

One of the simplest but very important special functions is the *gamma function*. It appears occasionally by itself in physical applications (mostly in the form of some integral), but much of its importance stems from its usefulness in developing other functions such as *Bessel functions* (Chapter 6) and *hypergeometric functions* (Chapters 8–10), which have more direct physical application.

The gamma function has several equivalent definitions, most of which are due to Euler. To begin, we define it by*

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)(x+2)\cdots(x+n)} \tag{2.1}$$

If x is not zero or a negative integer, it can be shown that the limit (2.1) exists.† It is apparent, however, that $\Gamma(x)$ cannot be defined at $x = 0, -1, -2, \ldots,$ since the limit becomes infinite for any of these values. Let us formalize this last statement as a theorem.

Theorem 2.1. If $x = -n$ $(n = 0, 1, 2, \ldots)$, then $|\Gamma(x)| = \infty$, or equivalently,

$$\frac{1}{\Gamma(-n)} = 0, \qquad n = 0, 1, 2, \ldots$$

By setting $x = 1$ in Equation (2.1), we see that

$$\Gamma(1) = \lim_{n\to\infty} \frac{n!n}{1\times 2\times 3\times\cdots\times n(n+1)} = \lim_{n\to\infty} \frac{n}{n+1}$$

*A variation of (2.1), called *Euler's infinite product* (see problem 43), was actually the starting point of Euler's work on the interpolation problem for $n!$.

†See E.D. Rainville, *Special Functions*. New York: Chelsea, 1960, p. 5.

from which we deduce the special value

$$\Gamma(1) = 1 \tag{2.2}$$

Other values of $\Gamma(x)$ are not so easily obtained, but the substitution of $x + 1$ for x in (2.1) leads to

$$\begin{aligned}\Gamma(x+1) &= \lim_{n\to\infty} \frac{n!n^{x+1}}{(x+1)(x+2)\cdots(x+n)(x+n+1)} \\ &= \lim_{n\to\infty} \frac{nx}{x+n+1} \cdot \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\cdots(x+n)}\end{aligned}$$

from which we deduce the *recurrence formula*

$$\Gamma(x+1) = x\Gamma(x) \tag{2.3}$$

Equation (2.3) is the basic functional relation for the gamma function; it is in the form of a *difference equation*. While many of the special functions satisfy some linear *differential equation*, it has been shown that the gamma function does not satisfy any linear differential equation with rational coefficients.*

A direct connection between the gamma function and factorials can be obtained from (2.2) and (2.3). That is, if we combine these relations, we have

$$\begin{aligned}\Gamma(2) &= 1 \times \Gamma(1) = 1 \\ \Gamma(3) &= 2 \times \Gamma(2) = 2 \times 1 = 2! \\ \Gamma(4) &= 3 \times \Gamma(3) = 3 \times 2! = 3! \\ &\vdots\end{aligned}$$

and through mathematical induction it can be shown that

$$\Gamma(n+1) = n!, \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

Thus the gamma function is a generalization of the factorial function from the domain of positive integers to the domain of all real numbers (except as noted in Theorem 2.1). Also, Equation (2.4) confirms a result which beginning algebra students often find puzzling to understand, viz., $0! = 1$.

It is sometimes considered a nuisance that $n!$ is not $\Gamma(n)$, but $\Gamma(n + 1)$. Because of this, some authors adopt the notation $x!$ for the gamma

*See R. Campbell, *Les intégrals Eulériennes et leurs applications*, Paris: Dunod, 1966, pp. 152–159.

function, whether or not x is an integer. C. Gauss (1777–1855) introduced the notation $\Pi(x)$, where $\Pi(x) = x!$, but this notation is seldom utilized. The symbol Γ, due to Legendre, is the most widely used today. We will not use the notation of Gauss, nor will we use the factorial notation except when dealing with nonnegative integer values.

2.2.1 *Integral Representations*

Our reason for using the limit definition (2.1) of the gamma function is mostly historical, but also that it defines the gamma function for negative values of x as well as positive values. The gamma function rarely appears in the form (2.1) in applications. Instead, it most often arises in the evaluation of certain integrals; for example, Euler was able to show that*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x > 0 \tag{2.5}$$

This *integral representation* of $\Gamma(x)$ is the most common way in which the gamma function is now defined. Since integrals are fairly easy to manipulate, (2.5) is often preferred to (2.1) for developing properties of this function. Equation (2.5) is less general than (2.1), however, since the variable x is restricted in (2.5) to positive values. Lastly, we note that (2.5) is an improper integral, due to the infinite limit of integration and also because the factor t^{x-1} becomes infinite at $t = 0$ for values of x in the interval $0 < x < 1$. Nonetheless, the integral (2.5) is *uniformly convergent* for all $a \le x \le b$, where $0 < a \le b < \infty$.

Let us first establish the equivalence of (2.1) and (2.5) for positive values of x. To do so, we set

$$\begin{aligned} F(x) &= \int_0^\infty e^{-t} t^{x-1}\, dt \\ &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt, \qquad x > 0 \end{aligned} \tag{2.6}$$

where we are making the observation

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n \tag{2.7}$$

Using successive integration by parts, after making the change of variable

*Legendre termed the right-hand side of (2.5) the *Eulerian integral of the second kind*.

$z = t/n$, we find

$$
\begin{aligned}
F(x) &= \lim_{n \to \infty} n^x \int_0^1 (1 - z)^n z^{x-1}\, dx \\
&= \lim_{n \to \infty} n^x \left[\cancelto{0}{(1 - z)^n \frac{z^x}{x} \bigg|_0^1} + \frac{n}{x} \int_0^1 (1 - z)^{n-1} z^x\, dz \right] \\
&= \cdots \\
&= \lim_{n \to \infty} n^x \left[\frac{n(n-1) \cdots 2 \times 1}{x(x+1) \cdots (x+n-1)} \int_0^1 z^{x+n-1}\, dz \right] \\
&= \lim_{n \to \infty} \frac{n! n^x}{x(x+1)(x+2) \cdots (x+n)}
\end{aligned}
\tag{2.8}
$$

and thus we have shown that

$$F(x) = \int_0^\infty e^{-t} t^{x-1}\, dt = \Gamma(x), \qquad x > 0 \tag{2.9}$$

It follows from the uniform convergence of the integral (2.5) that $\Gamma(x)$ is a continuous function for all $x > 0$ (see Theorem 1.19). To investigate the behavior of $\Gamma(x)$ as x approaches the value zero from the right, we use the recurrence formula (2.3) written in the form

$$\Gamma(x) = \frac{\Gamma(x + 1)}{x}$$

Thus, we see that

$$\lim_{x \to 0^+} \Gamma(x) = \lim_{x \to 0^+} \frac{\Gamma(x + 1)}{x} = +\infty \tag{2.10}$$

Another consequence of the uniform convergence of the defining integral for $\Gamma(x)$ is that we may differentiate the function under the integral sign to obtain*

$$\Gamma'(x) = \int_0^\infty e^{-t} t^{x-1} \log t\, dt, \qquad x > 0 \tag{2.11}$$

and

$$\Gamma''(x) = \int_0^\infty e^{-t} t^{x-1} (\log t)^2\, dt, \qquad x > 0 \tag{2.12}$$

*Actually, to completely justify the derivative relations (2.11) and (2.12) requires that we first establish the uniform convergence of the integrals in them. See Theorem 1.21 in Section 1.6.3.

The integrand in (2.12) is positive over the entire interval of integration, and thus it follows that $\Gamma''(x) > 0$. This implies that the graph of $y = \Gamma(x)$ is *concave upward* for all $x > 0$. While maxima and minima are ordinarily found by setting the derivative of the function to zero, here we make the observation that, since $\Gamma(1) = \Gamma(2) = 1$ and $\Gamma(x)$ is always concave upward, the gamma function has *only a minimum* on the interval $x > 0$. Moreover, the minimum occurs on the interval $1 < x < 2$. The exact position of the minimum was first computed by Gauss and found to be $x_0 = 1.4616\ldots$, which leads to the minimum value $\Gamma(x_0) = 0.8856\ldots$. Lastly, from the continuity of $\Gamma(x)$ and its concavity, we deduce that

$$\lim_{x \to +\infty} \Gamma(x) = +\infty \tag{2.13}$$

With this last result, we have determined the fundamental characteristics of the graph of the gamma function for $x > 0$ (see Fig. 2.1).

The gamma function is defined for negative values of x by Equation (2.1), but can be evaluated more conveniently by using the recurrence formula

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}, \qquad x \neq 0, -1, -2, \ldots \tag{2.14}$$

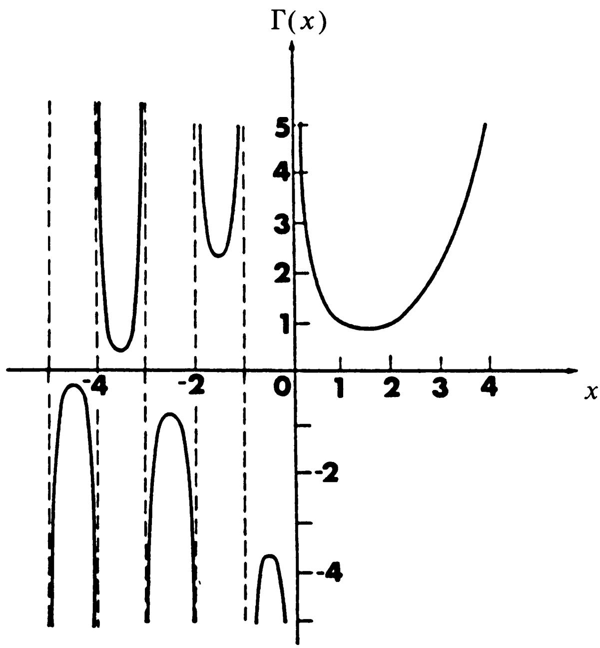


Figure 2.1 The Gamma Function

We are particularly interested in the behavior of the gamma function in the vicinity of the discontinuities at $x = 0, -1, -2, \ldots$. From the above expression, we immediately obtain

$$\lim_{x\to 0^-} \Gamma(x) = \lim_{x\to 0^-} \frac{\Gamma(x+1)}{x} = -\infty \tag{2.15}$$

and

$$\lim_{x\to -1^+} \Gamma(x) = \lim_{x\to -1^+} \frac{\Gamma(x+1)}{x} = -\infty \tag{2.16}$$

By replacing x with $x + 1$ in (2.14), we get

$$\Gamma(x+1) = \frac{\Gamma(x+2)}{x+1}$$

which leads to

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} = \frac{\Gamma(x+2)}{x(x+1)}$$

Using this last expression, we find the limiting values

$$\lim_{x\to -1^-} \Gamma(x) = \lim_{x\to -1^-} \frac{\Gamma(x+2)}{x(x+1)} = +\infty \tag{2.17}$$

and

$$\lim_{x\to -2^+} \Gamma(x) = \lim_{x\to -2^+} \frac{\Gamma(x+2)}{x(x+1)} = +\infty \tag{2.18}$$

Continuing this process, we finally derive the formula

$$\Gamma(x) = \frac{\Gamma(x+k)}{x(x+1)(x+2)\cdots(x+k-1)}, \qquad k = 1, 2, 3, \ldots \tag{2.19}$$

which defines the gamma fцnction over the interval $-k < x < 0$, except for $x = -1, -2, -3, \ldots, -k+1$.

Example 1: Evaluate $\Gamma(-\frac{3}{2})$.

Solution: Making use of (2.19) with $k = 2$ yields*

$$\Gamma(-\tfrac{3}{2}) = (-\tfrac{2}{3})(-2)\Gamma(\tfrac{1}{2}) = \tfrac{4}{3}\sqrt{\pi}$$

*$\Gamma(\frac{1}{2}) = \sqrt{\pi}$. See Equation (2.23).

If we now assemble all the information we have on the gamma function for both positive and negative values of x, we obtain the graph of this function shown in Fig. 2.1. Values of $\Gamma(x)$ are commonly tabulated for the interval $1 \leq x \leq 2$, and other values of $\Gamma(x)$ can then be generated through use of the recurrence formulas.

In addition to

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt, \qquad x > 0$$

there are a variety of other integral representations of $\Gamma(x)$, most of which can be derived from that one by simple changes of variable. For example, if we set $t = u^2$ in the above integral, we get

$$\Gamma(x) = 2\int_0^\infty e^{-u^2} u^{2x-1}\, du, \qquad x > 0 \tag{2.20}$$

whereas the substitution $t = \log(1/u)$ yields

$$\Gamma(x) = \int_0^1 \left(\log \frac{1}{u}\right)^{x-1} du, \qquad x > 0 \tag{2.21}$$

A slightly more complicated relation can be derived by using the representation (2.20) and forming the product

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 2\int_0^\infty e^{-u^2} u^{2x-1}\, du \cdot 2\int_0^\infty e^{-v^2} v^{2y-1}\, dv \\ &= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} u^{2x-1} v^{2y-1}\, du\, dv\end{aligned}$$

The presence of the term $u^2 + v^2$ in the integrand suggests the change of coordinates

$$u = r\cos\theta, \qquad v = r\sin\theta$$

which leads to

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r^{2x-1} \cos^{2x-1}\theta\, r^{2y-1} \sin^{2y-1}\theta\, r\, dr\, d\theta \\ &= 4\int_0^\infty e^{-r^2} r^{2(x+y)-1}\, dr \cdot \int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta \\ &= 2\Gamma(x+y)\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta\end{aligned}$$

Finally, solving for the integral, we get the interesting relation

$$\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}, \qquad x > 0, \quad y > 0 \tag{2.22}$$

By setting $x = y = \frac{1}{2}$ in (2.22), we have

$$\int_0^{\pi/2} d\theta = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(1)}$$

from which we deduce the special value

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \tag{2.23}$$

Example 2: Evaluate $\int_0^\infty e^{-t^2}\,dt$.

Solution: By comparison with (2.20), we see that

$$\int_0^\infty e^{-t^2}\,dt = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

Example 3: Evaluate $\int_0^\infty x^4 e^{-x^3}\,dx$.

Solution: Let $t = x^3$, and then

$$\int_0^\infty x^4 e^{-x^3}\,dx = \frac{1}{3}\int_0^\infty e^{-t} t^{2/3}\,dt = \tfrac{1}{3}\Gamma(\tfrac{5}{3})$$

2.2.2 *Legendre Duplication Formula*

A formula involving gamma functions that is somewhat comparable to the double-angle formulas for trigonometric functions is the *Legendre duplication formula*

$$2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2x) \tag{2.24}$$

In order to derive this relation, we first set $y = x$ in (2.22) to get

$$\begin{aligned}\frac{\Gamma(x)\Gamma(x)}{2\Gamma(2x)} &= \int_0^{\pi/2}\cos^{2x-1}\theta\,\sin^{2x-1}\theta\,d\theta\\ &= 2^{1-2x}\int_0^{\pi/2}\sin^{2x-1}2\theta\,d\theta\end{aligned}$$

where we have used the double-angle formula for the sine function. Next we make the variable change $\phi = 2\theta$, which yields

$$\begin{aligned}\frac{\Gamma(x)\Gamma(x)}{2\Gamma(2x)} &= 2^{-2x}\int_0^{\pi}\sin^{2x-1}\phi\,d\phi\\ &= 2^{1-2x}\int_0^{\pi/2}\sin^{2x-1}\phi\,d\phi\\ &= \frac{2^{1-2x}\Gamma(\frac{1}{2})\Gamma(x)}{2\Gamma(x+\frac{1}{2})}\end{aligned}$$

where the last step results from (2.22). Simplification of this identity leads to (2.24).

An important special case of (2.24) occurs when $x = n$ $(n = 0, 1, 2, \ldots)$, i.e.,

$$\Gamma(n + \tfrac{1}{2}) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi}, \qquad n = 0, 1, 2, \ldots \tag{2.25}$$

the verification of which is left to the exercises (see problem 39).

Example 4: Compute $\Gamma(\tfrac{3}{2})$.

Solution: The substitution of $n = 1$ in (2.25) yields

$$\Gamma(\tfrac{3}{2}) = \Gamma(1 + \tfrac{1}{2}) = \frac{2!\sqrt{\pi}}{2^2 \times 1!} = \tfrac{1}{2}\sqrt{\pi}$$

2.2.3 *The Weierstrass Infinite Product*

Although it was originally found by Schlömilch in 1844, thirty-two years before Weierstrass's famous work on entire functions, Weierstrass is usually credited with the infinite-product definition of the gamma function

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \tag{2.26}$$

where γ is the *Euler-Mascheroni constant* defined by*

$$\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \log n = 0.577215\ldots \tag{2.27}$$

We can derive this representation of $\Gamma(x)$ directly from (2.1) by first observing that

$$\begin{aligned} \frac{1}{\Gamma(x)} &= \lim_{n \to \infty} \frac{x(x+1)(x+2)\cdots(x+n)}{n!n^x} \\ &= x \lim_{n \to \infty} n^{-x}\left[\frac{(x+1)}{1} \cdot \frac{(x+2)}{2} \cdots \frac{(x+n)}{n}\right] \\ &= x \lim_{n \to \infty} \exp[-(\log n)x] \prod_{k=1}^{n} \left(1 + \frac{x}{k}\right) \end{aligned} \tag{2.28}$$

where we have written $n^{-x} = \exp[-(\log n)x]$. Next, relying on properties

*The constant γ is commonly called (simply) *Euler's constant*.

of exponentials, we recognize the identity

$$\exp\left[\left(\sum_{k=1}^{n} \frac{1}{k}\right)x\right] = \prod_{k=1}^{n} e^{x/k}$$

Thus, if we multiply (2.28) by the left-hand side of this expression and divide by the right-hand side, we arrive at

$$\frac{1}{\Gamma(x)} = x \lim_{n\to\infty} \exp\left[\left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right)x\right] \cdot \lim_{n\to\infty} \prod_{k=1}^{n}\left(1 + \frac{x}{k}\right)e^{-x/k}$$

which reduces to (2.26).

An important identity involving the gamma function and sine function can now be derived by using (2.26). We begin with the product of gamma functions

$$\frac{1}{\Gamma(x)\Gamma(-x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left(1 + \frac{x}{n}\right)e^{-x/n} \cdot (-x)e^{-\gamma x}\prod_{n=1}^{\infty}\left(1 - \frac{x}{n}\right)e^{x/n}$$

or

$$\frac{1}{\Gamma(x)\Gamma(-x)} = -x^2\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) \tag{2.29}$$

where we assume that x is nonintegral. Recalling Equation (1.92) in Section 1.7.2, which gives the infinite-product definition of the sine function, we have

$$\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x} \tag{2.30}$$

Comparison of (2.29) and (2.30) reveals that

$$\Gamma(x)\Gamma(-x) = -\frac{\pi}{x \sin \pi x} \qquad (x \text{ nonintegral}) \tag{2.31}$$

Also, by writing the recurrence formula (2.3) in the form

$$-x\Gamma(-x) = \Gamma(1 - x)$$

we deduce the identity

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x} \qquad (x \text{ nonintegral}) \tag{2.32}$$

Example 5: Evaluate the integral $\int_0^{\pi/2} \tan^{1/2}\theta \, d\theta$.

Solution: Making use of (2.22) and (2.32), we get

$$\begin{aligned}\int_0^{\pi/2} \tan^{1/2}\theta\,d\theta &= \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta\\ &= \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{2\Gamma(1)}\\ &= \frac{1}{2}\frac{\pi}{\sin(\pi/4)}\\ &= \frac{\pi}{\sqrt{2}}\end{aligned}$$

Remark: An *entire function* is one that is analytic for all finite values of its argument. Weierstrass was the first to show that any entire function (under appropriate restrictions) with an infinite number of zeros, such as $\sin x$ and $\cos x$, is essentially determined by its zeros. This result led to the infinite-product representations of such functions, and in particular, to the infinite-product representation of the gamma function.

2.2.4 *Fractional-Order Derivatives*

Besides generalizing the notion of factorials, the gamma function can be used in a variety of situations to generalize discrete processes into the continuum. Such generalizations are not new, however: mathematicians over the years have concerned themselves with this concept. In particular, the question concerning derivatives of nonintegral order was first raised by Leibniz in 1695, many years before Euler introduced the gamma function.

The general procedure for developing fractional derivatives is too involved for our purposes.* However, we can illustrate the concept by first recalling the familiar derivative formula from calculus,

$$D^n x^a = a(a-1)\cdots(a-n+1)x^{a-n}, \qquad a \geq 0 \tag{2.33}$$

where $D^n = d^n/dx^n$. In terms of the gamma function, we can rewrite (2.33) as (see problem 10)

$$D^n x^a = \frac{\Gamma(a+1)}{\Gamma(a-n+1)}x^{a-n}$$

The right-hand side of this expression is meaningful for any real number n for which $\Gamma(a-n+1)$ is defined. Hence, we will assume that the same is

*For a deeper discussion of fractional derivatives, see L. Debnath, Generalized Calculus and Its Applications, *Int. J. Math. Educ. Sci. Technol.*, **9**, No. 4, pp. 399–416 (1978).

true of the left-hand side and write

$$D^{\nu}x^{a} = \frac{\Gamma(a+1)}{\Gamma(a-\nu+1)}x^{a-\nu}, \ a \geq 0 \tag{2.34}$$

where ν is not restricted to integer values. Equation (2.34) provides a simple method of computing *fractional-order derivatives* of polynomials.

Example 6: Compute $D^{1/2}x^2$.

Solution: Directly from (2.34), we obtain

$$D^{1/2}x^2 = \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}x^{3/2}$$

the simplification of which yields

$$D^{1/2}x^2 = \frac{8}{3\sqrt{\pi}}x^{3/2}$$

Generalization of the differentiation formula for $D^n x^{-a}$, which covers the case of negative exponents, is left to the exercises (see problem 52).

EXERCISES 2.2

1. Use Equation (2.1) directly to evaluate

(a) $\Gamma(2)$. (b) $\Gamma(3)$.

In problems 2–7, give numerical values for the expressions.

2. $\Gamma(6)/\Gamma(3)$.

3. $\Gamma(7)/\Gamma(4)\Gamma(3)$.

4. $\Gamma(\frac{7}{2})$.

5. $\Gamma(-\frac{1}{2})$.

6. $\Gamma(-\frac{5}{2})/\Gamma(\frac{1}{2})$.

7. $\Gamma(\frac{8}{3})/\Gamma(\frac{2}{3})$.

In problems 8–14, verify the given identity.

8. $\Gamma(a+n) = a(a+1)(a+2)\cdots(a+n-1)\Gamma(a), \ n = 1,2,3,\ldots$.

9. $\dfrac{\Gamma(n-a)}{\Gamma(-a)} = (-1)^n a(a-1)(a-2)\cdots(a-n+1), \ n = 1,2,3,\ldots$.

10. $\dfrac{\Gamma(a)}{\Gamma(a-n)} = (a-1)(a-2)\cdots(a-n), \ n = 1,2,3,\ldots$.

11. $\dfrac{\Gamma(k-n)}{\Gamma(-n)} = \begin{cases} \dfrac{(-1)^k n!}{(n-k)!}, & 0 \le k \le n \ (k, n \text{ non-negative integers}), \\ 0, & k > n. \end{cases}$

Hint: See problem 9.

12. $\dbinom{a}{n} = \dfrac{\Gamma(a+1)}{n!\Gamma(a-n+1)}, \ n = 0, 1, 2, \ldots .$

Hint: See problem 10.

13. $\dbinom{-\frac{1}{2}}{n} = \dfrac{(-1)^n (2n)!}{2^{2n}(n!)^2}, \ n = 0, 1, 2, \ldots .$

14. $\dbinom{-2k-1}{m} = (-1)^m \dfrac{(m+2k)!}{(2k)!m!}, \ k, m = 0, 1, 2, \ldots .$

15. In problems in electromagnetic theory it is quite common to come across products like

$$2 \times 4 \times 6 \times \cdots \times 2n \equiv (2n)!!$$

and

$$1 \times 3 \times 5 \times \cdots \times (2n+1) \equiv (2n+1)!!$$

Use these definitions of the !! notation to show that

(a) $(2n)!! = 2^n n!$, (b) $(2n+1)!! = \dfrac{(2n+1)!}{2^n n!}$,

(c) $(-2n-1)!! = \dfrac{(-1)^n 2^n n!}{(2n)!}$, (d) $(-1)!! = 1$.

Hint: See problem 10 for (c) and (d).

16. Prove that $\int_0^\infty e^{-t} t^{x-1}\, dt$ converges uniformly in $1 \le x \le 2$.

In problems 17–20, verify the given integral representation.

17. $\Gamma(x) = s^x \int_0^\infty e^{-st} t^{x-1}\, dt, \ x, s > 0.$

18. $\Gamma(x) = \int_{-\infty}^\infty \exp(xt - e^t)\, dt, \ x > 0.$

Hint: Let $u = e^t$.

19. $\Gamma(x) = \int_1^\infty e^{-t} t^{x-1}\, dt + \sum_{n=0}^\infty \dfrac{(-1)^n}{n!(x+n)}, \ x > 0.$

20. $\Gamma(x) = (\log b)^x \int_0^\infty t^{x-1} b^{-t}\, dt, \ x > 0, \ b > 1.$

Hint: Let $u = t \log b$.

In problems 21–29, use properties of the gamma function to obtain the result.

21. $\int_a^\infty e^{2ax-x^2}\,dx = \frac{1}{2}\sqrt{\pi}\,e^{a^2}.$

Hint: $2ax - x^2 = -(x-a)^2 + a^2.$

22. $\int_0^\infty e^{-2x}x^6\,dx = \frac{45}{8}.$

23. $\int_0^\infty \sqrt{x}\,e^{-x^3}\,dx = \frac{\sqrt{\pi}}{3}.$

24. $\int_0^1 \frac{du}{\sqrt{-\log u}} = \sqrt{\pi}.$

25. $\int_0^1 x^k(\log x)^n\,dx = \frac{(-1)^n n!}{(k+1)^{n+1}},\ k > -1,\ n = 0, 1, 2, \ldots.$

26. $\int_0^{\pi/2} \cos^6\theta\,d\theta = \frac{5\pi}{32}.$

27. $\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta = \frac{2}{15}.$

28. $\int_0^{\pi} \cos^4 x\,dx = \frac{3\pi}{8}.$

29. $\int_0^{\pi/2} \sin^{2n+1}\theta\,d\theta = \int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta = \frac{2^{2n}(n!)^2}{(2n+1)!},\ n = 0, 1, 2, \ldots.$

In problems 30–35, evaluate the integral in terms of the gamma function and simplify when possible.

30. $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt,\ s > 0.$

31. $\int_0^\infty \frac{dx}{1+x^4}.$

Hint: Let $x^2 = \tan\theta$.

32. $\int_0^{\pi/2} \sqrt{\sin 2x}\,dx.$

33. $\int_0^1 t^{x-1}(\log\frac{1}{t})^{y-1}\,dt,\ x, y > 0.$

34. $\int_0^{\pi/2} \cot^{1/2}\theta\,d\theta.$

35. $\int_0^\infty e^{-st^p}t^{x-1}\,dt,\ p, s, x > 0.$

36. Using the recurrence formula (2.3), deduce that

(a) $\Gamma(x) = \Gamma'(x+1) - x\Gamma'(x),$

(b) $\Gamma(x) = \int_0^\infty e^{-t}(t-x)t^{x-1}\log t\,dt,\ x > 0.$

In problems 37 and 38, use the Euler formulas

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

and properties of the gamma function to derive the result. Assume that $b, x > 0$ and $-\frac{1}{2}\pi < a < \frac{1}{2}\pi$.

37. $\Gamma(x)\cos ax = b^x \int_0^\infty t^{x-1} e^{-bt\cos a} \cos(bt \sin a)\, dt.$

38. $\Gamma(x)\sin ax = b^x \int_0^\infty t^{x-1} e^{-bt\cos a} \sin(bt \sin a)\, dt.$

39. Based on the Legendre duplication formula, show that (for $n = 0, 1, 2, \ldots$)

(a) $\Gamma(n + \frac{1}{2}) = \dfrac{(2n)!\sqrt{\pi}}{2^{2n} n!}$,

(b) $\Gamma(\frac{1}{2} - n) = \dfrac{(-1)^n 2^{2n-1}(n-1)!\sqrt{\pi}}{(2n-1)!}$,

(c) $\Gamma(\frac{1}{2} + n)\Gamma(\frac{1}{2} - n) = (-1)^n \pi.$

40. Show that

$$\Gamma(3x) = \frac{1}{2\pi} 3^{3x-1/2} \Gamma(x)\Gamma(x + \tfrac{1}{3})\Gamma(x + \tfrac{2}{3})$$

41. Show that

$$|\Gamma'(x)|^2 \leq \Gamma(x)\Gamma''(x), \qquad x > 0$$

42. Show that

(a) $\Gamma(1 + x)\Gamma(1 - x) = \pi x \csc \pi x$ (x nonintegral),
(b) $\Gamma(\frac{1}{2} + x)\Gamma(\frac{1}{2} - x) = \pi \sec \pi x$, $x \neq n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$.

43. Derive Euler's infinite-product representation

$$\frac{1}{\Gamma(x)} = x \prod_{n=1}^{\infty} \frac{\left(1 + \dfrac{x}{n}\right)}{\left(1 + \dfrac{1}{n}\right)^x}$$

44. Derive the recurrence relation $\Gamma(x + 1) = x\Gamma(x)$, by use of the

(a) integral definition (2.5),
(b) Weierstrass infinite product (2.26).

45. A particle of mass m starts from rest at $r = 1$ and moves along a radial line toward the origin $r = 0$ under the reciprocal force law $f = -k/r$, where k is a positive constant. The energy equation of the particle is given by

$$\tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + k\log r = 0$$

(a) Show that the time required for the particle to reach the origin is $(m\pi/2k)^{1/2}$.
(b) If the particle starts from rest at $r = a$ $(a > 0)$, the energy equation becomes

$$\tfrac{1}{2}m\left(\frac{dr}{dt}\right)^2 + k\log r = k\log a$$

Again find the time required for the particle to reach the origin.

46. Find the area enclosed by the curve $x^4 + y^4 = 1$.

47. Find the total arclength of the lemniscate $r^2 = a^2\cos 2\theta$.

48. Find the area inside the curve $x^{2/3} + y^{2/3} = 1$.

49. Find the volume in the first octant below the surface

$$x^{1/2} + y^{1/2} + z^{1/2} = 1$$

50. Compute the fractional-order derivatives

(a) $D^{1/2}c$, where c is constant,
(b) $D^{1/2}(3x^2 - 7x + 4)$,
(c) $D^{3/2}x^2$,
(d) $D^{\nu}x^{\nu}$, where ν is not a positive integer.

51. Show that
(a) $D^{1/2}(D^{1/2}x^2) = Dx^2$,
(b) $D^{-1/2}(D^{1/2}x^2) = x^2$,
(c) $D^{\nu}(D^{\mu}x^a) = D^{\nu+\mu}x^a$.

52. By generalizing the formula for $D^n x^{-a}$, show that

$$D^{\nu}x^{-a} = (-1)^{\nu}\frac{\Gamma(\nu + a)}{\Gamma(a)}x^{-(a+\nu)}, \qquad a > 0$$

2.3 *Beta Function*

A useful function of two variables is the *beta function**

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1}\,dt, \qquad x > 0, \quad y > 0 \tag{2.35}$$

*This is called the *Eulerian integral of the first kind*.

The utility of the beta function is often overshadowed by that of the gamma function, partly perhaps because it can be evaluated in terms of the gamma function. However, since it occurs so frequently in practice, a special designation for it is widely accepted.

If we make the change of variable $u = 1 - t$ in (2.35), we find

$$B(x, y) = \int_0^1 (1-u)^{x-1} u^{y-1}\, du$$

from which we deduce the *symmetry property*

$$B(x, y) = B(y, x) \tag{2.36}$$

Another representation of the beta function results if we make the variable change $t = u/(1+u)$, leading to

$$B(x, y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\, du, \qquad x > 0, \quad y > 0 \tag{2.37}$$

Finally, to show how the beta function is related to the gamma function, we set $t = \cos^2\theta$ in (2.35) to find

$$B(x, y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta$$

and hence from (2.22) we obtain the relation

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x > 0, \quad y > 0 \tag{2.38}$$

Example 6: Evaluate the integral $I = \int_0^\infty x^{-1/2}(1+x)^{-2}\, dx$.

Solution: By comparison with (2.37), we recognize

$$\begin{aligned} I &= B\left(\tfrac{1}{2}, \tfrac{3}{2}\right) \\ &= \frac{\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{3}{2}\right)}{\Gamma(2)} \end{aligned}$$

Hence, we deduce that

$$\int_0^\infty x^{-1/2}(1+x)^{-2}\, dx = \frac{\pi}{2}$$

Example 7: Show that

$$\int_0^\infty \frac{\cos x}{x^p}\, dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \qquad 0 < p < 1$$

Solution: Making the observation (problem 17 in Exercises 2.2)

$$\frac{1}{x^p} = \frac{1}{\Gamma(p)} \int_0^\infty e^{-xt} t^{p-1}\, dt$$

it follows that

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{x^p}\, dx &= \frac{1}{\Gamma(p)} \int_0^\infty \cos x \int_0^\infty e^{-xt} t^{p-1}\, dt\, dx \\
&= \frac{1}{\Gamma(p)} \int_0^\infty t^{p-1} \int_0^\infty e^{-xt} \cos x\, dx\, dt \\
&= \frac{1}{\Gamma(p)} \int_0^\infty \frac{t^p}{1+t^2}\, dt
\end{aligned}$$

where we have reversed the order of integration. If we now let $u = t^2$, then

$$\begin{aligned}
\int_0^\infty \frac{\cos x}{x^p}\, dx &= \frac{1}{2\Gamma(p)} \int_0^\infty \frac{u^{\frac{1}{2}(p-1)}}{1+u}\, du \\
&= \frac{1}{2\Gamma(p)} B\left(\frac{1+p}{2}, \frac{1-p}{2}\right)
\end{aligned}$$

However (see problem 10),

$$B\left(\frac{1+p}{2}, \frac{1-p}{2}\right) = \pi \sec\left(\frac{p\pi}{2}\right)$$

and thus we have our result.

Example 7 illustrates one of the basic approaches we use in the evaluation of nonelementary integrals. That is, we replace part (or all) of the integrand by its series representation or integral representation and then interchange the order in which the operations are carried out.

EXERCISES 2.3

In problems 1–4, evaluate the beta function.

1. $B(\frac{2}{3}, \frac{1}{3})$.

2. $B(\frac{3}{4}, \frac{1}{4})$.

3. $B(\frac{1}{2}, 1)$.

4. $B(x, 1-x), 0 < x < 1$.

In problems 5–10, verify the identity.

5. $B(x+1, y) + B(x, y+1) = B(x, y),\ x, y > 0$.

6. $B(x, y+1) = \dfrac{y}{x} B(x+1, y) = \dfrac{y}{x+y} B(x, y),\ x, y > 0.$

7. $B(x, x) = 2^{1-2x} B(x, \frac{1}{2}),\ x > 0.$

8. $B(x, y) B(x+y, z) B(x+y+z, w) = \dfrac{\Gamma(x)\Gamma(y)\Gamma(z)\Gamma(w)}{\Gamma(x+y+z+w)},$
$x, y, z, w > 0.$

9. $B(n, n) B(n+\frac{1}{2}, n+\frac{1}{2}) = \pi 2^{1-4n} n^{-1},\ n = 1, 2, 3, \ldots.$

10. $B\left(\dfrac{1+p}{2}, \dfrac{1-p}{2}\right) = \pi \sec(p\pi/2),\ 0 < p < 1.$

In problems 11–18, use properties of the beta and gamma functions to evaluate the integral.

11. $\displaystyle\int_0^1 \sqrt{x(1-x)}\, dx.$

12. $\displaystyle\int_0^1 x^4 (1-x^2)^{-1/2}\, dx.$

13. $\displaystyle\int_0^\infty \frac{x}{(1+x^3)^2}\, dx.$

Hint: Set $t = x^3/(1+x^3)$.

14. $\displaystyle\int_{-1}^1 \left(\frac{1+x}{1-x}\right)^{1/2} dx.$

Hint: Set $x = 2t - 1$.

15. $\displaystyle\int_a^b (b-x)^{m-1}(x-a)^{n-1}\, dx$, where m, n are positive integers.

16. $\displaystyle\int_0^2 x^2 (2-x)^{-1/2}\, dx.$

17. $\displaystyle\int_0^a x^4 \sqrt{a^2 - x^2}\, dx.$

18. $\displaystyle\int_0^2 x\sqrt[3]{8 - x^3}\, dx.$

In problems 19–30, verify the integral formula.

19. $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \pi \csc p\pi,\ 0 < p < 1.$

20. $\displaystyle\int_0^\infty \frac{\sin x}{x^p}\, dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)},\ 0 < p < 1.$

21. $\displaystyle\int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$

Hint: Use problem 20.

22. $\displaystyle\int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

23. $\displaystyle\int_0^{\pi/2} \tan^p x\,dx = \int_0^{\pi/2} \cot^p x\,dx = \frac{\pi}{2\cos(p\pi/2)}$, $0 < p < 1$.

24. $\displaystyle\int_0^\infty \frac{x^{p-1}\log x}{1+x}\,dx = -\pi^2 \csc p\pi \cot p\pi$, $0 < p < 1$.

25. $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x^a}\,dx = \frac{\pi}{a\sin(p\pi/a)}$, $0 < p < a$.

26. $\displaystyle\int_0^\infty e^{-st}(1-e^{-t})^n\,dt = \frac{n!\Gamma(s)}{\Gamma(s+n+1)}$, where $s > 0$, $n = 0, 1, 2, \ldots$.

27. $\displaystyle\int_{-\infty}^\infty \frac{e^{2x}}{ae^{3x}+b}\,dx = \frac{2\pi}{3\sqrt{3}}a^{-2/3}b^{-1/3}$, where $a, b > 0$.

28. $\displaystyle\int_{-\infty}^\infty \frac{e^{2x}}{(e^{3x}+1)^2}\,dx = \frac{2\pi}{9\sqrt{3}}$.

Hint: Differentiate with respect to b in problem 27.

29. $\displaystyle\int_0^1 \frac{t^{x-1}+t^{y-1}}{(t+1)^{x+y}}\,dt = 2B(x,y)$, where $x, y > 0$.

30. $\displaystyle\int_0^1 \frac{t^{x-1}(1-t)^{y-1}}{(t+p)^{x+y}}\,dt = \frac{B(x,y)}{p^x(1+p)^{x+y}}$, where $x, y, p > 0$.

31. Using the notation of problem 15 in Exercises 2.2, show that

(a) $$\int_{-1}^1 (1-x^2)^{1/2}x^{2n}\,dx = \begin{cases} \dfrac{\pi}{2}, & n = 0, \\ \pi\dfrac{(2n-1)!!}{(2n+2)!!}, & n = 1, 2, 3, \ldots. \end{cases}$$

(b) $$\int_{-1}^1 (1-x^2)^{-1/2}x^{2n}\,dx = \begin{cases} \pi, & n = 0, \\ \pi\dfrac{(2n-1)!!}{(2n)!!}, & n = 1, 2, 3, \ldots. \end{cases}$$

32. Show that

$$\int_{-1}^1 (1-x^2)^n\,dx = 2^{2n+1}\frac{(n!)^2}{(2n+1)!}, \qquad n = 0, 1, 2, \ldots$$

33. The *incomplete beta function* is defined by

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt, \qquad 0 \le x \le 1, \quad p, q > 0$$

(a) Show that

$$B_x(p,q) = x^p \Gamma(q) \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\Gamma(q-n)(p+n)n!}, \; 0 \le x \le 1$$

(b) From (a), deduce that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(q-n)(p+n)n!} = \frac{\Gamma(p)}{\Gamma(p+q)}$$

The Hypergeometric Function

8.1 *Introduction*

Because of the many relations connecting the special functions to each other, and to the elementary functions, it is natural to inquire whether more general functions can be developed so that the special functions and elementary functions are merely specializations of these general functions. General functions of this nature have in fact been developed and are collectively referred to as *functions of the hypergeometric type*. There are several varieties of these functions, but the most common are the standard *hypergeometric function* (which we discuss in this chapter) and the *confluent hypergeometric function* (Chapter 9). Still, other generalizations exist, such as *MacRobert's E-function* and *Meijer's G-function*, for which even *generalized hypergeometric functions* are certain specializations (Chapter 10).

The major development of the theory of the hypergeometric function was carried out by Gauss and published in his famous memoir of 1812, a memoir that is also noted as being the real beginning of rigor in mathematics.* Some important results concerning the hypergeometric function had been developed earlier by Euler and others, but it was Gauss who made the first systematic study of the series that defines this function.

*C.F. Gauss, Disquisitiones Generales circa Seriem Infinitam..., *Comment. Soc. Reg. Sci. Gottingensis Recent.*, **2** (1812).

8.2 *The Pochhammer Symbol*

In dealing with certain product forms, factorials, and gamma functions, it is useful to introduce the abbreviation

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1,2,3,\ldots \tag{8.1}$$

called the *Pochhammer symbol.* Using properties of the gamma function, it follows that this symbol can also be defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n = 0,1,2,\ldots \tag{8.2}$$

Remark: For typographical convenience the symbol $(a)_n$ is sometimes replaced by *Appel's symbol* (a, n).

The Pochhammer symbol $(a)_n$ is important in most of the following material in this text. Because of its close association with the gamma function, it clearly satisfies a large number of identities. Some of the special properties are listed in Theorem 8.1 below, while other relations are taken up in the exercises.

Theorem 8.1. The Pochhammer symbol $(a)_n$ satisfies the identities:

(1) $(1)_n = n!$,

(2) $(a+n)(a)_n = a(a+1)_n$,

(3) $\binom{-a}{n} = \frac{(-1)^n}{n!}(a)_n$,

(4) $(a)_{n+k} = (a)_k(a+k)_n = (a)_n(a+n)_k$ (addition formula),

(5) $(a)_{k-n} = (-1)^n(a)_k/(1-a-k)_n$,

(6) $(a)_{2n} = 2^{2n}(\frac{1}{2}a)_n(\frac{1}{2}+\frac{1}{2}a)_n$ (duplication formula).

(Partial) proof: We will prove only parts (1), (2), and (3). The remaining proofs are left to the exercises.

From the definition, it follows that

(1):
$$(1)_n = 1 \times 2 \times \cdots \times n = n!,$$

(2):
$$\begin{aligned}(a+n)(a)_n &= a(a+1)\cdots(a+n-1)(a+n)\\ &= a(a+1)_n,\end{aligned}$$

(3):
$$\binom{-a}{n} = \frac{-a(-a-1)\cdots(-a-n+1)}{n!}$$
$$= \frac{(-1)^n}{n!}a(a+1)\cdots(a+n-1)$$
$$= \frac{(-1)^n}{n!}(a)_n. \qquad \blacksquare$$

From the definition, we see that the parameter a can be either positive or negative, but generally we assume $a \neq 0$. An exception to this is the special value $(0)_0 = 1$. If a is a negative integer, we find that (see problem 17)

$$(-k)_n = \begin{cases} \dfrac{(-1)^n k!}{(k-n)!}, & 0 \le n \le k \\ 0, & n > k \end{cases} \tag{8.3}$$

Part (5) of Theorem 8.1 can be used to give meaning to the Pochhammer symbol for negative index: by setting $k = 0$ we obtain

$$(a)_{-n} = \frac{(-1)^n}{(1-a)_n}, \qquad n = 1,2,3,\ldots \tag{8.4}$$

Like the binomial coefficient, the Pochhammer symbol plays a very important role in combinatorial problems, probability theory, and algorithm development. In developing certain relations it is more convenient to use the Pochhammer symbol than it is to use the binomial coefficient. The use of this symbol (and the hypergeometric function) in the evaluation of certain series and combinatorial relations is illustrated in Section 8.5.

The Pochhammer symbol and binomial coefficient are related directly by the formula given in part (3) of Theorem 8.1. A more complex relation between these symbols is developed in the next example.

Example 1: Based on the properties of the Pochhammer symbol listed in Theorem 8.1, show that

$$\binom{a+k-1}{n} = \frac{(-1)^n(1-a)_n(a)_k}{n!(a-n)_k}, \qquad k = 1,2,3,\ldots$$

Solution: From (3) and (5) of Theorem 8.1, we first obtain

$$\binom{a+k-1}{n} = \frac{(-1)^n}{n!}(1-a-k)_n$$
$$= \frac{(a)_k}{n!(a)_{k-n}}$$

Replacing n by $-n$ in part (4) of Theorem 8.1, we find

$$\begin{aligned}(a)_{k-n} &= (a)_{-n}(a-n)_k \\ &= \frac{(-1)^n(a-n)_k}{(1-a)_n}\end{aligned}$$

where the last step is a consequence of Equation (8.4). Combining the above results leads to the desired relation

$$\binom{a+k-1}{n} = \frac{(-1)^n(1-a)_n(a)_k}{n!(a-n)_k}$$

EXERCISES 8.2

In problems 1–16, verify the identity.

1. $(-n)_n = (-1)^n n!$.

2. $(a-n)_n = (-1)^n(1-a)_n$.

3. $(a)_{n+1} = a(a+1)_n$.

4. $(a)_{n+k} = (a)_k(a+k)_n$.

5. $(a+1)_n - n(a+1)_{n-1} = (a)_n$.

6. $(a-1)_n + n(a)_{n-1} = (a)_n$.

7. $(n+k)! = n!(n+1)_k$.

8. $\Gamma(a+1-n) = \dfrac{(-1)^n\Gamma(a+1)}{(-a)_n}$.

9. $(a+n)_{k-n}(a+k)_{n-k} = 1$.

10. $(a+k)_{n-k} = (-1)^{n-k}(1-a-n)_{n-k}$.

11. $(a)_{k-n} = \dfrac{(-1)^n(a)_k}{(1-a-k)_n}$.

12. $(a)_{2n} = 2^{2n}(\frac{1}{2}a)_n(\frac{1}{2}+\frac{1}{2}a)_n$.

13. $(2n)! = 2^{2n}(\frac{1}{2})_n n!$.

14. $(2n+1)! = 2^{2n}(\frac{3}{2})_n n!$.

15. $\dbinom{2a}{2n} = \dfrac{(-a)_n(\frac{1}{2}-a)_n}{(\frac{1}{2})_n n!}$.

16. $(a)_{3n} = 3^{3n}(\frac{1}{3}a)_n(\frac{1}{3}+\frac{1}{3}a)_n(\frac{2}{3}+\frac{1}{3}a)_n$.

17. Show that $(k = 1, 2, 3, \ldots)$

$$(-k)_n = \begin{cases} \dfrac{(-1)^n k!}{(k-n)!}, & 0 \le n \le k \\ 0, & n > k \end{cases}$$

18. Show that

$$(1+n)_n = 4^n\left(\tfrac{1}{2}\right)_n$$

19. Show that

$$\binom{n+a-1}{n} = \frac{(a)_n}{n!}$$

8.3 *The Function $F(a, b; c; x)$*

The series defined by*

$$1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!} \tag{8.5}$$

is called the *hypergeometric series*. It gets its name from the fact that for $a = 1$ and $c = b$ the series reduces to the elementary *geometric series*

$$1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \tag{8.6}$$

Denoting the general term of (8.5) by $u_n(x)$ and applying the ratio test, we see that

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| &= \lim_{n\to\infty}\left|\frac{(a)_{n+1}(b)_{n+1}x^{n+1}}{(c)_{n+1}(n+1)!}\cdot\frac{(c)_n n!}{(a)_n(b)_n x^n}\right| \\ &= |x|\lim_{n\to\infty}\left|\frac{(a+n)(b+n)}{(c+n)(n+1)}\right|\end{aligned}$$

where we have made use of property (4) of Theorem 8.1. Completing the limit process reveals that

$$\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = |x| \tag{8.7}$$

under the assumption that none of a, b, or c is zero or a negative integer. Therefore, we conclude that the series (8.5) converges under these circumstances for all $|x| < 1$ and diverges for all $|x| > 1$. For $|x| = 1$, it can be shown that a sufficient condition for convergence of the series is $c - a - b > 0$.†

*Throughout our discussion the parameters a, b, c are assumed to be real.
†See E.D. Rainville, *Special Functions*, New York: Chelsea, 1971, p. 46.

The function

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \qquad |x| < 1 \tag{8.8}$$

defined by the hypergeometric series is called the *hypergeometric function*. It is also commonly denoted by the symbol

$${}_2F_1(a,b;c;x) \equiv F(a,b;c;x) \tag{8.9}$$

where the 2 and 1 refer to the number of numerator and denominator parameters, respectively, in its series representation. The semicolons separate the numerator parameters a and b (which are themselves separated by a comma), the denominator parameter c, and the argument x.

If c is zero or a negative integer, the series (8.8) generally does not exist, and hence the function $F(a,b;c;x)$ is not defined. However, if either a or b (or both) is zero or a negative integer, the series is finite and thus converges for *all* x. That is, if $a = -m$ $(m = 0, 1, 2, \ldots)$ then $(-m)_n = 0$ when $n > m$, and in this case (8.8) reduces to the *hypergeometric polynomial* defined by

$$F(-m,b;c;x) = \sum_{n=0}^{m} \frac{(-m)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \qquad -\infty < x < \infty \tag{8.10}$$

8.3.1 *Elementary Properties*

There are several properties of the hypergeometric function that are immediate consequences of its definition (8.8). First, we note the *symmetry property* of the parameters a and b, i.e.,

$$F(a,b;c;x) = F(b,a;c;x). \tag{8.11}$$

Second, by differentiating the series (8.8) termwise, we find that

$$\begin{aligned}\frac{d}{dx}F(a,b;c;x) &= \underbrace{\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^{n-1}}{(n-1)!}}_{n \to n+1} \\ &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{x^n}{n!} \\ &= \frac{ab}{c}\sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n}\frac{x^n}{n!}\end{aligned}$$

and hence,

$$\frac{d}{dx}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x) \tag{8.12}$$

Repeated application of (8.12) leads to the general formula (see problem 1)

$$\frac{d^k}{dx^k}F(a,b;c;x) = \frac{(a)_k(b)_k}{(c)_k}F(a+k,b+k;c+k;x),$$
$$k = 1,2,3,\ldots \quad (8.13)$$

The parameters a, b, and c in the definition of the hypergeometric function play much the same role in the relationships of this function in that the parameters n or p did for the Legendre polynomials and Bessel functions. The usual nomenclature for the hypergeometric functions in which one parameter changes by $+1$ or -1 is "contiguous functions." There are six contiguous functions, defined by $F(a \pm 1, b; c; x)$, $F(a, b \pm 1; c; x)$, and $F(a, b; c \pm 1; x)$. Gauss was the first to show that between $F(a, b; c; x)$ and any two contiguous functions there exists a linear relation with coefficients at most linear in x. The six contiguous functions, taken two at a time, lead to a total of fifteen recurrence relations of this kind, i.e., $\binom{6}{2} = 15$.*

In order to derive one of the fifteen recurrence relations, we first observe that

$$\begin{aligned}
&x\frac{d}{dx}F(a,b;c;x) + aF(a,b;c;x) \\
&\quad = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{nx^n}{n!} + \sum_{n=0}^{\infty} \frac{a(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \\
&\quad = \sum_{n=0}^{\infty} \frac{(a+n)(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \\
&\quad = a\sum_{n=0}^{\infty} \frac{(a+1)_n(b)_n}{(c)_n}\frac{x^n}{n!}
\end{aligned}$$

from which we deduce

$$x\frac{d}{dx}F(a,b;c;x) + aF(a,b;c;x) = aF(a+1,b;c;x) \quad (8.14)$$

Similarly, from the symmetry property (8.11),

$$x\frac{d}{dx}F(a,b;c;x) + bF(a,b;c;x) = bF(a,b+1;c;x) \quad (8.15)$$

and by subtracting (8.15) from (8.14), it follows at once that

$$(a-b)F(a,b;c;x) = aF(a+1,b;c;x) - bF(a,b+1;c;x) \quad (8.16)$$

---

*For a listing of all 15 relations, see A. Erdelyi et al., *Higher Transcendental Functions*, Vol. I, New York: McGraw-Hill, 1953, pp. 103–104.

which is one of the simplest recurrence relations involving the contiguous functions. Some of the other recurrence relations are taken up in the exercises.

8.3.2 *Integral Representation*

To derive an integral representation for the hypergeometric function, we start with the beta-function relation (see Section 2.3)

$$B(n+b, c-b) = \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt, \qquad c > b > 0 \quad (8.17)$$

from which we deduce (for $n = 0, 1, 2, \dots$)

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt \quad (8.18)$$

The substitution of (8.18) into (8.8) yields

$$\begin{aligned} F(a,b;c;x) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\left(\sum_{n=0}^{\infty} \frac{(a)_n}{n!}(xt)^n\right) dt \end{aligned} \quad (8.19)$$

where we have reversed the order of integration and summation. Now, using the relation (from Theorem 8.1)

$$\frac{(a)_n}{n!} = \binom{-a}{n}(-1)^n \quad (8.20)$$

we recognize the series in (8.19) as a binomial series which has the sum

$$\sum_{n=0}^{\infty} \frac{(a)_n}{n!}(xt)^n = \sum_{n=0}^{\infty} \binom{-a}{n}(-xt)^n = (1-xt)^{-a} \quad (8.21)$$

provided $|xt| < 1$. Hence, (8.19) gives us the integral representation

$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt, \qquad c > b > 0 \quad (8.22)$$

Although (8.22) was derived under the assumption that $|xt| < 1$, it can be shown that the integral converges for all $|x| \le 1$.*. The convergence of (8.22) for $x = 1$ is important in our proof of the following useful theorem.

*See E.D. Rainville, *Special Functions*, New York: Chelsea, 1971, pp. 48–49.

Theorem 8.2. For $c \neq 0, -1, -2, \ldots$ and $c - a - b > 0$,

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

Proof: We will prove the theorem only with the added restriction $c > b > 0$, although it is valid without this restriction. We simply set $x = 1$ in (8.22) to get

$$\begin{aligned} F(a, b; c; 1) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}\, dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\, dt \end{aligned}$$

which, evaluated as a beta integral, yields our result, viz.,

$$\begin{aligned} F(a, b; c; 1) &= \frac{\Gamma(c)\Gamma(b)\Gamma(c-a-b)}{\Gamma(b)\Gamma(c-b)\Gamma(c-a)} \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \end{aligned}$$

■

8.3.3 *The Hypergeometric Equation*

The linear second-order DE

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0 \tag{8.23}$$

is called the *hypergeometric equation* of Gauss. It is so named because the function

$$y_1 = F(a, b; c; x), \qquad c \neq 0, -1, -2, \ldots \tag{8.24}$$

is a solution. To verify that (8.24) is indeed a solution, we can substitute the series for $F(a, b; c; x)$ directly into (8.23).

Examination of the coefficient of y'' reveals that both $x = 0$ and $x = 1$ are (finite) singular points of the equation. Therefore, to find a second series solution about $x = 0$ would normally require use of the Frobenius method.* Under special restrictions on the parameter c, however, we can produce a second (linearly independent) solution of (8.23) without resorting to this more general method. We simply make the change of dependent variable

$$y = x^{1-c}z \tag{8.25}$$

*For an introductory discussion of the Frobenius method, see L.C. Andrews, *Ordinary Differential Equations with Applications*, Glenview, Ill.: Scott, Foresman, 1982, Chapter 9.

from which we calculate

$$y' = x^{1-c}z' + (1-c)x^{-c}z \tag{8.26a}$$

$$y'' = x^{1-c}z'' + 2(1-c)x^{-c}z' - c(1-c)x^{-c-1}z \tag{8.26b}$$

The substitution of (8.25), (8.26a), and (8.26b) into (8.23) leads to (upon algebraic simplification)

$$x(1-x)z'' + [2 - c - (a + b - 2c + 3)x]z' \\ -(1 + a - c)(1 + b - c)z = 0 \tag{8.27}$$

which we recognize as another form of (8.23). Hence, Equation (8.27) has the solution

$$z = F(1 + a - c, 1 + b - c; 2 - c; x), \qquad c \neq 2, 3, 4, \ldots \tag{8.28}$$

and so we deduce that

$$y_2 = x^{1-c}F(1 + a - c, 1 + b - c; 2 - c; x), \qquad c \neq 2, 3, 4, \ldots \tag{8.29}$$

is a second solution of (8.23). For $c = 2, 3, 4, \ldots,$ the hypergeometric function in (8.29) does not usually exist, while for $c = 1$ the solutions (8.29) and (8.24) are identical. However, if we restrict c to $c \neq 0, \pm 1, \pm 2, \ldots,$ then (8.29) is linearly independent of (8.24) and

$$y = C_1F(a, b; c; x) + C_2x^{1-c}F(1 + a - c, 1 + b - c; 2 - c; x) \tag{8.30}$$

is a general solution of Equation (8.23).

To cover the cases when $c = 2, 3, 4, \ldots,$ a *hypergeometric function of the second kind* can be introduced (see problem 28). However, beyond its connection as a solution to the hypergeometric equation of Gauss, the hypergeometric function of the second kind has limited usefulness in applications.

Remark: Actually, $y_1 = F(a, b; c; x)$ and $y_2 = x^{1-c}F(1 + a - c, 1 + b - c; 2 - c; x)$ are only two of a total of 24 solutions of Equation (8.23) that can be expressed in terms of the hypergeometric function. For a listing of all 24 solutions, see W.W. Bell, *Special Functions for Scientists and Engineers*, London: Van Nostrand, 1968, pp. 208–209.

EXERCISES 8.3

1. Show that (for $k = 1, 2, 3, \dots$)

$$\frac{d^k}{dx^k} F(a, b; c; x) = \frac{(a)_k (b)_k}{(c)_k} F(a + k, b + k; c + k; x)$$

2. Show that (for $k = 1, 2, 3, \dots$)

(a) $\dfrac{d}{dx}[x^a F(a, b; c; x)] = a x^{a-1} F(a + 1, b; c; x)$,

(b) $\dfrac{d^k}{dx^k}\left[x^{a-1+k} F(a, b; c; x)\right] = (a)_k x^{a-1} F(a + k, b; c; x)$,

(c) $\dfrac{d^k}{dx^k}\left[x^{c-1} F(a, b; c; x)\right] = (c - k)_k x^{c-1-k} F(a, b; c - k; x)$.

In problems 3–6, verify the differentiation formula.

3. $x \dfrac{d}{dx} F(a, b; c; x) + (1 - c) F(a, b; c; x) = a F(a + 1, b; c; x)$
$+ (1 - c) F(a, b; c - 1; x)$.

4. $x \dfrac{d}{dx} F(a - 1, b; c; x) = (a - 1) F(a, b; c; x)$
$-(a - 1) F(a - 1, b; c; x)$.

5. $(1 - x) x \dfrac{d}{dx} F(a, b; c; x) = (a + b - c) x F(a, b; c; x)$
$+ c^{-1}(c - a)(c - b) x F(a, b; c + 1; x)$.

6. $x \dfrac{d}{dx} F(a - 1, b; c; x) = (a - 1) x F(a, b; c; x)$
$- c^{-1}(a - 1)(c - b) x F(a, b; c + 1; x)$.

In problems 7–13, verify the given contiguous relation by using the results of problems 3–6, or by series representations.

7. $(b - a)(1 - x) F(a, b; c; x) = (c - a) F(a - 1, b; c; x)$
$-(c - b) F(a, b - 1; c; x)$.

8. $(1 - x) F(a, b; c; x) = F(a - 1, b; c; x)$
$- c^{-1}(c - b) x F(a, b; c + 1; x)$.

9. $(1 - x) F(a, b; c; x) = F(a, b - 1; c; x)$
$- c^{-1}(c - a) x F(a, b; c + 1; x)$.

10. $(c - a - b) F(a, b; c; x) + a(1 - x) F(a + 1, b; c; x)$
$= (c - b) F(a, b - 1; c; x)$.

11. $(c-a-b)F(a,b;c;x)+b(1-x)F(a,b+1;c;x)$
$=(c-a)F(a-1,b;c;x).$

12. $(c-b-1)F(a,b;c;x)+bF(a,b+1;c;x)$
$=(c-1)F(a,b;c-1;x).$

13. $[2b-c+(a-b)x]F(a,b;c;x)=b(1-x)F(a,b+1;c;x)$
$-(c-b)F(a,b-1;c;x).$

In problems 14 and 15, verify the formula by direct substitution of the series representations.

14. $F(a,b+1;c;x)-F(a,b;c;x)=\dfrac{ax}{c}F(a+1,b+1;c+1;x).$

15. $F(a,b;c;x)-F(a,b;c-1;x)$
$$=-\frac{abx}{c(c-1)}F(a+1,b+1;c+1;x).$$

In problems 16 and 17, use termwise integration to derive the given integral representation.

16. $F(a,b;c;x)=\dfrac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)}\displaystyle\int_0^1 t^{d-1}(1-t)^{c-d-1}F(a,b;d;xt)\,dt,$
$c>d>0.$

17. $F(a,b;c+1;x)=c\displaystyle\int_0^1 F(a,b;c;xt)t^{c-1}\,dt,\ c>0.$

18. Show that $(s>0)$

(a) $\displaystyle\int_0^\infty e^{-st}F[a,b;1;x(1-e^{-t})]\,dt=\frac{1}{s}F(a,b;s+1;x),$

(b) $\displaystyle\int_0^\infty e^{-st}F(a,b;1;1-e^{-t})\,dt=\frac{\Gamma(s)\Gamma(s+1-a-b)}{\Gamma(s+1-a)\Gamma(s+1-b)}.$

Hint: Set $x=1$ in (a).

19. Show that (for $n=0,1,2,\ldots$)

(a) $F(-n,b;c;1)=\dfrac{(c-b)_n}{(c)_n},$

(b) $F(-n,a+n;c;1)=(-1)^n\dfrac{(1+a-c)_n}{(c)_n},$

(c) $F(-n,1-b-n;c;1)=\dfrac{(c+b-1)_{2n}}{(c)_n(c+b-1)_n}.$

20. Show that

$$F(-\tfrac{1}{2}n,\tfrac{1}{2}-\tfrac{1}{2}n;b+\tfrac{1}{2};1)=\frac{2^n(b)_n}{(2b)_n},\qquad n=0,1,2,\ldots$$

21. Using the result of problem 19(a), show that (for $p = 0, 1, 2, \ldots$)

(a) $$F(-p, a+n+1; a+1; 1) = \begin{cases} 0, & 0 \le n \le p-1, \\ \dfrac{(-1)^p p!}{(a+1)_p}, & n = p, \end{cases}$$

(b) $$F(-p, a+n+2; a+1; 1) = \begin{cases} 0, & 0 \le n \le p-2, \\ \dfrac{(-1)^p (n+1)!}{(a+1)_p (n+1-p)!}, & n = p-1, p. \end{cases}$$

22. Given the generating function

$$w(x,t) = (1-t)^{b-c}(1-t+xt)^{-b}, \qquad c \ne 0, -1, -2, \ldots$$

show that

$$w(x,t) = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} F(-n, b; c; x) t^n$$

where $F(-n, b; c; x)$ denotes the *hypergeometric polynomials* defined by Equation (8.10).

23. Show that, for $|x| < 1$ and $|x/(1-x)| < 1$,

$$(1-x)^{-a} F\left(a, c-b; c; \frac{-x}{1-x}\right) = F(a, b; c; x)$$

24. By substituting $y = x/(x-1)$ in problem 23, deduce that

(a) $F(a, c-b; c; x) = (1-x)^{b-c} F\left(c-a, c-b; c; \dfrac{-x}{1-x}\right)$,

(b) $F\left(a, c-b; c; \dfrac{-x}{1-x}\right) = (1-x)^{c-b} F(c-a, c-b; c; x)$,

(c) $F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x)$.

25. Show that

$$(1-x)^{-a} F\left[\tfrac{1}{2}a, \tfrac{1}{2} + \tfrac{1}{2}a - b; 1 + a - b; \frac{-4x}{(1-x)^2}\right] = F(a, b; 1 + a - b; x)$$

26. Use problems 23–25 to deduce that

(a) $$F(a, b; 1+a-b; -1) = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)},$$

(b) $$F(a, 1-a; c; \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2}c)\Gamma(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}c)},$$

(c) $$F(2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}) = \frac{\Gamma(a+b+\frac{1}{2})\sqrt{\pi}}{\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}.$$

27. By assuming a power-series solution of the form

$$y = \sum_{n=0}^{\infty} A_n x^n$$

show that $y = F(a, b; c; x)$ is a solution of the hypergeometric equation

$$x(1 - x)y'' + [c - (a + b + 1)x]y' - aby = 0$$

28. The *hypergeometric function of the second kind* is defined by

$$G(a, b; c; x) = \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} F(a, b; c; x) + \frac{\Gamma(c - 1)}{\Gamma(a)\Gamma(b)} x^{1-c} F(1 + a - c, 1 + b - c; 2 - c; x)$$

(a) Show that $G(a, b; c; x)$ is a solution of the hypergeometric equation in problem 27, $c \neq 0, \pm 1, \pm 2, \ldots$.
(b) Show that $G(a, b; c; x) = x^{1-c} G(1 + a - c, 1 + b - c; 2 - c; x)$.

29. Show that the Wronskian of $F(a, b; c; x)$ and $G(a, b; c; x)$ is given by (see problem 28)

$$W(F, G)(x) = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} x^{-c}(1 - x)^{c-a-b-1}$$

30. Derive the generating function relation

$$(1 - xt)^{-a} F\left[\tfrac{1}{2}a, \tfrac{1}{2}a + \tfrac{1}{2}; 1, \frac{t^2(x^2 - 1)}{(1 - xt)^2}\right] = \sum_{n=0}^{\infty} \frac{(a)_n P_n(x)}{n!} t^n$$

where $P_n(x)$ is the nth Legendre polynomial.

8.4 *Relation to Other Functions*

The hypergeometric function is important in many areas of mathematical analysis and its applications. Partly this is a consequence of the fact that so many elementary and special functions are simply special cases of the hypergeometric function. For example, the specialization

$$F(1, b; b; x) = \sum_{n=0}^{\infty} \frac{(1)_n}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

reveals that

$$F(1, b; b; x) = (1 - x)^{-1} \tag{8.31}$$

Similarly, it can be established that

$$\arcsin x = xF\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; x^2\right) \tag{8.32}$$

and

$$\log(1 - x) = -xF(1, 1; 2; x) \tag{8.33}$$

Example 2: Show that $\arcsin x = xF(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2)$.

Solution: From the calculus, we recall

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!x^{2n+1}}{2^{2n}(n!)^2(2n + 1)}$$

In order to recognize this series as a hypergeometric series, we need to express the coefficient of $x^{2n+1}/n!$ in terms of Pochhammer symbols. Thus, using the results of problems 13 and 14 in Exercises 8.2, we have

$$(2n)! = 2^{2n}\left(\tfrac{1}{2}\right)_n n!$$

$$(2n + 1) = \frac{(2n + 1)!}{(2n)!} = \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_n}$$

and making these substitutions leads to

$$\arcsin x = x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} \frac{x^{2n}}{n!}$$

from which we deduce

$$\arcsin x = xF\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; x^2\right)$$

The verification of (8.33) along with several other such relations involving elementary functions is left to the exercises.

A more involved relationship to establish is given by

$$P_n(x) = F\left(-n, n + 1; 1; \frac{1 - x}{2}\right) \tag{8.34}$$

where $P_n(x)$ is the nth Legendre polynomial. To prove (8.34), we first observe that

$$\begin{aligned}(1 - 2xt + t^2)^{-1/2} &= \left[(1 - t)^2 - 2t(x - 1)\right]^{-1/2} \\ &= (1 - t)^{-1}\left[1 - \frac{2t(x - 1)}{(1 - t)^2}\right]^{-1/2} \qquad (8.35)\end{aligned}$$

and thus we deduce the relation

$$\sum_{n=0}^{\infty} P_n(x)t^n = (1 - 2xt + t^2)^{-1/2}$$

$$= (1 - t)^{-1}\left[1 - \frac{2t(x - 1)}{(1 - t)^2}\right]^{-1/2}$$

$$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{(-1)^k(2t)^k(x - 1)^k}{(1 - t)^{2k+1}} \tag{8.36}$$

Our object now is to recognize the right-hand side of (8.36) as a power series in t which has the coefficient $F(-n, n + 1; 1; (1 - x)/2)$. To obtain powers of t, we further expand $(1 - t)^{-2k-1}$ in a binomial series and interchange the order of summation. Hence,

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k}\binom{-2k - 1}{m}(-1)^{m+k}2^k(x - 1)^k t^{k+m}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k}\binom{-2k - 1}{n - k}(-1)^n 2^k(x - 1)^k t^n \tag{8.37}$$

where the last step is a result of the index change $m = n - k$. Next, from part (3) of Theorem 8.1, we can write

$$\binom{-\frac{1}{2}}{k}\binom{-2k - 1}{n - k} = \frac{(-1)^n(\frac{1}{2})_k(2k + 1)_{n-k}}{k!(n - k)!} \tag{8.38}$$

but from problems 7 and 13 in Exercises 8.2, we further have

$$(2k + 1)_{n-k} = \frac{(n + k)!}{(2k)!} = \frac{n!(n + 1)_k}{2^{2k}(\frac{1}{2})_k k!} \tag{8.39}$$

Finally, setting $a = 1$ in problem 11 in Exercises 8.2 leads to

$$(n - k)! = \frac{(-1)^k n!}{(-n)_k} \tag{8.40}$$

so by combining the results of (8.38), (8.39), and (8.40), we find that (8.37) becomes

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{(-n)_k(n + 1)_k}{(1)_k k!}\left(\frac{1 - x}{2}\right)^k\right]t^n$$

$$= \sum_{n=0}^{\infty} F\left(-n, n + 1; 1; \frac{1 - x}{2}\right)t^n \tag{8.41}$$

from which (8.34) follows.

8.4.1 *Legendre Functions*

The relation (8.34) between the nth Legendre polynomial and hypergeometric function provides us with a natural way of introducing the more general function

$$P_\nu(x) = F\left(-\nu, \nu + 1; 1; \frac{1 - x}{2}\right) \tag{8.42}$$

where ν is not restricted to integer values. We call $P_\nu(x)$ a *Legendre function of the first kind* of degree ν; it is not a polynomial except in the special case when $\nu = n$ $(n = 0, 1, 2, \ldots)$. A *Legendre function of the second kind*, denoted by $Q_\nu(x)$, can also be defined in terms of the hypergeometric function, although we will not discuss it.*

The function $P_\nu(x)$ has many properties in common with the Legendre polynomial $P_n(x)$. For example, by setting $x = 1$ in (8.42), we obtain

$$P_\nu(1) = F(-\nu, \nu + 1; 1; 0) = 1 \tag{8.43}$$

The substitution of $x = 0$ in (8.42) leads to

$$P_\nu(0) = F(-\nu, \nu + 1; 1; \tfrac{1}{2}) \tag{8.44}$$

and by using the relation [see problem 26(b) in Exercises 8.3]

$$F(a, 1 - a; c; \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}c + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2}c)\Gamma(\frac{1}{2} - \frac{1}{2}a + \frac{1}{2}c)} \tag{8.45}$$

we deduce that

$$P_\nu(0) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2} - \frac{1}{2}\nu)\Gamma(\frac{1}{2}\nu + 1)} \tag{8.46}$$

Recalling the identity

$$\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x} \tag{8.47}$$

we can express (8.46) in the alternative form

$$P_\nu(0) = \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\frac{1}{2}\nu + 1)}\cos(\tfrac{1}{2}\nu\pi) \tag{8.48}$$

When ν is a nonnegative integer, we find that (8.48) reduces to the results that we previously derived for the Legendre polynomials (see problem 22).

Various recurrence formulas for $P_\nu(x)$ can be derived by expressing this function in its series representation, or by using properties of the hypergeo-

*See T.M. MacRobert, *Spherical Harmonics*, Oxford: Pergamon, 1967, Chapter VI.

metric function. For example, it can be verified that

$$(\nu+1)P_{\nu+1}(x)-(2\nu+1)xP_\nu(x)+\nu P_{\nu-1}(x)=0 \tag{8.49}$$

$$P'_{\nu+1}(x)-xP'_\nu(x)=(\nu+1)P_\nu(x) \tag{8.50}$$

$$xP'_\nu(x)-P'_{\nu-1}(x)=\nu P_\nu(x) \tag{8.51}$$

and so forth.

The Legendre functions $P_\nu(x)$ are important for theoretical purposes in the general study of spherical harmonics. Their properties are important also from a more practical point of view, since these functions are prominent in solving Laplace's equation in various coordinate systems, such as toroidal coordinates.*

EXERCISES 8.4

In problems 1–8, compare series to deduce the result.

1. $1 = F(0, b; c; x)$.

2. $(1-x)^{-a} = F(a, b; b; x)$.

3. $\log(1-x) = -xF(1, 1; 2; x)$.

4. $\log\dfrac{1+x}{1-x} = 2xF(\frac{1}{2}, 1; \frac{3}{2}; x^2)$.

5. $\arctan x = xF(\frac{1}{2}, 1; \frac{3}{2}; x^2)$.

6. $(1+x)(1-x)^{-2a-1} = F(2a, a+1; a; x)$.

7. $\frac{1}{2}(1+\sqrt{x})^{-2a} + \frac{1}{2}(1-\sqrt{x})^{-2a} = F(a, a+\frac{1}{2}; \frac{1}{2}; x)$.

8. $\left[\dfrac{1+\sqrt{1-x}}{2}\right]^{1-2a} = F(a-\frac{1}{2}, a; 2a; x)$.

9. Show that

$$K(x) = \tfrac{1}{2}\pi F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; x^2\right)$$

where $K(x)$ is the *complete elliptic integral of the first kind* defined by

$$K(x) = \int_0^{\pi/2} (1 - x^2\sin^2\phi)^{-1/2}\, d\phi$$

10. Show that

$$E(x) = \tfrac{1}{2}\pi F\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1; x^2\right)$$

*See Chapter 8 in N.N. Lebedev, *Special Functions and Their Applications*, Dover, 1972.

where $E(x)$ is the *complete elliptic integral of the second kind* defined by

$$E(x) = \int_0^{\pi/2} (1 - x^2 \sin^2\phi)^{1/2}\, d\phi$$

11. Show that the *associated Legendre functions*

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

satisfy the relation

$$P_n^m(x) = \frac{(n+m)!}{2^m (n-m)! m!} (1 - x^2)^{m/2} \times F\left(m - n, m + n + 1; m + 1; \frac{1-x}{2}\right)$$

12. Show that the *Chebyshev polynomials of the first kind*

$$T_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}, \quad n \geq 1$$

satisfy the relation

$$T_n(x) = F\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right)$$

13. Show that the *Chebyshev polynomials of the second kind*

$$U_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k (2x)^{n-2k}$$

satisfy the relation

$$U_n(x) = (n+1) F\left(-n, n + 2; \tfrac{3}{2}; \frac{1-x}{2}\right)$$

14. Show that the *Gegenbauer polynomials*

$$C_n^\lambda(x) = \sum_{k=0}^{[n/2]} \binom{-\lambda}{n-k} \binom{n-k}{k} (-1)^n (2x)^{n-2k}$$

satisfy the relations

(a) $C_{2n}^\lambda(x) = (-1)^n \dfrac{(\lambda)_n}{n!} F(-n, \lambda + n; \tfrac{1}{2}; x^2)$,

(b) $C_{2n+1}^\lambda(x) = (-1)^n \dfrac{(\lambda)_{n+1}}{n!} F(-n, \lambda + n; \tfrac{3}{2}; x^2)$,

(c) $C_n^\lambda(x) = \dbinom{n + 2\lambda - 1}{n} F\left(-n, 2\lambda + n; \lambda + \tfrac{1}{2}; \dfrac{1-x}{2}\right)$.

24. Show that $P_\nu(x) = P_{-\nu-1}(x)$.

Hint: Recall that $F(a, b; c; x) = F(b, a; c; x)$.

25. By making the substitution $x = 1 - 2z$ in the generalized form of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \nu(\nu + 1)y = 0$$

show that it transforms to Gauss' hypergeometric equation and thus deduce that $y = F\left(-\nu, \nu + 1; 1; \dfrac{1 - x}{2}\right)$ is one solution of the generalized Legendre equation.

26. Show that

$$\frac{1}{k!} = \frac{2}{\pi\left(\frac{1}{2}\right)_k} \int_0^{\pi/2} \sin^{2k}\phi \, d\phi, \qquad k = 0, 1, 2, \ldots$$

and then, by expressing $P_\nu(x)$ in its series representation (problem 18), deduce that

$$P_\nu(x) = \frac{2}{\pi} \int_0^{\pi/2} F\left(-\nu, \nu + 1; \tfrac{1}{2}; \frac{1 - x}{2} \sin^2\phi\right) d\phi$$

Legendre Polynomials and Related Functions

4.1 *Introduction*

The *Legendre polynomials* are closely associated with physical phenomena for which spherical geometry is important. In particular, these polynomials first arose in the problem of expressing the Newtonian potential of a conservative force field in an infinite series involving the distance variables of two points and their included central angle (see Section 4.2). Other similar problems dealing with either gravitational potentials or electrostatic potentials also lead to Legendre polynomials, as do certain steady-state heat-conduction problems in spherical-shaped solids, and so forth.

There exists a whole class of polynomial sets which have many properties in common, and for which the Legendre polynomials represent the simplest example. Each polynomial set satisfies several recurrence formulas, is involved in numerous integral relationships, and forms the basis for series expansions resembling Fourier trigonometric series where the sines and cosines are replaced by members of the polynomial set. Because of all the similarities in these polynomial sets, and because the Legendre polynomials are the simplest such set, our development of the properties associated with the Legendre polynomials will be more extensive than similar developments in Chapter 5, where we introduce other polynomial sets.

In addition to the Legendre polynomials, we will present a brief discussion of the *Legendre functions of the second kind* and *associated Legendre functions*. The Legendre functions of the second kind arise as a second solution set of Legendre's differential equation, and the associated functions are related to derivatives of the Legendre polynomials.

4.2 *The Generating Function*

Among other areas of application, the subject of potential theory is concerned with the forces of attraction due to the presence of a gravitational field. Central to the discussion of problems of gravitational attraction is *Newton's law of universal gravitation*:

> "Every particle of matter in the universe attracts every other particle with a force whose direction is that of the line joining the two, and whose magnitude is directly as the product of their masses and inversely as the square of their distance from each other."

The force field generated by a single particle is usually considered to be *conservative*. That is, there exists a potential function V such that the gravitational force $\boldsymbol{F}$ at a point of free space (i.e., free of point masses) is related to the potential function according to

$$\boldsymbol{F} = -\nabla V \tag{4.1}$$

where the minus sign is conventional. If r denotes the distance between a point mass and a point of free space, the potential function can be shown to have the form*

$$V(r) = \frac{k}{r} \tag{4.2}$$

where k is a constant whose numerical value does not concern us. Because of spherical symmetry of the gravitational field, the potential function V depends only upon the radial distance r.

Valuable information on the properties of potentials like (4.2) may be inferred from developments of the potential function into power series of certain types. In 1785, A.M. Legendre published his "Sur l'attraction des sphéroïdes," in which he developed the gravitational potential (4.2) in a power series involving the ratio of two distance variables. He found that the coefficients appearing in this expansion were polynomials that exhibited interesting properties.

In order to obtain Legendre's results, let us suppose that a particle of mass m is located at point P, which is a units from the origin of our

*See O.D. Kellogg, *Foundations of Potential Theory*, New York: Dover, 1953, Chapter III.

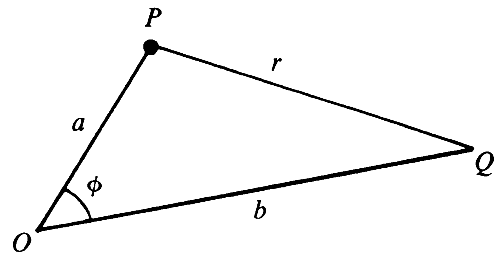


Figure 4.1

coordinate system (see Fig. 4.1). Let the point Q represent a point of free space r units from P and b units from the origin O. For the sake of definiteness, let us assume $b > a$. Then, from the law of cosines, we find the relation

$$r^2 = a^2 + b^2 - 2ab\cos\phi \tag{4.3}$$

where ϕ is the central angle between the rays $\overline{OP}$ and $\overline{OQ}$. By rearranging the terms and factoring out b^2, it follows that

$$r^2 = b^2\left[1 - 2\frac{a}{b}\cos\phi + \left(\frac{a}{b}\right)^2\right], \qquad a < b \tag{4.4}$$

For notational simplicity, we introduce the parameters

$$t = \frac{a}{b}, \qquad x = \cos\phi \tag{4.5}$$

and thus, upon taking the square root,

$$r = b(1 - 2xt + t^2)^{1/2} \tag{4.6}$$

Finally, the substitution of (4.6) into (4.2) leads to the expression

$$V = \frac{k}{b}(1 - 2xt + t^2)^{-1/2}, \qquad 0 < t < 1 \tag{4.7}$$

for the potential function. For reasons that will soon be clear, we refer to the function $w(x, t) = (1 - 2xt + t^2)^{-1/2}$ as the *generating function* of the Legendre polynomials. Our task at this point is to develop $w(x, t)$ in a power series in the variable t.

4.2.1 *Legendre Polynomials*

From Example 6 in Section 1.3.2, we recall the binomial series

$$(1 - u)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}(-1)^n u^n, \qquad |u| < 1 \tag{4.8}$$

Hence, by setting $u = t(2x - t)$, we find that

$$\begin{aligned} w(x,t) &= (1 - 2xt + t^2)^{-1/2} \\ &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^n (2x - t)^n \end{aligned} \tag{4.9}$$

which is valid for $|2xt - t^2| < 1$. For $|t| < 1$, it follows that $|x| \le 1$. The factor $(2x - t)^n$ is simply a finite binomial series, and thus (4.9) can further be expressed as

$$w(x,t) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n t^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k (2x)^{n-k} t^k$$

or

$$w(x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{-\frac{1}{2}}{n} \binom{n}{k} (-1)^{n+k} (2x)^{n-k} t^{n+k} \tag{4.10}$$

Since our goal is to obtain a power series involving powers of t to a single index, the change of indice $n \to n - k$ is suggested. Thus, recalling Equation (1.18) in Section 1.2.3, i.e.,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k,k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A_{n-2k,k}$$

we see that (4.10) can be written in the equivalent form

$$w(x,t) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{[n/2]} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} (-1)^n (2x)^{n-2k} \right\} t^n \tag{4.11}$$

The innermost summation in (4.11) is of finite length and therefore represents a polynomial in x, which happens to be of degree n. If we denote this polynomial by the symbol

$$P_n(x) = \sum_{k=0}^{[n/2]} \binom{-\frac{1}{2}}{n-k} \binom{n-k}{k} (-1)^n (2x)^{n-2k} \tag{4.12}$$

then (4.11) leads to the intended result

$$w(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n, \qquad |x| \le 1, \quad |t| < 1 \tag{4.13}$$

where $w(x,t) = (1 - 2xt + t^2)^{-1/2}$.

The polynomials $P_n(x)$ are called the *Legendre polynomials* in honor of their discoverer. By recognizing that [see Equation (1.27) in Section 1.2 and

Equation (2.25) in Section 2.2.2]

$$\begin{aligned}\binom{-\frac{1}{2}}{n} &= (-1)^n\binom{n-\frac{1}{2}}{n}\\ &= (-1)^n\frac{\Gamma(n+\frac{1}{2})}{n!\Gamma(\frac{1}{2})}\\ &= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\end{aligned} \tag{4.14}$$

it follows that the product of binomial coefficients in (4.12) is

$$\binom{-\frac{1}{2}}{n-k}\binom{n-k}{k} = \frac{(-1)^{n-k}(2n-2k)!}{2^{2n-2k}(n-k)!k!(n-2k)!} \tag{4.15}$$

and hence, (4.12) becomes

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!x^{n-2k}}{2^n k!(n-k)!(n-2k)!} \tag{4.16}$$

The first few Legendre polynomials are listed in Table 4.1.

Making an observation, we note that when n is an even number the polynomial $P_n(x)$ is an even function, and when n is odd the polynomial is an odd function. Therefore,

$$P_n(-x) = (-1)^n P_n(x), \qquad n = 0, 1, 2, \ldots \tag{4.17}$$

The graphs of $P_n(x)$, $n = 0, 1, 2, 3, 4$, are sketched in Fig. 4.2 over the interval $-1 \le x \le 1$.

Returning now to Equation (4.7) with $x = \cos\phi$ and $t = a/b$, we find that the potential function has the series expansion

$$V = \frac{k}{b}\sum_{n=0}^{\infty} P_n(\cos\phi)\left(\frac{a}{b}\right)^n, \qquad a < b \tag{4.18}$$

In terms of the argument $\cos\phi$, the Legendre polynomials can be expressed

Table 4.1 Legendre polynomials

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

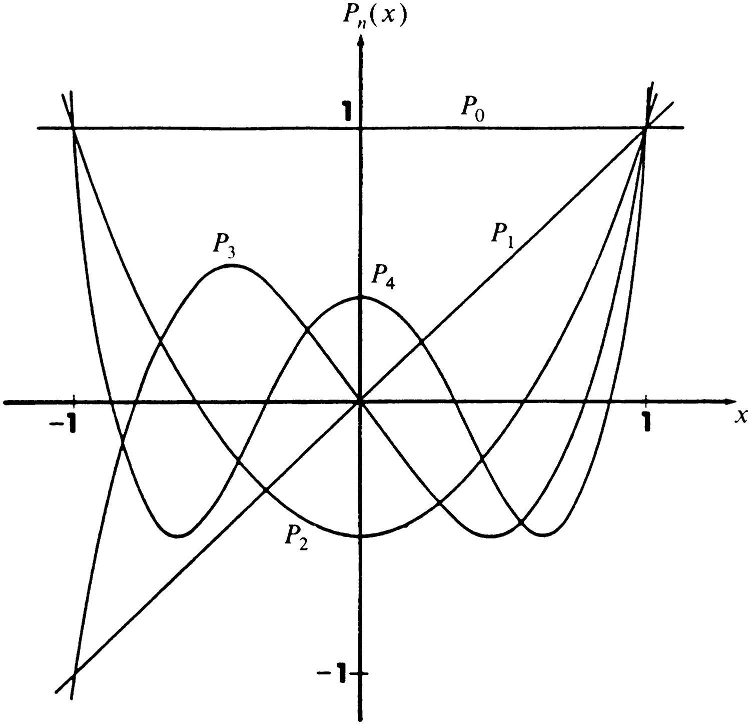


Figure 4.2 Graph of $P_n(x)$, $n = 0, 1, 2, 3, 4$

as trigonometric polynomials of the form shown in Table 4.2 (see problem 3).

In Fig. 4.3 the first few polynomials $P_n(\cos\phi)$ are plotted as a function of the angle ϕ.

4.2.2 *Special Values and Recurrence Formulas*

The Legendre polynomials are rich in recurrence relations and identities. Central to the development of many of these is the *generating-function*

Table 4.2 Legendre trigonometric polynomials.

$$
\begin{aligned}
P_0(\cos\phi) &= 1\\
P_1(\cos\phi) &= \cos\phi\\
P_2(\cos\phi) &= \tfrac{1}{2}(3\cos^2\phi - 1)\\
&= \tfrac{1}{4}(3\cos 2\phi + 1)\\
P_3(\cos\phi) &= \tfrac{1}{2}(5\cos^3\phi - 3\cos\phi)\\
&= \tfrac{1}{8}(5\cos 3\phi + 3\cos\phi)
\end{aligned}
$$

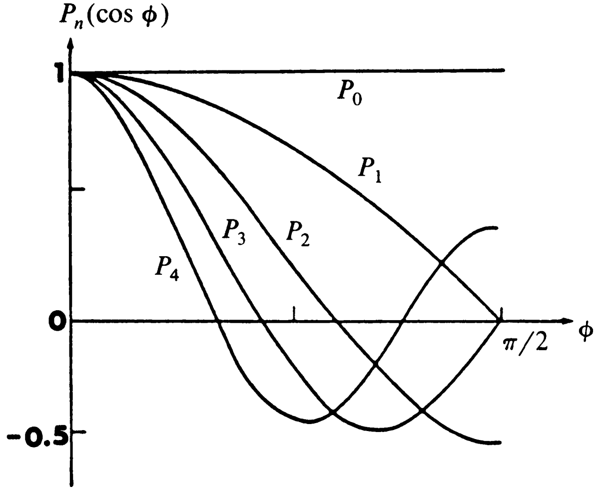


Figure 4.3 Graph of $P_n(\cos\phi)$, $n = 0, 1, 2, 3, 4$

relation

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n, \qquad |x| \le 1, \quad |t| < 1 \tag{4.19}$$

Special values of the Legendre polynomials can be derived directly from (4.19) by substituting particular values for x. For example, the substitution of $x = 1$ yields

$$(1 - 2t + t^2)^{-1/2} = (1 - t)^{-1} = \sum_{n=0}^{\infty} P_n(1)t^n \tag{4.20}$$

However, we recognize that $(1 - t)^{-1}$ is the sum of a geometric series, so that (4.20) is equivalent to

$$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)t^n \tag{4.21}$$

Hence, from the uniqueness theorem of power series (Theorem 1.13), we can compare like coefficients of t^n in (4.21) to deduce the result

$$P_n(1) = 1, \qquad n = 0, 1, 2, \ldots \tag{4.22}$$

Also, from (4.17) we see that

$$P_n(-1) = (-1)^n, \qquad n = 0, 1, 2, \ldots \tag{4.23}$$

The substitution of $x = 0$ into (4.19) leads to

$$(1 + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n \tag{4.24}$$

but the term on the left-hand side has the binomial series expansion

$$(1 + t^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} t^{2n} \tag{4.25}$$

Comparing terms of the series on the right in (4.24) and (4.25), we note that (4.25) has only *even* powers of t. Thus we conclude that $P_n(0) = 0$ for $n = 1, 3, 5, \ldots$, or equivalently,

$$P_{2n+1}(0) = 0, \qquad n = 0, 1, 2, \ldots \tag{4.26}$$

Since all odd terms in (4.24) are zero, we can replace n by $2n$ in the series and compare with (4.25), from which we deduce

$$P_{2n}(0) = \binom{-\frac{1}{2}}{n} = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}, \qquad n = 0, 1, 2, \ldots \tag{4.27}$$

where we are recalling (4.14).

Remark: Actually, (4.26) could have been deduced from the fact that $P_{2n+1}(x)$ is an odd (continuous) function, and therefore must necessarily pass through the origin. (Why?)

In order to obtain the desired recurrence relations, we first make the observation that the function $w(x, t) = (1 - 2xt + t^2)^{-1/2}$ satisfies the derivative relation

$$(1 - 2xt + t^2)\frac{\partial w}{\partial t} + (t - x)w = 0 \tag{4.28}$$

Direct substitution of the series (4.13) for $w(x, t)$ into (4.28) yields

$$(1 - 2xt + t^2) \sum_{n=0}^{\infty} nP_n(x)t^{n-1} + (t - x) \sum_{n=0}^{\infty} P_n(x)t^n = 0$$

Carrying out the indicated multiplications and simplifying gives us

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - 2x \sum_{n=0}^{\infty} \underbrace{nP_n(x)t^n}_{n \to n-1}$$
$$+ \sum_{n=0}^{\infty} \underbrace{nP_n(x)t^{n+1}}_{n \to n-2} + \sum_{n=0}^{\infty} \underbrace{P_n(x)t^{n+1}}_{n \to n-2} - x \sum_{n=0}^{\infty} \underbrace{P_n(x)t^n}_{n \to n-1} = 0 \tag{4.29}$$

We now wish to change indices so that powers of t are the same in each summation. We accomplish this by leaving the first sum in (4.29) as it is, replacing n with $n - 1$ in the second and last sums, and replacing n with $n - 2$ in the remaining sums; thus, (4.29) becomes

$$\sum_{n=0}^{\infty} nP_n(x)t^{n-1} - 2x\sum_{n=1}^{\infty}(n-1)P_{n-1}(x)t^{n-1} + \sum_{n=2}^{\infty}(n-2)P_{n-2}(x)t^{n-1}$$
$$+ \sum_{n=2}^{\infty} P_{n-2}(x)t^{n-1} - x\sum_{n=1}^{\infty} P_{n-1}(x)t^{n-1} = 0$$

Finally, combining all summations, we have

$$\sum_{n=2}^{\infty}\left[nP_n(x) - 2x(n-1)P_{n-1}(x) + (n-2)P_{n-2}(x)\right.$$
$$\left.+ P_{n-2}(x) - xP_{n-1}(x)\right]t^{n-1} + P_1(x) - xP_0(x) = 0 \quad (4.30)$$

But $P_1(x) - xP_0(x) = x - x = 0$, and the validity of (4.30) demands that the coefficient of t^{n-1} be zero for all x. Hence, after simplification we arrive at

$$nP_n(x) - (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x) = 0, \qquad n = 2, 3, 4, \ldots$$

or, replacing n by $n + 1$, we obtain the more conventional form

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad (4.31)$$

where $n = 1, 2, 3, \ldots$.

We refer to (4.31) as a *three-term recurrence formula*, since it forms a connecting relation between three successive Legendre polynomials. One of the primary uses of (4.31) in computations is to produce higher-order Legendre polynomials from lower-order ones by expressing them in the form

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right)xP_n(x) - \left(\frac{n}{n+1}\right)P_{n-1}(x) \quad (4.32)$$

where $n = 1, 2, 3, \ldots$. In practice, (4.32) is generally preferred to (4.16) in making computer calculations when several polynomials are involved.*

*Actually, to avoid excessive roundoff error in making computer calculations, Equation (4.32) should be rewritten in the form

$$P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x) - \frac{xP_n(x) - P_{n-1}(x)}{n+1}.$$

A relation similar to (4.31) involving derivatives of the Legendre polynomials can be derived in the same fashion by first making the observation that $w(x, t)$ satisfies

$$(1 - 2xt + t^2)\frac{\partial w}{\partial x} - tw = 0 \tag{4.33}$$

where this time the differentiation is with respect to x. Substituting the series for $w(x, t)$ directly into (4.33) leads to

$$(1 - 2xt + t^2)\sum_{n=0}^{\infty} P_n'(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = 0$$

or, after carrying out the multiplications,

$$\sum_{n=0}^{\infty} P_n'(x)t^n - 2x\sum_{n=0}^{\infty} \underbrace{P_n'(x)t^{n+1}}_{n \to n-1} + \sum_{n=0}^{\infty} \underbrace{P_n'(x)t^{n+2}}_{n \to n-2} - \sum_{n=0}^{\infty} \underbrace{P_n(x)t^{n+1}}_{n \to n-1} = 0 \tag{4.34}$$

Next, making an appropriate change of index in each summation, we get

$$\sum_{n=2}^{\infty} \left[P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x) - P_{n-1}(x)\right]t^n = 0 \tag{4.35}$$

where all terms outside this summation add to zero. Thus, by equating the coefficient of t^n to zero in (4.35), we find

$$P_n'(x) - 2xP_{n-1}'(x) + P_{n-2}'(x) - P_{n-1}(x) = 0, \qquad n = 2, 3, 4, \ldots$$

or, by a change of index,

$$P_{n+1}'(x) - 2xP_n'(x) + P_{n-1}'(x) - P_n(x) = 0 \tag{4.36}$$

for $n = 1, 2, 3, \ldots$.

Certain combinations of (4.21) and (4.36) can lead to further recurrence relations. For example, suppose we first differentiate (4.31), i.e.,

$$(n + 1)P_{n+1}'(x) - (2n + 1)P_n(x) - (2n + 1)xP_n'(x) + nP_{n-1}'(x) = 0 \tag{4.37}$$

From (4.36) we find

$$P_{n-1}'(x) = P_n(x) + 2xP_n'(x) - P_{n+1}'(x) \tag{4.38a}$$

$$P_{n+1}'(x) = P_n(x) + 2xP_n'(x) - P_{n-1}'(x) \tag{4.38b}$$

and the successive replacement of $P_{n-1}'(x)$ and $P_{n+1}'(x)$ in (4.37) by (4.38a) and (4.38b) leads to the two relations

$$P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x) \tag{4.39a}$$

$$xP'_n(x) - P'_{n-1}(x) = nP_n(x) \tag{4.39b}$$

The addition of (4.39a) and (4.39b) yields the more symmetric formula

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x) \tag{4.40}$$

Finally, replacing n by $n-1$ in (4.39a) and then eliminating the term $P'_{n-1}(x)$ by use of (4.39b), we obtain

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x) \tag{4.41}$$

This last relation allows us to express the *derivative* of a Legendre polynomial in terms of Legendre polynomials.

4.2.3 *Legendre's Differential Equation*

All the recurrence relations that we have derived thus far involve successive Legendre polynomials. We may well wonder if any relation exists between derivatives of the Legendre polynomials and Legendre polynomials of the same index. The answer is in the affirmative, but to derive this relation we must consider second derivatives of the polynomials.

By taking the derivative of both sides of (4.41), we get

$$\frac{d}{dx}\left[(1-x^2)P'_n(x)\right] = nP'_{n-1}(x) - nP_n(x) - nxP'_n(x)$$

and then, using (4.39b) to eliminate $P'_{n-1}(x)$, we arrive at the derivative relation

$$\frac{d}{dx}\left[(1-x^2)P'_n(x)\right] + n(n+1)P_n(x) = 0 \tag{4.42}$$

which holds for $n = 0, 1, 2, \ldots$. Expanding the product term in (4.42) yields

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0 \tag{4.43}$$

and thus we deduce that the Legendre polynomial $y = P_n(x)$ $(n = 0, 1, 2, \ldots)$ is a solution of the linear second-order DE

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{4.44}$$

called *Legendre's differential equation*.*

Perhaps the most natural way in which Legendre polynomials arise in practice is as solutions of Legendre's equation. In such problems the basic

*In Section 4.6 we will discuss other solutions of Legendre's equation.

model is generally a partial differential equation. Solving the partial DE by the separation-of-variables technique leads to a system of ordinary DEs, and sometimes one of these is Legendre's DE. This is precisely the case, for example, in solving for the steady-state temperature distribution (independent of the azimuthal angle) in a solid sphere. We will delay any further discussion of such problems, however, until Chapter 7.

Remark: Any function $f_n(x)$ that satisfies Legendre's equation, i.e.,

$$(1 - x^2)f_n''(x) - 2xf_n'(x) + n(n+1)f_n(x) = 0$$

will also satisfy *all* previous recurrence formulas given above, provided that $f_n(x)$ is properly normalized. Consequently, any further solutions of Legendre's equation can be selected in such a way that they automatically satisfy the whole set of recurrence relations already derived. The set of solutions $Q_n(x)$ introduced in Section 4.6 is a case in point.

EXERCISES 4.2

1. Use the series (4.16) to determine $P_n(x)$ directly for the specific cases $n = 0, 1, 2, 3, 4$, and 5.

2. Given that $P_0(x) = 1$ and $P_1(x) = x$, use the recurrence formula (4.33) to determine $P_2(x)$, $P_3(x)$, and $P_4(x)$.

3. Verify that

(a) $P_0(\cos\phi) = 1$.
(b) $P_1(\cos\phi) = \cos\phi$.
(c) $P_2(\cos\phi) = \frac{1}{4}(3\cos 2\phi + 1)$.
(d) $P_3(\cos\phi) = \frac{1}{8}(5\cos 3\phi + 3\cos\phi)$.

4. Given the function $w(x,t) = (1 - 2xt + t^2)^{-1/2}$,

(a) show that $w(-x,-t) = w(x,t)$.
(b) Use the result in (a) and the generating function relation (4.19) to deduce that (for $n = 0, 1, 2, \dots$)

$$P_n(-x) = (-1)^n P_n(x).$$

5. Verify the special values ($n = 0, 1, 2, \dots$)

(a) $P_n'(1) = \frac{1}{2}n(n+1)$, (b) $P_n'(-1) = (-1)^{n-1}\,\frac{1}{2}n(n+1)$.

6. Verify the special values ($n = 0, 1, 2, \dots$)

(a) $P_{2n}'(0) = 0$. (b) $P_{2n+1}'(0) = \dfrac{(-1)^n(2n+1)}{2^{2n}}\dbinom{2n}{n}$.

7. Establish the generating-function relation

$$(1 - 2xt + t^2)^{-1} = \sum_{n=0}^{\infty} U_n(x)t^n, \qquad |t| < 1, \quad |x| \le 1$$

where $U_n(x)$ is the nth *Chebyshev polynomial of the second kind** defined by

$$U_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k)!}{k!(n-2k)!}(2x)^{n-2k}$$

8. Given the generating function $w(x, t) = (1 - 2xt + t^2)^{-1}$,

(a) show that it satisfies the identity

$$(1 - 2xt + t^2)\frac{\partial w}{\partial t} + 2(t - x)w = 0$$

(b) Substitute the series in problem 7 into the identity in (a) and derive the recurrence formula (for $n = 1, 2, 3, \ldots$)

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0$$

9. Show that the generating function in problem 8 also satisfies the identity

$$(1 - 2xt + t^2)\frac{\partial w}{\partial x} - 2tw = 0$$

(a) and deduce the relation (for $n = 1, 2, 3, \ldots$)

$$U'_{n+1}(x) - 2xU'_n(x) + U'_{n-1}(x) - 2U_n(x) = 0$$

(b) Show that (a) can be obtained directly from problem 8(b) by differentiation.

10. Using the results of problems 7–9, show that

(a) $(1 - x^2)U_n'(x) = -nxU_n(x) + (n + 1)U_{n-1}(x)$,

(b) $(1 - x^2)U_n''(x) - 3xU_n'(x) + n(n + 2)U_n(x) = 0$.

11. Using the Cauchy product of two power series (Section 1.3.3), show that

$$\frac{e^{xt}}{1 - t} = \sum_{n=0}^{\infty} e_n(x)t^n, \qquad |t| < 1$$

where $e_n(x)$ is the polynomial equal to the first $n + 1$ terms of the

*We will discuss these polynomials further in Section 5.4.2.

Maclaurin series for e^x, i.e.,

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

12. Given the generating function $w(x, t) = e^{xt}/(1 - t)$,

(a) show that it satisfies the identity

$$(1 - t)\frac{\partial w}{\partial t} - [x(1 - t) + 1]\, w = 0$$

(b) Substitute the series in problem 11 into the identity in (a) and derive the recurrence formula ($n = 1, 2, 3, \ldots$)

$$(n + 1)e_{n+1}(x) - (n + 1 + x)e_n(x) + xe_{n-1}(x) = 0$$

(c) Show directly from the series definition of $e_n(x)$ that

$$e_n'(x) = e_{n-1}(x), \qquad n = 1, 2, 3, \ldots$$

13. Using the results of problems 11 and 12, show that $y = e_n(x)$ is a solution of the second-order linear DE

$$xy'' - (x + n)y' + ny = 0$$

14. Make the change of variable $x = \cos\phi$ in the DE

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dy}{d\phi}\right) + n(n + 1)y = 0$$

and show that it reduces to Legendre's DE (4.44).

15. Determine the values of n for which $y = P_n(x)$ is a solution of

(a) $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,\ y(0) = 0,\ y(1) = 1,$

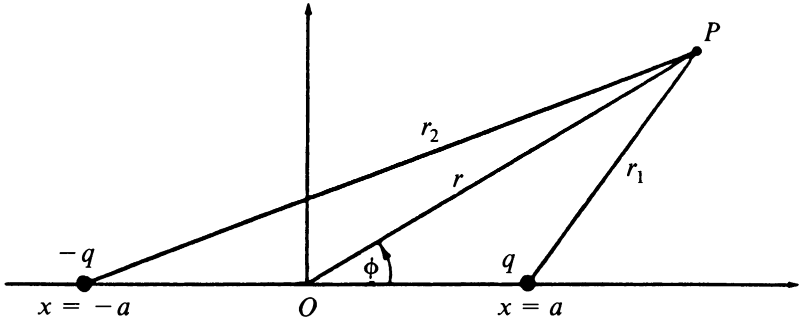
(b) $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,\ y'(0) = 0,\ y(1) = 1.$

16. When a tightly stretched string is rotating with uniform angular speed ω about its rest position along the x-axis, the DE governing the displacements of the string in the vertical plane is approximately

$$\frac{d}{dx}[T(x)y'] + \rho\omega^2 y = 0$$

where $T(x)$ is the tension in the string and ρ the linear density (constant) of the string. If $T(x) = 1 - x^2$ and the boundary condition $y(-1) = y(1)$ is prescribed, determine the two lowest possible critical speeds ω. What shape does the string assume in the vertical plane in each case?

Hint: Assume that $\rho\omega^2 = n(n + 1)$.



17. An electric dipole consists of electric charges q and $-q$ located along the x-axis as shown in the figure above. The potential induced at point P due to the charges is known to be ($r > a$)

$$V = kq\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

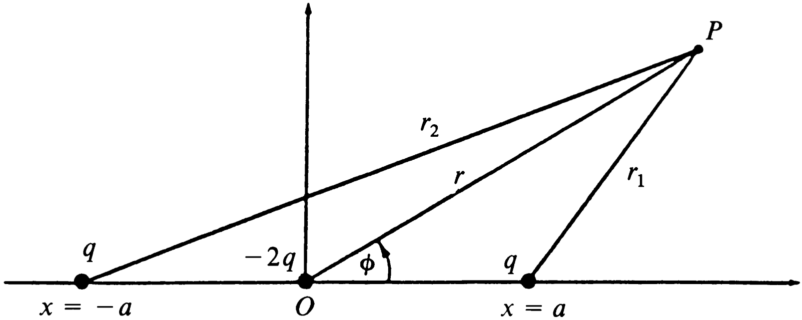
where k is a constant. Express the potential in terms of the coordinates r and ϕ and show that it leads to an infinite series involving Legendre polynomials. Also show that if only the first nonzero term of the series is retained, the dipole potential is

$$V \simeq \frac{2akq}{r^2}\cos\phi, \qquad r \gg a$$

18. The electrostatic potential induced at point P for the array of charges shown in the figure below is given by ($r > a$)

$$V = kq\left(\frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r}\right)$$

where k is constant. Expressing V entirely in terms of r and ϕ, show



that the first nonzero term of the resulting series is

$$V \simeq \frac{kqa^2}{2r^3}(3\cos 2\phi + 1), \qquad r \gg a$$

19. Show that the even and odd Legendre polynomials have the series representations (for $n = 0, 1, 2, \dots$)

(a) $P_{2n}(x) = \dfrac{(-1)^n}{2^{2n-1}} \displaystyle\sum_{k=0}^{n} \frac{(-1)^k (2n+2k-1)!}{(2k)!(n+k-1)!(n-k)!} x^{2k}$,

(b) $P_{2n+1}(x) = \dfrac{(-1)^n}{2^{2n}} \displaystyle\sum_{k=0}^{n} \frac{(-1)^k (2n+2k+1)!}{(2k+1)!(n+k)!(n-k)!} x^{2k+1}$.

20. Derive the identity ($n = 0, 1, 2, \dots$)

$$(1 - x^2)P_n'(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$$

21. Show that

(a) $\displaystyle\sum_{k=0}^{n} (2k+1)P_k(x) = P_{n+1}'(x) + P_n'(x)$,

(b) $(1 - x)\displaystyle\sum_{k=0}^{n} (2k+1)P_k(x) = (n+1)[P_n(x) - P_{n+1}(x)]$.

22. Show that

(a) $\displaystyle\sum_{n=0}^{\infty} [xP_n'(x) - nP_n(x)]t^n = t^2(1 - 2xt + t^2)^{-3/2}$,

(b) $\displaystyle\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} (2n - 4k + 1)P_{n-2k}(x)t^n = (1 - 2xt + t^2)^{-3/2}$.

23. Using the result of problem 22, deduce that

$$xP_n'(x) - nP_n(x) = \sum_{k=0}^{[\frac{1}{2}(n-2)]} (2n - 4k - 3)P_{n-2-2k}(x)$$

24. Show that

$$P_n'(x) = \sum_{k=0}^{[\frac{1}{2}(n-1)]} (2n - 4k - 1)P_{n-1-2k}(x)$$

25. Show that

$$\sum_{n=0}^{\infty} (2n+1)P_n(x)t^n = (1 - t^2)(1 - 2xt + t^2)^{-3/2}$$

4.3 *Other Representations of the Legendre Polynomials*

For each n, the Legendre polynomials can be defined either by the series

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \tag{4.45}$$

or by the recurrence formula

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right) x P_n(x) - \left(\frac{n}{n+1}\right) P_{n-1}(x) \tag{4.46}$$

where $P_0(x) = 1$ and $P_1(x) = x$. In some situations, however, it is advantageous to have other representations from which further properties of the polynomials are more readily found.

4.3.1 *Rodrigues's Formula*

A representation of the Legendre polynomials involving differentiation is given by the *Rodrigues formula*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right], \qquad n = 0, 1, 2, \ldots \tag{4.47}$$

In order to verify (4.47), we start with the binomial series

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} x^{2n-2k}$$

and differentiate n times. Noting that

$$\frac{d^n}{dx^n} x^m = \begin{cases} \dfrac{m!}{(m-n)!} x^{m-n}, & n \le m \\ 0, & n > m \end{cases}$$

we infer

$$\begin{aligned} \frac{d^n}{dx^n}\left[(x^2-1)^n\right] &= \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} \\ &= 2^n n! P_n(x) \end{aligned}$$

from which (4.47) now follows.

4.3.2 *Laplace Integral Formula*

An integral representation of $P_n(x)$ is given by

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[x + (x^2-1)^{1/2} \cos\phi\right]^n d\phi, \qquad n = 0, 1, 2, \ldots \tag{4.48}$$

which is called the *Laplace integral formula*. This relation is easily verified for $n = 0$ and $n = 1$, but more difficult to prove in the general case.

Let us call the integral I and expand the integrand in a finite binomial series to get

$$\begin{aligned} I &= \frac{1}{\pi}\int_0^{\pi}\left[x + (x^2 - 1)^{1/2}\cos\phi\right]^n d\phi \\ &= \frac{1}{\pi}\int_0^{\pi}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(x^2 - 1)^{k/2}\cos^k\phi\, d\phi \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}(x^2 - 1)^{k/2}\frac{1}{\pi}\int_0^{\pi}\cos^k\phi\, d\phi \end{aligned} \tag{4.49}$$

The residual integral in (4.49) can be shown to satisfy

$$\frac{1}{\pi}\int_0^{\pi}\cos^k\phi\, d\phi = 0, \qquad k = 1, 3, 5, \ldots \tag{4.50}$$

and for even values of k we set $k = 2j$ to find

$$\begin{aligned} \frac{1}{\pi}\int_0^{\pi}\cos^k\phi\, d\phi &= \frac{2}{\pi}\int_0^{\pi/2}\cos^{2j}\phi\, d\phi \\ &= \frac{(2j)!}{2^{2j}(j!)^2}, \qquad j = 0, 1, 2, \ldots \end{aligned} \tag{4.51}$$

The verification of (4.50) and (4.51) is left to the exercises (see problems 5 and 6). Thus, all odd terms in (4.49) are zero, and by setting $k = 2j$ and using (4.51), we see that

$$I = \sum_{j=0}^{[n/2]}\frac{n!x^{n-2j}(x^2 - 1)^j}{2^{2j}(n - 2j)!(j!)^2} \tag{4.52}$$

What remains now is to show that (4.52) is a series representation of $P_n(x)$, and this we leave also to the exercises (problem 7).

4.3.3 *Some Bounds on* $P_n(x)$

One of the uses of the Laplace integral formula (4.48) is to establish some inequalities for the Legendre polynomials which furnish certain bounds on them. Of particular interest is the interval $|x| \le 1$, but since the integrand in (4.48) is not real for this restriction on x, we first rewrite (4.48) in the form $(i^2 = -1)$

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + i(1 - x^2)^{1/2}\cos\phi\right]^n d\phi, \qquad |x| \le 1 \tag{4.53}$$

Now, using the fact that the absolute value of an integral is less than or equal to the integral of the absolute value of the integrand, we get

$$|P_n(x)| \leq \frac{1}{\pi}\int_0^{\pi}\left|x + i(1 - x^2)^{1/2}\cos\phi\right|^n d\phi \tag{4.54}$$

From the algebra of complex numbers, it is known that $|a + ib| = (a^2 + b^2)^{1/2}$, and thus for $|x| \leq 1$ it follows that

$$\begin{aligned}\left|x + i(1 - x^2)^{1/2}\cos\phi\right|^n &= \left[x^2 + (1 - x^2)\cos^2\phi\right]^{n/2}\\ &= \left(\cos^2\phi + x^2\sin^2\phi\right)^{n/2}\\ &\leq \left(\cos^2\phi + \sin^2\phi\right)^{n/2}\\ &\leq 1\end{aligned}$$

Returning now to (4.54), we have shown that

$$|P_n(x)| \leq \frac{1}{\pi}\int_0^{\pi} d\phi$$

or

$$|P_n(x)| \leq 1, \qquad |x| \leq 1, \quad n = 0, 1, 2, \ldots \tag{4.55}$$

which is our intended result. The equality in (4.55) holds only when $x = \pm 1$.

Another inequality, less obvious and more difficult to prove, is given by

$$P_n(x) < \left[\frac{\pi}{2n(1 - x^2)}\right]^{1/2}, \qquad |x| < 1, \quad n = 1, 2, 3, \ldots \tag{4.56}$$

Again the Legendre integral representation is used to derive this inequality, although we will not do so here (see problem 10).

EXERCISES 4.3

1. Using Rodrigues's formula (4.47), derive the identities ($n = 1, 2, 3, \ldots$)

(a) $(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$,

(b) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$,

(c) $xP_n'(x) = nP_n(x) + P_{n-1}'(x)$,

(d) $P_{n+1}'(x) - P_{n-1}'(x) = (2n + 1)P_n(x)$.

2. Representing $P_n(x)$ by Rodrigues's formula (4.47), show that

$$\int_{-1}^{1} P_n(x)\,dx = 0, \qquad n = 1, 2, 3, \ldots$$

3. Using Rodrigues's formula (4.47) and integration by parts, show that

$$\int_{-1}^{1} [P_n(x)]^2\,dx = \frac{2}{2n+1}, \qquad n = 0, 1, 2, \ldots$$

4. By defining $v = (x^2 - 1)^n$, show that

(a) $(1 - x^2)\dfrac{dv}{dx} + 2nxv = 0.$

(b) Differentiating the result in (a) $n + 1$ times and defining $u = v^{(n)}$, show that u satisfies Legendre's equation

$$(1 - x^2)u'' - 2xu' + n(n+1)u = 0$$

5. Verify that

$$\frac{1}{\pi}\int_0^{\pi} \cos^{2n+1}\theta\,d\theta = 0, \qquad n = 0, 1, 2, \ldots$$

6. Verify that

(a) $\dfrac{1}{\pi}\displaystyle\int_0^{\pi} \cos^{2n}\theta\,d\theta = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \cos^{2n}\theta\,d\theta.$

(b) Using properties of the gamma function, show that

$$\frac{2}{\pi}\int_0^{\pi/2} \cos^{2n}\theta\,d\theta = \frac{(2n)!}{2^{2n}(n!)^2}, \qquad n = 0, 1, 2, \ldots$$

7. Show that the generating function for the Legendre polynomials can be written in the form

(a) $(1 - 2xt + t^2)^{-1/2} = (1 - xt)^{-1}\left[1 - \dfrac{t^2(x^2 - 1)}{(1 - xt)^2}\right]^{-1/2}$

(b) Using the result in (a), expand the expression on the right in powers of t. Then, by comparing your result with Equation (4.19) in Section 4.2.1, deduce that

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{n!\,x^{n-2k}(x^2 - 1)^k}{2^{2k}(n - 2k)!(k!)^2}$$

8. (*Jordan inequality*) If $0 \le \phi \le \pi/2$, show that

$$\sin\phi \ge \frac{2\phi}{\pi}$$

Hint: Prove that $(\sin\phi)/\phi$ is a decreasing function on the given interval by showing its derivative is always negative. Hence, the minimum value occurs at $\phi = \pi/2$.

9. Derive the inequality

$$1 - y < e^{-y}, \qquad y > 0$$

10. By using the Laplace integral formula (4.48), show that for $|x| < 1$,

(a) $|P_n(x)| \le \frac{2}{\pi}\int_0^{\pi/2}[1 - (1 - x^2)\sin^2\phi]^{n/2}\,d\phi.$

(b) Show that application of the Jordan inequality (problem 8) reduces (a) to

$$|P_n(x)| \le \frac{2}{\pi}\int_0^{\pi/2}\left[1 - \frac{4\phi^2(1 - x^2)}{\pi^2}\right]^{n/2} d\phi$$

(c) Making use of the inequality in problem 9 together with an appropriate change of variables, show that

$$|P_n(x)| < \frac{2}{[2n(1 - x^2)]^{1/2}}\int_0^{\infty} e^{-t^2}\,dt$$

and from this result, deduce that ($n = 1, 2, 3, \ldots$)

$$|P_n(x)| < \left[\frac{\pi}{2n(1 - x^2)}\right]^{1/2}, \qquad |x| < 1$$

11. Starting with the identity

$$(1 - x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$$

show that

$$|P_n'(x)| \le \frac{n}{1 - |x|}, \qquad |x| < 1, \quad n = 1, 2, 3, \ldots$$

12. Starting with the identities

$$P_n(x) = xP_{n-1}(x) + \frac{x^2 - 1}{n}P_{n-1}'(x)$$

$$P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x)$$

(a) show that (for $n = 1, 2, 3, \ldots$)

$$\frac{1 - x^2}{n^2}[P_n'(x)]^2 + [P_n(x)]^2 = \frac{1 - x^2}{n^2}[P_{n-1}'(x)]^2 + [P_{n-1}(x)]^2$$

(b) From (a), establish the inequality

$$\frac{1 - x^2}{n^2}[P_n'(x)]^2 + [P_n(x)]^2 \le 1, \qquad |x| \le 1$$

(c) From (b), deduce that

$$|P_n(x)| \le 1, \qquad |x| \le 1$$

4.4 *Legendre Series*

In this section we wish to show how to represent certain functions by series of Legendre polynomials, called *Legendre series*. Because the general term in such series is a polynomial, we can interpret a Legendre series as some generalization of a power series for which the general term is also a polynomial, viz., $(x - a)^n$. However, to develop a given function f in a power series requires that the function f be at least continuous and differentiable in the interval of convergence. In the case of Legendre series we make no such requirement. In fact, many functions of practical interest exhibiting (finite) discontinuities may be represented by convergent Legendre series. Legendre series are only one member of a fairly large and special class of series collectively referred to as *generalized Fourier series*, all of which have many properties in common. In Section 1.5 we encountered *Fourier trigonometric series*, which are perhaps the best known members of this class, and in the following chapters we will come across several other members of this general class. Besides their obvious mathematical interest, it turns out that the applications of generalized Fourier series are very extensive—so much so, in fact, that they involve almost every facet of applied mathematics.

4.4.1 *Orthogonality*

Although we have already derived many identitities associated with the Legendre polynomials, none of these is so fundamental and far-reaching in practice as is the *orthogonality property*

$$\int_{-1}^{1} P_n(x)P_k(x)\,dx = 0, \qquad k \neq n \tag{4.57}$$

Remark: It is sometimes helpful to think of (4.57) as a generalization of the scalar (dot) product of vector analysis. In fact, much of the following discussion has a vector analog in three-dimensional vector space.

To prove (4.57), we first take note of the fact that both $P_k(x)$ and $P_n(x)$ satisfy Legendre's DE (4.42), and thus we write

$$\frac{d}{dx}\left[(1 - x^2)P_k'(x)\right] + k(k + 1)P_k(x) = 0 \tag{4.58a}$$

$$\frac{d}{dx}\left[(1 - x^2)P_n'(x)\right] + n(n + 1)P_n(x) = 0 \tag{4.58b}$$

If we multiply the first of these equations by $P_n(x)$ and the second by

$P_k(x)$, subtract the results, and integrate from -1 to 1, we find

$$\int_{-1}^{1} P_n(x)\frac{d}{dx}\left[(1-x^2)P_k'(x)\right] dx - \int_{-1}^{1} P_k(x)\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] dx$$
$$+\left[k(k+1)-n(n+1)\right]\int_{-1}^{1} P_n(x)P_k(x)\, dx = 0 \tag{4.59}$$

On integrating the first integral above by parts, we have

$$\int_{-1}^{1} P_n(x)\frac{d}{dx}\left[(1-x^2)P_k'(x)\right] dx = \cancelto{0}{P_n(x)(1-x^2)P_k'(x)\Big|_{-1}^{1}}$$
$$-\int_{-1}^{1}(1-x^2)P_n'(x)P_k'(x)\, dx \tag{4.60a}$$

and similarly for the second integral,

$$\int_{-1}^{1} P_k(x)\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] dx = -\int_{-1}^{1}(1-x^2)P_n'(x)P_k'(x)\, dx \tag{4.60b}$$

and therefore the difference of these two integrals is clearly zero. Hence, (4.59) reduces to

$$\left[k(k+1)-n(n+1)\right]\int_{-1}^{1} P_n(x)P_k(x)\, dx = 0$$

and since $k \neq n$ by hypothesis, the result (4.57) follows immediately.

When $k = n$, the situation is different. Let us define

$$A_n = \int_{-1}^{1}\left[P_n(x)\right]^2 dx \tag{4.61}$$

and replace one of the $P_n(x)$ in (4.61) by use of the identity [replace n with $n-1$ in (4.32)]

$$P_n(x) = \frac{2n-1}{n}xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x) \tag{4.62}$$

to get

$$A_n = \int_{-1}^{1} P_n(x)\left[\frac{2n-1}{n}xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x)\right] dx$$
$$= \frac{2n-1}{n}\int_{-1}^{1} xP_n(x)P_{n-1}(x)\, dx - \frac{n-1}{n}\cancelto{0}{\int_{-1}^{1} P_n(x)P_{n-2}(x)\, dx} \tag{4.63}$$

The second integral above vanishes because of the orthogonality property (4.57). To further simplify (4.63), we rewrite (4.62) in the form

$$xP_n(x) = \frac{1}{2n+1}\left[(n+1)P_{n+1}(x) + nP_{n-1}(x)\right]$$

and substitute it into (4.63), from which we deduce

$$\begin{aligned} A_n = {} & \left(\frac{2n-1}{n}\right)\left(\frac{n+1}{2n+1}\right)\int_{-1}^{1} \cancelto{0}{P_{n+1}(x)P_{n-1}(x)}\, dx \\ & + \frac{2n-1}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2\, dx \end{aligned}$$

or

$$A_n = \frac{2n-1}{2n+1}A_{n-1}, \qquad n = 2,3,4,\ldots \tag{4.64}$$

Equation (4.64) is simply a recurrence formula for A_n. Using the fact that

$$A_0 = \int_{-1}^{1}[P_0(x)]^2\, dx = \int_{-1}^{1} dx = 2$$

and

$$A_1 = \int_{-1}^{1}[P_1(x)]^2\, dx = \int_{-1}^{1} x^2\, dx = \tfrac{2}{3}$$

Equation (64) yields

$$\begin{aligned} A_2 &= \tfrac{3}{5} \times \tfrac{1}{3} \times 2 = \tfrac{2}{5} \\ A_3 &= \tfrac{5}{7} \times \tfrac{3}{5} \times \tfrac{1}{3} \times 2 = \tfrac{2}{7} \end{aligned}$$

while in general it can be verified by mathematical induction that

$$A_n = \frac{2n-1}{2n+1} \times \frac{2n-3}{2n-1} \times \frac{2n-5}{2n-3} \times \cdots \times \frac{1}{3} \times 2 = \frac{2}{2n+1}, \qquad n = 0,1,2,\ldots \tag{4.65}$$

Thus, we have derived the important result

$$\int_{-1}^{1}[P_n(x)]^2\, dx = \frac{2}{2n+1}, \qquad n = 0,1,2,\ldots \tag{4.66}$$

4.4.2 *Finite Legendre Series*

Because of the special properties associated with Legendre polynomials, it may be useful in certain situations to represent arbitrary polynomials as linear combinations of Legendre polynomials. For example, if $q_m(x)$ denotes an arbitrary polynomial of degree m, then, since $P_0(x)$, $P_1(x), \ldots, P_m(x)$ are all polynomials of degree m or less, we might expect to find a representation of the form*

$$q_m(x) = c_0 P_0(x) + c_1 P_1(x) + \cdots + c_m P_m(x) \tag{4.67}$$

Let us illustrate with a simple example.

Example 1: Express x^2 in a series of Legendre polynomials.

Solution: We write

$$\begin{aligned} x^2 &= c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) \\ &= c_0 + c_1 x + c_2 \tfrac{1}{2}(3x^2 - 1) \\ &= \left(c_0 - \tfrac{1}{2}c_2\right) + c_1 x + \tfrac{3}{2}c_2 x^2 \end{aligned}$$

Now equating like coefficients, we see that

$$c_0 - \tfrac{1}{2}c_2 = 0, \qquad c_1 = 0, \qquad \tfrac{3}{2}c_2 = 1$$

from which we deduce $c_0 = \frac{1}{3}$, $c_1 = 0$, and $c_2 = \frac{2}{3}$. Hence,

$$x^2 = \tfrac{1}{3}P_0(x) + \tfrac{2}{3}P_2(x)$$

When the polynomial $q_m(x)$ is of a high degree, solving a system of simultaneous equations for the c's as we did in Example 1 is very tedious. A more systematic procedure can be developed by using the orthogonality property (4.57). We begin by writing (4.67) in the form

$$q_m(x) = \sum_{n=0}^{m} c_n P_n(x) \tag{4.68}$$

Next, we multiply both sides of (4.68) by $P_k(x)$, $0 \le k \le m$, and integrate the result termwise (which is justified because the series is finite) from -1 to 1 to get

$$\int_{-1}^{1} q_m(x) P_k(x)\, dx = \sum_{n=0}^{m} c_n \cancelto{0\ (n \neq k)}{\int_{-1}^{1} P_n(x) P_k(x)\, dx} \tag{4.69}$$

Because of the orthogonality property (4.57), each term of the series in

*Two polynomials can be equated if and only if they are of the same degree.

(4.69) vanishes except the term corresponding to $n = k$, and here we find

$$\int_{-1}^{1} q_m(x) P_k(x)\,dx = c_k \int_{-1}^{1} [P_k(x)]^2\,dx$$
$$= c_k\left(\frac{2}{2k+1}\right)$$

where the last step is a consequence of (4.66). Hence, we deduce that (changing the dummy index back to n)

$$c_n = \left(n + \tfrac{1}{2}\right)\int_{-1}^{1} q_m(x) P_n(x)\,dx, \qquad n = 0, 1, 2, \ldots, m \qquad (4.70)$$

Remark: If the polynomial $q_m(x)$ in (4.70) is even (odd), then only those c_n with even (odd) suffixes are nonzero, due to the even-odd property of the Legendre polynomials (see problems 25 and 26).

As a consequence of the fact that a polynomial of degree m can be expressed as a Legendre series involving only $P_m(x)$ and lower-order Legendre polynomials, we have the following theorem.*

Theorem 4.1. If $q_m(x)$ is a polynomial of degree m and $m < r$, then

$$\int_{-1}^{1} q_m(x) P_r(x)\,dx = 0, \qquad m < r$$

Proof: Since $q_m(x)$ is a polynomial of degree m, we can write

$$q_m(x) = \sum_{n=0}^{m} c_n P_n(x)$$

Then, multiplying both sides of this expression by $P_r(x)$ and integrating from -1 to 1, we get

$$\int_{-1}^{1} q_m(x) P_r(x)\,dx = \sum_{n=0}^{m} c_n \int_{-1}^{1} \cancelto{0}{P_n(x) P_r(x)}\,dx$$

The largest value of n is m, and since $m < r$, the right-hand side is zero for each n [due to the orthogonality property (4.57)], and the theorem is proved. ■

*Theorem 4.1 says that $P_r(x)$ is orthogonal to *every* polynomial of degree less than r.

4.4.3 *Infinite Legendre Series*

In some applications we will find it necessary to represent a function f, other than a polynomial, as a linear combination of Legendre polynomials. Such a representation will lead to an *infinite series* of the general form

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \tag{4.71}$$

where the coefficients can be formally derived by a process similar to the derivation of (4.70), leading to

$$c_n = \left(n + \tfrac{1}{2}\right)\int_{-1}^{1} f(x) P_n(x)\, dx, \qquad n = 0, 1, 2, \ldots \tag{4.72}$$

Conditions under which the representation (4.71) and (4.72) is valid will be taken up in the next section. For now it suffices to say that for certain functions the series (4.71) will converge throughout the interval $-1 \le x \le 1$, even at points of finite discontinuities of the given function. Series of this type are called *Legendre series*, and because they belong to the larger class of generalized Fourier series, the coefficients (4.72) are commonly called the *Fourier coefficients* of the series.

In practice, the evaluation of integrals like (4.72) must be performed numerically. However, if the function f is not too complicated, we can sometimes use various properties of the Legendre polynomials to evaluate such integrals in closed form. The following example illustrates the point.

Remark: Because the interval of convergence of (4.71) is confined to $-1 \le x \le 1$, it really doesn't matter if the function f is defined outside this interval. That is, even if f is defined for *all* x, the representation will not be valid beyond the interval $-1 \le x \le 1$ (unless f is a polynomial).

Example 2: Find the Legendre series for

$$f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 < x \le 1 \end{cases}$$

Solution: The function f is an odd function. Hence, owing to the even-odd property of the Legendre polynomials depending upon the index n, we note that $f(x)P_n(x)$ is an odd function when n is even, and in this case it follows that (see problem 25)

$$c_n = \left(n + \tfrac{1}{2}\right)\int_{-1}^{1} f(x) P_n(x)\, dx = 0, \qquad n = 0, 2, 4, \ldots$$

For odd index n, the product $f(x)P_n(x)$ is even and therefore

$$\begin{aligned} c_n &= (n + \tfrac{1}{2})\int_{-1}^{1} f(x)P_n(x)\,dx \\ &= (2n+1)\int_0^1 P_n(x)\,dx, \qquad n = 1, 3, 5, \ldots \end{aligned}$$

Let us use the identity [see Equation (4.40)]

$$P_n(x) = \frac{1}{2n+1}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]$$

and set $n = 2k + 1$, thereby obtaining the result (for $k = 0, 1, 2, \ldots$)

$$\begin{aligned} c_{2k+1} &= (4k+3)\int_0^1 P_{2k+1}(x)\,dx \\ &= \int_0^1 \left[P'_{2k+2}(x) - P'_{2k}(x)\right] dx \\ &= \left[P_{2k+2}(x) - P_{2k}(x)\right]\Big|_0^1 \\ &= P_{2k}(0) - P_{2k+2}(0) \end{aligned}$$

where we have used the property $P_n(1) = 1$ for all n. Referring to Equation (4.27), we have

$$\begin{aligned} c_{2k+1} &= \frac{(-1)^k(2k)!}{2^{2k}(k!)^2} - \frac{(-1)^{k+1}(2k+2)!}{2^{2k+2}[(k+1)!]^2} \\ &= \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\left[1 + \frac{(2k+2)(2k+1)}{2^2(k+1)^2}\right] \\ &= \frac{(-1)^k(2k)!}{2^{2k}(k!)^2}\left[1 + \frac{2k+1}{2k+2}\right] \\ &= \frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!} \end{aligned}$$

and thus

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(2k)!(4k+3)}{2^{2k+1}k!(k+1)!} P_{2k+1}(x), \qquad -1 \le x \le 1$$

EXERCISES 4.4

In problems 1–15, use the orthogonality property and/or any other relations to derive the integral formula.

1. $\int_{-1}^{1} xP_n(x)\,dx = \begin{cases} 0, & n \neq 1, \\ \frac{2}{3}, & n = 1. \end{cases}$

2. $\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \dfrac{2n}{4n^2-1}, \quad n = 1, 2, 3, \ldots .$

3. $\int_{-1}^{1} P_n(x)P'_{n+1}(x)\,dx = 2, \quad n = 0, 1, 2, \ldots .$

4. $\int_{-1}^{1} xP'_n(x)P_n(x)\,dx = \dfrac{2n}{2n+1}, \quad n = 0, 1, 2, \ldots .$

5. $\int_{-1}^{1} (1-x^2)P'_n(x)P'_k(x)\,dx = 0, \quad k \neq n.$

6. $\int_{-1}^{1} (1-2xt+t^2)^{-1/2}P_n(x)\,dx = \dfrac{2t^n}{2n+1}, \quad n = 0, 1, 2, \ldots .$

7. $\int_{-1}^{1} (1-x)^{-1/2}P_n(x)\,dx = \dfrac{2\sqrt{2}}{2n+1}, \quad n = 0, 1, 2, \ldots .$

Hint: Let $t \to 1$ in problem 6.

8. $\int_{-1}^{1} x^2P_{n+1}(x)P_{n-1}(x)\,dx = \dfrac{2n(n+1)}{(4n^2-1)(2n+3)}, \quad n = 1, 2, 3, \ldots .$

9. $\int_{-1}^{1} (x^2-1)P_{n+1}(x)P'_n(x)\,dx = \dfrac{2n(n+1)}{(2n+1)(2n+3)}, \quad n = 1, 2, 3, \ldots .$

10. $\int_{-1}^{1} x^nP_n(x)\,dx = \dfrac{2^{n+1}(n!)^2}{(2n+1)!}, \quad n = 0, 1, 2, \ldots .$

Hint: Use problem 31.

11. If $k \leq n$, $\int_{-1}^{1} P'_n(x)P'_k(x)\,dx = \begin{cases} 0, & n+k \text{ even}, \\ k(k+1), & n+k \text{ odd}. \end{cases}$

12. $\int_{-1}^{1} P_n(x)P'_k(x)\,dx = \begin{cases} 0, & k \leq n, \\ 0, & k > n,\ k+n \text{ even}, \\ 2, & k > n,\ k+n \text{ odd}. \end{cases}$

13. $\int_{0}^{1} P_{2n}(x)\,dx = 0, \quad n = 1, 2, 3, \ldots .$

14. $\int_{-1}^{1} (1-x^2)[P'_n(x)]^2\,dx = \dfrac{2n(n+1)}{2n+1}, \quad n = 0, 1, 2, \ldots .$

15. $\int_{-1}^{1} x^2[P_n(x)]^2\,dx = \dfrac{2}{(2n+1)^2}\left[\dfrac{(n+1)^2}{2n+3} + \dfrac{n^2}{2n-1}\right],$
$n = 0, 1, 2, \ldots .$

16. Show that the orthogonality relation (4.57) for the functions $P_n(\cos\phi)$ is

$$\int_0^{\pi} P_n(\cos\phi)P_k(\cos\phi)\sin\phi\,d\phi = 0, \qquad k \neq n$$

In problems 17–21, derive the given integral formula.

17. $\int_0^{\pi} P_{2n}(\cos\phi)\,d\phi = \dfrac{\pi}{2^{2n}}\dbinom{2n}{n},\ n = 0, 1, 2, \ldots .$

18. $\int_0^{2\pi} P_{2n}(\cos\phi)\,d\phi = \dfrac{\pi}{2^{4n-1}}\dbinom{2n}{n}^2,\ n = 1, 2, 3, \ldots .$

19. $\int_0^{2\pi} P_{2n}(\cos\phi)\cos\phi\,d\phi = \dfrac{1}{2^{4n+1}}\dbinom{2n}{n}\dbinom{2n+2}{n+1},\ n = 1, 2, 3, \ldots .$

20. $\int_0^{\pi/2} P_{2n}(\cos\phi)\sin\phi\,d\phi = 0,\ n = 1, 2, 3, \ldots .$

21. $\int_0^{\pi} P_n(\cos\phi)\cos n\phi\,d\phi = B(n + \tfrac{1}{2}, \tfrac{1}{2}),\ n = 0, 1, 2, \ldots .$

22. Using Rodrigues's formula (4.47) for $P_n(x)$,

(a) show that integration by parts leads to

$$\int_{-1}^{1} P_n(x)P_k(x)\,dx = -\frac{1}{2^n n!}\int_{-1}^{1} P_k'(x)\frac{d^{n-1}}{dx^{n-1}}[(x^2-1)^n]\,dx$$

(b) Show, by continued integration by parts, that

$$\int_{-1}^{1} P_n(x)P_k(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}[P_k(x)](x^2-1)^n\,dx$$

(c) For $k \neq n$, show that the integral on the right in (b) is zero.

23. For $k = n$, show that problem 22(b) leads to $(n = 0, 1, 2, \ldots)$

(a) $\int_{-1}^{1} [P_n(x)]^2\,dx = \dfrac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx.$

(b) By making an appropriate change of variable, evaluate the integral in (a) through use of the gamma function and hence derive Equation (4.66).

24. Starting with the expression

$$(1 - 2xt + t^2)^{-1} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} P_n(x)P_k(x)t^{n+k}$$

use the orthogonality property (4.57) to deduce Equation (4.66).

Hint: $\log\dfrac{1+t}{1-t} = 2\sum_{n=0}^{\infty}\dfrac{t^{2n+1}}{2n+1}.$

25. Show that if f is an odd function

(a) $\int_{-1}^{1} f(x)P_n(x)\,dx = 0,\ n = 0, 2, 4, \ldots,$

(b) $\int_{-1}^{1} f(x)P_n(x)\,dx = 2\int_0^1 f(x)P_n(x)\,dx,\ n = 1, 3, 5, \ldots.$

26. Show that if f is an even function

(a) $\int_{-1}^{1} f(x)P_n(x)\,dx = 2\int_0^1 f(x)P_n(x)\,dx,\ n = 0, 2, 4, \ldots,$

(b) $\int_{-1}^{1} f(x)P_n(x)\,dx = 0,\ n = 1, 3, 5, \ldots.$

In problems 27–30, find the Legendre series for the given polynomial.

27. $q(x) = x^3.$

28. $q(x) = 9x^3 - 8x^2 + 7x - 6.$

29. $q(x) = 12x^4 - 8x^2 + 7.$

30. $q(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!}.$

31. Using Rodrigues's formula (4.47) and integration by parts, show that

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n\,dx$$

Hint: See problem 22.

32. From the result of problem 31, deduce that

$$\int_{-1}^{1} x^m P_n(x)\,dx = 0 \qquad \text{if} \quad m < n$$

33. From the result of problem 31, deduce that

$$\int_{-1}^{1} x^{n+2k} P_n(x)\,dx = \frac{(n+2k)!\Gamma(k+\frac{1}{2})}{2^n(2k)!\Gamma(n+k+\frac{3}{2})}, \qquad k = 0, 1, 2, \ldots$$

34. Show that

(a) $x^{2m} = \displaystyle\sum_{n=0}^{m} \frac{2^{2n}(4n+1)(2m)!(m+n)!}{(2m+2n+1)(m-n)!} P_{2n}(x),$

(b) $x^{2m+1} = \displaystyle\sum_{n=0}^{m} \frac{2^{2n+1}(4n+3)(2m+1)!(m+n+1)!}{(2m+2n+3)!(m-n)!} P_{2n+1}(x).$

Hint: Use problem 33.

In problems 35–40, develop the Legendre series for the given function.

35. $f(x) = P_6(x)$.

36. $f(x) = |x|, \ -1 \le x \le 1$.

37. $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ 1, & 0 < x \le 1. \end{cases}$

38. $f(x) = \begin{cases} 1, & -1 \le x < 0, \\ 0, & 0 < x \le 1. \end{cases}$

39. $f(x) = \begin{cases} 0, & -1 \le x < 0, \\ x, & 0 < x \le 1. \end{cases}$

40. $f(x) = \begin{cases} 1, & -1 \le x < 0, \\ x, & 0 < x \le 1. \end{cases}$

41. Show that the Legendre series of a function f defined in the interval $-a \le x \le a$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x/a), \qquad -a \le x \le a$$

where

$$c_n = \frac{2n+1}{2a} \int_{-a}^{a} f(x) P_n(x/a)\, dx, \qquad n = 0, 1, 2, \ldots$$

42. Making the change of variable $x = \cos\phi$, show that the Legendre series for a function $f(\phi)$ is given by

$$f(\phi) = \sum_{n=0}^{\infty} c_n P_n(\cos\phi), \qquad 0 \le \phi \le \pi$$

where

$$c_n = (n + \tfrac{1}{2}) \int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi \, d\phi, \qquad n = 0, 1, 2, \ldots$$

Hint: See problem 16.

43. Using the result of problem 42, find the Legendre series for

(a) $f(\phi) = \begin{cases} 0, & 0 \le \phi < \pi/2 \\ 1, & \pi/2 < \phi \le \pi. \end{cases}$ (b) $f(\phi) = \cos^2\phi, \ 0 \le \phi \le \pi$.

44. Show that

(a) $(1 - x)^n P_n\left(\dfrac{1+x}{1-x}\right) = \displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 x^k$.

(b) Letting $x \to 1$, use part (a) to derive the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

4.5 *Convergence of the Series*

Given the Legendre series of some function f, we now wish to discuss the validity of such a representation. What we mean is—if a value of x is selected in the chosen interval and each term of the series is evaluated for this value of x, will the sum of the series be $f(x)$? If so, we say the series *converges pointwise* to $f(x)$.* In order to establish pointwise convergence of the series, we need to obtain an expression for the partial sum†

$$S_n(x) = \sum_{k=0}^{n} c_k P_k(x) \tag{4.73}$$

and then for a fixed value of x, show that

$$\lim_{n\to\infty} S_n(x) = f(x) \tag{4.74}$$

4.5.1 *Piecewise Continuous and Piecewise Smooth Functions*

To be sure the Legendre series converges to the function which generates the series, it is essential to place certain restrictions on the function f. From a practical point of view, such conditions should be broad enough to cover most situations of concern and still simple enough to be easily checked for the given function.

Definition 4.1. A function f is said to be *piecewise continuous* in the interval $a \le x \le b$, provided that

(1) $f(x)$ is defined and continuous at all but a finite number of points in the interval, and
(2) the left-hand and right-hand limits exist at each point in the interval.

Remark: The left-hand and right-hand limits are defined, respectively, by

$$\lim_{\varepsilon\to 0^+} f(x-\varepsilon) = f(x^-), \qquad \lim_{\varepsilon\to 0^+} f(x+\varepsilon) = f(x^+)$$

Furthermore, when x is a point of continuity, $f(x^-) = f(x^+) = f(x)$.

It is not essential that a piecewise continuous function f be defined at every point in the interval of interest. In particular, it is often not defined at a point of discontinuity, and even when it is, it really doesn't matter what functional value is assigned at such a point. Also, the interval of interest may be open or closed, or open at one end and closed at the other (see Fig. 4.4).

---

*See also the discussion in Section 1.3.

†Although (4.73) has $n + 1$ terms, we still designate it by the symbol $S_n(x)$.

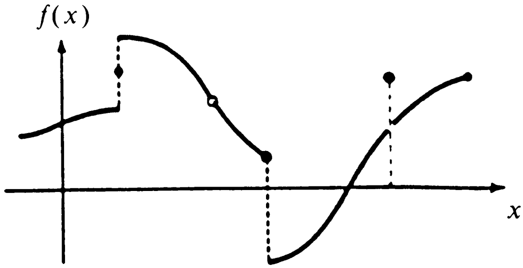


Figure 4.4 A Piecewise Continuous Function

Definition 4.2. A function f is said to be *smooth* in the interval $a \le x \le b$ if it has a continuous derivative there. We say the function is *piecewise smooth* if f and/or its derivative f' are only piecewise continuous in $a \le x \le b$.

Example 3: Classify the following functions as smooth, piecewise smooth, or neither in $-1 \le x \le 1$: (a) $f(x) = x$, (b) $f(x) = |x|$, (c) $f(x) = |x|^{1/2}$.

Solution: In (a), the function $f(x) = x$ and its derivative $f'(x) = 1$ are both continuous, and thus f is *smooth*. The function in (b) is also continuous, but because the derivative is discontinuous, i.e.,

$$f'(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

it is not smooth but only *piecewise smooth*. In (c), the function is once again continuous, but $|f'(x)| \to \infty$ as $x \to 0$, so it is *neither* smooth nor piecewise smooth.

4.5.2 *A Theorem on Pointwise Convergence*

Before stating and proving our main theorem on convergence, we must first establish two lemmas.

Lemma 4.1 (*Riemann*). If the function f is piecewise continuous in the closed interval $-1 \le x \le 1$, then

$$\lim_{n \to \infty} \left(n + \tfrac{1}{2}\right)^{1/2} \int_{-1}^{1} f(x) P_n(x)\, dx = 0$$

Proof: Let the nth partial sum be denoted by

$$S_n(x) = \sum_{k=0}^{n} c_k P_k(x)$$

and consider the nonnegative quantity

$$\int_{-1}^{1} [f(x) - S_n(x)]^2 \, dx \geq 0$$

or

$$\int_{-1}^{1} f^2(x) \, dx - 2\int_{-1}^{1} f(x)S_n(x) \, dx + \int_{-1}^{1} S_n^2(x) \, dx \geq 0$$

Now

$$\begin{aligned}\int_{-1}^{1} f(x)S_n(x) \, dx &= \sum_{k=0}^{n} c_k \int_{-1}^{1} f(x)P_k(x) \, dx \\ &= \sum_{k=0}^{n} \frac{c_k^2}{k + \frac{1}{2}}\end{aligned}$$

and

$0 \; (j \neq k)$

$$\begin{aligned}\int_{-1}^{1} S_n^2(x) \, dx &= \sum_{j=0}^{n} \sum_{k=0}^{n} c_j c_k \int_{-1}^{1} P_j(x)P_k(x) \, dx \\ &= \sum_{k=0}^{n} c_k^2 \int_{-1}^{1} [P_k(x)]^2 \, dx \\ &= \sum_{k=0}^{n} \frac{c_k^2}{k + \frac{1}{2}}\end{aligned}$$

Accordingly, we have

$$\int_{-1}^{1} f^2(x) \, dx - 2\sum_{k=0}^{n} \frac{c_k^2}{k + \frac{1}{2}} + \sum_{k=0}^{n} \frac{c_k^2}{k + \frac{1}{2}} \geq 0$$

from which we deduce

$$\sum_{k=0}^{n} \frac{c_k^2}{k + \frac{1}{2}} \leq \int_{-1}^{1} f^2(x) \, dx$$

Because this last inequality is valid for all n, we simply pass to the limit to get

$$\sum_{k=0}^{\infty} \frac{c_k^2}{k + \frac{1}{2}} \leq \int_{-1}^{1} f^2(x) \, dx$$

The integral on the right is necessarily bounded, since f is assumed to be piecewise continuous in the closed interval of integration. Hence, the series on the left is a convergent series (because its sum is finite), and therefore it follows that

$$\lim_{k\to\infty} \frac{c_k^2}{k+\frac{1}{2}} = 0$$

or equivalently (changing the index back to n),

$$\lim_{n\to\infty} \left(n+\tfrac{1}{2}\right)^{1/2} \int_{-1}^{1} f(x) P_n(x)\, dx = 0 \qquad \blacksquare$$

Lemma 4.2 (*Christoffel-Darboux*). The Legendre polynomials satisfy the identity

$$\sum_{k=0}^{n} (2k+1) P_k(t) P_k(x) = \frac{n+1}{t-x} \left[P_{n+1}(t) P_n(x) - P_n(t) P_{n+1}(x) \right]$$

Proof: We begin by multiplying the recurrence relation (4.31) by $P_k(t)$ to get

$$(2k+1) x P_k(t) P_k(x) = (k+1) P_k(t) P_{k+1}(x) + k P_k(t) P_{k-1}(x)$$

If we now interchange the roles of x and t in this expression and subtract the two results, we obtain

$$\begin{aligned}(2k+1)(t-x) P_k(t) P_k(x) &= (k+1)\left[P_{k+1}(t) P_k(x) - P_k(t) P_{k+1}(x) \right] \\ &\quad - k\left[P_k(t) P_{k-1}(x) - P_{k-1}(t) P_k(x) \right]\end{aligned}$$

Finally, summing both sides of this identity as k runs from 0 to n and setting $P_{-1}(x) = 0$, we find

$$\begin{aligned}&(t-x) \sum_{k=0}^{n} (2k+1) P_k(t) P_k(x) \\ &\quad = (n+1)\left[P_{n+1}(t) P_n(x) - P_n(t) P_{n+1}(x) \right]\end{aligned}$$

and the lemma is proved. $\blacksquare$

We note that integration of the Christoffel-Darboux formula leads to

$$\begin{aligned}&\sum_{k=0}^{n} (2k+1) P_k(x) \int_{-1}^{1} P_k(t)\, dt \\ &\quad = (n+1) \int_{-1}^{1} \frac{P_{n+1}(t) P_n(x) - P_n(t) P_{n+1}(x)}{t-x}\, dt\end{aligned}$$

from which we deduce

$$(n+1) \int_{-1}^{1} \frac{P_{n+1}(t) P_n(x) - P_n(t) P_{n+1}(x)}{t-x}\, dt = 2 \tag{4.75}$$

where we are using the orthogonality property

$$\int_{-1}^{1} P_k(t)\,dt = \begin{cases} 0, & k \neq 0 \\ 2, & k = 0 \end{cases} \tag{4.76}$$

We are now prepared to state and prove our main result.

Theorem 4.2 If the function f is piecewise smooth in the closed interval $-1 \le x \le 1$, then the Legendre series

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

where

$$c_n = \left(n + \tfrac{1}{2}\right)\int_{-1}^{1} f(x) P_n(x)\,dx, \qquad n = 0, 1, 2, \ldots$$

converges pointwise to $f(x)$ at every continuity point of the function f in the interval $-1 < x < 1$. At points of discontinuity of f in the interval $-1 < x < 1$, the series converges to the average value $\frac{1}{2}[f(x^+) + f(x^-)]$. Finally, at $x = -1$ the series converges to $f(-1^+)$, and at $x = 1$ it converges to $f(1^-)$.

***Proof (for a point of continuity)*:** Let us assume that x is a point of continuity of the function f, and consider the nth partial sum $(-1 < x < 1)$

$$\begin{aligned} S_n(x) &= \sum_{k=0}^{n} c_k P_k(x) \\ &= \sum_{k=0}^{n} \left[\left(k + \tfrac{1}{2}\right)\int_{-1}^{1} f(t) P_k(t)\,dt\right] P_k(x) \end{aligned}$$

where we have replaced the constants c_k by their integral representation. Interchanging the order of summation and integration, and recalling the Christoffel-Darboux formula (Lemma 4.2), we obtain

$$\begin{aligned} S_n(x) &= \frac{1}{2}\int_{-1}^{1} f(t) \sum_{k=0}^{n} (2k+1) P_k(t) P_k(x)\,dt \\ &= \tfrac{1}{2}(n+1)\int_{-1}^{1} f(t)\frac{P_{n+1}(t)P_n(x) - P_n(t)P_{n+1}(x)}{t-x}\,dt \end{aligned}$$

If we add and subtract the function $f(x)$ (which is independent of the variable of integration), we get

$$\begin{aligned} S_n(x) = {} & \tfrac{1}{2}(n+1) f(x) \int_{-1}^{1} \frac{P_{n+1}(t)P_n(x) - P_n(t)P_{n+1}(x)}{t-x}\,dt \\ & + \tfrac{1}{2}(n+1)\int_{-1}^{1} \frac{f(t) - f(x)}{t-x}\left[P_{n+1}(t)P_n(x) - P_n(t)P_{n+1}(x)\right] dt \end{aligned}$$

For notational convenience we introduce the function

$$g(t) = \frac{f(t) - f(x)}{t - x}$$

and use (4.75) to obtain

$$S_n(x) = f(x) + \tfrac{1}{2}(n+1)P_n(x)\int_{-1}^{1} g(t)P_{n+1}(t)\,dt$$
$$- \tfrac{1}{2}(n+1)P_{n+1}(x)\int_{-1}^{1} g(t)P_n(t)\,dt$$

At this point we wish to show that g satisfies the conditions of Riemann's lemma, i.e., that g is at least piecewise continuous. Because f is at least piecewise smooth, it follows that g is also piecewise smooth for all $t \neq x$. To investigate the behavior of g at $t = x$, we consider the limit (remembering that x is a point of continuity of f)

$$g(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x)$$

Since by hypothesis f' is at least piecewise continuous (why?), we see that g is indeed a piecewise continuous function.

Letting

$$b_n = \left(n + \tfrac{1}{2}\right)^{1/2}\int_{-1}^{1} g(t)P_n(t)\,dt$$

we can express the nth partial sum in the form

$$S_n(x) = f(x) + \frac{(n+1)P_n(x)}{2\left(n+\frac{3}{2}\right)^{1/2}} b_{n+1} - \frac{(n+1)P_{n+1}(x)}{2\left(n+\frac{1}{2}\right)^{1/2}} b_n$$

By recognizing that the Legendre polynomials are bounded on the interval $-1 < x < 1$ [see Equation (4.56)], and applying Riemann's lemma, it can now be shown that the last two terms in the expression for $S_n(x)$ vanish in the limit as $n \to \infty$ (see problem 10), and hence we deduce our intended result

$$\lim_{n \to \infty} S_n(x) = f(x)$$

at a point of continuity of f. ∎

To prove that*

$$\lim_{n \to \infty} S_n(x) = \tfrac{1}{2}\left[f(x^+) + f(x^-)\right]$$

---

*For details, see D. Jackson, *Fourier Series and Orthogonal Polynomials*, Carus Math. Monogr. **6**, LaSalle, Ill.: Math. Assoc. Amer., Open Court Publ. Co., 1941.

at a point of discontinuity of f requires only a slight modification of the above proof. Similar comments can be made about the points $x = \pm 1$.

EXERCISES 4.5

In problems 1–8, discuss whether the function is piecewise continuous, continuous, piecewise smooth, smooth, or none of these in the interval $-1 \leq x \leq 1$.

1. $f(x) = \tan 2x$.

2. $f(x) = \sin x$.

3. $f(x) = \dfrac{x^2 - 1}{x - 1}$, $x \neq 1$.

4. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational.} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

5. $f(x) = \dfrac{\sin x}{x}$, $x \neq 0$, $f(0) = 1$.

6. $f(x) = \dfrac{\sin x}{x}$, $x \neq 0$.

7. $f(x) = \sin(1/x)$, $x \neq 0$.

8. $f(x) = xe^{-1/x}$, $x \neq 0$.

9. Suppose that a piecewise smooth function f is to be approximated on the interval $-1 \leq x \leq 1$ by the finite sum

$$S_n(x) = \sum_{k=0}^{n} b_k P_k(x), \qquad -1 \leq x \leq 1$$

Determine the constants b_k so that the *mean square error* is minimized, i.e., minimize

$$E_n = \int_{-1}^{1} [f(x) - S_n(x)]^2 \, dx$$

Hint: Set $\partial E_n / \partial b_k = 0$, $k = 1, 2, \ldots, n$.

10. Given that

$$P_n(x) < \left[\frac{\pi}{2n(1 - x^2)} \right]^{1/2}, \qquad |x| < 1$$

and

$$b_n = (n + \tfrac{1}{2}) \int_{-1}^{1} g(t) P_n(t) \, dt$$

where $g(t)$ is piecewise continuous, deduce that

(a) $\displaystyle \lim_{n \to \infty} \frac{(n+1) P_n(x)}{2(n + \frac{3}{2})^{1/2}} b_{n+1} = 0,$

(b) $\displaystyle \lim_{n \to \infty} \frac{(n+1) P_{n+1}(x)}{2(n + \frac{1}{2})^{1/2}} b_n = 0.$

4.6 *Legendre Functions of the Second Kind*

The Legendre polynomial $P_n(x)$ represents only one solution of Legendre's equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \tag{4.77}$$

Because the equation is second-order, we know from the general theory of differential equations that there exists a second linearly independent solution $Q_n(x)$ such that the combination

$$y = C_1P_n(x) + C_2Q_n(x) \tag{4.78}$$

where C_1 and C_2 are arbitrary constants, is a general solution of (4.77).

Also from the theory of second-order linear DEs it is well known that if $y_1(x)$ is a nontrivial solution of

$$y'' + a(x)y' + b(x)y = 0 \tag{4.79}$$

then a second linearly independent solution can be defined by*

$$y_2(x) = y_1(x)\int \frac{\exp\left[-\int a(x)\,dx\right]}{y_1^2(x)}\,dx \tag{4.80}$$

Hence, if we express (4.77) in the form

$$y'' - \frac{2x}{1 - x^2}y' + \frac{n(n + 1)}{1 - x^2}y = 0$$

and let $y_1(x) = P_n(x)$, it follows that

$$y_2(x) = P_n(x)\int \frac{dx}{(1 - x^2)[P_n(x)]^2} \tag{4.81}$$

is a second solution, linearly independent of $P_n(x)$. Because any linear combination of solutions is also a solution of a homogeneous DE, it has become customary to define the second solution of (4.77), not by (4.81), but by

$$Q_n(x) = P_n(x)\left\{A_n + B_n\int \frac{dx}{(1 - x^2)[P_n(x)]^2}\right\} \tag{4.82}$$

where A_n and B_n are constants to be chosen for each n. We refer to $Q_n(x)$ as the *Legendre function of the second kind* of integral order.

Accordingly, when $n = 0$ we choose $A_0 = 0$ and $B_0 = 1$, and hence

$$Q_0(x) = \int \frac{dx}{1 - x^2}$$

---

*See Theorem 4.5 in L.C. Andrews, *Ordinary Differential Equations with Applications*, Glenview, Ill.: Scott, Foresman and Co., 1982.

which leads to

$$Q_0(x) = \tfrac{1}{2}\log\frac{1+x}{1-x}, \qquad |x| < 1 \tag{4.83}$$

For $n = 1$, we set $A_1 = 0$ and $B_1 = 1$, from which we obtain

$$\begin{aligned} Q_1(x) &= x\int\frac{dx}{(1-x^2)x^2} \\ &= x\int\left(\frac{1}{1-x^2} + \frac{1}{x^2}\right)dx \\ &= \tfrac{1}{2}x\log\frac{1+x}{1-x} - 1 \end{aligned} \tag{4.84}$$

or

$$Q_1(x) = xQ_0(x) - 1, \qquad |x| < 1 \tag{4.85}$$

Rather than continuing in this fashion, which leads to more difficult integrals to evaluate, we recall the Remark made at the end of Section 4.2.3 which stated that all (properly normalized) solutions of Legendre's equation automatically satisfy the recurrence formulas for $P_n(x)$. Hence, we will select the Legendre functions $Q_n(x)$ so that necessarily

$$Q_{n+1}(x) = \frac{2n+1}{n+1}xQ_n(x) - \frac{n}{n+1}Q_{n-1}(x) \tag{4.86}$$

for $n = 1, 2, 3, \ldots$. With $Q_0(x)$ and $Q_1(x)$ already defined, the substitution of $n = 1$ into (4.86) yields

$$\begin{aligned} Q_2(x) &= \tfrac{3}{2}xQ_1(x) - \tfrac{1}{2}Q_0(x) \\ &= \tfrac{1}{2}(3x^2 - 1)Q_0(x) - \tfrac{3}{2}x \end{aligned}$$

which we recognize as

$$Q_2(x) = P_2(x)Q_0(x) - \tfrac{3}{2}x, \qquad |x| < 1 \tag{4.87}$$

For $n = 2$, we find

$$Q_3(x) = P_3(x)Q_0(x) - \tfrac{5}{2}x^2 + \tfrac{2}{3}, \qquad |x| < 1 \tag{4.88}$$

whereas in general it has been shown that*

$$Q_n(x) = P_n(x)Q_0(x) - \sum_{k=0}^{[\frac{1}{2}(n-1)]}\frac{(2n-4k-1)}{(2k+1)(n-k)}P_{n-2k-1}(x), \qquad |x| < 1 \tag{4.89}$$

for $n = 1, 2, 3, \ldots$.

*See W.W. Bell, *Special Functions for Scientists and Engineers*, London: Van Nostrand, 1968, pp. 71–77.

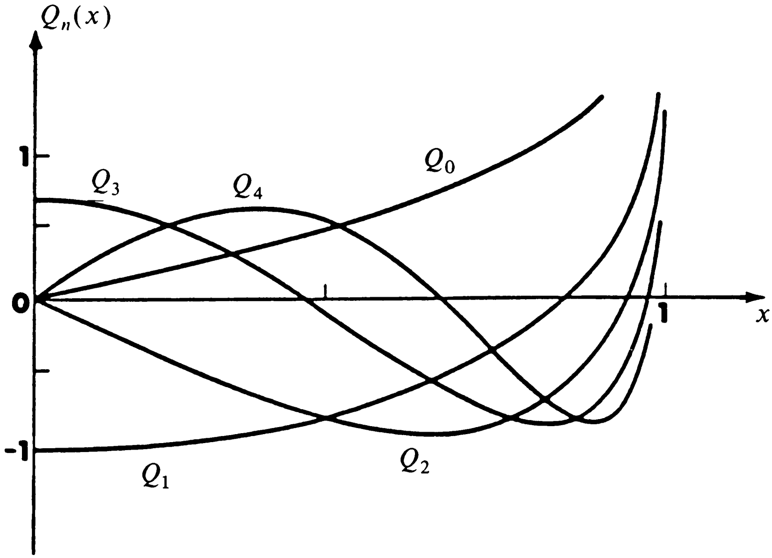


Figure 4.5 Graph of $Q_n(x)$, $n = 0, 1, 2, 3, 4$

Because of the logarithm term in $Q_0(x)$, it becomes clear that all $Q_n(x)$ have infinite discontinuities at $x = \pm 1$. However, within the interval $-1 < x < 1$ these functions are well defined. The first few Legendre functions of the second kind are sketched in Fig. 4.5 for the interval $0 \le x < 1$.

In some applications it is important to consider $Q_n(x)$ defined on the interval $x > 1$. While Equation (4.89) is not valid for $x > 1$, the functions, $Q_n(x)$ can be expanded in a convergent asymptotic series (problem 16). Based on this series, it can then be shown that all $Q_n(x)$ approach zero as $x \to \infty$. Such behavior for large x is quite distinct from that of the Legendre polynomials $P_n(x)$, which become unbounded as $x \to \infty$ except for $P_0(x) = 1$.

4.6.1 *Basic Properties*

We have already mentioned that the Legendre functions $Q_n(x)$ satisfy all recurrence relations given in Section 4.2.2 for $P_n(x)$. In addition, there are several relations that directly involve both $P_n(x)$ and $Q_n(x)$. For example, if $|t| < |x|$, then*

$$\frac{1}{x - t} = \sum_{n=0}^{\infty} (2n + 1) P_n(t) Q_n(x) \tag{4.90}$$

*See E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge U.P., 1965, pp. 321–322.

From this result, it is easily shown that (see problem 13)

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\,dt, \qquad n = 0,1,2,\ldots \tag{4.91}$$

which is called the *Neumann formula*. Other properties are taken up in the exercises.

EXERCISES 4.6

In problems 1–4, find a general solution of the DE in terms of $P_n(x)$ and $Q_n(x)$.

1. $(1-x^2)y'' - 2xy' = 0.$

2. $(1-x^2)y'' - 2xy' + 2y = 0.$

3. $(1-x^2)y'' - 2xy' + 12y = 0.$

4. $(1-x^2)y'' - 2xy' + 30y = 0.$

5. Given $P_0(x) = 1$ and $Q_0(x) = \frac{1}{2}\log[(1+x)/(1-x)]$, verify directly that their Wronskian* satisfies

$$W(P_0, Q_0)(x) = \frac{1}{1-x^2}$$

6. Use Equation (4.82) for $Q_n(x)$ to deduce that, in general, the Wronskian of $P_n(x)$ and $Q_n(x)$ is given by

$$W(P_n, Q_n)(x) = \frac{1}{1-x^2}, \qquad n = 0,1,2,\ldots$$

7. Show that $Q_n(x)$ satisfies the relations $(n = 1,2,3,\ldots)$

(a) $Q'_{n+1}(x) - 2xQ'_n(x) + Q'_{n-1}(x) - Q_n(x) = 0,$
(b) $Q'_{n+1}(x) - xQ'_n(x) - (n+1)Q_n(x) = 0,$
(c) $xQ'_n(x) - Q'_{n-1}(x) - nQ_n(x) = 0,$
(d) $Q'_{n+1}(x) - Q'_{n-1}(x) = (2n+1)Q_n(x),$
(e) $(1-x^2)Q'_n(x) = n[Q_{n-1}(x) - xQ_n(x)].$

8. Show that

(a) $Q_0(-x) = -Q_0(x),$
(b) $Q_n(-x) = (-1)^{n+1}Q_n(x),\ n = 1,2,3,\ldots.$

9. Show that (for $n = 1,2,3,\ldots$)

$$\begin{aligned} &n\left[Q_n(x)P_{n-1}(x) - Q_{n-1}(x)P_n(x)\right] \\ &\quad = (n-1)\left[Q_{n-1}(x)P_{n-2}(x) - Q_{n-2}(x)P_{n-1}(x)\right] \end{aligned}$$

---

*Recall that the Wronskian is defined by $W(y_1, y_2) = y_1y_2' - y_1'y_2$.

10. From the result of problem 9, deduce that ($n = 1, 2, 3, \ldots$)

$$Q_n(x)P_{n-1}(x) - Q_{n-1}(x)P_n(x) = -\frac{1}{n}$$

11. Deduce the result of problem 10 by using the Wronskian relation in problem 6 and appropriate recurrence relations.

12. Show that $Q_n(x)$ satisfies the Christoffel-Darboux formula

$$\sum_{k=0}^{n} (2k+1)Q_k(t)Q_k(x) = \frac{n+1}{t-x}[Q_{n+1}(t)Q_n(x) - Q_n(t)Q_{n+1}(x)]$$

13. Use the result of Equation (4.90) to deduce the Neumann formula

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{x-t}\,dt, \qquad |x| > 1$$

14. For $x > 1$, use the Neumann formula in problem 13 to show that

$$Q_n(x) = \frac{1}{2^{n+1}}\int_{-1}^{1} \frac{(1-t^2)^n}{(x-t)^{n+1}}\,dt$$

15. Using the result of problem 14, deduce that ($x > 1$)

(a) $Q_n(x) = \displaystyle\int_0^{\infty} \frac{d\theta}{\left[x + (x^2-1)^{1/2}\cosh\theta\right]^{n+1}}.$

Hint: Set $t = \dfrac{e^{\theta}(x+1)^{1/2} - (x-1)^{1/2}}{e^{\theta}(x+1)^{1/2} + (x-1)^{1/2}}.$

(b) $Q_n(x) \sim \dfrac{2^n}{x^{n+1}} \displaystyle\sum_{k=0}^{\infty} \frac{(n+k)!(n+2k)!}{k!(2n+2k+1)!}\frac{1}{x^{2k}}, \quad x \to \infty.$

16. Solve Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

by assuming a power series solution of the form $y = \sum_{m=0}^{\infty} c_m x^m$.

(a) Show that the general solution is

$$y = Ay_1(x) + By_2(x)$$

where A and B are any constants and

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots$$

and

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots$$

(b) For $n = 0$, show that

$$P_0(x) = \frac{y_1(x)}{y_1(1)}, \qquad Q_0(x) = y_1(1)y_2(x)$$

(c) For $n = 1$, show that

$$P_1(x) = \frac{y_2(x)}{y_2(1)}, \qquad Q_1(x) = -y_2(1)y_1(x)$$

4.7 *Associated Legendre Functions*

In applications involving either the Laplace or the Helmholtz equation in spherical, oblate spheroidal, or prolate spheroidal coordinates, it is not Legendre's equation (4.44) that ordinarily arises but rather the *associated Legendre equation*

$$(1 - x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \qquad (4.92)$$

Observe that for $m = 0$, (4.92) reduces to Legendre's equation (4.44). The DE (4.92) and its solutions, called *associated Legendre functions*, can be developed directly from Legendre's equation and its solutions. To show this we will need the *Leibniz formula* for the mth derivative of a product,

$$\frac{d^m}{dx^m}(fg) = \sum_{k=0}^{m}\binom{m}{k}\frac{d^{m-k}f}{dx^{m-k}}\frac{d^k g}{dx^k}, \qquad m = 1, 2, 3, \ldots \qquad (4.93)$$

If z is a solution of Legendre's equation, i.e., if

$$(1 - x^2)z'' - 2xz' + n(n+1)z = 0 \qquad (4.94)$$

we wish to show that

$$y = (1 - x^2)^{m/2}\frac{d^m z}{dx^m} \qquad (4.95)$$

is then a solution of (4.92). By taking m derivatives of (4.94), we get

$$\frac{d^m}{dx^m}\left[(1 - x^2)z''\right] - 2\frac{d^m}{dx^m}(xz') + n(n+1)\frac{d^m z}{dx^m} = 0$$

which, applying the Leibniz formula (4.93), becomes

$$(1-x^2)\frac{d^{m+2}z}{dx^{m+2}} - 2mx\frac{d^{m+1}z}{dx^{m+1}} - m(m-1)\frac{d^m z}{dx^m}$$
$$-2\left[x\frac{d^{m+1}z}{dx^{m+1}} + m\frac{d^m z}{dx^m}\right] + n(n+1)\frac{d^m z}{dx^m} = 0$$

Collecting like terms gives us

$$(1-x^2)\frac{d^2u}{dx^2} - 2(m+1)x\frac{du}{dx} + [n(n+1) - m(m+1)]u = 0 \tag{4.96}$$

where, for notational convenience, we have set $u = d^m z/dx^m$. Next, by introducing the new variable $y = (1-x^2)^{m/2}u$, or equivalently,

$$u = y(1-x^2)^{-m/2}$$

we find that (4.96) takes the form

$$(1-x^2)\frac{d^2}{dx^2}\left[y(1-x^2)^{-m/2}\right] - 2(m+1)x\frac{d}{dx}\left[y(1-x^2)^{-m/2}\right]$$
$$+[n(n+1) - m(m+1)]\,y(1-x^2)^{-m/2} = 0 \tag{4.97}$$

Carrying out the indicated derivatives in (4.97) leads to

$$\frac{d}{dx}\left[y(1-x^2)^{-m/2}\right] = y'(1-x^2)^{-m/2} + mxy(1-x^2)^{-1-m/2}$$
$$= \left[y' + \frac{mxy}{1-x^2}\right](1-x^2)^{-m/2} \tag{4.98}$$

and similarly

$$\frac{d^2}{dx^2}\left[y(1-x^2)^{-m/2}\right]$$
$$= \left[y'' + \frac{m(2xy'+y)}{1-x^2} + \frac{m(m+2)x^2y}{(1-x^2)^2}\right](1-x^2)^{-m/2} \tag{4.99}$$

Finally, the substitution of (4.98) and (4.99) into (4.97), and cancellation of the common factor $(1-x^2)^{-m/2}$, then yields

$$(1-x^2)\left[y'' + \frac{m(2xy'+y)}{1-x^2} + \frac{m(m+2)x^2y}{(1-x^2)^2}\right]$$
$$-2(m+1)x\left[y' + \frac{mxy}{1-x^2}\right] + [n(n+1) - m(m+1)]\,y = 0$$

which reduces to (4.92) upon algebraic simplification.

We define the *associated Legendre functions of the first and second kinds*, respectively, by ($m = 0, 1, 2, \dots, n$)

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) \tag{4.100}$$

and

$$Q_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} Q_n(x) \tag{4.101}$$

Since $P_n(x)$ and $Q_n(x)$ are solutions of Legendre's equation, it follows from (4.95) that $P_n^m(x)$ and $Q_n^m(x)$ are solutions of the associated Legendre equation (4.92).

The associated Legendre functions have many properties in common with the simpler Legendre polynomials $P_n(x)$ and Legendre functions of the second kind $Q_n(x)$. Many of these properties can be developed directly from the corresponding relation involving either $P_n(x)$ or $Q_n(x)$ by taking derivatives and applying the definitions (4.100) and (4.101).

4.7.1 *Basic Properties of $P_n^m(x)$*

Using the Rodrigues formula (4.47), it is possible to write (4.100) in the form

$$P_n^m(x) = \frac{1}{2^n n!}(1 - x^2)^{m/2} \frac{d^{n+m}}{dx^{n+m}}\left[(x^2 - 1)^n\right] \tag{4.102}$$

Here we make the interesting observation that the right-hand side of (4.102) is well defined for all values of m such that $n + m \geq 0$, i.e., for $m \geq -n$, whereas (4.100) is valid only for $m \geq 0$. Thus, (4.102) may be used to extend the definition of $P_n^m(x)$ to include all integer values of m such that $-n \leq m \leq n$. (If $m > n$, then necessarily $P_n^m(x) \equiv 0$, which we leave to the reader to prove.) Moreover, using the Leibniz formula (4.93) once again, it can be shown that (see problem 5)

$$P_n^{-m}(x) = (-1)^m \frac{(n - m)!}{(n + m)!} P_n^m(x) \tag{4.103}$$

Lastly, we note that for $m = 0$ we get the special case

$$P_n^0(x) = P_n(x) \tag{4.104}$$

The associated Legendre functions $P_n^m(x)$ satisfy many recurrence relations, several of which are generalizations of the recurrence formulas for $P_n(x)$. But because $P_n^m(x)$ has two indices instead of just one, there exists a wider variety of possible relations than for $P_n(x)$.

To derive the *three-term recurrence formula* for $P_n^m(x)$, we start with the known relation [see Equation (4.31)]

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad (4.105)$$

and differentiate it m times to obtain

$$(n+1)\frac{d^m}{dx^m}P_{n+1}(x) - (2n+1)x\frac{d^m}{dx^m}P_n(x)$$
$$-m(2n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x) + n\frac{d^m}{dx^m}P_{n-1}(x) = 0 \qquad (4.106)$$

Now recalling [Equation (4.40)]

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

we find that taking $m-1$ derivatives leads to

$$m(2n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x) = m\frac{d^m}{dx^m}P_{n+1}(x) - m\frac{d^m}{dx^m}P_{n-1}(x) \qquad (4.107)$$

and using this result, (4.106) becomes

$$(n-m+1)\frac{d^m}{dx^m}P_{n+1}(x) - (2n+1)x\frac{d^m}{dx^m}P_n(x)$$
$$+(n+m)\frac{d^m}{dx^m}P_{n-1}(x) = 0$$

Finally, multiplication of this last result by $(1-x^2)^{m/2}$ yields the desired recurrence formula

$$(n-m+1)P_{n+1}^m(x) - (2n+1)xP_n^m(x) + (n+m)P_{n-1}^m(x) = 0 \qquad (4.108)$$

Additional recurrence relations, which are left to the exercises for verification, include the following:

$$(1-x^2)^{1/2}P_n^m(x) = \frac{1}{2n+1}\left[P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)\right] \qquad (4.109)$$

$$(1-x^2)^{1/2}P_n^m(x) = \frac{1}{2n+1}\left[(n+m)(n+m-1)P_{n-1}^{m-1}(x)\right.$$
$$\left.-(n-m+1)(n-m+2)P_{n+1}^{m-1}(x)\right] \qquad (4.110)$$

$$P_n^{m+1}(x) = 2mx(1-x^2)^{-1/2}P_n^m(x)$$
$$-\left[n(n+1) - m(m-1)\right]P_n^{m-1}(x) \qquad (4.111)$$

By constructing a proof exactly analogous to the proof of orthogonality of the Legendre polynomials, it can be shown that

$$\int_{-1}^{1} P_n^m(x) P_k^m(x)\, dx = 0, \qquad k \neq n \tag{4.112}$$

Also, the evaluation of

$$\int_{-1}^{1} [P_n^m(x)]^2\, dx = \frac{2(n+m)!}{(2n+1)(n-m)!} \tag{4.113}$$

follows exactly our derivation of (4.66) given in Section 4.4.1. The details of proving (4.112) and (4.113) are left for the exercises.

As a final comment we mention that, although it is essentially only a mathematical curiosity, there is another orthogonality relation for the associated Legendre functions given by

$$\int_{-1}^{1} P_n^m(x) P_n^k(x)(1-x^2)^{-1}\, dx = \begin{cases} 0, & k \neq m \\ \dfrac{(n+m)!}{m(n-m)!}, & k = m \end{cases} \tag{4.114}$$

EXERCISES 4.7

1. Directly from Equation (4.100), show that

(a) $P_1^1(x) = (1-x^2)^{1/2}$, (b) $P_2^1(x) = 3x(1-x^2)^{1/2}$,
(c) $P_2^2(x) = 3(1-x^2)$. (d) $P_3^1(x) = \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}$,
(e) $P_3^2(x) = 15x(1-x^2)$,

2. Show that
(a) $P_n^m(-x) = (-1)^{n+m} P_n^m(x)$,
(b) $P_n^m(\pm 1) = 0$, $m > 0$.

3. Show that (for $n = 0, 1, 2, \ldots$)

(a) $P_{2n}^1(0) = 0$, (b) $P_{2n+1}^1(0) = \dfrac{(-1)^n (2n+1)!}{2^{2n}(n!)^2}$.

4. Show that
(a) $P_n^m(0) = 0$, $n + m$ odd,
(b) $P_n^m(0) = (-1)^{(n-m)/2} \dfrac{(n+m)!}{2^n[(n-m)/2]![(n+m)/2]!}$, $n + m$ even.

5. By applying the Leibniz formula (4.93) to the product $(x+1)^n(x-1)^n$ and using (4.102), verify that

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

6. Derive the generating function

$$\frac{(2m)!(1-x^2)^{m/2}}{2^m m!(1-2xt+t^2)^{m+\frac{1}{2}}} = \sum_{n=0}^{\infty} P_{n+m}^m(x)t^n$$

In problems 7–11, derive the given recurrence formula.

7. $(1-x^2)P_n^{m\prime}(x) = (n+m)P_{n-1}^m(x) - nxP_n^m(x)$.

8. $(1-x^2)P_n^{m\prime}(x) = (n+1)xP_n^m(x) - (n-m+1)P_{n+1}^m(x)$.

9. $(1-x^2)^{1/2}P_n^m(x) = \dfrac{1}{2n+1}[P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)]$.

10. $(1-x^2)^{1/2}P_n^m(x) = \dfrac{1}{2n+1}[(n+m)(n+m-1)P_{n-1}^{m-1}(x)$
$-(n-m+1)(n-m+2)P_{n+1}^{m-1}(x)]$.

11. $P_n^{m+1}(x) = 2mx(1-x^2)^{-1/2}P_n^m(x) - [n(n+1) - m(m-1)]P_n^{m-1}(x)$.

12. Prove the orthogonality relation

$$\int_{-1}^{1} P_n^m(x)P_k^m(x)\,dx = 0, \qquad k \neq n$$

13. Prove the orthogonality relation

$$\int_{-1}^{1} P_n^m(x)P_n^k(x)(1-x^2)^{-1}\,dx = 0, \qquad k \neq m$$

14. By defining

$$A_n = \int_{-1}^{1} [P_n^m(x)]^2\,dx, \qquad n = 0,1,2,\ldots$$

show that

(a) $A_n = \dfrac{(2n-1)(n+m)}{(2n+1)(n-m)}A_{n-1},\ n = 2,3,4,\ldots$.

(b) Evaluate A_0 and A_1 directly and use (a) to deduce that

$$A_n = \frac{2(n+m)!}{(2n+1)(n-m)!}, \qquad n = 0,1,2,\ldots$$

15. Show that

$$\int_{-1}^{1} [P_n^m(x)]^2(1-x^2)^{-1}\,dx = \frac{(n+m)!}{m(n-m)!}$$

Other Orthogonal Polynomials

5.1 *Introduction*

A set of functions $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$, is said to be *orthogonal* on the interval $a < x < b$, with respect to a weight function $r(x) > 0$, if*

$$\int_a^b r(x)\phi_n(x)\phi_k(x)\,dx = 0, \qquad k \neq n$$

Sets of orthogonal functions play an extremely important role in analysis, primarily because functions belonging to a very general class can be represented by series of orthogonal functions, called *generalized Fourier series*.

A special case of orthogonal functions consists of the sets of *orthogonal polynomials* $\{p_n(x)\}$, where n denotes the degree of the polynomial $p_n(x)$. The Legendre polynomials discussed in Chapter 4 are probably the simplest set of polynomials belonging to this class. Other polynomial sets which commonly occur in applications are the *Hermite*, *Laguerre*, and *Chebyshev polynomials*. More general polynomial sets are defined by the *Gegenbauer* and *Jacobi polynomials*, which include the others as special cases.

The study of general polynomial sets like the Jacobi polynomials facilitates the study of each polynomial set by focusing upon those properties

---

*In some cases the interval of orthogonality may be of infinite extent.

that are characteristic of all the individual sets. For example, the sets $\{p_n(x)\}$ that we will study all satisfy a second-order linear DE and Rodrigues formula, and the related set $\{(d^m/dx^m)p_n(x)\}$ (e.g., the associated Legendre functions) is also orthogonal. Moreover, it can be shown that any orthogonal polynomial set satisfying these three conditions is necessarily a member of the Jacobi polynomial set, or a limiting case such as the Hermite and Laguerre polynomials.

5.2 *Hermite Polynomials*

The *Hermite polynomials* play an important role in problems involving Laplace's equation in parabolic coordinates, in various problems in quantum mechanics, and in probability theory.

We define the Hermite polynomials $H_n(x)$ by means of the generating function*

$$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \qquad |t| < \infty, \quad |x| < \infty \tag{5.1}$$

By writing

$$\begin{aligned}\exp(2xt - t^2) &= e^{2xt} \cdot e^{-t^2} \\ &= \left(\sum_{m=0}^{\infty} \frac{(2xt)^m}{m!}\right)\left(\sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!}\right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!} t^n \end{aligned} \tag{5.2}$$

where the last step follows from the index change $m = n - 2k$ [see Equation (1.17) in Section 1.2.3], we identify

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k} \tag{5.3}$$

Examination of the series (5.3) reveals that $H_n(x)$ is a polynomial of degree n, and further, is an even function of x for even n and an odd function of x for odd n. Thus, it follows that

$$H_n(-x) = (-1)^n H_n(x) \tag{5.4}$$

The first few Hermite polynomials are listed in Table 5.1 for easy reference.

---

*There is another definition of the Hermite polynomials that uses the generating function $\exp(xt - \frac{1}{2}t^2)$. This definition occurs most often in statistical applications.

Table 5.1 Hermite polynomials

$H_0(x) = 1$
$H_1(x) = 2x$
$H_2(x) = 4x^2 - 2$
$H_3(x) = 8x^3 - 12x$
$H_4(x) = 16x^4 - 48x^2 + 12$
$H_5(x) = 32x^5 - 160x^3 + 120x$

In addition to the series (5.3), the Hermite polynomials can be defined by the *Rodrigues formula* (see problem 3)

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}), \qquad n = 0, 1, 2, \ldots \tag{5.5}$$

and the *integral representation* (see problem 5)

$$H_n(x) = \frac{(-i)^n 2^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2 + 2ixt} t^n \, dt, \qquad n = 0, 1, 2, \ldots \tag{5.6}$$

The Hermite polynomials have many properties in common with the Legendre polynomials, and in fact, there are many relations connecting the two sets of polynomials. For example, two of the simplest relations are given by $(n = 0, 1, 2, \ldots)$

$$\frac{2}{n!\sqrt{\pi}} \int_0^{\infty} e^{-t^2} t^n H_n(xt) \, dt = P_n(x) \tag{5.7}$$

and

$$2^{n+1} e^{x^2} \int_x^{\infty} e^{-t^2} t^{n+1} P_n(x/t) \, dt = H_n(x) \tag{5.8}$$

the verifications of which are left for the exercises.

Example 1: Use the generating function to derive the relation

$$x^n = \sum_{k=0}^{[n/2]} \frac{n! H_{n-2k}(x)}{2^n k!(n-2k)!}$$

Solution: From (5.1) we have

$$\exp(2xt - t^2) = \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$

or

$$e^{2xt} = e^{t^2} \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$

Expressing both exponentials in power series leads to

$$\sum_{n=0}^{\infty} \frac{(2x)^n t^n}{n!} = \sum_{m=0}^{\infty} \frac{t^{2m}}{m!} \cdot \sum_{k=0}^{\infty} H_k(x) \frac{t^k}{k!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(x) t^n}{k!(n-2k)!}$$

where the last step results from the change of index $m = n - 2k$. Finally, by comparing the coefficients of t^n in the two series, we deduce that

$$x^n = \sum_{k=0}^{[n/2]} \frac{n! H_{n-2k}(x)}{2^n k!(n-2k)!}$$

5.2.1 *Recurrence Relations*

By substituting the series for $w(x, t) = \exp(2xt - t^2)$ into the identity

$$\frac{\partial w}{\partial t} - 2(x - t)w = 0 \tag{5.9}$$

we obtain (after some manipulation)

$$\sum_{n=1}^{\infty} [H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x)] \frac{t^n}{n!} + H_1(x) - 2xH_0(x) = 0 \tag{5.10}$$

But $H_1(x) - 2xH_0(x) = 0$, and thus we deduce the recurrence formula

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \tag{5.11}$$

for $n = 1, 2, 3, \ldots$.

Another recurrence relation satisfied by the Hermite polynomials follows the substitution of the series for $w(x, t)$ into

$$\frac{\partial w}{\partial t} - 2tw = 0 \tag{5.12}$$

This time we find

$$\sum_{n=1}^{\infty} [H_n'(x) - 2nH_{n-1}(x)] \frac{t^n}{n!} = 0$$

which leads to

$$H_n'(x) = 2nH_{n-1}(x), \qquad n = 1, 2, 3, \ldots \tag{5.13}$$

The elimination of $H_{n-1}(x)$ from (5.11) and (5.13) yields

$$H_{n+1}(x) - 2xH_n(x) + H_n'(x) = 0 \tag{5.14}$$

and by differentiating this expression and using (5.13) once again, we find

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{5.15}$$

for $n = 0, 1, 2, \ldots$. Therefore we see that $y = H_n(x)$ $(n = 0, 1, 2, \ldots)$ is a solution of the linear second-order DE

$$y'' - 2xy' + 2ny = 0 \tag{5.16}$$

called *Hermite's equation*.

5.2.2 *Hermite Series*

The *orthogonality property* of the Hermite polynomials is given by*

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_k(x)\, dx = 0, \qquad k \neq n \tag{5.17}$$

We could construct a proof of (5.17) analogous to that given in Section 4.4.1 for the Legendre polynomials, but for the Hermite polynomials an interesting alternative proof exists.

Let us start with the generating-function relations

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2xt - t^2} \tag{5.18a}$$

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} H_k(x) = e^{2xs - s^2} \tag{5.18b}$$

and multiply these two series to obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^n}{n!} \frac{s^k}{k!} H_n(x) H_k(x) = \exp[-(t^2 + s^2) + 2x(t + s)] \tag{5.19}$$

Next, we multiply both sides of (5.19) by the weight function e^{-x^2} and integrate (assuming that termwise integration is permitted), to find

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^n}{n!} \frac{s^k}{k!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_k(x)\, dx &= e^{-(t^2+s^2)} \int_{-\infty}^{\infty} e^{-x^2 + 2x(t+s)}\, dx \\ &= \sqrt{\pi}\, e^{2ts} \end{aligned}$$

where we have made the observation (see Example 2 below)

$$\int_{-\infty}^{\infty} e^{-x^2 + 2bx}\, dx = \sqrt{\pi}\, e^{b^2} \tag{5.20}$$

Finally, expanding e^{2ts} in a power series, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^n}{n!} \frac{s^k}{k!} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_k(x)\, dx = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n t^n s^n}{n!} \tag{5.21}$$

---

*The function e^{-x^2} in (5.17) is called a *weight function*. In the case of the Legendre polynomials, the weight function is unity.

and by comparing like coefficients of $t^n s^k$, we deduce that

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_k(x)\, dx = 0, \quad k \neq n$$

As a bonus, we find that when $k = n$ in (5.21), we get the additional important result (for $n = 0, 1, 2, \ldots$)

$$\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2\, dx = 2^n n! \sqrt{\pi} \tag{5.22}$$

Based upon the relations (5.17) and (5.22), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in a series of Hermite polynomials. Specifically, if f is a suitable function defined for all x, we look for expansions of the general form

$$f(x) = \sum_{n=0}^{\infty} c_n H_n(x), \qquad -\infty < x < \infty \tag{5.23}$$

where the (Fourier) coefficients are given by*

$$c_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x)\, dx, \qquad n = 0, 1, 2, \ldots \tag{5.24}$$

Series of this type are called *Hermite series*. We have the following theorem for them.

Theorem 5.1. If f is piecewise smooth in every finite interval and

$$\int_{-\infty}^{\infty} e^{-x^2} f^2(x)\, dx < \infty$$

then the Hermite series (5.23) with constants defined by (5.24) converges pointwise to $f(x)$ at every continuity point of f. At points of discontinuity, the series converges to the average value $\frac{1}{2}[f(x^+) + f(x^-)]$.

The proof of Theorem 5.1 closely follows that of Theorem 4.1 [see N.N. Lebedev, *Special Functions and Their Applications*, New York: Dover, 1972, pp. 71–73].

Example 2: Express $f(x) = e^{2bx}$ in a Hermite series and use this result to deduce the value of the integral

$$\int_{-\infty}^{\infty} e^{-x^2 + 2bx} H_n(x)\, dx$$

---

*The constants c_n can be formally derived through use of the orthogonality property analogous to the technique used in Section 4.4.2.

Solution: In this case we can obtain the series in an indirect way. We simply set $t = b$ in the generating function (5.1) to obtain

$$\exp(2bx - b^2) = \sum_{n=0}^{\infty} \frac{b^n}{n!} H_n(x)$$

and hence we have our intended series

$$e^{2bx} = e^{b^2} \sum_{n=0}^{\infty} \frac{b^n}{n!} H_n(x)$$

The direct derivation of this result from (5.24) leads to

$$c_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2+2bx} H_n(x)\, dx, \qquad n = 0, 1, 2, \ldots$$

However, we have already shown that

$$c_n = \frac{b^n}{n!} e^{b^2}$$

and thus it follows that

$$\int_{-\infty}^{\infty} e^{-x^2+2bx} H_n(x)\, dx = \sqrt{\pi}\,(2b)^n e^{b^2}, \qquad n = 0, 1, 2, \ldots$$

In particular, for $n = 0$ we get the result of Equation (5.20).

5.2.3 *Simple Harmonic Oscillator*

A fundamental problem in quantum mechanics involving Schrödinger's equation concerns the one-dimensional motion of a particle bound in a potential well. It has been established that bounded solutions of Schrödinger's equation for such problems are obtainable only for certain discrete energy levels of the particle within the well. A particular example of this important class of problems is the *harmonic oscillator problem*, the solutions of which lead to Hermite polynomials.

In terms of dimensionless parameters, Schrödinger's equation for the harmonic-oscillator problem takes the form

$$\psi'' + (\lambda - x^2)\psi = 0, \qquad -\infty < x < \infty \tag{5.25}$$

The parameter λ is proportional to the possible energy levels of the oscillator and ψ is related to the corresponding wave function. In addition to (5.25), the solution ψ must satisfy the boundary condition

$$\lim_{|x| \to \infty} \psi(x) = 0 \tag{5.26}$$

In looking for bounded solutions of (5.25), we start with the observation that λ becomes negligible compared with x^2 for large values of x. Thus, asymptotically we expect the solution of (5.25) to behave like

$$\psi(x) \sim e^{\pm x^2/2}, \qquad |x| \to \infty \tag{5.27}$$

where only the negative sign in the exponent is appropriate in order that (5.26) be satisfied. Based upon this observation, we make the assumption that (5.25) has solutions of the form

$$\psi(x) = y(x)e^{-x^2/2} \tag{5.28}$$

for suitable y. The substitution of (5.28) into (5.25) yields the DE

$$y'' - 2xy' + (\lambda - 1)y = 0 \tag{5.29}$$

The boundary condition (5.26) suggests that whatever functional form y assumes, it must either be finite for all x or approach infinity at a rate slower than $e^{-x^2/2}$ approaches zero. It has been shown* that the only solutions of (5.29) satisfying this condition are those for which $\lambda - 1 = 2n$, or

$$\lambda \equiv \lambda_n = 2n + 1, \qquad n = 0, 1, 2, \ldots \tag{5.30}$$

These allowed values of λ are called *eigenvalues*, or energy levels, of the oscillator. With λ so restricted, we see that (5.29) becomes

$$y'' - 2xy' + 2ny = 0 \tag{5.31}$$

which is Hermite's equation with solutions $y = H_n(x)$. (The other solutions of Hermite's equation are not appropriate in this problem.) Hence, we conclude that to each eigenvalue λ_n given by (5.30), there corresponds the solution of (5.25) (called an *eigenfunction* or *eigenstate*) given by

$$\psi_n(x) = H_n(x)e^{-x^2/2}, \qquad n = 0, 1, 2, \ldots \tag{5.32}$$

EXERCISES 5.2

1. Show that (for $n = 0, 1, 2, \ldots$)

(a) $H_{2n}(0) = (-1)^n \dfrac{(2n)!}{n!}$, (b) $H_{2n+1}(0) = 0$,

(c) $H'_{2n}(0) = 0$, (d) $H'_{2n+1}(0) = (-1)^n \dfrac{(2n+2)!}{(n+1)!}$.

*See E.C. Kemble, *The Fundamental Principles of Quantum Mechanics with Elementary Applications*, New York: Dover, 1958, p. 87.

2. Derive the generating-function relations

(a) $e^{t^2}\cos 2xt = \sum_{n=0}^{\infty} (-1)^n H_{2n}(x)\frac{t^{2n}}{(2n)!}, \quad |t| < \infty,$

(b) $e^{t^2}\sin 2xt = \sum_{n=0}^{\infty} (-1)^n H_{2n+1}(x)\frac{t^{2n+1}}{(2n+1)!}, \quad |t| < \infty.$

3. Derive the Rodrigues formula (for $n = 0, 1, 2, \ldots$)

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

4. Starting with the integral formula

$$\int_{-\infty}^{\infty} e^{-t^2+2bt}\,dt = \sqrt{\pi}\,e^{b^2}$$

(a) show that differentiating both sides with respect to b leads to

$$\int_{-\infty}^{\infty} te^{-t^2+2bt}\,dt = \sqrt{\pi}\,be^{b^2}$$

(b) For $n = 1, 2, 3, \ldots,$ show that

$$\int_{-\infty}^{\infty} t^n e^{-t^2+2bt}\,dt = \frac{\sqrt{\pi}}{2^n}\frac{d^n}{db^n}(e^{b^2})$$

5. Set $b = ix$ in the result of problem 4(b) to deduce that (for $n = 0, 1, 2, \ldots$)

$$H_n(x) = \frac{(-i)^n 2^n e^{x^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2+2ixt}t^n\,dt$$

6. Using the result of problem 5, show that (for $n = 0, 1, 2, \ldots$)

(a) $H_{2n}(x) = \dfrac{(-1)^n 2^{2n+1}}{\sqrt{\pi}}e^{x^2}\displaystyle\int_0^{\infty} e^{-t^2}t^{2n}\cos 2xt\,dt,$

(b) $H_{2n+1}(x) = \dfrac{(-1)^n 2^{2n+2}}{\sqrt{\pi}}e^{x^2}\displaystyle\int_0^{\infty} e^{-t^2}t^{2n+1}\sin 2xt\,dt.$

7. Derive the Fourier transform relations

(a) $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2+ixt}H_n(t)\,dt = i^n e^{-x^2/2}H_n(x),$

(b) $\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} e^{-\frac{1}{2}t^2}H_{2n}(t)\cos xt\,dt = (-1)^n e^{-x^2/2}H_{2n}(x),$

(c) $\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty} e^{-\frac{1}{2}t^2}H_{2n+1}(t)\sin xt\,dt = (-1)^n e^{-x^2/2}H_{2n+1}(x).$

In problems 8–11, verify the integral relation.

8. $\int_{-\infty}^{\infty} x^k e^{-x^2} H_n(x)\,dx = 0,\ k = 0, 1, \ldots, n-1.$

9. $\int_{-\infty}^{\infty} x^2 e^{-x^2} [H_n(x)]^2\,dx = \sqrt{\pi}\, 2^n n!(n + \frac{1}{2}).$

10. $\int_0^{\infty} t^n e^{-t^2} H_n(xt)\,dt = \dfrac{\sqrt{\pi}\, n!}{2} P_n(x).$

11. $\int_x^{\infty} e^{-t^2} t^{n+1} P_n(x/t)\,dt = \dfrac{1}{2^{n+1}} e^{-x^2} H_n(x).$

12. Use the result of problem 5 to deduce that

(a) $(1 - t^2)^{-1/2} \exp\left[\dfrac{2xyt - (x^2 + y^2)t^2}{1 - t^2}\right] = \sum_{n=0}^{\infty} H_n(x) H_n(y) \dfrac{(t/2)^n}{n!},$

(b) $(1 - t^2)^{-1/2} \exp\left(\dfrac{2x^2 t}{1 + t}\right) = \sum_{n=0}^{\infty} [H_n(x)]^2 \dfrac{(t/2)^n}{n!}.$

13. Use problem 12 to show that $(n = 0, 1, 2, \ldots)$

$$\int_{-\infty}^{\infty} e^{-2x^2} [H_n(x)]^2\,dx = 2^{n-\frac{1}{2}} \Gamma(n + \tfrac{1}{2})$$

14. Derive the Hermite series relations

(a) $x^{2k} = \dfrac{(2k)!}{2^{2k}} \sum_{n=0}^{k} \dfrac{H_{2n}(x)}{(2n)!(k - n)!},$

(b) $x^{2k+1} = \dfrac{(2k + 1)!}{2^{2k+1}} \sum_{n=0}^{k} \dfrac{H_{2n+1}(x)}{(2n + 1)!(k - n)!}.$

15. Show that the functions $\psi_n(x) = H_n(x) e^{-x^2/2}$ satisfy the relations

(a) $2n\psi_{n-1}(x) = x\psi_n(x) + \psi_n'(x),$
(b) $2x\psi_n(x) - 2n\psi_{n-1}(x) = \psi_{n+1}(x),$
(c) $\psi_n'(x) = x\psi_n(x) - \psi_{n+1}(x).$

16. For $m < n$, prove that

$$\frac{d^m}{dx^m} H_n(x) = \frac{2^m n!}{(n - m)!} H_{n-m}(x)$$

In problems 17–20, derive the series relationship.

17. $[H_n(x)]^2 = 2^n (n!)^2 \sum_{k=0}^{n} \dfrac{H_{2k}(x)}{2^k (k!)^2 (n - k)!}.$

18. $\sum_{k=0}^{n} \dfrac{H_k(x) H_k(y)}{2^k k!} = \dfrac{H_n(x) H_{n+1}(y) - H_{n+1}(x) H_n(y)}{2^{n+1} n!(y - x)}.$

19. $H_n(x+y) = \dfrac{1}{2^{n/2}} \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(x\sqrt{2}) H_k(y\sqrt{2})$.

20. $H_n(x)H_{n+p}(x) = 2^n n!(n+p)! \sum_{k=0}^{n} \dfrac{H_{2k+p}(x)}{2^k k!(k+p)!(n-k)!}$.

5.3 *Laguerre Polynomials*

The generating function

$$(1-t)^{-1}\exp\left[-\frac{xt}{1-t}\right] = \sum_{n=0}^{\infty} L_n(x)t^n, \qquad |t| < 1, \quad 0 \le x < \infty \tag{5.33}$$

leads to yet another important class of polynomials, called *Laguerre polynomials*. By expressing the exponential function in a series, we have

$$\begin{aligned}(1-t)^{-1}\exp\left[-\frac{xt}{1-t}\right] &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(xt)^k(1-t)^{-k-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}(xt)^k \sum_{m=0}^{\infty} \binom{-k-1}{m}(-1)^m t^m\end{aligned} \tag{5.34}$$

but since [see Equation (1.27) in Section 1.2.4]

$$\binom{-k-1}{m} = (-1)^m \binom{k+m}{m}$$

it follows that (5.34) becomes

$$(1-t)^{-1}\exp\left[-\frac{xt}{1-t}\right] = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (k+m)! x^k}{(k!)^2 m!} t^{k+m} \tag{5.35}$$

where we have reversed the order of summation. Finally, the change of index $m = n - k$ leads to (5.33) where

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{(k!)^2 (n-k)!} \tag{5.36}$$

In Table 5.2 we have listed the first few Laguerre polynomials $L_n(x)$.

The *Rodrigues formula* for the polynomials $L_n(x)$ is given by

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}), \qquad n = 0,1,2,\ldots \tag{5.37}$$

Table 5.2 Laguerre polynomials

$L_0(x) = 1$
$L_1(x) = -x + 1$
$L_2(x) = \frac{1}{2!}(x^2 - 4x + 2)$
$L_3(x) = \frac{1}{3!}(-x^3 + 9x^2 - 18x + 6)$
$L_4(x) = \frac{1}{4!}(x^4 - 16x^3 + 72x^2 - 96x + 24)$

which can be verified by application of the Leibniz formula

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}f}{dx^{n-k}} \frac{d^k g}{dx^k}, \qquad n = 1,2,3,\ldots \tag{5.38}$$

5.3.1 *Recurrence Relations*

It is easily verified that the generating function

$$w(x,t) = (1-t)^{-1} \exp\left[-\frac{xt}{1-t}\right]$$

satisfies the identity

$$(1-t)^2 \frac{\partial w}{\partial t} + (x - 1 + t)w = 0 \tag{5.39}$$

By substituting the series (5.33) for $w(x,t)$ into (5.39), we find upon simplification that

$$\sum_{n=1}^{\infty} \left[(n+1)L_{n+1}(x) + (x - 1 - 2n)L_n(x) + nL_{n-1}(x)\right] t^n = 0 \tag{5.40}$$

Hence, equating the coefficient of t^n to zero, we obtain the recurrence formula

$$(n+1)L_{n+1}(x) + (x - 1 - 2n)L_n(x) + nL_{n-1}(x) = 0 \tag{5.41}$$

for $n = 1, 2, 3, \ldots$.

Similarly, substituting (5.33) into the identity

$$(1-t)\frac{\partial w}{\partial x} + tw = 0 \tag{5.42}$$

leads to the derivative relation

$$L_n'(x) - L_{n-1}'(x) + L_{n-1}(x) = 0 \tag{5.43}$$

where $n = 1, 2, 3, \ldots$.

If we now differentiate (5.41), we obtain

$$(n+1)L'_{n+1}(x) + (x-1-2n)L'_n(x) + L_n(x) + nL'_{n-1}(x) = 0 \tag{5.44}$$

and by writing (5.43) in the equivalent forms

$$L'_{n+1}(x) = L'_n(x) - L_n(x) \tag{5.45a}$$

$$L'_{n-1}(x) = L'_n(x) + L_{n-1}(x) \tag{5.45b}$$

we can eliminate $L'_{n+1}(x)$ and $L'_{n-1}(x)$ from (5.44), which yields

$$xL'_n(x) = nL_n(x) - nL_{n-1}(x) \tag{5.46}$$

This last relation allows us to express the derivative of a Laguerre polynomial in terms of Laguerre polynomials.

To obtain the governing DE for the Laguerre polynomials, we begin by differentiating (5.46) and using (5.43) to get

$$\begin{aligned} xL''_n(x) + L'_n(x) &= nL'_n(x) - nL'_{n-1}(x) \\ &= -nL_{n-1}(x) \end{aligned}$$

We can eliminate $L_{n-1}(x)$ by use of (5.46), which leads to

$$xL''_n(x) + (1-x)L'_n(x) + nL_n(x) = 0 \tag{5.47}$$

Hence we conclude that $y = L_n(x)$ $(n = 0, 1, 2, \ldots)$ is a solution of *Laguerre's equation*

$$xy'' + (1-x)y' + ny = 0 \tag{5.48}$$

5.3.2 *Laguerre Series*

Like the Legendre polynomials and Hermite polynomials, various functions satisfying rather general conditions can be expanded in a series of Laguerre polynomials. Fundamental to the theory of such series is the orthogonality property

$$\int_0^\infty e^{-x}L_n(x)L_k(x)\,dx = 0, \qquad k \neq n \tag{5.49}$$

Our proof of (5.49) will parallel that given for the Hermite polynomials.

We begin by multiplying the two series

$$\sum_{n=0}^{\infty} L_n(x)t^n = (1-t)^{-1}\exp\left[-\frac{xt}{1-t}\right] \tag{5.50a}$$

$$\sum_{k=0}^{\infty} L_k(x)s^k = (1-s)^{-1}\exp\left[-\frac{xs}{1-s}\right] \tag{5.50b}$$

to obtain

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} t^n s^k L_n(x) L_k(x) = \frac{\exp\left[-x\left(\frac{t}{1-t} + \frac{s}{1-s}\right)\right]}{(1-t)(1-s)} \tag{5.51}$$

Next, multiplication of both sides of (5.51) by the weight function e^{-x} and subsequent integration leads to (see problem 29)

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} t^n s^k \int_0^{\infty} e^{-x} L_n(x) L_k(x)\, dx &= (1-ts)^{-1} \\ &= \sum_{n=0}^{\infty} t^n s^n \end{aligned} \tag{5.52}$$

By comparing the coefficient of $t^n s^k$ on both sides of (5.52) we deduce the result (5.49), while for $k = n$, we also see that (for $n = 0, 1, 2, \ldots$)

$$\int_0^{\infty} e^{-x} [L_n(x)]^2\, dx = 1 \tag{5.53}$$

By a *Laguerre series*, we mean a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n L_n(x), \qquad 0 < x < \infty \tag{5.54}$$

where

$$c_n = \int_0^{\infty} e^{-x} f(x) L_n(x)\, dx, \qquad n = 0, 1, 2, \ldots \tag{5.55}$$

Without proof, we state the following theorem.

Theorem 5.2. If f is piecewise smooth in every finite interval $x_1 \le x \le x_2$, $0 < x_1 < x_2 < \infty$, and

$$\int_0^{\infty} e^{-x} f^2(x)\, dx < \infty$$

then the Laguerre series (5.54) with constants defined by (5.55) converges pointwise to $f(x)$ at every continuity point of f. At points of discontinuity, the series converges to the average value $\frac{1}{2}[f(x^+) + f(x^-)]$.

5.3.3 *Associated Laguerre Polynomials*

In many applications, particularly in quantum-mechanical problems, we need a generalization of the Laguerre polynomials called the *associated Laguerre polynomials*, i.e.,

$$L_n^{(m)}(x) = (-1)^m \frac{d^m}{dx^m}[L_{n+m}(x)], \qquad m = 0, 1, 2, \ldots \tag{5.56}$$

By repeated differentiation of the series representation (5.36), it readily follows that (see problem 4)

$$L_n^{(m)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (n+m)! x^k}{(n-k)!(m+k)!k!}, \qquad m = 0, 1, 2, \ldots \tag{5.57}$$

A generating function for the Laguerre polynomials $L_n^{(m)}(x)$ can be derived from that for $L_n(x)$. We first replace n by $n + m$ in (5.33) to get

$$(1-t)^{-1} \exp\left[-\frac{xt}{1-t}\right] = \sum_{n=-m}^{\infty} L_{n+m}(x) t^{n+m}$$

and then differentiate both sides m times with respect to x, i.e.,

$$(-1)^m t^m (1-t)^{-1-m} \exp\left[-\frac{xt}{1-t}\right] = \sum_{n=-m}^{\infty} \frac{d^m}{dx^m}\left[L_{n+m}(x)\right] t^{n+m}$$

The terms of the series for which $n = -1, -2, \ldots, -m$ are all zero, since the mth derivative of a polynomial of degree less than m is zero, and hence we deduce that

$$(1-t)^{-1-m} \exp\left[-\frac{xt}{1-t}\right] = \sum_{n=0}^{\infty} L_n^{(m)}(x) t^n, \qquad |t| < 1 \tag{5.58}$$

The associated Laguerre polynomials have many properties that are simple generalizations of those for the Laguerre polynomials. Among these are the recurrence relations*

$$(n+1) L_{n+1}^{(m)}(x) + (x - 1 - 2n - m) L_n^{(m)}(x) + (n+m) L_{n-1}^{(m)}(x) = 0 \tag{5.59}$$

$$x L_n^{(m)\prime}(x) - n L_n^{(m)}(x) + (n+m) L_{n-1}^{(m)}(x) = 0 \tag{5.60}$$

and the Rodrigues formula

$$L_n^{(m)}(x) = \frac{1}{n!} e^x x^{-m} \frac{d^n}{dx^n}\left(e^{-x} x^{n+m}\right) \tag{5.61}$$

The polynomials $L_n^{(m)}(x)$ also satisfy numerous relations where the upper index does not remain constant. Two such relations are given by

$$L_{n-1}^{(m)}(x) + L_n^{(m-1)}(x) - L_n^{(m)}(x) = 0 \tag{5.62}$$

and

$$L_n^{(m)\prime}(x) = -L_{n-1}^{(m+1)}(x) \tag{5.63}$$

*Note that for $m = 0$, (5.59) reduces to (5.41).

The second-order DE satisfied by the polynomials $L_n^{(m)}(x)$ is the *associated Laguerre's equation*

$$xy'' + (m + 1 - x)y' + ny = 0 \tag{5.64}$$

To show this, we first note that the polynomial $z = L_{n+m}(x)$ is a solution of Laguerre's equation

$$xz'' + (1 - x)z' + (n + m)z = 0 \tag{5.65}$$

By differentiating (5.65) m times, using the Leibniz rule (5.38), we obtain

$$x\frac{d^{m+2}z}{dx^{m+2}} + m\frac{d^{m+1}z}{dx^{m+1}} + (1 - x)\frac{d^{m+1}z}{dx^{m+1}} + n\frac{d^m z}{dx^m} = 0$$

or equivalently,

$$x\frac{d^2}{dx^2}\left(\frac{d^m z}{dx^m}\right) + (m + 1 - x)\frac{d}{dx}\left(\frac{d^m z}{dx^m}\right) + n\left(\frac{d^m z}{dx^m}\right) = 0 \tag{5.66}$$

Comparing (5.64) and (5.66), we see that any function $y = C_1(d^m z/dx^m)$ is a solution of (5.64) where C_1 is arbitrary. In particular, $y = L_n^{(m)}(x)$ is a solution.

Example 3: Prove the *addition formula*

$$L_n^{(a+b+1)}(x + y) = \sum_{k=0}^{n} L_k^{(a)}(x)L_{n-k}^{(b)}(y), \qquad a, b > -1$$

Solution: From the generating function (5.58), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n^{(a+b+1)}(x + y)t^n &= \frac{\exp[-(x + y)t/(1 - t)]}{(1 - t)^{a+b+2}} \\
&= \frac{\exp[-xt/(1 - t)]}{(1 - t)^{a+1}} \cdot \frac{\exp[-yt/(1 - t)]}{(1 - t)^{b+1}} \\
&= \sum_{k=0}^{\infty} L_k^{(a)}(x)t^k \cdot \sum_{m=0}^{\infty} L_m^{(b)}(y)t^m \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} L_k^{(a)}(x)L_m^{(b)}(y)t^{m+k}
\end{aligned}$$

Next, making the change of index $m = n - k$ leads to

$$\sum_{n=0}^{\infty} L_n^{(a+b+1)}(x + y)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n} L_k^{(a)}(x)L_{n-k}^{(b)}(y)t^n$$

and by comparing the coefficient of t^n in each series, we get our intended result.

Remark: The associated Laguerre polynomial $L_n^{(m)}(x)$ can be generalized to the case where m is not restricted to integer values by writing

$$L_n^{(a)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(n+a+1) x^k}{(n-k)! \Gamma(k+a+1) k!}, \qquad a > -1$$

Most of the above relations are also valid for this more general polynomial.

EXERCISES 5.3

1. Show that (for $n = 0, 1, 2, \ldots$)

(a) $L_n(0) = 1$,
(b) $L_n'(0) = -n$,
(c) $L_n''(0) = \frac{1}{2}n(n-1)$.

2. Derive the Rodrigues formula

(a) $L_n(x) = \dfrac{e^x}{n!} \dfrac{d^n}{dx^n}(x^n e^{-x})$,

(b) $L_n^{(m)}(x) = \dfrac{1}{n!} x^{-m} e^{-x} \dfrac{d^n}{dx^n}(x^{n+m} e^{-x})$.

Hint: Use the Leibniz formula (5.38).

3. Derive the recurrence formulas

(a) $L_n'(x) - L_{n-1}'(x) + L_{n-1}(x) = 0$.

(b) $L_n'(x) = -\sum_{k=0}^{n-1} L_k(x)$.

4. By repeated differentiation of the series (5.36), show that

$$L_n^{(m)}(x) = \sum_{k=0}^{n} \frac{(-1)^k (m+n)! x^k}{(n-k)!(m+k)! k!}, \qquad m = 0, 1, 2, \ldots$$

5. Show that

$$n! \frac{d^k}{dx^k}\left[e^{-x} x^m L_n^{(m)}(x)\right] = (n+k)! e^{-x} x^{m-k} L_{n+k}^{(m-k)}(x)$$

6. Show that

$$L_n^{(m)}(0) = \frac{(n+m)!}{n! m!}$$

In problems 7–10, verify the given recurrence relation.

7. $(n+1)L_{n+1}^{(m)}(x) + (x - 1 - 2n - m)L_n^{(m)}(x) + (n+m)L_{n-1}^{(m)}(x) = 0.$

8. $xL_n^{(m)\prime}(x) - nL_n^{(m)}(x) + (n+m)L_{n-1}^{(m)}(x) = 0.$

9. $L_{n-1}^{(m)}(x) + L_n^{(m-1)}(x) - L_n^{(m)}(x) = 0.$

10. $L_n^{(m)\prime}(x) = -L_{n-1}^{(m+1)}(x).$

In problems 11–18, verify the integral formula.

11. $\displaystyle\int_0^\infty e^{-x}x^k L_n(x)\,dx = \begin{cases} 0, & k < n, \\ (-1)^n n!, & k = n. \end{cases}$

12. $\displaystyle\int_0^x L_k(t)L_n(x-t)\,dt = \int_0^x L_{n+k}(t)\,dt = L_{n+k}(x) - L_{n+k+1}(x).$

13. $\displaystyle\int_x^\infty e^{-t}L_n^{(m)}(t)\,dt = e^{-x}[L_n^{(m)}(x) - L_{n-1}^{(m)}(x)],\ m = 0,1,2,\ldots .$

14. $\displaystyle\int_0^x (x-t)^m L_n(t)\,dt = \frac{m!n!}{(m+n+1)!}x^{m+1}L_n^{(m+1)}(x),\ m = 0,1,2,\ldots .$

15. $\displaystyle\int_0^1 t^a(1-t)^{b-1}L_n^{(a)}(xt)\,dt = \frac{\Gamma(b)\Gamma(n+a+1)}{\Gamma(n+a+b+1)}L_n^{(a+b)}(x),\quad a > -1,$
$b > 0.$

16. $\displaystyle\int_0^\infty e^{-x}x^a L_n^{(a)}(x)L_k^{(a)}(x)\,dx = 0,\ k \neq n,\ a > -1.$

17. $\displaystyle\int_0^\infty e^{-x}x^a[L_n^{(a)}(x)]^2\,dx = \frac{\Gamma(n+a+1)}{n!},\ a > -1.$

18. $\displaystyle\int_0^\infty e^{-x}x^{a+1}[L_n^{(a)}(x)]^2\,dx = \frac{\Gamma(n+a+1)}{n!}(2n+a+1),\ a > -1.$

In problems 19–23, derive the given relation between the Hermite and Laguerre polynomials.

19. $\displaystyle L_n^{(-1/2)}(x) = \frac{(-1)^n}{2^{2n}n!}H_{2n}(\sqrt{x}).$

20. $\displaystyle L_n^{(1/2)}(x) = \frac{(-1)^n}{2^{2n+1}n!\sqrt{x}}H_{2n+1}(\sqrt{x}).$

21. $\displaystyle\int_0^\infty e^{-t^2}[H_n(t)]^2\cos(\sqrt{2x}\,t)\,dt = \sqrt{\pi}\,2^{n-1}n!e^{-x/2}L_n(x).$

22. $\displaystyle\int_{-1}^1 (1-t^2)^{a-\frac{1}{2}}H_{2n}(\sqrt{x}\,t)\,dt = (-1)^n\sqrt{\pi}\,\frac{\Gamma(a+\frac{1}{2})(2n)!}{\Gamma(n+a+1)}L_n^{(a)}(x),$
$a > -\frac{1}{2}.$

23. $\displaystyle L_n(x^2+y^2) = \frac{(-1)^n}{2^{2n}}\sum_{k=0}^n \frac{H_{2k}(x)H_{2n-2k}(y)}{k!(n-k)!}.$

In problems 24 and 25, derive the Laguerre series.

24. $x^p = p! \sum_{n=0}^{p} \binom{p}{n}(-1)^n L_n(x).$

25. $e^{-ax} = (a+1)^{-1} \sum_{n=0}^{\infty} \left(\frac{a}{a+1}\right)^n L_n(x),\ a > -\frac{1}{2}.$

Hint: Set $t = a/(a+1)$ in the generating function.

26. Show that $(x > 0)$

$$\int_0^\infty \frac{e^{-xt}}{t+1}\,dt = \sum_{n=0}^{\infty} \frac{L_n(x)}{n+1}$$

Hint: Use problem 25.

27. Show that $(x > 0)$

$$e^t(xt)^{-m/2} J_m(2\sqrt{xt}) = \sum_{n=0}^{\infty} \frac{L_n^{(m)}(x)}{(n+m)!} t^n, \qquad m = 0, 1, 2, \ldots$$

where $J_m(x)$ is the *Bessel function* defined by (see Chapter 6)

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+m}}{k!(k+m)!}$$

28. Show that for $m > 1$,

$$\int_0^\infty t^{n+m/2} J_m(2\sqrt{xt}) e^{-t}\,dt = n! e^{-x} x^{m/2} L_n^{(m)}(x)$$

Hint: See problem 27.

29. Show that

$$\frac{1}{(1-t)(1-s)} \int_0^\infty \exp\left[-x\left(1 + \frac{t}{1-t} + \frac{s}{1-s}\right)\right] dx = \frac{1}{1-ts}$$

30. Show that the Laplace transform of $L_n(t)$ leads to

$$\int_0^\infty e^{-st} L_n(t)\,dt = \frac{1}{s}\left(1 - \frac{1}{s}\right)^n, \qquad s > 0$$

5.4 *Generalized Polynomial Sets*

The many properties that are shared by the Legendre, Hermite, and Laguerre polynomials suggest that there may exist more general polynomial sets of which these are certain specializations. Indeed, the *Gegenbauer* and *Jacobi polynomials* are two such generalizations. The Gegenbauer polynomials are closely connected with axially symmetric potentials in n dimensions and contain the Legendre, Hermite, and Chebyshev polynomials as special cases. The Jacobi polynomials are more general yet, as they contain the Gegenbauer polynomials as a special case.

5.4.1 *Gegenbauer Polynomials*

The *Gegenbauer polynomials** $C_n^\lambda(x)$ are defined by the generating function

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) t^n, \qquad |t| < 1, \quad |x| \le 1 \tag{5.67}$$

where $\lambda > -\frac{1}{2}$. By expanding the function $w(x,t) = (1 - 2xt + t^2)^{-\lambda}$ in a binomial series, and following our approach in Section 4.2.1, we find

$$\begin{aligned} w(x,t) &= \sum_{n=0}^{\infty} \binom{-\lambda}{n} (-1)^n t^n (2x - t)^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{-\lambda}{n} \binom{n}{k} (-1)^{n+k} (2x)^{n-k} t^{n+k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \binom{-\lambda}{n-k} \binom{n-k}{k} (-1)^n (2x)^{n-2k} t^n \end{aligned} \tag{5.68}$$

and thus deduce that

$$C_n^\lambda(x) = (-1)^n \sum_{k=0}^{[n/2]} \binom{-\lambda}{n-k} \binom{n-k}{k} (2x)^{n-2k} \tag{5.69}$$

By substituting the series (5.67) into the identity

$$(1 - 2xt + t^2)\frac{\partial w}{\partial t} + 2\lambda(t - x)w = 0 \tag{5.70}$$

where $w(x,t) = (1 - 2xt + t^2)^{-\lambda}$, we obtain the three-term recurrence formula ($n = 1, 2, 3, \ldots$)

$$(n + 1)C_{n+1}^\lambda(x) - 2(\lambda + n)xC_n^\lambda(x) + (2\lambda + n - 1)C_{n-1}^\lambda(x) = 0 \tag{5.71}$$

Other recurrence formulas satisfied by the Gegenbauer polynomials include

*The polynomials $C_n^\lambda(x)$ are also called *ultraspherical polynomials*.

the following:

$$(n+1)C_{n+1}^{\lambda}(x) - 2\lambda x C_n^{\lambda+1}(x) + 2\lambda C_{n-1}^{\lambda+1}(x) = 0 \tag{5.72}$$

$$(n+2\lambda)C_n^{\lambda}(x) - 2\lambda C_n^{\lambda+1}(x) + 2\lambda x C_{n-1}^{\lambda+1}(x) = 0 \tag{5.73}$$

$$C_n^{\lambda\prime}(x) = 2\lambda C_{n+1}^{\lambda+1}(x) \tag{5.74}$$

The orthogonality property is given by (see problem 13)

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_n^{\lambda}(x) C_k^{\lambda}(x)\,dx = 0, \qquad k \neq n \tag{5.75}$$

and the governing DE is

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0 \tag{5.76}$$

which can be verified by substituting the series (5.69) directly into (5.76).

One of the main advantages of developing properties of the Gegenbauer polynomials is that each recurrence formula, etc., becomes a master formula for all the polynomial sets that are generated as special cases. For example, when $\lambda = \frac{1}{2}$ we see that (5.67) is the generating function for the Legendre polynomials, and thus

$$P_n(x) = C_n^{1/2}(x), \qquad n = 0, 1, 2, \ldots \tag{5.77}$$

By setting $\lambda = \frac{1}{2}$ in (5.71), (5.75), and (5.76), we immediately obtain the recurrence formula, orthogonality property, and governing DE, respectively, for the Legendre polynomials.

The Hermite polynomials can also be generated from the Gegenbauer polynomials through the limit relation

$$H_n(x) = n! \lim_{\lambda\to\infty} \lambda^{-n/2} C_n^{\lambda}(x/\sqrt{\lambda}), \qquad n = 0, 1, 2, \ldots \tag{5.78}$$

To show this, we start with the series representation

$$\lambda^{-n/2} C_n^{\lambda}(x/\sqrt{\lambda}) = (-1)^n \sum_{k=0}^{[n/2]} \binom{-\lambda}{n-k}\binom{n-k}{k} \frac{(2x)^{n-2k}}{\lambda^{n-k}} \tag{5.79}$$

From Equation (1.27) in Section 1.2.4, we obtain the relation

$$\begin{aligned}\frac{(-1)^n}{\lambda^{n-k}}\binom{-\lambda}{n-k} &= \frac{(-1)^k}{\lambda^{n-k}}\binom{\lambda+n-k-1}{n-k}\\ &= \frac{(-1)^k\Gamma(\lambda+n-k)}{\lambda^{n-k}\Gamma(\lambda)(n-k)!}\end{aligned}$$

and thus establish that (see problem 3)

$$\lim_{\lambda\to\infty} \frac{(-1)^n}{\lambda^{n-k}} \binom{-\lambda}{n-k} = \frac{(-1)^k}{(n-k)!} \tag{5.80}$$

Hence, from (5.79) we now deduce our intended result

$$\begin{aligned} n! \lim_{\lambda\to\infty} \lambda^{-n/2} C_n^\lambda(x/\sqrt{\lambda}) &= \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2x)^{n-2k} \\ &= H_n(x) \end{aligned}$$

Properties of the Hermite polynomials can be obtained from properties of the Gegenbauer polynomials, although most such relations are more difficult to deduce than for the Legendre polynomials.

5.4.2 *Chebyshev Polynomials**

An important subclass of Gegenbauer polynomials are the Chebyshev polynomials, of which there are two kinds. The *Chebyshev polynomials of the first kind* are defined by

$$T_0(x) = 1, \qquad T_n(x) = \frac{n}{2} \lim_{\lambda\to 0} \frac{C_n^\lambda(x)}{\lambda}, \qquad n = 1,2,3,\ldots \tag{5.81}$$

Because the Gegenbauer polynomials vanish when $\lambda = 0$, we cannot just simply define the polynomials $T_n(x)$ by $C_n^0(x)$. The choice $T_0(x) = 1$ is made to preserve the recurrence relation (5.85) given below. By following a procedure similar to that used to verify the relation (5.78), it can be established that (see problem 15)

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!}(2x)^{n-2k} \tag{5.82}$$

The *Chebyshev polynomials of the second kind* are simply†

$$U_n(x) = C_n^1(x), \qquad n = 0,1,2,\ldots \tag{5.83}$$

and thus by setting $\lambda = 1$ in (5.69) we immediately deduce that

$$U_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k (2x)^{n-2k} \tag{5.84}$$

---

*There are numerous spellings of Chebyshev that occur throughout the literature, e.g., Tchebysheff, Tchebycheff, Tchebichef, and Chebysheff, among others.

†Some authors call $(1 - x^2)^{1/2} U_n(x)$ the Chebyshev functions of the second kind.

By using properties previously cited for the Gegenbauer polynomials, we readily obtain the recurrence formulas

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0 \tag{5.85}$$

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0 \tag{5.86}$$

orthogonality properties

$$\int_{-1}^{1} (1 - x^2)^{-1/2} T_n(x) T_k(x)\, dx = 0, \qquad k \neq n \tag{5.87}$$

$$\int_{-1}^{1} (1 - x^2)^{1/2} U_n(x) U_k(x)\, dx = 0, \qquad k \neq n \tag{5.88}$$

and governing DE for $T_n(x)$,

$$(1 - x^2)y'' - xy' + n^2 y = 0 \tag{5.89}$$

and for $U_n(x)$,

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0 \tag{5.90}$$

There are also several recurrence-type formulas connecting the polynomials $T_n(x)$ and $U_n(x)$, such as

$$T_n(x) = U_n(x) - xU_{n-1}(x) \tag{5.91}$$

and

$$(1 - x^2)U_n(x) = xT_n(x) - T_{n+1}(x) \tag{5.92}$$

the proofs of which are left for the exercises.

By making the substitution $x = \cos\phi$ in (5.89), we find it reduces to

$$\frac{d^2y}{d\phi^2} + n^2 y = 0$$

with solutions $\cos n\phi$ and $\sin n\phi$. Thus we speculate that

$$T_n(\cos\phi) = c_n \cos n\phi$$

for some constant c_n. But since $T_n(1) = 1$ for all n (see problem 26), it follows that $c_n = 1$ for all n. It turns out that this speculation is correct, and in general we write

$$T_n(x) = \cos n\phi = \cos(n \cos^{-1} x) \tag{5.93}$$

Similarly, it can be shown that

$$U_n(x) = \frac{\sin[(n + 1)\cos^{-1} x]}{\sqrt{1 - x^2}} \tag{5.94}$$

The significance of these observations is that the properties of sines and cosines can be used to establish many of the properties of the Chebyshev polynomials.

The Chebyshev polynomials have acquired great practical importance in polynomial approximation methods. Specifically, it has been shown that a series of Chebyshev polynomials converges more rapidly than any other series of Gegenbauer polynomials, and converges much more rapidly than power series.*

5.4.3 *Jacobi Polynomials*

The *Jacobi polynomials*, which are generalizations of the Gegenbauer polynomials, are defined by the generating function

$$\frac{2^{a+b}}{R}(1 - t + R)^{-a}(1 + t + R)^{-b} = \sum_{n=0}^{\infty} P_n^{(a,b)}(x)t^n, \qquad a > -1, \quad b > -1 \tag{5.95}$$

where

$$R = (1 - 2xt + t^2)^{1/2} \tag{5.96}$$

The Jacobi polynomials have the following three series representations (among others), which are somewhat involved to derive:

$$P_n^{(a,b)}(x) = \sum_{k=0}^{n} \binom{n+a}{n-k}\binom{n+b}{n-k}\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k} \tag{5.97}$$

$$P_n^{(a,b)}(x) = \sum_{k=0}^{n} \binom{n+a}{n-k}\binom{n+k+a+b}{k}\left(\frac{x-1}{2}\right)^k \tag{5.98}$$

$$P_n^{(a,b)}(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n+b}{n-k}\binom{n+k+a+b}{k}\left(\frac{x+1}{2}\right)^k \tag{5.99}$$

By examination of the generating function (5.95), we observe that the Legendre polynomials are a specialization of the Jacobi polynomials for which $a = b = 0$, i.e.,

$$P_n(x) = P_n^{(0,0)}(x), \qquad n = 0, 1, 2, \ldots \tag{5.100}$$

---

*For theory and applications involving the Chebyshev polynomials, see L. Fox and I.B. Parker, *Chebyshev Polynomials in Numerical Analysis*, London: Oxford U.P., 1968.

whereas the associated Laguerre polynomials arise as the limit (see problem 37)

$$L_n^{(a)}(x) = \lim_{b\to\infty} P_n^{(a,b)}(1 - 2x/b), \qquad n = 0,1,2,\ldots \tag{5.101}$$

In addition to the Legendre and Laguerre polynomials, the Gegenbauer polynomials are also a special case of the Jacobi polynomials. To derive the relation between the Gegenbauer and Jacobi polynomials, we start with the identity

$$(1 - 2xt + t^2)^{-\lambda} = (1 - t)^{-2\lambda}\left[1 - \frac{2t(x-1)}{(1-t)^2}\right]^{-\lambda} \tag{5.102}$$

and expand the right-hand side in a series. This action leads to

$$\begin{aligned}(1 - 2xt + t^2)^{-\lambda} &= \sum_{k=0}^{\infty}\binom{-\lambda}{k}\frac{(-1)^k(2t)^k(x-1)^k}{(1-t)^{2(k+\lambda)}}\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{-\lambda}{k}\binom{-2k-2\lambda}{m}(-1)^{m+k}2^k(x-1)^k t^{m+k}\end{aligned} \tag{5.103}$$

where we have expanded $(1-t)^{-2(k+\lambda)}$ in another binomial series and interchanged the order of summation. Next, replacing the left-hand side of (5.103) by the series (5.67) and making the change of index $m = n - k$, we get

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{-\lambda}{k}\binom{-2k-2\lambda}{n-k}(-1)^n 2^k(x-1)^k t^n$$

from which we deduce

$$C_n^{\lambda}(x) = (-1)^n\sum_{k=0}^{n}\binom{-\lambda}{k}\binom{-2k-2\lambda}{n-k}2^k(x-1)^k \tag{5.104}$$

Recalling Equation (1.27) in Section 1.2.4 and the Legendre duplication formula, we see that

$$\begin{aligned}(-1)^n\binom{-\lambda}{k}\binom{-2k-2\lambda}{n-k} &= \binom{\lambda+k-1}{k}\binom{n+k+2\lambda-1}{n-k}\\ &= \frac{\Gamma(\lambda+k)\Gamma(n+k+2\lambda)}{\Gamma(\lambda)k!\Gamma(2\lambda+2k)(n-k)!}\\ &= \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+k+2\lambda)}{\Gamma(2\lambda)\Gamma(\lambda+k+\frac{1}{2})k!(n-k)!2^{2k}}\end{aligned}$$

and hence (5.104) can be expressed in the form

$$C_n^\lambda(x) = \frac{\Gamma(\lambda + \frac{1}{2})\Gamma(n + 2\lambda)}{\Gamma(2\lambda)\Gamma(n + \lambda + \frac{1}{2})} \times \sum_{k=0}^{n} \binom{n + \lambda - \frac{1}{2}}{n - k}\binom{n + k + 2\lambda - 1}{k}\left(\frac{x - 1}{2}\right)^k \tag{5.105}$$

or, by comparing with (5.98),

$$C_n^\lambda(x) = \frac{\Gamma(\lambda + \frac{1}{2})\Gamma(n + 2\lambda)}{\Gamma(2\lambda)\Gamma(n + \lambda + \frac{1}{2})} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x) \tag{5.106}$$

The basic recurrence formula for the polynomials $P_n^{(a,b)}(x)$ is

$$\begin{aligned} 2(n + 1)(a + b + n + 1)(a + b + 2n)P_{n+1}^{(a,b)}(x) &= (a + b + 2n + 1) \\ &\quad \times \left[a^2 - b^2 + x(a + b + 2n + 2)(a + b + 2n)\right] P_n^{(a,b)}(x) \\ &\quad - 2(a + n)(b + n)(a + b + 2n + 2)P_{n-1}^{(a,b)}(x) \end{aligned} \tag{5.107}$$

for $n = 1, 2, 3, \ldots$. Also, the orthogonality property and governing DE are given respectively by

$$\int_{-1}^{1} (1 - x)^a(1 + x)^b P_n^{(a,b)}(x) P_k^{(a,b)}(x)\,dx = 0, \qquad k \neq n \tag{5.108}$$

and

$$(1 - x^2)y'' + [b - a - (a + b + 2)x]y' + n(n + a + b + 1)y = 0 \tag{5.109}$$

Some additional properties concerning the Jacobi polynomials are taken up in the exercises.

EXERCISES 5.4

1. Show that (for $n = 0, 1, 2, \ldots$)

$$C_n^\lambda(-x) = (-1)^n C_n^\lambda(x)$$

2. Show that (for $n = 0, 1, 2, \ldots$)

(a) $C_{2n}^\lambda(0) = \binom{-\lambda}{n}$, (b) $C_{2n+1}^\lambda(0) = 0$,

(c) $C_n^\lambda(1) = (-1)^n\binom{-2\lambda}{n}$, (d) $C_n^\lambda(-1) = \binom{-2\lambda}{n}$.

3. Show that

$$\lim_{\lambda \to \infty} \frac{\Gamma(\lambda + n - k)}{\lambda^{n-k}\Gamma(\lambda)} = 1$$

In problems 4–8, derive the given recurrence relation.

4. $xC_n^{\lambda\prime}(x) = nC_n^{\lambda}(x) + C_{n-1}^{\lambda\prime}(x)$.
5. $2(\lambda + n)C_n^{\lambda}(x) = C_{n+1}^{\lambda\prime}(x) - C_{n-1}^{\lambda\prime}(x)$.
6. $xC_n^{\lambda\prime}(x) = C_{n+1}^{\lambda\prime}(x) - (2\lambda + n)C_n^{\lambda}(x)$.
7. $(x^2 - 1)C_n^{\lambda\prime}(x) = nxC_n^{\lambda}(x) - (2\lambda - 1 + n)C_{n-1}^{\lambda}(x)$.
8. $nC_n^{\lambda}(x) = 2x(\lambda + n - 1)C_{n-1}^{\lambda}(x) - (2\lambda + n - 2)C_{n-2}^{\lambda}(x)$.
9. Use any of the results of problems 4–8 and the recurrence formula (5.71) to show that $y = C_n^{\lambda}(x)$ is a solution of

$$(1 - x^2)y'' - (2\lambda + 1)xy' + n(n + 2\lambda)y = 0$$

10. Show that (for $k = 1, 2, 3, \ldots$)

$$\frac{d^k}{dx^k}C_n^{\lambda}(x) = 2^k\frac{\Gamma(\lambda + k)}{\Gamma(\lambda)}C_{n-k}^{\lambda+k}(x)$$

Hint: Use Equation (5.74).

11. Verify that (for $k = 1, 2, 3, \ldots$)*

$$C_{n-k}^{k+\frac{1}{2}}(x) = \frac{1}{(2k - 1)!!}\frac{d^k}{dx^k}P_n(x)$$

12. Derive the recurrence relation

$$\sum_{k=0}^{n} (n + \lambda)C_k^{\lambda}(x) = \frac{(n + 2\lambda)C_n^{\lambda}(x) - (n + 1)C_{n+1}^{\lambda}(x)}{2(1 - x)}$$

13. Verify the orthogonality property

$$\int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}} C_n^{\lambda}(x)C_k^{\lambda}(x)\, dx = 0, \qquad k \neq n$$

14. Show that (for $n = 0, 1, 2, \ldots$)

$$\int_{-1}^{1} (1 - x^2)^{\lambda - \frac{1}{2}}[C_n^{\lambda}(x)]^2\, dx = \frac{2^{1-2\lambda}\pi}{(n + \lambda)}\frac{\Gamma(n + 2\lambda)}{[\Gamma(\lambda)]^2 n!}$$

*See problem 15 in Exercises 2.2 for definition of the symbol !!.

15. By using Equation (5.69) and the definition

$$T_n(x) = \frac{n}{2} \lim_{\lambda \to 0} \frac{C_n^\lambda(x)}{\lambda}, \qquad n = 1, 2, 3, \ldots$$

show that

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}$$

16. Using the recurrence formula (5.71), deduce the relations

(a) $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$
(b) $U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0.$

In problems 17–22, derive the given relation for the Chebyshev polynomials.

17. $T_n(x) = U_n(x) - xU_{n-1}(x).$

18. $(1 - x^2)U_n(x) = xT_n(x) - T_{n+1}(x).$

19. $T_n'(x) = nU_{n-1}(x).$

20. $2[T_n(x)]^2 = 1 + T_{2n}(x).$

21. $[T_n(x)]^2 - T_{n+1}(x)T_{n-1}(x) = 1 - x^2.$

22. $[U_n(x)]^2 - U_{n+1}(x)U_{n-1}(x) = 1.$

23. By making the substitution $x = \cos\phi$ in the orthogonality relation (5.87), show that

$$\int_0^\pi \cos n\phi \cos k\phi \, d\phi = 0, \qquad k \neq n$$

In problems 24 and 25, derive the generating-function relation.

24. $$\frac{1 - t^2}{1 - 2xt + t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)t^n.$$

25. $$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n.$$

26. Show that $T_n(1) = 1$, $n = 0, 1, 2, \ldots$, by using

(a) problem 24,
(b) problem 25.

27. Verify the special values (for $n = 0, 1, 2, \ldots$)

(a) $T_n(-1) = (-1)^n$,
(b) $T_{2n}(0) = (-1)^n$,
(c) $T_{2n+1}(0) = 0.$

15. Show that the *Jacobi polynomials*

$$P_n^{(a,b)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+a}{n-k}\binom{n+b}{n-k}(x+1)^{n-k}(x-1)^k$$

satisfy the relations

(a) $P_n^{(a,b)}(x) = (-1)^n\binom{n+b}{b}F\left(-n, n+a+b+1; 1+b; \frac{1+x}{2}\right)$,

(b) $P_n^{(a,b)}(x) = \binom{n+a}{a}F\left(-n, n+a+b+1; 1+a; \frac{1-x}{2}\right)$.

16. Given the *incomplete beta function*

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt, \qquad p,q > 0$$

show that

(a) $B_x(p,q) = \dfrac{x^p}{p}F(p, 1-q; 1+p; x)$,

(b) $B_1(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

17. Show that

$$\sum_{k=0}^{n-1}\binom{a}{k}x^k = (1+x)^a - \binom{a}{n}x^n F(n-a, 1; n+1; -x)$$

18. Verify that

$$P_\nu(x) = \sum_{k=0}^{\infty}\frac{(-\nu)_k(\nu+1)_k}{(k!)^2}\left(\frac{1-x}{2}\right)^k$$

In problems 19–21, use the series representation in problem 18 to deduce the given recurrence formula.

19. $(\nu+1)P_{\nu+1}(x) - (2\nu+1)xP_\nu(x) + \nu P_{\nu-1}(x) = 0$.

20. $P'_{\nu+1}(x) - xP'_\nu(x) = (\nu+1)P_\nu(x)$.

21. $xP'_\nu(x) - P'_{\nu-1}(x) = \nu P_\nu(x)$.

22. Using the relation (8.48), show that (for $k = 0, 1, 2, \ldots$)

(a) $P_{2k+1}(0) = 0$,

(b) $P_{2k}(0) = \dfrac{(-1)^k}{k!}\left(\frac{1}{2}\right)_k$.

23. Show that

$$P'_\nu(0) = \frac{2\Gamma\left(\frac{1}{2}\nu + 1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\right)}\sin\frac{\nu\pi}{2}$$

28. Verify the special values (for $n = 0, 1, 2, \dots$)

(a) $U_n(1) = n + 1$,
(b) $U_{2n}(0) = (-1)^n$,
(c) $U_{2n+1}(0) = 0$.

29. Show that

$$\int_{-1}^{1} (1 - x^2)^{-1/2} [T_n(x)]^2 \, dx = \begin{cases} \pi, & n = 0 \\ \dfrac{\pi}{2}, & n \geq 1 \end{cases}$$

30. Show that

$$\int_{-1}^{1} (1 - x^2)^{1/2} [U_n(x)]^2 \, dx = \frac{\pi}{2}$$

In problems 31–38, verify the given relation for the Jacobi polynomials.

31. $P_n^{(a,b)}(-x) = (-1)^n P_n^{(b,a)}(x)$.

32. $P_n^{(a,b)}(1) = \dbinom{a+n+1}{n}$.

33. $P_n^{(a,b)}(-1) = (-1)^n \dbinom{b+n+1}{n}$.

34. $P_n^{(a,b)}(x) = \dfrac{(-1)^n}{2^n n!}(1 - x)^{-a}(1 + x)^{-b} \dfrac{d^n}{dx^n}[(1 - x)^{a+n}(1 + x)^{b+n}]$.

35. $\dfrac{d^k}{dx^k} P_n^{(a,b)}(x) = \dfrac{\Gamma(k + n + a + b + 1)}{2^k \Gamma(n + a + b + 1)} P_{n-k}^{(a+k,b+k)}(x)$.

36. $P_n^{(a,b-1)}(x) - P_n^{(a-1,b)}(x) = P_{n-1}^{(a,b)}(x)$.

37. $L_n^{(a)}(x) = \lim\limits_{b \to \infty} P_n^{(a,b)}(1 - 2x/b)$

38. $T_n(x) = \dfrac{2^{2n}(n!)^2}{(2n)!} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x)$.

Bessel Functions

6.1 *Introduction*

The German astronomer F.W. Bessel (1784–1846) first achieved fame by computing the orbit of Halley's comet. In addition to many other accomplishments in connection with his studies of planetary motion, he is credited with deriving the differential equation bearing his name.* It is known, however, that Bessel's equation was first investigated in 1703 by J. Bernoulli, who was studying the oscillatory behavior of a hanging chain. In fact, Bernoulli solved Bessel's equation by an infinite series that now defines the *Bessel function of the first kind*. Bessel functions were also met with by Euler and others who were concerned with various problems in mechanics. Nonetheless, it was Bessel in 1824 who carried out the first systematic study of the properties of these functions, and thus they are named in his honor.

Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the study of free vibrations of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur in electromagnetic theory and numerous other areas of physics and engineering. In fact, Bessel functions occur so frequently in practice that they are undoubtedly the most important functions beyond the elementary ones.

---

*A short historical account of Bessel's problem of planetary motion is given in N.W. McLachlan, *Bessel Functions for Engineers*, London: Oxford, 1961, Chapter 1.

Because of their close association with cylindrical-shaped domains, all solutions of Bessel's equation are collectively called *cylinder functions*. The Bessel functions, of which there are several varieties, are certain special cases of cylinder functions. In addition to Bessel functions of the first kind, there are Bessel functions of the second and third kinds, modified Bessel functions of the first and second kinds, spherical Bessel functions, and so on.

6.2 *Bessel Functions of the First Kind*

Although Bessel functions arise in practice most frequently as solutions of certain DEs, it is both instructive and convenient to develop them from the same point of view that we adopted in introducing the orthogonal polynomials in Chapters 4 and 5, viz., by a generating function.

6.2.1 *The Generating Function*

By expanding the function

$$w(x,t) = \exp\left[\tfrac{1}{2}x\left(t - \frac{1}{t}\right)\right], \qquad t \neq 0 \tag{6.1}$$

in a series involving both positive and negative powers of t, we wish to establish the relation

$$w(x,t) = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{6.2}$$

where $J_n(x)$ denotes the Bessel function we want to define.

To begin, we write $w(x,t)$ as the product of two exponential functions and expand each in a Maclaurin series to get

$$\begin{aligned} w(x,t) &= e^{xt/2} \cdot e^{-x/2t} \\ &= \sum_{j=0}^{\infty} \frac{(xt/2)^j}{j!} \cdot \sum_{k=0}^{\infty} \frac{(-x/2t)^k}{k!} \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{j+k}}{j!k!} t^{j-k} \end{aligned}$$

We now make the change of index $n = j - k$. Because of the range of values on j and k, it follows that $-\infty < n < \infty$, and thus

$$w(x,t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!(k+n)!} t^n \tag{6.3}$$

By defining the *Bessel function of the first kind* of order n by the series

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!(k+n)!}, \qquad -\infty < x < \infty \tag{6.4}$$

we see that (6.3) leads to the desired generating-function relation

$$\exp\left[\tfrac{1}{2}x\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x) t^n, \qquad t \neq 0 \tag{6.5}$$

Since (6.5) involves both positive and negative values of n, we may wish to investigate the definition of $J_n(x)$ specifically when $n < 0$. The formal replacement of n with $-n$ in (6.4) yields

$$\begin{aligned} J_{-n}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!(k-n)!} \\ &= \sum_{k=n}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!(k-n)!} \end{aligned}$$

where we have used the fact that $1/(k-n)! = 0$ $(k = 0, 1, \ldots, n-1)$ by virtue of Theorem 2.1. Finally, the change of index $k = m + n$ gives us

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+n} (x/2)^{2m+n}}{m!(m+n)!} \tag{6.6}$$

from which it follows that

$$J_{-n}(x) = (-1)^n J_n(x), \qquad n = 0, 1, 2, \ldots \tag{6.7}$$

Graphs of $J_n(x)$ for certain values of n are provided in Fig. 6.1. Observe that only $J_0(x)$ is nonzero when $x = 0$. To prove this, we simply set $x = 0$ in the generating-function relation (6.5) to get

$$1 = \sum_{n=-\infty}^{\infty} J_n(0) t^n$$

and by comparing like terms we deduce the results

$$J_0(0) = 1, \qquad J_n(0) = 0, \quad n \neq 0 \tag{6.8}$$

6.2.2 *Bessel Functions of Nonintegral Order*

Thus far we have only discussed Bessel functions of integral order. We can generalize the series definition [Equation (6.4)] of the Bessel function $J_n(x)$

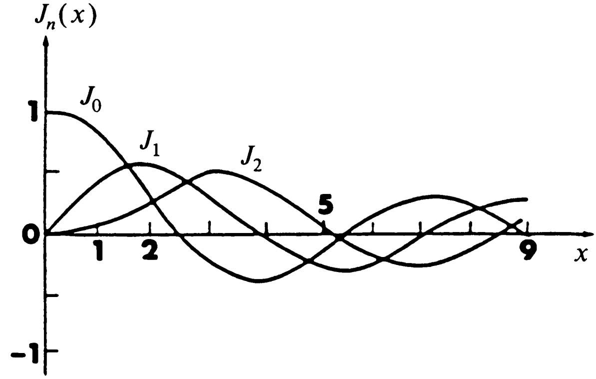


Figure 6.1 Graph of $J_n(x)$, $n = 0, 1, 2$

to include nonintegral values of n by replacing $(k+n)!$ with its gamma function equivalent. Hence, if p is any real number for which $p \geq 0$, we then define

$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!\Gamma(k+p+1)} \tag{6.9}$$

as the Bessel function of the first kind of order p.

The formal replacement of p with $-p$ in (6.9) yields

$$J_{-p}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-p}}{k!\Gamma(k-p+1)} \tag{6.10}$$

which for $p \neq 0, 1, 2, \ldots$ is not a multiple of $J_p(x)$. That is, since $J_{-p}(x)$ becomes infinite at $x = 0$ while $J_p(x)$ remains finite, the two functions are not proportional, and hence are linearly independent for nonintegral values of p. The ramifications of this observation will become clear in Section 6.5.

Although $J_p(x)$ and $J_{-p}(x)$ do not satisfy any generating-function relation, they are completely defined by their series representations and share most of the properties of $J_n(x)$ and $J_{-n}(x)$.

6.2.3 *Recurrence Relations*

There are many recurrence relations connecting the Bessel functions, analogous to those for the orthogonal polynomials. For example, suppose we multiply the series for $J_p(x)$ by x^p and then differentiate with respect to x.

This gives us

$$\begin{aligned}\frac{d}{dx}\left[x^p J_p(x)\right] &= \frac{d}{dx}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2p}}{2^{2k+p}k!\Gamma(k+p+1)}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k 2(k+p)x^{2k+2p-1}}{2^{2k+p}k!\Gamma(k+p+1)}\\
&= x^p\sum_{k=0}^{\infty}\frac{(-1)^k (x/2)^{2k+(p-1)}}{k!\Gamma(k+p)} \qquad (6.11)\end{aligned}$$

or

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \qquad (6.12)$$

Similarly, if we multiply $J_p(x)$ by x^{-p}, we find that (problem 14)

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x) \qquad (6.13)$$

If we carry out the differentiation in (6.12) and (6.13), and divide the results by the factors x^p and x^{-p}, respectively, we deduce that

$$J_p'(x) + \frac{p}{x}J_p(x) = J_{p-1}(x) \qquad (6.14)$$

and

$$J_p'(x) - \frac{p}{x}J_p(x) = -J_{p+1}(x) \qquad (6.15)$$

The substitution of $p = 0$ in (6.15) leads to the special result

$$J_0'(x) = -J_1(x) \qquad (6.16)$$

Finally, the sum of (6.14) and (6.15) yields the relation

$$2J_p'(x) = J_{p-1}(x) - J_{p+1}(x) \qquad (6.17)$$

whereas the difference of (6.14) and (6.15) gives us

$$\frac{2p}{x}J_p(x) = J_{p-1}(x) + J_{p+1}(x) \qquad (6.18)$$

This last relation is the *three-term recurrence formula* for the Bessel functions.

Repeated application of the above recurrence relations can lead to the additional results*

$$\left(\frac{d}{x\,dx}\right)^m \left[x^p J_p(x)\right] = x^{p-m} J_{p-m}(x) \tag{6.19}$$

and

$$\left(\frac{d}{x\,dx}\right)^m \left[x^{-p} J_p(x)\right] = (-1)^m x^{-p-m} J_{p+m}(x) \tag{6.20}$$

where $m = 1, 2, 3, \ldots$.

6.2.4 *Bessel's Differential Equation*

By using the above recurrence formulas, we can derive a derivative relation involving only the Bessel function $J_p(x)$. To start, we rewrite Equation (6.14) in the form

$$xJ_p'(x) - xJ_{p-1}(x) + pJ_p(x) = 0 \tag{6.21}$$

and differentiate to find

$$xJ_p''(x) + (p+1)J_p'(x) - xJ_{p-1}'(x) - J_{p-1}(x) = 0 \tag{6.22}$$

Multiplying (6.22) by x and subtracting (6.21) multiplied by p yields

$$x^2 J_p''(x) + xJ_p'(x) - p^2 J_p(x) + (p-1)xJ_{p-1}(x) - x^2 J_{p-1}'(x) = 0 \tag{6.23}$$

Now if we rewrite Equation (6.15) in the form

$$xJ_{p-1}'(x) = (p-1)J_{p-1}(x) - xJ_p(x)$$

and use it to eliminate $J_{p-1}'(x)$ and $J_{p-1}(x)$ from (6.23), we obtain

$$x^2 J_p''(x) + xJ_p'(x) + (x^2 - p^2)J_p(x) = 0 \tag{6.24}$$

Hence, we deduce that the Bessel function $J_p(x)$ is a solution of the second-order linear DE†

$$x^2 y'' + xy' + (x^2 - p^2)y = 0 \tag{6.25}$$

Equation (6.25) is called *Bessel's equation*. Among other areas of application, it arises in the solution of various partial differential equations

---

*We interpret $\left(\frac{d}{x\,dx}\right)^2 y = \frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right)$, and so on.

†Since only p^2 appears in (6.25), it is customary to make the assumption that $p \geq 0$.

of mathematical physics, particularly those problems displaying either circular or cylindrical symmetry (see Section 7.3).

EXERCISES 6.2

1. Show that the generating-function relation (6.5) can also be written in the form ($t \neq 0$)

$$\exp\left[\tfrac{1}{2}x\left(t - \frac{1}{t}\right)\right] = J_0(x) + \sum_{n=1}^{\infty} J_n(x)\left[t^n + (-1)^n t^{-n}\right]$$

2. Show that $J_n(x)$ is an even function for even n and an odd function foı odd n, i.e.,

$$J_n(-x) = (-1)^n J_n(x), \qquad n = 0, \pm 1, \pm 2, \ldots$$

(a) by using the generating function (6.5),
(b) by using the series representation (6.4).

3. By using the series representation (6.4), show that

(a) $J_1'(0) = \frac{1}{2}$, (b) $J_n'(0) = 0$ for $n > 1$.

4. For $w(x, t) = \exp[\frac{1}{2}x(t - 1/t)]$,

(a) show that $w(x + y, t) = w(x, t)w(y, t)$.
(b) From (a), deduce the *addition theorem*

$$J_n(x + y) = \sum_{k=-\infty}^{\infty} J_k(x)J_{n-k}(y)$$

(c) From (b), derive the result

$$J_0(2x) = [J_0(x)]^2 + 2\sum_{k=1}^{\infty} (-1)^k [J_k(x)]^2$$

5. Given the generating function $w(x, t) = \exp[\frac{1}{2}x(t - 1/t)]$,

(a) show that it satisfies the identity

$$\frac{\partial w}{\partial t} - \tfrac{1}{2}x\left(1 + \frac{1}{t^2}\right)w = 0$$

(b) Using (a), derive the recurrence relation

$$\frac{2n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x), \qquad n = 1, 2, 3, \ldots$$

6. Given the generating function $w(x,t) = \exp[\frac{1}{2}x(t - 1/t)]$,

(a) show that it satisfies the identity

$$\frac{\partial w}{\partial x} - \frac{1}{2}\left(t - \frac{1}{t}\right)w = 0$$

(b) Using (a), derive the relation

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \qquad n = 1,2,3,\ldots$$

7. Show that ($k \neq 0$, $t \neq 0$)

$$\exp\left[-\frac{x}{2t}\left(k - \frac{1}{k}\right)\right] \sum_{n=-\infty}^{\infty} J_n(x)k^n t^n = \sum_{n=-\infty}^{\infty} J_n(kx)t^n$$

8. From the product of the generating functions $w(x,t)w(-x,t)$,

(a) show that

$$1 = [J_0(x)]^2 + 2\sum_{n=1}^{\infty}[J_n(x)]^2$$

(b) From (a), deduce that (for all x)

$$|J_0(x)| \le 1 \quad \text{and} \quad |J_n(x)| \le \frac{1}{\sqrt{2}}, \qquad n = 1,2,3,\ldots$$

9. Use the generating function (6.5) to derive the *Jacobi-Anger expansion*

$$\exp(ix\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\theta}$$

10. Use the result of problem 9 to deduce that

(a) $\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta)$,

(b) $\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin[(2n-1)\theta]$,

(c) $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$,

(d) $\sin x = 2\sum_{n=1}^{\infty}(-1)^n J_{2n-1}(x)$.

11. Use the results of problem 10 to deduce that

(a) $x = 2\sum_{n=1}^{\infty}(2n-1)J_{2n-1}(x)$.

Hint: Differentiate problem 10(b).

(b) $x\sin x = 2\sum_{n=1}^{\infty}(2n)^2 J_{2n}(x)$.

12. Set $t = e^{\theta}$ in the generating function (6.5) and deduce that

(a) $\cosh(x \sinh\theta) = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x)\cosh(2n\theta)$,

(b) $\sinh(x \sinh\theta) = 2 \sum_{n=1}^{\infty} J_{2n-1}(x)\sinh[(2n-1)\theta]$.

13. Derive *Lommel's formula*

$$J_p(x)J_{1-p}(x) + J_{-p}(x)J_{p-1}(x) = \frac{2\sin p\pi}{\pi x}$$

14. Show that

$$\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$$

15. Show that

$$\frac{d}{dx}\left[J_p(kx)\right] = -kJ_{p+1}(kx) + \frac{p}{x}J_p(kx)$$

16. Show that

(a) $\dfrac{d}{dx}[xJ_p(x)J_{p+1}(x)] = x\{[J_p(x)]^2 - [J_{p+1}(x)]^2\}$,

(b) $\dfrac{d}{dx}[x^2J_{p-1}(x)J_{p+1}(x)] = 2x^2J_p(x)J_p'(x)$.

17. Show that

(a) $\dfrac{1}{x}\dfrac{d}{dx}\left(\frac{1}{2}x^2\left\{\left[J_p(x)\right]^2 - J_{p-1}(x)J_{p+1}(x)\right\}\right) = [J_p(x)]^2$.

(b) $\dfrac{d}{dx}\{[J_p(x)]^2 + [J_{p+1}(x)]^2\} = 2\left\{\dfrac{p}{x}\left[J_p(x)\right]^2 - \dfrac{p+1}{x}\right\}[J_{p+1}(x)]^2$.

18. Show directly that $y = J_{-p}(x)$ is a solution of

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

19. Show directly that $y = J_p(kx)$ is a solution of

$$x^2y'' + xy' + (k^2x^2 - p^2)y = 0$$

20. Establish the following identities:

(a) $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$, (b) $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$,

(c) $J_{1/2}(x)J_{-1/2}(x) = \dfrac{\sin 2x}{\pi x}$,

(d) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \dfrac{2}{\pi x}$.

21. By using the Cauchy product, show that

$$[J_0(x)]^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n!)^4} \left(\frac{x}{2}\right)^{2n}$$

Hint: $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$

In problems 22–24, derive the given identity.

22. $\displaystyle J_0\left(\sqrt{x^2 - 2xt}\right) = \sum_{n=0}^{\infty} J_n(x)\frac{t^n}{n!}.$

23. $\displaystyle \left(\frac{x-2t}{x}\right)^{-p/2} J_p\left(\sqrt{x^2 - 2xt}\right) = \sum_{n=0}^{\infty} J_{p+n}(x)\frac{t^n}{n!}.$

24. $\displaystyle e^{t\cos\phi} J_0(t\sin\phi) = \sum_{n=0}^{\infty} P_n(\cos\phi)\frac{t^n}{n!}$, where $P_n(x)$ is the nth Legendre polynomial.

25. A waveform with phase modulation distortion may be represented by*

$$s(t) = \cos[\omega_0 t + \varepsilon(t)]$$

where $\varepsilon(t)$ represents the "distortion term." In much of the analysis of such waveforms it suffices to approximate the distortion term by the first term of its Fourier series, i.e.,

$$\varepsilon(t) \simeq a \sin \omega_m t$$

where a denotes the peak phase error and ω_m is the fundamental frequency of the phase error. Thus, the original waveform becomes

$$s(t) \simeq \cos(\omega_0 t + a \sin \omega_m t)$$

(a) Show that this last form for $s(t)$ can be decomposed into its harmonic components with Bessel functions representing the corresponding amplitudes, i.e., show that

$$s(t) \simeq J_0(a)\cos\omega_0 t + \sum_{n=1}^{\infty} J_n(a)\left[\cos(\omega_0 t + n\omega_m t) + (-1)^n \cos(\omega_0 t - n\omega_m t)\right]$$

Hint: Use problem 10.

(b) Whenever the peak phase error satisfies $a \le 0.4$ radians, we can use the approximations

$$J_0(a) \simeq 1, \qquad J_1(a) \simeq \frac{a}{2}, \qquad J_n(a) \simeq 0 \quad (n = 2, 3, 4, \ldots)$$

*For a further discussion of this kind of problem, see C.E. Cook and M. Bernfeld, *Radar Signals*, New York: Academic, 1967.

Show that under these conditions the phase modulation error term produces only the effect of "paired sidebands" with a frequency displacement of $\pm\omega_m$ with respect to ω_0, and a relative amplitude of $a/2$.

6.3 *Integral Representations and Integrals of Bessel Functions*

There are several integral representations of $J_p(x)$ that are especially useful in practice. Foremost among these is one involving the Bessel function of integral order. To derive it, we start with the generating-function relation

$$e^{\frac{1}{2}x(t-1/t)} = \sum_{k=-\infty}^{\infty} J_k(x)t^k$$

and set $t = e^{-i\phi}$ to get

$$e^{-ix\sin\phi} = \sum_{k=-\infty}^{\infty} J_k(x)e^{-ik\phi} \tag{6.26}$$

where we have made the observation

$$t - \frac{1}{t} = e^{-i\phi} - e^{i\phi} = -2i\sin\phi$$

Next, we multiply both sides of (6.26) by $e^{in\phi}$ and integrate the result from 0 to π, obtaining

$$\int_0^\pi e^{i(n\phi - x\sin\phi)}\,d\phi = \sum_{k=-\infty}^{\infty} J_k(x)\int_0^\pi e^{i(n-k)\phi}\,d\phi \tag{6.27}$$

assuming that termwise integration is permitted. Now using Euler's formula, we can express (6.27) in terms of sines and cosines, i.e.,

$$\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi + i\int_0^\pi \sin(n\phi - x\sin\phi)\,d\phi$$
$$= \sum_{k=-\infty}^{\infty} J_k(x)\int_0^\pi \cos(n-k)\phi\,d\phi + i\sum_{k=-\infty}^{\infty} J_k(x)\int_0^\pi \sin(n-k)\phi\,d\phi \tag{6.28}$$

Equating the real parts of (6.28), and using the result

$$\int_0^\pi \cos(n-k)\phi\,d\phi = \begin{cases} 0, & k \neq n \\ \pi, & k = n \end{cases} \tag{6.29}$$

we find all terms of the infinite sum vanish except for the term corresponding to $k = n$, and thus we are left with the integral representation (for $n = 0, 1, 2, \ldots$)

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\,d\phi \tag{6.30}$$

When $n = 0$, we get the special case

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)\,d\phi \tag{6.31}$$

The representation (6.30) is restricted to Bessel functions of integral order. A less restrictive representation, due to S.D. Poisson (1781–1840), is given by

$$J_p(x) = \frac{(x/2)^p}{\sqrt{\pi}\,\Gamma(p+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{p-\frac{1}{2}}e^{ixt}\,dt, \qquad p > -\tfrac{1}{2}, \quad x > 0 \tag{6.32}$$

where p is not restricted to integral values. To derive (6.32), we start with the relation

$$\begin{aligned}\int_{-1}^{1}(1-t^2)^{p-\frac{1}{2}}e^{ixt}\,dt &= 2\int_0^1(1-t^2)^{p-\frac{1}{2}}\cos xt\,dt\\ &= 2\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\int_0^1(1-t^2)^{p-\frac{1}{2}}t^{2k}\,dt\end{aligned} \tag{6.33}$$

where we are using properties of even and odd functions and have expressed $\cos xt$ in a power series. The residual integral in (6.33) can be evaluated in terms of the beta function by making the change of variable $u = t^2$, from which we get (for $p > -\frac{1}{2}$)

$$\begin{aligned}\int_0^1(1-t^2)^{p-\frac{1}{2}}t^{2k}\,dt &= \tfrac{1}{2}\int_0^1(1-u)^{p-\frac{1}{2}}u^{k-\frac{1}{2}}\,du\\ &= \tfrac{1}{2}B(k+\tfrac{1}{2}, p+\tfrac{1}{2})\\ &= \frac{\Gamma(k+\frac{1}{2})\Gamma(p+\frac{1}{2})}{2\Gamma(k+p+1)}\end{aligned} \tag{6.34}$$

From the Legendre duplication formula, we have

$$\Gamma(k+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,(2k)!}{2^{2k}k!}, \qquad k = 0, 1, 2, \ldots \tag{6.35}$$

and by substituting the results of (6.34) and (6.35) into (6.33), we obtain

$$\int_{-1}^{1}(1-t^2)^{p-\frac{1}{2}}e^{ixt}\,dt = \sqrt{\pi}\,\Gamma\left(p+\tfrac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{2k}}{k!\Gamma(k+p+1)}$$

$$= \sqrt{\pi}\,\Gamma\left(p+\tfrac{1}{2}\right)\left(\frac{x}{2}\right)^{-p}J_p(x) \tag{6.36}$$

from which we deduce (6.32).

A variant of (6.32) results if we make the change of variable $t = \cos\theta$:

$$J_p(x) = \frac{(x/2)^p}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)}\int_0^{\pi}\cos(x\cos\theta)\sin^{2p}\theta\,d\theta, \qquad p > -\tfrac{1}{2}, \quad x > 0 \tag{6.37}$$

the verification of which is left to the reader (problem 2).

6.3.1 *Indefinite Integrals Involving Bessel Functions*

Many of the indefinite integrals that arise in practice are simple products of some Bessel function and x raised to a power. In particular, we find as a general rule that any integral of the form

$$I = \int x^m J_n(x)\,dx \tag{6.38}$$

where m and n are integers such that $m + n > 0$, can be integrated in closed form when $m + n$ is odd, but will ultimately depend upon the residual integral $\int J_0(x)\,dx$ when $m + n$ is even.*

By starting with the identities [see (6.12) and (6.13)]

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x) \tag{6.39}$$

and

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x) \tag{6.40}$$

we can derive two useful integration formulas for handling integrals of the form (6.38). Direct integration of (6.39) and (6.40) leads to

$$\int x^p J_{p-1}(x)\,dx = x^p J_p(x) + C \tag{6.41}$$

*The integral $\int_0^x J_0(t)\,dt$ has been tabulated. See, for example, M. Abramowitz and I. Stegun, (Eds.), *Handbook of Mathematical Functions*, New York: Dover, 1965.

and

$$\int x^{-p} J_{p+1}(x)\,dx = -x^{-p} J_p(x) + C \tag{6.42}$$

where C denotes a constant of integration.

Example 1: Reduce $\int x^2 J_2(x)\,dx$ to an integral involving only $J_0(x)$.

Solution: To use (6.42), we first write

$$\int x^2 J_2(x)\,dx = \int x^3 \left[x^{-1} J_2(x)\right] dx$$

and use integration by parts with

$$u = x^3, \qquad dv = x^{-1} J_2(x)\,dx$$

$$du = 3x^2\,dx, \qquad v = -x^{-1} J_1(x)$$

Thus we have

$$\int x^2 J_2(x)\,dx = -x^2 J_1(x) + 3\int x J_1(x)\,dx$$

and a second integration by parts finally gives

$$\int x^2 J_2(x)\,dx = -x^2 J_1(x) - 3x J_0(x) + 3\int J_0(x)\,dx$$

The last integral involving $J_0(x)$ cannot be evaluated in closed form, and so our integration is complete.

6.3.2 *Definite Integrals Involving Bessel Functions*

In practice we are often faced with the necessity of evaluating definite integrals involving Bessel functions in combinations with various elementary functions or, in some instances, special functions of other kinds. The usual procedure in such integrals is to replace the Bessel function by its series representation (or an integral representation) and then interchange the order in which the operations are carried out.

To illustrate the technique, let us consider the Laplace transform integral

$$I = \int_0^\infty e^{-ax} x^p J_p(bx)\,dx, \qquad p > -\tfrac{1}{2}, \quad a > 0, \quad b > 0 \tag{6.43}$$

Here we replace $J_p(bx)$ by its series representation (6.9) and integrate the resulting series termwise to get

$$I = \sum_{k=0}^{\infty} \frac{(-1)^k (b/2)^{2k+p}}{k!\Gamma(k+p+1)} \int_0^{\infty} e^{-ax} x^{2k+2p}\,dx$$

$$= b^p \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+2p+1)}{2^{2k+p} k!\Gamma(k+p+1)} (a^2)^{-(p+\frac{1}{2})-k} (b^2)^k \tag{6.44}$$

where the integral has been evaluated through properties of the gamma function. We wish to show that the series (6.44) is a binomial series, and hence it can be summed. Recalling the Legendre duplication formula and Equation (1.27) in Section 1.2.4, it follows that ($p > -\frac{1}{2}$)

$$\begin{aligned}
\frac{(-1)^k \Gamma(2k+2p+1)}{2^{2k+p} k!\Gamma(k+p+1)} &= \frac{(-1)^k 2^p}{\sqrt{\pi}} \frac{\Gamma(p+k+\frac{1}{2})}{k!} \\
&= \frac{(-1)^k}{\sqrt{\pi}} 2^p \Gamma(p+\tfrac{1}{2}) \binom{p+k-\frac{1}{2}}{k} \\
&= \frac{2^p \Gamma(p+\frac{1}{2})}{\sqrt{\pi}} \binom{-(p+\frac{1}{2})}{k}
\end{aligned} \tag{6.45}$$

Thus, (6.44) becomes

$$I = \frac{(2b)^p \Gamma(p+\frac{1}{2})}{\sqrt{\pi}} \sum_{k=0}^{\infty} \binom{-(p+\frac{1}{2})}{k} (a^2)^{-(p+\frac{1}{2})-k} (b^2)^k \tag{6.46}$$

and by summing this binomial series, we are led to*

$$\int_0^{\infty} e^{-ax} x^p J_p(bx)\,dx = \frac{(2b)^p \Gamma(p+\frac{1}{2})}{\sqrt{\pi}(a^2+b^2)^{p+\frac{1}{2}}}, \qquad p > -\tfrac{1}{2}, \quad a > 0, \quad b > 0 \tag{6.47}$$

Setting $p = 0$ in (6.47) yields the special result

$$\int_0^{\infty} e^{-ax} J_0(bx)\,dx = (a^2+b^2)^{-1/2}, \qquad a > 0, \quad b > 0 \tag{6.48}$$

Strictly speaking, the validity of (6.48) rests upon the condition that $a > 0$ (or at least the real part of a positive if a is complex). Yet it is possible to justify a limiting procedure whereby the real part of a approaches zero. Thus, if we formally replace a in (6.48) with the pure

*Summing the series (6.46) requires that $a \neq b$, although the result (6.47) is valid even when $a = b$.

imaginary number ia, we get

$$\int_0^\infty e^{-iax} J_0(bx)\,dx = (b^2 - a^2)^{-1/2}$$

The separation of this expression into real and imaginary parts leads to

$$\int_0^\infty \cos(ax)\,J_0(bx)\,dx - i\int_0^\infty \sin(ax)\,J_0(bx)\,dx$$

$$= \begin{cases} (b^2 - a^2)^{-1/2}, & b > a \\ -i(a^2 - b^2)^{-1/2}, & b < a \end{cases} \tag{6.49}$$

and by equating the real and imaginary parts of (6.49), we deduce the pair of integral formulas

$$\int_0^\infty \cos(ax)\,J_0(bx)\,dx = \begin{cases} (b^2 - a^2)^{-1/2}, & b > a \\ 0, & b < a \end{cases} \tag{6.50}$$

and

$$\int_0^\infty \sin(ax)\,J_0(bx)\,dx = \begin{cases} 0, & b > a \\ (a^2 - b^2)^{-1/2}, & b < a \end{cases} \tag{6.51}$$

These last two formulas are important in the theory of Fourier integrals. Both (6.50) and (6.51) diverge when $b = a$.

Example 2: Derive *Weber's integral formula*

$$\int_0^\infty x^{2m-p-1} J_p(x)\,dx = \frac{2^{2m-p-1}\Gamma(m)}{\Gamma(p - m + 1)}, \qquad 0 < m < \tfrac{1}{2}, \quad p > -\tfrac{1}{2}$$

Solution: Replacing $J_p(x)$ by its integral representation (6.37), we have

$$\int_0^\infty x^{2m-p-1} J_p(x)\,dx = \frac{2^{-p}}{\sqrt{\pi}\,\Gamma(p + \frac{1}{2})}$$
$$\times \int_0^\infty x^{2m-1} \int_0^\pi \cos(x\cos\theta)\sin^{2p}\theta\,d\theta\,dx$$
$$= \frac{2^{-p}}{\sqrt{\pi}\,\Gamma(p + \frac{1}{2})}$$
$$\times \int_0^\pi \sin^{2p}\theta \int_0^\infty x^{2m-1}\cos(x\cos\theta)\,dx\,d\theta$$

where we have reversed the order of integration. By making the substitution $t = x\cos\theta$ in the inner integral and using the result of problem 37 in Exercises 2.2, we obtain

$$\begin{aligned}\int_0^\infty x^{2m-1}\cos(x\cos\theta)\,dx &= \cos^{-2m}\theta\int_0^\infty t^{2m-1}\cos t\,dt\\ &= \cos^{-2m}\theta\,\Gamma(2m)\cos m\pi\\ &= \pi^{-1/2}2^{2m-1}\Gamma(m)\Gamma(m+\tfrac{1}{2})\cos m\pi\cos^{-2m}\theta\end{aligned}$$

The last step follows from the Legendre duplication formula. The remaining integral above now leads to

$$\begin{aligned}\int_0^\pi \sin^{2p}\theta\cos^{-2m}\theta\,d\theta &= 2\int_0^{\pi/2}\sin^{2p}\theta\cos^{-2m}\theta\,d\theta\\ &= \frac{\Gamma(p+\frac{1}{2})\Gamma(\frac{1}{2}-m)}{\Gamma(p-m+1)}\end{aligned}$$

and hence, we deduce that

$$\begin{aligned}\int_0^\infty x^{2m-p-1}J_p(x)\,dx &= \frac{2^{2m-p-1}\Gamma(m)\Gamma(m+\frac{1}{2})\Gamma(\frac{1}{2}-m)\cos m\pi}{\pi\Gamma(p-m+1)}\\ &= \frac{2^{2m-p-1}\Gamma(m)}{\Gamma(p-m+1)}\end{aligned}$$

where we are recalling the identity [problem 42(b) in Exercises 2.2]

$$\Gamma(\tfrac{1}{2}+m)\Gamma(\tfrac{1}{2}-m) = \pi\sec m\pi$$

(Although we won't show it, Weber's integral is valid for a much wider range of values on m and p than indicated above.)

EXERCISES 6.3

1. Using Equation (6.30), deduce the following results:

(a) $[1+(-1)^n]J_n(x) = \dfrac{2}{\pi}\displaystyle\int_0^\pi \cos n\theta\cos(x\sin\theta)\,d\theta$ $(n = 0,1,2,\ldots)$.

(b) $[1-(-1)^n]J_n(x) = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin n\theta\sin(x\sin\theta)\,d\theta$ $(n = 0,1,2,\ldots)$.

(c) $J_{2k}(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos 2k\theta\cos(x\sin\theta)\,d\theta$ $(k = 0,1,2,\ldots)$.

(d) $J_{2k+1}(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \sin[(2k+1)\theta]\sin(x\sin\theta)\,d\theta$ $(k = 0,1,2,\ldots)$.

(e) $\displaystyle\int_0^\pi \cos[(2k+1)\theta]\cos(x\sin\theta)\,d\theta = 0$ $(k = 0,1,2,\ldots)$.

(f) $\displaystyle\int_0^\pi \sin 2k\theta\sin(x\sin\theta)\,d\theta = 0$ $(k = 0,1,2,\ldots)$.

2. By setting $t = \cos\theta$ in (6.32), show that

$$J_p(x) = \frac{(x/2)^p}{\sqrt{\pi}\,\Gamma(p+\frac{1}{2})}\int_0^{\pi}\cos(x\cos\theta)\sin^{2p}\theta\,d\theta, \qquad p > -\tfrac{1}{2}, \quad x > 0$$

3. By writing $\cos xt$ in an infinite series and using termwise integration, deduce that

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\,dt$$

4. Replacing $J_m(xt)$ by its series representation and using termwise integration, deduce the integral relation

$$J_p(x) = \frac{2(x/2)^{p-m}}{\Gamma(p-m)}\int_0^1 (1-t^2)^{p-m-1}t^{m+1}J_m(xt)\,dt,$$
$$p > m > -1, \quad x > 0$$

In problems 5–16, use recurrence relations, integration by parts, etc., to verify the given result.

5. $\int xJ_0(x)\,dx = xJ_1(x) + C.$

6. $\int x^2J_0(x)\,dx = x^2J_1(x) + xJ_0(x) - \int J_0(x)\,dx + C.$

7. $\int x^3J_0(x)\,dx = (x^3 - 4x)J_1(x) + 2x^2J_0(x) + C.$

8. $\int J_1(x)\,dx = -J_0(x) + C.$

9. $\int xJ_1(x)\,dx = -xJ_0(x) + \int J_0(x)\,dx + C.$

10. $\int x^2J_1(x)\,dx = 2xJ_1(x) - x^2J_0(x) + C.$

11. $\int x^3J_1(x)\,dx = 3x^2J_1(x) - (x^3 - 3x)J_0(x) - 3\int J_0(x)\,dx + C.$

12. $\int J_3(x)\,dx = -J_2(x) - 2x^{-1}J_1(x) + C.$

13. $\int x^{-1}J_1(x)\,dx = -J_1(x) + \int J_0(x)\,dx + C.$

14. $\int x^{-2}J_2(x)\,dx = -\dfrac{2}{3x^2}J_1(x) - \tfrac{1}{3}J_1(x)$
$+ \dfrac{1}{3x}J_0(x) + \dfrac{1}{3}\int J_0(x)\,dx + C.$

15. $\int J_0(x)\cos x\,dx = xJ_0(x)\cos x + xJ_1(x)\sin x + C.$

16. $\int J_0(x)\sin x\,dx = xJ_0(x)\sin x - xJ_1(x)\cos x + C.$

17. Show that

$$\int x\left\{\left[J_p(x)\right]^2 - \left[J_{p+1}(x)\right]^2\right\} dx = xJ_p(x)J_{p+1}(x) + C$$

Hint: Use the result of problem 16(a) in Exercises 6.2.

18. Show that

$$\int x\left[J_p(x)\right]^2 dx = \tfrac{1}{2}x^2\left\{\left[J_p(x)\right]^2 - J_{p-1}(x)J_{p+1}(x)\right\} + C$$

Hint: Use the result of problem 17(a) in Exercises 6.2.

19. Show that (using repeated integration by parts)

$$\int J_0(x)\,dx = J_1(x) + \frac{J_2(x)}{x} + \frac{1\times 3}{x^2}J_3(x) + \cdots + \frac{(2n-2)!J_n(x)}{2^{n-1}(n-1)!x^{n-1}} + \frac{(2n)!}{2^n n!}\int \frac{J_n(x)}{x^n}\,dx$$

In problems 20–35, derive the given integral formula.

20. $\int_0^\infty J_0(bx)\,dx = \frac{1}{b}$, $b > 0$.

Hint: Let $a \to 0^+$ in Equation (6.48).

21. $\int_0^\infty J_{n+1}(x)\,dx = \int_0^\infty J_{n-1}(x)\,dx$, $n = 1, 2, 3, \ldots$.

Hint: Use Equation (6.17).

22. $\int_0^\infty J_n(x)\,dx = 1$, $n = 0, 1, 2, \ldots$.

Hint: Use problem 21.

23. $\int_0^\infty \frac{J_n(x)}{x}\,dx = \frac{1}{n}$, $n = 1, 2, 3, \ldots$.

Hint: Use problems 21 and 22.

24. $\int_0^\infty e^{-ax}x^{p+1}J_p(bx)\,dx = \frac{2^{p+1}\Gamma(p+\frac{3}{2})}{\sqrt{\pi}}\frac{ab^p}{(a^2+b^2)^{p+\frac{3}{2}}}$, $p > -1$, $a > 0$, $b > 0$.

Hint: Differentiate both sides of Equation (6.47) with respect to a.

25. $\displaystyle\int_0^\infty x^2 e^{-ax} J_0(bx)\,dx = \frac{(2a^2 - b^2)}{(a^2 + b^2)^{5/2}}$, $a > 0$, $b > 0$.

Hint: Differentiate both sides of Equation (6.48) with respect to a.

26. $\displaystyle\int_0^\infty e^{-ax^2} x^{p+1} J_p(bx)\,dx = \frac{b^p}{(2a)^{p+1}} e^{-b^2/4a}$, $p > -1$, $a > 0$, $b > 0$.

27. $\displaystyle\int_0^\infty e^{-ax^2} x^{p+3} J_p(bx)\,dx = \frac{b^p}{2^{p+1} a^{p+2}}\left(p + 1 - \frac{b^2}{4a}\right) e^{-b^2/4a}$, $p > -1$, $a > 0$, $b > 0$.

Hint: Differentiate both sides of problem 26 with respect to a.

28. $\displaystyle\int_0^\infty x^{-1} \sin x\, J_0(bx)\,dx = \arcsin\left(\frac{1}{b}\right)$, $b > 1$.

Hint: Integrate Equation (6.50) with respect to a.

29. $\displaystyle\int_0^{\pi/2} J_0(x\cos\phi)\cos\phi\,d\phi = \frac{\sin x}{x}$.

30. $\displaystyle\int_0^{\pi/2} J_1(x\cos\phi)\,d\phi = \frac{1 - \cos x}{x}$.

31. $\displaystyle\int_0^a x(a^2 - x^2)^{-1/2} J_0(kx\sin\phi)\,d\phi = \frac{\sin(ka\sin\phi)}{k\sin\phi}$.

32. $\displaystyle\int_0^\pi e^{t\cos\phi} J_0(t\sin\phi)\sin\phi\,d\phi = 2$.

Hint: Use problem 24 in Exercises 6.2.

33. $\displaystyle\int_0^\infty e^{-t\cos\phi} J_0(t\sin\phi) t^n\,dt = n! P_n(\cos\phi)$, $0 \le \phi < \pi$, where $P_n(x)$ is the nth Legendre polynomial.

Hint: Use problem 24 in Exercises 6.2.

34. $\displaystyle\int_0^\infty x(x^2 + a^2)^{-1/2} J_0(bx)\,dx = \frac{1}{b} e^{-ab}$, $a \ge 0$, $b > 0$.

Hint: Use the integral representation

$$(x^2 + a^2)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(x^2+a^2)t} t^{-1/2}\,dt$$

and then interchange the order of integration.

35. $\displaystyle\int_0^\infty \frac{J_p(x)}{x^m}\,dx = \frac{\Gamma\left(\frac{p+1-m}{2}\right)}{2^m \Gamma\left(\frac{p+1+m}{2}\right)}$, $m > \frac{1}{2}$, $p - m > -1$.

36. The amplitude of a diffracted wave through a circular aperture is given by

$$U = k\int_0^a \int_0^{2\pi} e^{ibr\sin\theta} r\,d\theta\,dr$$

where k is a physical constant, a is the radius of the aperture, θ is the azimuthal angle in the plane of the aperture, and b is a constant inversely proportional to the wavelength of the incident wave. Show that the intensity of light in the diffraction pattern is given by

$$I = |U|^2 = \frac{4\pi^2 k^2 a^2}{b^2}[J_1(ab)]^2$$

6.4 *Bessel Series*

A *Bessel series*, which is a member of the class of generalized Fourier series,* has the form

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(k_n x), \qquad 0 < x < b, \quad p > -\tfrac{1}{2} \tag{6.52}$$

where the c's are constants to be determined and the k_n $(n = 1, 2, 3, \ldots)$ are solutions of the equation†

$$J_p(k_n b) = 0, \qquad n = 1, 2, 3, \ldots \tag{6.53}$$

The theory of Bessel series closely parallels that of Legendre series. For example, the Bessel functions satisfy an orthogonality relation, and the constants (Fourier coefficients) c_n are defined by a formula similar to that for Legendre series. The conditions under which the series (6.52) converges will be stated (see Theorem 6.1 below), although we will not present the formal proof.

6.4.1 *Orthogonality*

The theory of all generalized Fourier series rests heavily upon the *orthogonality property* of the particular special functions. In the case of Bessel functions, we have $(p > -1)$

$$\int_0^b x J_p(k_m x) J_p(k_n x)\,dx = 0, \qquad m \neq n \tag{6.54}$$

where k_m and k_n are distinct roots satisfying (6.53).

---

*See the discussion in Section 4.4 on generalized Fourier series.

†The Bessel function $J_p(x)$ has an infinite number of zeros for $x > 0$. See Theorem 5.6 in L.C. Andrews, *Ordinary Differential Equations with Applications*, Glenview, Ill.: Scott, Foresman, 1982.

In order to prove (6.54), we first note that since $y = J_p(x)$ is a solution of Bessel's equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \tag{6.55}$$

it follows that $y = J_p(kx)$ satisfies the more general equation (see problem 19 in Exercises 6.2)

$$x^2y'' + xy' + (k^2x^2 - p^2)y = 0 \tag{6.56}$$

For our purposes we wish to rewrite (6.56) in the more useful form

$$x\frac{d}{dx}(xy') + (k^2x^2 - p^2)y = 0 \tag{6.57}$$

and hence, $J_p(k_m x)$ and $J_p(k_n x)$ satisfy respectively the DEs

$$x\frac{d}{dx}\left[x\frac{d}{dx}J_p(k_m x)\right] + (k_m^2x^2 - p^2)J_p(k_m x) = 0 \tag{6.58}$$

$$x\frac{d}{dx}\left[x\frac{d}{dx}J_p(k_n x)\right] + (k_n^2x^2 - p^2)J_p(k_n x) = 0 \tag{6.59}$$

If we multiply (6.58) by $x^{-1}J_p(k_n x)$ and (6.59) by $x^{-1}J_p(k_m x)$, subtract the resulting equations, and integrate from 0 to b, we find upon rearranging the terms

$$\begin{aligned}(k_m^2 - k_n^2)\int_0^b xJ_p(k_m x)J_p(k_n x)\,dx &= \int_0^b J_p(k_m x)\frac{d}{dx}\left[x\frac{d}{dx}J_p(k_n x)\right]dx \\ &\quad - \int_0^b J_p(k_n x)\frac{d}{dx}\left[x\frac{d}{dx}J_p(k_m x)\right]dx\end{aligned}$$

Carrying out the integrations (by parts) on the right-hand side and dividing by the factor $k_m^2 - k_n^2$ leads to

$$\begin{aligned}&\int_0^b xJ_p(k_m x)J_p(k_n x)\,dx \\ &\qquad = \frac{x\left[J_p(k_m x)\frac{d}{dx}J_p(k_n x) - J_p(k_n x)\frac{d}{dx}J_p(k_m x)\right]\Big|_{x=0}^{x=b}}{k_m^2 - k_n^2}\end{aligned} \tag{6.60}$$

By hypothesis, $k_m \neq k_n$ and $J_p(k_m b) = J_p(k_n b) = 0$, and thus the right-hand side of (6.60) vanishes, which proves the orthogonality property (6.54).

When $k_m = k_n$, the resulting integral

$$I = \int_0^b x\left[J_p(k_n x)\right]^2 dx$$

is also of interest to us. To deduce its value we take the limit of (6.60) as

$k_m \to k_n$. Because the right-hand side of (6.60) approaches the indeterminate form $0/0$ in the limit, we need to employ L'Hôpital's rule, which leads to (treating k_m as the variable and all other parameters constant)

$$I = \frac{x}{2k_n}\left[\frac{d}{dx}J_p(k_n x)\frac{d}{dk_n}J_p(k_n x) - J_p(k_n x)\frac{d}{dk_n}\frac{d}{dx}J_p(k_n x)\right]\Bigg|_{x=0}^{x=b} \tag{6.61}$$

Now, using the recurrence relations (see problem 15 in Exercises 6.2)

$$\frac{d}{dx}J_p(kx) = \frac{p}{x}J_p(kx) - kJ_{p+1}(kx) \tag{6.62a}$$

$$\frac{d}{dk}J_p(kx) = \frac{p}{k}J_p(kx) - xJ_{p+1}(kx) \tag{6.62b}$$

we find that (6.61) reduces to

$$I = \left\{\frac{1}{2}\frac{p^2}{k_n^2}\left[J_p(k_n x)\right]^2 + \frac{1}{2}x^2\left[J_{p+1}(k_n x)\right]^2 - \frac{p}{k_n}J_p(k_n x)J_{p+1}(k_n x)\right\}\Bigg|_{x=0}^{x=b}$$

or finally

$$\int_0^b x\left[J_p(k_n x)\right]^2 dx = \tfrac{1}{2}b^2\left[J_{p+1}(k_n b)\right]^2 \tag{6.63}$$

6.4.2 *A Convergence Theorem*

Returning now to the series

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(k_n x), \qquad 0 < x < b, \quad p > -\tfrac{1}{2} \tag{6.64}$$

where $J_p(k_n b) = 0$ $(n = 1, 2, 3, \ldots)$, let us assume the validity of this representation and attempt to formally find the Fourier coefficients. To begin, we multiply both sides of (6.64) by $xJ_p(k_m x)$ and integrate from 0 to b. Under the assumption that termwise integration is permitted, we obtain

$$\begin{aligned}\int_0^b xf(x)J_p(k_m x)\,dx &= \sum_{n=1}^{\infty} c_n \underbrace{\int_0^b xJ_p(k_m x)J_p(k_n x)\,dx}_{0(n \neq m)} \\ &= c_m\int_0^b x\left[J_p(k_m x)\right]^2 dx \end{aligned} \tag{6.65}$$

and hence deduce that (changing the index back to n)

$$c_n = \frac{2}{b^2\left[J_{p+1}(k_n b)\right]^2} \int_0^b x f(x) J_p(k_n x)\, dx, \qquad n = 1, 2, 3, \ldots \tag{6.66}$$

Theorem 6.1. If f is a piecewise smooth function in the interval $0 \le x \le b$, then the Bessel series (6.64) with constants defined by (6.66) converges pointwise to $f(x)$ at points of continuity of f, and to $\frac{1}{2}[f(x^+) + f(x^-)]$ at points of discontinuity of f.*

Example 3: Find the Bessel series for

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

corresponding to the set of functions $\{J_1(k_n x)\}$, where k_n satisfies $J_1(2k_n) = 0$ $(n = 1, 2, 3, \ldots)$.

Solution: The series we seek is

$$f(x) = \sum_{n=1}^{\infty} c_n J_1(k_n x), \qquad 0 < x < 2$$

where

$$\begin{aligned} c_n &= \frac{1}{2[J_2(2k_n)]^2} \int_0^2 x f(x) J_1(k_n x)\, dx \\ &= \frac{1}{2[J_2(2k_n)]^2} \int_0^1 x^2 J_1(k_n x)\, dx \qquad (\text{let } t = k_n x) \\ &= \frac{1}{2[J_2(2k_n)]^2} \frac{1}{k_n^3} \int_0^{k_n} t^2 J_1(t)\, dt \end{aligned}$$

Recalling the formula $t^2 J_1(t) = (d/dt)[t^2 J_2(t)]$, we find that

$$\int_0^{k_n} t^2 J_1(t)\, dt = \int_0^{k_n} \frac{d}{dt}\left[t^2 J_2(t)\right] dt = k_n^2 J_2(k_n)$$

and thus

$$c_n = \frac{J_2(k_n)}{2k_n[J_2(2k_n)]^2}, \qquad n = 1, 2, 3, \ldots$$

*The series always converges to zero for $x = b$, and converges to zero at $x = 0$ if $p > 0$.

The desired Bessel series is therefore given by

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{J_2(k_n)}{k_n[J_2(2k_n)]^2} J_1(k_n x)$$

Generalizations of the Bessel series can be developed where the k_n $(n = 1, 2, 3, \dots)$ satisfy the more general condition

$$hJ_p(k_n b) + k_n J_p'(k_n b) = 0 \qquad (h \text{ constant}) \tag{6.67}$$

The theory in such cases requires only a slight modification of that presented here and is taken up in the exercises.

EXERCISES 6.4

In problems 1 and 2, verify the series relation given that $J_0(k_n) = 0$ $(n = 1, 2, 3, \dots)$.

1. $\frac{1}{8}(1 - x^2) = \sum_{n=1}^{\infty} \frac{J_0(k_n x)}{k_n^3 J_1(k_n)}, \; 0 \le x \le 1.$

2. $\log x = -2 \sum_{n=1}^{\infty} \frac{J_0(k_n x)}{[k_n J_1(k_n)]^2}, \; 0 < x \le 1.$

In problems 3–5, find the Bessel series for $f(x)$ in terms of $\{J_0(k_n x)\}$, given that $J_0(k_n) = 0$ $(n = 1, 2, 3, \dots)$.

3. $f(x) = 0.1 J_0(k_3 x), \; 0 < x < 1.$

4. $f(x) = 1, \; 0 < x < 1.$

5. $f(x) = x^4, \; 0 < x < 1.$

6. If $p \ge -\frac{1}{2}$ and $J_p(k_n) = 0$ $(n = 1, 2, 3, \dots)$, show that

$$x^p = 2 \sum_{n=1}^{\infty} \frac{J_p(k_n x)}{k_n J_{p+1}(k_n)}, \qquad 0 < x < 1$$

7. If $p > -\frac{1}{2}$ and $J_p(k_n) = 0$ $(n = 1, 2, 3, \dots)$, show that

(a) $x^{p+1} = 2^2(p+1) \sum_{n=1}^{\infty} \frac{J_{p+1}(k_n x)}{k_n^2 J_{p+1}(k_n)}, \; 0 < x < 1,$

(b) $x^{p+2} = 2^3(p+1)(p+2) \sum_{k=1}^{\infty} \frac{J_{p+2}(k_n x)}{k_n^3 J_{p+1}(k_n)}, \; 0 < x < 1.$

8. Expand $f(x) = x^{-p}$, $0 < x < 1$, in the series

$$x^{-p} = \sum_{n=1}^{\infty} c_n J_p(k_n x), \qquad 0 < x < 1$$

where $J_p(k_n) = 0$ ($n = 1, 2, 3, \dots$ and $p \geq 0$).

9. Given that ($p > -\frac{1}{2}$)

$$J_p'(k_n b) = 0, \qquad n = 1, 2, 3, \dots$$

show that

(a) $\int_0^b x J_p(k_m x) J_p(k_n x)\, dx = 0$, $m \neq n$,

(b) $\int_0^b x[J_p(k_n x)]^2\, dx = \dfrac{k_n^2 b^2 - p^2}{2k_n^2}[J_p(k_n b)]^2$.

10. Given that ($p > -\frac{1}{2}$)

$$hJ_p(k_n b) + k_n J_p'(k_n b) = 0, \qquad n = 1, 2, 3, \dots \quad (h \text{ constant})$$

show that

(a) $\int_0^b x J_p(k_m x) J_p(k_n x)\, dx = 0$, $m \neq n$,

(b) $\int_0^b x[J_p(k_n x)]^2\, dx = \dfrac{(k_n^2 + h^2)b^2 - p^2}{2k_n^2}[J_p(k_n b)]^2$.

11. Under the assumption that ($p > 0$)

$$J_p'(k_n) = 0, \qquad n = 1, 2, 3, \dots$$

use the result of problem 9 to derive the Bessel series

$$x^p = 2\sum_{n=1}^{\infty} \frac{k_n J_{p+1}(k_n)}{(k_n^2 - p^2)[J_p(k_n)]^2} J_p(k_n x), \qquad 0 < x < 1$$

12. Does the expansion in problem 11 hold when $p = 0$? Explain.

6.5 *Bessel Functions of the Second and Third Kinds*

We have previously shown that

$$y_1 = J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!\Gamma(k+p+1)}, \qquad p \geq 0 \tag{6.68}$$

is a solution of Bessel's equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0 \tag{6.69}$$

Because $J_{-p}(x)$ satisfies the same recurrence relations as $J_p(x)$, it follows that $J_{-p}(x)$ is also a solution of (6.69). Moreover, for p not an integer, we have already established that $J_{-p}(x)$ is linearly independent of $J_p(x)$, and hence, under these conditions a general solution of (6.69) is given by

$$y = C_1J_p(x) + C_2J_{-p}(x), \qquad p \neq n \quad (n = 0, 1, 2, \ldots) \tag{6.70}$$

where C_1 and C_2 are arbitrary constants.

For $p = n$ ($n = 0, 1, 2, \ldots$), the solutions $J_n(x)$ and $J_{-n}(x)$ are related by [see Section 6.2.1]

$$J_{-n}(x) = (-1)^n J_n(x), \qquad n = 0, 1, 2, \ldots \tag{6.71}$$

and thus are *not* linearly independent. Therefore, (6.70) cannot represent a general solution of (6.69) in this case.

For purposes of constructing a general solution of (6.69), it is preferable to find a second solution whose independence of $J_p(x)$ is not restricted to certain values of p. Hence, we introduce the function

$$Y_p(x) = \frac{(\cos p\pi)J_p(x) - J_{-p}(x)}{\sin p\pi} \tag{6.72}$$

called the *Bessel function of the second kind* of order p. Because $Y_p(x)$ is a linear combination of $J_p(x)$ and $J_{-p}(x)$, it is clearly a solution of (6.69). Furthermore, it is linearly independent of $J_p(x)$ when p is not an integer. (Why?) When $p = n$ ($n = 0, 1, 2, \ldots$), however, it requires further investigation. That is, when $p = n$ we find that (6.72) assumes the indeterminate form $0/0$. Nonetheless, the limit as $p \to n$ does exist and we define (see Section 6.5.1)

$$Y_n(x) = \lim_{p \to n} Y_p(x) \tag{6.73}$$

The function $Y_n(x)$ is linearly independent of $J_n(x)$,* and we conclude therefore that for arbitrary values of p, the general solution of (6.69) is

$$y = C_1J_p(x) + C_2Y_p(x) \tag{6.74}$$

6.5.1 *Series Expansion for $Y_n(x)$*

We wish to derive an expression for the Bessel function of the second kind when p takes on integer values. Because the limit (6.73) leads to the

*The Wronskian of J_p and Y_p is $2/\pi x$ (see problem 8), and thus the functions are linearly independent for all p.

indeterminate form 0/0, we must apply L'Hôpital's rule, from which we deduce

$$\begin{aligned} Y_n(x) &= \lim_{p\to n} Y_p(x) \\ &= \lim_{p\to n} \frac{(-\pi \sin p\pi) J_p(x) + (\cos p\pi)\dfrac{\partial}{\partial p} J_p(x) - \dfrac{\partial}{\partial p} J_{-p}(x)}{\pi \cos p\pi} \\ &= \lim_{p\to n} \frac{1}{\pi}\left[\frac{\partial}{\partial p} J_p(x) - (-1)^n \frac{\partial}{\partial p} J_{-p}(x)\right] \end{aligned} \tag{6.75}$$

The derivative of the Bessel function with respect to its order leads to $(x > 0)$

$$\begin{aligned} &\frac{\partial}{\partial p} J_p(x) \\ &\quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left\{\frac{(x/2)^{2k+p}\log(x/2)}{\Gamma(k+p+1)} - \frac{(x/2)^{2k+p}\Gamma'(k+p+1)}{[\Gamma(k+p+1)]^2}\right\} \\ &\quad = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!\Gamma(k+p+1)}[\log(x/2) - \psi(k+p+1)] \end{aligned}$$

where $\psi(x)$ is the digamma function (see Section 2.5). We can further write this last expression as

$$\frac{\partial}{\partial p} J_p(x) = J_p(x)\log(x/2) - \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+p}}{k!\Gamma(k+p+1)}\psi(k+p+1) \tag{6.76}$$

By a similar analysis, it follows that

$$\frac{\partial}{\partial p} J_{-p}(x) = -J_{-p}(x)\log(x/2) + \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-p}}{k!\Gamma(k-p+1)}\psi(k-p+1) \tag{6.77}$$

At this point we wish to first consider the special case when $p \to 0$. Here we see that (6.75) reduces to

$$Y_0(x) = \frac{2}{\pi} \lim_{p\to 0} \frac{\partial}{\partial p} J_p(x),$$

or by using (6.76), we obtain $(x > 0)$

$$Y_0(x) = \frac{2}{\pi} J_0(x)\log(x/2) - \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2}\psi(k+1) \tag{6.78}$$

Another form of (6.78) can be obtained by making the observation*

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2} \psi(k+1)$$

$$= -\gamma + \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2} \left(-\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)$$

$$= -\gamma \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2}$$

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \tag{6.79}$$

from which we deduce ($x > 0$)

$$Y_0(x) = \frac{2}{\pi} J_0(x)[\log(x/2) + \gamma]$$

$$- \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \tag{6.80}$$

The derivation of the series for $Y_n(x)$, $n = 1, 2, 3, \ldots$, is a little more difficult to obtain. Proceeding as before and taking the limit in (6.75) by using (6.76) and (6.77), we find

$$Y_n(x) = \frac{1}{\pi}\left[J_n(x) + (-1)^n J_{-n}(x)\right] \log(x/2)$$

$$- \frac{1}{\pi}\left[\sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!\Gamma(k+n+1)} \psi(k+n+1)\right.$$

$$\left. + (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\Gamma(k-n+1)} \psi(k-n+1)\right] \tag{6.81}$$

Recalling that

$$|\Gamma(k-n+1)| \to \infty, \qquad k = 0, 1, \ldots, n-1$$

and

$$|\psi(k-n+1)| \to \infty, \qquad k = 0, 1, \ldots, n-1$$

we see that the first n terms in the last series in (6.81) become inde-

* γ is Euler's constant.

terminate. However, it can be shown that (see problem 9)

$$\lim_{p \to n} \frac{\psi(k-p+1)}{\Gamma(k-p+1)} = (-1)^{n-k}(n-k-1)!, \qquad k = 0, 1, \ldots, n-1 \tag{6.82}$$

and therefore

$$\begin{aligned}(-1)^n \sum_{k=0}^{\infty} &\frac{(-1)^k (x/2)^{2k-n}}{k!\Gamma(k-n+1)} \psi(k-n+1) \\ &= \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(x/2)^{2k-n} \\ &\quad + (-1)^n \sum_{k=n}^{\infty} \frac{(-1)^k (x/2)^{2k-n}}{k!\Gamma(k-n+1)} \psi(k-n+1)\end{aligned} \tag{6.83}$$

Finally, making the change of index $m = k - n$ in the last sum in (6.83), we obtain the desired result (for $n = 1, 2, 3, \ldots$ and $x > 0$)

$$\begin{aligned}Y_n(x) = \frac{2}{\pi} J_n(x) \log(x/2) &- \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(x/2)^{2k-n} \\ - \frac{1}{\pi} \sum_{m=0}^{\infty} &\frac{(-1)^m (x/2)^{2m+n}}{m!(m+n)!}[\psi(m+n+1) + \psi(m+1)]\end{aligned} \tag{6.84}$$

Graphs of $Y_n(x)$ for various values of n are shown in Fig. 6.2. Observe the logarithmic behavior as $x \to 0^+$. Also note that these functions have oscillatory characteristics similar to those of $J_n(x)$.

6.5.2 *Hankel Functions*

Another class of Bessel functions is the class of *Bessel functions of the third kind*, or *Hankel functions*, defined by

$$H_p^{(1)}(x) = J_p(x) + iY_p(x) \tag{6.85}$$

and

$$H_p^{(2)}(x) = J_p(x) - iY_p(x) \tag{6.86}$$

The primary motivation for introducing the Hankel functions is that these linear combinations of $J_p(x)$ and $Y_p(x)$ lend themselves more readily to the

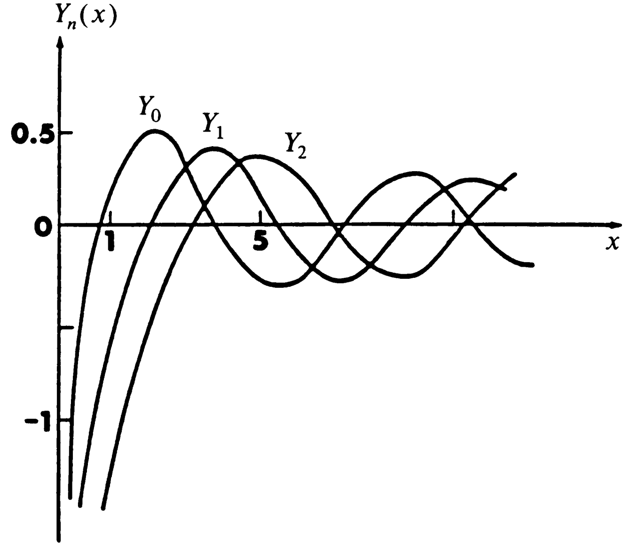


Figure 6.2 Graph of $Y_n(x)$, $n = 0, 1, 2$

development of asymptotic formulas for large x, from which we can deduce the asymptotic formulas for $J_p(x)$ and $Y_p(x)$ (see Section 6.9.2). Also, the Hankel functions are occasionally encountered directly in applications.

It follows from their definition that the Hankel functions are solutions of Bessel's equation (6.69). Moreover, they are linear independent solutions of this DE. Thus we can choose to write the general solution of Bessel's equation in the alternate form

$$y = C_1 H_p^{(1)}(x) + C_2 H_p^{(2)}(x) \tag{6.87}$$

where C_1 and C_2 are arbitrary constants.

6.5.3 *Recurrence Relations*

Because $Y_p(x)$ is a linear combination of $J_p(x)$ and $J_{-p}(x)$ for nonintegral p, it follows that $Y_p(x)$ satisfies the same recurrence formulas as $J_p(x)$ and $J_{-p}(x)$. For example, it is easily established that

$$\frac{d}{dx}\left[x^p Y_p(x)\right] = x^p Y_{p-1}(x) \tag{6.88}$$

$$\frac{d}{dx}\left[x^{-p} Y_p(x)\right] = -x^{-p} Y_{p+1}(x) \tag{6.89}$$

and also that

$$Y_{p-1}(x) + Y_{p+1}(x) = \frac{2p}{x} Y_p(x) \tag{6.90}$$

$$Y_{p-1}(x) - Y_{p+1}(x) = 2Y_p'(x) \tag{6.91}$$

For p equal to an integer n, the validity of these formulas can be deduced by considering the limit $p \to n$, noting that all functions are continuous with respect to the index p. Furthermore, it can be shown that (problem 14)

$$Y_{-n}(x) = (-1)^n Y_n(x), \qquad n = 0, 1, 2, \ldots \tag{6.92}$$

The Hankel functions $H_p^{(1)}(x)$ and $H_p^{(2)}(x)$ are simply linear combinations of $J_p(x)$ and $Y_p(x)$. Therefore it follows that they too satisfy the same recurrence formulas as $J_p(x)$ and $Y_p(x)$ (see problems 12 and 13).

EXERCISES 6.5

In problems 1–4, write the general solution of the DE in terms of Bessel functions.

1. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0.$
2. $xy'' + y' + xy = 0.$
3. $16x^2y'' + 16xy' + (16x^2 - 1)y = 0.$
4. $x^2y'' + xy' + (4x^2 - 1)y = 0.$

   Hint: Let $t = 2x$.

5. Show that the change of variable $y = u(x)/\sqrt{x}$ reduces Bessel's equation (6.69) to

$$u'' + \left[1 + \frac{1 - 4p^2}{4x^2}\right]u = 0$$

6. Use the result of problem 5 to find a general solution of Bessel's equation (6.69) when $p = \frac{1}{2}$ that does not involve Bessel functions.
7. The Wronskian of the solutions of the second-order DE

$$y'' + a(x)y' + b(x)y = 0$$

   is given by (*Abel's formula*)

$$W(y_1, y_2)(x) = C\exp\left(-\int a(x)\,dx\right)$$

   for some constant C. Use this result to deduce that the Wronskian of the solutions of Bessel's equation is

$$W(y_1, y_2)(x) = \frac{C}{x}$$

8. From the result of problem 7, show that

(a) $W(J_p, J_{-p})(x) = -\dfrac{2\sin p\pi}{\pi x}$, $p \neq$ integer.

Hint: Use the relation $C = \lim_{x\to 0^+} xW(J_p, J_{-p})(x)$.

(b) From (a), deduce that $W(J_p, Y_p)(x) = 2/\pi x$.

9. Using the identities $\Gamma(x)\Gamma(1-x) = \pi \csc \pi x$ and $\psi(1-x) - \psi(x) = \pi \cot \pi x$, show that

$$\lim_{p\to n} \frac{\psi(k-p+1)}{\Gamma(k-p+1)} = (-1)^{n-k}(n-k-1)!,$$

$$k = 0, 1, \ldots, n-1$$

10. Show that

(a) $\dfrac{d}{dx}[x^p Y_p(x)] = x^p Y_{p-1}(x)$,

(b) $\dfrac{d}{dx}[x^{-p} Y_p(x)] = -x^{-p} Y_{p+1}(x)$.

11. From the results of problem 10, deduce that

(a) $Y_{p-1}(x) + Y_{p+1}(x) = \dfrac{2p}{x} Y_p(x)$,

(b) $Y_{p-1}(x) - Y_{p+1}(x) = 2Y_p'(x)$.

12. Show that the identities in problem 10 for $Y_p(x)$ are also true for $H_p^{(1)}(x)$ and $H_p^{(2)}(x)$.

13. Show that the identities in problem 11 for $Y_p(x)$ are also true for $H_p^{(1)}(x)$ and $H_p^{(2)}(x)$.

14. Verify that

$$Y_{-n}(x) = (-1)^n Y_n(x), \qquad n = 0, 1, 2, \ldots$$

15. By making the change of variable $t = bx$, show that $(b > 0)$

$$y = C_1 J_p(bx) + C_2 Y_p(bx)$$

is the general solution of

$$x^2 y'' + xy' + (b^2x^2 - p^2)y = 0, \qquad p \geq 0$$

16. Show that the boundary value problem ($p \geq 0$)

$$x^2y'' + xy' + (k^2x^2 - p^2)y = 0, \qquad 0 < x < 1$$
$$y(x) \text{ finite as } x \to 0^+, \qquad y(1) = 0$$

has only the set of solutions $y_n(x) = J_p(k_n x)$, $n = 1, 2, 3, \ldots$, where the k's are chosen to satisfy the relation

$$J_p(k) = 0, \qquad k > 0$$

Hint: See problem 15.

17. Solve Bessel's equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad p \geq 0$$

by assuming a power-series solution of the form (*Frobenius method*)*

$$y = x^s \sum_{n=0}^{\infty} c_n x^n$$

and

(a) show that one solution corresponding to $s = p$ is

$$y_1(x) = J_p(x)$$

(b) For $p = 0$, show that the method of Frobenius leads to the general solution

$$y = (A + B\log x) \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{(k!)^2}$$
$$+ B \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x/2)^{2k}}{(k!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)$$

where A and B are arbitrary constants.

6.6 *Differential Equations Related to Bessel's Equation*

Elementary problems are regarded as solved when their solutions can be expressed in terms of tabulated functions, such as trigonometric and exponential functions. The same can be said of many problems of a more complicated nature when their solutions can be expressed in terms of Bessel functions, since extensive tables of Bessel functions have been compiled for various values of x and p.†

---

*For an introductory discussion of the Frobenius method, see L.C. Andrews, *Ordinary Differential Equations with Applications*, Glenview, Ill.: Scott, Foresman, 1982, Chapter 9.

†For example, see M. Abramowitz and I. Stegun (eds.), *Handbook of Mathematical Functions*, Dover Pub. Co., New York (1965), Chapters 9 and 10.

A fairly large number of DEs occurring in physics and engineering are specializations of the form

$$x^2y'' + (1 - 2a)xy' + \left[b^2c^2x^{2c} + (a^2 - c^2p^2)\right] y = 0, \qquad p \geq 0, \quad b > 0 \tag{6.93}$$

the general solution of which, expressed in terms of Bessel functions, is

$$y = x^a\left[C_1J_p(bx^c) + C_2Y_p(bx^c)\right] \tag{6.94}$$

where C_1 and C_2 are arbitrary constants.

To derive the solution formula (6.94) requires two transformations of variables. First, let us set

$$y = x^az \tag{6.95}$$

from which we obtain

$$xy' = x^{a+1}z' + ax^az$$
$$x^2y'' = x^{a+2}z'' + 2ax^{a+1}z' + a(a - 1)x^az$$

Then substituting these expressions into (6.93) and simplifying, we get

$$x^2z'' + xz' + (b^2c^2x^{2c} - c^2p^2)z = 0 \tag{6.96}$$

Next, we make the change of independent variable

$$t = x^c \tag{6.97}$$

from which it follows, through application of the chain rule, that

$$xz' = cx^c\frac{dz}{dt}$$

$$x^2z'' = c(c - 1)x^c\frac{dz}{dt} + c^2x^{2c}\frac{d^2z}{dt^2}$$

Hence, Equation (6.96) becomes

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + (b^2t^2 - p^2)z = 0 \tag{6.98}$$

whose general solution is (see problem 15 in Exercises 6.5)

$$z(t) = C_1J_p(bt) + C_2Y_p(bt) \tag{6.99}$$

Transforming back to the original variables x and y leads us to the desired result (6.94).

Remark: For those cases when p is not an integer, we can express the general solution (6.94) in the alternate form

$$y = x^a\left[C_1 J_p(bx^c) + C_2 J_{-p}(bx^c)\right]$$

Example 4: Find the general solution of *Airy's equation**

$$y'' + xy = 0$$

Solution: In order to compare this equation with (6.93), we must multiply through by x^2, putting it in the form

$$x^2y'' + x^3y = 0$$

Thus, we see that

$$1 - 2a = 0, \quad b^2c^2 = 1, \quad 2c = 3, \quad a^2 - c^2p^2 = 0$$

from which we calculate $a = \frac{1}{2}$, $b = \frac{2}{3}$, $c = \frac{3}{2}$, and $p = \frac{1}{3}$. The general solution therefore has the form

$$y = x^{1/2}\left[C_1 J_{1/3}\left(\tfrac{2}{3}x^{3/2}\right) + C_2 Y_{1/3}\left(\tfrac{2}{3}x^{3/2}\right)\right]$$

or, since p is not an integer, we also can represent the general solution in the form

$$y = x^{1/2}\left[C_1 J_{1/3}\left(\tfrac{2}{3}x^{3/2}\right) + C_2 J_{-1/3}\left(\tfrac{2}{3}x^{3/2}\right)\right]$$

EXERCISES 6.6

In problems 1–12, express the general solution in terms of Bessel functions.

1. $xy'' + y' + \frac{1}{4}y = 0.$

2. $4x^2y'' + 4xy' + (x^2 - n^2)y = 0.$

3. $x^2y'' + xy' + 4(x^4 - k^2)y = 0.$

4. $xy'' - y' + xy = 0.$

5. $xy'' + (1 + 2n)y' + xy = 0.$

6. $x^2y'' + (x^2 + \frac{1}{4})y = 0.$

7. $x^2y'' - 7xy' + (36x^6 - \frac{175}{16})y = 0.$

*The solutions of this DE, called *Airy functions*, are important in the theory of diffraction of radio waves around the earth's surface.

8. $y'' + y = 0.$

9. $y'' + k^2x^2y = 0.$

10. $y'' + k^2x^4y = 0.$

11. $4x^2y'' + (1 + 4x)y = 0.$

12. $x^2y'' + 5xy' + (9x^2 - 12)y = 0.$

13. Given the DE

$$y'' + ae^{mx}y = 0, \qquad m > 0$$

(a) show that the substitution $t = e^{mx}$ transforms it into

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{a}{m^2}y = 0$$

(b) Solve the DE in (a) in terms of Bessel functions.
(c) Write the general solution of the original DE in terms of Bessel functions.

14. Given the DE

$$x^2y'' + x(1 - 2x\tan x)y' - (x\tan x + n^2)y = 0$$

(a) show that the transformation $y = u(x)\sec x$ leads to an equation in u solvable in terms of Bessel functions.
(b) Write the general solution for y in terms of Bessel functions.

15. A particle of variable mass $m = (a + bt)^{-1}$, where a and b are positive constants, starting from rest at a distance r_0 from the origin O, is attracted to O by a force always directed toward O and whose magnitude is k^2mr ($k > 0$). The equation of motion is given by

$$\frac{d}{dt}\left(m\frac{dr}{dt}\right) = -k^2mr$$

Solve this equation for r subject to the prescribed initial conditions.

Hint: Make the change of variable $bx = a + bt$, transforming the equation of motion to $x^2r'' - xr' + k^2x^2r = 0$.

16. In a problem on the stability of a tapered strut, the displacement y satisfies the boundary-value problem

$$y'' + \left(\frac{K^2}{4x}\right)y = 0, \qquad y'(a) = 0, \ y'(b) = 0 \quad (0 < a < b)$$

For solutions to exist, show that the constant K must satisfy

$$J_0(K\sqrt{a})Y_0(K\sqrt{b}) = J_0(K\sqrt{b})Y_0(K\sqrt{a}), \qquad K > 0$$

17. The small deflections of a uniform column of length b bending under its own weight are governed by

$$\theta'' + K^2 x\theta = 0, \qquad \theta'(0) = 0, \quad \theta(b) = 0$$

where θ is the angle of deflection from the vertical and K is a positive constant.

(a) Show that the solution of the DE satisfying the first boundary condition at $x = 0$ is

$$\theta(x) = Cx^{1/2}J_{-1/3}\left(\tfrac{2}{3}Kx^{3/2}\right)$$

where C is an arbitrary constant.

(b) Show that the shortest column length for which buckling may occur (denoted by b_0) is $b_0 \simeq 1.99K^{-2/3}$.

Hint: The first zero of $J_{-1/3}(u)$ is $u \simeq 1.87$.

18. An axial load P is applied to a column whose circular cross section is tapered so that the moment of inertia is $I(x) = (x/a)^4$. If the column is simply supported at the ends $x = 1$ and $x = a$ $(a > 1)$, the deflections are governed by

$$x^4 y'' + k^2 y = 0, \qquad y(1) = 0, \quad y(a) = 0$$

where $k^2 = Pa^4/E$ (constant).

(a) Express the general solution (not satisfying the boundary conditions) in terms of Bessel functions.

(b) By making the substitution $y = xu(x)$ followed by $x = 1/t$, show that the general solution of the DE can also be expressed in terms of sines and cosines.

(c) Apply the prescribed boundary conditions to the solution in (b) and show that the first buckling mode is described by

$$y(x) = x \sin\left[\frac{a\pi}{a-1}\left(1 - \frac{1}{x}\right)\right]$$

Remark: For additional applications like problems 16–18, consult N.W. McLachlan, *Bessel Functions for Engineers*, 2nd ed., London: Oxford U.P., 1961.