APPLIED MATHEMATICS 1

STATICS AND DYNAMICS

First Year
Mathematics Dept.
Faculty of Education

Dynamics

INTRODUCTION

The subject of Dynamics is generally divided into two branches: the first one, is called *Kinematics*, is concerned with the geometry of motion apart from all considerations of force, mass or energy; the second, is called *Kinetics*, is concerned with the effects of forces on the motion of bodies.

In order to describe the motion of a particle (or point) two things are needed,

(i) a frame of reference,
(ii) a time-keeper.

It is not possible to describe absolute motion, but only motion relative to surrounding objects; and a suitable frame of reference depends on the kind of motion that it is desired to describe. Thus if the motion is rectilinear the distance from a fixed point on the line is a sufficient description of the position of the moving point; and in more general cases systems of two or of three rectangular axes may be chosen as a frame of reference. For example, in the case of a body projected from the surface of the Earth a set of axes with the origin at the point of projection would be suitable for the description of motion relative to the Earth. But, for the description of the motion of the planets, it would be more convenient to take a frame of axes with an origin at the Sun's center (Polar co-ordinates).

■ Definitions

1. Mass: The mass of a body is the quantity of matter in the body. The unit of mass used in England is a pound and is defined to be the mass of a certain piece of platinum kept in the Exchequer Office.

2. A Particle (point): is a portion of matter which is indefinitely small in size, or which, for the purpose of our investigations, is so small that the distances between its different parts may be neglected.

3. A Body: may be regarded as an indefinitely large number of indefinitely small portions, or as a conglomeration of particles.

4. A Rigid Body is a body whose parts always preserve an invariable position with respect to one another.

5. Space is the boundless, three-dimensional extent in which objects and events occur and have relative position and direction. Two-dimensional space is described with two coordinates (x, y), while three-dimensional space (physical reality) is described in three coordinates (x, y, z).

6. Time is a part of the measuring system used to sequence events, to compare the durations of events and the intervals between them, and to quantify rates of change such as the motions of object (not related to analysis of statics problems).

7. Force is any influence that causes an object to undergo a change in speed, a change in direction, or in a change in shape. Force can also be described by intuitive concepts such as a push or pull that can cause an object with mass to change its velocity, i.e. accelerate. A force has both magnitude and direction, which is a vector quantity.

KINEMATICS IN ONE DIMENSION

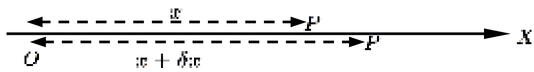
RECTILINEAR MOTION

Although motion in a straight line or rectilinear motion constitute the simplest of dynamical problems, yet it is very important because many physical problems reduce to this category, e.g., simple harmonic motion, motion under inverse square law, motion in a resisting medium and motion of a rocket. Therefore, in this chapter, we first proceed to determine the solution of the one dimensional equation of motion with subject to initial conditions. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical.

■ Velocity and Acceleration

Suppose a particle moves along a straight line $\mathbf{OX}$ where $\mathbf{O}$ represents a fixed point on the line. Let $\mathbf{P}$ be the position of the particle at time t, where $\mathbf{OP} = x$ and $\mathbf{P'}$ be the position of the particle at time $t + \delta t$, with $\mathbf{OP'} = x + \delta x$. Therefore $\delta x/\delta t$ represents the average rate of displacement or the average velocity during the interval δt. If this ratio be independent of the interval δt, i.e. if it has the same value for all intervals of time, then the velocity is constant or uniform, and equal distances will be traversed in equal times. Whether the ratio $\delta x / \delta t$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the *velocity* (also known as instantaneous velocity) of the moving point at time t. But this limiting value is the differential coefficient of x with regard to t, so that if we denote the velocity by v, we have

$$v = \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \dot{x}$$



Again, *Acceleration* is similarly defined as the rate of change of velocity. Thus, if $v, v + \delta v$ denote the velocities of the moving point at times $t, t + \delta t$, then δv is the change of velocity in time δt and $\delta v/\delta t$ is the average rate of change of velocity during the interval δt. If this ratio is independent of the interval δt, then the acceleration is constant or uniform, or equal increments of velocity take place in equal intervals. Whether the ratio $\delta v/\delta t$ be constant or not, its limiting value as δt tends to zero is defined to be the measure of the acceleration of the moving point at time t. But this limiting value is the differential coefficient of v with regard to t, so that if we denote the acceleration by a, we have

$$\begin{aligned} a &= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} \\ &= \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x} \end{aligned}$$

■ <u>Other Expression for Acceleration</u>

Let $v = \dfrac{dx}{dt}$. We can write (using chain rule in Differentiation)

$$\begin{aligned} a = \ddot{x} &= \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) \\ &= \frac{dv}{dt} = \frac{dv}{dx} \times \underbrace{\frac{dx}{dt}}_{v} = v\frac{dv}{dx} \end{aligned}$$

Therefore, $\dfrac{d^2x}{dt^2}$, $\dfrac{dv}{dt}$ and $v\dfrac{dv}{dx}$ are three expressions for representing the acceleration and any one of them can be used to suit the convenience in working out the problems.

■ Remember

The law of acceleration in a particular problem may be given by expressing the acceleration as a function of the time t, or the distance x, or the velocity v. The problem of further investigating the motion can then be solved as follows:

➤If acceleration is given as a function of the time t say $\varphi(t)$ so

$$a = \varphi(t) \qquad \Rightarrow \frac{dv}{dt} = \varphi(t)$$
$$\Rightarrow dv = \varphi(t)dt$$
$$\Rightarrow v = \int \varphi(t)dt + c_1$$

And then $\qquad \because \quad v = \int \varphi(t)dt + c_1 \qquad \Rightarrow \dfrac{dx}{dt} = \int \varphi(t)dt + c_1$

$$\Rightarrow dx = \quad \int \varphi(t)dt + c_1 \quad dt$$

$$\therefore x \; = \int \quad \int \varphi(t)dt + c_1 \quad dt + c_2$$

➤If acceleration is given as a function of the distance x say $f(x)$ so

$$a = f(x) \qquad \Rightarrow v\frac{dv}{dx} = f(x) \qquad \Rightarrow vdv = f(x)dx$$
$$\Rightarrow v^2 \; = 2\int f(x)dx + c_3$$

Further, $\qquad \because v^2 \; = 2\int f(x)dx + c_3$

$$\Rightarrow \frac{dx}{dt} = \mp\sqrt{2\int f(x)dx + c_3}$$

$$\Rightarrow \mp \frac{dx}{\sqrt{2\int f(x)dx + c_3}} = dt$$

$$\Rightarrow t + c_4 \; = \mp\int \frac{dx}{\sqrt{2\int f(x)dx + c_3}}$$

➤Again, Acceleration is given as a function of velocity v say $\varphi(v)$

$$a = \varphi(v) \qquad \Rightarrow \frac{dv}{dt} = \varphi(v)$$

$$\Rightarrow \frac{dv}{\varphi(v)} = dt \qquad \text{by integrating}$$

$$\Rightarrow t = \int \frac{dv}{\varphi(v)} + c_5$$

or we may connect velocity with distance by writing

$$v\frac{dv}{dx} = \varphi(v) \quad \Rightarrow \frac{vdv}{\varphi(v)} = dx \qquad \therefore x = \int \frac{vdv}{\varphi(v)} + c_6$$

where, $c_1 - c_6$ are constants of integration.

■ Illustrative Examples ■

‖ Example ▸

A car moves along a straight line such that its displacement x from a fixed point on the line (origin) at time t is given by $x = t^3 - 9t^2 + 24t + 6$. Determine the instant when the acceleration becomes zero, the position of the car at this instant and the velocity of the particle then.

‖ Solution ▸

Since, $x = t^3 - 9t^2 + 24t + 6$. Differentiating with respect to time (w.r.t),

the velocity $$v = \frac{dx}{dt} = 3t^2 - 18t + 24,$$

and the acceleration is $$a = \frac{dv}{dt} = 6t - 18$$

Now the acceleration vanishes i.e. $a = 0$ when $6t - 18 = 0 \quad \Rightarrow t = 3$

When $t = 3$, the position is given by $x = 3^3 - 9(3^2) + 24(3) + 6 = 24$ units.

Again when $t = 3$ the velocity is given by $v = 3(3^2) - 18(3) + 24 = -3$, this means that at $t = 3$ the velocity of the particle equals 3 units and in the opposite direction of x.

‖ Example ▸

If at time t the displacement x of a particle moving away from the origin is given by $x = A\cos t + B\sin t$, where A, B are constants. Find the velocity and acceleration of the particle at in terms of time.

‖ Solution ▸

Given that $x = A\cos t + B\sin t$

Differentiating with respect to time (w.r.t), we obtain the velocity of the particle

$$v = \frac{dx}{dt} = B\cos t - A\sin t$$

Differentiating again, one get the acceleration at any time,

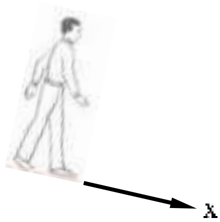
$$\begin{aligned} a = \frac{dv}{dt} &= -A\cos t - B\sin t \\ &= -(\underbrace{A\cos t + B\sin t}_{x}) \\ &= -x \end{aligned}$$

<u>Note that</u> the acceleration proportional to the displacement.

Example

A man moves along a straight line where its distance x from a fixed point on the line is given by $x = A\cos(\mu t + \epsilon)$. Prove that its acceleration varies as the distance measured from the origin and is directed towards the origin.

Solution



Since we have $x = A\cos(\mu t + \epsilon)$

Differentiating w.r.t $x = A\cos(\mu t + \epsilon)$, we get

$$\frac{dx}{dt} = -\mu A\sin(\mu t + \epsilon)$$

Differentiation again $\dfrac{d^2x}{dt^2} = -\mu^2 \underbrace{A\cos(\mu t + \epsilon)}_{x} = -\mu^2 x$

That is the acceleration varies as the distance x from the origin. The negative sign “-“ indicates that it is in the negative sense of the x-axis, i.e., towards the origin.

Example

A truck moves along a straight line such that its distance x from a fixed point on it and the velocity v are related by $v^2 = \mu(b^2 - x^2)$. Prove that the acceleration varies as the distance from the origin and is directed towards the origin.

Solution

Since we have $v^2 = \mu(b^2 - x^2)$

Differentiating w.r.t x, we obtain

$$2v\frac{dv}{dx} = \mu(-2x) \qquad \therefore\ v\frac{dv}{dx} = a = -\mu x$$

Hence the acceleration varies as the distance x from the origin. The negative sign “-“ indicates that it is in the direction of x decreasing, i.e., towards the origin.

Example

A particle moves along a straight line such that its distance x from a fixed point on it and the time at any time t are related by $x = 2(1 - e^{-t})$. Find the velocity in terms of distance and the acceleration in terms of velocity.

Solution

In order to obtain the velocity with differentiating the function of position x with respect to time, we get

$$x = 2\;\;1 - e^{-t} \qquad \Rightarrow v = \frac{dx}{dt} = 2e^{-t} \qquad \textbf{Note} \quad \frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\because\ x - 2 = -2e^{-t} \qquad \Rightarrow v = 2 - x$$

This equation illustrates the relation between velocity and distance.

Now to get the relation between acceleration and velocity

$$\because\ a = v\frac{dv}{dx} = v(-1) = -v \qquad \textbf{Note} \quad \frac{dv}{dx} = -1 \quad \therefore\ a = -v$$

Example

A car moves along a straight line such that its acceleration at any time t is given by $6t + 2$. Initially the mass at rest placed at the origin point. Determine the velocity and distance as a function of time. Determine the position of the car after 5 sec.

Solution

Since we have $a = 6t + 2,$ $\quad a = \dfrac{dv}{dt} \quad \Rightarrow \dfrac{dv}{dt} = 6t + 2$

Thus, by separation of variables we get

X

$$dv = \ 6t + 2 \ \ dt \qquad \Rightarrow \int dv = \int \ 6t + 2 \ \ dt$$

$$\therefore \ v = 3t^2 + 2t + c_1$$

From initial conditions at $t = 0$, $v = 1$ then $c_1 = 0$

Again, $\therefore \ v = 3t^2 + 2t$ this equation gives the relation between velocity and time. Since $v = \dfrac{dx}{dt}$ that is

$\dfrac{dx}{dt} = 3t^2 + 2t \qquad \Rightarrow dx = \ 3t^2 + 2t \ \ dt$ (Separation variables)

$\int dx = \int \ 3t^2 + 2t \ \ dt$ or $x = t^3 + t^2 + c_2$

From initial conditions at $t = 0$, $x = 0$ then $c_2 = 0$, i.e.

$x = t^3 + t^2$

this equation gives the relation between distance and time.

The position at $t = 5$ is $x\Big|_{t=5} = 5^3 + 5^2 = 150$

Example

A point moves along a straight line according to $v = u + bx$, where u, b are constants. Find the velocity and acceleration in terms of time and the acceleration in terms of distance and also as a function of velocity.

Solution

Velocity and acceleration can be obtained by differentiation the function of position and then velocity with respect to time, therefore

$$v = u + bx \quad \Rightarrow a = \frac{dv}{dt} = b\frac{dx}{dt} = bv = b(u + bx) \qquad \Rightarrow a = b(u + bx)$$

This equation gives the acceleration as a function of velocity $a = bv$ and as a function of distance $a = b(u + bx)$

Again to get the velocity and acceleration as functions of time

$$\because v = u + bx \quad \Rightarrow \frac{dx}{dt} = b(u + bx) \qquad \Rightarrow \frac{dx}{u + bx} = bdt$$

Multiply the previous relation by b and then integrate

$$\int \frac{bdx}{u + bx} = \int b^2 dt \quad \Rightarrow \ln(u + bx) = b^2\, t + C$$

Where C is integration constant , the last relation can be rewritten as

$$\because \ln(u + bx) = b^2\, t + C \qquad \Rightarrow \ln v = b^2\, t + C \quad \textbf{Or}$$
$$\Rightarrow v = Ae^{b^2 t}, \quad A = e^C$$

This is the relation between velocity and time, also the acceleration given by

$$a = bv = bAe^{b^2 t}$$

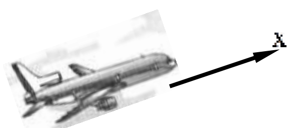
‖ Example ▸

A plane flies along a straight line with retardation $a = -2v^2$. Find the position at any instance if the point starts from origin with initial velocity equals unity.

‖ Solution ▸

The motion under retardation where $a = -2v^2$ but we know $a = \dfrac{dv}{dt}$, so

$$\because a = -2v^2 \qquad \Rightarrow \frac{dv}{dt} = -2v^2$$



By separation of variables and integrate, we obtain

$$-\int \frac{dv}{v^2} = \int 2dt + c_1 \qquad \Rightarrow \frac{1}{v} = 2t + c_1$$

The integration constant c_1 can be evaluated as $v = 1$ when $t = 0$, hence $1 = 2(0) + c_1 \quad \therefore c_1 = 1$ then the velocity can be obtained by

$$\frac{1}{v} = 2t + 1 \quad \text{but} \quad v = \frac{dx}{dt} \quad \therefore \frac{dt}{dx} = 2t + 1 \quad \text{Or} \quad \frac{dt}{2t+1} = dx$$

Again by integrating we get

$$\frac{2dt}{2t+1} = 2dx \quad \Rightarrow \ln(2t+1) = 2x + c_2$$

From initial condition $x = 0$ when $t = 0$ then $c_2 = 0$ and the relation between distance and time becomes

$$x = \frac{1}{2}\ln(2t+1)$$

Example

A particle starts from rest at a distance h from the origin $\mathbf{O}$ with retardation $-4x^{-3}$. Prove that the particle reach to distance ℓ from $\mathbf{O}$ in time $\frac{h}{2}\sqrt{h^2 - \ell^2}$ and then find its velocity at this position.

Solution

Since we have been given the retardation as $a = -16x^{-3}$ and $a = v\frac{dv}{dx}$ therefore,

$$\therefore v\frac{dv}{dx} = -4x^{-3} \qquad \Rightarrow vdv = -4x^{-3}\ dx$$

By integrating, we obtain

$$\therefore \int vdv = -\int 4x^{-3}\ dx + c_1 \qquad \text{Or} \qquad \frac{1}{2}v^2 = \frac{2}{x^2} + c_1 \quad \text{Or} \quad v^2 = \frac{4}{x^2} + c$$

The integration constant c can be evaluated as $v = 0$ when $x = h$, hence $0 = \frac{4}{h^2} + c$ i.e. $c_1 = -\frac{4}{h^2}$ and then we get

$$v^2 = \frac{4}{x^2} - \frac{4}{h^2} = \frac{4(h^2 - x^2)}{x^2 h^2} \qquad \therefore\ v = \pm\frac{2}{h}\frac{\sqrt{h^2 - x^2}}{x}$$

We will consider the minus sign since the motion of the particle towards the origin –in decreasing x - and use $v = \frac{dx}{dt}$

$$\therefore\ \frac{dx}{dt} = -\frac{2}{h}\frac{\sqrt{h^2 - x^2}}{x} \qquad \Rightarrow -\frac{x dx}{\sqrt{h^2 - x^2}} = \frac{2}{h}dt \quad \textbf{Or}$$

$$\Rightarrow -\int \frac{x dx}{\sqrt{h^2 - x^2}} = \int \frac{2}{h}dt + c_2$$

$$\Rightarrow \sqrt{h^2 - x^2} = \frac{2}{h}t + c_2$$

To obtain the constant c_2 when $x = h$ as $t = 0$ and then $c_2 = 0$ so

$$\therefore\ \sqrt{h^2 - x^2} = \frac{2}{h}t \qquad \textbf{Or} \qquad t = \frac{h}{2}\sqrt{h^2 - x^2}$$

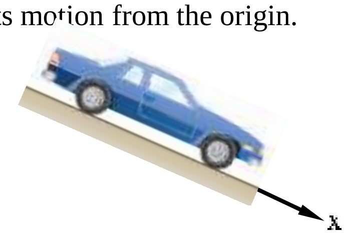
The spent time to reach to a distance ℓ from origin point is $t = \frac{h}{2}\sqrt{h^2 - \ell^2}$,

to determine the velocity at this position, we put $x = \ell$ in velocity relation, that is

$$v\Big|_{x=\ell} = \frac{2\sqrt{h^2 - \ell^2}}{h\ell}$$

Example

A car moves along a straight line according to the relation $v = (1 + x^2)t$. Find the distance as a function of time if the point starts its motion from the origin.



Solution

Since $v = (1 + x^2)t$ thus

$$\frac{dx}{dt} = (1 + x^2)t \qquad \Rightarrow \frac{dx}{1 + x^2} = t dt$$

$$\therefore \int \frac{dx}{1+x^2} = \int t\,dt + c_1 \qquad \Rightarrow \tan^{-1} x = \frac{1}{2}t^2 + c_1$$

From initial condition where the point starts its motion at origin

$$\therefore \tan^{-1} 0 = \frac{1}{2}0^2 + c_1 \quad \Rightarrow 0 = 0 + c_1 \therefore c_1 = 0 \qquad \therefore x = \tan\left(\frac{1}{2}t^2\right)$$

Note that

$$\int \frac{f'dx}{1+f^2} = \tan^{-1} f$$

Example

If t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $a^3\,\dfrac{d^2t}{dv^2}$, a being the acceleration.

Solution

Let a be the acceleration at time t. Then $a = \dfrac{dv}{dt}$. Now the rate of decrease of acceleration = $-\dfrac{da}{at}$

$$= -\frac{d}{at}\left(\frac{dv}{dt}\right) = -\frac{d}{at}\left(\frac{dt}{dv}\right)^{-1} \text{ regarded } t \text{ as a function of } v$$

$$= -\left(\frac{d}{av}\left(\frac{dt}{dv}\right)^{-1}\right)\frac{dv}{dt} = \left(\left(\frac{dt}{dv}\right)^{-2}\frac{d^2t}{dv^2}\right)\frac{dv}{dt}$$

$$= \left(\left(\frac{dv}{dt}\right)^{2}\frac{d^2t}{dv^2}\right)\frac{dv}{dt} = \left(\underbrace{\frac{dv}{dt}}_{a}\right)^{3}\frac{d^2t}{dv^2} = a^3\,\frac{d^2t}{dv^2}$$

Example

Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.

Solution

If x is the distance of the particle from the fixed point **O** at any time t, then its speed v at this time is given by $v = kx^n$, where k is a constant and n is not less than 1. Since the particle is moving towards the fixed point i.e., in the direction decreasing, therefore

$$\frac{dx}{dt} = -v \qquad \text{or} \qquad \frac{dx}{dt} = -kx^n \qquad \text{....(1)}$$

Case 1. If $n = 1$, then from (1), we have

$$\frac{dx}{dt} = -kx \qquad \text{or} \qquad dt = -\frac{1}{k}\frac{dx}{x}$$

Integrating, $t = -\frac{1}{k}\ln x + A$ where A is a constant.

Putting $x = 0$ then the time t to reach the fixed point **O** is given by

$$t = -\frac{1}{k}\ln 0 + A = \infty$$

i.e., the particle will never reach the fixed point **O**

Case 2. If $n > 1$, then from (1), we have

$$dt = -\frac{1}{k}x^{-n}dx$$

Integrating, $t = -\frac{1}{k}\frac{x^{1-n}}{1-n} + B$ where B is a constant.

Or
$$t = \frac{1}{k(n-1)x^{n-1}} + B$$

Putting $x = 0$ then the time t to reach the fixed point **O** is given by

$$t = \infty + B = \infty$$

i.e., the particle will never reach the fixed point **O**

Hence if $n \geq 1$, the particle will never reach the fixed point, it is approaching.

PROBLEMS

❒ A particle moving in a straight line is subject to a resistance which produces the retardation kv^3, where v is the velocity and k is a constant. Show that v and t (the time) are given in terms of x (the distance) by the equations $v = \dfrac{u}{kux+1}$, $t = \dfrac{1}{2}kx^2 + \dfrac{x}{u}$, where u is the initial velocity.

❒ If the relation between x and t is of the form $t = bx^2 + kx$, find the velocity v as a function of x, and prove that the retardation of the particle is $2bv^3$.

❒ A particle is projected vertically upwards with speed u and moves in a vertical straight line under uniform gravity with no air resistance. Find the maximum height achieved by the particle and the time taken for it to return to its starting point.

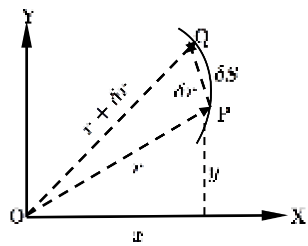
Kinematics in Two Dimensions

■ Velocity in Cartesian Coordinates

The velocity vector of a particle (or point) moving along a curve is the rate of change of its displacement with respect to time.

Let P and Q be the positions of a particle moving along a curve at times t and $t+\delta t$ respectively. With respect to **O** as the origin of vectors, let $\underline{\mathbf{O}P} = \underline{r}$ and $\underline{\mathbf{O}Q} = \underline{r} + \delta\underline{r}$ Then $\underline{\mathbf{P}Q} = \underline{\mathbf{O}Q} - \underline{\mathbf{OP}} = \delta\underline{r}$ represents the displacement of the particle in time δt and $\dfrac{\delta\underline{r}}{\delta t}$ indicates the average rate of displacement (or average velocity) during the interval δt. The limiting value of the average velocity $\dfrac{\delta\underline{r}}{\delta t}$ as δt tends to zero ($\delta t \to 0$) is the velocity. Therefore if the vector $\underline{v}$ represents the velocity of the particle at time t then

$$\underline{v} = \lim_{\delta t \to 0} \frac{\delta\underline{r}}{\delta t} = \frac{d\underline{r}}{dt} = \dot{\underline{r}}$$



Where $\underline{r}$ is the position vector of the particle.

Now, if $\qquad \underline{r} = x\hat{i} + y\hat{j}$

Then $\qquad \underline{v} = \dfrac{d\underline{r}}{dt} = \underbrace{\dfrac{dx}{dt}}_{v_x}\hat{i} + \underbrace{\dfrac{dy}{dt}}_{v_y}\hat{j} = \dot{x}\hat{i} + \dot{y}\hat{j}$

Note that $(\dot{x}, \dot{y})$ are called the components or resolved parts of the velocity $\underline{v}$ along the axes x and y respectively. The speed of the particle at P is given by

$$|\underline{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{ds}{dt}$$

Also the angle θ which the direction of $\underline{v}$ makes with **OX** is

$$\tan\theta = \frac{dy}{dt} \,/\, \frac{dx}{dt} = \frac{dy}{dx}$$

■ Acceleration in Cartesian Coordinates

The acceleration vector of a particle moving along a curve is defined as the rate of change of its velocity vector.

if $\underline{v}$ and $\underline{v} + \delta\underline{v}$ are the velocities of a particle moving along a curve at times t and $t + \delta t$ respectively, then $\delta\underline{v}$ is the change in velocity vector in time δt and $\frac{\delta \underline{v}}{\delta t}$ is the average and then

$$\underline{a} = \lim_{\delta t \to 0} \frac{\delta \underline{v}}{\delta t} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\left(\frac{d\underline{r}}{dt}\right) = \frac{d^2\underline{r}}{dt^2}$$

Substituting for $\underline{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$ we have,

$$\underline{a} = \frac{d}{dt}\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}\right) = \underbrace{\frac{d^2x}{dt^2}}_{a_x}\hat{i} + \underbrace{\frac{d^2y}{dt^2}}_{a_y}\hat{j} = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j}$$

Here, $(\ddot{x}, \ddot{y})$ are called the components of the acceleration $\underline{a}$ along the axes x and y respectively. The magnitude of the acceleration is given by

$$|\underline{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

Again, the angle φ which the direction of $\underline{a}$ makes with OX is

$$\tan\varphi = \frac{d^2y}{dt^2} \,/\, \frac{d^2x}{dt^2}$$

■ Illustrative Examples ■

Example

A point moves along the curve $x = t^3 + 1, \quad y = t^2$ where, t is the time. Determine the components of velocity and acceleration at $t = 1$

Solution

Let $\underline{r}$ be the position vector of the particle at time t, therefore

$$\underline{r} = x\,\hat{i} + y\,\hat{j} = (t^3 + 1)\,\hat{i} + t^2\,\hat{j}$$

Then the velocity vector is

$$\underline{v} = \frac{d\underline{r}}{dt} = 3t^2\,\hat{i} + 2t\,\hat{j} \quad \text{and} \quad \underline{v}\Big|_{t=1} = 3(1)^2\,\hat{i} + 2(1)\,\hat{j} = 3\,\hat{i} + 2\,\hat{j}$$

Again the vector of acceleration is

$$\underline{a} = \frac{d\underline{v}}{dt} = 6t\,\hat{i} + 2\,\hat{j} \quad \text{and} \quad \underline{a}\Big|_{t=1} = 6(1)\,\hat{i} + 2\,\hat{j} = 6\,\hat{i} + 2\,\hat{j}$$

Example

The position of a moving point at time t is given by $x = 3\cos t, \quad y = 2\sin t$

Find its path velocity and acceleration vectors.

Solution

Since the parametric equations are $x = 3\cos t, \quad y = 2\sin t$ then

$$\left(\frac{x}{3}\right)^2 = \cos^2 t, \quad \left(\frac{y}{2}\right)^2 = \sin^2 t \quad \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \text{or } 4x^2 + 9y^2 = 36$$

This is a the path equation which represents an Ellipse

Velocity vector is $\underline{v} = -3\sin t\ \hat{i} + 2\cos t\ \hat{j}$

While the acceleration vector is

$$\underline{a} = -3\cos t\ \hat{i} - 2\sin t\ \hat{j} = -\underbrace{(3\cos t\ \hat{i} + 2\sin t\ \hat{j})}_{\underline{r}} = -\underline{r}$$

Example

A particle moves along the curve $y = 2x^2$ such that its horizontal component of velocity is constant and equals 2. Calculate the components of acceleration and velocity when $y = 8$.

Solution

Since the horizontal component of velocity equals 2, i.e. $\dot{x} = 2$, therefore by differentiating w.r.t t we get

$\ddot{x} = 0$ and $y = 2x^2 \quad \Rightarrow \dot{y} = 4x\dot{x} = 8x \quad \therefore \ddot{y} = 8\dot{x} = 16$

That is the acceleration vector is given by

$\underline{a} = 16\,\hat{j}$

and the velocity components are $\dot{x} = 2$ and $\dot{y} = 8x$

Since as $y = 8$ gives $x = \pm 2$ thus, $\quad \underline{v} = 2\,\hat{i} + 8(\pm 2)\,\hat{j}, \quad \left|\underline{v}\right| = \sqrt{260}$

Example

A particle describes a plane curve such that its components of acceleration equal $(0, -\mu / y^2)$ with initial velocity $\sqrt{2\mu / b}$ parallel to X-axis and the initial position $(0, b)$. Find the path equation.

Solution

Here we are given that

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -\frac{\mu}{y^2}$$

Note that $\dfrac{d^2y}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d}{dy}\left(\dfrac{dy}{dt}\right) \times \dfrac{dy}{dt} = \dot{y}\dfrac{d\dot{y}}{dy}$ chain rule

Then $\dot{y}\dfrac{d\dot{y}}{dy} = -\dfrac{\mu}{y^2}$ $\Rightarrow \dot{y}d\dot{y} = -\dfrac{\mu}{y^2}dy$ $\Rightarrow \int \dot{y}d\dot{y} = -\int \dfrac{\mu}{y^2}dy$

$\dot{y}^2 = \dfrac{2\mu}{y} + c_1$ $\left(\dot{y} = \dfrac{dy}{dt}\right)$

Initially $\dfrac{dy}{dt} = 0$ when $y = b$, thus $c_1 = -\dfrac{2\mu}{b}$

$\dot{y}^2 = \dfrac{2\mu}{y} - \dfrac{2\mu}{b}$ $\Rightarrow \dot{y}^2 = \dfrac{2\mu}{y} - \dfrac{2\mu}{b} = 2\mu\left(\dfrac{1}{y} - \dfrac{1}{b}\right) = \dfrac{2\mu}{b}\dfrac{b-y}{y}$

Hence

$$\frac{dy}{dt} = -\sqrt{\frac{2\mu}{b}}\sqrt{\frac{b-y}{y}} \tag{1}$$

(Negative sign has been taken because the particle is moving in the direction of y decreasing)

Again from $\dfrac{d^2x}{dt^2} = 0$ $\Rightarrow \dfrac{dx}{dt} = c_3$

Initially when $t = 0$, $\dfrac{dx}{dt} = \sqrt{\dfrac{2\mu}{b}}$ thus $c_3 = \sqrt{\dfrac{2\mu}{b}}$

$$\therefore \frac{dx}{dt} = \sqrt{\frac{2\mu}{b}} \tag{2}$$

By dividing the two equations (1) and (2) we get

$\dfrac{dy}{dx} = -\sqrt{\dfrac{b-y}{y}}$ $\Rightarrow \sqrt{\dfrac{y}{b-y}}\,dy = -dx$, then by integrating

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}} - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = -x + c_2$$

Hint to get the integration $\int \sqrt{\dfrac{y}{b-y}}\,dy$ let us use the transformation

$$y = b\sin^2\theta \qquad \Rightarrow dy = 2b\sin\theta\cos\theta\, d\theta$$

$$\therefore \int\sqrt{\frac{y}{b-y}}\,dy = \int\sqrt{\frac{b\sin^2\theta}{b-b\sin^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$= \int\sqrt{\frac{b\sin^2\theta}{b\cos^2\theta}}\,2b\sin\theta\cos\theta\,d\theta$$

$$= \int\frac{\sin\theta}{\cancel{\cos\theta}}\,2b\sin\theta\,\cancel{\cos\theta}\,d\theta = 2b\int\sin^2\theta\,d\theta$$

$$\because \sin^2\theta = \frac{1}{2}\ \ 1-\cos 2\theta$$

$$\Rightarrow 2b\int\sin^2\theta\,d\theta = 2b\int\frac{1}{2}\ \ 1-\cos 2\theta\ \ d\theta$$

$$= b\int\ \ 1-\cos 2\theta\ \ d\theta = b\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$\therefore \int\sqrt{\frac{y}{b-y}}\,dy = b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right) - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right)$$

The initial condition is $t = 0\ \ x = 0,\ \ y = b$ then from the equation

$$b\left(\sin^{-1}\sqrt{\frac{y}{b}} - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = -x + c_2 \qquad \Rightarrow c_2 = b\frac{\pi}{2}$$

$$\therefore b\left(\sin^{-1}\left(\sqrt{\frac{y}{b}}\right) - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\right) = b\frac{\pi}{2} - x$$

$$\Rightarrow \sin^{-1}\sqrt{\frac{y}{b}} = \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}} + \frac{\pi}{2} - \frac{x}{b}$$

$$\sqrt{\frac{y}{b}} = \sin\left(\sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}} + \frac{\pi}{2} - \sqrt{2\mu b}\,t\ \right)$$

$$= \cos\left(x - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\ \right)$$

$$y = b\cos^2\left(x - \sqrt{\frac{y}{b}}\sqrt{1-\frac{y}{b}}\ \right)$$

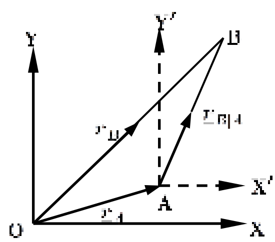
■ Relative motion of two particles

Motion does not happen in isolation. If you're riding in a train moving at 10 ms^{-1} east, this velocity is measured relative to the ground on which you're traveling. However, if another train passes you at 15 ms^{-1} east, your velocity relative to this other train is different from your velocity relative to the ground. Your velocity relative to the other train is 5 ms^{-1} west. To explore this idea further, we first need to establish some terminology.

♦ Reference Frames

To discuss relative motion in one or more dimensions, we first introduce the concept of reference frames. When we say an object has a certain velocity, we must state it has a velocity with respect to a given reference frame. In most examples we have examined so far, this reference frame has been Earth. If you say a person is sitting in a train moving at 10 m/s east, then you imply the person on the train is moving relative to the surface of Earth at this velocity, and Earth is the reference frame. We can expand our view of the motion of the person on the train and say Earth is spinning in its orbit around the Sun, in which case the motion becomes more complicated. In this case, the solar system is the reference frame. In summary, all discussion of relative motion must define the reference frames involved. We now develop a method to refer to reference frames in relative motion.

For two particles A and B moving in plane as shown, we considered the relative motion of B with respect to A, or more precisely, with respect to a moving frame attached to A and in translation with A. Denoting by $\underline{r}_{B|A}$ the relative position vector of B with respect to A, we had



$$\underline{r}_B = \underline{r}_A + \underline{r}_{B|A} \qquad \text{or} \qquad \underline{r}_{B|A} = \underline{r}_B - \underline{r}_A$$

Denoting by $\underline{v}_{B|A}$ and $\underline{a}_{B|A}$, respectively, the relative velocity and the relative acceleration of B with respect to A, we also showed that

Differentiating previous equation with respect to time

$$\frac{d\underline{r}_{B|A}}{dt} = \frac{d\underline{r}_B}{dt} - \frac{d\underline{r}_A}{dt} \quad \textbf{or} \quad \underline{v}_{B|A} = \underline{v}_B - \underline{v}_A$$

Differentiating previous equation with respect to time

$$\frac{d\underline{v}_{B|A}}{dt} = \frac{d\underline{v}_B}{dt} - \frac{d\underline{v}_A}{dt} \quad \textbf{or} \quad \underline{a}_{B|A} = \underline{a}_B - \underline{a}_A$$

■ Illustrative Examples ■

Example

Two points A and B are moving along a straight line such that $\underline{x}_A = t^3 - 2t$ and $\underline{x}_B = 2t^3 + t^2 - 5$. Find the relative velocity $\underline{v}_{BA}$ and acceleration $\underline{a}_{BA}$.

Solution

Since the relative position of point B with respect to point A, $\underline{x}_{B|A}$, is given by

$$\underline{x}_{B|A} = \underline{x}_B - \underline{x}_A$$

$$\Rightarrow \underline{x}_{B|A} = (2t^3 + t^2 - 5) - (t^3 - 2t) = t^3 + t^2 + 2t - 5$$

Hence the relative velocity $\underline{v}_{B|A}$ is obtained by

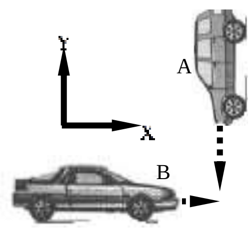
$$\underline{v}_{B|A} = \frac{d\underline{x}_{B|A}}{dt} = 3t^2 + 2t + 2$$

Again the relative acceleration $\underline{a}_{BA}$ is given by

$$\underline{a}_{B|A} = \frac{d\underline{v}_{B|A}}{dt} = 6t + 2$$

Example

A car A is traveling south at a speed of 70 km/h toward an intersection. A car B is traveling east toward the intersection at a speed of 80 km/h, as shown. Determine the velocity of the car B relative to the car A.

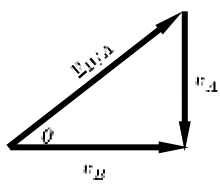


Solution

According to the given data the velocity of car A is $\underline{v}_A = -70\,\hat{j}$ and velocity of car B is $\underline{v}_B = 80\,\hat{i}$ then

$$\begin{aligned}\underline{v}_{B|A} &= \underline{v}_B - \underline{v}_A \\ &= 80\hat{i} - (-70\hat{j}) \\ &= 80\hat{i} + 70\hat{j}\end{aligned}$$

$$\Rightarrow \left|\underline{v}_{B|A}\right| = \sqrt{(80)^2 + (70)^2} = \sqrt{11300} \simeq 106.3 \text{ km h}^{-1}$$



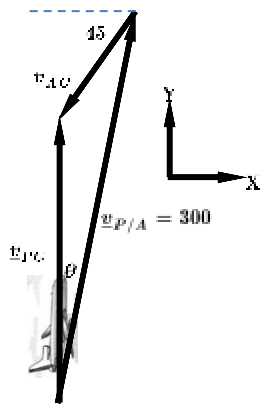
And make an angle θ with the velocity direction of car B obtained by

$$\tan\theta = \frac{70}{80} = \frac{7}{8} \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{7}{8}\right)$$

Example

A pilot must fly his plane due north to reach his destination. The plane can fly at 300 km/h in still air. A wind is blowing out of the northeast at 90 km/h. Calculate the speed of the plane relative to the ground and in what direction must the pilot head her plane to fly due north.

Solution



The pilot must point her plane somewhat east of north to compensate for the wind velocity. We need to construct a vector equation that contains the velocity of the plane with respect to the ground, the velocity of the plane with respect to the air, and the velocity of the air with respect to the ground. Since these last two quantities are known, we can solve for the velocity of the plane with respect to the ground. We can graph the vectors and use this diagram to evaluate the magnitude of the plane's velocity with respect to the ground. The diagram will also tell us the angle the plane's velocity makes with north with respect to the air, which is the direction the pilot must head her plane.

From the given data the velocity of plane P is $\underline{v}_{P|A} = 300(\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$ and

velocity of air A is $\underline{v}_{A|G} = -90(\cos 45^0\,\hat{i} + \sin 45^0\,\hat{j})$ and $\underline{v}_{P|G} = v_{P|G}\,\hat{j}$ then

$$\begin{aligned}
\underline{v}_{P|G} &= \underline{v}_{P|A} + \underline{v}_{A|G}\\
&= 300(\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) + -90(\cos 45^0\,\hat{i} + \sin 45^0\,\hat{j})\\
&= (300\sin\theta - 90\cos 45^0)\hat{i} + (300\cos\theta - 90\sin 45^0)\hat{j}
\end{aligned}$$

$$\Rightarrow 300\sin\theta - 90\cos 45^0 = 0$$

$$\sin\theta = \frac{45\sqrt{2}}{300} \qquad \text{And } v_{P|G} = 300\cos\theta - 90\sin 45^0 \simeq 230\ \text{km h}^{-1}$$

Problems

❒ The position of a moving point at time t is given by $x = at^2, \; y = 2at$

Find its velocity and acceleration

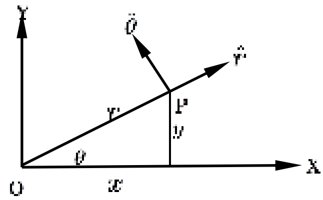
❒ A particle moves with constant velocity parallel to the axis of $\mathbf{Y}$ and a velocity proportional to y parallel to the axis of $\mathbf{X}$. Prove that it will describe a parabola

❒ A particle is acted on by a force parallel to the axis of $\mathbf{Y}$ whose acceleration is λy and is initially projected with a velocity $a\sqrt{\lambda}$ parallel to the axis of $\mathbf{X}$ at a point where $y = a$. Prove that it will describe the catenary $y = a\cosh(x / a)$

❒ A boat heads north in still water at 4.5 ms^{-1} directly across a river that is running east at 3.0 ms^{-1}. Find the velocity of the boat with respect to Earth.

POLAR COORDINATES

In some problems it is convenient to employ another coordinates not Cartesian coordinates as polar coordinates. Let the position of a point $\mathbf{P}$ be defined by its distance r from a fixed origin $\mathbf{O}$ and the angle θ that OP makes with a fixed axis $\mathbf{OX}$.



The Cartesian coordinates (x, y) of P are connected with the polar coordinates (r, θ) by the relations $x = r\cos\theta, \quad y = r\sin\theta$.

Note that $\hat{r}$ and $\hat{\theta}$ represent unit vectors in direction of increasing r and normal to r in the direction of increasing θ as illustrated in the figure.

■ Angular Velocity and Acceleration

Let $\mathbf{P}$ be a moving point in a plane. If $\mathbf{O}$ be a fixed point (pole) and $\mathbf{O}X$ is a fixed line through $\mathbf{O}$ in the plane of motion, then the angular velocity of the moving point $\mathbf{P}$ about $\mathbf{O}$ (or the line $\mathbf{OP}$ in the plane $\mathbf{XOP}$) is the rate of change of the angle $\mathbf{XOP}$ Figure.

Let $\mathbf{P}$ and $\mathbf{Q}$ be the positions of a moving particle at times t and $t + \delta t$ respectively such that $\angle\mathbf{POX} = \theta$ and $\angle\mathbf{QOX} = \theta + \delta\theta$. Therefore, the angle turned by the particle in time δt is $\delta\theta$. That is the average rate of change of the angle of $\mathbf{P}$ about $\mathbf{O}$ is $\dfrac{\delta\theta}{\delta t}$

Then the angular velocity of the point $\mathbf{P}$ about $\mathbf{O}$ is

$$\lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta}$$

Where the dot placed over θ denotes differentiation with respect to time, and the units of angular velocity is radian/sec.

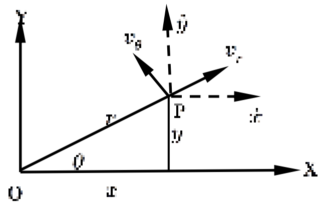
Now the rate of change of angular velocity is called angular acceleration

That is the angular acceleration

$$\lim_{\delta t \to 0} \frac{\delta \dot{\theta}}{\delta t} = \frac{d\dot{\theta}}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

The units of angular acceleration is radian/sec^2

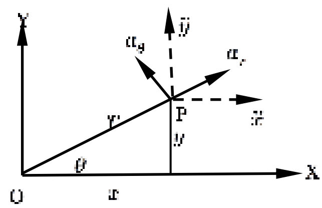
■ Velocity and Acceleration in Polar Coordinates



Let the position of a point **P** be defined by its distance r from a fixed origin **O** and the angle θ that OP makes with a fixed axis **OX** .

The Cartesian coordinates (x, y) of P are connected with the polar coordinates (r, θ) by the relations $x = r\cos\theta, \;\; y = r\sin\theta$.

Let v_r, v_θ denote the components of velocity of **P** in the direction **OP** and at right angles to **OP** in the sense in which θ increases. The resultant of the components v_r, v_θ is also the resultant of the components $\dot{x}$, $\dot{y}$. Therefore by resolving parallel to **OX** and **OY** we get



$$\begin{aligned} v_r\cos\theta - v_\theta\sin\theta = \dot{x} &= \frac{d}{dt}(r\cos\theta) \\ &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \end{aligned}$$

And

$$\begin{aligned} v_r\sin\theta + v_\theta\cos\theta = \dot{y} &= \frac{d}{dt}(r\sin\theta) \\ &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}$$

Solving these equations for u and v clearly gives (comparing)

$$v_r = \dot{r}, \qquad v_\theta = r\dot{\theta},$$

and these are the polar components of velocity.

In like manner if a_r, a_θ denote the components of acceleration along and at right angles to **OP**, since these have the same resultant as $\ddot{x}$ and $\ddot{y}$, we get

$$a_r \cos\theta - a_\theta \sin\theta = \ddot{x} = \frac{d^2}{dt^2}(r\cos\theta)$$
$$= \ddot{r} - r\dot{\theta}^2 \;\; \cos\theta - \;\; r\ddot{\theta} + 2\dot{r}\dot{\theta} \;\; \sin\theta$$

Again for $\ddot{y}$ we have

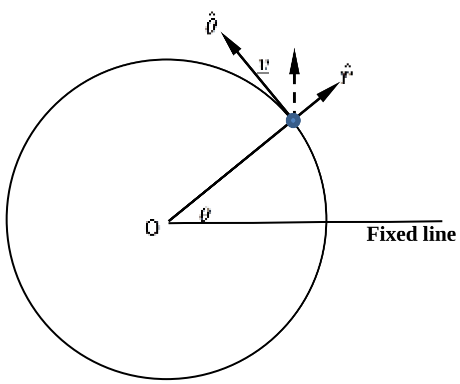
$$a_r \sin\theta + a_\theta \cos\theta = \ddot{y} = \frac{d^2}{dt^2}(r\sin\theta)$$
$$= \ddot{r} - r\dot{\theta}^2 \;\; \sin\theta + \;\; r\ddot{\theta} + 2\dot{r}\dot{\theta} \;\; \cos\theta$$

giving on solution $a_r = \ddot{r} - r\dot{\theta}^2$ and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

These components constitute a third representation of the velocity and acceleration of a point moving in a plane; they are sometimes called <u>radial and transverse components</u>**,** and we note that the transverse component of acceleration may also be written

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt} \;\; r^2\dot{\theta}$$

<u>■ Special Case:</u> If the particle moves in a circle with radius ℓ**,** i.e. $r = \ell$ then $\dot{r} = \ddot{r} = 0$ and hence the velocity of the particle is given by $\underline{v} = \ell\dot{\theta}\,\hat{\theta}$ and its direction will be along the normal to tangent to the circle and also the acceleration will be $\underline{a} = -\ell\dot{\theta}^2\,\hat{r} + \ell\ddot{\theta}\,\hat{\theta}$.



■ Another method for Velocity and Acceleration in Polar Coordinates

We will now evaluate the two derivatives $\dfrac{d\hat{r}}{d\theta}$ and $\dfrac{d\hat{\theta}}{d\theta}$. These will be needed when we derive the formulae for the velocity and acceleration of *P* in polar co-ordinates. First we expand $(\hat{r},\, \hat{\theta})$ in terms of the Cartesian basis vectors $(\hat{i},\, \hat{j})$. This gives

$$\hat{r} = \cos\theta\, \hat{i} + \sin\theta \hat{j}, \qquad \hat{\theta} = -\sin\theta\, \hat{i} + \cos\theta \hat{j}$$

Since $\hat{r}, \hat{\theta}$ are now expressed in terms of the constant vectors $(\hat{i},\, \hat{j})$, the differentiations with respect to θ are simple and give

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}, \qquad \frac{d\hat{\theta}}{d\theta} = -\hat{r} \tag{1}$$

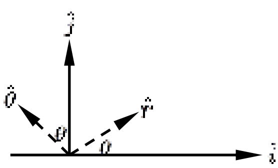
Suppose now that *P* is a moving particle with polar co-ordinates r, θ that are functions of the time t. The position vector of **P** relative to **O** has magnitude $\mathbf{OP} = r$ and direction $\hat{r}$ and can therefore be written

$$\underline{r} = r\,\hat{r} \tag{2}$$

In what follows, one must distinguish carefully between the position vector $\underline{r}$, which is the vector $\underline{\mathbf{OP}}$, the co-ordinate r, which is the *distance* **OP**, and the polar unit vector $\hat{r}$.

To obtain the polar formula for the velocity of **P**, we differentiate formula (2) with respect to time t. This gives

$$\begin{aligned} \underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\hat{r}) &= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \\ &= \dot{r}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} \\ &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \end{aligned}$$



We used the chain rule and formula (1), which is the polar formula for the velocity of **P**.

In order to obtain the polar formula for acceleration, we differentiate the velocity formula $\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ with respect to t again. This gives

$$\begin{aligned}
\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}\ \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \\
= \ddot{r}\,\hat{r} + \dot{r}\underbrace{\frac{d\hat{r}}{d\theta}}_{\hat{\theta}}\underbrace{\frac{d\theta}{dt}}_{\dot{\theta}} + (r\ddot{\theta} + \dot{r}\dot{\theta})\hat{\theta} + r\dot{\theta}\underbrace{\frac{d\hat{\theta}}{d\theta}}_{-\hat{r}}\underbrace{\frac{d\theta}{dt}}_{\dot{\theta}} \\
= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}
\end{aligned}$$

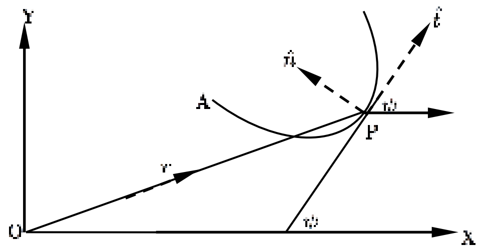
which is the polar formula for the acceleration of **P**.

The formula $\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ shows that the velocity of **P** is the vector sum of an outward radial velocity $\dot{r}$ and a transverse velocity $r\dot{\theta}$; in other words $\boldsymbol{v}$ is just the sum of the velocities that **P** would have if r and θ varied separately. This is *not* true for the acceleration as it will be observed that adding together the separate accelerations would not yield the term $2\dot{r}\dot{\theta}\,\hat{\theta}$. This 'Coriolis term' is certainly present however, but is difficult to interpret intuitively.

INTRINSIC COORDINATES

Let **P** be the position of a moving particle at time t and $\underline{r}$ its position vector with respect to the origin **O**. Let $\mathbf{AP} = S$ and let ψ be the angle which the tangent at **P** to the path of the particle makes with **OX**. Then $(\mathbf{S}, \psi)$ are the intrinsic coordinates of **P**.

Let $\hat{t}$ denote the unit vector along the tangent at in the direction of S increasing and $\hat{n}$ be the unit vector normal at **P** in the direction of ψ increasing i.e., in the direction of inwards drawn normal.



In the same manner we will now evaluate the two derivatives $\dfrac{d\hat{t}}{d\psi}$ and $\dfrac{d\hat{n}}{d\psi}$.

These will be needed when we derive the formulae for the velocity and acceleration of **P** in intrinsic co-ordinates. First we expand $(\hat{t}, \hat{n})$ in terms of the Cartesian basis vectors $(\hat{i}, \hat{j})$. This gives

$$\hat{t} = \cos\psi\,\hat{i} + \sin\psi\,\hat{j}, \qquad \hat{n} = -\sin\psi\,\hat{i} + \cos\psi\,\hat{j}$$

Since $(\hat{t}, \hat{n})$ are now expressed in terms of the constant vectors $(\hat{i}, \hat{j})$, the differentiations with respect to ψ are simple and give

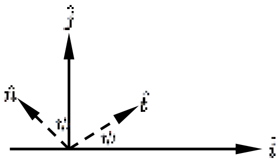
$$\frac{d\hat{t}}{d\psi} = \hat{n}, \qquad \frac{d\hat{n}}{d\psi} = -\hat{t} \tag{1}$$

Since the velocity be in tangent so

$$\underline{v} = \frac{dS}{dt}\hat{t} \tag{2}$$

Now, in order to obtain the polar formula for acceleration, we differentiate the velocity formula $\underline{v} = \frac{dS}{dt}\hat{t}$ with respect to time t. This leads to

$$\begin{aligned}
\underline{a} = \frac{d\underline{v}}{dt} &= \frac{d}{dt}\left(\frac{dS}{dt}\hat{t}\right) \\
&= \frac{d^2S}{dt^2}\hat{t} + \frac{dS}{dt}\frac{d\hat{t}}{dt} \\
&= \frac{d^2S}{dt^2}\hat{t} + \frac{dS}{dt}\frac{d\hat{t}}{d\psi}\frac{d\psi}{dS}\frac{dS}{dt} \\
&= \frac{d^2S}{dt^2}\hat{t} + \left(\frac{dS}{dt}\right)^2 \underbrace{\frac{d\hat{t}}{d\psi}}_{\hat{n}}\underbrace{\frac{d\psi}{dS}}_{1/\rho} \\
&= \frac{d^2S}{dt^2}\hat{t} + \frac{v^2}{\rho}\hat{n}
\end{aligned}$$



Where $\rho\left(= \frac{dS}{d\psi}\right)$ is the radius of curvature at the point $\mathbf{P}$ which is the tangential $a_t = \frac{d^2S}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dS}$ and normal $a_n = \frac{v^2}{\rho}$ acceleration at $\mathbf{P}$.

The formula $\underline{v} = \frac{dS}{dt}\hat{t}$ illustrates that the velocity of $\mathbf{P}$ is in the tangent at $\mathbf{P}$, while the acceleration has two components $(\mathbf{a}_t, a_n)$ and the resultant of acceleration is $a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \left(\frac{v^2}{\rho}\right)^2}$

■ ILLUSTRATIVE EXAMPLES ■

Example

A point **P** describes, with a constant angular velocity about the origin and $r = ae^{\theta}$ Obtain the radial and transverse acceleration of **P**.

Solution

Since $\because \underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}$ and given $\frac{d\theta}{dt} = \omega$ **(constant)**

then by differentiating $r = ae^{\theta}$ with respect to time we have

$\dot{r} = ae^{\theta}\dot{\theta} = a\omega e^{\theta} = \omega r$ **and** $\Rightarrow \ddot{r} = a\omega\dot{\theta}e^{\theta} = a\omega^2 e^{\theta} = \omega^2 r$

Also $\frac{d\theta}{dt} = \omega \Rightarrow \frac{d^2\theta}{dt^2} = 0$ then the radial a_r and transverse a_{θ} acceleration are

$\because a_r = \omega^2 r - r\omega^2 = 0,\quad a_{\theta} = 0 + 2\omega^2 r$

That is $\underline{a} = 2\omega^2 r\hat{\theta}$

Example

The velocities of a particle along and perpendicular to the radius vector are constants. Prove that the acceleration inversely varies as the radius r.

Solution

Since $\dot{r} = A$ and $r\dot{\theta} = B$ where A, B are constants then by differentiating with respect to time we have

$\dot{r} = A \Rightarrow \ddot{r} = 0$ and $r\dot{\theta} = B \Rightarrow r\ddot{\theta} + \dot{r}\dot{\theta} = 0$

then, the radial acceleration is $\because a_r = \ddot{r} - r\dot{\theta}^2$ thus $a_r = 0 - \frac{B^2}{r} = -\frac{B^2}{r}$

And for transverse acceleration a_{θ}

$$\because\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad \text{thus} \qquad a_\theta = \dot{r}\dot{\theta} = \frac{AB}{r}$$

The magnitude of acceleration is given by

$$\begin{aligned} \left|\underline{a}\right| = \sqrt{a_r^2 + a_\theta^2} &= \sqrt{\frac{B^4}{r^2} + \frac{A^2B^2}{r^2}} \\ &= \sqrt{\frac{B^4 + A^2B^2}{r^2}} \\ &= \frac{C}{r}, \qquad\qquad C = \sqrt{B^4 + A^2B^2} \end{aligned}$$

that is the acceleration inversely varies as the radius vector r.

Example

A particle moves along the circle $r = 2\cos\theta$ in such a way that its acceleration perpendicular to the radius vector is always zero. Find the velocity of the moving particle in terms of r.

Solution

Since $r = 2\cos\theta$ then by differentiating with respect to time we have

$$r = 2\cos\theta \qquad \Rightarrow \dot{r} = -2\sin\theta\ \dot{\theta}$$

Also we have that the acceleration towards the origin is always zero i.e.,

$$\begin{aligned} a_\theta = 0 \qquad &\Rightarrow \frac{1}{r}\frac{d}{dt}\ r^2\dot{\theta}\ = 0 \\ &\Rightarrow d\ r^2\dot{\theta}\ = 0 \\ &\Rightarrow r^2\dot{\theta} = h\ \text{(const.)} \end{aligned}$$

Therefore the velocity magnitude is

$$\begin{aligned} \left|\underline{v}\right| = \sqrt{v_r^2 + v_\theta^2} &= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} \\ &= \sqrt{(-2\sin\theta\,\dot{\theta})^2 + (r\dot{\theta})^2} \\ &= \sqrt{4\sin^2\theta + r^2}\ \dot{\theta} \\ &= \sqrt{4(1-\cos^2\theta) + r^2}\ \dot{\theta} \\ &= \sqrt{4 - 4\cos^2\theta + r^2}\ \dot{\theta} \\ &= \sqrt{4 - r^2 + r^2}\ \dot{\theta} = 2\ \underbrace{\dot{\theta}}_{h/r^2} = \frac{2h}{r^2} \end{aligned}$$

Example

A particle describes a curve with constant velocity and its angular velocity about a given fixed point $\mathbf{O}$ varies inversely as its distance from $\mathbf{O}$. Find the path equation.

Solution

Let the velocity of the particle be equal to v (constant). Given that the angular velocity $d\theta / dt$ of the particle about a fixed point $\mathbf{O}$ varies inversely as its distance r from $\mathbf{O}$, we have

$$\frac{d\theta}{dt} \propto \frac{1}{r} \qquad \Rightarrow \frac{d\theta}{dt} = \frac{k}{r} \ \ (k \text{ is constant})$$

Since $v = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2} = \lambda$ (λ is constant)

$$\Rightarrow v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2 = \lambda^2$$

$$\Rightarrow v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{k}{r}\right)^2 = \lambda^2$$

$$\Rightarrow \left(\frac{dr}{dt}\right)^2 = \lambda^2 - k^2$$

$$\Rightarrow \frac{dr}{dt} = \sqrt{\lambda^2 - k^2} = \mu$$

$$\Rightarrow \frac{dr}{d\theta}\frac{d\theta}{dt} = \mu \qquad \Rightarrow \frac{dr}{d\theta}\frac{k}{r} = \mu \qquad \Rightarrow \frac{dr}{r} = \frac{\mu}{k}d\theta$$

By integrating

$$\therefore \ln r = \frac{\mu}{k}\theta + \ln c \ \ (\ln c \text{ is integration constant})$$

$$\therefore \ln\frac{r}{c} = \frac{\mu}{k}\theta \qquad \Rightarrow r = ce^{\mu/k\,\theta}$$

This is the equation of equiangular spiral

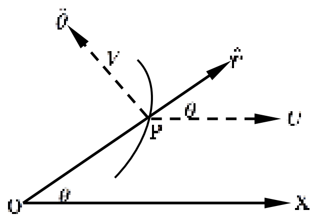
Example

Find the path equation of a point P which possesses two constant velocities U and $\mathbf{V}$, the first of which is in $\mathbf{OX}$ direction and the other is perpendicular to the radius $\mathbf{O}P$ drown from a fixed point $\mathbf{O}$.

Solution

Take the fixed point $\mathbf{O}$ as the pole and the fixed direction as the initial line $\mathbf{OX}$. Let $P(r,\theta)$ be the position of the particle at any time. Resolve the velocities in the direction of and perpendicular to the radius $\mathbf{O}P$ we have

$$\frac{dr}{dt} = U\cos\theta \qquad \text{and} \qquad r\frac{d\theta}{dt} = V - U\sin\theta$$



Dividing these two equations we have

$$\frac{dr}{rd\theta} = \frac{U\cos\theta}{V - U\sin\theta} \qquad \Rightarrow \frac{dr}{r} = \frac{U\cos\theta}{V - U\sin\theta}d\theta$$

By integrating we get

$$\ln r = -\ln(V - U\sin\theta) + \ln c$$

or $\qquad \ln\frac{c}{r} = \ln(V - U\sin\theta) \qquad$ ($\ln c$ is integration constant)

$$\therefore \ \frac{c}{r} = V - U\sin\theta$$

Example

A particle moves in a plane curve so that its tangential acceleration is constant, and the magnitude of tangential velocity and normal acceleration are in a constant ratio. Find the intrinsic equation of the curve.

Solution

In our problem it is given that

tangential acceleration $\frac{dv}{dt} = \lambda$ (a constant) and (1)

$$\frac{\textbf{tangential velocity}}{\textbf{normal acceleration}} = \frac{v}{v^2 / \rho} = \frac{\rho}{v} = \mu \text{ (a constant)} \qquad (2)$$

Since $$v = \frac{dS}{dt}, \qquad \rho = \frac{dS}{d\psi}$$

Then from formula (2)

$$\frac{\rho}{v} = \frac{dS / d\psi}{dS / dt} = \mu \quad \Rightarrow \frac{dt}{d\psi} = \mu \qquad \textbf{Or} \qquad \frac{d\psi}{dt} = \frac{1}{\mu}$$

From formula (1)

$$\frac{dv}{dt} = \frac{dv}{d\psi} \times \underbrace{\frac{d\psi}{dt}}_{1/\mu} = \lambda \qquad \textbf{Or} \qquad \frac{dv}{d\psi} = \lambda\mu \qquad \Rightarrow dv = \mu\lambda d\psi$$

By integrating $$v = \mu\lambda\psi + c_1$$

Where c_1 is a constant, again from equation (2)

$$\rho = \mu v \quad \Rightarrow \frac{dS}{d\psi} = \mu\ \ \mu\lambda\psi + c_1 \qquad \textbf{Or} \quad dS = \mu\ \ \mu\lambda\psi + c_1\ \ d\psi$$

Integrating

$$S = \frac{1}{2}\mu^2\lambda\psi^2 + \mu c_1\psi + C \quad \textbf{Or}$$
$$S = A\psi^2 + B\psi + C, \qquad A = \frac{1}{2}\lambda\mu^2,\ B = \mu c_1$$

‖ Example ▸

A particle moves in a catenary $s = c\tan\psi$, the direction of its acceleration at any point makes equal angles with the tangent and normal to the path at the point. If the speed at the vertex ($\psi = 0$) be u, show that the velocity and acceleration at any other point ψ are ue^{ψ} and $(\sqrt{2} / c)u^2e^{2\psi}\cos^2\psi$.

‖ Solution ▸

It is given that the direction of acceleration at any point makes equal angles with the tangent and normal to the path at the point. Therefore the tangential and normal accelerations will be equal at any time i.e. $v\frac{dv}{ds}=\frac{v^2}{\rho}$

$$v\frac{dv}{ds}=\frac{v^2}{\rho} \qquad \Rightarrow \frac{dv}{ds}\rho = v \quad \text{or}$$

$$\frac{dv}{ds}\frac{ds}{d\psi}=v \qquad \Rightarrow \frac{dv}{d\psi}=v \qquad \Rightarrow \frac{dv}{v}=d\psi$$

$\ln v=\psi+c$ from the initial conditions ($v=u, \quad \psi=0 \qquad \therefore c=\ln u$)

$$\therefore \ln v=\psi+\ln u \qquad \Rightarrow \ln\frac{v}{u}=\psi \qquad \Rightarrow v=ue^{\psi}$$

which gives the velocity of the particle at any point ψ .

Further it is given that the path of the particle is the catenary $s=c\tan\psi$

$$\therefore \rho=\frac{ds}{d\psi}=c\sec^2\psi$$

And since the acceleration magnitude is given by

$$=\sqrt{\left(v\frac{dv}{ds}\right)^2+\left(\frac{v^2}{\rho}\right)^2}=\sqrt{\left(\frac{v^2}{\rho}\right)^2+\left(\frac{v^2}{\rho}\right)^2}$$
$$=\sqrt{2}\left(\frac{v^2}{\rho}\right)=\sqrt{2}\,u^2e^{2\psi}\frac{1}{c\sec^2\psi}=\frac{\sqrt{2}}{c}u^2e^{2\psi}\cos^2\psi$$

Example

The relation between the velocity of a particle moving in a plane v and its tangent acceleration a_t is $a_t=\frac{1}{1+v}$. Find the relation between v,S and v,t if the particle start from rest at the position $S=0$.

Solution

To obtain the relation between v,t where $a_t=\frac{dv}{dt}$ then

$$\frac{dv}{dt}=\frac{1}{1+v} \qquad \Rightarrow (1+v)dv=dt$$

By integrating $$v + \frac{1}{2}v^2 = t + c_1$$

The integration constant c_1 can be evaluated from initial conditions, i.e. $v = 0$ at $t = 0$ and hence $c_1 = 0$ then the last formula becomes

$$v + \frac{1}{2}v^2 = t$$

Again since, $a_t = v\frac{dv}{dS}$ and therefore,

$$v\frac{dv}{dS} = \frac{1}{1+v} \qquad \Rightarrow (v + v^2)dv = dS$$

Integration again $$\frac{1}{2}v^2 + \frac{1}{3}v^3 = S + c_2$$

Where c_2 is a constant where $v = 0$ at $S = 0$ and hence $c_2 = 0$ then the relation between v, S is $\frac{1}{2}v^2 + \frac{1}{3}v^3 = S$

Example

A particle describes a curve (for which and vanish simultaneously) with uniform v, if acceleration at any point S be $\mu v^2 / (S^2 + \mu^2)$. Find the intrinsic equation of the curve.

Solution

It is given that the velocity is constant i.e.,

$$v = \frac{dS}{dt} = c \qquad \Rightarrow \frac{d^2S}{dt^2} = 0$$

And since the resultant of acceleration is

$$a = \sqrt{\left(\frac{d^2S}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2} = \frac{v^2}{\rho}$$

But it is given that

$$a = \frac{\mu v^2}{S^2 + \mu^2} \qquad \Rightarrow \frac{v^2}{\rho} = \frac{\mu v^2}{S^2 + \mu^2}$$

$$\Rightarrow \frac{\cancel{v^2} d\psi}{dS} = \frac{\mu \cancel{v^2}}{S^2 + \mu^2}$$

$$\Rightarrow \frac{\mu dS}{S^2 + \mu^2} = d\psi$$

$$\Rightarrow \frac{1}{\mu} \frac{dS}{1 + \left(\frac{S}{\mu}\right)^2} = d\psi$$

By integrating

$$\tan^{-1}\left(\frac{S}{\mu}\right) = \psi + C \qquad \text{C is a constant}$$

Given that $\psi = 0$ when $S = 0$, gives $C = 0$

Therefore, $\tan^{-1}\left(\frac{S}{\mu}\right) = \psi$ **Or** $S = \mu \tan(\psi)$

Example

A particle moves over a circle with radius 2 ft according to a constant tangent acceleration 4 ft sec^{-2}. If initially, the particle at the point A on a circumference and have zero velocity. Find the velocity of the particle after it returns to the point A and time spent to reach. Find its acceleration after return to point a.

Solution

It’s given that $a_t = 4$ thus

$$\frac{dv}{dt} = 4 \qquad \Rightarrow dv = 4dt \qquad \Rightarrow v = 4t + c_1$$

To get the constant c_1 we apply the initial conditions i.e. $v = 0$ when $t = 0$ so $c_1 = 0$

Then the last equation turns into $v = 4t$

Again since $v = \frac{dS}{dt}$ hence

$\frac{dS}{dt} = 4t \quad \Rightarrow dS = 4tdt \quad \Rightarrow S = 2t^2 + c_2$

Again To get the constant c_2 we apply the initial conditions i.e. $S = 0$ when $t = 0$ so $c_1 = 0$ (consider A be the fixed point) then the relation between S, t is $\quad S = 2t^2$

From this equation we can obtain the tome spent to reach to the point A again – note $S = 4\pi$ - thus the time $\quad 4\pi = 2t^2 \quad \therefore t = \sqrt{2\pi}$

And its velocity is $v = 4\sqrt{2\pi}$

Moreover the acceleration has two components namely tangential component $a_t = 4\,\text{ft}\sec^{-2}$ and normal component a_n, where

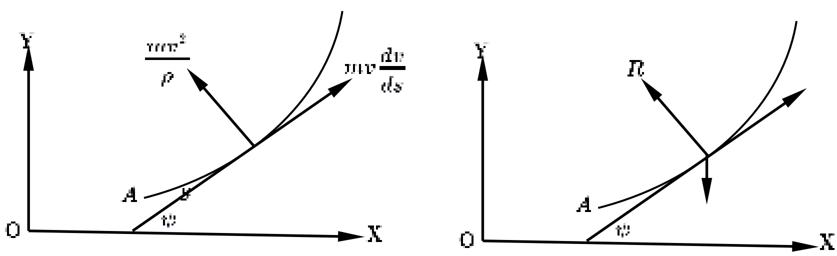
$$a_n = \frac{v^2}{\rho} = \frac{16(2\pi)}{2} = 16\pi \ \text{ft}\sec^{-2} \quad (\text{note } \rho = 2 \ \text{ft})$$

MOTION UNDER CONSTRAINT

A particle may be constrained to move along a given curve or surface, and the constraint may be one-sided, as for example when a heavy particle slides on the inside of a spherical surface and is free to break contact with the surface on the inside of the sphere but cannot get outside. There will then be a normal pressure inwards exerted by the sphere on the particle so long as contact persists, and the pressure will vanish at the point where the particle leaves the surface. On the other hand if the constraint is two-sided as when a particle moves in a fine tube, or a bead moves along a wire, then the normal reaction may vanish and change sign but the particle persists in the prescribed path.

■ Motion of a Heavy Particle on a Smooth Curve in a Vertical Plane

The motion is determined by the tangential and normal components of acceleration. The beginner may find it useful in such problems as this to make two diagrams, one showing the components of acceleration multiplied by the mass and the other showing the forces. It is then only necessary to realize that the two diagrams are equivalent representations of the same vector, so that the resolved parts in any assigned direction in the two diagrams are equal.



If m is the mass of the particle, the forces acting on it are the weight mg and the reaction R along the normal. The components of acceleration are $v\dfrac{dv}{ds}$ along the tangent and $\dfrac{v^2}{\rho}$ along the inward normal. Hence, by resolving along the tangent, we get

$$mv\frac{dv}{dS} = -mg\sin\psi = -mg\frac{dy}{ds},$$

therefore, by integration,

$$\frac{1}{2}mv^2 = c - mgy$$

or, if u is the velocity when the ordinate is y_0, we have

$$\frac{1}{2}m\ \ v^2 - u^2\ \ = mg\ \ y_0 - y \qquad (1)$$

This is the equation of energy and might have been written down at once; for since the curve is smooth no work is done by the reaction R in any displacement, so the increase in kinetic energy is equal to the work done by the weight.

Again, resolving along the normal, we get

$$m\frac{v^2}{\rho} = R - mg\cos\psi \qquad (2)$$

Substituting for v from equation (1), we have

$$R = mg\cos\psi + m\ \ u^2 + 2g(y_0 - y)\ \ /\ \rho \qquad (3)$$

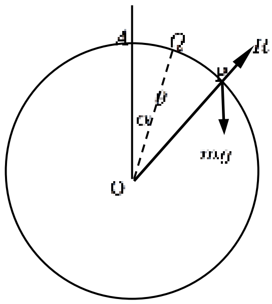
Assuming that the form of the curve is given, the values of ρ and ψ at any point can be determined, and thus R is known; and if we equate to zero the value of R we shall have an equation to determine the point, if any, at which the particle leaves the curve.

■ Motion of a Heavy Particle, placed on the outside of a Smooth Circle in a Vertical Plane and allowed to slide down

If the particle starts with zero initial velocity from position *Q* at an angular distance $\alpha = \angle \mathbf{AOQ}$ from the highest point *A*, and a is the radius of the circle and v the velocity at *P* where the angular distance from *A* is $\theta = \angle \mathbf{AOP}$, then,

$$v^2 = 2ga(\cos\alpha - \cos\theta)$$

Also by resolving along the inward normal



$$m\frac{v^2}{a} = R - mg\cos\theta$$

where R is the outward reaction of the curve.

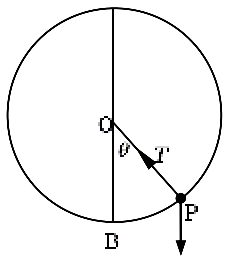
Therefore $\qquad R = mg(3\cos\theta - 2\cos\alpha)$

showing that the pressure vanishes, and that the particle flies off the curve, when $\cos\theta = \frac{2}{3}\cos\alpha$.

■ Motion in a Vertical Plane of a Heavy Particle attached by a Pine String to a Fixed Point

Suppose that the particle starts with velocity u from its lowest position *B*. If v is the velocity at $\mathbf{P}$ and θ is the angle that the string makes with the vertical, the equation of energy is

$$\frac{1}{2}m\left(v^2 - u^2\right) = -mga\left(1 - \cos\theta\right) \qquad (1)$$



and by resolving along the inward normal

$$m\frac{v^2}{a} = T - mg\cos\theta$$

where T is the tension of the string.

Therefore

$$T = m(3g\cos\theta - 2g + u^2 / a) \qquad (2)$$

In order to find the height of ascent we put $v = 0$ in (1), and get

$$2ga\cos\theta = 2ga - u^2 \qquad (3)$$

and by putting $T = 0$ in equation (2), we find that the tension vanishes when

$$3ga\cos\theta = 2ga - u^2 \qquad (4)$$

Now we have the following cases:

(i) If $u^2 < 2ga$, the string does not reach the horizontal position and the tension does not vanish.

(ii) If $u^2 = 2ga$, the string just reaches the horizontal position, the tension vanishes for $\theta = \frac{\pi}{2}$, and the particle swings through a quadrant on each side of the vertical.

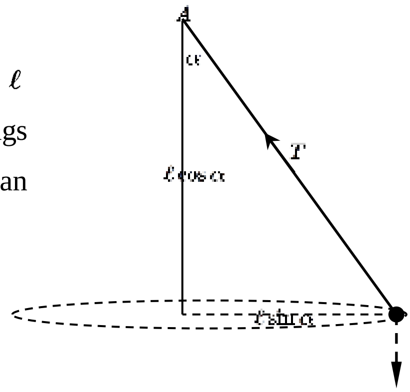
(iii) If $2ga < u^2 < 5ga$, we find that there is a value of θ, an obtuse angle, given by (4) smaller than that given by (3), so that the string becomes slack before the velocity vanishes and the particle will fall away from the circular path and move in a parabola till the string again becomes taut.

(iv) If $u^2 = 5ga$, the tension just vanishes in the highest position, but v does not vanish, so that circular motion persists.

(v) If $u^2 > 5ga$, neither v nor T vanish. This is an example of a one-sided constraint; if instead of the problem of a particle attached to a string, we consider that of a bead sliding on a wire, we find that if $u^2 = 4ga$ the bead will reach the highest point of the wire and for any greater value of u it will describe the complete circle.

■ Circular pendulum:

A mass hangs from a massless string of length ℓ Conditions have been set up so that the mass swings around in a horizontal circle, with the string making an angle α with the vertical (see Figure).



The mass travels in a circle, so the horizontal radial
force must be

$$F_n = m\left(\frac{v^2}{\ell \sin\alpha}\right) = m\left(\frac{\omega^2 \ell^2 \sin^2\alpha}{\ell \sin\alpha}\right) = \omega^2 \ell \sin\alpha$$

directed radially inward. The forces on the mass are the tension in the string, T , and gravity, mg as illustrated. There is no acceleration in the vertical direction, so $F = ma$ in the vertical and radial directions give, respectively,

$T\cos\alpha = mg$ and $T\sin\alpha = m\omega^2 \ell \sin\alpha$

Solving for ω gives

$$\omega = \sqrt{\frac{g}{\ell \cos\alpha}}$$

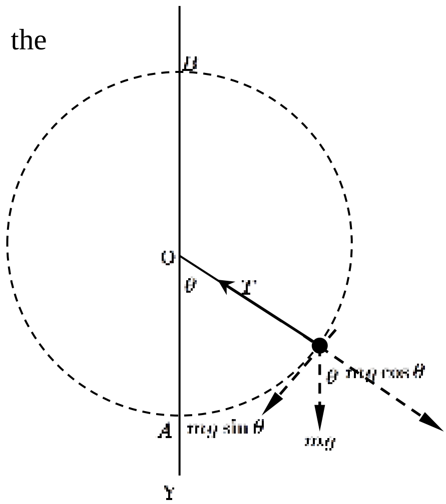
Note that if $\alpha \approx 0$, then $\omega = \sqrt{\frac{g}{\ell}}$, which equals the frequency of a plane pendulum of length ℓ. And if $\alpha \approx 90$, then $\omega \to \infty$, which makes sense.

■ Illustrative Examples ■

Example

A heavy particle of weight mg, attached to a fixed point by a light inextensible string with length ℓ, describes a circle in a vertical plane. The tension in the string has the values $n\,mg$ and $n'\,mg$, respectively, when the particle is at the highest and lowest points in its path. Show that $n' = n + 6$.

Solution



The equation of motion along and perpendicular to the radius r

$$m\ell\dot{\theta}^2 = T - mg\cos\theta \qquad (1)$$

$$m\ell\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \ell\ddot{\theta} = -g\sin\theta \qquad (2)$$

Since we have $\ddot{\theta} = \dfrac{d\dot{\theta}}{dt} = \dfrac{d\dot{\theta}}{d\theta}\dfrac{d\theta}{dt} = \dot{\theta}\dfrac{d\dot{\theta}}{d\theta}$ and substituting in formula (1) and integrate

$$\Rightarrow \ell\dot{\theta}\frac{d\dot{\theta}}{d\theta} = -g\sin\theta$$
$$\Rightarrow \ell\dot{\theta}d\dot{\theta} = -g\sin\theta d\theta$$

$$\Rightarrow \ell\dot{\theta}^2 = 2g\cos\theta + c \qquad (3)$$

Where c the integration constant and substituting from equation (3) in equation (1) we get

$$T = m(2g\cos\theta + c) + mg\cos\theta = 3mg\cos\theta + C \qquad (4)$$

It's given that the tension at highest position is $n\,mg$ i.e. at point B and at the lowest position $n'\,mg$ i.e. at point A then

$$n\ mg = 3mg\cos\pi + C \quad \Rightarrow C = (n+3)mg \qquad \text{and}$$

$$n'\ mg = 3mg\cos 0 + C \quad \Rightarrow C = (n'-3)mg$$

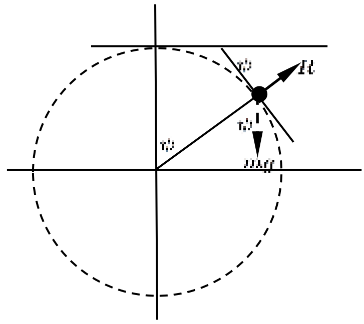
Therefore, by subtracting $n - n' + 6 = 0$ **Or** $n' = n + 6$

<u>The reader can resolve this problem by using intrinsic coordinates.</u>

Example

A particle slides outside a smooth vertical circle with radius. At time $t = 0$ the particle was at the top of the circle and has zero initial velocity. Determine the velocity at any position and the reaction of the circle the find the position that the particle will leave the circle.

Solution



The forces acting on the particle are mg the weight and reaction R as illustrated in the figure

The equation of motion along perpendicular to the radius is

$$mv\frac{dv}{dS} = mg\sin\psi \qquad \text{Or} \quad v\frac{dv}{d\psi}\frac{d\psi}{dS} = g\sin\psi \quad \left(\frac{dS}{d\psi} = b\right)$$

$$\Rightarrow vdv = bg\sin\psi \; d\psi$$

By integration

$$v^2 = C - 2bg\cos\psi$$

Since $v = 0$ at $t = 0$ thus $C = 2bg$ then $\qquad v^2 = 2bg(1 - \cos\psi)$

The equation of motion along the radius we get

$$m\frac{v^2}{b} = mg\cos\psi - R \qquad \Rightarrow R = mg\cos\psi - m\frac{v^2}{b}$$

$$\begin{aligned} \Rightarrow \quad R &= mg\cos\psi - 2mg(1 - \cos\psi) \\ &= mg(3\cos\psi - 2) \end{aligned}$$

The last equation gives the reaction of the circle at any position and the particle will leave the circle when the reaction vanishes, i.e.

$$mg(3\cos\psi - 2) = 0 \qquad \Rightarrow \cos\psi = \frac{2}{3}$$

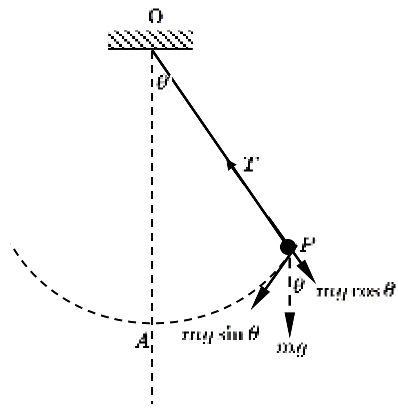
That is the particle will leave the circle after sliding a vertical distance equals

Example

Prove that the simple pendulum executes simple harmonic motion and determine the periodic time.

Solution

If a heavy particle is tied to one end of a light inextensible string the other end of which is fixed, and oscillates in a vertical circle, we have what is called a Simple Pendulum



We now obtain the time of oscillation of such a pendulum when it is allowed to oscillate through a small angle only. Let O be the fixed point, A the lowest position of the particle, and P any position such that $\angle \mathbf{AOP} = \theta$

Since in polar co-ordinate

$$(a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = L\ddot{\theta} \quad (r = L \therefore \dot{r} = 0))$$

The equations of motion in θ direction is

$$mL\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \ddot{\theta} = -\frac{g}{L}\sin\theta$$

When the angle θ is small enough so the approximation $\sin\theta \cong \theta$ can be applied and the equation of motion, $\ddot{\theta} = -\frac{g}{L}\sin\theta$ becomes $\ddot{\theta} = -\frac{g}{L}\theta$

Which is similar to $\ddot{\theta} = -w^2\theta$, with $w^2 = \frac{g}{L}$ **Or** $w=\sqrt{\frac{g}{L}}$

So, a simple pendulum moves like a SHM with periodic time of motion equals

$$2\pi\sqrt{\frac{L}{g}}$$

PROBLEMS

❒ A point starts from the origin in the direction of the initial line with velocity $\frac{u}{\omega}$ and moves with angular velocity ω about the origin and with constant negative radial acceleration u. Find the equation of path

❒ A point describes the cycloid $S = 4a\sin\psi$ with uniform speed v. Find its acceleration at any point.

❒ If the tangential and normal acceleration of a particle describing a plane curve be constant throughout, prove that the radius of curvature ρ at any point is given by $\rho = (at + b)^2$, where a, b are constants

❒ The velocities of a particle along and perpendicular to the radius vector are λr and $\mu\theta$ respectively. Find the path equation and obtain the accelerations along and perpendicular to the radius vector.

❒ The velocities of a particle along and perpendicular to the radius vector are λr^2 and $\mu\theta^2$ respectively. Find the path equation and obtain the accelerations along and perpendicular to the radius vector.

KINETICS OF A PARTICLE

This chapter is concerned with the foundations of dynamics and gravitation. Kinematics is concerned purely with geometry of motion, but dynamics seeks to answer the question as to what motion will actually occur when specified forces act on a body. The rules that allow one to make this connection are Newton's laws of motion. These are laws of physics that are founded upon experimental evidence and stand or fall according to the accuracy of their predictions. In fact, Newton's formulation of mechanics has been astonishingly successful in its accuracy and breadth of application, and has survived, essentially intact, for more than three centuries. The same is true for Newton's universal law of gravitation which specifies the forces that all masses exert upon each other.

Taken together, these laws represent virtually the entire foundation of classical mechanics and provide an accurate explanation for a vast range of motions from large molecules to entire galaxies.

■ Newton's Laws

Isaac Newton's∗ three famous laws of motion were laid down in Principia, written in Latin and published in 1687. These laws set out the founding principles of mechanics and have survived, essentially unchanged, to the present day. Even when translated into English, Newton's original words are hard to understand, mainly because the terminology of the seventeenth century is now archaic. Also, the laws are now formulated as applying to particles, a concept never used by Newton. A particle is an idealized body that occupies only a single point of space and has no internal structure. True particles do not exist in nature, but it is convenient to regard realistic bodies as being made up of particles. Using modern terminology, Newton's laws may be stated as follows:

➢**First Law:** When all external influences on a particle are removed, the particle moves with constant velocity. {This velocity may be zero in which case the particle remains at rest.}

➢**Second Law:** When a force $\boldsymbol{F}$ acts on a particle of mass m, the particle moves with instantaneous acceleration $\underline{\boldsymbol{a}}$ given by the formula

$$\boldsymbol{F} = m\underline{\boldsymbol{a}}$$

where the unit of force is implied by the units of mass and acceleration.

➢**Third Law:** When two particles exert forces upon each other, these forces are (i) equal in magnitude, (ii) opposite in direction, and (iii) parallel to the straight line joining the two particles.

■ The Law of Gravitation

Physicists recognize only four distinct kinds of interaction forces that exist in nature. These are gravitational forces, electromagnetic forces and weak/strong nuclear forces. The nuclear forces are important only within the atomic nucleus and will not concern us at all. The electromagnetic forces include electrostatic attraction and repulsion, but we will encounter them mainly as 'forces of contact' between material bodies. Since such forces are intermolecular, they are ultimately electromagnetic although we will make no use of this fact! The present section however is concerned with gravitation.

It is an observed fact that any object with mass attracts any other object with mass with a force called gravitation. When gravitational interaction occurs between particles, the Third Law implies that the interaction forces must be equal in magnitude, opposite in direction and parallel to the straight line joining the particles. The magnitude of the gravitational interaction forces is given by:

The gravitational forces that two particles exert upon each other each have magnitude

$$\underline{\boldsymbol{F}} = \gamma \frac{\boldsymbol{Mm}}{\boldsymbol{R}^2} \hat{\boldsymbol{F}} \qquad (1)$$

where $\boldsymbol{M}, \boldsymbol{m}$ are the particle masses, $\boldsymbol{R}$ is the distance between the particles, and γ, the constant of gravitation, is a universal constant. Since γ is not dimensionless, its numerical value depends on the units of mass, length and force.

This is the famous inverse square law of gravitation originally suggested by Robert Hooke, a scientific contemporary (and adversary) of Newton. In SI units, the constant of gravitation is given approximately by

$$\gamma = \mathbf{6.67 \times 10^{-11}\ Nm^2kg^{-2}}$$

this value being determined by observation and experiment. There is presently no theory (general relativity included) that is able to predict the value of γ. Indeed, the theory of general relativity does not exclude *repulsion* between masses!

To give some idea of the magnitudes of the forces involved, suppose we have two uniform spheres of lead, each with mass 5000 kg (five metric tons). Their common radius is about 47 cm which means that they can be placed with their centers 1 m apart. What gravitational force do they exert upon each other when they are in this position? We will show later that the gravitational force between uniform spheres of matter is exactly the same *as if* all the mass of each sphere were concentrated at its center. Given that this result is true, we can find the force that each sphere exerts on the other simply by substituting $\boldsymbol{M} = \boldsymbol{m} = \mathbf{5000}$ and $\boldsymbol{R} = \mathbf{1}$ into equation (1). This gives $\boldsymbol{F} = \mathbf{0.00167\,N}$ approximately, the weight of a few grains of salt! Such forces seem insignificant, but gravitation is the force that keeps the Moon in orbit around the Earth, and the Earth in orbit around the Sun. The reason for this disparity is that the masses involved are so much larger than those of the lead spheres in our example. For instance, the mass of the Sun is about 2×10^{30} kg.

■ Motion Through a Resisting Medium

When a body moves in a medium like air or any other fluid, it experiences a resistance to its motion. The resistance which we have been neglecting so far, generally varies with the velocity. For small velocities the resistance is approximately proportional to the velocity, for greater velocities it varies as the square of the velocity and for still greater velocities, the resistance varies as the cube or even a higher power of the velocity. The forces of resistance being non-conservative, the principle of Conservation of Energy is not applicable to such cases.

■ Bodies Falling Vertically in a Resisting Medium

Suppose a particle with mass m is allowed to fall vertically subject to a resistance proportional to some power of the velocity v, e.g. a resistance force $\mu m v$, then we have the equation of motion

$$m\frac{dv}{dt} = mg - \mu m v \qquad \textbf{Or} \qquad \frac{dv}{dt} = g - \mu v$$

where μm is the constant of proportionality and g, the acceleration due to gravity, is supposed to remain constant. The equation shows that the acceleration of the particle decreases as its velocity increases and that it vanishes when g / μ . Separation of variables for the previous equation we get

$$\frac{dv}{g - \mu v} = dt \qquad \Rightarrow \frac{-\mu dv}{g - \mu v} = -\mu dt$$

Integration we have

$$\ln(g - \mu v) = -\mu t + c_1$$

If the initial velocity is u therefore, the constant c_1 may be obtained as $\ln(g - \mu u) = c_1$ then

$$\ln(g - \mu v) = -\mu t + \ln(g - \mu u) \qquad \Rightarrow \ln\frac{g - \mu v}{g - \mu u} = -\mu t$$

$$\Rightarrow g - \mu v = (g - \mu u)e^{-\mu t} \qquad Or \quad v = \frac{g}{\mu} - \frac{1}{\mu}(g - \mu u)e^{-\mu t}$$

The value $\frac{g}{\mu}$ is the greatest velocity attainable by the particle and is called the limiting or terminal velocity.

To get the height since $v = \frac{dy}{dt}$ then

$$\frac{dy}{dt} = \frac{g}{\mu} - \frac{1}{\mu}(g - \mu u)e^{-\mu t} \qquad \Rightarrow dy = \left(\frac{g}{\mu} - \frac{1}{\mu}(g - \mu u)e^{-\mu t}\right)dt$$

And integrate we get

$$y = \frac{g}{\mu}t + \frac{1}{\mu^2}(g - \mu u)e^{-\mu t} + c_2$$

Where $c_2 = -\frac{1}{\mu^2}(g - \mu u)$ sine $y = 0$ when $t = 0$ that is

$$y = \frac{g}{\mu}t + \frac{1}{\mu^2}(g - \mu u) \; e^{-\mu t} - 1$$

Subsequently the particle moves uniformly with this limiting velocity. The velocity for the rain drops at the surface of the earth cannot, therefore, give us any idea of the height from which they might have fallen, for after moving for some time they acquire the terminal velocity and continue to move uniformly with that velocity.

■ Illustrative Examples ■

Example

A particle with mass m moves horizontally through a resisting medium where its resistance proportional to v and the proportional constant is αm. If the particle starts its motion from the origin point with initial velocity u. Find the distance after time t.

Solution

The equation of motion of the particle is (horizontally)

$$m\frac{dv}{dt} = -\alpha m v \quad \Rightarrow \frac{dv}{v} = -\alpha\, dt$$

By integrating we have $\ln(v) = c_1 - \alpha t$ **(1)**

The constant c_1 can be determined from the initial conditions, $v = u$ at $t = 0$, therefore $c_1 = \ln u$ and equation (1) becomes

$$\ln(v) = \ln u - \alpha t \qquad \textbf{Or} \qquad v = ue^{-\alpha t} \qquad \textbf{(2)}$$

Equation (2) gives the velocity of the particle at any instance, and the position of the particle x can be obtained as follows

$$\frac{dx}{dt} = ue^{-\alpha t} \quad \Rightarrow \quad dx = ue^{-\alpha t} dt$$

v

$$\Rightarrow \int dx = \int ue^{-\alpha t} dt + c_2 \qquad \textbf{Or}$$

$$x = -\frac{u}{\alpha} e^{-\alpha t} + c_2 \qquad \textbf{(3)}$$

Where c_2 is integration constant that can be calculated from the initial conditions, $x = 0$ at $t = 0$, therefore $c_2 = \frac{u}{\alpha}$ and equation (3) turns into

$$x = \frac{u}{\alpha}\ 1 - e^{-\alpha t}$$

Example

A moving point with mass equals unity subject to a resistance $\lambda v + \mu v^2$ If the resisting force is the only force acting on the point. Find the distance where u is the initial velocity of the point.

Solution

Equation of motion is ($m = 1$) – Note resisting force is the only acting force-

$$v\frac{dv}{dx} = -(\lambda v + \mu v^2) \qquad \Rightarrow \frac{\mu dv}{\lambda + \mu v} = -\mu dx$$

By integration we get

$$\ln(\lambda + \mu v) = c_1 - \mu x \qquad \textbf{(1)}$$

Where c_1 represents integration constant and can be obtained from the initial conditions, $v = u$ at $x = 0$, therefore $c_1 = \ln(\lambda + \mu u)$ and equation (1) turns into

$$\ln(\lambda + \mu v) = \ln(\lambda + \mu u) - \mu x \qquad \text{Or} \qquad \ln\left(\frac{\lambda + \mu u}{\lambda + \mu v}\right) = \mu x \qquad \textbf{(2)}$$

Again from the last equation we can obtain the position of the point as the velocity vanishes

$$x\Big|_{v=0} = \frac{1}{\mu}\ln\left(\frac{\lambda + \mu u}{\lambda}\right) = \frac{1}{\mu}\ln\left(1 + \frac{\mu}{\lambda}u\right)$$

Example

Two equal particles with mass m projected downwards from the same point and at the same instance with initial velocities u_1, u_2 subject to a resistance $\mu m v$ If u_1', u_2' are the velocities of the particles after time T. Prove that $u_1' - u_2' = (u_1 - u_2)e^{-\mu T}$.

Solution

With respect the first particle we suppose that its velocity at any time is v therefore, the equation of motion is

$$m\frac{dv}{dt} = mg - \mu mv \quad \Rightarrow \frac{dv}{g - \mu v} = dt \quad \textbf{Or} \quad \frac{-\mu dv}{g - \mu v} = -\mu dt$$

By integration we have

$$\ln(g - \mu v) = c - \mu t$$

look c indicates the integration constant which can be calculated from the initial conditions, $v = u_1$ when $t = 0$, therefore $c = \ln(g - \mu u_1)$ and the last equation become

$$\ln(g - \mu v) = \ln(g - \mu u_1) - \mu t \qquad \textbf{Or} \qquad g - \mu v = (g - \mu u_1)e^{-\mu t}$$

Now after time T, the velocity become u_1' that is

$$g - \mu u_1' = (g - \mu u_1)e^{-\mu T} \tag{1}$$

Now with respect the second particle we suppose that its velocity at any time is v' therefore, the equation of motion is

$$m\frac{dv'}{dt} = mg - \mu mv' \quad \Rightarrow \frac{dv'}{g - \mu v} = dt \quad \textbf{Or} \quad \frac{-\mu dv'}{g - \mu v'} = -\mu dt$$

By integration we have

$$\ln(g - \mu v') = c' - \mu t$$

where c refers to the integration constant which can be obtained from the initial conditions, $v' = u_2$ when $t = 0$, therefore $c' = \ln(g - \mu u_2)$ and the previous equation converted to

$$\ln(g - \mu v') = \ln(g - \mu u_2) - \mu t \qquad \textbf{Or} \qquad g - \mu v' = (g - \mu u_2)e^{-\mu t}$$

Again, Now after time T, the velocity become u_2' that is

$$g - \mu u_2' = (g - \mu u_2)e^{-\mu T} \tag{2}$$

By subtracting Equations (1) and (2) we obtain

$$\mu \quad u_1' - u_2' \quad = \mu(u_1 - u_2)e^{-\mu T} \qquad \text{Or} \qquad u_1' - u_2' = (u_1 - u_2)e^{-\mu T}$$

Example

A point with mass m is projected vertically upwards with initial velocity $\sqrt{g\mu^{-1}}$ and the resistance of air produces retardation per unit mass μv^2 where v is the velocity and μ is constant. Find the highest position and the time spent to reach is $\frac{\pi}{4\sqrt{g\mu}}$.

Solution

The equation of motion – let the projection point be the origin-then

$$mv\frac{dv}{dy} = -mg - \mu m v^2 \quad \Rightarrow \frac{vdv}{g + \mu v^2} = -dy \quad \text{Or} \quad \frac{2\mu v dv}{g + \mu v^2} = -2\mu dy$$

By integration we get

$$\ln(g + \mu v^2) = c_1 - 2\mu y \tag{1}$$

Note c_1 indicates the integration constant which can be obtained from the initial conditions, $v = \sqrt{g\mu^{-1}}$ when $y = 0$, therefore $c_1 = \ln 2g$ and equation (1) be

$$\ln(g + \mu v) = \ln 2g - 2\mu y \qquad \text{Or} \quad y = \frac{1}{2\mu}\ln\frac{2g}{g + \mu v} \tag{2}$$

Equation (2) gives the position of the point at any instance t and at highest position the velocity is zero $v = 0$ and then

$$y = \frac{1}{2\mu}\ln\frac{2g}{g} \qquad \Rightarrow Y = \frac{1}{2\mu}\ln 2$$

And this is the highest position and to evaluate the spent time to reach since

$$m\frac{dv}{dt} = -mg - \mu m v^2 \quad \Rightarrow \frac{dv}{g+\mu v^2} = -dt \quad \text{Or} \quad \frac{\sqrt{\frac{\mu}{g}}dv}{1+\left(\sqrt{\frac{\mu}{g}}\,v\right)^2} = -\sqrt{g\mu}\,dt$$

By integration we obtain

$$\tan^{-1}\left(\sqrt{\frac{\mu}{g}}\,v\right) = c_2 - \sqrt{g\mu}\,t \qquad \textbf{(3)}$$

Note c_2 is the integration constant which its value can be evaluated by the initial conditions, $v = \sqrt{g\mu^{-1}}$ when $t = 0$, therefore $c_2 = \frac{\pi}{4}$ and equation (3) turn into

$$\tan^{-1}\left(\sqrt{\frac{\mu}{g}}\,v\right) = \frac{\pi}{4} - \sqrt{g\mu}\,t \qquad \text{Or} \qquad v = \sqrt{\frac{g}{\mu}}\tan\left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right)$$

This equation gives the velocity at any time t, and when $v = 0$ then t

$$\Rightarrow 0 = \sqrt{\frac{g}{\mu}}\tan\left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right)$$
$$\Rightarrow \quad \left(\frac{\pi}{4} - \sqrt{g\mu}\,t\right) = 0 \qquad \text{Or} \quad t = \frac{\pi}{4\sqrt{g\mu}}$$

Example

A point with mass m is projected vertically upwards where the resistance of air produces a retardation $m\mu v$ where v is the velocity and μ is constant. If the velocity vanish at time T with a height ℓ from the point of projection Show that the initial velocity of the point is $\mu\ell + gT$.

Solution

The equation of motion –the point of projection is chosen to be the origin point-

$$m\frac{dv}{dt} = -mg - \mu mv \quad \Rightarrow \frac{\mu dv}{g+\mu v} = -\mu dt$$

By integration we get

$$\ln(g+\gamma v) = c_1 - \mu t \tag{1}$$

here c_1 gives the integration constant which can be obtained from the initial conditions, $v = u$ when $t = 0$, -we suppose that the initial velocity is u which we need to obtain- therefore $c_1 = \ln(g+\mu u)$ and equation (1) takes the following formula

$$\ln(g+\mu v) = \ln(g+\mu u) - \mu t \qquad \text{Or} \quad g+\mu v = \ g+\mu u \ \ e^{-\mu t}$$

$$\text{at } \ t = T, \ v = 0 \qquad \Rightarrow \ g = \ g+\mu u \ \ e^{-\mu T} \tag{2}$$

In order to determine the height of the point we have

$$\because \ g+\mu v = (g+\mu u)e^{-\mu t} \qquad \text{Or} \qquad v = \frac{1}{\mu}\ (g+\mu u)e^{-\mu t} \ - g$$

But $v = \dfrac{dy}{dt}$ then

$$\frac{dy}{dt} = \frac{1}{\mu}\ (g+\mu u)e^{-\mu t} \ - g \qquad \Rightarrow dy = \frac{1}{\mu}\ (g+\mu u)e^{-\mu t} \ - g \ \ dt$$

By integration we get

$$y = -\frac{1}{\mu}\left(\frac{(g+\mu u)}{\mu}e^{-\mu t} \ + gt\right) + c_2 \tag{3}$$

here c_2 gives the integration constant which can be obtained from the initial conditions, $y = 0$ when $t = 0$, therefore $c_2 = \dfrac{(g+\mu u)}{\mu^2}$ and equation (2) become

$$y = \frac{g+\mu u}{\mu^2} - \frac{1}{\mu}\left(\frac{(g+\mu u)}{\mu}e^{-\mu t} \ + gt\right)$$

Now let $y = \ell$ when $t = T$

$$\ell = \frac{g+\mu u}{\mu^2} - \frac{(g+\mu u)}{\mu^2} e^{-\mu T} - \frac{gT}{\mu}$$

$$\Rightarrow \frac{(g+\mu u)}{\mu^2} e^{-\mu T} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell \quad \textbf{Or} \quad \frac{g}{\mu^2} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell$$

We use equation (2)

$$\frac{g}{\mu^2} = \frac{g+\mu u}{\mu^2} - \frac{gT}{\mu} - \ell \qquad \Rightarrow u = gT + \mu\ell$$

Example

A point with mass m is projected vertically upwards with initial velocity u and the resistance of air produces a retardation $m\gamma v^2$ where v is the velocity and γ is constant. Show that the velocity with which the point will return to the point of projection is $\frac{uu'}{\sqrt{u^2 + u'^2}}$ where $u' = \sqrt{g\gamma^{-1}}$.

Solution

To determine the velocity with which the point will return to the point of projection, we will consider the motion of the point upwards until it stop then it return.

The equation of motion of the point – consider $\mathbf{Y}$ axis to be vertically and the point of projection is chosen to be the origin point-

$$mv\frac{dv}{dy} = -mg - \gamma m v^2 \quad \Rightarrow \frac{2\gamma v dv}{g+\gamma v^2} = -2\gamma dy$$

By integration we get

$$\ln(g+\gamma v^2) = c_1 - 2\gamma y \tag{1}$$

Where c_1 points out integration constant which can be obtained from the initial conditions, $v = u$ at $y = 0$, therefore $c_1 = \ln(g+\gamma u^2)$ and equation (1) takes the following formula

$$\ln(g+\gamma v^2) = \ln(g+\gamma u^2) - 2\gamma y \quad Or \quad y = \frac{1}{2\gamma}\ln\left(\frac{g+\gamma u^2}{g+\gamma v^2}\right)$$

The point will stop as $v = 0$, therefore

$$y\big|_{v=0} = Y = \frac{1}{2\gamma}\ln\left(\frac{g+\gamma u^2}{g}\right) = \frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right), \quad (u'^2 = \frac{g}{\gamma})$$

Now by taking the motion where the point moves downwards, let the highest position represents the new origin point and the **Y** axis is chosen to be vertically downward. Moreover, the initial condition will be $v = 0$ when $y = 0$ where v is the velocity. The equation of motion

$$mv\frac{dv}{dy} = mg - \gamma mv^2 \quad \Rightarrow \frac{2\gamma v dv}{g-\gamma v^2} = -2\gamma dy$$

By integration we get

$$\ln(g-\gamma v^2) = c_2 - 2\gamma y \qquad \textbf{(2)}$$

Constant of integration c_2 can be obtained from the initial conditions, $v = 0$ at $y = 0$, therefore $c_2 = \ln g$ and equation (2) becomes

$$\ln(g-\gamma v^2) = \ln g - 2\gamma y \quad \textbf{Or} \quad y = \frac{1}{2\gamma}\ln\left(\frac{g}{g-\gamma v^2}\right)$$

And the velocity of the point with which the point will return to the point of projection is that is at $y = Y = \frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right)$ hence

$$\frac{1}{2\gamma}\ln\left(1+\frac{u^2}{u'^2}\right) = \frac{1}{2\gamma}\ln\left(\frac{g}{g-\gamma v^2}\right) \quad \textbf{Or} \quad 1+\frac{u^2}{u'^2} = \frac{g}{g-\gamma v^2}$$

$$\therefore \frac{u'^2+u^2}{u'^2} = \frac{g}{g-\gamma v^2} \quad \Rightarrow g-\gamma v^2 = \frac{gu'^2}{u'^2+u^2} \quad \Rightarrow \gamma v^2 = g - \frac{gu'^2}{u'^2+u^2}$$

$$
\begin{aligned}
v^2 &= u'^2 - \frac{u'^4}{u'^2 + u^2} \\
&= \frac{u'^2(u'^2 + u^2)}{u'^2 + u^2} - \frac{u'^4}{u'^2 + u^2} \\
&= \frac{u'^2 u^2}{u'^2 + u^2} \\
\therefore v &= \frac{uu'}{\sqrt{u^2 + u'^2}} \qquad \left(u'^2 = \frac{g}{\gamma}\right)
\end{aligned}
$$

PROBLEMS

❒ A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is γv, γ is a constant. Obtain the velocity v and the distance after a time t .

❒ A particle is projected vertically upwards with velocity u and the resistance of the air produces a retardation kv where v is the velocity. Determine the velocity with which the particle will return to the point of projection.

❒ A particle **P** moving along a horizontal straight line has retardation μv, where v is the velocity at time t. When $t = 0$, the particle is at **O** and has velocity u. Show that $u - v$ is proportional to **OP**.

❒ A particle subject to gravity describes a curved path in a resisting medium which causes retardation hv. Show that the resultant acceleration has a constant direction, and equals $a_0 e^{-ht}$ where a_0 is the acceleration when $t = 0$.

PROJECTILE MOTION

Let us consider that u, v denote the resolved parts of the velocity of the particle parallel to the axes at time t and $u+\delta u$, $v+\delta v$ refer to the resolved parts at time $t+\delta t$ then the resolved parts of the acceleration are given as

$$a_x = \lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \ddot{x}$$

$$a_y = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{du}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} = \ddot{y}$$

The consideration of component velocities and accelerations is of great importance when we have to deal with cases of motion where the path is not a straight line.

■ Equations of Motion of a Particle Moving in a Plane

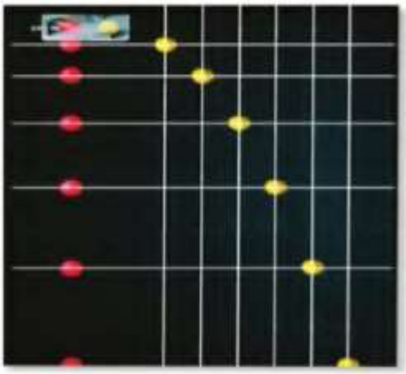
The position of a point in a straight line being determined by one co-ordinate, only one equation of motion is sufficient to determine the motion completely. In the case of a particle moving in a plane, two equations of motion are required in order to obtain the two co-ordinates which define the position of a point in a plane. The two equations of motion are obtained by resolving the forces in any two convenient directions at right angles to one another. If the two directions are taken parallel to the co-ordinate axes the equations of motion, as deduced from the second law of motion, will be of the form

$$m\frac{d^2x}{dt^2} = F_x \quad \text{and} \quad m\frac{d^2y}{dt^2} = F_y$$

where F_x, F_y are the sums of the resolved parts of the forces parallel to the axes of x and y:

■ Projectiles

As an example of motion in two dimension is the projectile motion. Recall that a particle has a mass but negligible size and shape. Therefore, we must limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. In most problems, we will be focused in bodies of finite size, such as rockets, projectiles, or vehicles. Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass center and any rotation of the body is neglected. The free-flight motion of a projectile is often studied in terms of its rectangular components. The acceleration is of approximately 9.81 ms^{-2} or 32.2 ft s^{-2}.

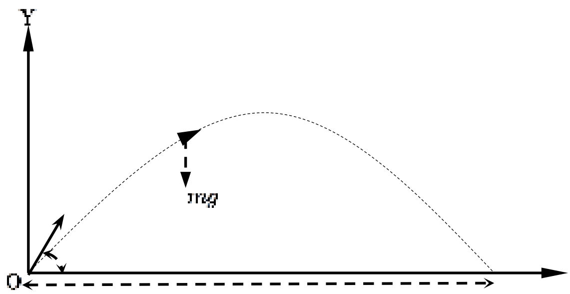


Each picture in this sequence is taken after the same time interval. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity when released. Both balls accelerate downward at the same rate, and so they remain at the same elevation at any instant. This acceleration causes the difference in elevation between the balls to increase between successive photos. Also, note the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction remains constant.

We will discuss the motion of a particle projected in the field of gravity. We now consider the motion of a ***projectile***, that is, the motion of a body which is small enough to be regarded as a particle and which is projected in a direction oblique to the direction of gravity. A body that moves freely under uniform gravity, and possibly air resistance, is called a projectile. Projectile motion is very common. In ball games, the ball is a projectile, and controlling its trajectory is a large part of the skill of the game. On a larger scale, artillery shells are projectiles, but guided missiles, which have rocket propulsion, are not.

Note: Near the Earth's surface, we assume that the downward acceleration due to gravity is constant and the effect of air resistance is negligible.
We shall suppose the body to be projected in *vacuum* near the surface of the earth or, in other words, we shall suppose the resistance due to air and the slight variation in the force of gravity to be negligible. A particle of mass m is projected into the air with velocity u, in a direction making an angle α with the horizontal, to find its motion and the path described.

Let O, the point of projection, be taken as the origin and let the horizontal and the vertical lines through be taken as the axes of $\mathbf{X}$ and $\mathbf{Y}$. Again, let P be the position of the moving point, after time t. During the motion of the projectile, the only force acting on it is its weight acting downwards. The equations of motion, therefore, are



$$m\frac{d^2x}{dt^2} = 0 \qquad \text{and} \qquad m\frac{d^2y}{dt^2} = -mg$$

Or in other formula

$$\frac{d^2x}{dt^2} = 0 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = -g$$

Integrating these equations, we get

$$\frac{dx}{dt} = C_1 \qquad \text{and} \qquad \frac{d^2y}{dt^2} = C_2 - gt \tag{1}$$

where C_1, C_2 are integration constants

Initially at O when $t = 0$, $\frac{dx}{dt} = u\cos\alpha$ and $\frac{dy}{dt} = u\sin\alpha$ then Equation (1) becomes

$$\frac{dx}{dt} = u\cos\alpha \quad \text{and} \quad \frac{dy}{dt} = u\sin\alpha - gt \qquad (2)$$

Integrating these equations again and applying initial conditions, *viz.*, when $t = 0,\ x = y = 0$, we obtain

$$x = u\cos\alpha\ t \quad \text{and} \quad y = u\sin\alpha\ t - \frac{1}{2}gt^2 \qquad (3)$$

Equation (2) gives the components of the velocity and (3) the displacements of the particle in the horizontal and vertical directions at any time t. These equations could also be written down at once by regarding the particle to be projected with a constant velocity $u\cos\alpha$ in the horizontal direction and with an initial velocity $u\sin\alpha$ under a retardation g in the vertical direction. Eliminating the time t the two parts of Equation (3) we have,

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha} \qquad (4)$$

We now deduce the following facts from the five equations just obtained:

■ The Path Equation of Projectile

Equation (4) is of the second degree and the second degree term x^2 is a perfect square. It follow, therefore, that the path of the particle is a parabola. Equation (4) can be re-written in the form

$$y - \frac{u^2\sin^2\alpha}{2g} = \frac{-g}{2u^2\cos^2\alpha}\left(x - \frac{u^2\sin 2\alpha}{2g}\right)^2$$

It shows that the latus-rectum of the parabolic path $= 2u^2\cos^2\alpha\ /\ g$.

In the particular case when the particle is projected horizontally, $\alpha = 0$, and the Equation (4) of the path reduces to

$$y = -\frac{g}{2u^2}x^2$$

which is obviously a parabola the length of whose latus-rectum is $2u^2\ /\ g$. The path of a projectile is called its ***trajectory***.

■ The Time of Flight

Let T, represents the time which the particle takes in reaching the horizontal plane through the point of projection.

Putting $y = 0$, in the second part of Equation (3) we get either $t = 0$ (at **O**)

And $t = \frac{2u\sin\alpha}{g} \qquad \Rightarrow T = \frac{2u\sin\alpha}{g}$

■ Greatest Height

This is also obtained either by finding by differentiation, the, maximum value of y from the second part of Equation (3) or by the fact that at the greatest height the vertical component of the velocity must vanish, i.e. from the second part of Equation (2)

$$\frac{dy}{dt} = u\sin\alpha - gt = 0 \qquad \Rightarrow t = \frac{u\sin\alpha}{g}$$

Substituting this in Equation (3) and simplifying we get

$$Y = \frac{u^2\sin^2\alpha}{2g}$$

■ Horizontal Range

The range $R = \mathrm{OB}$, on the horizontal plane through the point of projection the horizontal distance described by the particle in the time of flight T.

$$R = u\cos\alpha.T = u\cos\alpha\frac{2u\sin\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

R can also be obtained by putting $y = 0$ in Equation (4).

Since, $R = \frac{u^2 \sin 2\alpha}{g}$ so R can be obtained by two values of projected angles

because $\sin 2\alpha = \sin(\pi - 2\alpha) = \sin 2\left(\frac{\pi}{2} - \alpha\right) \quad (\alpha, \frac{\pi}{2} - \alpha)$

■ Maximum Horizontal Range

The range R is maximum when $\sin 2\alpha = 1$, i.e., when $\alpha = \frac{\pi}{4}$ Or $\alpha = 45^O$

therefore, the maximum range $R_{\max} = \frac{u^2}{g}$.

For a given velocity of projection, the horizontal range is the greatest when the angle of projection is 45°.

■ Range on an Inclined Plane

Let a particle be projected from a point O on a plane of inclination β, in the vertical plane through OP, the line of greatest slope of the inclined plane.

Let the velocity of projection be u at an elevation α to the horizontal. The equation to the path of the particle is

$$y = x \tan\alpha - \frac{1}{2}\frac{gx^2}{u^2 \cos^2\alpha} \qquad (10)$$

If the particle strikes the inclined Q plane at the point P, the distance, OP is called the range on the inclined plane. If OP$= R$ then the co-ordinates of P $(R\cos\beta, R\sin\beta)$ must satisfy Equation (10).

$$R\sin\beta = R\cos\beta\tan\alpha - \frac{1}{2}\frac{gr^2\cos^2\beta}{u^2\cos^2\alpha}$$

Then the range r

$$R = \frac{2u^2}{g}.\frac{(\cos\beta\tan\alpha - \sin\beta)\cos^2\alpha}{\cos^2\beta} = \frac{2u^2}{g}.\frac{\sin(\alpha - \beta)\cos\alpha}{\cos^2\beta}$$

$$= \frac{u^2}{g\cos^2\beta}\ \sin(2\alpha - \beta) - \sin\beta$$

The range down the plane may be obtained by putting $-\beta$ for β in this case the slope of the inclined plane is downwards.

■ Maximum Range on an Inclined Plane

u and β being known, the range varies with α, and it will be maximum when $\sin(2\alpha - \beta)$ is maximum. When $2\alpha - \beta = \frac{\pi}{2}$ Or $\alpha - \beta = \frac{\pi}{2} - \alpha$

Hence for maximum range, the direction of projection must bisect the angle between the vertical and the inclined plane. If OT be the direction of projection, then OT is tangent to the path at O, and the vertical through is perpendicular to the directrix. OT being equally inclined to OP and the vertical, the focus to the path must, therefore, lie on the line OP of the inclined plane, i.e., in the case of maximum range the focus lies in the range. The value of the maximum range is

$$R_{\max} = \frac{u^2(1 - \sin\beta)}{g\cos^2\beta}$$

$$= \frac{u^2(1 - \sin\beta)}{g(1 - \sin^2\beta)}$$

$$= \frac{u^2\,\cancel{(1 - \sin\beta)}}{g(1 + \sin\beta)\cancel{(1 - \sin\beta)}}$$

$$= \frac{u^2}{g(1 + \sin\beta)}$$

■ Illustrative Examples ■

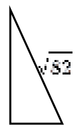
Example

If the maximum height for a projectile is **900 ft** and the horizontal range is **400 ft**. Find the velocity and its direction.

Solution

Since the maximum height and horizontal range are given by formulas

$$Y = \frac{u^2 \sin^2 \alpha}{2g}, \qquad R = \frac{u^2 \sin 2\alpha}{g}$$



Then using given values we get

$$900 = \frac{u^2 \sin^2 \alpha}{2g}, \qquad 400 = \frac{2u^2 \sin\alpha \cos\alpha}{g}$$

Then by dividing these two equations

$$\frac{9}{4} = \frac{u^2 \sin^2 \alpha}{2g} \Big/ \frac{2u^2 \sin\alpha\cos\alpha}{g} \Rightarrow \frac{9}{4} = \frac{\tan\alpha}{4} \quad \therefore \alpha = \tan^{-1} \; 9$$

which gives the angle of projection and the magnitude of the velocity of projection by using first equation

$$900 = \frac{u^2}{2g} \times \frac{81}{82} \Rightarrow u^2 = \frac{1800 \times 82 \times 32.2}{81} \quad \text{Or} \quad u \cong 242.23 \quad (g = 32.2 \ \text{ft}\,\text{sec}^{-2})$$

Example

If the ratio between the magnitude of the velocity at maximum height and a height equals half of maximum height is $\sqrt{\frac{6}{7}}$. Show that the angle of projection is 30^0.

Example

As it is obtained that $y = (u\sin\alpha)\, t - \frac{1}{2}gt^2$

Let the point A be the maximum height and hence $Y_A = \frac{u^2\sin^2\alpha}{2g}$

And **B** be the point where its height equals half of maximum height i.e.,

$$Y_B = \frac{1}{2}Y_A = \frac{u^2\sin^2\alpha}{4g}$$

The time spent from the projection of the particle reach point **B** is given by

$$\frac{u^2\sin^2\alpha}{4g} = (u\sin\alpha)\, t - \frac{1}{2}gt^2$$

Rewrite this equation again as (multiply by $4g$)

$$2(gt)^2 - 4(gt)u\sin\alpha + u^2\sin^2\alpha = 0 \qquad \Rightarrow gt = \left(1 - \frac{1}{\sqrt{2}}\right)u\sin\alpha$$

The components of velocity at point B are

$$\dot{x}_B = u\cos\alpha, \quad \dot{y}_B = u\sin\alpha - gt = u\sin\alpha - \left(1 - \frac{1}{\sqrt{2}}\right)u\sin\alpha = \frac{1}{\sqrt{2}}u\sin\alpha$$

The resultant of the velocity at point B

$$v_B = \sqrt{\dot{x}_B^2 + \dot{y}_B^2} = \sqrt{(u\cos\alpha)^2 + \left(\frac{1}{\sqrt{2}}u\sin\alpha\right)^2} = \frac{u}{\sqrt{2}}\sqrt{1+\cos^2\alpha}$$

Since at the maximum height $\dot{x}_A = u\cos\alpha, \quad \dot{y}_A = 0$ then

$$v_A = \sqrt{\dot{x}_A^2 + \dot{y}_A^2} = u\cos\alpha$$

But as given $\frac{v_A}{v_B} = \sqrt{\frac{6}{7}}$ therefore,

$$\Rightarrow \frac{\sqrt{2}u\cos\alpha}{u\sqrt{1+\cos^2\alpha}} = \sqrt{\frac{6}{7}}$$
$$\Rightarrow \frac{\cos\alpha}{\sqrt{1+\cos^2\alpha}} = \sqrt{\frac{3}{7}}$$
$$\Rightarrow \frac{\cos^2\alpha}{1+\cos^2\alpha} = \frac{3}{7}$$

$7\cos^2\alpha = 3 + 3\cos^2\alpha \qquad \Rightarrow 4\cos^2\alpha = 3$

$$\Rightarrow \cos\alpha = \frac{\sqrt{3}}{2} \quad \text{Or} \quad \alpha = 30^0$$

Example

A particle is projected with a velocity of 24 ft sec^{-1} at an angle of elevation 60. Find (a) the equation to its path, (b) the greatest height attained, (c) the time for the range, (d) the length of the range.,

Solution

Since $u = 24$ and $\alpha = 60^o$, $\quad g \simeq 32.2$ ftsec^{-2}

(a) the equation to the path is

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}, \qquad \text{therefore} \qquad y = \sqrt{3}\,x - \frac{1}{9}x^2$$

(b) The maximum height $= \dfrac{u^2\sin^2\alpha}{2g} = \dfrac{24\times 24}{2\times 32.2}\times\dfrac{3}{4} \simeq 6.71$ ft

(c) The time for the range $= \dfrac{2u\sin\alpha}{g} = \dfrac{2\times 24}{32.2}\times\dfrac{\sqrt{3}}{2} \simeq 1.29$ sec

(d) the length of the ran $= u\cos\alpha\; T = 24\times\dfrac{1}{2}\times\dfrac{3\sqrt{3}}{4} \simeq 15.49$ ft

Example

Find the maximum horizontal range of cricket ball projected with a velocity of 48 ft. per sec. If the ball is to have a range of $36\sqrt{3}$ ft., find the least angle of projection and the least time taken (let $g \simeq 32$ ftsec^{-2}).

Solution

We have $u = 48$ and $\alpha = 45^o$, $\quad g \simeq 32$ ftsec^{-2}

$$R_{\max} = \frac{u^2}{g} \qquad \Rightarrow R_{\max} = \frac{48\times 48}{32} \simeq 72 \text{ ft}$$

If $R = \dfrac{u^2\sin 2\alpha}{g} = 36\sqrt{3} \qquad \Rightarrow \sin 2\alpha = \dfrac{36\sqrt{3}\times 32}{48\times 48} = \dfrac{\sqrt{3}}{2}$

Then $2\alpha = 60^o$ or 120^o that is $\alpha = 30^o$ or 60^o

Thus, the least angle of projection $\alpha = 30^o$

and the least time taken $= \dfrac{2u\sin\alpha}{g} = \dfrac{2\times 48}{32}\times\dfrac{1}{2} \simeq 1.5$ sec

Example

A ball is projected from a point on the ground distant a from the foot of a vertical wall of height b, the angle of projection being α to the horizontal. If the ball just clears the wall prove that the greatest height reached is

$$\frac{a^2\tan^2\alpha}{4(a\tan\alpha - b)}$$

Solution

Let u be the velocity of projection, then since the ball passes through the top of the wall, a point (a,b), we have

$$b = a\tan\alpha - \frac{ga^2}{2u^2\cos^2\alpha} \qquad \text{Or} \qquad a\tan\alpha - b = \frac{ga^2}{2u^2\cos^2\alpha}$$

$$\therefore\ u^2 = \frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha}$$

Now the greatest height Y reached by the ball

$$\begin{aligned} Y &= \frac{u^2\sin^2\alpha}{2g} \\ &= \frac{\sin^2\alpha}{2g}\,\frac{ga^2}{2(a\tan\alpha - b)\cos^2\alpha} \\ &= \frac{a^2\tan^2\alpha}{4(a\tan\alpha - b)} \end{aligned}$$

Example

If T be the time taken to reach the other common point A of its path and T' the time to reach the horizontal plane through the point of projection. Find the height of the point A.

Solution

Since $x = u\cos\alpha\ t$ and the time of flight is $T + T'$ also $R = \dfrac{u^2 \sin 2\alpha}{g}$

Hence $u\cos\alpha\ (T + T') = \dfrac{2u^2 \cos\alpha \sin\alpha}{g} \quad \Rightarrow u\sin\alpha = \dfrac{1}{2}g(T + T')$

$$\because y\big|_A = u\sin\alpha\ T - \frac{1}{2}gT^2 \quad \Rightarrow \quad y\big|_A = \frac{1}{2}gT(T + T') - \frac{1}{2}gT^2 = \frac{1}{2}gTT'$$

Example

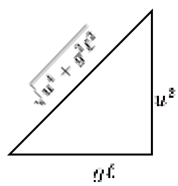
A particle is projected with a velocity u so as just to pass over the highest possible post at a horizontal distance ℓ from the point of projection $\mathbf{O}$. Prove that the greatest height above O attained by the particle in its flight is $\dfrac{u^6}{2g(u^4 + g^2\ell^2)}$.

Solution

Taking θ as the angle of projection and substituting ℓ for x the equation to the path, we have

$$y = \ell\tan\theta - \frac{g\ell^2}{2u^2\cos^2\theta} = \ell\tan\theta - \frac{g\ell^2}{2u^2}(1 + \tan^2\theta)$$

$$\therefore \frac{dy}{d\theta} = \ell\sec^2\theta - \frac{g\ell^2}{u^2}\tan\theta\sec^2\theta = \ell\sec^2\theta\left(1 - \frac{gd^2}{u^2}\tan\theta\right)$$



$$\frac{dy}{d\theta} = 0 \Rightarrow \tan\theta = \frac{u^2}{g\ell} \qquad \text{or} \qquad \sin^2\theta = \frac{u^4}{u^4 + g^2\ell^2}$$

y being positive and its minimum value being zero, the value of θ given in previous equation gives the maximum value of y. Now the greatest height attained by the particle

$$Y = \frac{u^2\sin^2\theta}{2g} = \frac{u^2}{2g}\left(\frac{u^4}{u^4 + g^2\ell^2}\right) = \frac{u^6}{2g\ \ u^4 + g^2\ell^2}$$

Example

Two particles are projected from the same point in the same vertical plane with equal velocities. If t, t' be the times taken to reach the common point of their paths and T, T' the times to the highest point, show that $tT + t'T'$ is independent of the directions of projection

Solution

Let α, β be the directions of projection

$$T = \frac{u\sin\alpha}{g}, \quad T' = \frac{u\sin\beta}{g}$$

If x is the horizontal distance of the common point, then

$$x = u\cos\alpha\ t,\ x = u\cos\beta\ t'$$

$$\therefore tT + t'T' = \frac{x}{u\cos\alpha}\frac{u\sin\alpha}{g} + \frac{x}{u\cos\beta}\ \frac{u\sin\beta}{g} = \frac{x}{g}(\tan\alpha + \tan\beta) \qquad (*)$$

Now the equations of the two- paths arc

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2\sec^2\alpha}{u^2}, \quad y = x\tan\beta - \frac{1}{2}\frac{gx^2\sec^2\beta}{u^2}$$

Subtracting we have,

$$x(\tan\alpha - \tan\beta) = \frac{1}{2}\frac{gx^2}{u^2}\ \sec^2\alpha - \sec^2\beta \ = \frac{1}{2}\frac{gx^2}{u^2}\ \tan^2\alpha - \tan^2\beta$$

$$\frac{x}{g}(\tan\alpha + \tan\beta) = \frac{2u^2}{g^2}$$

Hence from Equation (*)

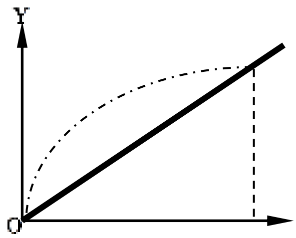
$\therefore tT + t'T' = \dfrac{2u^2}{g^2}$ which is independent of the directions of projection.

Example

A particle is projected with velocity u from a point on an inclined plate. If v_1 be its velocity on striking the plane when the range up the plane is maximum and v_2 the velocity on striking the plane when the range down the plane is maximum, prove that $u^2 = v_1 v_2$

Solution

Let R be the maximum range up the plane and α be the inclination of the plane, then



$$R = \frac{u^2}{g(1+\sin\alpha)}, \text{ and } v_1^2 = u^2 - 2gy = u^2 - 2gR\sin\alpha$$

$$\therefore\ v_1^2 = u^2 - 2g\sin\alpha \times \frac{u^2}{g(1+\sin\alpha)} = u^2 \times \frac{1-\sin\alpha}{1+\sin\alpha}$$

Similarly, by changing the sign of α, we have

$$\therefore\ v_2^2 = u^2 \times \frac{1+\sin\alpha}{1-\sin\alpha} \quad \text{Hence} \quad u^4 = v_1^2 v_2^2 \quad \textbf{Or} \quad u^2 = v_1 v_2$$

Example

A particle is projected and it paths through the two points $(12,12)$ **and** $(36,12)$

Find its velocity and the direction of projection.

Solution

The trajectory or path equation is $$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

The two points $(12,12)$ **and** $(36,12)$ lies on the path so that

With regard the point $(36,12)$ $$12 = 36\tan\alpha - \frac{g(36)^2}{2u^2\cos^2\alpha}$$

With regard the point $(12,12)$ $$12 = 12\tan\alpha - \frac{g(12)^2}{2u^2\cos^2\alpha}$$

By multiplying the second equation by 9 then subtracting, we have

$$96 = 72\tan\alpha \qquad \Rightarrow \tan\alpha = \frac{96}{72} = \frac{4}{3}$$

which gives the direction of velocity of projection, and to obtain the magnitude of the projection velocity, from first equation

$$\Rightarrow 12 = 36\left(\frac{4}{3}\right) - \frac{g(36)^2}{2u^2\left(\frac{3}{5}\right)^2} \qquad \Rightarrow \frac{g(36)^2}{2u^2\left(\frac{3}{5}\right)^2} = 36$$

$$\Rightarrow u^2 = 50\,g \qquad \text{Or} \quad u = 5\sqrt{2g}$$

Example

A particle is projected and it paths through the two points (a,b) **and** (b,a) where (a,b) **and** (b,a) Prove that the range is given by $\dfrac{a^2+ab+b^2}{a+b}$.

Solution

The trajectory or path equation is $\qquad y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

The two points (a,b) **and** (b,a) lies on the path so that

With regard the point (a,b) $\qquad a = b\tan\alpha - \dfrac{gb^2}{2u^2\cos^2\alpha}$

With regard the point (b,a) $\qquad b = a\tan\alpha - \dfrac{ga^2}{2u^2\cos^2\alpha}$

By multiplying the first equation by a and the second by b then subtracting, we have

$$\underbrace{a^2-b^2}_{(a+b)\cancel{(a-b)}} = \frac{gab}{2u^2\cos^2\alpha}\cancel{(a-b)} \qquad \Rightarrow \quad a+b = \frac{gab}{2u^2\cos^2\alpha}$$

$$\text{Or} \qquad \frac{ab}{a+b} = \frac{2u^2\cos^2\alpha}{g}$$

Once again by multiplying the first equation by a^2 and the second by b^2 then subtracting, we have

$$\underbrace{a^3 - b^3}_{(a^2+ab+b^2)\cancel{(a-b)}} = ab\cancel{(a-b)}\tan\alpha \qquad \Rightarrow ab\tan\alpha = a^2 + ab + b^2$$

Since the range is given by $R = \dfrac{u^2 \sin 2\alpha}{g}$ therefore,

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{2u^2 \cos\alpha \sin\alpha}{g}$$

$$= \underbrace{\frac{2u^2 \cos^2\alpha}{g}}_{ab/(a+b)} \tan\alpha = \frac{ab}{a+b}\tan\alpha = \frac{a^2 + ab + b^2}{a+b}$$

$$\therefore R = \frac{a^2 + ab + b^2}{a+b}$$

Example

A particle is projected to reach a certain object located in the same horizontal plate of projection point, when it projected with angle α it falls down before the object by distance ℓ and when it projected with angle β it falls down after the object by distance ℓ. Find the exact angle to reach the object.

Solution

Let u be the velocity of projection and R is the exact range of the object then the range in first case is $R - \ell$ and the range in second case is $R + \ell$ therefore

$$R - \ell = \frac{u^2 \sin 2\alpha}{g} \qquad \text{and} \qquad R + \ell = \frac{u^2 \sin 2\beta}{g}$$

By addition the two equations, we get

$$\therefore 2R = \frac{u^2}{g}\ \sin 2\alpha + \sin 2\beta \qquad \Rightarrow R = \frac{u^2}{2g}\ \sin 2\alpha + \sin 2\beta$$

Now, let θ be the exact angle to reach the object so $R = \dfrac{u^2 \sin 2\theta}{g}$

By comparing (or dividing) the last two equations then

$$\Rightarrow \frac{u^2 \sin 2\theta}{g} = \frac{u^2}{2g}\ \sin 2\alpha + \sin 2\beta$$

$$\Rightarrow \sin 2\theta = \frac{\sin 2\alpha + \sin 2\beta}{2} \qquad \Rightarrow \theta = \frac{1}{2}\sin^{-1}\left(\frac{\sin 2\alpha + \sin 2\beta}{2}\right)$$

■ Projectiles with Resistance

We now proceed to include the effect of air resistance. From our earlier discussion of fluid drag, it is evident that in most practical instances of projectile motion through the Earth's atmosphere, it is the **quadratic law** of resistance that is appropriate. On the other hand, only the **linear law** of resistance gives rise to linear equations of motion and simple analytical solutions. This explains why mechanics textbooks contain extensive coverage of the linear case, even though this case is almost never appropriate in practice; the case that is appropriate cannot be solved! In the following example, we treat the linear resistance case.

Now suppose that the motion is opposed by a force proportional to the velocity. Thus if m denote the mass and v the velocity, let $m\gamma v$ denote the magnitude of the resistance. Therefore the components of the resistance parallel to horizontal and vertical axes **OX** , **OY** are

$$-m\gamma\dot{x}, \ -m\gamma\dot{y}$$

Let u denote the initial velocity in a direction making an angle α with the horizontal. The equations of motion give

$$\ddot{x} = -\gamma\dot{x} \qquad \text{and} \qquad \ddot{y} = -g - \gamma\dot{y}$$

By integrating we obtain

$$\ln\dot{x} = c_1 - \gamma t \qquad \text{and} \qquad \ln\left(\dot{y} + \frac{g}{\gamma}\right) = c_2 - \gamma t$$

since initially $x = y = 0$ and $\dot{x} = u\cos\alpha, \quad \dot{y} = u\sin\alpha$,then $c_1 = \ln u\cos\alpha$

and $c_2 = \ln\left(u\sin\alpha + \dfrac{g}{\gamma}\right)$, and hence

$$\dot{x} = u\cos\alpha \ e^{-\gamma t} \qquad \text{and} \qquad \dot{y} = \left(u\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t} - \frac{g}{\gamma}$$

Once again integrate the previous formula

$$x = -\frac{u\cos\alpha}{\gamma}\ e^{-\gamma t} + c_3 \quad \text{and} \quad y = -\frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)e^{-\gamma t} - \frac{g}{\gamma}t + c_4$$

Where, c_4, c_3 are constant, and $x = y = 0$ at $t = 0$ so that

$$c_3 = \frac{u\cos\alpha}{\gamma}, \qquad c_4 = \frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)$$

So the last equation becomes

$$x = \frac{u\cos\alpha}{\gamma}\ \ 1 - e^{-\gamma t} \quad \text{and} \quad y = \frac{1}{\gamma}\left(u\sin\alpha + \frac{g}{\gamma}\right)\ 1 - e^{-\gamma t}\ \ - \frac{g}{\gamma}t$$

- The time spent to reach the maximum height is

$$T = \frac{1}{\gamma}\ln\left(\frac{\gamma u\sin\alpha}{g} + 1\right)$$

- The maximum height is

$$y = \frac{u\sin\alpha}{\gamma} - \frac{g}{\gamma^2}\ln\left(1 + \frac{\gamma u\sin\alpha}{g}\right)$$

- The time of flight is

$$T' = \frac{1}{\gamma}\left(\frac{\gamma u\sin\alpha}{g} + 1\right)\ 1 - e^{-\gamma T'}$$

- The path equation is

$$y = \frac{g}{\gamma u\cos\alpha}\left(\frac{\gamma u\sin\alpha}{g} + 1\right)x + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x}{u\cos\alpha}\right)$$

For instance to evaluate the spent time to reach the maximum height

Since $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .....$$

this is true for $|x| < 1$, and now let $\gamma \to 0$ in formula

$$T = \frac{1}{\gamma}\ln\left(\frac{\gamma u\sin\alpha}{g} + 1\right)$$

We get

$$T = \lim_{\gamma \to 0} \frac{1}{\gamma}\left(\frac{\gamma u \sin\alpha}{g} - \frac{\gamma^2 u^2 \sin^2\alpha}{2g^2} + \frac{\gamma^3 u^3 \sin^3\alpha}{3g} + \ldots \right)$$
$$= \lim_{\gamma \to 0}\left(\frac{u \sin\alpha}{g} - \frac{\gamma u^2 \sin^2\alpha}{2g^2} + \frac{\gamma u^3 \sin^3\alpha}{3g} + \ldots \right) = \frac{u \sin\alpha}{g}$$

This result obtained before when we neglected the resistance of air.

Example

A particle of mass m is projected with initial velocity u at an angle of elevation α through a resisting medium where its resistance proportional to v and the proportional constant is μm. Prove that the direction of the velocity makes an angle α with the horizontal $\frac{1}{\mu}\ln\left(1 + \frac{\mu u}{g}(\sin\alpha + \cos\alpha)\right)$

Solution

By writing the equation of motion in **OX, OY** and then integrating and use the initial conditions as illustrated before we obtain the components of velocity of the particle at any instance

$$\dot{x} = u\cos\alpha \; e^{-\mu t} \qquad \text{and} \qquad \dot{y} = \left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu}$$

Since the angle of projection is α and the angle that the direction of velocity makes with the horizontal axis decreases until vanish at the highest position then it reverse to be α again downwards after time t which determines from

$$\tan\; -\alpha \; = \frac{\dot{y}}{\dot{x}} = \frac{\left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu}}{u\cos\alpha \; e^{-\mu t}} = -\tan\;\alpha$$

That is

$$\left(u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} - \frac{g}{\mu} = -u\sin\alpha \; e^{-\mu t} \;\Rightarrow\; \left(2u\sin\alpha + \frac{g}{\mu}\right)e^{-\mu t} = \frac{g}{\mu}$$
$$\left(\frac{2\mu u\sin\alpha}{g} + 1\right) = e^{\mu t} \qquad \Rightarrow t = \frac{1}{\mu}\ln\left(\frac{2\mu u\sin\alpha}{g} + 1\right)$$

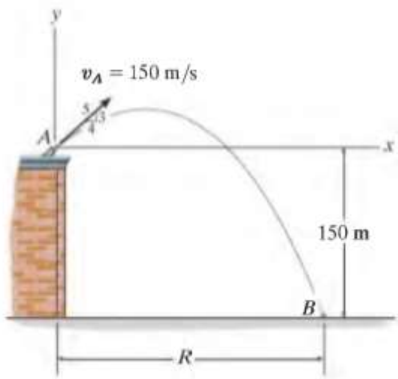
PROBLEMS

❒ A body, projected with a velocity of 120 ft sec^{-1} just clears a vertical wall 72 ft high and 360 ft. distant, find the two possible angles of projection and the corresponding horizontal ranges.

❒ A particle is projected so as just to clear a wall of height b at a horizontal distance a, and to have a range c from the point of projection, show that the velocity of projection V is given by

$$\frac{2V^2}{g} = \frac{a^2(c-a)^2 + b^2c^2}{ab(c-a)} .$$

❒ A projectile is fired with an initial velocity of V_A = 150 m/s off the roof of the building. Determine the range R where it strikes the ground at B.



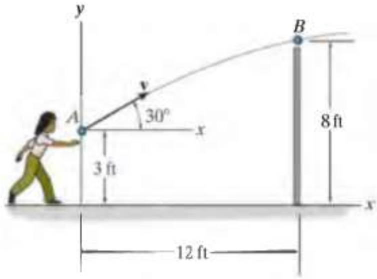
❒ A stone is projected with velocity V and elevation from a point O on level ground so as to hit a mark P on a wall whose distance from O is a, the height of P above the ground being 6. Prove that

$$2V^2(a\sin\theta\cos\theta - b\cos^2\theta) = ga^2 .$$

❒ A particle in projected with a velocity of 120 ft. per sec. at an angle of 60 with the horizontal from the foot of an inclined plane of inclination 30. Find the time of flight and the range on the inclined plane.

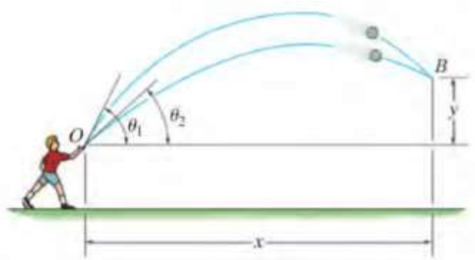
❒ A particle is projected from a point on a plane of inclination β with velocity u. Show that the maximum range down the plane is

$$\frac{u^2}{g}\left(\frac{1+\sin\beta}{\cos^2\beta}\right).$$



❒ A ball is thrown from A. If it is required to clear the wall at B, determine the minimum magnitude of its initial velocity V_A.

❒ A boy throws a ball at 0 in the air with a speed Vo at an angle θ_1. If he then throws another ball with the same speed v_0 at an angle $\theta_2 < \theta_1$ determine the time between the throws so that the balls collide in midair at B.



SIMPLE HARMONIC MOTION

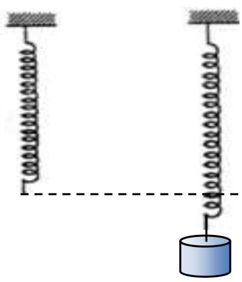
Oscillations are a particularly important part of mechanics and indeed of physics as a whole. This is because of their widespread occurrence and the practical importance of oscillation problems.

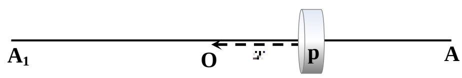
Most engineering materials are nearly elastic under working conditions. And, of course, all real things have mass. These ingredients, elasticity and mass, are what make vibration possible. Even structures which are fairly rigid will vibrate if encouraged to do so by the shaking of a rotating motor, the rough rolling of a truck, or the ground motion of an earthquake. The vibrations of a moving structure can also excite oscillations in flowing air which can in turn excite the structure further. This mutual excitement of fluids and solids is the cause of the vibrations in a clarinet reed, and may have been the source of the wild oscillations in the famous collapse of the Tacoma Narrows bridge. Mechanical vibrations are not only the source of most music but also of most annoying sounds. They are the main function of a vibrating massager, and the main defect of a squeaking hinge. Mechanical vibrations in pendulum or quartz crystals are used to measure time. Vibrations can cause a machine to go out of control, or a building to collapse. So, the study of vibrations, for better or for worse, is not surprisingly one of the most common applications of dynamics.

When an engineer attempts to understand the oscillatory motion of a machine or structure, she undertakes a vibration analysis. A vibration analysis is a study of the motions that are associated with vibrations. Study of motion is what dynamics is all about, so vibration analysis is just a part of dynamics. A vibration analysis could mean the making of a dynamical model of the structure one is studying, writing equations of motion using the momentum balance or energy equations and then looking at the solution of these equations. But, in practice, the motions associated with vibrations have features which are common to a wide class of structures and machines. For this reason, a special vocabulary and special methods of approach have been developed for vibration analysis. For example, one can usefully discuss resonance, normal modes, and frequency response, concepts which we will soon discuss, without ever writing down any equations of motion. We will first approach these concepts within the framework of the differential equations of motion and their solutions. But after the concepts have been learned, we can use them without necessarily referring directly to the governing differential equations.

■ **Definition**

A particle is said to execute Simple Harmonic Motion if it moves such that its acceleration is always directed towards a fixed point, and is proportional to the distance of the particle from the faced point.





The expressions for velocity and position of the particle at any instant are obtained as follows:

Suppose O be the fixed point in the line A_1OA and let P denote the particle after time t from moving with a velocity v in the positive direction from O to A. Let OP $= x$, then the acceleration is kx where k is a constant. Since the acceleration is in the direction opposite to that in which x increases, the equation of motion of the particle is given as

$$m\frac{d^2x}{dt^2} = -kx$$

Rearranging this equation, we get one of the most famous and useful differential equations of all time:

$$\ddot{x} + \frac{k}{m}x = 0$$

This equation appears in many contexts both in and out of dynamics. In non-mechanical contexts the variable x and the parameter combination k/m are replaced by other physical quantities. In an electrical circuit, for example, x might represent a voltage and the term corresponding to k/m might be 1/LC, where C is a capacitance and L an inductance. But even in dynamics the equation appears with other physical quantities besides k/m multiplying the x, and x itself could represent rotation, say, instead of displacement. In order to avoid being specific about the physical system being modeled, the harmonic oscillator equation is often written as

$$\ddot{x} + w^2x \ = 0$$

The constant in front of the x is called w^2 instead of just, say, w, for two reasons:

(i) This convention shows that w^2 is positive,

(ii) In the solution we need the square root of this coefficient, so it is convenient to have $\sqrt{w^2} = w$.

For the spring-block system, w^2 is k/m and in other problems w^2 is some other combination of physical quantities.

■ Solution of harmonic oscillator differential equation

$\because \dfrac{d^2x}{dt^2} = -w^2 x$ or $v\dfrac{dv}{dx} = -w^2 x \quad \Rightarrow vdv = -w^2 x dx$

Integrating previous equation, we have

$$\frac{1}{2}v^2 = c_1 - \frac{1}{2}w^2 x^2$$

where c_1 is an integral constant. As P is supposed to be moving in the direction OA and as the acceleration is given to be taking place in the opposite direction, the particle P must come to rest at some point in OA say at A, i.e., suppose $v = 0$ where $x = a$, so that

$$0 = c_1 - \frac{1}{2}w^2 a^2 \qquad \Rightarrow c_1 = \frac{1}{2}w^2 a^2$$

Therefore

$v^2 = w^2 a^2 - w^2 x^2 = w^2(a^2 - x^2)$ or $v = \pm w\sqrt{a^2 - x^2}$

This equation gives the value of the velocity v for any displacement x

As P is moving in the positive direction $v = w\sqrt{a^2 - x^2}$

$$\frac{dx}{dt} = w\sqrt{a^2 - x^2} \qquad \frac{dx}{\sqrt{a^2 - x^2}} = w\,dt$$

By integrating

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = wdt \qquad \Rightarrow \sin^{-1}\left(\frac{x}{a}\right) = wt + \epsilon \quad \text{Or}$$

$$x = a\sin(wt + \epsilon)$$

where ϵ is integration constant to be determined from the initial conditions. If t is measured from the instant when P is at O, i.e., if $x = 0$ when $t = 0$, then $\epsilon = 0$.

➢ Note 1 Velocity in terms of time t can be obtained by differentiating any of these equations involving x and t.

➢ Note 2 When the particle is on the left-hand side of O, the equation of motion is $\ddot{x} = -w^2 x$ acceleration in the direction of $P_1A = w^2\, OP_1$

$= w^2(-x) = -w^2x$ Hence the same equation that holds on the right-hand side of O, holds also on the left hand side.

The Equation $\pm w\sqrt{a^2 - x^2}$ gives the velocity of P in terms of its distance from O. Initially, when $x = 0$ at the point O, the velocity is maximum and equal to wa. As As the particle proceeds towards A, the acceleration being towards O, the velocity goes on decreasing as x increases. At A where $x = a$, it vanishes and the particle is, for an instant, at rest. Then owing to the acceleration towards O the particle moves in the negative direction with a velocity which increases numerically as x decreases and is the greatest at O where it is $-wa$. Due to this velocity, the particle proceeds further to the negative side of O, the velocity remaining negative and decreasing gradually in magnitude till the particle comes to rest at A_1 where $x = -a$. The acceleration being towards O, the particle then starts and moves towards O with a positive velocity which increases gradually till it is again maximum at O. The same motion is repeated again and again and the particle goes on oscillating indefinitely between A and A_1, the two positions of momentary rest.

The motion of the particle is oscillatory. All oscillatory motions are, however, not necessarily simple harmonic. In fact, simple harmonic motion is the simplest and most important case of oscillatory motion which occurs in nature and it is always dominated by the differential equation

$$\frac{d^2x}{dt^2} = -w^2x$$

The distance OA or OA_1 i.e., the distance of the center from one of the positions of rest is called the ***Amplitude***.

■ The Periodic time of Motion

The equation $x = a\sin wt$ gives the time form in terms of x, the distance of the particle measured from O. Since

$$x = a\sin wt = a\sin(wt + 2\pi) = a\sin\left(wt + \frac{2\pi}{w}\right)$$

And $\dfrac{dx}{dt} = aw\cos wt = aw\cos(wt + 2\pi) = aw\cos\left(wt + \dfrac{2\pi}{w}\right)$

the particle has the same position, velocity and direction after time $t + \dfrac{2\pi}{w}$, $t + \dfrac{4\pi}{w}$ etc., as it had at the time t, i.e., the particle has a periodic motion, its periodic time τ being $\dfrac{2\pi}{w}$

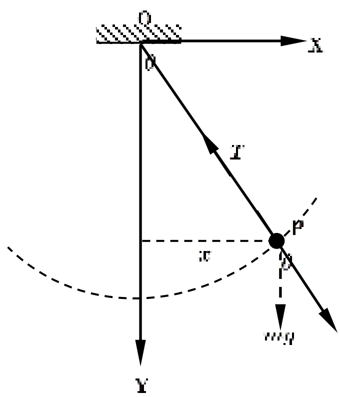
■ The Frequency

The frequency of SHM is the number of complete oscillations in one second, so that if n denotes the frequency and τ the periodic time of the motion,

$$n\tau = 1 \quad \Rightarrow n = \frac{1}{\tau} = \frac{w}{2\pi}$$

■ Simple Pendulum

If a heavy particle is tied to one end of a light inextensible string with length and the other end of which is fixed, and oscillates in a vertical circle, we have what is called a Simple Pendulum. We now obtain the time of oscillation of such a pendulum when it is allowed to oscillate through a small angle only. Let O be the fixed point, A the lowest position of the particle, and P any position such that $\angle$**YOP**$=\theta$



The equations of motion in horizontal direction is (resolve the tension)

$$m\ddot{x} = -T\sin\theta \qquad \textbf{and} \qquad T\cos\theta = mg$$

Here we suppose that the motion of mass m in X direction only. Now, when the angle θ is small enough so the approximations $\cos\theta \cong 1$ and $\sin\theta \cong \theta$ can be applied and the equation of motion, $m\ddot{x} = -T\sin\theta$ becomes

$$m\ddot{x} = -mg\frac{x}{L} \qquad \textbf{or} \qquad \ddot{x} = -\frac{g}{L}x$$

which is similar to $\ddot{x} = -w^2 x$ with $w^2 = \dfrac{g}{L}$ **Or** $w = \sqrt{\dfrac{g}{L}}$

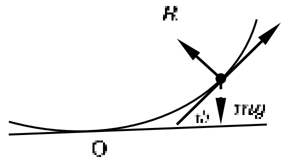
So, a simple pendulum moves like a SHM with periodic time of motion equals

$$2\pi\sqrt{\frac{L}{g}}$$

■ The Cycloid Pendulum

We have illustrated that the motion of a simple pendulum is simple harmonic motion only when the angle of swing is so small that $\sin\theta$ is very nearly equal to θ and the amplitude to the motion is so small that it may be treated as infinitesimal. If, however, the amplitude of motion is not small and the particle supposed to be constrained to move under gravity, along the arc of a smooth cycloid in a vertical plane, the equation of motion of the particle along the tangent to the curve is

$$m\frac{d^2S}{dt^2} = -mg\sin\psi \qquad (1)$$



where ψ is the angle which the tangent to the curve makes with the horizontal and S the length of its arc measured from the vertex, the cycloid being placed with its vertex downwards and axis vertical. We know by the Calculus that the intrinsic equation of the cycloid is

$$S = 4a\sin\psi \qquad (2)$$

Note S being measured from the vertex where $\psi = 0$, and a being the radius of the generating circle. From equations (1) and (2), we have

$$\frac{d^2S}{dt^2} = -\frac{g}{4a}S \tag{3}$$

this formula shows that the motion represents simple harmonic no matter how great the amplitude. The time of a complete oscillation is given as $2\pi\sqrt{4a / g}$ which is constant for oscillations, small or large. Thus, if a particle is constrained to move along a smooth cycloid curve, its period of motion is absolutely independent of the amplitude. (This is an answer to the question which interested the mathematicians of the 18th century in what curve should that the bob of a pendulum swing in order that the period of oscillation may be absolutely independent of the amplitude?)

The oscillations on a cycloid are called isochronous because the period is the same for large or small oscillations. This important property of a cycloid finds its application in the formation of clocks. A cycloid pendulum may be constructed by causing the cord of the pendulum to wind and unwind itself on the evaluate of the path.

In order to find the pressure of the curve on the particle, we write its equation of motion in the direction of the normal at the point, namely

$$m\frac{v^2}{\rho} = R - mg\cos\psi$$

where ρ is the radius of curvature of the curve, R the normal pressure and v the velocity of the particle obtained from equation (3) by integration.

Note (i). The students acquainted with elements of differential equation will note that

$$S = A\cos wt + B\sin wt \text{ where } w = g / 4a$$

is the most general solution of differential equation (3).

Note a. Since $S = 4a\sin\psi = 4a\frac{dy}{ds} \Rightarrow S^2 = 8ay$, y and S being measured from the vertex of the cycloid

■ Hooke's Law

The ' extension' of a stretched elastic string means the ratio of the increment in length to the unstretched length. Thus if ℓ, is the natural or unstretched length and the stretched length is ℓ' then the extension is $(\ell' - \ell)\ /\ \ell$.

Hooke's Law is that the *tension of the string is proportional to the extension.* If T denote the tension and we state the law in the form

$$T = \lambda\left(\frac{\ell' - \ell}{\ell}\right)$$

where λ is called the modulus of elasticity of the string.
The extension or compression of a spiral spring follows the same law, but in this' case the length is measured along the axis of the helix and not along the wire that forms the spring; and when the spring is extended or compressed the force exerted by the spring is a tension or a thrust in the direction of the axis. The formula above may be used for compression as well as extension provided we regard a negative tension as a thrust. For when the spring is compressed the length ℓ' is less than the natural length ℓ, so that the formula would give a negative tension, i.e. a thrust of magnitude $T = \lambda\left(\frac{\ell - \ell'}{\ell}\right)$.

■ Motion of a Particle Attached to an Elastic String

Elastic Strings. If an elastic string or wire or a spiral spring is fixed at one point and pulled within limits at the other, it is found to increase in length, the extension being proportional to the tension of the string.

If different wires of the same material are considered, the extension is directly proportional to the product of the tension and the natural length and inversely as the area A of the cross-section. Thus if x denotes the extension, l the natural length and T the tension (in absolute units), then,

$$x = \frac{T\ell}{\lambda A} \quad \text{Or} \quad T = \lambda\frac{Ax}{\ell}$$

where λ is a constant depending on the material of the wire. If we take A =unit area, we have $T=\lambda\frac{x}{\ell}$

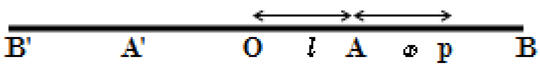
If ℓ is the natural length of an elastic string and l' the stretched length, then

$$T=\frac{\lambda}{\ell}(\ell'-\ell)$$

i.e., tension of an elastic string or a spring is proportional to the extension of the spring beyond its natural length. This is ***Hooke's law*** of elastic string and A is called the ***Modulus of Elasticity***.

When $x=\ell,\ T=\lambda$, so that λ for a string of unit cross-section is equal to the amount of force which would stretch it to twice its natural length.

Let one end of an elastic string be fixed to a point O on a smooth horizontal table and let OA=ℓ be its natural length.



If a particle of mass m is attached to the other end and if the particle is displaced along the line OA, a distance AB=b and P be position of the particle at any subsequent time so that AP=x, then the tension in the string is $T=\lambda\frac{x}{\ell}$; which acts in the direction PA and is directed towards A. The tension of the string being the only force which tends to move the particle, its equation of motion is

$$m\frac{d^2x}{dt^2}=-T=-\lambda\frac{x}{\ell} \quad \text{Or} \quad \frac{d^2x}{dt^2}=-\frac{\lambda}{\ell m}x\equiv-w^2x$$

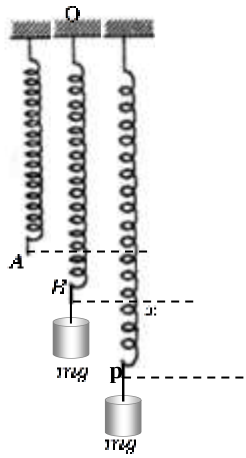
which shows that the motion about A is simple harmonic, the constant w equals $\sqrt{\frac{\lambda}{\ell m}}$. The periodic time of oscillation is $2\pi\sqrt{\frac{\ell m}{\lambda}}$

The particle will further move through to a point B' at an equal distance on the other side of and then back again and so on. The distance from A to A'

(OA= OA') and back to A is moved with the velocity which the particle acquires at A. The string being slack this velocity remains the same throughout this part. The periodic time obtained above refers to the time which the particle takes in moving from B to A, from A' to B' and then from B' to A' and from A to B. This is the only part where motion is simple harmonic.

■ Vertical Elastic string

Suppose that a particle of mass m is suspended from a fixed point by a string (or spring) OA of a natural length ℓ. Let OB be the length of the string when the mass hangs in equilibrium, then AB(=e), the extension of the string is given by



$$mg = \lambda \frac{AB}{\ell} = \lambda \frac{e}{\ell}$$

Now if the particle is displaced vertically from B it will oscillate in a vertical line about B and it will execute SHM which can be proved as follows:

Let P be the displaced position of the particle during its motion and let PB$= x$, then the tension, T, of the string in this position is given by

$T = \frac{\lambda}{\ell}(BA + x) = mg + \frac{\lambda}{\ell}x$ (from previous equation)

Then the resultant force acting on the particle in the direction BP

$$= mg - T = mg - (mg + \frac{\lambda}{\ell}x) = -\frac{\lambda}{\ell}x$$

Hence the equation of motion of the particle

$$m\frac{d^2x}{dt^2} = -\lambda\frac{x}{\ell} \quad \textbf{Or} \quad \frac{d^2x}{dt^2} = -\frac{\lambda}{\ell m}x \equiv -w^2x$$

which shows that the particle moves with simple harmonic motion having B, the position of equilibrium, as the center of oscillation. The period of motion is

$$2\pi\sqrt{\frac{\ell m}{\lambda}} = 2\pi\sqrt{\frac{e}{g}}$$

e being the extension of the string in the equilibrium position of the particle. By Equation $mg = \lambda \frac{AB}{\ell} = \lambda \frac{e}{\ell}$, e being proportional to $\frac{m}{\lambda}$, the A period depends on the weight which is hung on, and on the stiffness of the string or spring to which the particle is attached.

Note (i). At B, the ultimate position of equilibrium of the particle, the forces acting on it, viz., its weight and the tension of the string, balance. In all problems of this typo the position of this point must be obtained first.

Note (ii). The particle moves with Simple Harmonic motion only so long as the particle is below A, i.e., so long as the string remains stretched. If the particle rises above A (it will do so, for example when it is pulled down below, B, a distance greater than AB) the string will become slack and the part of the motion above A will be simply free vertical motion under gravity.

■ ILLUSTRATIVE EXAMPLES ■

Example

A point moves along a straight line such that its distance given by $x = 3\cos 2t + 4\sin 2t$. Prove that the motion of the point is simple harmonic motion and find its periodic time and amplitude.

Solution

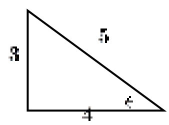
Since the position of the point is given by $x = 3\cos 2t + 4\sin 2t$ and by differentiating w.r.t t we get

$$\dot{x} = -6\sin 2t + 8\cos 2t,$$

again differentiating $\therefore \ddot{x} = -12\cos 2t - 16\sin 2t$

$$\text{Or } \therefore \ddot{x} = -4(\underbrace{3\cos 2t + 4\sin 2t}_{x}) = -2^2 x$$

This equation represents a simple harmonic motion with $\omega = 2$ since the acceleration varies with distance, where the periodic time is τ and given by $\tau = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ and the amplitude may be calculated as



$$3\cos 2t + 4\sin 2t = 5\left(\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t\right)$$
$$= 5 \ \sin\epsilon\, cos 2t + \cos\epsilon \sin 2t \ = 5\sin(2t+\epsilon)$$

that is the amplitude is $a = 5$

Example

A moving particle along a straight line where μ, $x = \mu - \mu\cos 2t$ is constant. Show that the motion of the point is simple harmonic motion and find its periodic time and amplitude.

Solution

Since the position of the point is given by $x = \mu - \mu\cos 2t$ and by differentiating twice w.r.t t we get

$$\frac{dx}{dt} = 2\mu\sin 2t \qquad \text{and} \qquad \frac{d^2x}{dt^2} = 4\underbrace{\mu\cos 2t}_{\mu - x} = 4(\mu - x) = -4(x - \mu)$$

This equation indicates a simple harmonic motion (SHM) with center $x = \mu$ and $\omega^2 = 4$. The periodic time is τ and given by $\tau = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$.

(Hint: Let $y = x - \mu$ then the previous equation turn into $\ddot{y} = -2^2 y$, which represents a simple harmonic motion with center $y = 0$ ($x = \mu$)).

(Readers have to calculate the amplitude)

Example

A particle moves with SHM in a straight line. In the first second after starting from rest, it travels a distance a and in the next second it travels a distance b in the same direction. Prove that the amplitude of the motion is $2a^2 / (3a - b)$?

Solution

Measuring time t from the starting point and the distance x of the particle from the center of motion and denoting the amplitude by A, we have

$x = a\cos wt$

Now by the question when $t = 1, \quad x = A - a$

And when $t = 2, \quad x = A - a - b$

$$A - a = A\cos w \qquad \text{and} \qquad A - a - b = A\cos 2w = A\ \ 2\cos^2 w - 1$$

From these two equations we have

$$A - a - b = A\left(2\frac{(A-a)^2}{A^2} - 1\right) = \frac{1}{A}\left[A^2 - 4aA + 2a^2\right]$$

$$A^2 - aA - bA = A^2 - 4aA + 2a^2 \quad \text{Or} \quad (3a - b)A = 2a^2$$

$$\therefore A = \frac{2a^2}{3a - b}$$

Example

A point executing SHM has velocities u, u' and positions in two of its positions b, b' respectively. Show that the periodic time of motion is

$$2\pi\sqrt{\frac{b^2 - b'^2}{u'^2 - u^2}}$$

Solution

Let a be the amplitude of the simple harmonic motion then

$$v^2 = w^2(a^2 - x^2)$$

Therefore,

$$u^2 = w^2(a^2 - b^2) \qquad \textbf{and} \qquad u'^2 = w^2(a^2 - b'^2)$$

By subtracting

$$u'^2 - u^2 = w^2(b^2 - b'^2) \qquad \Rightarrow w^2 = \frac{u'^2 - u^2}{b^2 - b'^2} \qquad \textbf{Or} \qquad w = \sqrt{\frac{u'^2 - u^2}{b^2 - b'^2}}$$

Since $\tau = \dfrac{2\pi}{w}$ $\qquad \Rightarrow \tau = 2\pi\sqrt{\dfrac{b^2 - b'^2}{u'^2 - u^2}}$

Example

A body moving with SHM has an amplitude a and period T. Show that the velocity v at a distance x from the mean position is given by $v^2T^2 = 4\pi^2(a^2 - x^2)$

Solution

As we have $v^2 = w^2(a^2 - x^2)$ where a represents the amplitude

Also $\qquad \because T = \dfrac{2\pi}{w} \qquad \therefore w = \dfrac{2\pi}{T}$

$$v^2 = \left(\frac{2\pi}{T}\right)^2 (a^2 - x^2) \quad \Rightarrow v^2T^2 = 4\pi^2(a^2 - x^2)$$

Example

The speed v of a particle moving along the axis of x is given by the relation $v^2 = n^2(8bx - x^2 - 12b^2)$. Show that the motion is simple harmonic with its center at $x = 4b$ and amplitude $2b$. Find the time from $x = 5b$ to or $x = 6b$.

Solution

A particle is said to be its motion as simple harmonic motion if

$\ddot{x} = -w^2 x$

From the question we have $v^2 = n^2(8bx - x^2 - 12b^2)$ thus by differentiation

$$2v\frac{dv}{dx} = n^2(8b - 2x) \qquad \text{Or} \qquad v\frac{dv}{dx} = -n^2(x - 4b)$$

$\ddot{x}$

So the particle moves as a SHM with center $x = 4b$

$$v = 0 \qquad \therefore 8bx - x^2 - 12b^2 = 0 \quad \Rightarrow (x - 6b)(x - 2b) = 0$$

Therefore $\qquad x = 6b$ and $x = 2b$

which gives the ended points of SHM and the amplitude is $2b$.

Example

At the ends of three successive seconds, the distances of a point moving with SHM, from its mean position, measured in the same direction are X_1, X_2, X_3. Find the periodic time of motion.

Solution

As known the general solution of simple harmonic motion is $x = a\sin(\omega t + \epsilon)$

Let the time to reach position X_1 is t and thus the time to reach position X_2 is $t + 1$ and $t + 2$ is the time to reach position X_3 and therefore,

$$\begin{aligned} X_1 &= a\sin(\omega t + \epsilon) \\ X_2 &= a\sin(\omega(t + 1) + \epsilon) \\ X_3 &= a\sin(\omega(t + 2) + \epsilon) \end{aligned}$$

From previous equations – by adding first and third equations- we have

$$X_1 + X_3 = a \quad \sin(\omega t + \epsilon) + \sin(\omega(t + 2) + \epsilon)$$

$$= 2\underbrace{a\sin(\omega(t + 1) + \epsilon)}_{X_2}\cos\omega$$

$$= 2X_2 \cos\omega$$

Here we use the triangle relation

$$\sin x + \sin y = 2\sin\left(\frac{x + y}{2}\right)\cos\left(\frac{x - y}{2}\right)$$

$$\because X_1 + X_3 = 2X_2 \cos\omega \qquad \Rightarrow \cos\omega = \frac{X_1 + X_3}{2X_2} \qquad \text{Or} \quad \omega = \cos^{-1}\left(\frac{X_1 + X_3}{2X_2}\right)$$

But the periodic time is given by $\tau = \frac{2\pi}{\omega}$ therefore,

$$\tau = \frac{2\pi}{\cos^{-1}\left(\frac{X_1 + X_3}{2X_2}\right)}$$

Where $-2X_2 \le X_1 + X_3 \le 2X_2$

Example

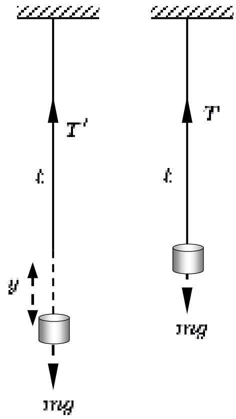
An elastic string supporting a heavy particle with mass m hangs in equilibrium. The particle is now pulled down below the equilibrium position through a small distance and let go then the particle done n complete oscillations per second. If ℓ represents the natural length of the string in the case of equilibrium. Find the natural length of string and evaluate the tension when the equals natural length.

Solution

Suppose that ℓ_0 represents natural length of string and T gives the tension in equilibrium after hangs mass m**,** in equilibrium case and from Hooke's law

$$mg = T = \frac{\lambda}{\ell_0}(\ell - \ell_0) \qquad (1)$$

After particle is pulled down below the equilibrium position a distance x then equation of motion becomes



$$m\ddot{x} = mg - T'$$

Where $T' = \frac{\lambda}{\ell_0}(\ell + x - \ell_0)$

$$\therefore\ m\ddot{x} = mg - \frac{\lambda}{\ell_0}(\ell + x - \ell_0)$$

$$= mg - \underbrace{\frac{\lambda}{\ell_0}(\ell - \ell_0)}_{mg} - \frac{\lambda}{\ell_0}x = -\frac{\lambda}{\ell_0}x$$

$$\ddot{x} = -\frac{\lambda}{\ell_0 m}x = -w^2 x$$

Which shows that the motion about a point of equilibrium is simple harmonic motion, the constant w equals $\sqrt{\frac{\lambda}{m\ell_0}}$. The periodic time is $2\pi\sqrt{\frac{\ell_0 m}{\lambda}}$

Now since $n = \frac{w}{2\pi}\ \Rightarrow n = \frac{1}{2\pi}\sqrt{\frac{\lambda}{m\ell_0}}\ \Rightarrow 4\pi^2 n^2 m = \frac{\lambda}{\ell_0}$

Therefore, from Equation (1)

$$\cancel{m}g = \frac{\lambda}{\ell_0}(\ell - \ell_0) = 4\pi^2 n^2 \cancel{m}(\ell - \ell_0) \qquad \therefore\ \ell - \ell_0 = \frac{g}{4\pi^2 n^2}$$

Or $\ell_0 = \ell - \frac{g}{4\pi^2 n^2}$

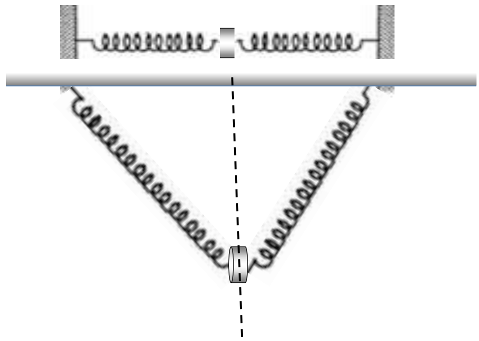
Which evaluate the natural length of string. To obtain the tension from Hooke's law

$$T = \frac{\lambda}{\ell_0}\ell_0 = 4\pi^2 n^2 m\ell_0 = 4\pi^2 n^2 m\left(\ell - \frac{g}{4\pi^2 n^2}\right) = m\ \ 4\pi^2 n^2 \ell - g$$

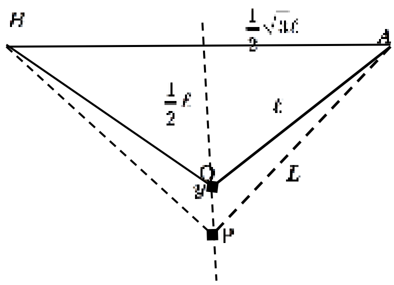
Example

A heavy particle is supported in equilibrium by two equal elastic strings with their other ends attached to two points in a horizontal plane and each inclined at an angle of 60° to the vertical. The modulus of elasticity is such that when the particle is suspended from any portion of the string its extension is equal to its natural length. The particle is displaced vertically a small distance and then released. Prove that the period of its small oscillations is $2\pi\sqrt{2\ell / 5g}$, where ℓ is the stretched length of either string in equilibrium.

Solution



Let m be the mass of the particle and λ the modulus of elasticity. Then by supposing the particle to be suspended from any portion of the string, since the extended length is double the natural length we find
that $\lambda = mg$.



If ℓ_0 be the natural length of either string, we have, in the equilibrium position,

$$mg = 2\lambda \frac{\ell - \ell_0}{\ell_0} \cos 60^0 = \lambda \frac{\ell - \ell_0}{\ell_0}$$

but $\lambda = mg$, therefore $\ell_0 = \frac{1}{2}\ell$

Let y denote the vertical displacement and L the length of either string at time t. To find the period of small oscillations we want to obtain an equation of the form

$$\ddot{x} = -\omega^2 x$$

where ω is a constant. It will therefore be sufficient for our purpose to write down the equation of motion at time t and neglect all powers of x higher than the first.

We have

$$m\ddot{y} = mg - 2\lambda \frac{L - \ell_0}{\ell_0} \cos \angle \mathrm{OPA}$$

where $\mathbf{P}$ is the particle at time t, $\mathbf{O}$ is its equilibrium position and $\mathbf{PA}=\mathbf{PB} = L$ are the strings.

Now

$$L^2 = \left(y + \frac{1}{2}\ell\right)^2 + \frac{3}{4}\ell^2 = \ell^2 + \ell y + y^2$$

Therefore $$L = \ell\left(1 + \frac{y}{\ell}\right)^{1/2} = \ell + \frac{1}{2}y$$

correct to the first power of x, and

$$\cos\angle\mathrm{OPA} = \frac{\frac{1}{2}\ell + y}{L} = \frac{\frac{1}{2}\ell + y}{\ell + \frac{1}{2}y}$$
$$= \frac{1}{2}\left(1 + \frac{2y}{\ell}\right)\left(1 - \frac{y}{2\ell}\right)$$
$$= \frac{1}{2}\left(1 + \frac{3y}{2\ell}\right)$$

to the first power of y . And hence

$$m\ddot{y} = mg - \frac{2\lambda\left(\ell + \frac{1}{2}y - \frac{1}{2}\ell\right)}{\frac{1}{2}\ell} \times \frac{1}{2}\left(1 + \frac{3y}{2\ell}\right)$$

Therefore,

$$\ddot{y} = g - g\left(1 + \frac{y}{\ell}\right)\left(1 + \frac{3y}{2\ell}\right) \qquad \text{Or} \qquad \ddot{y} = -\frac{5g}{2\ell}y$$

which represents a simple harmonic motion of period $2\pi\sqrt{2\ell / 5g}$.

PROBLEMS

❒ A point executing SHM has velocities u and v and accelerations a and b in two of its positions. Find the distance between the two positions and that the periodic time of motion

❒ If the displacement, velocity and acceleration at a particular instant of a particle describing SHM are respectively 3 in., 3 in./sec. and 3 in./sec^2, Find the greatest velocity of the particle and the period of motion.

❒ A point moving with SHM has a period of oscillation of л sec. and its greatest acceleration is 5 ft. /sec^2. Find the amplitude and the velocity when the particle is at a distance 1 ft. from the center of oscillation.

❒ A particle describing simple harmonic motion executes 100 complete Vibrations per minute and its speed at its mean position is 15 ft. per sec. What is the length of its path?

❒ A particle oscillates in a cycloid under gravity the amplitude of the motion being l and the periodic time being T. Show that its velocity at a time t measured from a position of rest is $w = \frac{2\pi l}{T} \sin \frac{2\pi t}{T}$

❒ A body is suspended from a fixed point by a light elastic string of natural length ℓ whose modulus of elasticity is equal to the weight of the body and makes vertical oscillations of amplitude a. Show that, if as the body rises through its equilibrium position it picks up another body of equal weight, the amplitude of the oscillation becomes $\left(\ell^2 + \frac{1}{2}a^2\right)^{1/4}$

IMPACT AND COLLISION OF ELASTIC BODIES

In this section we will integrate the equation of motion with respect to time and thereby obtain the principle of impulse and momentum. The resulting equation will be useful for solving problems involving force, velocity, and time. Using kinematics, the equation of motion for a particle of mass m can be written as

$$\sum F = ma = m\frac{dv}{dt}$$

where a and v are both measured from an inertial frame of reference. Rearranging the terms and integrating between the limits $v = v_1$ at $t = t_1$ and $v = v_2$, at $t = t_2$ we have

$$\sum\int_{t_1}^{t_2} F dt = m\int_{v_1}^{v_2} dv \qquad \Rightarrow \sum\int_{t_1}^{t_2} F dt = mv_2 - mv_1 \qquad \textbf{(*)}$$

This equation is referred to as the *principle of linear impulse and momentum*. From the derivation it can be seen that it is simply a time integration of the equation of motion. It provides a direct means of obtaining the particle's final velocity v_2 after a specified time period when the particle's initial velocity is known and the forces acting on the particle are either constant or can be expressed as functions of time. By comparison, if v_2 was determined using the equation of motion, a two-step process would be necessary; i.e., apply $\sum F = ma$ to obtain a, then integrate $a = \frac{dv}{dt}$ to obtain v_2

■ Linear Momentum

Each of the two vectors of the form $\boldsymbol{L} = m\boldsymbol{v}$ in Equation (*) is referred to as the particle's linear momentum. Since m is a positive scalar, the linear-momentum vector has the same direction as $\boldsymbol{v}$ and its magnitude mv has units of mass-velocity, e.g., kg.m/s, or slug. ft/s.

■ Linear Impulse

The integral $\boldsymbol{I} = \int \boldsymbol{F}dt$ in Equation (0) is referred to as the *linear impulse*. This term is a vector quantity which measures the effect of a force during the time the force acts. Since time is a positive scalar, the impulse acts in the same direction as the force, and its magnitude has units of force-time, e.g., N.s or lb·s. If the force is expressed as a function of time, the impulse can be determined by direct evaluation of the integral. In particular, if the force is constant in both magnitude and direction, the resulting impulse becomes

$$\boldsymbol{I} = \int_{t_1}^{t_2} \boldsymbol{F}_c dt = \boldsymbol{F}_c (t_2 - t_1)$$

■ Impact

This action occurs when two bodies collide with each other during a very short period of time, causing relatively large (impulsive) forces to be exerted between the bodies. The striking of a hammer on a nail, or a golf club on a ball, are common examples of impact loadings. In general, there are two types of impact. *Central impact* occurs when the direction of motion of the mass centers of the two colliding particles is along a line passing through the mass centers of the particles. This line is called the line of impact, which is perpendicular to the plane of contact. When the motion of one or both of the particles makes an angle with the line of impact, the impact is said to be *oblique impact.*

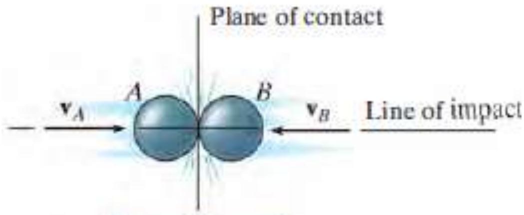
➢ **Elasticity** If we drop a ball of glass on to a marble floor, it rebounds almost to its original height but if the same ball were dropped on to a wooden floor, the distance through which it rebounds is much smaller. If further we allow an ivory ball and a wooden ball to drop from the same height upon a hard floor the heights through which they rebound are quite different. The velocities of these balls are the same when they reach the floor but since they rebound to different heights their velocities on leaving the floor are different.

Again, when a ball strikes against a floor or when two balls of any hard material collide, the balls are slightly compressed and when "they tend to recover their original shape, they rebound. The property of the bodies which causes these differences in velocities and which makes them rebound after collision is called ***Elasticity***. If a body does not tend to return to its original shape and does not rebound after collision, it is said to be ***Inelastic***.

In considering impact of elastic bodies, we suppose that they are smooth, so that the mutual action between them takes place only in the direction of their common normal at the point where they meet, there being no force in the direction perpendicular to their common normal.

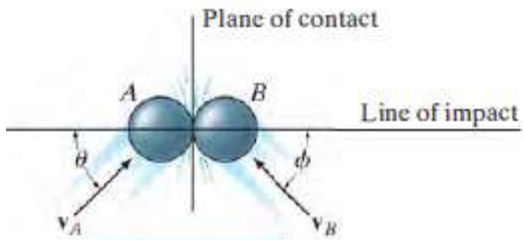
■ **Definitions**

*When the, direction of each body is along the common normal at the point where they touch, the impact is said to be **direct**.*



Central impact

*When the direction of motion of either or both, is not along the common normal at the point of contact the impact is said to be **oblique.***



Oblique impact

■ Direct Impact of two Smooth Spheres

Suppose two smooth spheres of masses m and m' moving in the same straight line with velocities u and u', collide and stick together. The forces which act between them during the collision act equally but in opposite directions on the two spheres so that the total momentum of the spheres remain unaltered by the impact. If U be the common velocity of the spheres after the collision and if the velocities are all measured in the same direction, we have

$$(m+m')U = mu + m'u'$$

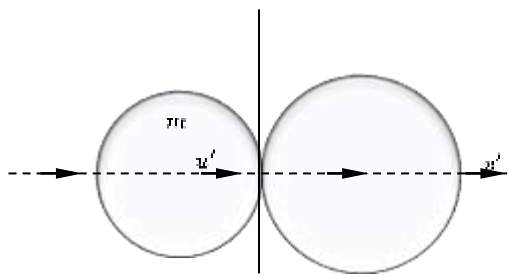
This equation is sufficient to determine the one unknown quantity U.

But we know, as a matter of ordinary experience, that when two bodies of any hard material impinge on each other, they separate almost immediately and a finite change of velocity is generated in each by their mutual action depending on the material of the bodies. Hence the spheres, if free to move, will have after impact, different velocities say v and v'.

The equation of momentum now becomes

$$mv + m'v' = mu + m'u' \qquad (1)$$

This single equation is not sufficient to determine the two unknown quantities v and v'.



Another relation between the velocities is supplied by ***Newton's Experimental Law*** which states that *when two bodies impinge directly, their relative velocity after impact is in a constant ratio to their relative velocity before impact, and is in the opposite direction.*

If bodies impinge obliquely, the same fact holds for their component velocities along the common nominal at the point of contact. The equation derived from this law for the above spheres is,

$$\frac{v-v'}{u-u'}=-e \qquad \text{Or} \qquad v-v'=-e(u-u') \qquad (2)$$

v, v', u and u' being all measured in the same direction.

The constant ratio, e is called ***the co-efficient of elasticity*** or ***restitution.*** It depends on the substances of which the bodies are made and is independent of the masses of the bodies and their velocities before impact. The value of e differs considerably for different bodies and varies from 0 to 1.

(i) When $e=\mathbf{0}$, the bodies are said to be inelastic (Plastic impact). In this case we have from Equation (2) $v=v'$ i.e., if two inelastic spheres impinge they move with the same velocity after impact.

(ii) When $e=\mathbf{1}$ the bodies are said to be perfectly elastic.

Both these are ideal eases never actually realized in nature.

In order to evaluate the velocities of the spheres after direct impact we solve Equations (1) and (2) and get

$$v=\frac{mu+m'u'-em'(u-u')}{m+m'} \quad \text{and} \quad v'=\frac{mu+m'u'+em(u-u')}{m+m'}$$

When $m=m'$ and $e=\mathbf{1}$, we have $v=u'$ and $v'=u$ i.e., if two equal perfectly elastic spheres impinge directly they interchange their velocities after impact.

■ Kinetic energy lost by direct impact

In general, there is always a loss of kinetic energy whenever two bodies impinge. Since we have by algebra

$$(m+m')(mu^2+m'u'^2)=(mu+m'u')^2+mm'(u-u')^2$$

And $\quad (m+m')(mv^2+m'v'^2)=(mv+m'v')^2+mm'(v-v')^2$

Subtracting these two equations and divide by $\mathbf{2}(m+m')$ and using

$$mv + m'v' = mu + m'u' \text{ and } v - v' = -e(u - u')$$

Therefore, Loss in K.E. is

$$= \frac{1}{2}mu^2 + \frac{1}{2}m'u'^2 - \left(\frac{1}{2}mv^2 + \frac{1}{2}m'v'^2\right) = \frac{1}{2}\frac{mm'}{m+m'}(u-u')^2(1-e^2)$$

■ Oblique Impact of two smooth spheres

Suppose that at the moment of impact the direction of motion of the spheres is not along the line joining their centers. Let m, m' be the masses of the two spheres with centers A and B at the time of impact, u, u' the velocities just before impact, α, β the angles the directions of motion make with AB before impact, v, v' the velocities after impact, and θ, φ angles the directions of motion make with AB after impact.

Since the spheres are smooth, there is no impulse perpendicular to the line of centers and hence the resolved parts of velocities of the two spheres in the direction perpendicular to AB remain unaltered.

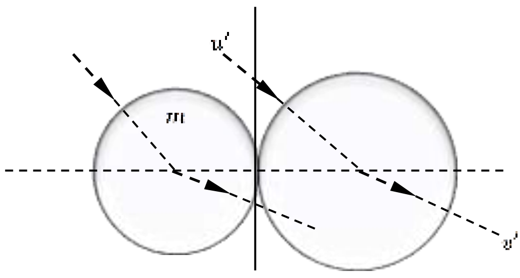
$$v\sin\theta = u\sin\alpha \quad \text{and} \quad v'\sin\varphi = u'\sin\beta \tag{3}$$

Since the impulsive forces acting during the collision on the two spheres along their line of centers are equal and opposite, the total momentum along AB remains unchanged.

$$mv\cos\theta + m'v'\cos\varphi = mu\cos\alpha + m'u'\cos\beta \tag{4}$$

By Newton's experimental law for relative velocities resolved along the common normal AB, we have

$$v\cos\theta - v'\cos\varphi = -e(u\cos\alpha - u'\cos\beta) \tag{5}$$



We deduce the following particular cases from the above equations:

(i) If $u' = 0$, from Equation (3) $\varphi = 0$, ($\because v' \neq 0$), i.e., if the sphere of mass m' were at rest, it will move along the line of centers after impact.

(ii) If $u' = 0$ and $m = em'$ from Equation (3) $\varphi = 0$ and then $\theta = 90^o$, so that if a sphere of mass m impinges obliquely on a sphere of mass m' at rest, the directions of motion of the spheres after impact will be at right angles if $m = em'$. This evidently holds true when the spheres are equal and perfectly elastic i.e. , when $u' = 0,\ e = 1$ and $m = m'$.

(iii) If $m = m'$ and $e = 1$ then, we have

$$v\cos\theta = u'\cos\beta \quad \text{and} \quad v'\cos\varphi = u\cos\alpha$$

i.e., if two equal-and perfectly elastic spheres impinge they interchange their velocities in the direction of their line of centers. Also in this case, by using Equation (3), we get: $\tan\theta\tan\varphi = \tan\alpha\tan\beta$

It follows that if two equal and perfectly elastic spheres impinge at right angles, their directions after impact will still be at right angles.

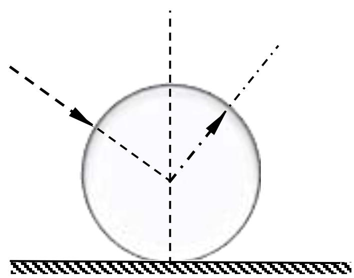
The students advised to prove this particular case independently.

Obtain the relation that describes the loss of kinetic energy in Oblique Impact

$$\therefore \text{Loss K.E.} = \frac{1}{2}\frac{mm'}{m+m'}(u\cos\alpha - u'\cos\beta)^2(1-e^2)$$

■ Impact against a Fixed Plane

Suppose a smooth sphere (or particle) of mass m, moving with a velocity u, strikes a smooth fixed plane in a direction making an angle α with the normal to the plane, and that it rebounds with velocity v making an angle θ with the normal. Then, since the plane is smooth, the component of the velocity along the plane must remain unaltered



$\therefore \ v\sin\theta = u\sin\alpha$ (6)

The plane being fixed its velocity is taken as zero. by Newton's experimental law for relative velocity along the common normal AN, we have

$\therefore \ v\cos\theta - 0 = -e(-u\cos\alpha - 0) \quad \Rightarrow \quad v\cos\theta = eu\cos\alpha$ (7)

Squaring and adding (6) and (7), we get

$$\therefore \ v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$$

Dividing (7) by (6) we have: $\cot\theta = e\cot\alpha$

These equations give the velocity and direction of motion of the sphere after impact. The following facts may be noted:

(1) If $\alpha = 0$ then by Equation (6), $\theta = 0$ and by Equation (7), $v = eu$ i.e., when the impact is direct, the direction of motion of the sphere is reversed after impact and its velocity is reduced in the ratio $e:1$.

(2) If $e = 1$, therefore $\alpha = \theta$ and then $u = v$, i.e., when the plane is perfectly elastic, the angle of reflection is equal to the angle of incidence, and the velocity remains unchanged in magnitude.

(3) If $e = 0$, thus $\theta = 90^o$ and then $v = u\sin\alpha$, i.e., when the plane i perfectly inelastic, the sphere simply slides along the plane, its velocity parallel to the plane remaining unaltered.

(4) Loss of Kinetic energy ΔE

$$\begin{aligned}\Delta E &= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 \\ &= \frac{1}{2}mu^2 - \frac{1}{2}mu^2(\sin^2\alpha + e^2\cos^2\alpha) \\ &= \frac{1}{2}mu^2(1-e^2)\cos^2\alpha\end{aligned}$$

■ ILLUSTRATIVE EXAMPLES ■

Example

A ball of mass $\mathbf{8\ Ib}$ moving with a velocity of $\mathbf{4\ ft\,sec^{-1}}$ is overtaken by a ball, of mass $\mathbf{12\ Ib}$ moving with a velocity of $\mathbf{9\ ft\,sec^{-1}}$, (i) in the same direction as the first, (ii) in the opposite direction. If $e = \mathbf{0.2}$ find the velocities of the balls after impact. Find also the loss of Kinetic energy in the first case.

Solution

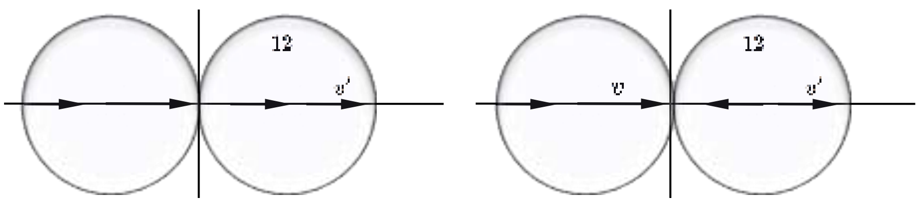
(i) Let the direction of motion of the first ball be taken as positive and let v, v' be the velocities after impact, then with consideration conservation of momentum.

$\mathbf{8v + 12v' = 8 \times 4 + 12 \times 9 = 140}$ and $\mathbf{v - v' = -0.2(4 - 9) = 1}$

which give $v = \mathbf{7.6}$ ft./s. and $v' = \mathbf{6.6}$ ft./s.

(ii) $\mathbf{8v + 12v' = 8 \times 4 - 12 \times 9 = -76}$ and $\mathbf{v - v' = -\frac{1}{5}(4 - (-9)) = -2.6}$

which give $v = \mathbf{-5.36}$ ft./s. and $v' = \mathbf{-2.76}$ ft/s



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In this case the first ball turns back after impact. It should be noted that the velocities are measured algebraically, that is, all velocities in one direction cert taken as positive while those in the opposite direction as negative.

Example

A ball A, moving with velocity u impinges directly on an equal ball B moving with velocity v in the opposite direction. If A be brought to rest by the impact, show that $\frac{u}{v} = \frac{1+e}{1-e}$ where e is the co-efficient of restitution.

Solution

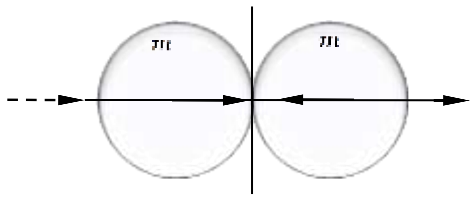
Let V be the velocity of B after impact and let m be the mass of each, then since A is reduced to rest after the impact, according to Conservation of momentum we obtain

$m \times 0 + m \times V = mu + m(-v)$ **Or** $V = u - v$ and

$0 - V = -e(u - (-v))$ **Or** $V = e(u + v)$

$\therefore u - v = V = e(u + v)$ **Or** $(1 - e)u = (1 + e)v$

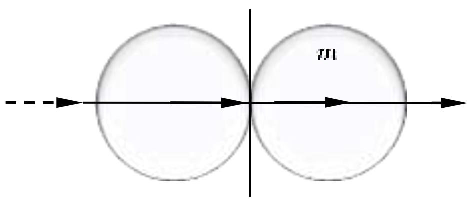
Hence $\frac{u}{v} = \frac{1+e}{1-e}$



Example

A ball with mass nm moving with velocity ua^{-1} impinges directly on another ball with mass m moving with velocity u in the same direction. If the ball with mass m be brought to rest by the impact, determine the co-efficient of restitution.

Solution



Let V be the velocity of the mass nm after impact (along the impact line since the balls impinge directly). According to the principle of the momentum along the impact line, we get

$$m(0)+nmV=mu+nm\left(\frac{u}{a}\right)\quad\Rightarrow nV=\left(1+\frac{n}{a}\right)u \qquad \textbf{(1)}$$

From Newton's Experimental Law, we obtain

$$V-0=-e\left(\frac{u}{a}-u\right)\quad\Rightarrow V=eu\left(1-\frac{1}{a}\right) \qquad \textbf{(2)}$$

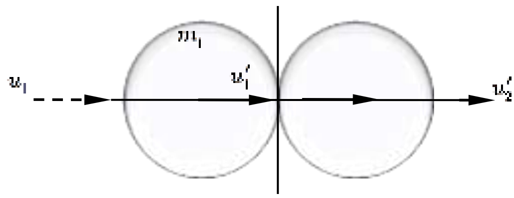
From these two equations (1) and (2)

$$ne\cancel{u}\left(1-\frac{1}{a}\right)=\left(1+\frac{n}{a}\right)\cancel{u}\quad\Rightarrow e=\frac{a+n}{n(a-1)}$$

Example

Let m_1, m_2 be the masses of two spheres impinge directly with velocities u_1, u_2 in the same direction. If e be the co-efficient of restitution. Prove that the loss of kinetic energy by impact is $\dfrac{(1-e^2)m_1m_2}{2(m_1+m_2)}(u_1-u_2)^2$

Solution



From the figure and according to the principle of the momentum along the impact line, we get Let u_1', u_2' be the velocities of the spheres after impact.

$$m_1u_1'+m_2u_2'=m_1u_1+m_2u_2 \qquad \textbf{(1)}$$

By Newton's experimental law

$$u_1'-u_2'=-e(u_1-u_2) \qquad \textbf{(2)}$$

Squaring equations (1) and (2) and multiply equation (2) by $m_1 m_2$ then adding we get

$$\left(m_1 u_1' + m_2 u_2'\right)^2 + m_1 m_2 \left(u_1' - u_2'\right)^2 = \left(m_1 u_1 + m_2 u_2\right)^2 + m_1 m_2 e^2 (u_1 - u_2)^2$$

By adding and subtracting the value $m_1 m_2 (u_1 - u_2)^2$ to the R.H.S. of previous equation

$$(m_1 + m_2)\left(m_1 u_1'^2 + m_2 u_2'^2\right) = \left(m_1 u_1 + m_2 u_2\right)^2 \pm m_1 m_2 (u_1 - u_2)^2 + m_1 m_2 e^2 (u_1 - u_2)^2$$

Or

$$(m_1 + m_2)\left(m_1 u_1'^2 + m_2 u_2'^2\right) = (m_1 + m_2)\left(m_1 u_1^2 + m_2 u_2^2\right) - m_1 m_2 \left(1 - e^2\right)(u_1 - u_2)^2$$

Dividing the last equation by $\frac{1}{2}\left(m_1 + m_2\right)$, we have

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{m_1 m_2 \left(1 - e^2\right)(u_1 - u_2)^2}{2(m_1 + m_2)}$$

$$\Delta E = \left(\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2\right) - \left(\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2\right) = \frac{m_1 m_2 \left(1 - e^2\right)(u_1 - u_2)^2}{2(m_1 + m_2)}$$

This relation illustrates that the total of kinetic energies of the two spheres after impact is less that the total of kinetic energies before impact by the value $\frac{m_1 m_2 \left(1 - e^2\right)(u_1 - u_2)^2}{2(m_1 + m_2)}$ and this values represents the loss of the kinetic energy by collision.

Example

A ball weighting one pound and moving with a velocity 8 ftsec^{-1}, impinges on a smooth fixed horizontal plane in a direction making 60^0 with the plane; find its velocity and direction of motion after impact, the co-efficient of restitution being 0.5. Find also the loss in Kinetic energy due to the impact.

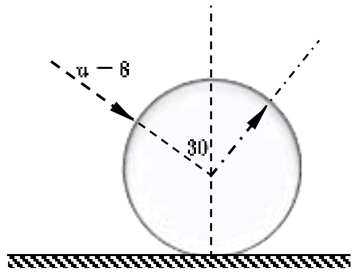
Solution

The direction of motion of the ball makes an angle of 30^0 with the normal to the plane. If after impact the ball moves in a direction making an angle θ with the normal with velocity v, then

$$v\cos\theta = eu\cos\alpha = \frac{1}{2}\times 8\times \cos 30 = 2\sqrt{3}$$

$$v\sin\theta = u\sin\alpha = 8\times \sin 30 = 4$$

$$\Rightarrow v^2 = 12 + 16 = 28 \quad \Rightarrow v = 5.29 \text{ ft sec}^{-1}$$



and

$$\cot\theta = \frac{\sqrt{3}}{2} \text{ i.e. } \theta = 49^o\, 6'$$

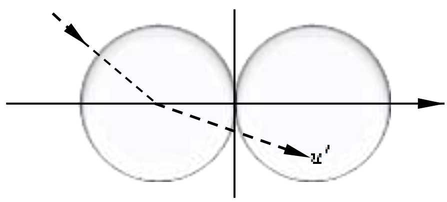
Loss of K.E. ΔE is

$$\Delta E = \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(8\times 8 - 28) = 18 \qquad (m = 1)$$

Example

A smooth ball A , collides Obliquely with an equal smooth ball B. Just before impact B is stationary and A makes an angle of α with the line joining the centers of the spheres with velocity v in a direction making an angle of α at the instant of impact. If e is the co-efficient of restitution, find the resulting motion of the sphere A?

Solution



Since the momentum after impact along the line of centers = momentum before impact, we have, let be the velocity of the rest ball after collision

$$mu'\cos\theta + mV = mu\cos\alpha + 0 \qquad \therefore\ u'\cos\theta + V = u\cos\alpha \tag{1}$$

Again by Newton's experimental law

$$u'\cos\theta - V = -e(u\cos\alpha - 0) \tag{2}$$

By adding the equations (1) and (2)

$$\therefore\ 2u'\cos\theta = (1-e)u\cos\alpha \tag{3}$$

Now since the velocity of the sphere A perpendicular to AB remains the same, we have

$$u'\sin\theta = u\sin\alpha \tag{4}$$

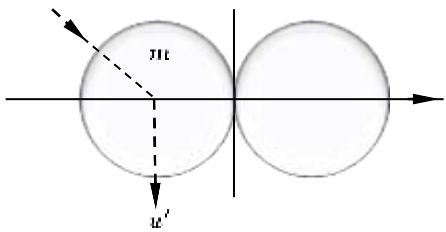
By dividing the equations (3) and (4) therefore

$$\frac{1}{2}\tan\theta = \frac{\tan\alpha}{1-e} \quad \Rightarrow \tan\theta = \frac{2\tan\alpha}{1-e} \quad \text{Or} \quad \theta = \tan^{-1}\left(\frac{2\tan\alpha}{1-e}\right)$$

Example

A sphere A, impinges obliquely on another sphere B at rest. If the direction of ball A after impact is perpendicular to the direction of ball B and the balls are perfectly elastic. Prove that the masses of the spheres are equivalent.

Solution



Let m' be the mass of the sphere B and hence its motion after impact will be along the line of impact and suppose its velocity will be V. Since the directions after impact make right angle, that is the velocity of the sphere A will be perpendicular to the line of impact. Let the sphere has a mass m and velocity u with an angle of α before the impact and has velocity u' after impact (perpendicular to the line of impact). According to the principle of constant of momentum along the line of impact we have

$$mu'\cos 90 + m'V = mu\cos\alpha + 0 \quad \Rightarrow m'V = mu\cos\alpha \qquad (1)$$

According to Newton's experimental law

$$u'\cos 90 - V = -e(u\cos\alpha - 0) \quad \Rightarrow V = u\cos\alpha \quad (e = 1) \qquad (2)$$

Substituting Equation (1) into Equation (2) we have $\therefore m = m'$

Example

A smooth sphere A moving with speed *u*, collides with an identical smooth sphere B which is moving in a perpendicular direction with the same speed *u*

The line of centers at the instant of impact is perpendicular to the direction of motion of sphere B. If the coefficient of restitution between the spheres is e. Prove that $\tan\varphi = \left(\frac{1+e}{2}\right)$, where φ is the angle through which sphere B is turned as a result of the impact.

Solution

Let V, u' be the velocities of the spheres after impact. From the figure and according to the principle of the momentum along the line of impact, we have,

$$mV - mu'\cos\theta = mu\cos 90 + mu \quad \Rightarrow V - u'\cos\theta = u \qquad (1)$$

Again from Newton's experimental law

$$V - (-u'\cos\theta) = -e(u - u\cos 90) \quad \Rightarrow V + u'\cos\theta = -eu \qquad (2)$$

Subtracting Equations (1) and (2)

$$\therefore 2u'\cos\theta = (1+e)u \qquad (3)$$

Since the resolved parts of velocities of the two spheres in the direction perpendicular to the line of impact remain unaltered.

$$u'\sin\theta = u \qquad (4)$$

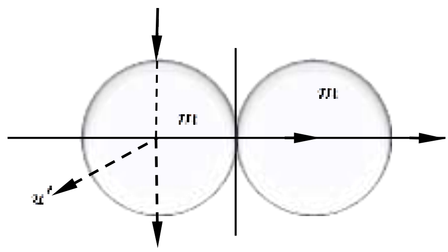
Now by dividing the equations (3) and (4)

$$\tan\theta = \frac{2}{1+e}$$

In order to determine the deviates of the velocity at an angle say φ where $\varphi = \frac{\pi}{2} - \theta$ therefore

$$\tan\varphi = \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta = \frac{1+e}{2}$$

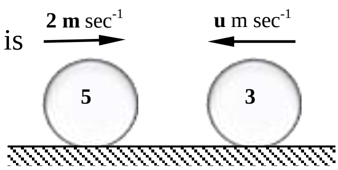
$$\tan\varphi = \frac{1+e}{2} \;\; \text{Or} \;\; \varphi = \tan^{-1}\left(\frac{1+e}{2}\right)$$



PROBLEMS

❒ A smooth sphere A of mass 5 kg is moving on a smooth horizontal surface with velocity (2i+3j) m s1. Another smooth sphere B of mass 3 kg and the same radius as A is moving on the same surface with velocity (4i-2j) m s1. The spheres collide when their line of centres is parallel to j. The coeffi cient of restitution between the spheres is 3/5. Find the velocities of both spheres after the impact.

❒ A smooth sphere P, of mass 5 kg, moving with a speed of 2 m/s collides directly with a smooth sphere Q, of mass 3 kg, moving in the opposite direction with a speed of *u* m/s on a smooth horizontal table. The coefficient of restitution for the collision is 0.5. As a result of the collision, sphere P is brought to rest.



(i) Find the value of *u*.

(ii) Find the speed of Q after the collision.

❒ An imperfectly elastic sphere whose elasticity is equal to tan 30 impinges upon a plane with a velocity such that the velocity after impact equals the velocity before impact × sin 45. Calculate the angles of incidence and reflection.

❒ If the masses of two balls be as 2:1 and their respective velocities before impact be as 1 : 2 in opposite directions. Evaluate the co-efficient of restitution, each ball moves back, after impact, with $\mathbf{5} / \mathbf{6}$ of its original velocity.

sphere impinges directly on an equal sphere at rest; if the coefficient of restitution is e show that their velocities after the impact are as $\frac{1-e}{1+e}$.

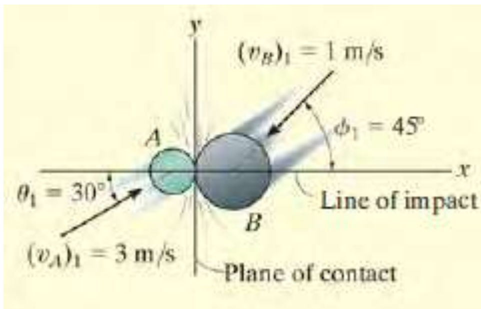
❒ Two bodies A and B whose elasticity is e, moving in opposite directions with velocities a and b, impinge directly upon each other ; determine their distance at time t after impact.

❒ Two equal balls moving with equal speeds impinge, their directions bring inclined at 30 and 60 to the line of centers at the time of impact; show that if $e = 1$, the balls move in parallel directions after the impact, inclined at 45 to the line of centers

❒ body of moss M moving with a velocity v collides with another of mass m which rests on a table. Both are perfectly elastic and smooth and the body m is driven in a direction making an angle θ with the previous line of motion of the body M, show that its velocity is $\frac{2M}{M+m} v \cos\theta$

❒Two equal smooth spheres moving along parallel lines in opposite directions with velocities u and v. collide with the line of centers at an angle α with their direction of motion. If after impact their lines of motion are at right angles to one another, show that $\frac{(u-v)^2}{(u+v)^2} = \sin^2\alpha + e^2\cos^2\alpha$

❒ Two smooth disks A and E, having a mass of 1 kg and 2 kg, respectively, collide with the velocities shown in the Figure. If the coefficient of restitution for the disks is e = 0.75, determine the x and y components of the final velocity of each disk just after collision.



❒ Determine the coefficient of restitution e between ball A and ball B. The velocities of A and B before and after the collision are shown

ORBITAL MOTION

We have already illustrated the motion of a particle in a plane by writing down its equations of motion either in the directions of two fixed co-ordinate axes or in the direction of the tangent and normal to the path described by the particle. However a large number of dynamical problems, where a particle moves under a central force, are readily solved, as already pointed out, by writing the equations of motion in the direction of the radius vector and in a direction perpendicular to it. These equations are of the form (using polar coordinates)

$$F_r = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)$$

and

$$F_\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) = m\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

where m is the mass of the particle and F_r and F_θ denote the sums of the components of the forces in the radial and transverse directions

Now, if a particle is moving only under the influence of a force having a line of action which is always directed toward a fixed point called the centre of force, the motion is called central-force motion. The path described by the particle is called a central orbit. This type of motion is commonly caused by electrostatic and gravitational forces. The position of the particle at any instant is defined by the polar co-ordinates r and θ referred to the centre of force $\mathbf{O}$ as the origin and any fixed line OX through O as the initial line.

♦ Definitions:

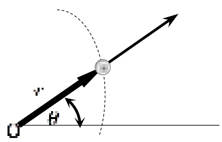
Central force. A force whose line of action always passes through a fixed point, is called a central force. The fixed point is known as the center of force

Central orbit. A central orbit is the path described by a particle moving under the action of a central force. The motion of a planet about the sun is an important example of a central orbit.

❒ **Theorem**. A central orbit is always a plane curve.

Proof.

Take the center of force $\mathbf{O}$ as the origin of vectors. Let P be the position of a particle moving in a central orbit at any time t and let $\underline{\boldsymbol{OP}} = \boldsymbol{r}$ .Then is the expression for the acceleration vector of the particle at the point P. Since the particle moves under the action of a central force with center at $\mathbf{O}$, therefore the only force acting on the particle at P is along the line OP or PO. So the acceleration vector of P is parallel to the vector $\underline{\boldsymbol{OP}}$



$$\therefore \frac{d^2\boldsymbol{r}}{dt^2} \text{ is parallel to } \boldsymbol{r}$$

$$\Rightarrow \frac{d^2\boldsymbol{r}}{dt^2} \wedge \boldsymbol{r} = \mathbf{0}$$

$$\frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt} \wedge \boldsymbol{r}\right) = \mathbf{0} \qquad \left(\frac{d\boldsymbol{r}}{dt} \wedge \frac{d\boldsymbol{r}}{dt} = \mathbf{0}\right)$$

Integration we have $\frac{d\boldsymbol{r}}{dt} \wedge \boldsymbol{r} = \textbf{Const vector} = \boldsymbol{h}$ (say) $\quad$ ....(1)

Taking dot product of both sides of Eq. (1) with the vector $\boldsymbol{r}$, we get

$$\boldsymbol{r}\bullet\left(\frac{d\boldsymbol{r}}{dt} \wedge \boldsymbol{r}\right) = \boldsymbol{r}\bullet\boldsymbol{h}$$

But the left hand member is a scalar triple product involving two equal vectors, and so it vanishes

$$\boldsymbol{r}\bullet\boldsymbol{h} = \mathbf{0}$$

Which shows that $\boldsymbol{r}$ is always perpendicular to a constant vector $\boldsymbol{h}$. Thus the radius vector $\boldsymbol{OP}$ is always perpendicular to a fixed direction and hence lies in a plane. Therefore the path of P is a plane curve

■ Differential Equation of Orbital Path

In order to find the differential equation of the path of a particle moving in a plane under a force which is directed to a fixed centre, we will consider the particle P shown in Fig. 1, which has a mass *m* and is acted upon only by the central force $\boldsymbol{F}$. The free-body diagram for the particle is shown in Fig. 2. Using polar coordinates (r,θ) the equations of motion are

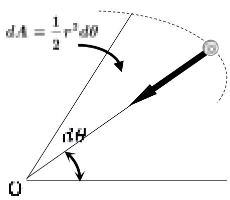


Fig. 1

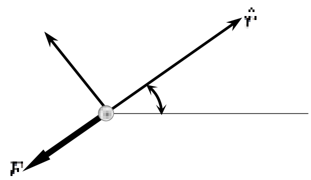


Fig. 2

$$F_r = ma_r, \qquad -F = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) \qquad \text{....(1)}$$

$$F_\theta = ma_\theta, \qquad 0 = m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right) \qquad \text{....(2)}$$

The Equation (2) may be re-written in the form

$$\frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right] = 0$$

so that integrating yields

$$r^2\frac{d\theta}{dt} = h \qquad \text{....(3)}$$

Here h represents the constant of integration.

To obtain the *path of motion,* $r = f(\theta)$, the independent variable t must be eliminated from Equations (1) and (2). Using the chain rule of calculus and Equation (3), the time derivatives of Equations (1) and (2) may be replaced by

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta}$$

$$\begin{aligned}\frac{d^2r}{dt^2} &= \frac{d}{dt}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\\ &= \frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\frac{d\theta}{dt}\\ &= \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right)\end{aligned}$$

Substituting a new dependent variable $r = u^{-1}$ into the Equation (2), we have

$$\frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}$$

$$\Rightarrow \frac{d^2r}{dt^2} = -h^2u^2\frac{d^2u}{d\theta^2} \qquad \text{....(4)}$$

As well as, the square of Equation (3) becomes

$$\left(\frac{d\theta}{dt}\right)^2 = h^2u^4 \qquad \text{...(5)}$$

Substituting these two Equations (4) and (5) into Equation (1) yields

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -\frac{F}{m}$$

Or $$F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \qquad \text{.....(6)}$$

This differential equation defines the path over which the particle travels when it is subjected to the central force $d\theta$. Equation (6) is important for the solution of two problems:

(i) Given the orbit, to determine the law of central force.

(ii) Given the law of central force, to determine the orbit.

■ Velocity Law

Since, $v_r = \dfrac{dr}{dt},\ v_\theta = r\dfrac{d\theta}{dt}$ then $v_r = -h\dfrac{du}{d\theta},\ v_\theta = h^2u^2$

Therefore, the velocity law describes as $v^2 = h^2\left(\left(\dfrac{du}{d\theta}\right)^2 + u^2\right)$ ......(7)

which gives the velocity when the path is known.

■ Areal Velocity

When a particle moves along a plane curve, the rate of change of the area traced out by the radius vector joining the particle to a fixed point is called the areal velocity of the particle. Let the particle moves along the curve APQ and let it describes the arc PQ$= \delta s$ in time δt.

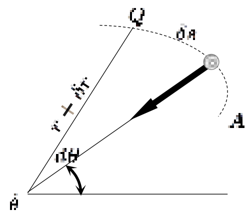


Fig. 3

Let (r,θ) be the co-ordinates of P and $(r+\delta r, \theta+\delta\theta)$ be those of Q, therefore the areal velocity $\dot{A}$ at P is given by

$$\begin{aligned}\dot{A} = \frac{dA}{dt} = \lim_{\delta t\to 0}\frac{\delta OPQ}{\delta t} &= \lim_{\delta t\to 0}\frac{1}{2}\frac{r(r+\delta r)\sin\delta\theta}{\delta t} \\ &= \lim_{\delta t\to 0}\frac{1}{2}\frac{r^2\sin\delta\theta}{\delta t} = \frac{1}{2}r^2\lim_{\delta t\to 0}\frac{\delta\theta}{\delta t} \\ &= \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\underset{h}{r^2\dot{\theta}} = \frac{h}{2}\end{aligned} \qquad (7)$$

From Fig. 3 notice that the shaded area described by the radius r, as r moves through an angle $\delta\theta$. In other words, the particle will sweep out equal segments of area per unit of time as it travels along the path.

■ Apse and Apsidal Distance

An apse is a point on central orbit at which the radius vector drawn from the center of a force is a maximum or minimum. The length of the radius vector at such a point is known as the *apsidal distance*. The analytical condition for a maximum or minimum value of the length of the radius vector is that $du/d\theta$

shall vanish and that the first differential co-efficient which does not vanish shall be of an even order.

Now if φ be the angle between the radius vector and the tangent to the curve, then by the Calculus,

$$\tan\varphi = -u\frac{d\theta}{du} \qquad \text{Or} \qquad \frac{du}{d\theta} = -u\cot\varphi$$

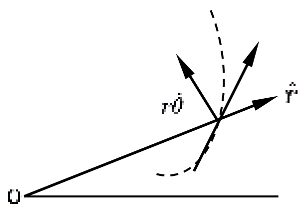
so that when $$\frac{du}{d\theta} = 0 \Rightarrow \varphi = \frac{\pi}{2}$$

Hence the tangent at an apse is perpendicular to the radius vector. In the case of a planet moving round the sun in an ellipse, the ends of the major axis are the two apses, the one nearer to the sun is called, perihelion and the further one is called aphelion.

■ Conservation of Angular Momentum

The angular of momentum about the center of **O** represents by the moment of linear momentum about **O** – remember that $v \equiv (\dot{r}, r\dot{\theta})$ -

$$m\dot{r}(0) + mr\dot{\theta}(r) = m\underbrace{r^2\dot{\theta}}_{h} = mh = \text{constant}$$



That is the angular of momentum about **O** remains constant during the motion, this called the principle of Conservation of Momentum

■ Planetary Motion (Three Kepler's Laws)

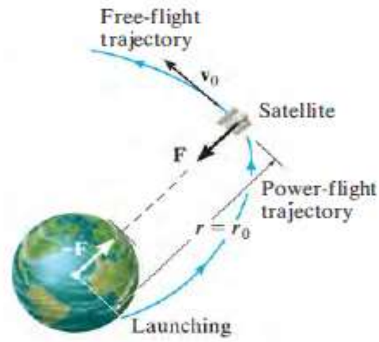
The laws according to which planets move round the sun are stated as follows:

(i) *The orbit of a planet round the sun is an ellipse, in one focus of which the center of the sun is situated.*

(ii) *The radius vector, drawn from the center of the sun to the planet describes equal areas in equal times.*

(iii) *The square of the periodic times of the various planets are proportional* *to the cubes of the semi-major axes of their orbits.*

These three laws were discovered by Kepler (1571-1630) and were deduced by him entirely from observations of the movements of the planets without any reference to the nature of the forces which control these movements.



For application, the force of gravitational attraction will be considered. Some common examples of central-force systems which depend on gravitation include the motion of the moon and artificial satellites about the earth, and the motion of the planets about the sun. As a typical problem in space mechanics, consider the trajectory of a space satellite or space vehicle launched into free-flight orbit with an initial velocity V_0 , see the figure. It will be assumed that this velocity is initially parallel to the tangent at the surface of the earth, as shown in the figure. Just after the satellite is released into free flight, the only force acting on it is the gravitational force of the earth. (Gravitational attractions involving other bodies such as the moon or sun will be neglected, since for orbits close to the earth their effect is small in comparison with the earth's gravitation.) According to Newton's law of gravitation, force F will always act between the mass centers of the earth and the satellite, Figure 3. From Equation 1, this force of attraction has a magnitude of

$$F = G\frac{M_e m}{r^2}$$

where M_e and m represent the mass of the earth and the satellite, respectively, G is the gravitational constant, and r is the distance between the mass centers. To obtain the orbital path, we set $r = u^{-1}$ in the foregoing equation and substitute the result into Equation 6. We get

$$\frac{d^2u}{d\theta^2} + u = \frac{GM_e}{h^2}$$

This second-order differential equation has constant coefficients and is nonhomogeneous. The solution is the sum of the complementary and particular solutions given by

$$u = \frac{1}{r} = C\cos(\theta - \phi) + \frac{GM_e}{h^2}$$

This equation represents the free-flight trajectory of the satellite. It is the equation of a conic section expressed in terms of polar coordinates.

The type of path traveled by the satellite is determined from the value of the eccentricity of the conic section as

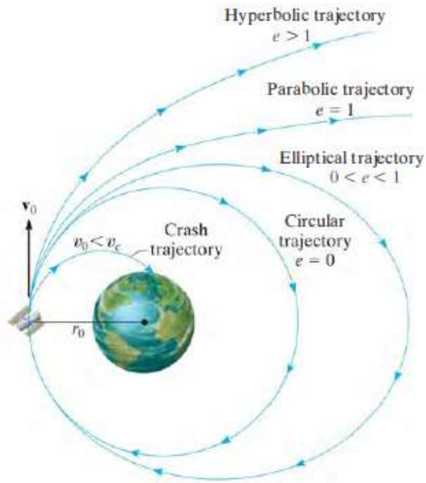
$$e = \frac{Ch^2}{GM_e}$$

$e = 0$ free-flight trajectory is a circle,

$e = 1$ free-flight trajectory is a parabola

$e < 1$ free-flight trajectory is an ellipse

$e > 1$ free-flight trajectory is a hyperbola



■ Illustrative Examples ■

Example

A particle describes the path $r = a\tan\theta$ under a force to the origin. Find its acceleration and velocity in terms of r.

Solution

Since, $r = a\tan\theta$ and let us consider $r = \dfrac{1}{u}$ then $au = \cot\theta$

By differentiation with respect to θ

$$a\frac{du}{d\theta} = -\csc^2\theta = -\ \ 1+\cot^2\theta\ \ = -\ \ 1 + a^2u^2$$

Again

$$a\frac{d^2u}{d\theta^2} = 2\csc^2\theta\cot\theta = 2au\ \ 1 + a^2u^2$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\therefore F = mh^2u^2\ \ 2u\ \ 1 + a^2u^2\ \ + u\ = mh^2u^3\ \ 2a^2u^2 + 3\ \ = \frac{h^2}{r^2}\left(3 + \frac{3a^2}{r^2}\right)$$

Also to get the velocity law

$$\because v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) \quad \Rightarrow v^2 = h^2\left(\frac{\ 1 + a^2u^2\ \ ^{2}}{a^2} + u^2\right)$$

$$\therefore v^2 = \frac{h^2}{a^2r^4}\ \ a^4 + 3a^2r^2 + r^4$$

Example

Determine the law of force in the following orbits, the pole being the centre of attraction

(i) $r^2 = a^2\cos 2\theta$

(ii) $r = \dfrac{a}{\theta^2 + b}$

Solution

(i) Due to $r^2 = a^2 \cos 2\theta$ and let us choose $r = u^{-1}$ therefore $\frac{1}{u^2} = a^2 \cos 2\theta$

Now differentiate with respect to θ

$$-\cancel{2}\frac{1}{u^3}\frac{du}{d\theta} = -\cancel{2}a^2 \sin 2\theta \quad \Rightarrow \frac{du}{d\theta} = a^2u^3 \sin 2\theta$$

Again

$$\frac{d^2u}{d\theta^2} = 2a^2u^3 \cos 2\theta + 3a^2u^2 \frac{du}{d\theta}\sin 2\theta = 2u^3 \underbrace{a^2 \cos 2\theta}_{1/u^2} + 3a^2u^2 \underbrace{\frac{du}{d\theta}}_{a^2u^3 \sin 2\theta} \sin 2\theta$$

$$\begin{aligned}\therefore \frac{d^2u}{d\theta^2} = 2u + 3a^4u^5 \sin^2 2\theta &= 2u + 3a^4u^5 \ \ 1 - \cos^2 2\theta \\ &= 2u + 3a^4u^5 \ - 3a^4u^5 \cos^2 2\theta \\ &= 2u + 3a^4u^5 \ - 3u = 3a^4u^5 \ - u\end{aligned}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = 3a^4u^5$$

$$\because F = mh^2u^2 \left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\therefore F = mh^2u^2 \ \ 3a^4u^5 \ \ = 3ma^4h^2u^7 \ = \frac{3ma^4h^2}{r^7}$$

(ii) In the similar manner, we have $r = \frac{a}{\theta^2 + b}$ assume that $r = u^{-1}$ hence

$$\frac{1}{u} = \frac{a}{\theta^2 + b} \qquad \Rightarrow -\frac{1}{u^2}\frac{du}{d\theta} = -\frac{2a\theta}{{\theta^2 + b}^{\ 2}}$$

$$\Rightarrow \frac{du}{d\theta} = \frac{2}{a}\cancel{u^2}\theta \times \frac{1}{\cancel{u^2}} = \frac{2}{a}\theta \qquad \therefore \frac{du}{d\theta} = \frac{2}{a}\theta$$

Now differentiate again with respect to variable θ

$$\therefore \frac{d^2u}{d\theta^2} = \frac{2}{a}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{2}{a} + u$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$
$$\therefore F = mh^2u^2\left(\frac{2}{a} + u\right) = \frac{mh^2}{r^3}\left(\frac{2}{a}r + 1\right)$$

Example

If a particle describes the cardioid $r = a(1 - \cos\theta)$ under a force to the pole, show that the force is proportional to the inverse fourth power of the distance. If P be the force at the apse ($\theta = \pi$) and V represents the velocity, prove that $3V^2 = 4aP$.

Solution

Since we have $r = a(1 - \cos\theta)$ and let us choose $r = u^{-1}$ therefore

$$\frac{1}{u} = a(1 - \cos\theta)$$

Now differentiate with respect to θ

$$-\frac{1}{u^2}\frac{du}{d\theta} = a\sin\theta \quad \Rightarrow \frac{du}{d\theta} = -au^2\sin\theta$$

Once again

$$\frac{d^2u}{d\theta^2} = -2au\frac{du}{d\theta}\sin\theta - au^2\cos\theta = 2a^2u^3\sin^2\theta - au^2\cos\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = -au^2 \;\; \cos\theta - 2au\sin^2\theta \;\; = -au^2 \;\; \cos\theta - 2au \;\; 1 - \cos^2\theta$$

$$\begin{aligned}
\therefore \frac{d^2u}{d\theta^2} &= -au^2 \;\; \cos\theta + 2au \;\; 1 - \cos^2\theta \\
&= -au^2\left(\cos\theta - 2u\underbrace{a \;\; 1 - \cos\theta}_{1/u}\;(1 + \cos\theta)\right) \\
&= -au^2 \;\; \cos\theta - 2(1 + \cos\theta) \\
&= -au^2(-2 - \cos\theta) = -au^2(-3 + \underbrace{1 - \cos\theta}_{1/au}) \\
&= 3au^2 - u
\end{aligned}$$

$$\therefore \frac{d^2u}{d\theta^2} + u = 3au^2$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$\therefore F = mh^2u^2 \;\; 3au^2 \; = 3mah^2u^4 \; = \frac{3mah^2}{r^4}$$

At apse we have

$$\dot{r} = 0 \;\; \text{Or} \;\; \frac{du}{d\theta} = 0 \qquad \Rightarrow -au^2 \sin\theta = 0 \qquad \therefore \sin\theta = 0 \quad \Rightarrow \theta = \pi$$

$$\because h = r^2\dot{\theta} = \underbrace{(r\dot{\theta})}_{V}r = 2aV \quad \text{Note} \;\; r\Big|_{\theta=\pi} = a(1 - \cos\pi) = 2a$$

But we derived the law of force

$$\therefore F = \frac{3mah^2}{r^4} \qquad \Rightarrow F\Big|_{\theta=\pi} = P = \frac{3ma(2aV)^2}{(2a)^4} \quad \therefore 3mV^2 = 4aP$$

Example

Show that the curve $r^n = a^n \cos n\theta$ can be described under a force to the pole varying inversely as $2n+3$ power of the distance

Solution

Since, $r^n = a^n \cos n\theta$ and let us take $r = u^{-1}$ thus $\dfrac{1}{u^n} = a^n \cos n\theta$

Now differentiate with respect to θ

$$-\cancel{n}\frac{1}{u^{n+1}}\frac{du}{d\theta} = -\cancel{n}a^n \sin n\theta \;\; \Rightarrow \frac{du}{d\theta} = a^n u^{n+1} \sin n\theta$$

Once again

$$\frac{d^2u}{d\theta^2} = na^n u^{n+1}\cos n\theta + (n+1)a^n u^n \frac{du}{d\theta}\sin n\theta$$

$$= nu^{n+1}\underbrace{a^n \cos n\theta}_{1/u^n} + (n+1)a^n u^n \underbrace{\frac{du}{d\theta}}_{a^n u^{n+1}\sin n\theta} \sin n\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = nu + (n+1)a^{2n}u^{2n+1}\sin^2 n\theta = nu + (n+1)a^{2n}u^{2n+1}(1-\cos^2 n\theta)$$
$$= nu + (n+1)a^{2n}u^{2n+1} - (n+1)u^{2n+1}\underbrace{a^{2n}\cos^2 n\theta}_{1/u^{2n}}$$

$$\therefore \frac{d^2u}{d\theta^2} = nu + (n+1)a^{2n}u^{2n+1} - (n+1)u = (n+1)a^{2n}u^{2n+1} - u$$

$$\therefore \frac{d^2u}{d\theta^2} + u = (n+1)a^{2n}u^{2n+1}$$

$$\because F = mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$
$$\therefore F = mh^2u^2\ \ (n+1)a^{2n}u^{2n+1}\ = (n+1)ma^{2n}h^2u^{2n+3} = \frac{3ma^{2n}h^2}{r^{2n+3}}$$

Example

A particle moves under the action of a force to a fixed point varying inversely as the square of the distance r. Prove that the orbit is a conic section with one focus at the center of force.

Solution

Since, $F = \frac{\mu}{r^2} \equiv \mu u^2$ then

$$\because \frac{F}{mh^2u^2} = \frac{d^2u}{d\theta^2} + u \qquad \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu u^2}{mh^2u^2} = \lambda \quad (\lambda = \frac{\mu}{mh^2})$$

$$\therefore \frac{d^2u}{d\theta^2} + u = \lambda$$

This is a differential equation which its general solution is

$$u = \frac{1}{r} = \mu(1 + \epsilon\cos(\theta - \alpha))$$

Where ϵ and α represent the constants of integration.

Example

A particle with mass $1\,\mathbf{gr}$ moves under an attractive force varies inversely as r^3 where the force equals 1 **Dyne** when $r = 1$ **cm**. Find the path equation if

$\theta = 0$ when $r = 2$ cm and velocity $\frac{1}{2}$ cm sec^{-1} with direction makes an angle of $\frac{\pi}{4}$ with constant line.

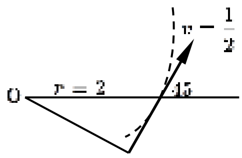
Solution

Since the attractive force varies inversely as cub of r, i.e. $F = \frac{\mu}{r^3}$ where μ is constant of proportional which can be evaluated from the condition $F = 1$ when $r = 1$ then $\mu = 1$, therefore $F = \frac{1}{r^3}$ and the path differential equation is

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = u^3 \quad \Rightarrow h^2\left(\frac{d^2u}{d\theta^2} + u\right) = u \quad \text{Or} \quad \frac{d^2u}{d\theta^2} + \left(1 - \frac{1}{h^2}\right)u = 0 \quad \textbf{(1)}$$

The constant h can be obtained from the principle of conservation of angular momentum about the attractive point and then

$$\Rightarrow h = \frac{1}{2}(2\sin\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$



Substituting in differential equation (1) we obtain

$$\frac{d^2u}{d\theta^2} - u = 0$$

$$\frac{du}{d\theta}\frac{d}{du}\left(\frac{du}{d\theta}\right) - u = 0 \quad \text{Or} \quad \left(\frac{du}{d\theta}\right)d\left(\frac{du}{d\theta}\right) - udu = 0$$

Then by integration

$$\left(\frac{du}{d\theta}\right)^2 = u^2 + c_1 \qquad \textbf{(2)}$$

Where c_1 is constant of integration and to determine c_1 we have to evaluate $\frac{du}{d\theta}$ as $r = 2$ which can be evaluated from velocity law

$$v^2 = h^2\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right) = \frac{1}{2}\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)$$

Since $v = \frac{1}{2}$ when $u = \frac{1}{2}$ then

$$\frac{1}{4} = \frac{1}{2}\left(\left(\frac{du}{d\theta}\right)^2 + \frac{1}{4}\right) \qquad \Rightarrow \left(\frac{du}{d\theta}\right)^2\Bigg|_{r=2} = \frac{1}{4}$$

That is as $u = \frac{1}{2}$ we have $\left(\frac{du}{d\theta}\right)^2 = \frac{1}{4}$ therefore, the value of integration constant $c_1 = 0$ and from equation (2)

$$\left(\frac{du}{d\theta}\right)^2 = u^2 \quad \Rightarrow \frac{du}{d\theta} = -u \qquad \textbf{Or} \qquad \frac{du}{u} = -d\theta$$

Then by integration we have

$$\ln u = -\theta + c_2$$

Now from the initial condition $u = \frac{1}{2}$ as $\theta = 0$ we get $c_2 = \ln\frac{1}{2}$ **Or** $c_2 = -\ln 2$ and then

$$\ln u = -\theta - \ln 2 \quad \Rightarrow \ln r = \theta + \ln 2 \quad \textbf{Or} \quad r = 2e^{\theta}$$

Which gives the path equation.

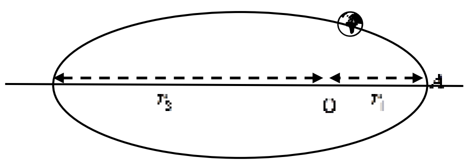
Example

If the ratio between the maximum value of angular velocity of a planet and the minimum value is γ^2. Prove that eccentricity of the planet trajectory is $\frac{\gamma - 1}{\gamma + 1}$.

Solution

According Kepler's law the planet moves around the sun in an ellipse path, in one focus of which the center of the sun is situated, where

$$r^2\dot{\theta} = h \quad \therefore \dot{\theta} = \frac{h}{r^2}$$



It's clear that the angular velocity $\dot{\theta}$ varies inversely as the square of distance of the sun r, therefore the greatest angular velocity occurs as r be smallest say $r = r_1$ where $r_1 = OA = a - ae$ and again the lowest angular velocity occurs as $r = r_2$ where $r_2 = OB = a + ae$

$$\Rightarrow \frac{\dot{\theta}_A}{\dot{\theta}_B} = \gamma^2 = \frac{r_2^2}{r_1^2}$$
$$= \frac{(1+e)^2}{(1-e)^2}$$

$$\Rightarrow \gamma = \frac{1+e}{1-e} \qquad \textbf{Or} \qquad e = \frac{\gamma - 1}{\gamma + 1}$$

Example

If a particle moves under the effect of a detractive central force to outside such that its path equation is $\theta = \theta(r)$. Prove that the force law is given by

$$-\frac{mh^2(2\theta' + r\theta'' + r^2\theta'^3)}{r^5\theta'^3}$$

where ' indicates differentiations with respect to r.

Solution

The law of detractive force is given by

$$F = -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

from the path equation θ is a function of r let as usual $r = \frac{1}{u}$ hence

$$\theta = \theta(r) \qquad \Rightarrow \frac{d\theta}{du} = \underbrace{\frac{d\theta}{dr}}_{\theta'}\frac{dr}{du} = \theta'\left(-\frac{1}{u^2}\right)$$

$$\Rightarrow \frac{du}{d\theta} = -\frac{u^2}{\theta'}$$

Once time differentiate we have

$$\begin{aligned}\frac{d^2u}{d\theta^2} &= \frac{d}{du}\left(-\frac{u^2}{\theta'}\right)\frac{du}{d\theta} \\ &= -\frac{2u\theta' - u^2\left(-\frac{1}{u^2}\right)\theta''}{\theta'^2}\frac{du}{d\theta} \qquad \left(\theta'' = \frac{d^2\theta}{dr^2}\right) \\ &= -\frac{2u\theta' + \theta''}{\theta'^2}\left(-\frac{u^2}{\theta'}\right) \\ &= \frac{2u^3\theta' + u^2\theta''}{\theta'^3}\end{aligned}$$

$$\begin{aligned}\therefore F &= -mh^2u^2\left(\frac{2u^3\theta' + u^2\theta''}{\theta'^3} + u\right) \qquad \left(u = \frac{1}{r}\right) \\ &= -mh^2\left(\frac{2\theta' + r\theta'' + r^2\theta'^3}{r^5\theta'^3}\right)\end{aligned}$$

Example

Prove that the areal velocity in Cartesian coordinate is $\frac{1}{2}(x\dot{y} - y\dot{x})$.

Solution

Since the relation between Cartesian (x, y) and Polar (r, θ) coordinates are

$$\tan\theta = \frac{y}{x}, \qquad r^2 = x^2 + y^2$$

And the areal velocity is given by $\dot{A} = \frac{1}{2}h = \frac{1}{2}r^2\dot{\theta}$

Then differentiating

$$\begin{aligned}\tan\theta = \frac{y}{x} \qquad &\Rightarrow \dot{\theta}\sec^2\theta = \frac{\dot{y}x - \dot{x}y}{x^2} \\ &\Rightarrow \dot{\theta} = \frac{\dot{y}x - \dot{x}y}{x^2}\cos^2\theta\end{aligned}$$

but $\cos^2\theta = \dfrac{x^2}{x^2 + y^2}$

$$\begin{aligned}\dot{A} = \frac{1}{2}r^2\dot{\theta} &= \frac{1}{2}(x^2 + y^2)\frac{\dot{y}x - \dot{x}y}{x^2}\left(\frac{x^2}{x^2 + y^2}\right) \\ &= \frac{1}{2}\ \dot{y}x - \dot{x}y\end{aligned}$$

PROBLEMS

❒ A particle is attracted to a point by a central force, and it is observed that the orbit of the particle is the spiral $r = e^{\theta}$. Determine the force that is acting.

❒ A particle moving under the influence of a central force, describes a circle through the center of the force. Prove that the force is attractive and inversely proportional to the fifth power of the distance

[Hint. Equation of the circle is $r = 2a\cos\theta$].

❒ If in a central orbit under the force ($\mu u^3(3+2a^2u^2)$), a particle be projected at a distance a with a velocity $\sqrt{5\mu}\ /\ a$ in a direction making $\tan^{-1}\dfrac{1}{2}$ with the radius, show that the equation to the path is $r = a\cot\left(\theta+\dfrac{\pi}{4}\right)$.

❒ Show that the curve $r = \dfrac{\ell}{1+\epsilon\cos\theta}$ can be described under a force to the pole varying inversely as 2 power of the distance.