CHAPTER 2

SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

1 Homogeneous Linear Equations of the Second Order

1.1 Linear Differential Equation of the Second Order

$$y'' + p(x)\, y' + q(x)\, y = r(x) \qquad \text{Linear}$$

where $p(x)$, $q(x)$: coefficients of the equation

if
$r(x) = 0 \quad \Rightarrow$ homogeneous
$r(x) \neq 0 \quad \Rightarrow$ nonhomogeneous
$p(x)$, $q(x)$ are constants $\Rightarrow$ constant coefficients

[Example]

(i) $$(1 - x^2)\, y'' - 2x\, y' + 6y = 0$$

$$\Downarrow$$

$$y'' - \frac{2x}{1 - x^2} y' + \frac{6}{1 - x^2} y = 0$$

homogeneous
variable coefficients
linear

(ii) $$y'' + 4y' + 3y = e^x$$

nonhomogeneous
constant coefficients
linear

(iii) $$y''\, y + y' = 0$$

nonlinear

(iv) $y'' + (\sin x)\, y' + y = 0$ linear,homogeneous,variable coefficients

1.2 Second–Order Differential Equations Reducible to the First Order

Case I: $F(x, y', y'') = 0$ —— y does not appear explicitly

[Example] $y'' = y' \tanh x$

[Solution] **Set $y' = z$ and** $y'' = \dfrac{dz}{dx}$

Thus, the differential equation becomes first order

$$z' = z \tanh x$$

which can be solved by the method of separation of variables

$$\frac{dz}{z} = \tanh x \, dx = \frac{\sinh x}{\cosh x} dx$$

or $\ln|z| = \ln|\cosh x| + c'$

$\Rightarrow$ $z = c_1 \cosh x$

or $y' = c_1 \cosh x$

Again, the above equation can be solved by separation of variables:

$$dy = c_1 \cosh x \, dx$$

$\Rightarrow$ $y = c_1 \sinh x + c_2$ #

Case II: $F(y, y', y'') = 0$ – x does not appear explicitly

[Example] $y'' + y'^3 \cos y = 0$

[Solution] Again, set $z = y' = dy/dx$

$$\text{thus, } y'' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy} y' = \frac{dz}{dy} z$$

Thus, the above equation becomes a first–order differential equation of z (dependent variable) with respect to y (independent variable):

$$\frac{dz}{dy} z + z^3 \cos y = 0$$

which can be solved by *separation of variables:*

$$-\frac{dz}{z^2} = \cos y \, dy \qquad \text{or} \qquad \frac{1}{z} = \sin y + c_1$$

$$\text{or} \qquad z = y' = dy/dx = \frac{1}{\sin y + c_1}$$

which can be solved by *separation of variables* again

$$(\sin y + c_1)\, dy = dx \quad \Rightarrow \quad -\cos y + c_1 y + c_2 = x_{\#}$$

[Exercise] Solve $y'' + e^y (y')^3 = 0$

[Answer] $e^y - c_1 y = x + c_2$ (Check with your answer!)

[Exercise] Solve $y\, y'' = (y')^2$

2 General Solutions

2.1 Superposition Principle

[Example] Show that (1) $y = e^{-5x}$, (2) $y = e^{2x}$ and (3) $y = c_1 e^{-5x} + c_2 e^{2x}$ are all solutions to the 2nd-order linear equation

$$y'' + 3y' - 10y = 0$$

[Solution]

$$(e^{-5x})'' + 3(e^{-5x})' - 10e^{-5x}$$
$$= 25e^{-5x} - 15e^{-5x} - 10e^{-5x} = 0$$

$$(e^{2x})'' + 3(e^{2x})' - 10e^{2x}$$
$$= 4e^{2x} + 6e^{2x} - 10e^{2x} = 0$$

$$(c_1 e^{-5x} + c_2 e^{2x})'' + 3(c_1 e^{-5x} + c_2 e^{2x})' - 10(c_1 e^{-5x} + c_2 e^{2x})$$
$$= c_1(25e^{-5x} - 15e^{-5x} - 10e^{-5x})$$
$$+ c_2(4e^{2x} + 6e^{2x} - 10e^{2x}) = 0$$

Thus, we have the following *superposition principle*:

[Theorem]

*Let y_1 and y_2 be two solutions of the **homogeneous linear** differential equation*

$$y'' + p(x)\, y' + q(x)\, y = 0$$

then the linear combination of y_1 and y_2, i.e.,

$$y_3 = c_1 y_1 + c_2 y_2$$

is also a solution of the differential equation, where c_1 and c_2 are arbitrary constants.

[Proof]

$$(c_1 y_1 + c_2 y_2)'' + p(x)\,(c_1 y_1 + c_2 y_2)' + q(x)\,(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + p(x)\, c_1 y_1' + p(x)\, c_2 y_2' + q(x)\, c_1 y_1 + q(x)\, c_2 y_2$$

$$= c_1 (y_1'' + p(x)\, y_1' + q(x)\, y_1) + c_2 (y_2'' + p(x)\, y_2' + q(x)\, y_2)$$

$$= c_1 (0) \quad \text{(since } y_1 \text{ is a solution)}$$
$$\quad + c_2 (0) \quad \text{(since } y_2 \text{ is a solution)}$$

$$= 0$$

Remarks: The above theorem applies only to the homogeneous linear differential equations

2.2 Linear Independence

Two functions, $y_1(x)$ and $y_2(x)$, are ***linearly independent*** on an interval $[x_0, x_1]$ whenever the relation $c_1 y_1(x) + c_2 y_2(x) = 0$ for all x in the interval implies that

$$c_1 = c_2 = 0.$$

Otherwise, they are ***linearly dependent.***

There is an easier way to see if two functions y_1 and y_2 are linearly independent. If $c_1 y_1(x) + c_2 y_2(x) = 0$ (where c_1 and c_2 are not both zero), we may suppose that $c_1 \neq 0$. Then

$$y_1(x) + \frac{c_2}{c_1} y_2(x) = 0 \quad \text{or} \quad y_1(x) = -\frac{c_2}{c_1} y_2(x) = C\, y_2(x)$$

Therefore:

*Two functions are **linearly dependent** on the interval if and only if one of the functions is a constant multiple of the other.*

2.3 General Solution

Consider the second–order homogeneous linear differential equation:

$$y'' + p(x)\, y' + q(x)\, y = 0$$

where p(x) and q(x) are continuous functions, then

(1) **Two linearly independent solutions** of the equation can always be found.

(2) Let $y_1(x)$ and $y_2(x)$ be any two solutions of the homogeneous equation, then any linear combination of them (i.e., $c_1\, y_1 + c_2\, y_2$) is also a solution.

(3) The ***general solution*** of the differential equation is given by the linear combination

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

where c_1 and c_2 are arbitrary constants, and $y_1(x)$ and $y_2(x)$ are two linearly independent solutions. (In other words, y_1 and y_2 form a **basis** of the solution on the interval I)

(4) A ***particular solution*** of the differential equation on I is obtained if we assign specific values to c_1 and c_2 in the general solution.

[Example] Verify that $y_1 = e^{-5x}$, and $y_2 = e^{2x}$ are linearly independent solutions to the equation

$$y'' + 3y' - 10y = 0$$

[Solution]

It has already been shown that $y = e^{-5x}$ and $y = e^{2x}$ are solutions to the differential equation. In addition

$$y_1 = e^{-5x} = e^{-7x} e^{2x} = e^{-7x} y_2$$

and e^{-7x} is not a constant, we see that e^{-5x} and e^{2x} are linearly independent and form the basis of the general solution. The general solution is then

$$y = c_1 e^{-5x} + c_2 e^{2x}$$

2.4 Initial Value Problems and Boundary Value Problems

Initial Value Problems (IVP)

	Differential Equation	$y'' + p(x)\, y' + q(x)\, y = 0$
with	**Initial Conditions**	$y(\mathbf{x_0}) = k_0, \quad y'(\mathbf{x_0}) = k_1$

$\Rightarrow$ Particular solutions with c_1 and c_2 evaluated from the initial conditions.

Boundary Value Problems (BVP)

	Differential Equation	$y'' + p(x)\, y' + q(x)\, y = 0$
with	**Boundary Conditions**	$y(\mathbf{x_0}) = k_0, \quad y(\mathbf{x_1}) = k_1$ where x_0 and x_1 are boundary points.

$\Rightarrow$ Particular solution with c_1 and c_2 evaluated from the boundary conditions.

2.5 Using One Solution to Find Another (Reduction of Order)

If y_1 is a nonzero solution of the equation $y'' + p(x)\,y' + q(x)\,y = 0$, we want to seek another solution y_2 such that y_1 and y_2 are linearly independent. Since y_1 and y_2 are linearly independent, the ratio

$$\frac{y_2}{y_1} = u(x) \neq \textbf{constant}$$

must be a non-constant function of x, and $\mathbf{y_2 = u\,y_1}$ must satisfy the differential equation. Now

$$(u\,y_1)' = u'\,y_1 + u\,y_1'$$
$$(u\,y_1)'' = u\,y_1'' + 2\,u'\,y_1' + u''\,y_1$$

Put the above equations into the differential equation and collect terms, we have

$$u''\,y_1 + u'\,(2\,y_1' + p\,y_1) + u\,(y_1'' + p\,y_1' + q\,y_1) = 0$$

Since y_1 is a solution of the differential equation, $y_1'' + p\,y_1' + q\,y_1 = 0$

$$\Rightarrow \qquad u''\,y_1 + u'\,(2y_1' + p\,y_1) = 0 \quad \text{or} \quad u'' + u'\left[2\,\frac{y_1'}{y_1} + p\right] = 0$$

Note that the above equation is of the form $F(u'', u', x) = 0$ which can be solved by setting $U = u'$ $\therefore$ $U' + \left[2\,\frac{y_1'}{y_1} + p\right]U = 0$

which can be solved by separation of variables:

$$U = \frac{c}{y_1^2} e^{-\int p(x)\,dx}$$

where c is an arbitrary constant. Take simply (by **setting c = 1**)

$$du/dx = U = \frac{1}{y_1^2} e^{-\int p(x)\,dx}$$

and perform another integration to obtain u, we have

$$y_2 = u\,y_1 = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx$$

Note that $e^{-\int p(x)\,dx}$ is never zero, i.e., u is non-constant. Thus, y_1 and y_2 form a basis.

[Example] $y_1 = x$ is a solution to

$$x^2 y'' - x y' + y = 0 \quad ; \quad x > 0$$

Find a second, linearly independent solution.

[Solution] **Method 1: Use $y_2 = u\, y_1$**

Let $y_2 = u\, y_1 = u\, x$

then $y_2' = u' x + u$ and $y_2'' = u'' x + 2u'$

$$x^2 y_2'' - x y_2' + y_2 = x^3 u'' + 2x^2 u' - x^2 u' - x u + x u = x^3 u'' + x^2 u' = 0$$

or $x u'' + u' = 0$

Set $U = u'$, then $U' = -\dfrac{1}{x} U \quad \Rightarrow \dfrac{dU}{U} = -\dfrac{dx}{x}$

$$\therefore \quad U = e^{-\int 1/x\, dx} = e^{-\ln x} = \frac{1}{x}$$

Since $U = u'$, $\therefore \quad u = \int U\, dx = \int 1/x\, dx = \ln x$

Therefore, $y_2(x) = u\, y_1 = x \ln x$ (You should verify that y_2 is indeed a solution.)

Method II: Use formula.

To use the formula, we need to write the differential equation in the following standard form:

$$y'' - \frac{1}{x} y' + \frac{1}{x^2} y = 0$$

$$y_2 = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)} dx$$

$$= x \int \frac{e^{\int \frac{1}{x} dx}}{x^2} dx$$

$$= x \int \frac{x}{x^2} dx = x \ln x$$

[Exercise 1] Given that $y_1 = x$, find the second linearly independent solution to

$$(1 - x^2)\, y'' - 2\, x\, y' + 2\, y = 0$$

$$\text{Hint: } \int \frac{dx}{1 - x^2} = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right)$$

[Exercise 2] Given that $y_1 = x$, find the second linearly independent solution to

$$y'' - \frac{y'}{x^2} + \frac{y}{x^3} = 0$$

[Exercise 3] Verify that $y = \tan x$ satisfies the equation

$$y'' \cos^2 x = 2y$$

and obtain the general solution to the above differential equation.

3 Homogeneous Equations with Constant Coefficients

$$y'' + a\,y' + b\,y = 0$$

where a and b are real constants.

Try the solution

$$y = e^{\lambda x} \qquad \text{—— trial solution}$$

Put the above equation into the differential equation, we have

$$(\lambda^2 + a\,\lambda + b)\,e^{\lambda x} = 0$$

Hence, if $y = e^{\lambda x}$ be the solution of the differential equation, λ must be a solution of the quadratic equation

$$\lambda^2 + a\,\lambda + b = 0 \qquad \text{—— characteristic equation}$$

Since the characteristic equation is quadratic, we have two roots:

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}$$

$$\lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

Thus, there are three possible situations for the roots of λ_1 and λ_2 of the characteristic equation:

<u>Case I</u>	$a^2 - 4b > 0$	λ_1 and λ_2 are distinct real roots
<u>Case II</u>	$a^2 - 4b = 0$	$\lambda_1 = \lambda_2$, a real double root
<u>Case III</u>	$a^2 - 4b < 0$	λ_1 and λ_2 are two complex conjugate roots

We now discuss each case in the following:

Case I Two Distinct Real Roots, λ_1 and λ_2

Since $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are ***linearly independent***, we have the general solution $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

[Example] $y'' + 3y' - 10y = 0 \quad ; \qquad y(0) = 1, \; y'(0) = 3$

The characteristic equation is

$$\lambda^2 + 3\lambda - 10 = (\lambda - 2)(\lambda + 5) = 0$$

we have two distinct roots

$$\lambda_1 = 2 \quad ; \qquad \lambda_2 = -5$$

$\Rightarrow \quad y(x) = c_1 e^{2x} + c_2 e^{-5x}$ —— general solution

The initial conditions can be used to evaluate c_1 and c_2:

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - 5c_2 = 3$$

$\Rightarrow \quad c_1 = 8/7 \; , \qquad c_2 = -1/7$

$\therefore \quad y(x) = \dfrac{1}{7}\left(8e^{2x} - e^{-5x}\right)$ -- particular solution

<u>Case II</u> Real Double Roots ($a^2 - 4b = 0$)

Since $\lambda_1 = \lambda_2 = -\frac{a}{2}$, $y_1(x) = e^{-ax/2}$ should be the first solution of the differential equation.

The second linearly independent solution can be obtained by the procedure of reduction of order: $y_2 = x\, e^{-ax/2}$

[Derivation]

Let $y_2 = u\, y_1 = u\, e^{-ax/2}$

then $y_2' = u'\, e^{-ax/2} - \frac{a}{2}\, u\, e^{-ax/2}$ and

$$y_2'' = u''\, e^{-ax/2} - a\, u'\, e^{-ax/2} + \frac{a^2}{4}\, u\, e^{-ax/2}$$

so that the differential equation becomes

$$y'' + a\, y' + b\, y = \left(u'' - a\, u' + \frac{a^2}{4}\, u\right) e^{-ax/2} + a\left(u' - \frac{a}{2}\, u\right) e^{-ax/2} + b\, u\, e^{-ax/2} = 0$$

or $u'' + \left[b - \frac{a^2}{4}\right] u = 0$

But since $a^2 = 4\, b$, we have $u'' = 0$. Thus, u' is a constant which can be chosen to be 1. ∴ $u = x$.

Hence $y_2 = x\, e^{-ax/2}$

Thus, the general solution for this case is

$$y(x) = (c_1 + c_2\, x)\, e^{-ax/2} \quad \text{—— general solution}$$

[Example] Solve $y'' - 6y' + 9y = 0$

[Solution]

The characteristic equation is

$$\lambda^2 - 6\lambda + 9 = 0 \quad \text{or} \quad (\lambda - 3)^2 = 0$$

and

$$\lambda_1 = \lambda_2 = 3$$

Thus, the general solution is

$$y = (c_1 + c_2 x)\, e^{3x}$$

Case III *Complex Conjugate Roots* λ_1 *and* λ_2 **($a^2 - 4b < 0$)**

$$\lambda_1 = -\frac{1}{2}a + i\omega$$

$$\lambda_2 = -\frac{1}{2}a - i\omega$$

where $\omega = \sqrt{b - \frac{a^2}{4}}$ and $i = \sqrt{-1}$

Thus, $Y_1 = e^{\lambda_1 x}$ and $Y_2 = e^{\lambda_2 x}$ are solutions (which are complex functions) of the differential equation, i.e.

$$y = C_1 Y_1 + C_2 Y_2$$

Note that we have proven that any linear combination of solutions is also a solution. This is also valid if the constants are complex numbers. Thus, we consider the solutions (which are **real functions** as shown later):

$$y_1 = \frac{1}{2}(Y_1 + Y_2) \quad \text{and} \quad y_2 = \frac{1}{2i}(Y_1 - Y_2)$$

From the complex variable analysis[1], we have Euler Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Thus,

$$Y_1 = e^{\lambda_1 x} = e^{-ax/2}(\cos\omega x + i\sin\omega x)$$

$$Y_2 = e^{\lambda_2 x} = e^{-ax/2}(\cos\omega x - i\sin\omega x)$$

or

$$y_1 = e^{-ax/2}\cos\omega x$$

$$y_2 = e^{-ax/2}\sin\omega x$$

Therefore, $y = Ay_1 + By_2$, where $C_1 = \frac{1}{2}(A - iB)$ and $C_2 = \frac{1}{2}(A + iB)$

Since $y_1/y_2 = \cot \omega x$, $\omega \neq 0$, is not constant, y_1 and y_2 are linearly independent. We therefore have the following general solution:

$$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$$

where A and B are arbitrary constants.

[Example] Solve $y'' + y' + y = 0$; $y(0) = 1,\ y'(0) = 3$

[Solution]

The characteristic equation is $\lambda^2 + \lambda + 1 = 0$, which has the solutions

$$\lambda_1 = \frac{-1 + i\sqrt{3}}{2} \qquad \lambda_2 = \frac{-1 - i\sqrt{3}}{2}$$

Thus, the general solution is $y(x) = e^{-x/2}\left[A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right]$

The constants A and B can be evaluated by considering the initial conditions:

$$y(0) = 1 \Rightarrow A = 1$$

$$y'(0) = 3 \Rightarrow \frac{\sqrt{3}}{2} B - \frac{1}{2} A = 3$$

$$\Rightarrow \quad A = 1 \quad ; \quad B = \frac{7}{\sqrt{3}}$$

Thus

$$y(x) = e^{-x/2}\left[\cos \frac{\sqrt{3}}{2} x + \frac{7}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} x \right]$$

Complex Exponential Function

Let $z = s + it \Rightarrow e^{z_1 + z_2} = e^{z_1} e^{z_2}$

$\therefore e^{z} = e^{s+it} = e^{s} e^{it}$

Expand e^{it} in Maclaurin series:

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$$

$$= \cos t + i \sin t$$

$$\therefore e^{z} = e^{s}(\cos t + i \sin t)$$

Summary

For the second–order homogeneous linear differential equation

$$y'' + a\,y' + b\,y = 0$$

the characteristic equation is

$$\lambda^2 + a\,\lambda + b = 0$$

The general solution of the differential equation can be classified by the types of the roots of the characteristic equation:

Case	Roots of λ	General Solution
I	Distinct real $\lambda_1,\ \lambda_2$	$y = c_1\, e^{\lambda_1 x} + c_2\, e^{\lambda_2 x}$
II	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\,\omega$, $\lambda_2 = -\frac{1}{2}a - i\,\omega$	$y = e^{-ax/2}\,(A\cos\omega x + B\sin\omega x)$
III	Real double root $\lambda_1 = \lambda_2 = -\frac{1}{2}a$	$y = (c_1 + c_2\,x)\,e^{-ax/2}$

Riccati Equation (Nonlinear 1st-order ODE)

Linear 2nd order ODEs may also be used in finding the solution to a **special form** of Riccati Equation:

Original: $y' + g(x)\,y + h(x)\,y^2 = k(x)$

Special Case: $y' + y^2 + p(x)\,y + q(x) = 0$

Let $y = \dfrac{z'}{z}$ then $y' = \dfrac{z''}{z} - \left[\dfrac{z'}{z}\right]^2$

thus the special Riccati equation becomes

$$\frac{z''}{z} - \left[\frac{z'}{z}\right]^2 + \left[\frac{z'}{z}\right]^2 + p(x)\frac{z'}{z} + q(x) = 0$$

or $z'' + p(x)\,z' + q(x)\,z = 0$

If the general solution to the above equation can be found, then

$$y = \frac{z'}{z}$$

is the general solution to the Riccati equation.

[Exercise 1] Solve $y' + y^2 + 2y + 1 = 0$, $y(0) = 0$

[Exercise 2] Solve $x^2 y' + x\,y + x^2 y^2 = 1$

Differential Operators

The symbol of differentiation d/dx can be replaced by D, i.e.,

$$Dy = \frac{dy}{dx} = y'$$

where D is called *the* ***differential operator*** *which transforms y into its derivative y'*. For example:

$$D(x^2) = 2x$$

$$D(\sin x) = \cos x$$

$$D^2y = D(Dy) = D(y') = y''$$

$$D^3y = y'''$$

In addition, $y'' + a y' + b y$ (where a, b are constant) can be written as

$D^2y + a Dy + b y$ or $L[y] = P(D)[y] = (D^2 + aD + b)[y] = y'' + ay' + by$

where P(D) is called a second–order (linear) differential operator. The homogeneous linear differential equation, $y'' + a y' + b y = 0$, may be written as

$$(D^2 + a D + b)y = 0 \quad \text{or} \quad L[y] = P(D)[y] = 0$$

[Example]

Calculate $(3D^2 - 10D - 8)\,x^2$, $(3D+2)\,(D-4)x^2$, and $(D-4)\,(3D+2)\,x^2$

[Solution]

$$
\begin{aligned}
(3D^2 - 10D - 8)\,x^2 &= 3D^2x^2 - 10Dx^2 - 8x^2 \\
&= 6 - 20x - 8x^2
\end{aligned}
$$

$$
\begin{aligned}
(3D + 2)(D - 4)x^2 &= (3D + 2)\,(Dx^2 - 4x^2) \\
&= (3D + 2)\,(2x - 4x^2) \\
&= 3D(2x - 4x^2) + 2(2x - 4x^2) \\
&= 6 - 24x + 4x - 8x^2 \\
&= 6 - 20x - 8x^2
\end{aligned}
$$

$$
\begin{aligned}
(D - 4)(3D + 2)x^2 &= (D - 4)\,(3Dx^2 + 2x^2) \\
&= (D - 4)\,(6x + 2x^2) \\
&= D(6x + 2x^2) - 4(6x + 2x^2) \\
&= 6 + 4x - 24x - 8x^2 \\
&= 6 - 20x - 8x^2
\end{aligned}
$$

Note that $(3D^2 - 10D - 8) = (3D + 2)\,(D - 4) = (D-4)\,(3D + 2)$

The above example seems to imply that *the operator D can be handled as though it were a* ***simple algebraic quantity***.

But...

[Example] Is $(D+1)(D+x)e^x = (D+x)(D+1)e^x$?

[Solution]

$$
\begin{aligned}
(D+1)(D+x)e^x &= (D+1)(De^x + x\,e^x) \\
&= (D+1)(e^x + x\,e^x) \\
&= D(e^x + x\,e^x) + (e^x + x\,e^x) \\
&= e^x + e^x + x\,e^x + e^x + x\,e^x \\
&= \underline{3\,e^x} + 2\,x\,e^x
\end{aligned}
$$

$$
\begin{aligned}
(D+x)(D+1)e^x &= (D+x)(De^x + e^x) \\
&= (D+x)(e^x + e^x) \\
&= (D+x)(2e^x) \\
&= D(2e^x) + 2\,x\,e^x \\
&= \underline{2e^x} + 2\,x\,e^x
\end{aligned}
$$

Thus, $(D+1)(D+x)\,e^x \neq (D+x)(D+1)\,e^x$

This example illustrates that *interchange of the order of factors containing **variable co-efficients** are not allowed.* e.g., $xDy \neq Dxy$, or in general, $P_1(D)\, P_2(D) \neq P_2(D)\, P_1(D)$

[Question] Is $(x^2 D)(x D)\, y = (x D)(x^2 D)\, y$?

[Example] Factor $L(D) = D^2 + D - 6$ and solve $L(D)y = 0$

[Solution]

$$L(D) = D^2 + D - 6 = (D+3)(D-2)$$
$$L(D)y = y'' + y' - 6y = 0$$

has the linearly independent solutions

$$y_1 = e^{-3x} \quad \text{and} \quad y_2 = e^{2x}$$

Note that

$$(D+3)(D-2)\, y = 0$$

can be factored as

$$(D+3)\, y = 0 \quad \Rightarrow \quad y = e^{-3x}$$
$$(D-2)\, y = 0 \quad \Rightarrow \quad y = e^{2x}$$

which also form the basis of $L(D)y = 0$.

4 **Euler Equations** (Linear 2nd-order ODE with variable coefficients)

For most linear second–order equations with variable coefficients, it is necessary to use techniques such as the power series method to obtain information about solutions. However, there is one class of such equations for which closed–form solutions can be obtained – the *Euler equation*:

$$x^2 y'' + a x y' + b y = 0, \qquad x \neq 0$$

We now *guess* that the form of the solutions of the above equation be

$$y = x^m$$

and put the derivatives of y into the Euler equation, we have

$$x^2 m (m-1) x^{m-2} + a x m x^{m-1} + b x^m = 0$$

If $x \neq 0$, we can divide the above equation by x^m to obtain the characteristic equation for Euler equation:

$$m (m-1) + a m + b = 0 \qquad \text{or}$$

$$m^2 + (a-1) m + b = 0 \quad \textbf{(Characteristic Equation)}$$

As with the constant–coefficient equations, there are three cases to consider:

Case I *Two Distinct Real Roots m_1 and m_2*

In this case, x^{m1} and x^{m2} constitute a basis of the Euler equation. Thus, the general solution is

$$y = c_1 x^{m1} + c_2 x^{m2}$$

Case II *The Roots are Real and Equal $m_1 = m_2 = m = (1-a)/2$*

In this case, x^m is a solution of the Euler equation. To find a second solution, we can use the method of reduction of order and obtain (Exercise!):

$$y_2 = x^m \ln |x|$$

Thus, the general solution is

$$y = x^m (c_1 + c_2 \ln |x|)$$

Case III The Roots are Complex Conjugates $\mu \pm i\,\nu$

This case is of **no great practical importance.** The two linearly independent solutions of the Euler equation are

$$x^{i\nu} = \left(e^{\ln x}\right)^{i\nu} = e^{i\nu \ln x} = \cos(\nu \ln x) + i\sin(\nu \ln x)$$

$$x^{m_1} = x^{\mu+i\nu} = x^{\mu}\left[\cos(\nu \ln x) + i\sin(\nu \ln x)\right]$$

$$x^{m_2} = x^{\mu-i\nu} = x^{\mu}\left[\cos(\nu \ln x) - i\sin(\nu \ln x)\right]$$

By adding and subtracting these two equations

$$x^{\mu} \cos(\nu \ln |x|) \quad \text{and} \quad x^{\mu} \sin(\nu \ln |x|)$$

Thus, the general solution is

$$y = x^{\mu} \left[A\cos(\nu \ln |x|) + B\sin(\nu \ln |x|) \right]$$

[Example] $x^2 y'' + 2xy' - 12y = 0$

[Solution] The characteristic equation is

$$m(m-1) + 2m - 12 = 0$$

with roots $m = -4$ and 3

Thus, the general solution is

$$y = c_1 x^{-4} + c_2 x^3$$

[Example] $x^2 y'' - 3xy' + 4y = 0$

[Solution] The characteristic equation is

$$m(m-1) - 3m + 4 = 0$$

$$m = 2,\ 2 \quad \text{(double roots)}$$

Thus, the general solution is

$$y = x^2 (c_1 + c_2 \ln|x|)$$

[Example] $x^2 y'' + 5 x y' + 13 y = 0$

[Solution] The characteristic equation is

$$m (m - 1) + 5 m + 13 = 0$$

or $m = -2 + 3i$ and $-2 - 3i$

Thus, the general solution is

$$y = x^{-2} [c_1 \cos (3 \ln|x|) + c_2 \sin (3 \ln|x|)]$$

[Exercise 1] The Euler equation of the third order is

$$x^3 y''' + a x^2 y'' + b x y' + c y = 0$$

Show that $y = x^m$ is a solution of the equation if and only if m is a root of the characteristic equation

$$m^3 + (a - 3) m^2 + (b - a + 2) m + c = 0$$

What is the characteristic equation for the n^{th} order Euler equation?

[Exercise 2] An alternative method to solve the Euler equation is by making the substitution

$$x = e^z \quad \text{or} \quad z = \ln x$$

Show that he homogeneous second–order Euler equation

$$x^2 y'' + a x y' + b y = 0, \; x \neq 0$$

can be transformed into the constant–coefficient equation

$$\frac{d^2 y}{dz^2} + (a-1)\frac{dy}{dz} + by = 0$$

[Exercise 3] $(x^2 + 2x + 1) y'' - 2(x+1) y' + 2y = 0$

[Exercise 4] $(3x+4)^2 y'' - 6(3x+4) y' + 18y = 0$

[Exercise 5] $y'' + (2e^x - 1) y' + e^{2x} y = 0$ (Hint: Let $z = e^x$)

5 Existence and Uniqueness of Solutions

5.1 Second–Order Differential Equations

Consider the *initial value problem* (IVP):

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad (1a)$$

with $$y(x_0) = k_0 \;,\; y'(x_0) = k_1 \quad (1b)$$

Note that (1a) is a 2^{nd}-order, linear homogeneous differential equation.

Theorem–Existence and Uniqueness Theorem

If $p(x)$ and $q(x)$ are continuous functions on an open interval I and x_0 is in I, then the initial value problem, (1a) and (1b), has a unique solution $y(x)$ on the interval.

Wronskian–Definition

The *Wronskian* of two solutions y_1 and y_2 of (1a) is defined as

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Theorem–Linear Dependence and Independence of Solutions

If $p(x)$ and $q(x)$ of the equation

$$y'' + p(x)\,y' + q(x)\,y = 0$$

are continuous on an open interval I, then the two solutions $y_1(x)$ and $y_2(x)$ on I are linearly dependent, iff (if and only if) $W(y_1, y_2) = 0$ for some $x = x_0$ in I.

Furthermore, if $W=0$ for $x = x_0$, then $W \equiv 0$ on I; hence if there is an x_1 in I at which W is not zero, then y_1 and y_2 are linearly independent on I.

[Proof]:

(1) If solutions y_1 and y_2 are linearly *dependent* on I $\Rightarrow$ $W(y_1, y_2) = 0$

If y_1 and y_2 are linearly dependent on I, then

$$y_1 = c\, y_2 \quad \text{or} \quad y_2 = k\, y_1$$

This is true for any two linearly-dependent functions!

If we take $y_1 = c\, y_2$, then

$$W(y_1, y_2) = W(cy_2, y_2) = \begin{vmatrix} cy_2 & y_2 \\ cy_2' & y_2' \end{vmatrix} = 0$$

Similarly, when $y_2 = k\, y_1$, $W(y_1, y_2) = 0$.

(2) $W(y_1, y_2) = 0$ at $x = x_0 \quad \Rightarrow \quad y_1, y_2$ linearly dependent

We need to prove that if $W(y_1, y_2) = 0$ for some $x = x_0$ on I, then y_1 and y_2 are linearly dependent.

- Determine nontrivial constants $\bar{c}_1$ and $\bar{c}_1$ at $x = x_0$:

We consider the system of linear equations:

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

where c_1 and c_2 are constants to be determined. Since the determinant of the above set of equations is

$$y_1(x_0)\, y_2'(x_0) - y_1'(x_0)\, y_2(x_0) \quad = \quad W(y_1(x_0), y_2(x_0)\,) \quad = \quad 0$$

we have a **nontrivial** solution for c_1 and c_2; that is, $\bar{c}_1$ and $\bar{c}_2$ are not both zero.

- Show that $y = \bar{c}_1 y_1 + \bar{c}_2 y_2 \equiv 0$ on I

Using these numbers $\bar{c}_1$ and $\bar{c}_2$, we define

$$y = \bar{c}_1 y_1(x) + \bar{c}_2 y_2(x) \qquad (*)$$

Since $y_1(x)$ and $y_2(x)$ are solutions to the differential equation, y is also a solution. Note that

$$y(x_0) = \bar{c}_1 y_1(x_0) + \bar{c}_2 y_2(x_0) = 0$$
$$y'(x_0) = \bar{c}_1 y_1'(x_0) + \bar{c}_2 y_2'(x_0) = 0$$

Thus, y(x) in equation (*) solves the initial value problem

$$y'' + p(x) y' + q(x) y = 0,$$

$$\text{IC:} \quad y(x_0) = y'(x_0) = 0$$

But this initial value problem also has the solution $y^*(x) = 0$ for all values on I. From the existence and uniqueness theorem, the solution of this initial value problem is unique so that

$$y(x) = y^*(x) = \bar{c}_1 y_1(x) + \bar{c}_2 y_2(x) = 0$$

for all values on I.

- Establish linear dependence between y_1 and y_2

Now since $\overline{c}_1$ and $\overline{c}_2$ are not both zero, this proves that y_1 and y_2 are linearly dependent.

Implication:

> If $W(y_1, y_2) \neq 0$ at $x = x_1$ in I, then
> $y_1(x)$ and $y_2(x)$ are linearly independent!

Alternative Proof by Abel's Formula

$$W = y_1 y_2' - y_2 y_1'$$
$$W' = (y_1 y_2' - y_2 y_1')' = y_1'y_2' + y_1'y_2'' - y_2'y_1' - y_2y_1''$$
$$= y_1 y_2'' - y_2 y_1''$$

Since y_1 and y_2 are solutions to $y'' + p(x)\, y' + q(x)\, y = 0$, we have

$$y_1'' + p(x)\, y_1' + q(x)\, y_1 = 0$$

and $$y_2'' + p(x)\, y_2' + q(x)\, y_2 = 0$$

Multiplying the first of these equations by y_2 and the second by y_1 and subtracting, we obtain

$$y_1y_2'' - y_2y_1'' + p(x)(y_1y_2' - y_2y_1') = 0$$

or $$W' + p(x)\, W = 0$$

Thus,

$$W(y_1, y_2) = C\, e^{-\int p(x)\, dx}$$ **Abel's Formula**

where C is an arbitrary constant.

Since an exponential is never zero, we see that $W(y_1, y_2)$ is either always zero (when $C = 0$) or never zero (when $C \neq 0$).

Thus, if W = 0 for some $x = x_0$ in I, then W = 0 on the entire I. In addition, if there is an x_1 on I at which $W \neq 0$, then y_1 and y_2 are linearly independent on I.

[Example] $y_1 = \cos \omega x$, $y_2 = \sin \omega x$ $\omega \neq 0$

$$W(y_1, y_2) = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} = \omega \neq 0$$

thus, y_1 and y_2 are linearly independent.

Theorem–Existence of a General Solution

If $p(x)$ and $q(x)$ are continuous on an open interval I, then $y'' + p(x)y' + q(x)y = 0$ has a general solution.

Theorem–General Solution

Suppose that $y'' + p(x)y' + q(x)y = 0$ has continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then every solution $Y(x)$ of this equation on I is of the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

where y_1, y_2 form a basis of solution on I and $C_1,\ C_2$ are suitable constants. Hence, the above equation does not have singular solution.

6 Nonhomogeneous Linear Differential equations

6.1 General Concepts

A general solution of the nonhomogeneous linear differential equation

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = r(x)$$

on some interval I is a solution of the form

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x) = c_1\, y_1(x) + \ldots + c_n\, y_n(x)$ is a solution of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)\, y' + p_0(x)\, y = 0$$

and $y_p(x)$ is a particular solution of the nonhomogeneous equation.

$$y'' + p(x)y' + q(x)y = r(x) \text{ ---------------(1)}$$

$$y'' + p(x)y' + q(x)y = 0 \text{ -------------------(2)}$$

Relations between solutions of (1) and (2):

- The difference of two solutions of (1) on some open interval I is a solution of (2) on I.
- The sum of a solution of (1) on I and a solution of (2) on I is a solution of (1) on I.

[Example]

$$y(x) = c_1 e^x + c_2 e^{3x} + \frac{2}{3} e^{-2x}$$

is the solution of

$$y'' - 4y' + 3y = 10 e^{-2x}$$

where $y_h(x) = c_1 e^x + c_2 e^{3x}$ is the general solution of

$$y'' - 4y' + 3y = 0$$

and $y_p(x) = \dfrac{2}{3} e^{-2x}$ satisfies the nonhomogeneous equation, i.e., $y_p(x)$ is a particular solution of the nonhomogeneous equation.

There are two methods to obtain the particular solution $y_p(x)$: (1) *Method of Undetermined Coefficients* and (2) *Method of Variation of Parameters*. Our main task in the following is to discuss these two methods for finding $y_p(x)$.

6.2 Method of Undetermined Coefficients

[Example 1] $y'' + 4y = 12$

The general solution of $y'' + 4y = 0$ is

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x$$

If we assume the particular solution

$$y_p(x) = k$$

then we have $y_p'' = 0$, and

$$4k = 12 \quad \text{or} \quad k = 3 \qquad \textbf{ok!}$$

Thus the general solution of the nonhomogeneous equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + 3$$

[Example 2] $y'' + 4y = 8x^2$

If we now **assume** the particular solution is of the form

$$y_p(x) = m x^2$$

then $y_p''(x) = 2m$

and $2m + 4mx^2 = 8x^2$

However, since the above equation is valid for any value of x, we need

$$m = 0 \quad \text{and} \quad m = 2$$

which is **not possible**.

If we now **assume** the particular solution is of the form

$$y_p(x) = m\,x^2 + n\,x + q$$

then $y_p' = 2\,m\,x + n$

$y_p'' = 2\,m$

thus $2\,m + 4\,(m\,x^2 + n\,x + q) = 8\,x^2$

or $4\,m\,x^2 + 4\,n\,x + (2\,m + 4\,q) = 8\,x^2$

or $$\begin{cases} 4\,m = 8 \\ 4\,n = 0 \\ 2\,m + 4\,q = 0 \end{cases}$$

or $m = 2 \quad n = 0 \quad q = -1$

$y_p(x) = 2\,x^2 - 1$

and $y(x) = c_1 \cos 2x + c_2 \sin 2x + 2\,x^2 - 1$

[Example 3] $y'' - 4y' + 3y = 10e^{-2x}$

The general solution of the homogeneous equation

$$y'' - 4y' + 3y = 0$$

is $\quad y_h(x) = c_1 e^x + c_2 e^{3x}$

If we assume a particular solution of the nonhomogeneous equation is of the form

$$y_p(x) = k e^{-2x}$$

then $\quad y_p' = -2k e^{-2x} \qquad y_p'' = 4k e^{-2x}$

and $\quad 4k e^{-2x} - 4(-2k e^{-2x}) + 3(k e^{-2x}) = 10e^{-2x}$

or $\quad 15k e^{-2x} = 10e^{-2x}$

or $\quad k = 2/3$

Thus $\quad y(x) = c_1 e^x + c_2 e^{3x} + \frac{2}{3} e^{-2x}$

[Example 4] $y'' + y = x e^{2x}$

The general solution to the homogeneous equation is

$$y_h = c_1 \sin x + c_2 \cos x$$

Since the nonhomogeneous term is of the form

$$x e^{2x}$$

If we assume the particular solution be

$$y_p = k x e^{2x}$$

we will have

$$k (4e^{2x} + 4 x e^{2x}) + k x e^{2x} = x e^{2x}$$

or $k = 0$ and $5k = 1$

which is not possible.

So we try a solution of the form

$$y_p = e^{2x}(m + n\,x)$$

we will have

$$y_p = \frac{e^{2x}}{25}\,(5\,x - 4)$$

Therefore, the general solution of this example is

$$y(x) = c_1 \sin x + c_2 \cos x + \frac{e^{2x}}{25}\,(5\,x - 4)$$

[Example 5] $y'' + 4y' + 3y = 5 \sin 2x$

The general solution of the homogeneous equation is

$$y_h = c_1 e^{-x} + c_2 e^{-3x}$$

If we assume the particular solution be of the form

$$y_p = k \sin 2x$$

then $y_p' = 2k \cos 2x \qquad y_p'' = -4k \sin 2x$

$$-4k \sin 2x + 4(2k \cos 2x) + 3k \sin 2x = 5 \sin 2x$$

or $\quad -k \sin 2x + 8k \cos 2x = 5 \sin 2x$

since the above equation is valid for any values of x, we need

$$-k = 5 \qquad \underline{\text{and}} \qquad 8k = 0$$

which is not possible.

We now assume

$$y_p = m \sin 2x + n \cos 2x$$

and substitute y_p, y_p' and y_p'' into the nonhomogeneous equation, we have

$$m = -\frac{1}{13} \quad \text{and} \quad n = -\frac{8}{13}$$

Thus $$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{13}(\sin 2x + 8 \cos 2x)$$

[Example 6] $y'' - 3y' + 2y = e^x \sin x$

The general solution to the homogeneous equation is

$$y_h = c_1 e^x + c_2 e^{2x}$$

Since the $r(x) = e^x \sin x$, we assume the particular solution of the form

$$y_p = m e^x \sin x + n e^x \cos x$$

Substituting the above equation into the differential equation and equating the coefficients of $e^x \sin x$ and $e^x \cos x$, we have

$$y_p = \frac{e^x}{2}(\cos x - \sin x)$$

and $$y(x) = c_1 e^x + c_2 e^{2x} + \frac{e^x}{2}(\cos x - \sin x)$$

[Example 7] $y'' + 2y' + 5y = 16e^x + \sin 2x$

The general solution of the homogeneous equation is

$$y_h = e^{-x}(c_1 \sin 2x + c_2 \cos 2x)$$

Since the nonhomogeneous term r(x) contains terms of e^x and sin 2x, we can assume the particular solution of the form

$$y_p = c\,e^x + m \sin 2x + n \cos 2x$$

After substitution the above y_p into the nonhomogeneous equation, we arrive

$$y_p = 2e^x - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

Thus

$$y(x) = e^{-x}(c_1 \sin 2x + c_2 \cos 2x) + 2e^x - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

[Example 8] $y'' - 3y' + 2y = e^x$

The general solution of the homogeneous equation is

$$y_h(x) = c_1 e^x + c_2 e^{2x}$$

If we assume the particular solution be of the form

$$y_p = k e^x$$

we would have

$$k - 3k + 2k = 1$$

or $$0 = 1$$

which is not possible (Recall that $k e^x$ satisfies the homogeneous equation). We need to try a different form for y_p.

Assume

$$y_p = k\,x\,e^x$$

then $y_p' = k\,(e^x + x\,e^x)$ $y_p'' = k\,(2\,e^x + x\,e^x)$

and $k\,(2\,e^x + x\,e^x) - 3\,k\,(e^x + x\,e^x) + 2\,k\,x\,e^x = e^x$

or $-k = 1$ or $k = -1$

Thus, $y = c_1\,e^x + c_2\,e^{2x} - x\,e^x$

[Example 9] $y'' - 2y' + y = e^x$

The general solution of the homogeneous equation is

$$y_h = (c_1 + c_2 x)\, e^x = c_1 e^x + c_2 x e^x$$

If we assume the particular solution of the nonhomogeneous equation be

$$y_p = k e^x \qquad \text{or} \qquad y_p = k x e^x$$

we would arrive some conflict equations for k.

If we assume $y_p = k x^2 e^x$

then we have $k = \frac{1}{2}$

thus $\quad y(x) = (c_1 + c_2 x)\, e^x + \frac{1}{2} x^2 e^x$

In summary, for a constant coefficient nonhomogeneous linear differential equation of the form

$$y^{(n)} + a\,y^{(n-1)} + \ldots + f\,y' + g\,y \;=\; r(x)$$

we have the following rules for the method of undetermined coefficients:

(A) Basic Rule: *If $r(x)$ in the nonhomogeneous differential equation is one of the functions in the first column in the following table, choose the corresponding function y_p in the second column and determine its undetermined coefficients by substituting y_p and its derivatives into the nonhomogeneous equation.*

(B) Modification Rule: *If any term of the suggested solution $y_p(x)$ is the solution of the corresponding homogeneous equation, multiply y_p by x repeatedly until no term of the product $x^k y_p$ is a solution of the homogeneous equation. Then use the product $x^k y_p$ to solve the nonhomogeneous equation.*

(C) Sum Rule: *If $r(x)$ is sum of functions listed in several lines of the first column of the following table, then choose for y_p the sum of the functions in the corresponding lines of the second column.*

Table for Choosing the Particular Solution

$r(x)$		$y_p(x)$	
$P_n(x)$		$a_0 + a_1 x + \dots + a_n x^n$	
$P_n(x)\, e^{ax}$		$(a_0 + a_1 x + \dots + a_n x^n)\, e^{ax}$	
$P_n(x)\, e^{ax} \sin bx$		$(a_0 + a_1 x + \dots + a_n x^n)\, e^{ax} \sin bx$	
+	***and/or***	+	***and***
$Q_n(x)\, e^{ax} \cos bx$		$(c_0 + c_1 x + \dots + c_n x^n)\, e^{ax} \cos bx$	

where $P_n(x)$ and $Q_n(x)$ are polynomials in x of degree n ($n \varepsilon 0$).

[Example 10] $y'' - 4y' + 4y = 6xe^{2x}$

[Solution] $y_h = c_1 e^{2x} + c_2 x e^{2x}$

y_p first guess: $y_p = (a + bx)e^{2x}$ No!

$y_p = x(a + bx)e^{2x}$ No!

$y_p = x^2(a + bx)e^{2x}$ O.K.

[Example 11] $y'' - 2y' + y = e^x + x$

[Solution] $y_h = (c_1 + c_2 x)e^x$

Guess of y_p: $y_p = a + bx + ce^x$ No!

$y_p = a + bx + cxe^x$ No!

$y_p = a + bx + cx^2e^x$ O.K.

.... $\Rightarrow$ $y_p = 2 + x + \frac{1}{2}x^2e^x$

[Example 12] $x^2 y'' - 5xy' + 8y = 2\ln x, \quad x > 0$

[Solution] Note that the above equation is not of constant coefficient type!

Let $z = \ln x$, or $x = e^z$, then

$$x^2 y'' + axy' + by = 0 \Rightarrow \frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = 0$$

thus, $x^2 y'' - 5xy' + 8y = 2\ln x$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = 2z \therefore \frac{d^2y}{dz^2} - 6\frac{dy}{dz} + 8y = 2z$$

$$y_h = c_1 e^{4z} + c_2 e^{2z} \quad \text{and} \quad y_p = cz + d = \frac{1}{4}z + \frac{3}{16}$$

$$\therefore \quad y(z) = c_1 e^{4z} + c_2 e^{2z} + \frac{1}{4}z + \frac{3}{16}$$

$$\Rightarrow \quad y(x) = c_1 x^4 + c_2 x^2 + \frac{1}{4}\ln x + \frac{3}{16}$$

[Exercise 1] (a) $x^2 y'' - 4xy' + 6y = x^2 - x$

[Answer] $y = c_1 x^2 + c_2 x^3 - \dfrac{x}{2} - x^2 \ln x$

(b) $y'' - y = x \sin x$

(c) $y'' - y = x e^x \sin x$

(d) $y'' + y = -2 \sin x + 4 x \cos x$

(e) $(D^2 + 1)(D - 1)y = x e^{2x} + \cos x$

(f) $y'' - 4y' + 4y = x e^{2x}$, with $y(0) = y'(0) = 0$

[Exercise 2] Transform the following Euler differential equation into a constant coefficient linear differential equation by the substitution $z = \ln(x)$ and find the particular solution $y_p(z)$ of the transformed equation by the <u>method of undetermined coefficients</u>:

$$x^2 y'' - x y' - 8y = x^4 - 3\ln(x) \quad ; \quad x > 0$$

6.2 Method of Variation of Parameters

In this section, we shall consider a procedure for finding a particular solution of *any* nonhomogeneous second–order linear differential equation

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

where p(x), q(x) and r(x) are continuous on an open interval I.

Assume that the general solution of the corresponding homogeneous equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

is **given** $y_h = c_1 y_1 + c_2 y_2$

where, y_1 and y_2 are *linearly independent* **known functions**, c_1 and c_2 are arbitrary constants.

Suppose that the particular solution of the nonhomogeneous equation is of the form

$$y_p = u(x)\, y_1(x) + v(x)\, y_2(x)$$

This replacement of constants or parameters by variables gives the method name "Variation of Parameters".

Notice that the assumed particular solution y_p contains two unknown functions u and v. The requirement that the particular solution satisfies the non-homogeneous differential equation imposes only one condition on u and v.

It seems plausible we can impose a second *arbitrary* condition. By differentiating y_p, we have

$$y_p' = u'\, y_1 + u\, y_1' + v'\, y_2 + v\, y_2'$$

To simplify this expression, it is convenient to set

$$u'\, y_1 + v'\, y_2 = 0$$

(Condition 1)

This reduces the expression for y_p' to

$$y_p' = u\, y_1' + v\, y_2'$$

Differentiating once again, we have

$$y_p'' = u'\, y_1' + u\, y_1'' + v'\, y_2' + v\, y_2''$$

Putting y_p'', y_p' and y_p into the nonhomogeneous equation and collecting terms, we have

$$u\,(y_1'' + p\, y_1' + q\, y_1) + v\,(y_2'' + p\, y_2' + q\, y_2) + u'\, y_1' + v'\, y_2' = r$$

Since y_1 and y_2 are the solutions of the homogeneous equation, we have

$$u'\, y_1' + v'\, y_2' = r$$

(Condition 2)

This gives a second equation relating u' and v', and we have the simultaneous equations

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ y_1' u' + y_2' v' &= r \end{aligned}$$

which has the solution

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = -\frac{y_2 r}{W} \qquad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 r}{W}$$

where $W = y_1 y_2' - y_1' y_2 \neq 0$

is the Wronskian of y_1 and y_2. **Notice that y_1 and y_2 are linearly independent!**

After integration, we have

$$u = -\int \frac{y_2\, r}{W}\, dx \qquad v = \int \frac{y_1\, r}{W}\, dx$$

Thus, the particular solution y_p is

$$y_p(x) = -y_1 \int \frac{y_2\, r}{W}\, dx + y_2 \int \frac{y_1\, r}{W}\, dx$$

[Example 1] $y'' - y = e^{2x}$

The general solution to the homogeneous equation is

$$y_h = c_1 e^{-x} + c_2 e^{x}$$

i.e., $y_1 = e^{-x} \qquad y_2 = e^{x}$

The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 2$$

thus, $$u' = -\frac{y_2 r}{W} = -\frac{e^{x} e^{2x}}{2} = \frac{-e^{3x}}{2}$$

$$v' = \frac{y_1 r}{W} = \frac{e^{-x} e^{2x}}{2} = \frac{e^{x}}{2}$$

Integrating these functions, we obtain

$$u = -\frac{e^{3x}}{6} \qquad v = \frac{e^{x}}{2}$$

A particular solution is therefore

$$y_p = u\,y_1 + v\,y_2 = -\frac{e^{3x}}{6}e^{-x} + \frac{e^{x}}{2}e^{x} = \frac{e^{2x}}{3}$$

and the general solution is

$$y(x) = y_h + y_p = c_1\,e^{-x} + c_2\,e^{x} + \frac{e^{2x}}{3}$$

[Example 2] $y'' + y = \tan x$

The general solution to the homogeneous equation is

$$y_h = c_1 \cos x + c_2 \sin x$$

thus, $y_1 = \cos x \quad y_2 = \sin x$

Also $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

so that $u' = -\dfrac{y_2 r}{W} = -\sin x \tan x$

$$v' = \frac{y_1 r}{W} = \cos x \tan x = \sin x$$

Hence $$u = \int -\frac{\sin^2 x}{\cos x} dx = \int \frac{\cos^2 x - 1}{\cos x} dx = \int \cos x dx - \int \sec x dx$$

Since by looking up table

$$\int \sec dx = \ln|\sec x + \tan x| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$$

Thus,

$$u = \sin x - \ln|\sec x + \tan x|$$

$$v = -\cos x$$

Thus, the particular solution is

$$y_p = u y_1 + v y_2 = -\cos x \ln|\sec x + \tan x|$$

and the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$$

[Example 3] $x^2 y'' + 2xy' - 12y = \sqrt{x}$

The homogeneous part is a variable-coefficient **Euler equation**. The general solution is

$$y_h = c_1 x^{-4} + c_2 x^3$$

or $\quad y_1 = x^{-4} \qquad y_2 = x^3$

and $\quad W = \begin{vmatrix} x^{-4} & x^3 \\ -4x^{-5} & 3x^2 \end{vmatrix} = 7x^{-2}$

or $\quad \dfrac{1}{W} = \dfrac{x^2}{7}$

In order to use the method of variation of parameters, we must write the differential equation in the standard form in order to obtain the correct r(x), i.e.,

$$y'' + \frac{2}{x} y' - \frac{12}{x^2} y = x^{-3/2} \quad \text{or} \quad r(x) = x^{-3/2}$$

Thus, $u' = -\dfrac{y_2\, r}{W} = -x^3\, x^{-3/2}\, \dfrac{x^2}{7} = -\dfrac{x^{7/2}}{7}$

and $v' = \dfrac{y_1\, r}{W} = x^{-4}\, x^{-3/2}\, \dfrac{x^2}{7} = \dfrac{x^{-7/2}}{7}$

Hence $u = -\dfrac{1}{7}\dfrac{2}{9}\, x^{9/2} \qquad v = -\dfrac{1}{7}\dfrac{2}{5}\, x^{-5/2}$

so that $y_p = u\, y_1 + v\, y_2$

$$= -\frac{2}{63} x^{9/2} x^{-4} - \frac{2}{35} x^{-5/2} x^3$$

$$= -\frac{4}{45}\, x^{1/2}$$

Thus, the general solution is given by

$$y(x) = c_1\, x^{-4} + c_2\, x^3 - \frac{4}{45}\, x^{1/2}$$

[Example 4] $(D^2 + 2D + 1)\, y = e^{-x} \ln x$

[Solution] $y = y_h + y_p$

where y_h is the solution of $(D^2 + 2D + 1)\, y = 0$

or $y_h = c_1 e^{-x} + c_2\, x\, e^{-x}$ $\therefore$ $y_1 = e^{-x}$, $y_2 = x\, e^{-x}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x}+e^{-x} \end{vmatrix} = e^{-2x}$$

$$\therefore \quad y_p(x) = -y_1 \int \frac{y_2\, r}{W}\, dx + y_2 \int \frac{y_1\, r}{W}\, dx$$

$$= -e^{-x} \int (x\, e^{-x})(e^{-x} \ln x)(e^{2x})\, dx + x\, e^{-x} \int (e^{-x})(e^{-x} \ln x)\, (e^{2x})\, dx$$

$$= -e^{-x} \int x \ln x\, dx + xe^{-x} \int \ln x\, dx$$

From Table:

$$\int \ln x dx = x \ln x - x$$

$$\int x \ln x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4}$$

$$y_p(x) = -e^{-x}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + xe^{-x}\left(x \ln x - x\right)$$

$$= e^{-x}\left(\frac{x^2}{2} \ln x - \frac{3}{4} x^2\right)$$

$$\therefore \quad y = c_1 e^{-x} + c_2 x e^{-x} + e^{-x}\left(\frac{x^2}{2} \ln x - \frac{3}{4} x^2\right)$$

[Exercise 1]

(a) Solve $x^2 y'' - 2 x y' + 2 y = x^2 + 2$

(b) $x^2 y'' - x y' - 8 y = x^4 - 3 \ln (x) \quad ; \quad x > 0$

(c) Solve $x y'' + y' - \dfrac{y}{x} = x e^x$

(d) Solve $y'' - 3y' + 2y = \cos(e^{-x})$

[Exercise 2][2] Consider the third–order equation

$$y''' + a(x)\, y'' + b(x)\, y' + c(x)\, y = f(x) \quad (1)$$

Let $y_1(x)$, $y_2(x)$ and $y_3(x)$ be three linearly independent solutions of the associated homogeneous equation. Assume that there is a solution of equation (1) of the form

$$y_p(x) = u(x)\, y_1(x) + v(x)\, y_2(x) + w(x)\, y_3(x)$$

(a) Following the steps used in deriving the variation of parameters procedure for second–order equations, derive a method for solving third–order equations.

$$\begin{aligned} y_1 u' + y_2 v' + y_3 w' &= 0 \\ y_1' u' + y_2' v' + y_3' w' &= 0 \\ y_1'' u' + y_2'' v' + y_3'' w' &= f \end{aligned}$$

(b) Find a particular solution of the equation

$$y''' - 2y' - 4y = e^{-x} \tan x$$

2 Grossman, S. I. and Derrick, W. R., Advanced Engineering Mathematics, p. 123, 1988.