Chapter 1

Indefinite Integration

1.1 Introduction

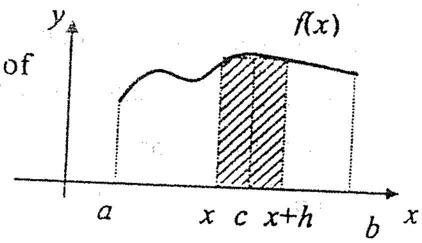
In the course of differential calculus, we studied the mathematical process of finding the derivative of a function, and we considered various applications of derivatives. In this course we will study another branch of calculus, called integral calculus. In differential calculus, the tangent problem led us to formulate, in terms of limits, the idea of a derivative, which later turned out to be applicable, through velocities and other rates of change, to a variety of applied problems.

In integral calculus, the area problem will lead us to formulate, again in terms of limits, the idea of an integral, which will later be used to find volumes, lengths of curves, work, and forces. The area problem states that: Given a function that is continuous and nonnegative in an interval

$[a, b]$, it is required to find the area of the region bounded by the graph of the curve, the interval $[a, b]$ on the x-axis and the vertical lines $x = a$ and $x = b$. The major development of solving this area problem was made independently by I. Newton and G. Leibniz in 1675. They discovered that areas could be obtained by reversing the process of differentiation. Their idea is that, to find the area $A(x)$ of the region bounded by the graph of a nonnegative and continuous function $f(x)$ on the interval $[a, x]$ and the x-axis (where x is any point on the x-axis), we first find the derivative of the area function $A(x)$, then we use the value of the derivative $A'(x)$ to determine $A(x)$ itself.

It is clear from the definition of the derivative that

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$



But, it can be seen from the opposite figure that

$$\frac{A(x+h) - A(x)}{h} \approx \frac{h.f(c)}{h} = f(c),$$

where c is a point between and $x + h$, and when $h \to 0, c \to x$. This implies that

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} f(c) = f(x).$$

The above result means that $A(x)$, which we are looking for, is a function whose derivative is $f(x)$.

The above discussion shows that there is a strong connection between the two branches of calculus. Namely, the differential and integral calculus. This connection is shown via the Fundamental Theorems of Calculus. These theorems greatly simplify the solution of many mathematical problems.

1.2 Antiderivative

We already discussed methods of finding derivatives of functions in the course of differential calculus. We will now turn our attention towards reversing the operation of differentiation. Given the derivative of a function, we are looking to find the function itself. This process is called ***antidifferentiation***. For example, if the derivative of the function is $3x^2$, we know that the function would be

$F(x)=x^3$ because $\frac{d}{dx}\left(x^3\right)=3x^2$. But the function could also be $F(x)=x^3+4$ because $\frac{d}{dx}\left(x^3+4\right)=3x^2$. It is clear that any function of the form $F(x)=x^3+C$, where C is a constant, will have $F'(x)=3x^2$ as its derivative. Thus, we say that the ***antiderivative*** of $f(x)=3x^2$ is $F(x)=x^3+C$, where C is an ***arbitrary constant***.

It is easily seen that

If $f(x)=x^4$, then $F(x)=\frac{x^5}{5}+C$

and,

If $f(x)=x^7$, then $F(x)=\frac{x^8}{8}+C$

In general,

$$\text{If } f(x)=x^n, \text{ then } F(x)=\frac{x^{n+1}}{n+1}+C, \quad \text{for } n\neq -1$$

<u>Example</u> (1): What are the antiderivatives of $f(x)=x^{11}$, $g(x)=x^{1/3}$ and $h(x)=x^{-1/2}$?

Solution: Using the above formula, we get

$$F(x) = \frac{x^{12}}{12} + C,$$

$$G(x) = \frac{x^{4/3}}{\frac{4}{3}} + C = \frac{3}{4}x^{4/3} + C,$$

and,

$$H(x) = \frac{x^{1/2}}{1/2} + C = 2x^{1/2} + C.$$

From the above discussion we reach the following definition of the antiderivative of a function.

Definition 1.1 (Antiderivative): The differentiable function $F(x)$ is called an ***antiderivative*** of $f(x)$ on a given interval, if $F'(x) = f(x)$, on that interval.

The following theorem gives the relation between different antiderivatives of a given function.

Theorem 1.1: If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then there is a constant C such that

$$F(x) - G(x) = C$$

(Two antiderivatives of a function can differ only by a constant.)

For example,

$$F(x) = x^3 + 2, \qquad G(x) = x^3, \qquad H(x) = x^3 - 6$$

are all antiderivatives of the function $f(x) = 3x^2$.

1.3 The Indefinite Integral

The process of finding an antiderivative is called ***integration***. The function that results when integration takes place is called ***indefinite integral***, or more simply, an ***integral***. We denote the indefinite integral of a function $f(x)$ by $\int f(x)dx$. The symbol $\int$ is called the ***integral sign*** and the function $f(x)$ is called the ***integrand***. The dx in the indefinite integral means that $\int f(x)\,dx$ is the integral of $f(x)$ with respect to the variable x just as the symbol $df(x)/dx$ means the derivative of $f(x)$ with respect to x. Thus we have the following definition

Definition 1.2 (Indefinite integral):

If $F(x)$ is any antidervative of a given function $f(x)$

i.e. $$F'(x) = f(x)$$

then, the indefinite integral of $f(x)$ with respect to the variable x is given by

$$\int f(x)\,dx = F(x) + C$$

The constant C is called "***the constant of integration***".

For example, using this notation,

$$\int 3x^2\,dx = x^3 + C$$

where C is the constant of integration.

1.4 Basic Integration Rules

Since integration is the inverse operation of differentiation so we have the following trivial rule.

Rule 1

$$\int \frac{df}{dx}\,dx = \int df = f(x) + C,$$

$$\frac{d}{dx}\int f(x)dx = f(x).$$

That is to say, the two symbols "$\frac{d}{dx}$" and "$\int$" are opposite operators, any one of them cancels the other.

Also, as shown earlier, the derivative of the product of a constant and a function is the product of the constant and the derivative of the function. A similar rule applies to indefinite integral. Moreover, since derivative of sums or differences are found term by term, indefinite integrals can also be found term by term. This is described by the following two rules.

Rule 2 **(Scalar Multiplication Rule)** The constant factor can be taken outside the integral sign. That is:
For any constant k, $\int kf(x)\,dx = k\int f(x)\,dx$.

Example (1)

$$\int 3x^5\,dx = 3\int x^5\,dx = 3\left(\frac{x^6}{6} + C_1\right) = \frac{1}{2}x^6 + C$$

Rule 3 (Sum or Difference Rule): The indefinite integral of the algebraic sum (or difference) of two functions equals the sum of their integrals:

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Example (2)

$$\int (x - x^4) dx = \int x dx - \int x^4 dx = \frac{x^2}{2} - \frac{x^5}{5} + C$$

Example (3)

$$\int (21x^{3/4} + 8x^3) dx = \int 21x^{3/4} dx + \int 8x^3 dx$$
$$= 21 \cdot \frac{x^{7/4}}{7/4} + 8 \cdot \frac{x^4}{4} + C = 12x^{7/4} + 2x^4 + C$$

Example (4)

$$\int (x-3)^2 dx = \int (x^2 - 6x + 9) dx$$
$$= \frac{x^3}{3} - 6 \cdot \frac{x^2}{2} + 9x + C = \frac{x^3}{3} - 3x^2 + 9x + C$$

1.5 Table of Famous Integrals

The following table summarizes the integral formulas for some elementary functions .

The Integrand	The Integral
$x^n, \quad n \neq -1$	$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$
$1/x$	$\int \frac{1}{x} \, dx = \ln\lvert x \rvert + C$
e^x	$\int e^x \, dx = e^x + C$
a^x	$\int a^x \, dx = \frac{a^x}{\ln a} + C$
$\cos x$	$\int \cos x \, dx = \sin x + C$
$\sin x$	$\int \sin x \, dx = -\cos x + C$
$\sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\cosh x$	$\int \cosh x \, dx = \sinh x + C$
$\sinh x$	$\int \sinh x \, dx = \cosh x + C$
$\operatorname{sech}^2 x$	$\int \operatorname{sech}^2 x \, dx = \tanh x + C$

The Integrand	The Integral
$\operatorname{csch}^2 x$	$\int \operatorname{csch}^2 x\, dx = -\coth x + C$
$\operatorname{sech} x \tanh x$	$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$
$\operatorname{csch} x \coth x$	$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C = -\cos^{-1} x + C$
$\dfrac{1}{1+x^2}$	$\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + C = -\cot^{-1} x + C$
$\dfrac{1}{x\sqrt{x^2-1}}$	$\int \dfrac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C = -\csc^{-1} x + C$
$\dfrac{1}{\sqrt{1+x^2}}$	$\int \dfrac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + C$
$\dfrac{1}{\sqrt{x^2-1}}$	$\int \dfrac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C$
$\dfrac{1}{1-x^2}, \lvert x\rvert < 1$	$\int \dfrac{1}{1-x^2} dx = \tanh^{-1} x + C$
$\dfrac{1}{1-x^2}, \quad \lvert x\rvert > 1$	$\int \dfrac{1}{1-x^2} dx = \coth^{-1} x + C$

The Integrand	The Integral
$\frac{1}{x\sqrt{1-x^2}}, \lvert x \rvert<1$	$\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1}x + C$
$\frac{1}{x\sqrt{1+x^2}}$	$\int \frac{1}{x\sqrt{1+x^2}}\,dx = -\operatorname{csch}^{-1}x + C$

Using rules 1,2 and 3 together with the above table we are able to solve the following example:

Example (1): Find each of the following integrals

(i) $\int\left(-\frac{4}{x} + 4e^x\right)dx$ (ii) $\int\left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{x\sqrt{1-x^2}}\right)dx$

(iii) $\int \frac{10x^2 - 1}{\sqrt{x}}\,dx$ (iv) $\int\left(\sec^2 x + \cos x - 3^x\right)dx$

(v) $\int\left[\frac{3}{\sqrt[3]{x}} + \frac{1}{2\sqrt{x}} + x\sqrt[4]{x}\right]dx$

Solution:

(i) $\int\left(-\frac{4}{x} + 4e^x\right)dx = -4\ln\lvert x\rvert + 4e^x + C$

(ii) $\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{x\sqrt{1-x^2}}\right)dx=\sin^{-1}x-\text{sech}^{-1}x+C$

(iii) $\int\frac{10x^2-1}{\sqrt{x}}dx=\int\left(\frac{10x^2}{\sqrt{x}}-\frac{1}{\sqrt{x}}\right)dx$

$$=\int\left(10x^{3/2}-x^{-1/2}\right)dx$$

$$=10\frac{x^{5/2}}{5/2}-\frac{x^{1/2}}{1/2}+C$$

$$=4x^{5/2}-2\sqrt{x}+C$$

(iv) $\int\left(\sec^2x+\cos x-3^x\right)dx=\tan x+\sin x-\frac{3^x}{\ln 3}+C$

(v) $\int\left[\frac{3}{\sqrt[3]{x}}+\frac{1}{2\sqrt{x}}+x\sqrt[4]{x}\right]dx=\int\left[3x^{-1/3}+\frac{1}{2}x^{-1/2}+x^{5/4}\right]dx$

$$=3\int x^{-1/3}dx+\frac{1}{2}\int x^{-1/2}dx+\int x^{5/4}dx$$

$$=3\frac{x^{2/3}}{2/3}+\frac{1}{2}\frac{x^{1/2}}{1/2}+\frac{x^{9/4}}{9/4}+C$$

$$=\frac{9}{2}x^{2/3}+x^{1/2}+\frac{4}{9}x^2\sqrt[4]{x}+C$$

$$=\frac{9}{2}\sqrt[3]{x^2}+\sqrt{x}+\frac{4}{9}x^2\sqrt[4]{x}+C$$

Rule 4

If $$\int f(x)\,dx = F(x) + C,$$

then,

(a) $\int f(ax)dx = \frac{1}{a}F(ax) + C$

(b) $\int f(x+b)dx = F(x+b) + C$

(c) $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

Applying Rules 1,2,3 and 4 we can solve the following examples:

Example (2):

$$\int co3xdx = \frac{1}{3}\sin 3x + C$$

$$\int e^{7x}dx = \frac{1}{7}e^{7x} + C$$

$$\int \sin(x+5)dx = -\cos(x+5) + C,$$

$$\int e^{x-1}dx = e^{x-1} + C$$

$$\int \sin(3x-1)dx = -\frac{1}{3}\cos(3x-1) + C,$$

$$\int e^{5x+2}dx = \frac{1}{5}e^{5x+2} + C,$$

$$\int \frac{dx}{4x-7} = \frac{1}{4}\ln|4x-7| + C.$$

Example (3): If $\frac{d^2y}{dx^2} = 4\pi(\cos 2x - \sin 2x)$, find $\frac{dy}{dx}$, then find y.

Solution: Since $\int \frac{df}{dx}dx = f(x) + C$ then

$$\frac{dy}{dx} = \int \frac{d^2y}{dx^2}dx = \int 4\pi(\cos 2x - \sin 2x)dx$$
$$= 2\pi(\sin 2x + \cos 2x) + C_1,$$

and,

$$y = \int \frac{dy}{dx}dx = \int ((2\pi(\sin 2x + \cos 2x)) + C_1)dx$$
$$= \pi(-\cos 2x + \sin 2x) + C_1 x + C_2,$$

where, C_1 and C_2 are constants.

Using Rule 4 we get the following more general table of basic integrals.

$$\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{(ax+b)} \, dx = \frac{1}{a} \ln|(ax+b)| + C$$

$$\int e^{(ax+b)} \, dx = \frac{1}{a} e^{(ax+b)} + C$$

$$\int A^{(ax+b)} dx = \frac{1}{a}\frac{A^{(ax+b)}}{\ln A} + C$$

$$\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \sec^2(ax+b) \, dx = \frac{1}{a}\tan(ax+b) + C$$

$$\int \csc^2(ax+b) \, dx = -\frac{1}{a}\cot(ax+b) + C$$

$$\int \sec(ax+b)\tan(ax+b) \, dx = \frac{1}{a}\sec(ax+b) + C$$

$$\int \csc(ax+b)\cot(ax+b) \, dx = -\frac{1}{a}\csc(ax+b) + C$$

$$\int \cosh(ax+b)\,dx = \frac{1}{a}\sinh(ax+b) + C$$

$$\int \sinh(ax+b)\,dx = \frac{1}{a}\cosh(ax+b) + C$$

$$\int \operatorname{sech}^2(ax+b)\,dx = \frac{1}{a}\tanh(ax+b) + C$$

$$\int \operatorname{csch}^2(ax+b)\,dx = -\frac{1}{a}\coth(ax+b) + C$$

$$\int \operatorname{sech}(ax+b)\tanh(ax+b)\,dx = -\frac{1}{a}\operatorname{sech}(ax+b) + C$$

$$\int \operatorname{csch}(ax+b)\coth(ax+b)\,dx = -\frac{1}{a}\operatorname{csch}(ax+b) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 + x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \cosh^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C, \quad |x| < a$$

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C, \quad |x| > a,$$

$$\int \frac{1}{x\sqrt{a^2 - x^2}}\, dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C, \qquad |x| < a$$

$$\int \frac{1}{x\sqrt{a^2 + x^2}}\, dx = -\frac{1}{a}\operatorname{csch}^{-1}\left(\frac{x}{a}\right) + C$$

Using the above table we can solve the following examples:

<u>Example</u> (4): Find each of the following integrals

(i) $\int e^{(3x-4)}\, dx$ (ii) $\int \frac{1}{\sqrt{4 - x^2}}\, dx$

(iii) $\int \frac{1}{9 + 16x^2}\, dx$ (iv) $\int \frac{1}{(3x + 7)}\, dx$

(v) $\int \operatorname{sech}(2x - 3)\tanh(2x - 3)\, dx$

Solution

(i) $\int e^{(3x-4)}\,dx = \frac{1}{3}e^{(3x-4)} + C$

(ii) $\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\left(\frac{x}{2}\right) + C$

(iii) $\int \frac{1}{9+16x^2}\,dx = \frac{1}{16}\int \frac{1}{9/16+x^2}\,dx$

$$= \frac{1}{16}\int \frac{1}{(3/4)^2+x^2}\,dx$$

$$= \frac{1}{16}\,\frac{1}{(3/4)}\tan^{-1}\left(\frac{x}{3/4}\right) + C$$

$$= \frac{1}{12}\tan^{-1}\left(\frac{4x}{3}\right) + C$$

(iv) $\int \frac{1}{(3x+7)}\,dx = \frac{1}{3}\ln|3x+7| + C$

(v) $\int \operatorname{sech}(2x-3)\tanh(2x-3)\,dx = -\frac{1}{2}\operatorname{sech}(2x-3) + C$

1.6 Determination of the Constant of Integration

In order to find the value of the constant of integration we need an auxiliary condition to be satisfied, as in the following example:

Example (1): Find the equation of the curve whose slope at the point (x,y) is $3x^2$ if the curve is required to pass through the point $(1,-1)$.

Solution: The slope of the curve at any point (x,y) is $\frac{dy}{dx}$.

But,

$$\frac{dy}{dx} = 3x^2$$

Integrating both sides with respect to x, we get

$$\int \frac{dy}{dx} dx = \int 3x^2 dx$$

or

$$y = x^3 + C$$

This last equation is the equation of the curve passing through a general point (x,y).

Now, if the curve has to pass through the point (1,−1) we must have that

$$y = x^3 + C$$
$$-1 = 1 + C,$$

from which we get

$$C = -2.$$

Therefore, the equation of the required curve is

$$y = x^3 - 2$$

Exercise 1.1

Integrate each of the following functions with respect to x:

1. $\left(2\sqrt{x}+\frac{4}{\sqrt{x}}\right)^3$ 2. $\sin(5x-1)$ 3. $\cosh 6x$

4. $\sec^2(6x-5)$ 5. $6^x 5^{2x}$ 6. $\frac{1}{4x+5}$

7. $(2x+7)^{-7}$ 8. $\sqrt{2x+5}$ 9. $e^{-2(-3x+8)}dx$

10. If $y''=12t^2-6t+5$, find y' and y given that $y=y'=5$ when $t=0$, ($y'=\frac{dy}{dt}$).

11. Find the equation of the curve whose slope is $-12x^3$ and passing through the point (1,8). Find y when $x=3$.

12. Evaluate each of the following integrals:

(a) $\int\frac{1}{x\sqrt{a^2+x^2}}dx$ (b) $\int\frac{1}{\sqrt{14-x^2}}dx$

(c) $\int \operatorname{csch}(3x+1)\coth(3x+1)dx$ (d) $\int \csc^2(4x-9)dx$

Chapter (2)

Methods of Integration

Chapter 2

Methods of Integration

2.1 Introduction

In this chapter we shall develop techniques for obtaining indefinite integrals of more complicated functions. The most important integration techniques that will be considered here are the following:

Integration by Substitution (Change of Variables), Integration of Trigonometric Functions, Integration by Removing Roots, Integration by Parts, Integration by Reduction, Integration using Partial Fractions, Miscellaneous Method.

Before introducing these techniques we have the following important rules:

2.2 Two Important Rules

Using the chain rule for the derivatives, we get

$$\frac{d}{dx}[f(x)]^{n+1} = (n+1)[f(x)]^n f'(x) \qquad (1)$$

Integrate both sides of (1) with respect to x, we obtain

$$[f(x)]^{n+1} = (n+1)\int [f(x)]^n \cdot f'(x)\,dx \qquad (2)$$

which implies

Rule 1

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, n \neq -1 \qquad (3)$$

Remark: The integration of a function raised to a given power, multiplied by the derivative of this function is computed easily from (3).

For example,

$$\int (1+\sin x)^2 \cos x\,dx = \frac{1}{3}(1+\sin x)^3 + C,$$

$$\int \frac{(\ln x)^4}{x}\,dx = \int (\ln x)^4 \cdot \left(\frac{1}{x}\right) dx = \frac{1}{5}(\ln x)^5 + C$$

Also, since $\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)}$ then we have:

Rule 2

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Using the above two rules we can solve the following examples:

Example (1): Evaluate each of the following integrals

(i) $\int (x^3+1)^9 x^2\, dx$ (ii) $\int x\sqrt{7x^2+12}\, dx$

(iii) $\int \frac{\sin 2x}{\sqrt{1+5\cos 2x}}dx$ (iv) $\int \frac{x}{x^2+5}dx$

(v) $\int \frac{1}{x\ln x}dx$ (vi) $\int \tan x\, dx$

(vii) $\int \coth x\, dx$ (viii) $\int \sec x\, dx$

(ix) $\int \frac{\tan^{-1}x}{1+x^2}dx$ (x) $\int \frac{48\,\text{csch}6x\,\coth 6x}{(1+\text{csch}6x)^3}$

(xi) $\int \left(\ln x + \frac{1}{\ln x}\right)\frac{1}{x}dx$ (xii) $\int \sqrt{\sin x}\,\cos x\, dx$

(xiii) $\int (6x-1)\sqrt{3x^2-x+5}\, dx$ (ivx) $\int \frac{6x-1}{3x^2-x+5}dx$

Solution

(i) $\int (x^3+1)^9 x^2\,dx = \frac{1}{3}\int (x^3+1)^9 3x^2\,dx$ $\qquad f(x)=(x^3+1)$

$f'(x)=3x^2$

then by Rule 1

$$\int (x^3+1)^9 x^2\,dx = \frac{1}{3}\cdot\frac{1}{10}(x^3+1)^{10}+C$$

(ii) $\int x\sqrt{7x^2+12}\,dx = \frac{1}{14}\int \sqrt{7x^2+12}\cdot 14x\,dx$

$$= \frac{1}{14}\int (7x^2+12)^{\frac{1}{2}}\cdot 14x\,dx \qquad f(x)=7x^2+12$$

$$f'(x)=14x$$

$$= \frac{1}{14}\cdot\frac{2}{3}(7x^2+12)^{\frac{3}{2}}+C$$

(iii) $\int \frac{\sin 2x}{\sqrt{1+5\cos 2x}}dx = \int (1+5\cos 2x)^{-\frac{1}{2}}\sin 2x\,dx$

$$= \frac{-1}{10}\int (1+5\cos 2x)^{-\frac{1}{2}}(-10\sin 2x)$$

$$= \frac{-2}{10}(1+5\cos 2x)^{\frac{1}{2}}+C$$

(iv) $\int \frac{x}{x^2+5}dx = \frac{1}{2}\int \frac{2x}{x^2+5}dx$ $\qquad f(x)=x^2+5$

$f'(x)=2x$

then by Rule 2

$$\int \frac{x}{x^2+5}dx = \frac{1}{2}\ln(x^2+5)+C$$

(v) $\int \frac{1}{x\ln x}dx = \int \frac{1/x}{\ln x}dx$ $\qquad f(x)=\ln x$

$f'(x)=\frac{1}{x}$

then by Rule 2

$$\int \frac{1}{x\ln x}dx = \ln|\ln x| + C$$

(vi) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}dx$

$$= -\ln|\cos x| + C = \ln|\sec x| + C$$

(vii) $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}dx = \ln|\sinh x| + C$

(viii) $\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}dx$

$$= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}dx = \ln|\sec x + \tan x| + C$$

(ix) $\int \frac{\tan^{-1} x}{1+x^2}dx = \int \tan^{-1} x \cdot \frac{1}{1+x^2}dx = \frac{1}{2}\left(\tan^{-1} x\right)^2 + C$

(x) $\int \frac{48\,\text{csch}6x\coth 6x}{(1+\text{csch}6x)^3}$

$$= -8\int (1+\text{csch}6x)^{-3}\cdot(-6\,\text{csch}6x\coth 6x)dx$$

$$= \frac{-8}{-2}(1+\text{csch}6x)^{-2} + C = \frac{4}{(1+\text{csch}6x)^2} + C$$

(xi) $\int\left(\ln x+\frac{1}{\ln x}\right)\frac{1}{x}dx=\int(\ln x)\frac{1}{x}dx+\int\frac{1/x}{\ln x}dx$

$$=\frac{1}{2}(\ln x)^2+\ln|\ln x|+C$$

(xii) $\int\sqrt{\sin x}\,\cos x\,dx=\int(\sin x)^{\frac{1}{2}}\cos x\,dx=\frac{2}{3}(\sin x)^{\frac{3}{2}}+C$

(xiii) $\int(6x-1)\sqrt{3x^2-x+5}\,dx=\int(3x^2-x+5)^{\frac{1}{2}}(6x-1)dx$

$$=\frac{2}{3}(3x^2-x+5)^{\frac{3}{2}}+C$$

(ivx) $\int\frac{6x-1}{3x^2-x+5}\,dx=\ln\left|3x^2-x+5\right|+C$

2.3 Integration by Simple Substitution (Change of Variables)

Consider the integral

$$\int f(x)\,dx \qquad (1)$$

Sometimes it is difficult to evaluate this integral directly, so we introduce a new variable to get an easier integral or we get directly one of the standard integrals. Of course, dx must also be replaced by the appropriate differential.

Let us replace the variable x by an appropriate function of another variable t, say. For example, let

$$x = \varphi(t)$$

then,

$$dx = \varphi'(t)\,dt$$

Using these substitutions in (1), we get

$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt, \qquad (2)$$

which is assumed to be an easier integral, and so we can evaluate it directly. The following examples illustrate the above idea:

Example (1): Evaluate the following integrals

(i) $\displaystyle\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$ (ii) $\displaystyle\int \frac{x^2}{5-x^6}\,dx$

(iii) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ (iv) $\displaystyle\int x^2\sqrt{x+2}\,dx$

(v) $\displaystyle\int \frac{\cos x}{\sin^2 x - 2\sin x + 1}\,dx$ (vi) $\displaystyle\int x e^{3x^2}\,dx$

(vii) $\displaystyle\int \frac{\sinh(\ln x)}{x}\,dx$ (viii) $\displaystyle\int x^2 \cos x^3\,dx$

Solution

(i) **For evaluating the integral**

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}},$$

let $\ln x = t \Rightarrow \frac{1}{x}dx = dt$, so that we obtain

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{dx}{x}$$

$$= \int \frac{1}{\sqrt{1-t^2}} dt = \sin^{-1} t + C$$

$$= \sin^{-1}(\ln x) + C$$

(ii) For the integral $\int \frac{x^2}{5-x^6}dx = \int \frac{x^2}{5-(x^3)^2}dx$

let, $x^3 = t \Rightarrow 3x^2 dx = dt$, so, we obtain

$$\int \frac{x^2}{5-x^6}dx = \frac{1}{3}\int \frac{3x^2}{5-x^6}dx = \frac{1}{3}\frac{3x^2}{5-(x^3)^2}dx$$

$$= \frac{1}{3}\int \frac{1}{5-t^2}dt = \frac{1}{3}\cdot\frac{1}{\sqrt{5}}\tanh^{-1}\left(\frac{t}{\sqrt{5}}\right) + C$$

$$= \frac{1}{3\sqrt{5}}\tanh^{-1}\left(\frac{x^3}{\sqrt{5}}\right) + C$$

(iii) For the integral $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx$,

let, $e^x = t \Rightarrow e^x dx = dt$, so, we get

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{1}{\sqrt{1-t^2}}dt$$

$$= \sin^{-1} t + C = \sin^{-1}(e^x) + C$$

(iv) For the integral $\int x^2\sqrt{x+2}\,dx$,

let, $\sqrt{x+2} = t \Rightarrow x+2 = t^2 \Rightarrow dx = 2tdt$,

Then,

$$\int x^2\sqrt{x+2}\,dx = \int (t^2-2)^2\, 2t^2\, dt$$

$$= \int (t^4 - 4t^2 + 4)\, 2t^2\, dt$$

$$= \int 2(t^6 - 4t^4 + 4t^2)\, dt$$

$$= 2[\frac{t^7}{7} - 4\frac{t^5}{5} + 4\frac{t^3}{3}] + C$$

$$= \frac{2}{7}(x+2)^{7/2} - \frac{8}{5}(x+2)^{5/2} + \frac{8}{3}(x+2)^{3/2} + C$$

The above integral can be solved by using another substitution as follows:

let, $x+2 = t \Rightarrow dx = dt, \quad x = t-2$

Then,

$$\int x^2\sqrt{x+2}\,dx = \int (t-2)^2 t^{1/2}\,dt$$

$$= \int (t^2 - 4t + 4)t^{1/2}\,dt$$

$$= \int (t^{5/2} - 4t^{3/2} + 4t^{1/2})\,dt$$

$$= \frac{t^{7/2}}{7/2} - 4\frac{t^{5/2}}{5/2} + 4\frac{t^{3/2}}{3/2} + C$$

$$= \frac{2}{7}(x+2)^{7/2} - \frac{8}{5}(x+2)^{5/2} + \frac{8}{3}(x+2)^{3/2} + C$$

(v) For the integral $\int \frac{\cos x}{\sin^2 x - 2\sin x + 1}\,dx$, we use the substitution

$$\sin x = t \quad \Rightarrow \quad \cos x\,dx = dt$$

So we have

$$\int \frac{\cos x}{\sin^2 x - 2\sin x + 1}\,dx = \int \frac{1}{t^2 - 2t + 1}\,dt$$

$$= \int \frac{1}{(t-1)^2}\,dt = \int (t-1)^{-2}\,dt = \frac{(t-1)^{-1}}{-1} + C$$

$$= \frac{-1}{(t-1)} + C = \frac{-1}{(\sin x - 1)} + C$$

The above integral can be evaluated directly as follows:

$$\int \frac{\cos x}{\sin^2 x - 2\sin x + 1}\,dx = \int \frac{\cos x}{(\sin x - 1)^2}\,dx$$

$$\int (\sin x - 1)^{-2} \cos x\,dx = \frac{(\sin x - 1)^{-1}}{-1} + C$$

$$= \frac{-1}{(\sin x - 1)} + C$$

(vi) For the integral $\int x e^{3x^2}\,dx$ we use the substitution

$t = 3x^2 \Rightarrow dt = 6x\,dx$ then

$$\int x e^{3x^2}\,dx = \int \frac{1}{6} e^t\,dt = \frac{1}{6} e^t + C = \frac{1}{6} e^{3x^2} + C$$

(vii) For the integral $\int \frac{\sinh(\ln x)}{x}\,dx$ we use the substitution

$t = \ln x \Rightarrow dt = \frac{1}{x}\,dx$ then

$$\int \frac{\sinh(\ln x)}{x}\,dx = \int \sinh t\,dt = \coth t + C = \cosh(\ln x) + C)$$

(viii) For the integral $\int x^2 \cos x^3\,dx$ we use the substitution

$t = x^3 \Rightarrow dt = 3x^2\,dx$ then

$$\int x^2 \cos x^3\,dx = \int \frac{1}{3} \cos t\,dt = \frac{1}{3} \sin t + C = \frac{1}{3} \sin x^3 + C$$

Exercises (2)

Evaluate each of the following integrals:

(1) $\int x\left(2x^2+3\right)^{10} dx$

(2) $\int x^2\sqrt[3]{3x^3+7}\, dx$

(3) $\int \frac{\left(1+\sqrt{x}\right)^3}{\sqrt{x}} dx$

(4) $\int \sqrt{x}\cos\sqrt{x^3}\, dx$

(5) $\int \tan x \sec^2 x\, dx$

(6) $\int \sqrt[4]{2x+5}\, dx$

(7) $\int \frac{dx}{\sqrt{4-5x}}$

(8) $\int \frac{\sin 2x}{\sqrt{1-\cos 2x}} dx$

(9) $\int x \cot x^2 \csc x^2\, dx$

(10) $\int \left(1+\frac{1}{x}\right)^2 \frac{1}{x^2} dx$

(11) $\int \sin x(1+\cos x)^2 dx$

(12) $\int \sin^3 x \cos x\, dx$

(13) $\int \frac{\tanh^{-1} x}{1-x^2} dx$

(14) $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx$

(15) $\int \frac{\cos x}{\sqrt{9-\sin^2 x}} dx$

(16) $\int \frac{1}{x\sqrt{x^6-1}} dx$

(17) $\int \frac{dx}{\sqrt{e^{2x}-25}}$

(18) $\int \frac{dx}{\sqrt{81+16x^2}}$

(19) $\int e^{\sin x} \cos x\, dx$

(20) $\int \frac{\left(2^x+1\right)^2}{2^x} dx$

(21) $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ (22) $\int \frac{x^3}{\sqrt[3]{x^2+4}}dx$

(23) $\int \sqrt{\frac{1}{x}-\frac{1}{\sqrt{x}}}\, dx$ (24) $\int \sqrt{\sqrt{x}-1}\, dx$

2.4 Integration of Trigonometric Functions

The following trigonometric identities are useful for evaluating some integrals involving trigonometric functions:

$\sin^2 x + \cos^2 x = 1$	$\cos^2 x = \frac{1+\cos 2x}{2}$
$\tan^2 x + 1 = \sec^2 x$	$\sin^2 x = \frac{1-\cos 2x}{2}$
$1 + \cot^2 x = \csc^2 x$	

$$\sin x \cos y = \frac{1}{2}\left[\sin(x-y)+\sin(x+y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y)+\cos(x+y)\right]$$

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$$

(a) Integration of $\int \sin^n x dx$ & $\int \cos^n x dx$ where n is a positive even integer

For these types of integrals we proceed as in the following examples:

Example (1): Evaluate $\int \sin^2 x\, dx$, $\int \cos^2 x\, dx$ and $\int \sin^4 x dx$

Solution:

$$\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$\int \cos^2 x\, dx = \int \frac{1}{2}(1 + \cos 2x)dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$\int \sin^4 x\, dx = \int \left(\sin^2 x\right)^2 dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{1}{4}\left(x - 2\frac{\sin 2x}{2} + \frac{1}{2}\left(x + \frac{\sin 4x}{4}\right)\right) + C$$

$$=\frac{1}{4}\left(\frac{3}{2}x-\sin 2x+\frac{\sin 4x}{8}\right)+C$$

Example (2): Find $\int (2 + 3\cos 2x)^2\,dx$

Solution:

$$\int (2+3\cos 2x)^2\,dx = \int (4+12\cos 2x+9\cos^2 2x)\,dx$$
$$= 4x+6\sin 2x+\frac{9}{2}\int (1+\cos 4x)\,dx$$
$$= 4x+6\sin 2x+\frac{9}{2}x+\frac{9}{8}\sin 4x+C$$
$$= \frac{17}{2}x+6\sin 2x+\frac{9}{8}\sin 4x+C$$

Example (3): Find $\int \frac{dx}{\sin^2 x\cos^2 x}$

Solution:

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx = \int \frac{dx}{\cos^2 x}+\int \frac{dx}{\sin^2 x}$$
$$= \int (\sec^2 x+\operatorname{cosec}^2 x)dx$$
$$= \tan x-\cot x+C$$

Notice that the above integral can be evaluated by another method as follows:

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \sec^2 x \csc^2 x dx = \int \sec^2 x(1+\cot^2 x)dx$$

$$= \int (\sec^2 x + \sec^2 x \cot^2 x)dx$$

$$= \int (\sec^2 x + \csc^2 x)dx$$

$$= \tan x - \cot x + C$$

(b) Integration of $\int \sin^n x dx$ & $\int \cos^n x dx$ where n is a positive odd integer

In this case, we follow the procedure shown in the following examples:

Example (1): Evaluate $\int \sin^3 x\, dx$, $\int \cos^5 x\, dx$

Solution

Separate one $\cos x$ to get

$$\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x\, dx = \int (1-\cos^2 x)\sin x\, dx$$

$$(\text{put } t = \cos x) \Rightarrow dt = -\sin x\, dx$$

$$= \int (1-t^2)(-dt) = -t + \frac{1}{3}t^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C$$

$$\int \cos^5 x\,dx = \int \cos^4 x \cdot \cos x\,dx$$

$$= \int \left(1-\sin^2 x\right)^2 \cos x\,dx, \quad (\text{put } t = \sin x)$$

$$= \int \left(1-2t^2+t^4\right) dt$$

$$= t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

(c) Integration of $\int \sin^n x \cos^m x\,dx$ where at least one of n or m is a positive odd integer

We have the following two cases:

(a) If n is odd, use $\sin^2 x = 1 - \cos^2 x$

(b) If m is odd, use $\cos^2 x = 1 - \sin^2 x$

Example (1): Evaluate $\int \sin^2 x \cos^3 x\,dx$

Solution:

$$\int \sin^2 x \cos^3 x\,dx = \int \sin^2 x \cos^2 x \cdot \underline{\cos x\,dx}$$

$$= \int \sin^2 x\left(1-\sin^2 x\right)\cos x\,dx$$

$$= \int \left(\sin^2 x - \sin^4 x\right)\cos x\,dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

(d) Integration of $\int \sin^n x \cos^m x\, dx$ where n and m are positive even integers

We express each of $\sin x$ and $\cos x$ in terms of the double angle trigonometric identities, so that we reduce this type of integrals into integrals of different powers of $\sin x$ and $\cos x$ as in the following examples:

Example (1): Evaluate $\int \sin^2 x \cos^4 x\, dx$

Solution:

$$\int \sin^2 x \cos^4 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int \left(1-\cos^2 2x\right)\left(1+\cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+\cos 2x - \cos^2 2x - \cos^3 2x\right) dx$$

$$= \frac{1}{8}\int \left(1+\cos 2x - \frac{1+\cos 4x}{2} - \cos^2 2x \cos 2x\right) dx$$

$$= \frac{1}{8}\int \left(\frac{1}{2}+\cos 2x - \frac{1}{2}\cos 4x - \left(1-\sin^2 2x\right)\cos 2x\right) dx$$

$$=\frac{1}{8}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x+\sin^2 2x\cos 2x\right)dx$$

$$=\frac{1}{8}\left(\frac{1}{2}x-\frac{1}{2}\frac{\sin 4x}{4}+\frac{1}{3\cdot 2}\sin^3 2x\right)+C$$

$$=\frac{1}{8}\left(\frac{1}{2}x-\frac{1}{8}\sin 4x+\frac{1}{6}\sin^3 2x\right)+C$$

(e) Integration of product of trigonometric functions with different arguments

(i) $$\int \sin ax\cos bx\,dx=\frac{1}{2}\int[\sin(a-b)x+\sin(a+b)x]dx$$

$$=-\frac{1}{2}\left[\frac{\cos(a-b)x}{(a-b)}+\frac{\cos(a+b)x}{(a+b)}\right]+C$$

(ii) $$\int\cos ax\cos bx dx=\frac{1}{2}\int[\cos(a-b)x+\cos(a+b)x]dx$$

$$=\frac{1}{2}\left[\frac{\sin(a-b)x}{(a-b)}+\frac{\sin(a+b)x}{(a+b)}\right]+C$$

(iii) $$\int\sin ax\sin bx\,dx=\frac{1}{2}\int[\cos(a-b)x-\cos(a+b)x]dx$$

$$=\frac{1}{2}\left[\frac{\sin(a-b)x}{(a-b)}-\frac{\sin(a+b)x}{(a+b)}\right]+C$$

Example (1): Find each of the following integrals

(i) $I_1 = \int \sin 4x \cos 5x\,dx$, (ii) $I_2 = \int \sin 7x \sin 4x\,dx$,

(iii) $I_3 = \int \cos 5x \cos 3x\,dx$.

Solution:

(i) $I_1 = \int \sin 4x \cos 5x\,dx = \frac{1}{2}\int (\sin 9x - \sin x)dx$

$$= \frac{1}{2}\left[\frac{-1}{9}\cos 9x + \cos x\right] + C = \frac{-1}{18}\cos 9x + \frac{1}{2}\cos x + C$$

(ii) $I_2 = \int \sin 7x \sin 4x\,dx = \frac{1}{2}\int (\cos 3x - \cos 11x)dx$

$$= \frac{1}{2}\left[\frac{1}{3}\sin 3x - \frac{1}{11}\sin 11x\right] + C = \frac{1}{6}\sin 3x - \frac{1}{22}\sin 11x + C$$

(iii) $I_3 = \int \cos 5x \cos 3x\,dx = \frac{1}{2}\int (\cos 2x + \cos 8x)dx$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + \frac{1}{8}\sin 8x\right] + C = \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C$$

2.5 Evaluating Integrals of the Form

$$\int R(\sin x,\ \cos x)\,dx$$

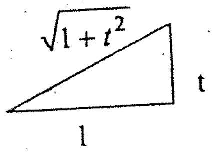
where, $R(\sin x,\ \cos x)$ is a rational function of $\sin x$ and $\cos x$, which may take one of the following forms:

$$\int\frac{dx}{a \pm b\sin x},\quad \int\frac{dx}{a \pm b\cos x},\quad \int\frac{dx}{a + \sin x + \cos x},$$

$$\int\frac{dx}{a\sin x + b\cos x},\quad \int\frac{dx}{a + b\sec x}$$

To solve any of the above integrals we use the substitution

$$\tan\frac{x}{2} = t \quad \Rightarrow \quad x = 2\tan^{-1} t \Rightarrow$$



Then,

$$dx = \frac{2}{1+t^2}dt,$$

$$\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}, \qquad \cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}},$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{1+t^2},$$

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{1-t^2}{1+t^2},$$

Example (1): Find $\int \frac{dx}{5+4\cos x}$

Solution: Put $t=\tan\frac{x}{2}$, then,

$$dx=\frac{2}{1+t^2}dt,\ \cos x=\frac{1-t^2}{1+t^2}.$$

Substituting we get

$$\int\frac{dx}{5+4\cos x}=\int\frac{\frac{2dt}{(1+t^2)}}{5+4\frac{(1-t^2)}{1+t^2}}=\int\frac{2}{5(1+t^2)+4(1-t^2)}dt$$

$$=\int\frac{2}{9+t^2}dt=\frac{2}{3}\tan^{-1}\frac{t}{3}+C=\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right)+C$$

Example (2): Evaluate $\int\frac{dx}{3\sin x+4\cos x}$

Solution: Put $t=\tan\frac{x}{2}$, then,

$$dx=\frac{2}{1+t^2}dt,\ \sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2},$$

substituting we get

$$\int\frac{dx}{3\sin x+4\cos x}=\int\frac{\frac{2dt}{1+t^2}}{3\frac{2t}{1+t^2}+4\frac{1-t^2}{1+t^2}}=\int\frac{2\,dt}{6t+4-4t^2}$$

$$=\frac{1}{2}\int\frac{dt}{\frac{6}{4}t+1-t^2}=-\frac{1}{2}\int\frac{dt}{t^2-\frac{3}{2}t-1}$$

$$=-\frac{1}{2}\int\frac{dt}{\left(t-\frac{3}{4}\right)^2-\frac{9}{16}-1}=\frac{1}{2}\int\frac{dt}{\frac{25}{16}-\left(t-\frac{3}{4}\right)^2}$$

$$=\frac{1}{2}\left(\tanh^{-1}\frac{t-\frac{3}{4}}{\frac{5}{4}}\right)\times\frac{4}{5}+C=\frac{2}{5}\tanh^{-1}\left(\frac{4t-3}{5}\right)+C$$

$$=\frac{2}{5}\tanh^{-1}\left(\frac{4\tan\left(\frac{x}{2}\right)-3}{5}\right)+C$$

2.6 Integration of Hyperbolic Functions

The hyperbolic integrals are computed with the aid of the following hyperbolic identities

$\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \text{sech}^2 x$ $\coth^2 x - 1 = \text{csch}^2 x$	$\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$ $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

$$\sinh x \cosh y = \frac{1}{2}\left[\sinh(x+y) + \sinh(x-y)\right]$$

$$\cosh x \cosh y = \frac{1}{2}\left[\cosh(x+y) + \cosh(x-y)\right]$$

$$\sinh x \sinh y = \frac{1}{2}\left[\cosh(x+y) - \cosh(x-y)\right]$$

Using the above identities and following the order in which integration of trigonometric functions are computed, we can easily evaluate integrals involving hyperbolic functions as illustrated in the following examples:

1. $\int \cosh x dx = \sinh x + C$
2. $\int \sinh x dx = \cosh x + C$
3. $\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \ln|\cosh x| + C$
4. $\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} dx = \ln|\sinh x| + C$
5. $\int \operatorname{sech} x dx = \int \frac{1}{\cosh x} \cdot \frac{\cosh x}{\cosh x} dx$

$= \int \frac{\cosh x}{1+\sinh^2 x} dx$, using the substitution $t = \sinh x$ we get

$= \int \frac{dt}{1+t^2} = \tan^{-1} t + C = \tan^{-1}(\sinh x) + C$

Notice that the above integral can be evaluated by two other methods (try to find them). Similarly, we can deal with $\int \operatorname{csch} x dx$.

Example (1): Evaluate each of the following integrals

1. $\int \cosh^2 x dx = \int \frac{1 + \cosh 2x}{2} dx = \frac{1}{2}\left(x + \frac{\sinh 2x}{2}\right) + C$

2. $\int \sinh^2 x \, dx = \int \frac{\cosh 2x - 1}{2} dx = \frac{1}{2}\left(\frac{\sinh 2x}{2} - x\right) + C$

3. $\int \operatorname{sech}^2 x dx = \tanh x + C$

4. $\int \operatorname{csch}^2 x dx = -\coth x + C$

5. $\int \tanh^2 x dx = \int\left(1 - \operatorname{sech}^2 x\right) dx = x - \tanh x + C$

6. $\int \coth^2 x dx = \int\left(\operatorname{csch}^2 x + 1\right) dx = -\coth x + x + C$

Example (2):

(i) $\int \cosh^3 x\, dx$ (ii) $\int \sinh^3 x \cosh^2 x\, dx$

Solution

(i) $\int \cosh^3 x\, dx = \int \cosh^2 x \cosh dx$

$$= \int \left(1 + \sinh^2 x\right) \cosh dx$$

$$= \sinh x + \frac{1}{3} \sinh^3 x + C$$

(ii) $\int \sinh^3 x \cosh^2 x\, dx = \int \sinh^2 x \cosh^2 x \cdot \sinh x\, dx$

$$= \int \left(\cosh^2 x - 1\right) \cosh^2 x \cdot \sinh x dx$$

$$= \int \left(\cosh^4 x - \cosh^2 x\right) \cdot \sinh x dx$$

$$= \frac{1}{5} \cosh^5 x - \frac{1}{3} \cosh^3 x + C$$

Example (3): Find $\int (5-2\sinh 3x)^2\,dx$

Solution:

$$\int (5-2\sinh 3x)^2\,dx = \int (25-20\sinh 3x+4\sinh^2 3x)\,dx$$
$$= 25x-\frac{20}{3}\cosh 3x + 2\int (\cosh 6x - 1)\,dx$$
$$= 25x-\frac{20}{3}\cosh 3x + \frac{1}{3}\sinh 6x - 2x + C$$
$$= 23x-\frac{20}{3}\cosh 3x + \frac{1}{3}\sinh 6x + C$$

Example (4): Find $\int (2\operatorname{sech} x - \tanh x)^2\,dx$

Solution:

$$\int (2\operatorname{sech} x - \tanh x)^2\,dx =$$
$$= \int (4\operatorname{sech}^2 x - 4\operatorname{sech} x \tanh x + \tanh^2 x)\,dx$$
$$= \int (4\operatorname{sech}^2 x - 4\operatorname{sech} x \tanh x + 1 - \operatorname{sech}^2 x)\,dx$$
$$= \int (3\operatorname{sech}^2 x - 4\operatorname{sech} x \tanh x + 1)\,dx$$
$$= 3\tanh x + 4\operatorname{sech} x + x + C$$

Example (5): Find

(i) $I_1 = \int \cosh 8x \sinh 6x\,dx$ (ii) $I_2 = \int \sinh 6x \sinh 8x\,dx$

(iii) $I_3 = \int \cosh 3x \cosh 6x\, dx$

Solution:

(i) $I_1 = \int \sinh 6x \cosh 8x dx$

$$= \frac{1}{2} \int [\sinh 14x + \sinh(-2x)] \; dx$$

$$= \frac{1}{2}\left[\frac{1}{14}\cosh 14x - \frac{1}{2}\cosh 2x\right] + C$$

$$= \frac{1}{28}\cosh 14x - \frac{1}{4}\cosh 2x + C$$

(ii) $I_2 = \int \sinh 6x \sinh 8x dx$

$$= \frac{1}{2} \int [\cosh 14x - \cosh 2x] dx$$

$$= \frac{1}{2}\left[\frac{1}{14}\sinh 14x - \frac{1}{2}\sinh 2x\right] + C$$

$$= \frac{1}{28}\sinh 14x - \frac{1}{4}\sinh 2x + C$$

(iii) $I_3 = \int \cosh 3x \cosh 6x dx$

$$= \frac{1}{2} \int [\cosh 9x + \cosh 3x] dx$$

$$= \frac{1}{2}\left[\frac{1}{9}\sinh 9x + \frac{1}{3}\sinh 3x\right] + C$$

$$= \frac{1}{18}\sinh 9x + \frac{1}{6}\sinh 3x + C$$

Exercises (3)

Evaluate each of the following integrals:

(1) $\int \sin^2 2x\, dx$ (2) $\int \cos^7 x\, dx$

(3) $\int \sin^5 x \cos^2 x\, dx$ (4) $\int \sin^4 x \cos^2 x\, dx$

(5) $\int \sqrt{\sin x}\, \cos^3 x\, dx$ (6) $\int \frac{\cos^3 x}{\sqrt{\sin x}}\, dx$

(7) $\int \sin 5x \cos 3x\, dx$ (8) $\int \cos x \cos 5x\, dx$

(9) $\int \sin 5x \sin 3x\, dx$ (10) $\int \left(1+\sqrt{\cos x}\right)^2 \sin x\, dx$

(11) $\int \frac{\cos x}{2-\sin x}\, dx$ (12) $\int \frac{dx}{2+\sin x}$

(13) $\int \frac{dx}{3+2\cos x}$ (14) $\int \frac{1}{1+\sin x+\cos x}\, dx$

(15) $\int \frac{1}{\tan x+\sin x}\, dx$ (16) $\int \frac{dx}{\sin x-\sqrt{3}\cos x}$

(17) $\int \sinh^4 x\, dx$ (18) $\int \cosh^5 2x\, dx$

(19) $\int \left(1+\sinh^2 x\right)\cosh^3 x\, dx$ (20) $\int \sinh^4 x \cosh^4 x\, dx$

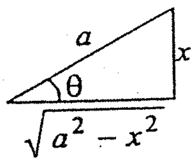
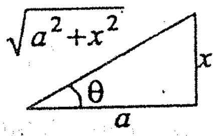
(21) $\int \sinh 2x \cosh 3x\, dx$ (22) $\int \sinh 3x \sinh 7x\, dx$

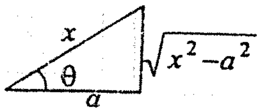
2.7 Integration by Removing Roots

Sometimes the integrands of certain integrals involve radical expressions in the forms

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2}$$

Each of such integrals can be evaluated by using a suitable trigonometric or hyperbolic substitution. The following table shows some of these substitutions:

Form	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,$ $dx = a\cos\theta d\theta$  	$1 - \sin^2\theta = \cos^2\theta$ $a^2 - x^2 = a^2 - a^2\sin^2\theta$ $= a^2\cos^2\theta$ $\boxed{\sqrt{a^2 - x^2} = a\cos\theta}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,$ $dx = a\sec^2\theta d\theta$  	$1 + \tan^2\theta = \sec^2\theta$ $a^2 + x^2 = a^2 + a^2\tan^2\theta$ $= a^2\sec^2\theta$ $\boxed{\sqrt{a^2 + x^2} = a\sec\theta}$

$\sqrt{x^2-a^2}$	$x = a\sec\theta,$ $dx = a\sec\theta\tan\theta d\theta$ 	$\sec^2\theta - 1 = \tan^2\theta$ $x^2 - a^2 = a^2\sec^2\theta - a^2$ $= a^2\tan^2\theta$ $\boxed{\sqrt{x^2-a^2} = a\tan\theta}$

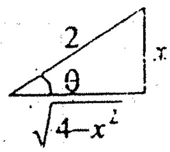
Other substitutions can also be used in order to remove the roots. These substitutions are listed in the following table.

Form	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\tanh\theta,$ $dx = a\text{sech}^2\theta\, d\theta$	$1-\tanh^2\theta = \text{sech}^2\theta$ $a^2 - x^2 = a^2 - a^2\tanh^2\theta$ $= a^2\text{sech}^2\theta$ $\boxed{\sqrt{a^2-x^2} = a\,\text{sech}\theta}$
$\sqrt{a^2+x^2}$	$x = a\sinh\theta,$ $dx = a\cosh d\theta$	$1+\sinh^2\theta = \cosh^2\theta$ $a^2 + x^2 = a^2 + a^2\sinh^2\theta$ $= a^2\cosh^2\theta$ $\boxed{\sqrt{a^2+x^2} = a\cosh\theta}$

$\sqrt{x^2-a^2}$	$x=a\cosh\theta$, $dx=a\sinh\theta d\theta$	$\cosh^2\theta-1=\sinh^2\theta$ $x^2-a^2=a^2\cosh^2\theta-a^2$ $=a^2\sinh^2\theta$ $\boxed{\sqrt{x^2-a^2}=a\sinh\theta}$

It must be noticed that each of these substitutions reduces the integral to an easier one that can be evaluated directly.

Example (1): Find $\int\frac{\sqrt{4-x^2}}{x^2}dx$.



Solution: Put

$$x=2\sin\theta \Rightarrow dx=2\cos\theta d\theta,$$

and,

$$4-x^2=4-4\sin^2\theta=4(1-\sin^2\theta)=4\cos^2\theta$$

So we have

$$\int\frac{\sqrt{4-x^2}}{x^2}dx=\int\frac{2\cos\theta}{4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$=\int\frac{\cos\theta}{\sin^2\theta}\cos\theta\, d\theta=\int\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta \, d\theta = \int \left(\csc^2\theta - 1\right) d\theta$$

$$= -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

Example (2): Find $\int \frac{1}{x^2\sqrt{5-x^2}} dx$.

Solution: Put $x = \sqrt{5}\sin\theta \Rightarrow dx = \sqrt{5}\cos\theta d\theta$,

and,

$$5 - x^2 = 5 - 5\sin^2\theta = 5\left(1 - \sin^2\theta\right) = 5\cos^2\theta$$

So we have

$$\int \frac{1}{x^2\sqrt{5-x^2}} dx = \int \frac{1}{5\sqrt{5}\sin^2\theta\cos\theta} \cdot \sqrt{5}\cos\theta d\theta$$

$$= \frac{1}{5}\int \frac{1}{\sin^2\theta} d\theta = \frac{1}{5}\int \csc^2\theta d\theta = -\frac{1}{5}\cot\theta + C$$

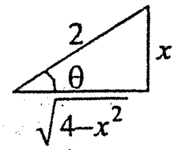
$$= -\frac{1}{5}\cot(\sin^{-1}\frac{x}{\sqrt{5}}) + C = -\frac{\sqrt{5-x^2}}{5x} + C$$

Example (3): Evaluate $\int x^3\sqrt{4-x^2}\, dx$

Solution: Put $x = 2\sin\theta \quad \Rightarrow \quad dx = 2\cos\theta \, d\theta$

and $\quad 4-x^2 = 4-4\sin^2\theta$

$\quad\quad = 4\left(1-\sin^2\theta\right) = 4\cos^2\theta$



so we have

$$\int x^3\sqrt{4-x^2}\;dx = \int 8\sin^3\theta\sqrt{4\cos^2\theta}\cdot 2\cos\theta\;d\theta$$

$$= \int 32\sin^3\theta\cos^2\theta\;d\theta = 32\int \sin^2\theta\cos^2\theta\cdot\sin\theta\;d\theta$$

$$= 32\int\left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\;d\theta$$

$$= 32\int\left(\cos^2\theta-\cos^4\theta\right)\sin\theta\;d\theta$$

$$= 32\left[\frac{1}{3}\cos^3\theta-\frac{1}{5}\cos^5\theta\right]+C$$

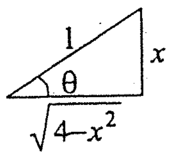
$$= 32\left[\frac{1}{3}\cos^3\left(\sin^{-1}\left(\frac{x}{2}\right)\right)-\frac{1}{5}\cos^5\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\right]+C$$

$$= 32\left[\frac{1}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3-\frac{1}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5\right]+C$$

Example (4): Evaluate $\int x^4\sqrt{1-x^2}\;dx$

Solution: Put $\quad x=\sin\theta \quad \Rightarrow \quad dx = \cos\theta\;d\theta$

so we have



$$\int x^4\sqrt{1-x^2}\;dx = \int \sin^4\theta\sqrt{1-\sin^2\theta}\cdot\cos\theta\;d\theta$$

$$= \int \sin^4\theta\cos^2\theta\;d\theta = \int\left(\sin^2\theta\right)^2\cos^2\theta\;d\theta$$

$$= \int \left[\frac{1}{2}(1-\cos 2\theta)\right]^2 \left[\frac{1}{2}(1+\cos 2\theta)\right] d\theta$$

$$= \frac{1}{8}\int \left(1 - 2\cos 2\theta + \cos^2 2\theta\right)(1+\cos 2\theta)d\theta$$

$$= \frac{1}{8}\int \left(1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta\right) d\theta$$

$$= \frac{1}{8}\int \left[1 - \cos 2\theta - \frac{1}{2}(1+\cos 4\theta) + \cos^2 2\theta \cos 2\theta\right] d\theta$$

$$= \frac{1}{8}\int \left[1 - \cos 2\theta - \frac{1}{2} - \frac{1}{2}\cos 4\theta + \left(1 - \sin^2 2\theta\right)\cos 2\theta\right] d\theta$$

$$= \frac{1}{8}\left[\frac{1}{2}\theta - \frac{1}{2}\sin 2\theta - \frac{1}{8}\sin 4\theta + \frac{1}{2}\sin 2\theta - \frac{1}{6}\sin^3 2\theta\right] + C$$

$$= \frac{1}{8}\left[\frac{1}{2}\sin^{-1} x - \frac{1}{8}\sin\left(4\sin^{-1} x\right) - \frac{1}{6}\sin^3\left(2\sin^{-1} x\right)\right] + C$$

Example (5): Find $\int \frac{\left(1-x^2\right)^{3/2}}{x^6}\,dx$

Solution: Put

$$x = \sin\theta \Rightarrow dx = \cos\theta d\theta,$$

and,

$$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$$

So we have

$$\int\frac{(1-x^2)^{3/2}}{x^6}dx = \int\frac{\cos^3\theta}{\sin^6\theta}\cdot\cos\theta d\theta$$

$$= \int\frac{\cos^4\theta}{\sin^6\theta}d\theta = \int\cot^4\theta\csc^2\theta d\theta$$

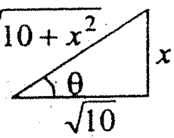
$$= -\frac{1}{5}\cot^5\theta + C = -\frac{1}{5}\cot^5(\sin^{-1}x) + C$$

$$= -\frac{1}{5}\left(\frac{\sqrt{1-x2}}{x}\right)^5 + C$$

Example (6): Evaluate $\int\frac{1}{\sqrt{\left(10+x^2\right)^3}}dx$.

Solution: Put $x=\sqrt{10}\tan\theta \Rightarrow dx=\sqrt{10}\sec^2\theta d\theta$,

i.e.



$$10+x^2 = 10+10\tan^2\theta$$

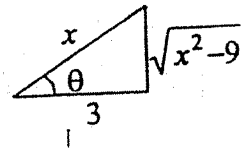
$$= 10\left(1+\tan^2\theta\right) = 10\sec^2\theta$$

So we have

$$\int\frac{1}{\sqrt{\left(10+x^2\right)^3}}dx=\int\frac{\sqrt{10}\sec^2\theta}{\sqrt{\left(10\sec^2\theta\right)^3}}d\theta=\int\frac{\sqrt{10}\sec^2\theta}{(10)^{3/2}\sec^3\theta}d\theta$$

$$=\frac{1}{10}\int\frac{1}{\sec\theta}d\theta=\frac{1}{10}\int\cos\theta d\theta=\frac{1}{10}\sin\theta+C=\frac{1}{10}\frac{x}{\sqrt{10+x^2}}+C$$

Example (7): Evaluate $\int \frac{1}{\sqrt{x^2-9}}dx$.



Solution: Put $x = 3\sec\theta \Rightarrow dx = 3\sec\theta\tan\theta\, d\theta$,

and,

$$x^2-9=9\sec^2\theta-9=9(\sec^2\theta-1)=9\tan^2\theta$$

Thus we have

$$\int\frac{1}{\sqrt{x^2-9}}dx=\int\frac{3\sec\theta\tan\theta}{3\tan\theta}d\theta=\int\sec\theta\, d\theta$$

$$=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{x}{3}+\frac{\sqrt{x^2-9}}{3}\right|+C$$

$$=\ln\left|\frac{x+\sqrt{x^2-9}}{3}\right|+C=\ln\left|x+\sqrt{x^2-9}\right|+C$$

Example (8): Evaluate $\int\frac{\sqrt{x^2-9}}{x}dx$

Solution: Put

$$x=3\sec\theta \Rightarrow dx=3\sec\theta\tan\theta\, d\theta,$$

then

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{3\tan\theta}{3\sec\theta}(3\sec\theta\tan\theta)d\theta$$
$$= \int 3\tan^2\theta\, d\theta = 3\int(\sec^2\theta - 1)d\theta$$
$$= 3(\tan\theta - \theta) + C = \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C$$

Example (9): Evaluate $\int\sqrt{a^2+x^2}\,dx$

Solution: Let $x = a\sinh\theta \Rightarrow dx = a\cosh\theta d\theta$,

and,

$$a^2 + x^2 = a^2 + a^2\sinh^2\theta$$
$$= a^2(1+\sinh^2\theta) = a^2\cosh^2\theta$$

So we have

$$\int\sqrt{a^2+x^2}\,dx = a^2\int\sqrt{1+\sinh^2\theta}\,.\cosh\theta\, d\theta$$
$$= a^2\int\cosh^2\theta\, d\theta$$
$$= a^2\int\frac{1}{2}(\cosh 2\theta + 1)\,d\theta$$
$$= \frac{1}{2}a^2\left[\frac{\sinh 2\theta}{2} + \theta\right] + C \qquad (1)$$

Since, $\sinh\theta = \frac{x}{a}$, then,

$$\cosh\theta = \sqrt{1+\left(\frac{x}{a}\right)^2},$$

$$\sinh 2\theta = 2\sinh\theta\cosh\theta = 2\,\frac{x\sqrt{a^2+x^2}}{a^2}$$

Substitute in (1), we get

$$\int\sqrt{a^2+x^2}\,dx = \frac{1}{2}a^2\left[\frac{x\sqrt{a^2+x^2}}{a^2} + \sinh^{-1}\left(\frac{x}{a}\right)\right]+C$$

Notice that the above example can be solved using trigonometric substitution. But this will be done after introducing the method of integration by parts as we shall see in the next section.

Exercises (4)

Evaluate each of the following integrals:

(1) $\int \frac{dx}{x\sqrt{4-x^2}}$

(2) $\int \frac{\sqrt{4-x^2}}{x^2}\, dx$

(3) $\int \frac{dx}{x^2\sqrt{9+x^2}}$

(4) $\int \frac{dx}{x^3\sqrt{x^2-25}}$

(5) $\int \frac{1}{\left(x^2-1\right)^{3/2}}$

(6) $\int \frac{dx}{\left(36+x^2\right)^2}$

(7) $\int \frac{x^3}{\sqrt{9x^2+49}}\, dx$

(8) $\int \frac{dx}{x\sqrt{25x^2+16}}$

(9) $\int \frac{x^2}{\left(1-9x^2\right)^{3/2}}\, dx$

(10) $\int \frac{\left(4+x^2\right)^2}{x^3}\, dx$

(11) $\int \frac{3x-5}{\sqrt{1-x^2}}\, dx$

(12) $\int \sqrt{25+x^2}\, dx$

(13) $\int \frac{x^2}{\sqrt{x^2-1}}\, dx$

(14) $\int \frac{\sqrt{x^2-4}}{x}\, dx$

(15) $\int \frac{dx}{x^2\sqrt{x^2-25}}$

(16) $\int x^2\sqrt{9-x^2}\, dx$

2.8 Integration by Parts

2.8.1 Standard Formula of Integration by Parts

For most differentiation rules, introduced in the previous course, there are corresponding integration rules. For instance, the chain rule for differentiation corresponds to the substitution rule for integration. The integration rule that corresponds to the product rule for differentiation is the rule of integration by parts. In order to see this correspondence let u and v be continuously differentiable functions of x, then

$$\frac{d}{dx}(uv)=uv'+vu'$$

Integrate both sides with respect to x, we get

$$uv = \int (uv' + vu')\,dx$$

or

$$uv=\int uv'dx+\int vu'dx$$

This last equation can be rearranged in the form

$$\int uv'\,dx = uv - \int vu'\,dx,$$

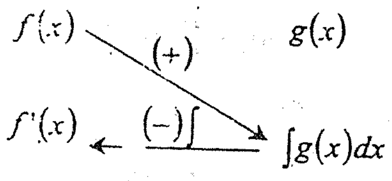
or $$\int u\,dv = uv - \int v\,du \qquad (1)$$

The above formula is called the ***integration by parts*** formula, which is used to evaluate the integral of a product of two functions. It shows that the integral $\int uv'dx$ is reduced to another integral $\int vu'\,dx$ which is supposed to be simpler than the one we started with.

For a product of two functions, which is usually applied for, the integration by parts formula can be expressed in the form:

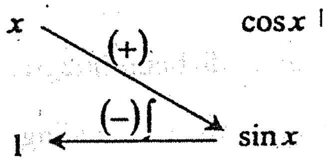
$$\begin{aligned}\int f(x)g(x)dx &= \left(\int f(x)dx\right)g(x) - \int\left[\int f(x)dx\right]g'(x)dx \\ &= \left(\int g(x)dx\right)f(x) - \int\left[\int g(x)dx\right]f'(x)dx\end{aligned}$$

This formula can be tabulated as follows:



Example (1): Find $\int x\cos x\,dx$.

Solution:



Thus, we have

$$\int x \cos x \, dx = x \sin x - \int (\sin x) \, dx$$
$$= x \sin x - (-\cos x) + C = x \sin x + \cos x + C$$

Remark: The main reason for using integration by parts is to obtain a simpler integral than the one we started with. Thus, in Example 1 we started with $\int x \cos x \, dx$ and expressed it in terms of the simpler integral $\int \sin x \, dx$. If we had chosen $f(x) = \cos x$ and $g(x) = x$, then $f'(x) = -\sin x$ and $\int g(x) \, dx = \frac{x^2}{2}$, so the integration by parts formula gives

$$\int x \cos x \, dx = \left(\frac{x^2}{2}\right) \cos x - \int \frac{x^2}{2} - \sin x \, dx$$
$$= \frac{1}{2} x^2 \cos x + \frac{1}{2} \int x^2 \sin x \, dx$$

But $\int x^2 \sin x\,dx$ is a more difficult integral than the one we started with. Therefore, when deciding to choose the function to be differentiated and that to be integrated, we have to choose them in such a way that the resulting integral is easier than the given one.

Example (2): Evaluate each of the following integrals

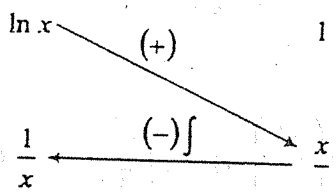
(i) $\int \ln x\,dx$ (ii) $\int \tan^{-1} x\,dx$

Solution:

(i) $\int \ln x\,dx = \int 1 \ \ln x\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx$

$$= x \ln x - \int dx = x \ln x - x + C$$

or using the tabular form we get



Then,

$$\int \ln x\,dx = (\ln x)(x) - \int (x)\left(\frac{1}{x}\right)dx = (\ln x)(x) - \int dx$$

$$= x \ln x - x + C$$

(ii) Again,.

$\tan^{-1} x$	$(+)$ ↘	1
$\dfrac{1}{1+x^2}$	← $(-)\int$	x

Then,

$$\int \tan^{-1} x \, dx = (\tan^{-1} x)(x) - \int (x)\left(\frac{1}{1+x^2}\right) dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

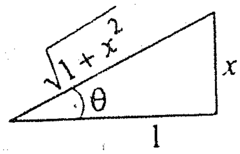
$$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

Example (3): Evaluate $\int x^2 \tan^{-1} x \, dx$

Solution:

put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$



put $t = \cos\theta$

$dt = -\sin\theta \, d\theta$

$$\int x^2 \tan^{-1} x \, dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \int \frac{1}{3} x^3 \cdot \frac{1}{1+x^2} dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{\tan^3\theta}{\sec^2\theta} \cdot \sec^2\theta \, d\theta$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \tan^3 \theta \, d\theta = \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} \int \frac{\sin^3 \theta}{\cos^3 \theta} \, d\theta$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \int \frac{\sin^2 \theta}{\cos^3 \theta} \sin \theta \, d\theta$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \int \frac{1 - \cos^2 \theta}{\cos^3 \theta} \sin \theta \, d\theta = \frac{1}{3} x^3 \tan^{-1} x + \int \frac{1 - t^2}{t^3} \, dt$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \int \left(\frac{1}{t} - \frac{1}{t^3} \right) dt = \frac{1}{3} x^3 \tan^{-1} x - \ln|t| + \frac{1}{2t^2} + C$$

$$= \frac{1}{3} x^3 \tan^{-1} x - \ln|\cos\theta| + \frac{1}{2} \sec^2 \theta + C$$

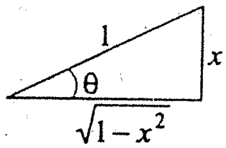
$$= \frac{1}{3} x^3 \tan^{-1} x - \ln\left|\cos\left(\tan^{-1} x\right)\right| + \frac{1}{2} \sqrt{1 + x^2} + C$$

Example (4): Evaluate $\int x \sin^{-1} x \, dx$

Solution:

put $x = \sin\theta$

$dx = \cos\theta \, d\theta$



$$\int x \sin^{-1} x \, dx = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int x^2 \frac{1}{\sqrt{1 - x^2}} dx$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \sin^2 \theta \frac{1}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \sin^2 \theta \, d\theta = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \int (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \left[\theta - \frac{1}{2} \sin 2\theta \right] + C$$

$$= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\left[\sin^{-1} x - \frac{1}{2}\sin\left(2\sin^{-1} x\right)\right] + C$$

Example (5): Evaluate $\int \sec^3 x\, dx$

Solution: The given integral may be written as

$$\int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx$$
$$= \sec x \tan x - \int (\tan x)\,(\sec x \tan x)\,dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$
$$= \sec x \tan x - \int \sec x\left(\sec^2 x - 1\right)dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \ln\left|\sec x + \tan x\right| + C$$

From which we get

$$2\int \sec^3 x\, dx = \sec x \tan x + \ln\left|\sec x + \tan x\right| + C$$

Thus,

$$\boxed{\int \sec^3 x\, dx = \frac{1}{2}\left[\sec x \tan x + \ln\left|\sec x + \tan x\right|\right] + C}$$

Notice that this last integral is very important because it appears quite frequently in applications.

Example (6): Evaluate $\int\sqrt{a^2+x^2}\,dx$

Solution: Put $x = a\tan\theta \Rightarrow dx = a\sec^2\theta d\theta$,

then

$$\int\sqrt{a^2+x^2}\,dx = \int a\sec\theta\cdot a\sec^2\theta\,d\theta$$

$$= a^2\int\sec^3\theta\,d\theta = \frac{1}{2}a^2\left[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right]$$

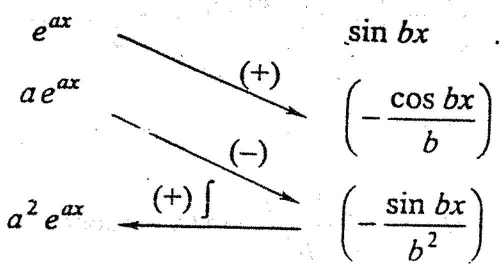
$$= \frac{a^2}{2}\left[\frac{x\sqrt{a^2+x^2}}{a^2} + \ln\left|\frac{\sqrt{a^2+x^2}}{a}\right| + \frac{x}{a}\right] + C$$

Example (7): Find each of the following integrals

(i) $\int e^{ax}\sin bx\,dx$ (ii) $\int e^{ax}\cos bx\,dx$

Solution:

(i) From the tabular formula of integration by parts we obtain



From this pattern we find the given integral as

$$I=\int e^{ax}\sin bx\ dx=\left(e^{ax}\right)\left(-\frac{\cos bx}{b}\right)-\left(ae^{ax}\right)\left(-\frac{\sin bx}{b^2}\right)$$
$$+\int\left(a^2e^{ax}\right)\left(-\frac{\sin bx}{b^2}\right)dx$$

or,

$$I=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

or,

$$I=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}I$$

From which we obtain

$$I\left(1+\frac{a^2}{b^2}\right)=-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b^2}e^{ax}\sin bx$$

or,

$$I\left(\frac{a^2+b^2}{b^2}\right)=\frac{a}{b^2}e^{ax}\sin bx-\frac{1}{b}e^{ax}\cos bx$$

Thus, we have

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$$

(ii) Similarly, we can prove that

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$$

Tabular Formula of Integration by Parts

Consider the integral

$$\int f(x)g(x)dx \tag{2}$$

We can evaluate this integral by the use of the tabular formula of the integration by parts, especially if one of the two functions is a power function. In fact, the tabular formula of the integration by parts is a successive application of the main formula. This tabular formula evaluates the integral of two functions as illustrated by the following pattern

Repeated differentiation		Repeated integration
$f(x)$		$g(x)$
	$(+)$	
$f'(x)$		$\int g(x)\,dx$
	$(-)$	
$f''(x)$		$\int\left[\int g(x)\,dx\right]dx$
	$(+)$	
$f'''(x)$		$\int\left[\int\left[\int g(x)\,dx\right]dx\right]dx$
$\vdots$		$\vdots$

If the function $f(x)$ becomes zero after a finite number of differentiation this table terminates and the given integral is then evaluated. If the two functions do not vanish with differentiation, then we terminate the given integral with another integral as in the case of integration of the functions

$$\int e^{ax}\cos x\,dx,\ \int e^{ax}\sin x\,dx,\ \int \sec^3 x\,dx,\cdots.$$

Example (1): Evaluate $\int x^3 \cos 2x\,dx$

Solution: Using the tabular formula of integration by parts, we get

x^3		$\cos 2x$
$3x^2$	(+)	$\left(\frac{\sin 2x}{2}\right)$
$6x$	(−)	$\left(-\frac{\cos 2x}{4}\right)$
6	(+)	$\left(-\frac{\sin 2x}{8}\right)$
$\boxed{0}$	(−)	$\left(\frac{\cos 2x}{16}\right)$

From this pattern we find the given integral as

$$\int x^3 \cos 2x \, dx = (x^3)\left(\frac{\sin 2x}{2}\right) - (3x^2)\left(-\frac{\cos 2x}{4}\right) + (6x)\left(-\frac{\sin 2x}{8}\right) - (6)\left(\frac{\cos 2x}{16}\right) + C$$

or,

$$\int x^3 \cos 2x \, dx = \frac{1}{2}x^3 \sin 2x + \frac{3}{4}x^2 \cos 2x - \frac{3}{4}x \sin 2x - \frac{3}{8}\cos 2x + C$$

2.9 Integration by Successive Reduction

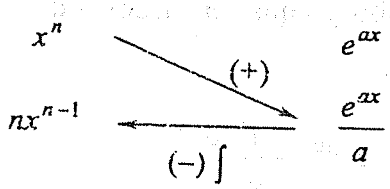
Integration by successive reduction is one way of simplifying complicated integrals. The basic idea of this method is to obtain a recurrence formula for the given integral. By this we mean, to obtain a formula expressing the original integral I_n, say, in terms of a lower order integral I_{n-1} or I_{n-2}, say. We explain this idea by the following examples:

Example (1): Find a reduction formula for the integral

$$I_n = \int x^n e^{ax} dx,$$

where, n is a positive integer. Hence, find $\int x^3 e^{ax} dx$.

Solution: Integrating by parts, we get



$$I_n = \int x^n e^{ax} dx = (x^n)\left(\frac{e^{ax}}{a}\right) - \frac{n}{a}\int x^{n-1} e^{ax} dx$$

Hence,

$$I_n = \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}$$

Substitute in the reduction formula to evaluate I_3; we get respectively,

$$n = 3 \Rightarrow I_3 = \frac{1}{a}x^3 e^{ax} - \frac{3}{a}I_2$$

$$n = 2 \Rightarrow I_2 = \frac{1}{a}x^2 e^{ax} - \frac{2}{a}I_1$$

$$n = 1 \Rightarrow I_1 = \frac{1}{a}xe^{ax} - \frac{1}{a}I_0$$

From the given integral we can get I_0 by setting $n = 0$, so we have

$$I_0 = \int x^0 e^{ax} dx = \int e^{ax} dx = \frac{e^{ax}}{a}$$

Substitute in the above equations from bottom to top to find I_3, thus we obtain

$$I_1 = \frac{1}{a}xe^{ax} - \frac{1}{a} \cdot \frac{e^{ax}}{a}$$

$$I_2 = \frac{1}{a}x^2 e^{ax} - \frac{2}{a}\left(\frac{1}{a}xe^{ax} - \frac{1}{a} \cdot \frac{e^{ax}}{a}\right)$$

$$= \frac{1}{a}x^2 e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax},$$

and thus we have

$$I_3 = \frac{1}{a}x^3e^{ax} - \frac{3}{a}\left(\frac{1}{a}x^2e^{ax} - \frac{2}{a^2}xe^{ax} + \frac{2}{a^3}e^{ax}\right)$$

$$= \frac{1}{a}x^3e^{ax} - \frac{3}{a^2}x^2e^{ax} + \frac{6}{a^3}xe^{ax} - \frac{6}{a^4}e^{ax} + C$$

$$= e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right) + C$$

Example (2): Find a reduction formula for the following integrals, and hence evaluate the indicated integral

(i) $I_n = \int \left(\sin^{-1} x\right)^n dx, \quad I_4$

(ii) $I_m = \int \left(\cos^{-1} x\right)^m dx, \quad I_5$

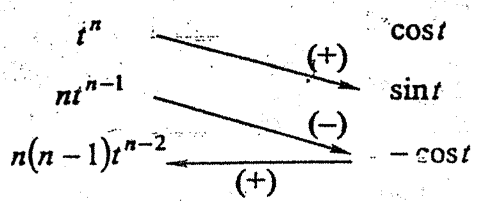
Solution:

(i) Put $t = \sin^{-1} x$,

then, we have $\sin t = x$ and $dx = \cos t\, dt$. Substitute in the given integral we get

$$I_n = \int t^n \cos t\, dt.$$

Integrate by parts two times, we get



From the above table we obtain

$$I_n = t^n \sin t + nt^{n-1} \cos t - \int n(n-1)t^{n-2} \cos t \, dt$$

which implies that

$$\begin{aligned} I_n &= t^n \sin t + nt^{n-1} \cos t - n(n-1)I_{n-2} \\ &= x\left(\sin^{-1} x\right)^n + n\left(\sin^{-1} x\right)^{n-1} \sqrt{1-x^2} - n(n-1)I_{n-2} \end{aligned}$$

To evaluate I_4, we substitute successively in the derived reduction formula. Therefore, we get

$$n = 4 \quad \Rightarrow \quad I_4 = t^4 \sin t + 4t^3 \cos t - 4 \cdot 3I_2,$$
$$n = 2 \quad \Rightarrow \quad I_2 = t^2 \sin t + 2t \cos t - 2 \cdot 1I_0,$$

But

$$I_0 = \int \cos t \, dt = \sin t$$

Thus we have

$$I_2 = t^2 \sin t + 2t \cos t - 2 \sin t,$$

and,

$$I_4 = t^4 \sin t + 4t^3 \cos t - 12\left(t^2 \sin t + 2t \cos t - 2\sin t\right)$$

$$= t^4 \sin t + 4t^3 \cos t - 12t^2 \sin t - 24t \cos t + 24 \sin t + C$$

$$= x\left(\sin^{-1} x\right)^4 + 4\sqrt{1-x^2}\left(\sin^{-1} x\right)^3 - 12x\left(\sin^{-1} x\right)^2$$

$$-24\sqrt{1-x^2}\left(\sin^{-1} x\right) + 24x + C$$

(ii) We leave this part as an exercise.

<u>Example</u> (3): Evaluate $I_n = \int \sec^n x\,dx$

<u>**Solution:**</u>

<u>**Case**</u> **(1): n is a positive even integer**

In this case the integral can be evaluated directly as follows:

Let $n = 2m$, then

$$\int \sec^n x\,dx = \int \sec^{2m} x\,dx = \int \sec^{2m-2} \sec^2 x\,dx$$

$$= \int (1 + \tan^2 x)^{m-1} \sec^2 x\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$, then

$$\int \sec^n x\, dx = \int (1+t^2)^{m-1} dt.$$

We then apply the binomial theorem and integrate term by term. For a numerical example, let us evaluate,

$$\int \sec^8 x dx = \int \sec^6 x \sec^2 x dx \quad (t = \tan x \Rightarrow dt = \sec^2 x dx)$$
$$= \int (1+\tan^2 x)^3 \sec^2 x dx$$
$$= \int (1+t^2)^3 dt = (1+3t^2+3t^4+t^6) dt$$
$$= t + t^3 + \frac{3}{5} t^5 + \frac{1}{7} t^7 + C$$
$$= \tan x + \tan^3 x + \frac{3}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C$$

Case (2): n **is a positive odd integer**

In this case we get a reduction formula as follows:

Let

$$I_n = \int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$$
$$= (\sec^{n-2} x)(\tan x) - \int (\tan x)(n-2)\sec^{n-3} x \sec x \tan x dx$$
$$= (\tan x)\sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x dx$$
$$= (\tan x)\sec^{n-2} x - (n-2)\left(\int \sec^{n-2} x(\sec^2 x - 1) dx\right)$$
$$= (\tan x)\sec^{n-2} x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} x dx$$
$$= (\tan x)\sec^{n-2} x - (n-2)I_n + (n-2)I_{n-2}$$

$$= \frac{1}{n-1}\left[(\tan x)\sec^{n-2} x + (n-2)I_{n-2}\right],$$

which is a reduction formula for I_n; namely

$$I_n = \frac{1}{n-1}\left[(\tan x)\sec^{n-2} x + (n-2)I_{n-2}\right]$$

For a numerical example let us evaluate, $I_7 = \int \sec^7 x dx$.

$$I_7 = \int \sec^7 x dx$$

$$I_7 = \frac{1}{6}\left[(\tan x)(\sec^5 x) + (5)I_5\right]$$

$$I_5 = \frac{1}{4}\left[(\tan x)(\sec^3 x) + (3)I_3\right]$$

$$I_3 = \frac{1}{2}\left[(\tan x)(\sec\ x) + I_1\right]$$

$$I_1 = \int \sec\ x dx = \ln|\tan x + \sec x| + C$$

Thus we have

$$I_7 = \frac{1}{6}(\tan x)(\sec^5 x) + \frac{5}{24}(\tan x)(\sec^3 x) +$$

$$+ \frac{15}{48}(\tan x)(\sec x) + \frac{15}{48}\ln|\tan x + \sec x| + C$$

<u>Example</u> (4): Evaluate $\int \tan^n x\, dx$.

Solution:

Case (1): n is an odd positive integer

In this case the integral can be evaluated directly as follows:

Put $n = 2m + 1$,

then,

$$\int \tan^n x dx = \int \tan^{2m+1} x dx$$
$$= \int \tan^{2m} x \tan x dx = \int (\sec^2 x - 1)^m \tan x dx$$
$$= \int \frac{(\sec^2 x - 1)^m}{\sec x} \tan x \sec x dx$$
$$= \int \frac{(t^2 - 1)^m}{t} dt. \qquad (t = \sec x)$$

Then. applying the binomial theorem and integrate term by term to obtain the final result. For a numerical example, let us evaluate:

Example (5): Evaluate $\int \tan^7 x dx$

Solution:

$$\int \tan^7 x dx = \int \tan^6 x \tan x dx = \int \frac{(\sec^2 x - 1)^3}{\sec x} \tan x \sec x dx$$
$$= \int \frac{(t^2 - 1)^3}{t} dt, \qquad \text{where } t = \sec x$$

$$\int \tan^7 x dx = \int \frac{(t^6 - 3t^4 + 3t^2 - 1)}{t} dt$$

$$= \frac{\sec^6 x}{6} - \frac{3}{4}\sec^4 x + \frac{3}{2}\sec^2 x - \ln|\sec x| + C$$

Case (2): n is an even positive integer

In this case we get a reduction formula as follows:

Let

$$I_n = \int \tan^n x dx = \int (\tan^{n-2} x)(\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2},$$

which is the required reduction formula. As a numerical example, let us have the following:

Example (6): Evaluate $I_6 = \int \tan^6 x dx$

Solution:

$$I_6 = \int \tan^6 x dx = \frac{\tan^5 x}{5} - I_4,$$

$$I_4 = \frac{\tan^3 x}{3} - I_2,$$

$$I_2 = \tan x - I_0,$$

$$I_0 = \int \tan^0 x dx = \int 1 dx = x$$

Thus, we have

$$I_6 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$$

Integrals of the form $\int \sec^m x \tan^n x dx$

Case (1): m is an even integer

In this case we separate one $\sec^2 x$ from $\sec^m x$ and then proceed as in the following examples:

Example (7): Evaluate the following integral

$$\int \sec^6 x \tan^6 x \, dx .$$

Solution:

$$\begin{aligned}\int \sec^6 x \tan^6 x \, dx &= \int \sec^4 x \tan^6 x \underline{\sec^2 x \, dx} \\ &= \int (\sec^2 x)^2 \tan^6 x \underline{\sec^2 x \, dx} \\ &= \int (1+\tan^2 x)^2 \tan^6 x \underline{\sec^2 x \, dx} \\ &= \int (\tan^6 x + 2\tan^8 x + \tan^{10} x) \underline{\sec^2 x \, dx} \\ &= \frac{1}{7}\tan^7 x + \frac{2}{9}\tan^9 x + \frac{1}{11}\tan^{11} x + C\end{aligned}$$

Example (8): Evaluate the following integral

$$\int \sec^4 x \tan^3 x \, dx.$$

Solution:

$$\begin{aligned}\int \sec^4 x \tan^3 x \, dx &= \int \sec^2 x \tan^3 x \underline{\sec^2 x \, dx} \\ &= \int \sec^2 x \tan^3 x \underline{\sec^2 x} \, dx \\ &= \int (1+\tan^2 x) \tan^3 x \underline{\sec^2 x} \, dx \\ &= \int (\tan^3 x + \tan^5 x) \underline{\sec^2 x} \, dx \\ &= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C\end{aligned}$$

Case (2): **n is an odd integer**

In this case we separate one $\sec x \tan x$ from $\sec^m x \tan^n x$ and then proceed as in the following example:

Example (9): Evaluate the following integral

$$\int \sec^3 x \tan^3 x \, dx.$$

Solution:

$$\begin{aligned}\int \sec^3 x \tan^3 x \, dx &= \int \underline{\sec^2} x \tan^2 x \underline{\sec x \tan x \, dx} \\ &= \int \sec^2 x \left(\sec^2 x - 1\right) \underline{\sec x \tan x \, dx} \\ &= \int \left(\sec^4 x - \sec^2 x\right) \underline{\sec x \tan x \, dx} \\ &= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 + C\end{aligned}$$

Case (3): m is an odd integer and n is an even integer

In this case we express the whole integral in terms of $\int \sec^k x dx$ and proceed as in the case of integration of the form $\int \sec^k x dx$.

Example (10): Evaluate the following integral

$$\int \sec x \tan^2 x \, dx .$$

Solution:

$$\int \sec x \tan^2 x \, dx = \int \sec x (\sec^2 x - 1) \, dx$$

$$= \int (\sec^3 x - \sec x) \, dx$$

$$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$$

$$= \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C.$$

Reduction formulas involving two parameters

Example (11): Find a reduction formula for

$$I_{m,n} = \int x^m (\ln x)^n \, dx .$$

Solution: Using integration by parts we directly obtain

$$I_{m,n} = \int x^m (\ln x)^n \, dx$$

$$= (\ln x)^n \left(\frac{x^{m+1}}{m+1}\right) - \int \left(\frac{x^{m+1}}{m+1}\right) n (\ln x)^{n-1} \cdot \frac{1}{x} \, dx$$

$$= (\ln x)^n \left(\frac{x^{m+1}}{m+1}\right) - \frac{n}{m+1} \int x^m (\ln x)^{n-1} \, dx$$

$$= \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}$$

Hence, the required reduction formula is

$$I_{m,n} = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} I_{m,n-1}.$$

Example (12): Find a reduction formula for

$$I_{n,m} = \int x^n \sin mx \, dx.$$

Solution:

$$I_{n,m} = \int x^n \sin mx \, dx$$

$$= x^n \left(\frac{-\cos mx}{m}\right) - \int \left(\frac{-\cos mx}{m}\right) n x^{n-1} \, dx$$

$$= x^n \left(\frac{-\cos mx}{m}\right) + \frac{n}{m} \int x^{n-1} \cos mx \, dx$$

$$= \frac{-\cos mx}{m} x^n + \frac{n}{m} \left[x^{n-1} \frac{\sin mx}{m} - \int \left(\frac{\sin mx}{m}\right) (n-1) x^{n-2} \, dx \right]$$

$$= \frac{-\cos mx}{m} x^n + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} \int x^{n-2} \sin mx dx$$

$$= \frac{-\cos mx}{m} x^n + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2,m}$$

or,

$$I_{n,m} = \frac{-\cos mx}{m} x^n + \frac{n}{m^2} x^{n-1} \sin mx - \frac{n(n-1)}{m^2} I_{n-2,m}$$

Example (13): Find a reduction formula for

$I_{m,n} = \int \cos^m x \sin nx dx$, ($m$ and n are positive integrs)

Solution:

$$I_{m,n} = \int \cos^m x \sin n x \, dx$$

$$= \cos^m x \left(-\frac{\cos nx}{n} \right) - \int \left(-\frac{\cos nx}{n} \right) m \cos^{m-1} x (-\sin x) dx$$

$$= -\frac{1}{n} \cos nx \cos^m x - \frac{m}{n} \int \cos^{m-1} x \sin x \cos nx dx$$

But,

$$\sin(n-1)x = \sin nx \cos x - \cos nx \sin x.$$

Then,

$$\sin x \cos nx = \sin nx \cos x - \sin(n-1)x$$

Therefore,

$$I_{m,n} = -\frac{1}{n}\cos nx\cos^m x - \frac{m}{n}\int \cos^{m-1} x\sin nx\cos x dx$$

$$+\frac{m}{n}\int \cos^{m-1} x\sin(n-1)dx$$

$$= -\frac{1}{n}\cos nx\cos^m x - \frac{m}{n}I_{m,n} + \frac{m}{n}I_{m-1,n-1}$$

or

$$I_{m,n} = \frac{1}{m+n}\left[-\cos nx\cos^m x + mI_{(m-1,n-1)}\right]$$

Exercises (5)

Evaluate each of the following integrals:

(1) $\int x^2 e^{3x}\,dx$ (2) $\int x\cos 5x\,dx$

(3) $\int x\sec x\tan x\,dx$ (4) $\int \sqrt{x}\ln x\,dx$

(5) $\int x\csc^2 x\,dx$ (6) $\int \sin(\ln x)\,dx$

(7) $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ (8) $\int x(2x+3)^{99}\,dx$

(9) $\int \frac{x^5}{\sqrt{1-x^3}}\,dx$ (10) $\int (\ln x)^2\,dx$

(11) $\int (x+4)\cosh 4x\,dx$ (12) $\int \cos\sqrt{x}\,dx$

(13) $\int \frac{x\,e^x}{(x+1)^2}\,dx$ (14) $\int \sec^{-1}\sqrt{x}\,dx$

(15) $\int \ln(x^2+1)\,dx$ (16) $\int \sqrt{x}\tan^{-1}\sqrt{x}\,dx$

(17) Find a reduction formula for

(i) $\int x^n\cos mx\,dx$ (ii) $\int x^n\sinh mx\,dx$

(iii) $\int \sec^m x\tan^n x\,dx$ (iv) $\int \frac{dx}{(x^2+a^2)^n}$

2.10 Integrals Involving Quadratic Polynomial Functions

We have three cases to consider

(a) Integrals of the Form

$$\int\sqrt{ax^2+bx+c}\ dx$$

In this case we first take the coefficient of x^2 outside the square root sign and then complete the square of the quantity under the square root sign to get one of the three standard integrals $\int k\sqrt{L^2-t^2}\ dt$, $\int k\sqrt{L^2+t^2}\ dt$ or $\int k\sqrt{t^2-L^2}\ dt$ which have been treated previously. The following example clarify this case:

<u>Example</u> (1): Evaluate the integrals

(i) $\int\sqrt{x^2-4x+13}\ dx$ (ii) $\int\sqrt{6-4x-2x^2}\ dx$

Solution

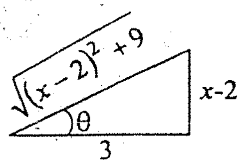
(i) $\int\sqrt{x^2-4x+13}\ dx=\int\sqrt{(x-2)^2-4+13}\ dx$

$$=\int\sqrt{(x-2)^2+9}\ dx$$

Put,

$$(x-2)=3\tan\theta \quad \Rightarrow \quad \begin{cases} dx = 3\sec^2\theta\, d\theta \\ (x-2)^2+9 = 9\sec^2\theta \end{cases}$$

Therefore, we have



$$\int\sqrt{x^2-4x+13}\,dx = \int\sqrt{(x-2)^2+9}\,dx$$

$$=\int 3\sec\theta \cdot 3\sec^2\theta\, d\theta = 9\int\sec^3\theta\, d\theta$$

$$=9\cdot\frac{1}{2}\left[\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)\right]+C$$

$$=\frac{9}{2}\left[\sec\theta\tan\theta+\ln(\sec\theta+\tan\theta)\right]+C$$

$$=\frac{9}{2}\left[\frac{\sqrt{(x-2)^2+9}}{3}\;\frac{(x-2)}{3}+\ln\left|\frac{\sqrt{(x-2)^2+9}}{3}+\frac{(x-2)}{3}\right|\right]+C$$

$$=\frac{1}{2}\left[(x-2)\sqrt{(x-2)^2+9}+9\ln\left|\sqrt{(x-2)^2+9}+(x-2)\right|\right]+C$$

(ii) Since,

$$6-4x-2x^2 = -2\left(x^2+2x-3\right)$$

$$=-2\left((x+1)^2-1-3\right) = -2\left[(x+1)^2-4\right]$$

$$=2\left[4-(x+1)^2\right]$$

Then,

$$\int\sqrt{6-4x-2x^2}\,dx=\sqrt{2}\int\sqrt{4-(x+1)^2}\,dx$$

Using the substitution

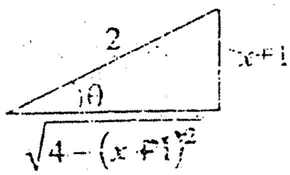
$$(x+1)=2\sin\theta \quad\Rightarrow\quad \begin{cases} dx=2\cos\theta\,d\theta \\ 4-(x+1)^2=4\cos^2\theta \end{cases}$$

We obtain,

$$\int\sqrt{6-4x-2x^2}\,dx=\sqrt{2}\int\sqrt{4-(x+1)^2}\,dx$$

$$=\sqrt{2}\int 2\cos\theta\times 2\cos\theta\,d\theta=4\sqrt{2}\int\cos^2\theta\,d\theta$$

$$=4\sqrt{2}\int\frac{1+\cos 2\theta}{2}\,d\theta$$



$$=2\sqrt{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+C$$

$$=2\sqrt{2}(\theta+\sin\theta\cos\theta)+C$$

$$=2\sqrt{2}\left(\sin^{-1}\left(\frac{x+1}{2}\right)+\left(\frac{x+1}{2}\right)\left(\frac{\sqrt{4-(x+1)^2}}{2}\right)\right)+C$$

b) Integrals of the Forms

$$\int\frac{dx}{ax^2+bx+c},\qquad \int\frac{dx}{\sqrt{ax^2+bx+c}}$$

These forms are reduced to the standard integrals by completing the square of the expression in the denominator.

Example (2): Evaluate the integrals

(i) $\int \frac{1}{2x^2 - 8x + 5}\,dx$ (ii) $\int \frac{1}{\sqrt{x^2 + 2x + 5}}\,dx$

Solution:

(i) Since,

$$2x^2 - 8x + 5 = 2\left(x^2 - 4x + \frac{5}{2}\right) = 2\left((x-2)^2 - 4 + \frac{5}{2}\right)$$

$$= 2\left[(x-2)^2 - \frac{3}{2}\right]$$

Then,

$$\int \frac{1}{2x^2 - 8x + 5}\,dx = \frac{1}{2}\int \frac{1}{(x-2)^2 - \frac{3}{2}}\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{\sqrt{3/2}}\tanh^{-1}\left(\frac{x-2}{\sqrt{3/2}}\right) + C$$

$$= -\frac{1}{\sqrt{6}}\tanh^{-1}\left(\frac{x-2}{\sqrt{3/2}}\right) + C$$

(ii) For the integral $\int \frac{1}{\sqrt{x^2 + 2x + 5}}\,dx$, we complete the square of the quantity in the denominator to get

$$x^2 + 2x + 5 = \left((x+1)^2 - 1 + 5\right) = (x+1)^2 + 4$$

Thus we obtain

$$\int \frac{1}{\sqrt{x^2+2x+5}}dx=\int \frac{1}{\sqrt{(x+1)^2+4}}dx=\sinh^{-1}\left(\frac{x+1}{2}\right)+C$$

(c) Integrals of the Form

$$\int \frac{Ax+B}{ax^2+bx+c}dx, \qquad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

Each of these integrals is reduced to two integrals; in the first integral the numerator will be the derivative of the denominator or we shall have the integral of a function multiplied by its derivative. The other integral is of the previous form, i.e. we get a standard integration by completing the square of the quantity in the denominator.

Example (3): Evaluate the integrals

(i) $\int \frac{3x-6}{x^2+4x+5}dx$ (ii) $\int \frac{x+4}{\sqrt{2x+x^2}}dx$

Solution:

(i) For the first integral, we have

$$\int \frac{3x-6}{x^2+4x+5}dx = 3\int \frac{x-2}{x^2+4x+5}dx$$

$$= \frac{3}{2}\int \frac{2x-4}{x^2+4x+5}dx = \frac{3}{2}\int \frac{2x+4-4-4}{x^2+4x+5}dx$$

$$= \frac{3}{2}\int \frac{2x+4}{x^2+4x+5}dx + \frac{3}{2}\int \frac{-8}{(x+2)^2+1}dx$$

$$= \frac{3}{2}\ln\left|x^2+4x+5\right| - 12\tan^{-1}(x+2) + C$$

(ii) $$\int \frac{x+4}{\sqrt{2x+x^2}}dx = \frac{1}{2}\int \frac{2x+8}{\sqrt{2x+x^2}}dx$$

$$= \frac{1}{2}\int \frac{2x+2-2+8}{\sqrt{2x+x^2}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{2x+x^2}}dx + \frac{1}{2}\int \frac{6}{\sqrt{x^2+2x+1-1}}dx$$

$$= \frac{1}{2}\int \left(2x+x^2\right)^{-1/2}\cdot(2x+2)dx + 3\int \frac{1}{\sqrt{(x+1)^2-1}}dx$$

$$= \frac{1}{2}\frac{\left(2x+x^2\right)^{1/2}}{1/2} + 3\cosh^{-1}(x+1) + C$$

$$= \sqrt{2x+x^2} + 3\cosh^{-1}(x+1) + C$$

(d) Integrals of the Form

$$\int\sqrt{\frac{ax+b}{Ax+B}}\,dx$$

In this case, multiply the integrand by $\frac{\sqrt{ax+b}}{\sqrt{ax+b}}$, so we get

$$\int\sqrt{\frac{ax+b}{Ax+B}}\,dx = \int\frac{ax+b}{\sqrt{Lx^2+Mx+N}}\,dx,$$

which is the integral studied in the previous case.

Example (4): Evaluate $\int\sqrt{\frac{x+3}{x+2}}\,dx$

Solution:

$$\int\sqrt{\frac{x+3}{x+2}}dx = \int\sqrt{\frac{x+3}{x+2}}\cdot\frac{\sqrt{x+3}}{\sqrt{x+3}}\,dx = \int\frac{x+3}{\sqrt{(x+2)(x+3)}}dx$$

$$= \int\frac{x+3}{\sqrt{x^2+5x+6}}dx$$

$$= \frac{1}{2}\left[\int\frac{2x+5}{\sqrt{x^2+5x+6}}dx + \int\frac{dx}{\sqrt{x^2+5x+6}}\right]$$

$$= \sqrt{x^2+5x+6} + \frac{1}{2}\int\frac{dx}{\sqrt{x^2+5x+6}}$$

$$=\sqrt{x^2+5x+6}+\frac{1}{2}\int\frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\frac{1}{4}}}$$

$$=\sqrt{x^2+5x+6}+\frac{1}{2}\cosh^{-1}2\left(x+\frac{5}{2}\right)+C$$

$$=\sqrt{x^2+5x+6}+\frac{1}{2}\cosh^{-1}(2x+5)+C.$$

Example (5): Evaluate $\int\sqrt{\frac{1-x}{2x+3}}\,dx, \qquad -\frac{3}{2}<x<1$

Solution:

$$\int\sqrt{\frac{1-x}{2x+3}}\,dx=\int\frac{(1-x)}{\sqrt{-2x^2-x+3}}\,dx$$

The first derivative of $-2x^2-x+3$ is $-4x-1$, thus we have

$$\int\sqrt{\frac{1-x}{2x+3}}\,dx=\frac{1}{4}\int\frac{-4x+4}{\sqrt{-2x^2-x+3}}dx$$

$$=\frac{1}{4}\times2\sqrt{-2x^2-x+3}+\frac{1}{4}\int\frac{5}{\sqrt{-2\left(x^2+\frac{x}{2}-\frac{3}{2}\right)}}dx$$

$$=\frac{1}{2}\sqrt{-2x^2-x+3}+\int\frac{\frac{5}{4}\sqrt{2}}{\sqrt{-\left(\left(x+\frac{1}{4}\right)^2-\frac{1}{16}-\frac{3}{2}\right)}}dx$$

$$=\frac{1}{2}\sqrt{-2x^2-x+3}+\int\frac{\frac{5}{4}\sqrt{2}}{\sqrt{\frac{25}{16}-\left(x+\frac{1}{4}\right)^2}}dx$$

$$=\frac{1}{2}\int\sqrt{-2x^2-x+3}+\frac{5}{4\sqrt{2}}\sin^{-1}\frac{\left(x+\frac{1}{4}\right)}{5/4}+C$$

$$=\frac{1}{2}\int\sqrt{-2x^2-x+3}+\frac{5}{4\sqrt{2}}\sin^{-1}\frac{(4x+1)}{5}+C$$

Exercises (6)

Evaluate each of the following integrals:

(1) $\int \frac{x\,dx}{x^2-4x+8}$

(2) $\int \frac{dx}{\sqrt{5-4x-2x^2}}$

(3) $\int \frac{dx}{x^2-4x+13}$

(4) $\int \frac{dx}{\sqrt{2x-x^2}}$

(5) $\int \frac{x\,dx}{x^2+6x+10}$

(6) $\int \frac{dx}{\sqrt{x^2-6x+10}}$

(7) $\int \sqrt{3-2x-x^2}\,dx$

(8) $\int \frac{e^x\,dx}{\sqrt{1+e^x+e^{2x}}}$

(9) $\int \frac{dx}{\left(x^2+6x+13\right)^{3/2}}$

(10) $\int \sqrt{x(6-x)}\,dx$

(11) $\int \frac{\cos x\,dx}{\sin^2 x-\sin x-2}$

(12) $\int \frac{\sin 2x\,dx}{\sin^2 x-2\sin x-8}$

(13) $\int \frac{\sin x\,dx}{5\cos x+\cos^2 x}$

(14) $\int \sqrt{\frac{x+2}{x+3}}\,dx$

2.12 Using Partial Fraction Decomposition to Evaluate Integral of Rational Functions

Consider we want to evaluate integrals of the form

$$\int \frac{a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \ldots\ldots + b_n x^n}\, dx,$$

<u>Case</u> (1): If the degree of the numerator is less than the degree of the denominator we proceed as in the following examples:

<u>Example</u> (5): Evaluate $\int \frac{8x+3}{x^2-2x-3} dx$.

<u>Solution</u>: we decompose the expression $\frac{8x+3}{x^2-2x-3}$ as in example (2), section (2.11) to get

$$\frac{8x+3}{x^2-2x-3} = \frac{5}{4(x+1)} + \frac{27}{4(x-3)},$$

then,

$$\int \frac{8x+3}{x^2-2x-3}dx = \frac{5}{4}\int \frac{1}{1+x}dx + \frac{27}{4}\int \frac{1}{x-3}dx$$

$$= \frac{5}{4}\ln|x+1| + \frac{27}{4}\ln|x-3| + C$$

Example (6): Evaluate $\int \frac{6x^2+x+10}{x^3-3x-2}dx$

Solution: In example (3), section (2.11), we have seen that

$$\frac{6x^2+x+10}{x^3-3x-2} = \frac{-5}{(x+1)^2} + \frac{2}{(x+1)} + \frac{4}{(x-2)}$$

then,

$$\int \frac{6x^2+x+10}{x^3-3x-2}dx = -5\int \frac{1}{(x+1)^2}dx + 2\int \frac{1}{(x+1)}dx + 4\int \frac{1}{(x-2)}dx$$

$$= -5\frac{(x+1)^{-1}}{-1} + 2\ln|x+1| + 4\ln|x-2| + C$$

$$= \frac{5}{x+1} + 2\ln|x+1| + 4\ln|x-2| + C$$

Example (7): Evaluate $\int \frac{7x^2+8}{x^3-1}dx$

Solution: In example (4), section (2.11), we have seen that

$$\frac{7x^2+8}{x^3-1}=\frac{5}{x-1}+\frac{2x-3}{x^2+x+1}$$

$$\therefore \int\frac{7x^2+8}{x^3-1}dx=5\int\frac{1}{x-1}dx+\int\frac{2x-3}{x^2+x+1}dx$$

$$=5\ln(x-1)+\int\frac{2x+1-4}{x^2+x+1}dx$$

$$=5\ln(x-1)+\int\frac{2x+1}{x^2+x+1}dx-4\int\frac{1}{x^2+x+1}dx$$

$$=5\ln(x-1)+\ln(x^2+x+1)-4\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$=\ln|x-1|^5+\ln(x^2+x+1)-4\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)+C$$

$$=\ln|x-1|^5+\ln(x^2+x+1)-\frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C.$$

Case (2): If the degree of numerator m is greater than the degree of denominator n we divide using the prolongation method of division to express the integrand as the sum of a polynomial and a proper rational function as we have done in example (1), section (2.11). Then

$$\int \frac{a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n} dx$$

$$= \int (a'_0 + a'_1 x + a'_2 x^2 + \dots + a'_p x^p) dx$$

$$+ \int \frac{C_0 + C_1 x + C_2 x^2 + \dots + C_q x^q}{b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n} dx,$$

where $p + q = m$ and $q < n$. Then, the first part can be integrated without difficulties, but for the second part we follow the method of partial fractions mentioned above and then integrate each fraction separately.

Example (8): Evaluate $\int \frac{x^4 - 5x - 7}{x^2 - 4} dx$

Solution: From example (1), we see that

$$\frac{x^4 - 5x - 7}{x^2 - 4} = x^2 + 4 + \frac{-5x + 9}{x^2 - 4}$$

Then,

$$\int \frac{x^4 - 5x - 7}{x^2 - 4} dx = \int (x^2 + 4) dx + \int \frac{-5x + 9}{x^2 - 4} dx$$

But,

$$\frac{-5x+9}{x^2-4}=\frac{A}{x-2}+\frac{B}{x+2}$$

$$-5x+9=A(x+2)+B(x-2)$$

$$-5=A+B\,,\quad 9=2A-2B,$$

then,

$$A=-\frac{1}{4},\quad B=-\frac{19}{4}.$$

Hence,

$$\frac{-5x+9}{x^2-4}=\frac{-1}{4(x-2)}+\frac{-19}{4(x+2)},$$

and

$$\int\frac{x^4-5x-7}{x^2-4}dx=\frac{x^3}{3}+4x-\frac{1}{4}\int\frac{1}{x-2}dx-\frac{19}{4}\int\frac{1}{x+2}dx$$

$$=\frac{x^3}{3}+4x-\frac{1}{4}\ln|x-2|-\frac{19}{4}\ln|x+2|+C$$

Example (9): Evaluate $\int\frac{e^x}{(e^{2x}+2e^x+1)(e^x+2)}dx$

Solution: We first use the substitution

$$e^x=u\Rightarrow e^{2x}=u^2,\ e^x dx=du$$

Then,

$$\int \frac{e^x}{\left(e^{2x}+2e^x+1\right)\left(e^x+2\right)}dx = \int \frac{du}{\left(u^2+2u+1\right)\left(u+2\right)}$$

$$= \int \frac{du}{(u+1)^2(u+2)}$$

Using partial fractions decomposition we get

$$\frac{1}{(u+1)^2(u+2)} = \frac{A}{(u+1)^2} + \frac{B}{(u+1)} + \frac{C}{(u+2)}$$

Then,

$$1 = A(u+2) + B(u+2)(u+1) + C(u+1)^2$$

Put u = −1: we get $A = -1$.

Put u = −2: we get $C = 1$

Put u = 0: we get $1 = A + 2B + C \Rightarrow 1 = -1 + 2B + 1 \Rightarrow B = \frac{1}{2}$

Hence,

$$\frac{1}{(u+1)^2(u+2)} = \frac{-1}{(u+1)^2} + \frac{1/2}{(u+1)} + \frac{1}{(u+2)}$$

and,

$$\int \frac{1}{(u+1)^2(u+2)}du = \int\left(\frac{-1}{(u+1)^2} + \frac{1/2}{(u+1)} + \frac{1}{(u+2)}\right)du$$

$$= \frac{1}{u+1} + \frac{1}{2}\ln|u+1| + \ln|u+2| + C$$

Hence,

$$\int \frac{e^x}{(e^{2x}+2e^x+1)(e^x+2)}dx$$

$$= \frac{1}{e^x+1} + \frac{1}{2}\ln\left|e^x+1\right| + \ln\left|e^x+2\right| + C$$

Exercises (7)

Evaluate each of the following integrals:

(1) $\int \frac{dx}{x^2 + x - 2}$ (2) $\int \frac{2x + 4}{x^3 - 2x^2}\, dx$

(3) $\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1}\, dx$ (4) $\int \frac{3x^2 - 10}{x^2 - 4x + 4}\, dx$

(5) $\int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x + 2)(x^2 + 3)^2}\, dx$

(6) $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\, dx$

(7) $\int \frac{2x^2 + 3}{x(x-1)^2}\, dx$ (8) $\int \frac{dx}{x^3 + x}$

(9) $\int \frac{2x^5 - x^3 - 1}{x^3 - 4x}\, dx$ (10) $\int \frac{x^2 + x - 16}{(x+1)(x-3)^2}\, dx$

(11) $\int \frac{x^3 + 3x^2 + x + 9}{(x^2 + 1)(x^2 + 3)}\, dx$ (12) $\int \frac{x^2}{(x + 2)^3}\, dx$

(13) $\int \frac{\cos\theta\, d\theta}{\sin^2\theta + 4\sin\theta - 5}$ (14) $\int \frac{e^t}{e^{2t} - 4}\, dt$

(15) $\int \frac{x^5}{(x^2 + 4)^2}\, dx$ (16) $\int \frac{x^6 - x^3 + 1}{x^4 + 9x^2}\, dx$

Review Exercises on Chapter (2)

1. Evaluate each of the following integrals:

$\int x^2\sqrt{7+3x}\,dx$, $\int\sqrt{x-4x^2}\,dx$, $\int e^{\pi x}\sin^{-1}(e^{\pi x})\,dx$

$\int\frac{3x}{4x-1}\,dx$, $\int\sqrt{x^2-3}\,dx$, $\int\frac{\sqrt{x^2+5}}{x^2}\,dx$,

$\int\frac{dx}{x^2\sqrt{x^2-2}}$, $\int\frac{\sqrt{3-x^2}}{x}\,dx$, $\int x^3\ln x\,dx$,

$\int\frac{\ln x}{\sqrt{x}}\,dx$ $\int e^{-2x}\sin 3x\,dx$ $\int\frac{\sin^2(\ln x)}{x}\,dx$

$\int\frac{dx}{x^2+4x-5}$ $\int\sqrt{3-2x-x^2}\,dx$ $\int\frac{x\,dx}{\sqrt{5+4x-x^2}}$

$\int\frac{x\,dx}{x^2+6x+13}$ $\int\sin 3x\sin 2x\,dx$ $\int\sin 2x\cos 5x\,dx$

$\int\cos 3x\cos 2x\,dx$ $\int\sin^2 x\cos^2 x\,dx$ $\int\sin^2 x\cos^4 x\,dx$

$\int\sin^3 2x\cos^2 2x\,dx$ $\int\frac{\sin x}{\cos^8 x}\,dx$ $\int\tan^5 x\,dx$

$\int\tan^5 x\sec x\,dx$ $\int\tan^5 x\sec^4 x\,dx$ $\int\tan^4 x\sec^4 x\,dx$

$\int\frac{dx}{x^2+x-2}$ $\int\frac{2x+4}{x^3-2x^2}\,dx$ $\int\frac{dx}{1+e^x}$

$$\int \frac{x^2+x-2}{3x^3-x^2+3x-1}\,dx$$

$$\int \frac{\cos\theta}{\sin^2\theta+4\sin\theta-5}\,d\theta$$

$$\int \frac{5x^3-3x^2+7x-3}{(x^2+1)^2}$$

$$\int \frac{x^{2/3}}{x+1}\,dx$$

$$\int \frac{e^t}{e^{2t}-4}\,dt$$

$$\int \frac{e^t}{e^{2t}-4e^t+3}\,dt$$

$$\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx$$

$$\int \frac{\ln x\,dx}{x\sqrt{4\ln x-1}}$$

$$\int e^x\sqrt{3-4e^{2x}}\,dx$$

$$\int x\sin 3x\,dx$$

$$\int e^{-\sqrt{x}}\,dx$$

$$\int \frac{e^{4x}}{\left(4-3e^{2x}\right)^2}\,dx$$

$$\int \frac{\cos x}{\sqrt{2-\sin^2 x}}\,dx$$

$$\int \frac{x^3}{\left(3+x^2\right)^{5/2}}\,dx$$

$$\int \frac{e^x\,dx}{\sqrt{1+e^x+e^{2x}}}$$

$$\int \sqrt{1+e^x}\,dx$$

$$\int \frac{dx}{1+\sin x}$$

$$\int \sqrt{x(4-x)}\,dx$$

$$\int \frac{dx}{4\sin x-3\cos x}$$

$$\int \frac{dx}{1-\sin x+\cos x}$$

$\int \frac{\cos x\, dx}{2-\cos x}$

2. Derive a reduction formula in part (a) and use it to evaluate the integral in (b)

(i) (a) $\int x^n e^x dx$, (b) $\int x^3 e^x dx$

(ii) (a) $\int \tan^n x dx$, (b) $\int \tan^6 x dx$

(iii) (a) $\int (\ln x)^n dx$, (b) $\int (\ln x)^5 dx$

(iv) (a) $\int \sin^n x \cos^m x dx$, (b) $\int \sin^6 x \cos^4 x dx$

Chapter (3)

Definite and Improper Integrals

Chapter 3

Definite and Improper Integrals

Definite integration is a fundamental concept of mathematical analysis. It is a powerful tool in mathematics, physics, mechanics and other disciplines. It is used for calculation of areas of regions bounded by curves, arc lengths, volumes, work, velocity, and others.

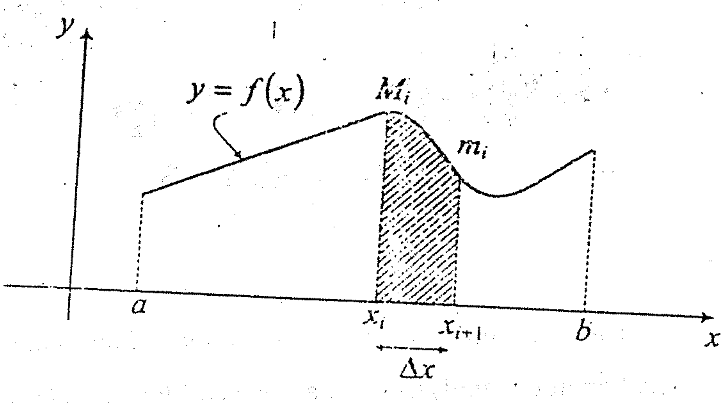
3.1 Definition and Geometric Interpretation of Definite Integrals

Let $f(x)$ be a continuous nonnegative function defined on the interval $[a, b]$. Divide the interval $[a, b]$ into n-equal subintervals by the points:

$$x_0 = a,\ x_1,\ x_2,\ \cdots,\ x_n = b,$$

and let

$$\Delta x = x_{i+1} - x_i, \quad i = 0, 1, \cdots, n-1.$$



Let M_i and m_i be the maximum and minimum values of the function $f(x)$ in the interval $[x_i, x_{i+1}]$. Define the lower sums L_n by:

$$L_n = \sum_{i=0}^{n-1} m_i \Delta x,$$

and the upper sums U_n by:

$$U_n = \sum_{i=0}^{n-1} M_i \Delta x.$$

If we take limits as $n \to \infty$ $(\Delta x \to 0)$ we get

$$U = \lim_{n\to\infty} U_n, \quad L = \lim_{n\to\infty} L_n$$

These two limits exist and they are called the upper and lower integrals of the function $f(x)$ over the interval $[a, b]$, respectively. If $f(x)$ is continuous and nonnegative on $[a, b]$, then U = L. In this case we define the definite integral of $f(x)$ over $[a, b]$ as follows:

$$\int_a^b f(x)\,dx = U = L$$

We shall also define,

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Remark: If $f(x) \geq 0$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = A,$$

where A is the area bounded by the curve $y = f(x)$, the x-axis, and the two vertical lines $x = a$ and $x = b$.

3.2 Basic Rules on Definite Integration

Rule 1 **(Scalar Multiplication Rule)**

The constant factor may be taken outside the sign of the definite integral. That is

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx,\ \text{for any constant } k,$$

For example, $\int_0^\pi 3\sin x\,dx = 3\int_0^\pi \sin x\,dx$.

Rule 2 **(Sum and Difference Rule)**

The definite integral of an algebraic sum (difference) of several functions is equal to the algebraic sum (difference) of the integrals of the summands. Thus, in the case of two terms

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

For example, $\int_a^b [x + \sin x]\,dx = \int_a^b x\,dx + \int_a^b \sin x\,dx$.

Rule 3

If the function $f(x) \geq 0$ on the interval $[a, b]$ then,

$$\int_a^b f(x)\,dx \geq 0$$

Rule 4

If on the interval $[a, b]$ where $(a < b)$, the function $f(x)$ and $g(x)$ satisfy the condition $f(x) \le g(x)$ then,

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Rule 5

If m and M are the smallest and greatest values of the function $f(x)$ on the interval $[a, b]$ and $(a \le b)$ then,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

The above rule is usualy used to estimate the value of the integral without calculating it as seen in the following example:

Example (1): Evaluate the upper and lower values of the integral $\int_0^{2\pi} \frac{dx}{\sqrt{10+6\sin x}}$.

Solution: By using rule (5),

$$m = \min_{0 \le x \le 2\pi} \frac{1}{\sqrt{10+6\sin x}} = \left.\frac{1}{\sqrt{10+6\sin x}}\right|_{x=\pi/2} = 0.25$$

$$M = \max_{0 \le x \le 2\pi} \frac{1}{\sqrt{10 + 6\sin x}} = \left. \frac{1}{\sqrt{10 + 6\sin x}} \right|_{x=3\pi/2} = 0.5$$

Thus we have

$$2\pi(0.25) \le \int_0^{2\pi} \frac{dx}{\sqrt{10 + 6\sin x}} \le 2\pi(0.5)$$

$$\frac{\pi}{2} \le \int_0^{2\pi} \frac{dx}{\sqrt{10 + 6\sin x}} \le \pi$$

Rule 6

If the two integration limits are equal then the corresponding integral vanishes, i. e.

$$\int_a^a f(x)\,dx = 0$$

Rule 7

For any three numbers a, b and c we have that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

For example, $\int_0^{\pi} \sin x\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^{\pi} \sin x\,dx.$

Rule 8

If the function $f(x)$ is even ($f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

Rule 9

If the function $f(x)$ is odd ($f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 0,$$

Remark: Rules 8 and 9 can be written in the following compact form

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

Rule 10

$$\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$$

Rule 11

$$\int_{0}^{2a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

Example (2): Find

(i) $\int_{-3}^{3} x^5 \, dx$, (ii) $\int_{-2}^{2} x^4 \, dx$, (iii) $\int_{-\pi}^{\pi} \sin^7 x \, dx$

Solution:

(i) Since x^5 is an odd function, then

$$I_1 = \int_{-3}^{3} x^5 \, dx = 0$$

(ii) Since x^4 is an even function, then

$$I_2 = \int_{-2}^{2} x^4 \, dx = 2\int_{0}^{2} x^4 \, dx = 2\frac{x^5}{5}\Big|_0^2 = \frac{64}{5}.$$

(iii) Since $\sin^7 x$ is an odd function, then (by rule 9)

$$I_3 = \int_{-\pi}^{\pi} \sin^7 x dx = 0.$$

3.3 The Fundamental Theorems of Integral Calculus

In the previous section we introduced the properties of definite integration without showing how to calculate it. In this section, in addition to giving the relation between differentiation and definite integration and the relation between definite and indefinite integration we show how to calculate definite integration in terms of indefinite integration. The following theorems are called the fundamental theorem of calculus because, as early mentioned, they establish the relationship between differentiation and integration.

Theorem 3.1: if $f(x)$ is continuous on $[a,\ b]$ and $F(x)$ is defined on $[a,\ b]$ by

$$F(x)=\int_a^x f(t)\,dt\,,$$

then $F(x)$ is differentiable on $[a,\ b]$ and

$$\frac{dF(x)}{dx}=\frac{d}{dx}\int_a^x f(t)\,dt=f(x), \qquad a<x<b\,.$$

Example (1): Find $S'(x)$ of the Fresnel function $S(x)=\int_0^x \sin(\frac{\pi t^2}{2})\,dt$. This function appears in the study of differaction of light waves and recently in the design of highways.

Solution: Applying Theorem (3.1), we get

$$S'(x)=\sin(\frac{\pi x^2}{2}).$$

The above theorem has the following generalization.

Theorem (3.2): If $f(x)$ is continuous function on $[a,\ b]$ and $u(x)$ and $v(x)$ are differentable functions of x whose values lie in $[a,\ b]$ and let $y=\int_{u(x)}^{v(x)} f(t)dt$, then

$$\frac{dy}{dx}=\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x)$$

Example (2): Find $\frac{dy}{dx}$ if:

(i) $y = \int_0^x \sin t \, dt$ (ii) $y = \int_{2x}^{x^2} \cos t \, dt$

Solution:

(i) Using Theorem (3.1), we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x \sin t \, dt\right) = \sin x$$

(ii) The direct application of Theorem (3.2) yields

$$\frac{dy}{dx} = \frac{d}{dx}\left(\int_{2x}^{x^2} \cos t \, dt\right)$$

$$= (\cos(x^2)).(2x) - (\cos(2x)).(2) = 2x\cos x^2 - 2\cos 2x$$

Example (3): Find the equation of the tangent line to the curve $y = F(x)$ at the point on the curve, where $x = 1$ and

$$F(x) = \int_1^{x^2} \frac{\sin(\frac{\pi}{2}t^2)}{e^{t-1}} dt.$$

Solution:

At $x = 1$ $y(1) = F(1) = \int_1^1 \frac{\sin(\frac{\pi}{2}t^2)}{e^{t-1}} dt = 0$.

Then, the slope m of the tangent line to the curve $y = F(x)$ is given by

$$y'(1) = \frac{d}{dx} F(x)\Big|_{x=1} = \frac{\sin(\frac{\pi}{2}x^4)}{e^{x^4-1}}(2x)\Bigg|_{x=1} = 2.$$

Hence, the equation of the tangent line at the point (1,0) is

$$y - y_1 = m(x - x_1)$$

or,

$$y - 0 = 2(x - 1)$$

Hence, the required equation is

$$y = 2(x - 1)$$

Example (4): Find $f(4)$ if $\int_0^x f(t)dt = x\cos(\pi x)$.

Solution: We have that

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$$

$$= \frac{d}{dx}(x\cos(\pi x)) = \cos(\pi x) - (\pi)\,x\sin(\pi x),$$

from which we get

$$f(x) = \cos(\pi x) - (\pi)x\sin(\pi x)$$

Therefore,

$$f(4) = \cos(4\pi) - 4\pi\sin(4\pi) = 1$$

The following theorem shows how to calculate definite integration in terms of indefinite integration

Theorem (3.3): (*The First Fundamental Theorem of Calculus*). If $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The following example is a direct application of the above theorem.

Example (5): Find

(i) $\int_0^2 x^3\,dx$, (ii) $\int_0^{\pi/2} (6\cos 4x + 7)dx$ (iii) $\int_1^4 \frac{1+\sqrt{x}}{x^2}dx$

Solution:

(i) $I_1 = \int_0^2 x^3\,dx = \left.\frac{x^4}{4}\right|_0^2 = \frac{16}{4} - \frac{0}{4} = 4$

(ii) $I_2 = \int_0^{\pi/2} (6\cos 4x + 7)dx = \left[\frac{6}{4}\sin 4x + 7x\right]_0^{\pi/2}$

$$= \left[\frac{3}{2}\sin 2\pi + 7\left(\frac{\pi}{2}\right)\right] - \left[\frac{3}{2}\sin 0 + 7(0)\right] = \frac{7\pi}{2}$$

(iii) $I_3 = \int_1^4 \frac{(1+\sqrt{x})}{x^2}\,dx = \int_1^4 x^{-2}\,dx + \int_1^4 x^{-3/2}\,dx$

$$= -\left.\frac{1}{x}\right|_1^4 + \left[\frac{x^{-1/2}}{-1/2}\right]_1^4$$

$$= -\left(\frac{1}{4} - 1\right) - 2\left(\frac{1}{\sqrt{4}} - 1\right) = \frac{3}{4} + 1 = \frac{7}{4}$$

3.4 Integration by Substitution

The following theorem is analogous to rule (2) (the rule of integration by substitution) of section 2.2

<u>Theorem</u> (3.4): Given the integral $\int_a^b f(x)dx$, where $f(x)$ is continuous on the interval $[a,\ b]$ and let $x = \varphi(t)$. Suppose that the function $\varphi(t)$ satisfies the following conditions:

(I) The value of $\varphi(t)$ varies from a to b when t varies from α to β so that $\varphi(\alpha) = a$, $\varphi(\beta) = b$ and all intermediate values of $\varphi(t)$ are in $[a,\ b]$.

(II) The derivative $\varphi'(t)$ of $\varphi(t)$ is a continuous function on the closed interval $[\alpha,\ \beta]$.

Then,

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt$$

In the following examples the above conditions are automatically satisfied, therefore we shall not verify them.

<u>Example</u> (1): Find $I = \int_0^a \sqrt{a^2 - x^2}\,dx$, $(a > 0)$

<u>Solution</u>: By the substitution

$$x = a\sin t \quad \Rightarrow \quad dx = a\cos t\,dt,$$

we get the following new limits.

When $x=0 \Rightarrow 0=a\sin t \Rightarrow t=0$;

when $x=a \Rightarrow a=a\sin t \Rightarrow 1=\sin t \Rightarrow t=\frac{\pi}{2}$.

Then,

$$I=\int_0^a \sqrt{a^2-x^2}\,dx=a^2\int_0^{\pi/2}\cos^2 t\,dt$$

$$=a^2\int_0^{\pi/2}\frac{1+\cos 2t}{2}\,dt$$

$$=\frac{a^2}{2}\left[\left.t\right|_0^{\pi/2}+\left.\frac{1}{2}\sin 2t\right|_0^{\pi/2}\right]=\frac{\pi a^2}{4}$$

Example (2): Find $I=\int_0^{\ln 2}\sqrt{e^x-1}\,dx$

Solution: Put

$$t=\sqrt{e^x-1} \Rightarrow t^2=e^x-1 \Rightarrow e^x=t^2+1$$

Then,

$$e^x dx=2t\,dt \Rightarrow dx=\frac{2t}{1+t^2}\,dt$$

when $x=0$, $t=\sqrt{e^0-1}=0$,

and when $x = \ln 2$, $t = \sqrt{e^{\ln 2} - 1} = \sqrt{2-1} = 1$.

Hence,

$$I = \int_0^{\ln 2} \sqrt{e^x - 1}\, dx = \int_0^1 \frac{2t^2}{1+t^2}\, dt$$

$$= 2\int_0^1 \frac{(1+t^2)-1}{1+t^2}\, dt = 2\int_0^1 (1 - \frac{1}{1+t^2})\, dt$$

$$= 2t \Big|_0^1 - 2\tan^{-1} t \Big|_0^1 = 2 - \frac{\pi}{2}$$

3.5 Integration by Parts

Theorem (3.5): Let the functions $u = u(x)$ and $v = v(x)$ have continuous derivatives $u'(x)$ and $v'(x)$ on the interval $[a, b]$. Then,

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v\, du$$

Example (1): Evaluate each of the following integrals:

(i) $I_1 = \int_0^1 xe^x dx$ (ii) $I_2 = \int_0^1 \tan^{-1} x dx$

(iii) $I_3 = \int_1^e \frac{\ln x}{x^2} dx$

Solution:

Using integration by parts we have that

(i) $I_1 = \int_0^1 xe^x dx = xe^x\Big|_0^1 - \int_0^1 e^x dx$

$$= xe^x\Big|_0^1 - e^x\Big|_0^1 = e - e + 1 = 1$$

(ii) $I_2 = \int_0^1 \tan^{-1} x dx = x \tan^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2} dx$

$= \frac{\pi}{4} - \frac{1}{2} \ln(1+x^2) \Big|_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2.$

(iii) $I_3 = -\frac{1}{x} \ln x \Big|_1^e + \int_1^e \frac{dx}{x^2}$

$= -\frac{\ln x}{x} \Big|_1^e - \frac{1}{x} \Big|_1^e = (\frac{-\ln e}{e} - \frac{1}{e}) - (-\frac{\ln 1}{1} - \frac{1}{1})$

$= 1 - \frac{2}{e}.$

Exercises (1)

Find the value of each of the following integrals:

(1) $\int_0^1 x e^{-5x} dx$ (2) $\int_{\sqrt{e}}^{e} \frac{\ln x}{x^2} dx$

(3) $\int_0^{1/2} \sin^{-1} x \, dx$ (4) $\int_0^{\pi} (x + x \cos x) dx$

(5) $\int_0^{\pi/4} \sin^4 x \, dx$ (6) $\int_0^{\pi/2} \cos^6 x \, dx$

(7) $\int_{0}^{\pi/2} \tan^3\left(\frac{x}{2}\right) dx$

(8) $\int_{0}^{1/4} \sec \pi x \ \tan \pi x \ dx$

(9) $\int_{0}^{1} \sqrt{1+x^2} \ dx$

(10) $\int_{\sqrt{2}}^{2} \frac{\sqrt{2x^2-4}}{x} dx$

(11) $\int_{0}^{3} \frac{x^3}{\left(3+x^2\right)^{5/2}} dx$

(12) $\int_{1}^{3} \frac{dx}{x^4\sqrt{x^2+3}}$

(13) $\int_{2}^{3} \frac{x \ dx}{(x-1)^6}$

(14) $\int_{0}^{25} \frac{dx}{\sqrt{4+\sqrt{x}}}$

(15) $\int_{0}^{1} \frac{x-1}{x^2+x+1} dx$

(16) $\int_{0}^{\pi/4} \cos x \ \cos 5x \ dx$

(17) Find the value of x if

(i) $\int_{2}^{x} \frac{dt}{t(4-t)} = \frac{1}{2}$

(ii) $\int_{1}^{x} \frac{dt}{t\sqrt{2t-1}} = 1$

<u>Cartesian & Polar Coordinates</u>:

- The distance between two points $P_1(x_1, y_1), P_2(x_2, y_2)$ is:

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

<u>Example</u>: Find the lengths of the sides of the triangle whose vertices are $(5,1), (-3,7)$ and $(8,5)$,and prove that one of the angles is a right angle.

<u>Solution</u>: Let $P_1(5,1), P_2(-3,7), P_3(8,5)$

$$\therefore \overline{P_1P_2} = \sqrt{(-3-5)^2 + (7-1)^2} = \sqrt{64+36} = \sqrt{100} = 10 ,$$

$$\overline{P_1P_3} = \sqrt{(8-5)^2 + (5-1)^2} = \sqrt{9+16} = \sqrt{25} = 5 ,$$

$$\overline{P_2P_3} = \sqrt{(8-(-3))^2 + (5-7)^2} = \sqrt{121+4} = \sqrt{125} = 5\sqrt{5} ,$$

$$\therefore \overline{P_1P_2}^2 + \overline{P_1P_3}^2 = \overline{P_2P_3}^2$$

Hence the angle at P_1 is a right angle.

- The coordinates of a point (x, y) which divides the straight line joining two given points $P_1(x_1, y_1), P_2(x_2, y_2)$ internally$^+$ (externally$^-$) in the ratio $m_1 : m_2$ is: $(x = \frac{m_1x_2 \pm m_2x_1}{m_1 \pm m_2}, y = \frac{m_1y_2 \pm m_2y_1}{m_1 \pm m_2})$.

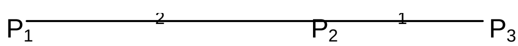
<u>Example1</u>: Find the coordinates of the point which divide the line joining the points $(2,-8)$ and $(-5,6)$ internally in the ratio $3:4$.

<u>Solution</u>:

$$(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}) = (\frac{3(-5) + 4(2)}{3+4}, \frac{3(6) + 4(-8)}{3+4}) = (-1,-2).$$

<u>Example2</u>: Find the coordinates of the point P_3 which divides the line joining the points $P_1(-3,-2), P_2(1,2)$ externally from the side of P_2 such that $\overline{P_1P_2} = 2\overline{P_2P_3}$.

<u>Solution</u>:



$$\frac{\overline{P_1P_3}}{\overline{P_2P_3}} = \frac{m_1}{m_2} = \frac{3}{1} ,$$

$$P_3\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right) = \left(\frac{3(1) - 1(-3)}{3-1}, \frac{3(2) - 1(-2)}{3-1}\right) = (3,4).$$

Example3: Find the coordinates of the two points P_3, P_4 which divides the line joining the points $P_1(2,-1), P_2(-1,5)$ into three equal parts.

Solution:

P_1 —1— P_3 —1— P_4 —1— P_2

$$\frac{\overline{P_1P_3}}{\overline{P_2P_3}} = \frac{m_1}{m_2} = \frac{1}{2} \quad ,$$

$$\therefore P_3\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right) = \left(\frac{1(-1) + 2(2)}{1+2}, \frac{1(5) + 2(-1)}{1+2}\right) = (1,1).$$

P_4 is the middle point between $P_3(1,1), P_2(-1,5)$,

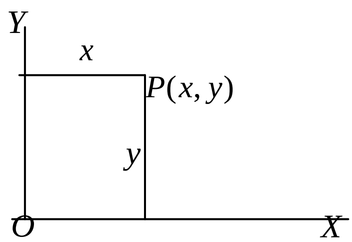
$$\therefore P_4\left(\frac{-1+1}{2}, \frac{5+1}{2}\right) = (0,3).$$

H.W: In what ratio does the point $(-1,-1)$ divide the join of $(-5,-3)$ and $(5,2)$?.

Coordinates System in a plane

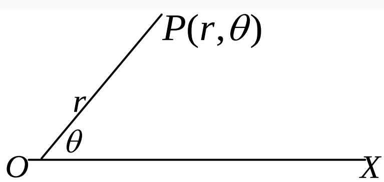
(1)- Cartesian Coordinates:

From a fixed point 0 at the plane is called the origin point we draw orthogonal straight lines **OX,OY** they are called axis coordinates If it is **P** at some point in the plane, P is completely determined by two number quantities (x, y) called point coordinates in the plane, where x represents the vertical dimension of the point P from the Y axis, and y represents the vertical dimension of the point P from the X (See figure):



(2)- Polar Coordinates:

Let O be a fixed point on the plane. From this fixed point we draw a straight horizontal constant that applies to the OX axis (See figure):



if P is a point in the plane, then P must be completely defined if we know the distance OP (i.e. the distance P from O), and if we also know the angle that the rectal OP makes with the OX axis

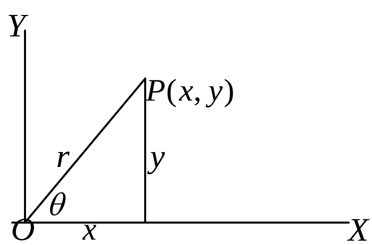
A fixed point, O, is called the starting line.

The OP dimension is called the polar dimension and symbolized by r, and the angle at which the OP straight from its original position applied to the OX axis to the OP position is called the polar angle of point P and is denoted by the symbol θ. The polar coordinates of point P in this case are the arranged two (r, θ).

The polar dimension OP is considered positive if measured from the O electrode in the straight direction that defines the polar angle θ, and is considered negative if measured in the opposite direction. The polar angle θ is considered positive if measured in an anti-clockwise direction, and is considered negative if measured in clockwise direction, and is: $(-\pi \leq \theta \leq)$

(3)-The relation between Cartesian and Polar Coordinates:

Let P be a point in the plane of its polar coordinates (r, θ) and its Cartesian coordinates (x, y).as shone:



From the figure we see that:

$x = r\cos\theta$ (1) , $y = r\sin\theta$ (2)

These two expressions x, y in terms of :(r, θ)

square the relations (1) and (2) and add them, we get:

$r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2}$ (3).

dividing (2) by (1), we get:

$y/x = \tan\theta \Rightarrow \theta = \tan^{-1}(y/x)$ (4)

These two relations (3), (4) express r, θ in terms of x, y

Example:1 Find the Polar Coordinates of the point: $P(\sqrt{3}, 1)$

Set the position of this point.

Solution:

the point is given in Cartesian coordinates (x, y) = (√3, 1), so:

$$r = \sqrt{x^2+y^2} = \sqrt{3+1} = \sqrt{4} = 2 \quad , \; x = r\cos\theta, \; y = r\sin\theta$$

$$\sqrt{3} = 2\cos\theta, 1 = 2\sin\theta$$

$$\cos\theta = \frac{\sqrt{3}}{2}, \sin\theta = \frac{1}{2}, \; \theta = \frac{\pi}{6} \quad \text{then: } (r,\theta) = \left(2, \frac{\pi}{6}\right)$$

Then the angle θ is in the first quadrant of the plane

Example:2

(i) Transform : $x^2+y^2-2x+2y=0$ into polar form.

(ii) Transform : $r = 4a\cos\theta$ into Cartesian form.

Solution:

(i) put : $x = r\cos\theta, \; y = r\sin\theta$

$$\therefore (r\cos\theta)^2 + (r\sin\theta)^2 - 2(r\cos\theta) + 2(r\sin\theta) = 0$$

$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) - 2r(\cos\theta - \sin\theta) = 0$$

$$\Rightarrow r = 2(\cos\theta - \sin\theta).$$

(ii) $r = 4a\cos\theta \Rightarrow r^2 = 4ar\cos\theta \Rightarrow x^2+y^2 = 4ax.$

Example:3

(i) Transform : $r^2 = a^2\cos 2\theta$ into Cartesian form.

(ii) Transform : $x^3 = y^2(2-x)$ into polar form.

Solution:

$(i) - r^2 = a^2\cos 2\theta = a^2(\cos^2\theta - \sin^2\theta) \Rightarrow r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$

$\Rightarrow (x^2+y^2)^2 = a^2(x^2-y^2)$

(ii) put $x = r\cos\theta,\ y = r\sin\theta$

$\therefore (r\cos\theta)^3 = (r\sin\theta)^2(2 - r\cos\theta)$

$\Rightarrow r^3\cos^3\theta s = r^2\sin^2\theta(2 - r\cos\theta)$

$\Rightarrow r^3\cos^3\theta + r^3\sin^2\theta\cos\theta = 2r^2\sin^2\theta$

$\Rightarrow r^3\cos\theta(\cos^2\theta + \sin^2\theta) = 2r^2\sin^2\theta$

$\Rightarrow r^3\cos\theta = 2r^2\sin^2\theta \Rightarrow r5\cos\theta = 2\sin^2\theta$

$r = 2\tan\theta\sin\theta$

Exercises:

1- Find the coordinates of the point P_3 which divides the line joining the points $P_1(0,-1), P_2(2,3)$ externally from the side of P_2 such that $\overline{P_1P_2} = 2\overline{P_2P_3}$.

2- Find the coordinates of the two points P_3, P_4 which divides the line joining the points $P_1(1,1), P_2(-2,-5)$ into three equal parts.

3- Prove that the medians of a triangle with vertices $P_1(x_1,y_1), P_2(x_2,y_2), P_3(x_3,y_3)$ is $M(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3})$.

4- Show that the distance between the two points $P_1(x_1,y_1), P_2(x_2,y_2)$ in polar coordinates is: $\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2-\theta_1)}$.

5- Find the Polar Coordinates for each of the following points::

$$P_1(-\sqrt{3},1), P_2(-1,\sqrt{3}), P_3(-1,1), P_4(-3,3\sqrt{3}), P_5(1,-\sqrt{3})$$

6- Find the Cartesian Coordinates for each of the following points:

$$P_1(2,-\frac{\pi}{2}), P_2(1,\frac{\pi}{3}), P_3(3,\frac{\pi}{4}), P_4(4,\frac{\pi}{3}), P_5(2,-\frac{\pi}{6})$$

7- Transform the following equations to the polar Coordinate:

(1) $(x^2+y^2)^2 = 2a^2xy$ (2) $y^{2\,2} = x^3/(2a-x)$

(3) $x^4+y^4 = a^2xy$ (4) $2x^2-2y^2 = 9$

8- Transform the following equations to the Cartesian Coordinate:

(1) $r = 1-\cos\theta$ (2) $r^2 = 9\cos 2\theta$

(3) $r = 3/(2+3\sin\theta)$ (4) $r(2-\cos\theta) = 2$

Analytical Geometry Of Three Dimensions

There are three ways to specification each point in R^3 :

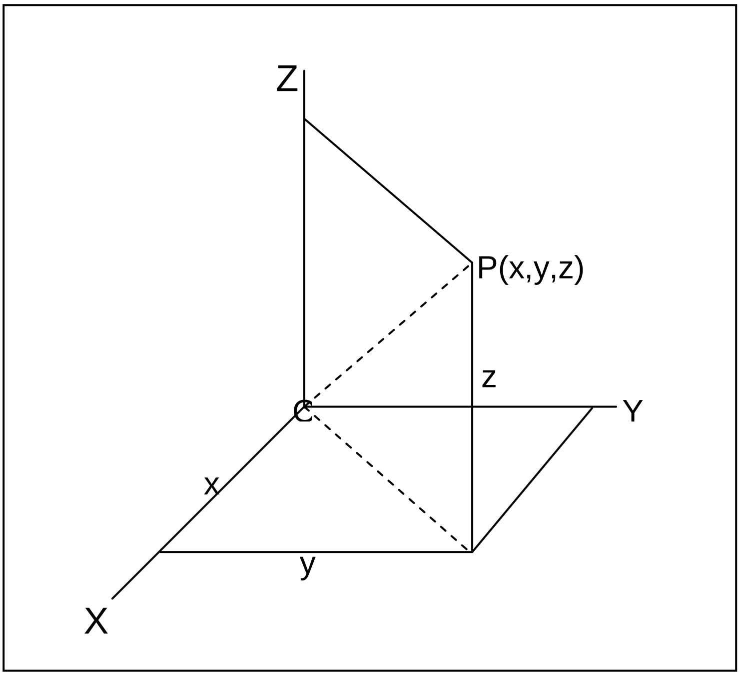
1. **1- Cartesian Coordinates.**
2. **2- Cylindrical Coordinates.**
3. **3- Spherical Coordinates.**

1- Cartesian Coordinates: (x, y, z)

The Orthogonal straight lines $\overleftrightarrow{OX}, \overleftrightarrow{OY}, \overleftrightarrow{OZ}$ are called *coordinates axes*, and XOY, YOZ, ZOX are called *coordinates planes* ;
(such that $O(0,0,0)$ is the origin point).
And as a result of the intersection of the coodinates axes produce eight zones in R^3 :

1st zone	$X > 0, Y > 0, Z > 0$
2nd zone	$X > 0, Y > 0, Z < 0$
3rd zone	$X > 0, Y < 0, Z > 0$
4th zone	$X > 0, Y < 0, Z < 0$
5th zone	$X < 0, Y > 0, Z > 0$
6th zone	$X < 0, Y > 0, Z < 0$
7th zone	$X < 0, Y < 0, Z > 0$
8th zone	$X < 0, Y < 0, Z < 0$

Therefore, each point in R^3 is similar in position to seven points.



- Determine the zone where the point $(-1,2,3)$ is located?.

In other words, each point in R^3 correspondes to seven points with respect to:

1- Coordinates Axes OX, OY, OZ.

2- Coordinates Planes XOY, YOZ, ZOX.

3- The origin point O.

Example: Determine the points corresponding to the point (a,b,c) with respect to:

1- Coordinates Axes OX, OY, OZ.

2- Coordinates Planes XOY, YOZ, ZOX.

3- The origin point O.

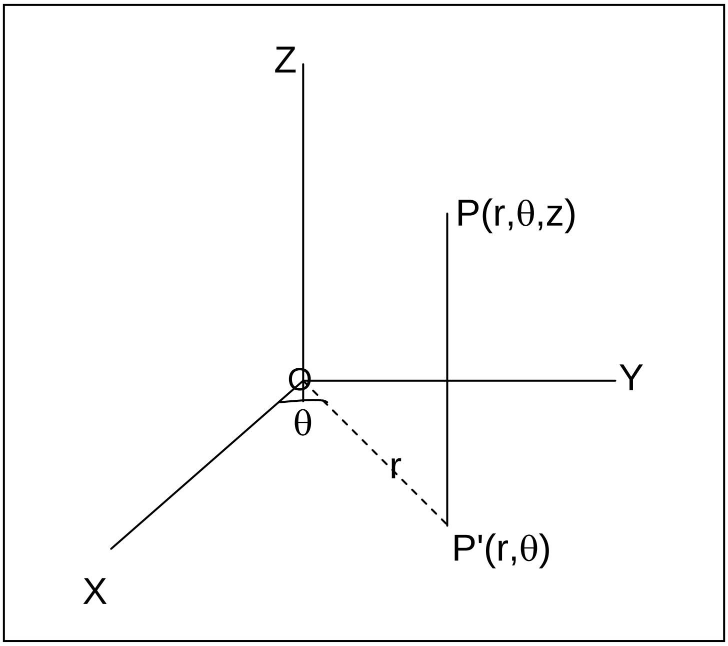
Solution:

1- $(a,-b,-c),(-a,b,-c),(-a,-b,c)$ respectively.

2- $(a,b,-c),(-a,b,c),(a,-b,c)$ respectively.

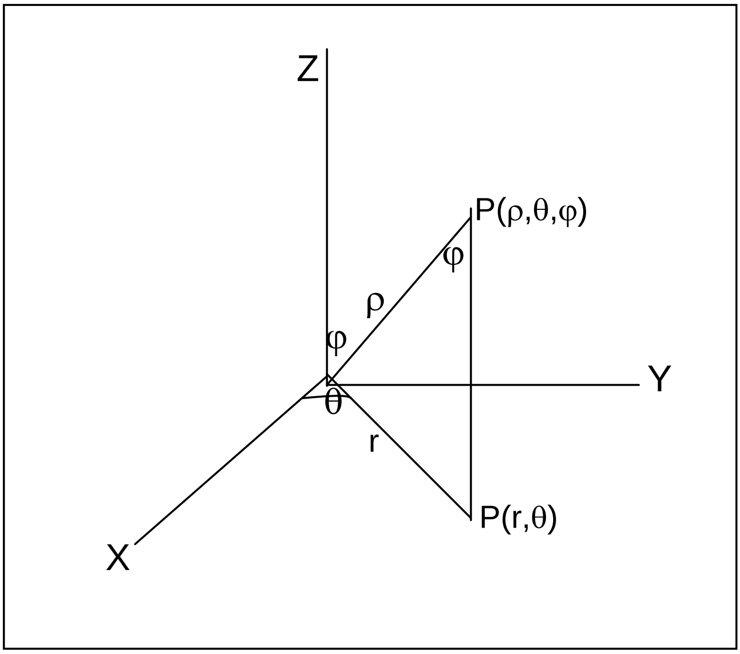
3- $(-a,-b,-c)$.

2- Cylindrical Coordinates: (r,θ,z)



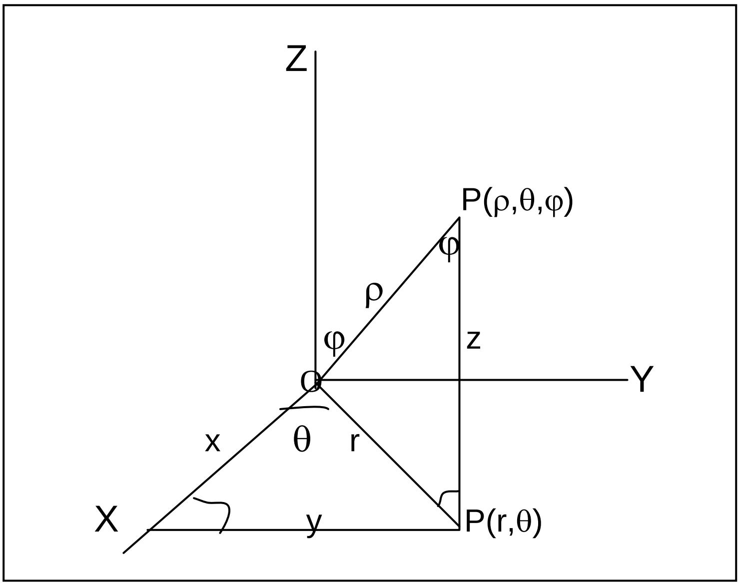
$$0 \le r < \infty \ , \ 0 \le \theta \le 2\pi \ , \ -\infty < z < \infty$$

3- Spherical Coordinates: (ρ,θ,φ)



$$0 \le \rho < \infty \ , \ 0 \le \theta \le 2\pi \ , \ 0 \le \varphi \le 2\pi$$

The relation between Cartesian, Cylindrical, and Spherical Coordinates:



From the figure above, we have the following relations:

$$x = r\cos\theta \quad , \qquad y = r\sin\theta \qquad (1)$$

$$r = \rho\sin\varphi \quad , \qquad z = \rho\cos\varphi \qquad (2)$$

And From (1) we have:

$$r = \sqrt{x^2 + y^2} \quad , \qquad \theta = \tan^{-1}\frac{y}{x} \qquad (3).$$

And From (2) we have:

$$\rho = \sqrt{r^2 + z^2} \quad , \qquad \varphi = \tan^{-1}\frac{r}{z} \qquad (4).$$

And From (1),(2) we have:

$$x = \rho \sin\varphi \cos\theta \ ,$$

$$y = \rho \sin\varphi \sin\theta \ , \qquad (5).$$

$$z = \rho \cos\varphi$$

And From (3),(4) we have:

$$\rho = \sqrt{x^2 + y^2 + z^2} \ ,$$

$$\theta = \tan^{-1}\frac{y}{x} \ , \qquad (6).$$

$$\varphi = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

Examples:

1- Find the Cylindrical Coordinates, and Spherical Coordinates of a point $(1,-\sqrt{3},2)$.

Solution:

$$r=\sqrt{x^2+y^2}=\sqrt{1+3}=2,$$

$$\theta=\tan^{-1}(\frac{-\sqrt{3}}{1})=-\frac{\pi}{3}$$

Therefor, the Cylindrical Coordinates of a point $(1,-\sqrt{3},2)$ is $(2,-\frac{\pi}{3},2)$,

$$\rho=\sqrt{x^2+y^2+z^2}=\sqrt{1+3+4}=2\sqrt{2},$$

$$\varphi=\tan^{-1}\frac{\sqrt{x^2+y^2}}{z}=\tan^{-1}\frac{2}{2}=\tan^{-1}1=\frac{\pi}{4}$$

Therefor, the Spherical Coordinates of a point $(1,-\sqrt{3},2)$ is $(2\sqrt{2},-\frac{\pi}{3},\frac{\pi}{4})$.

2- Find the Cartesian Coordinates of a point corresponding to the point $(2\sqrt{2},\frac{\pi}{3},\frac{\pi}{4})$ with respect to ZOX _ plane.

Solution: $(2\sqrt{2},\frac{\pi}{3},\frac{\pi}{4})\equiv(\rho,\theta,\varphi)$

$$x=\rho\sin\varphi\cos\theta=(2\sqrt{2})(\frac{1}{\sqrt{2}})(\frac{1}{2})=1,$$

$$y=\rho\sin\varphi\sin\theta=(2\sqrt{2})(\frac{1}{\sqrt{2}})(\frac{\sqrt{3}}{2})=\sqrt{3},$$

$$z=\rho\cos\varphi=(2\sqrt{2})(\frac{1}{\sqrt{2}})=2$$

$\therefore$ the Cartesian Coordinates of a point $(2\sqrt{2},\frac{\pi}{3},\frac{\pi}{4})$ is $(1,\sqrt{3},2)$, therefor, the Cartesian Coordinates of a point corresponding to the point $(2\sqrt{2},\frac{\pi}{3},\frac{\pi}{4})$ with respect to ZOX _ plane is $(1,-\sqrt{3},2)$.

Exercises:

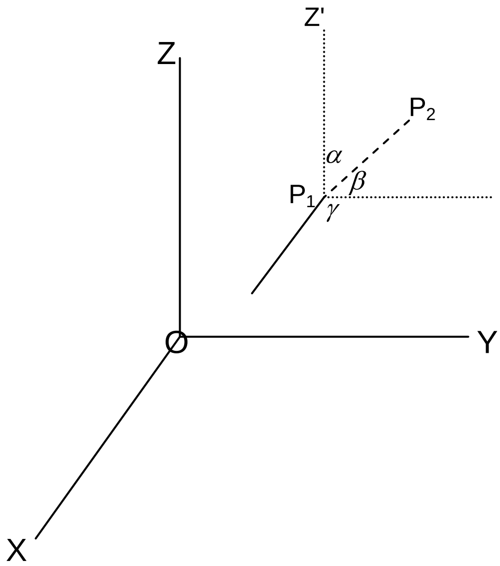
1- Using geometric figure, find the relation between Cartesian, Cylindrical, and Spherical Coordinates of a point P in R^3.
And verify that the Cylindrical, and Spherical Coordinates of a point $(-1,\sqrt{3},-2)$ are $(2,\frac{2\pi}{3},-2)$,and $(2\sqrt{2},\frac{2\pi}{3},\frac{3\pi}{4})$ respectively.

2- Find the Cartesian Coordinates of a point corresponding to each of the points $P_1(2,\frac{-\pi}{2},0)$, $P_2(1,\frac{-\pi}{3},1)$, $P_3(3,\frac{\pi}{4},1)$ with respect to the origin point.

3- Find the Spherical Coordinates of a point corresponding to each of the points $P_1(-1,\sqrt{3},-2)$, $P_2(\sqrt{3},1,2)$ with respect to the OX _ axis.

Direction Angles and Direction Ratios:

The angles α, β, γ made by the straight line $\overleftrightarrow{P_1P_2}$ with the positive direction of the coordinate axes in R^3
are called the ***direction angles*** of $\overleftrightarrow{P_1P_2}$,
as it shown in the following figure:



$(\cos\alpha, \cos\beta, \cos\gamma)$ are called ***cosines*** of the direction angles of $\overleftrightarrow{P_1P_2}$.

The direction angles of $\overleftrightarrow{OX}$ (X_axis) , $\overleftrightarrow{OY}$ (Y_axis), and $\overleftrightarrow{OZ}$ (Z_axis) are $(0, \frac{\pi}{2}, \frac{\pi}{2}), (\frac{\pi}{2}, 0, \frac{\pi}{2})$, and $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ respectively , and its cosines are $(1,0,0), (0,1,0), (0,0,1)$ respectively.

Theorem: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ if and only if α, β, γ are direction angles for a line in R^3 .

We denote $L = \cos\alpha$, $M = \cos\beta$, $N = \cos\gamma$
and any three quantities a, b, c are called ***direction ratios*** for the line $\overleftrightarrow{P_1P_2}$ in R^3 if and only if $L : M : N \equiv a : b : c$

Results:

(1) If $P_1(x_1,y_1,z_1), P_2(x_2,y_2,z_2)$, then the direction ratios of $\overleftrightarrow{P_1P_2}$ are:
x_2-x_1, y_2-y_1, z_2-z_1

and the cosines of the direction angles of $\overleftrightarrow{P_1P_2}$ are:

$$\cos\alpha=\frac{x_2-x_1}{\overline{P_1P_2}},\cos\beta=\frac{y_2-y_1}{\overline{P_1P_2}},\cos\gamma=\frac{z_2-z_1}{\overline{P_1P_2}}$$

---

(2) The cosines of the direction angles of $\overleftrightarrow{P_1P_2}$ whose direction ratios a,b,c are:

$$L=\frac{a}{\sqrt{a^2+b^2+c^2}}\ ,\ M=\frac{b}{\sqrt{a^2+b^2+c^2}}\ ,\ N=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

---

(3) The angle between two straight lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ which cosines are L_1,M_1,N_1 and L_2,M_2,N_2 respectively , (which direction ratios are a_1,b_1,c_1 and a_2,b_2,c_2 respectively) is:

$$\theta=\cos^{-1}[L_1L_2+M_1M_2+N_1N_2]=\cos^{-1}[\mp\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}],$$

and $\overleftrightarrow{P_1P_2}\perp\overleftrightarrow{P_3P_4}$ if $L_1L_2+M_1M_2+N_1N_2=0$.

---

(4) The direction ratios a,b,c for the vertical on two straight lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ which direction ratios are a_1,b_1,c_1 and a_2,b_2,c_2 respectively are:

$$a=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},\ b=\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix},\ c=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

---

(5) The length of the shortest distance K between two straight lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ (not parallel in R^3) determine as follows:

$K=\left|\overline{P_1P_3}(L_1L_2+M_1M_2+N_1N_2)\right|$ such that:

$\overline{P_1P_3}$ the distance between P_1 and P_3, and

L_1,M_1,N_1 the cosines of $\overleftrightarrow{P_1P_3}$, and

L_2,M_2,N_2 the cosines of the vertical on $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$.

Examples:

1- determine in which case of the following α,β,γ are direction angles for a line in R^3 ?

(i) $\alpha=\frac{\pi}{4},\beta=\frac{\pi}{3},\gamma=\frac{2\pi}{3}$

(ii) $\alpha=\frac{\pi}{3},\beta=\frac{\pi}{4},\gamma=\frac{3\pi}{4}$

Solution:

(i) $\cos^2\frac{\pi}{4}+\cos^2\frac{\pi}{3}+\cos^2\frac{2\pi}{3}=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{-1}{2}\right)^2=1$

Then α,β,γ are direction angles for a line in R^3.

(ii) $\cos^2\frac{\pi}{3}+\cos^2\frac{\pi}{4}+\cos^2\frac{3\pi}{4}=\left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{-1}{\sqrt{2}}\right)^2=\frac{5}{4}\neq 1$

Then α,β,γ are not direction angles for a line in R^3.

2- Find cosines of the straight line joining two given points $P_1(1,-2,3), P_2(2,-3,5)$.

Solution:

Let a,b,c are direction ratios for $\overleftrightarrow{P_1P_2}$.

$\therefore a=2-1=1$, $b=-3-(-2)=-1$, $c=5-3=2$,

Therefore, cosines of $\overleftrightarrow{P_1P_2}$ are:

$$L=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{1}{\sqrt{6}},$$

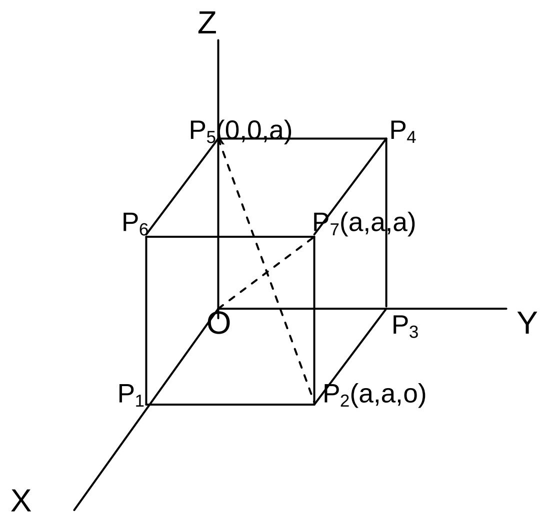
$$M=\frac{b}{\sqrt{a^2+b^2+c^2}}=-\frac{1}{\sqrt{6}},$$

$$N=\frac{c}{\sqrt{a^2+b^2+c^2}}=\frac{2}{\sqrt{6}}.$$

3- Verify that the angle θ between two diagonals of the cubic is $\theta = \cos^{-1}[\mp \frac{1}{3}]$?.

<u>Solution</u>:

Let a be the length of the cubic side, and three faces of the cubic are applicable at the coordinates planes XOY, YOZ, ZOX
as shown in the following figure:



Consequently, the diagonals of the cubic are $\overline{P_1P_4}, \overline{P_3P_6}, \overline{OP_7}, \overline{P_2P_5}$,
we find the angle θ between the two diagonals $\overline{OP_7}$ and $\overline{P_2P_5}$
as follow:
the direction ratios of $\overline{OP_7}$ is "a,a,a" and the direction ratios of $\overline{P_2P_5}$ is "$-a,-a,a$" then:

$$\theta = \cos^{-1}[\pm \frac{a(-a)+a(-a)+a(a)}{\sqrt{a^2+a^2+a^2}\,\sqrt{a^2+a^2+a^2}}] = \cos^{-1}[\pm \frac{-a^2}{3a^2}] = \cos^{-1}[\mp \frac{1}{3}].$$

4- Find cosines of the vertical on the plane passing the points $P_1(2,3,-2), P_2(1,-1,-1), P_3(0,1,2)$.

Solution:

The vertical on the plane passing the given points, is also vertical on the two straight lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_1P_3}$.

Let a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_1P_3}$ respectively, and a, b, c the direction ratios of the vertical on $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_1P_3}$,

$\therefore a_1 = 1-2 = -1$, $b_1 = -1-3 = -4$, $c_1 = -1-(-2) = 1$,

$a_2 = 0-2 = -2$, $b_2 = 1-3 = -2$, $c_2 = 2-(-2) = 4$,

Then, the direction ratios a, b, c of the vertical on $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_1P_3}$ are:

$$a = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -4 & 1 \\ -2 & 4 \end{vmatrix} = -14,$$

$$b = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} = 2,$$

$$c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} -1 & -4 \\ -2 & -2 \end{vmatrix} = -6.$$

Therefor, cosines of the vertical on the plane passing the given points are:

$$L = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{-14}{\sqrt{196+4+36}} = \frac{-14}{\sqrt{236}},$$

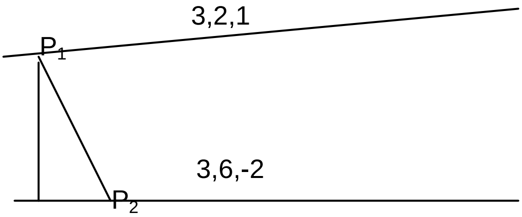
$$M = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{236}},$$

$$N = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{-6}{\sqrt{236}}.$$

5- Determine the length of the shortest distance between two straight lines; direction ratios of one of them are $3,2,1$ and passes the point $(3,4,5)$, and direction ratios of the other are $3,6,-2$ and passes the point $(4,6,3)$.

Solution:

Let $P_1(3,4,5), P_2(4,6,3)$,



The length of the shortest distance between two given lines is:

$K = \left|\overline{P_1P_2}(L_1L_2 + M_1M_2 + N_1N_2)\right|$ such that:

$\overline{P_1P_2}$ the distance between P_1 and P_2, and

L_1, M_1, N_1 the cosines of $\overleftrightarrow{P_1P_2}$, and

L_2, M_2, N_2 the cosines of the vertical on $\overleftrightarrow{P_1P_3}$ and $\overleftrightarrow{P_2P_4}$.

$\overline{P_1P_2} = \sqrt{(4-3)^2 + (6-4)^2 + (3-5)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$,

$\therefore L_1 = \frac{4-3}{3} = \frac{1}{3}$, $M_1 = \frac{6-4}{3} = \frac{2}{3}$, $N_1 = \frac{3-5}{3} = \frac{-2}{3}$,

$a_1, b_1, c_1 \equiv 3,2,1$, $a_2, b_2, c_2 \equiv 3,6,-2$,

Then, the direction ratios a, b, c of the vertical on the two given lines are:

$$a = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 6 & -2 \end{vmatrix} = -10,$$

$$b = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} = 9,$$

$$c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 3 & 6 \end{vmatrix} = 12.$$

Then, the cosines of the vertical on $\overleftrightarrow{P_1P_3}$ and $\overleftrightarrow{P_2P_4}$ are:

$$L_2=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{-10}{\sqrt{100+81+144}}=\frac{-10}{\sqrt{325}}=\frac{-10}{\sqrt{(13)(25)}}=\frac{-10}{5\sqrt{13}}\ ,$$

$$M_2=\frac{b}{\sqrt{a^2+b^2+c^2}}=\frac{9}{\sqrt{325}}=\frac{9}{5\sqrt{13}}\ ,$$

$$N_2=\frac{c}{\sqrt{a^2+b^2+c^2}}=\frac{12}{\sqrt{325}}=\frac{12}{5\sqrt{13}}\ ,$$

Therefor, the length of the shortest distance K between $\overline{P_1P_2}$ and $\overline{P_3P_4}$ is:

$$K=\left|\overline{P_1P_2}\left(L_1L_2+M_1M_2+N_1N_2\right)\right|$$

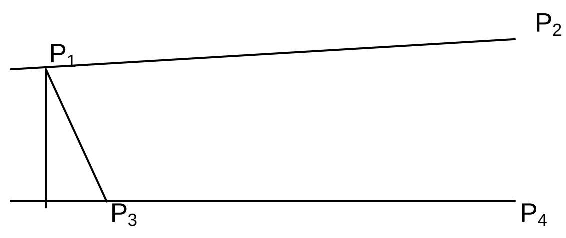
$$=\left|3\left[(\frac{1}{3})(\frac{-10}{5\sqrt{13}})+(\frac{2}{3})(\frac{9}{5\sqrt{13}})+(\frac{-2}{3})(\frac{12}{5\sqrt{13}})\right]\right|$$

$$=\left|\frac{-16}{5\sqrt{13}}\right|=\frac{16}{5\sqrt{13}}.$$

6- Determine the length of the shortest distance between the two straight lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ such that:

$P_1(-4,-1,2), P_2(2,-3,5), P_3(0,3,-5), P_4(2,4,-4)$.

<u>Solution</u>:



The length of the shortest distance between the two straight lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ is $K=\left|\overline{P_1P_3}\left(L_1L_2+M_1M_2+N_1N_2\right)\right|$ such that:

$\overline{P_1P_3}$ the distance between P_1 and P_3, and

L_1, M_1, N_1 the cosines of $\overleftrightarrow{P_1P_3}$, and

L_2, M_2, N_2 the cosines of the vertical on $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$,

$\overline{P_1P_3} = \sqrt{16+16+49} = \sqrt{81} = 9$, and cosines of $\overleftrightarrow{P_1P_3}$ is:

$L_1 = \frac{0-(-4)}{9} = \frac{4}{9}$, $M_1 = \frac{3-(-1)}{9} = \frac{4}{9}$, $N_1 = \frac{-5-2}{9} = \frac{-7}{9}$,

let a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ respectively, and a, b, c the direction ratios of the vertical on $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$,

$\therefore a_1 = 2-(-4) = 6$, $b_1 = -3-(-1) = -2$, $c_1 = 5-2 = 3$,

$a_2 = 2-0 = 2$, $b_2 = 4-3 = 1$, $c_2 = -4-(-5) = 1$,

$$a = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = -5,$$

$$b = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} = 0,$$

$$c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 2 & 1 \end{vmatrix} = 10.$$

Then, the cosines of the vertical on $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ are:

$$L_2 = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{-5}{\sqrt{25+0+100}} = \frac{-5}{5\sqrt{5}} = \frac{-1}{\sqrt{5}} ,$$

$$M_2 = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{0}{5\sqrt{5}} = 0 ,$$

$$N_2 = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{10}{5\sqrt{5}} = \frac{2}{\sqrt{5}} ,$$

Therefor, the length of the shortest distance between $\overline{P_1P_2}$ and $\overline{P_3P_4}$ is:

$$K = \left|\overline{P_1P_3}\left(L_1L_2 + M_1M_2 + N_1N_2\right)\right| = \left|9\left[(\frac{4}{9})(\frac{-1}{\sqrt{5}}) + (\frac{4}{9})(0) + (\frac{-7}{9})(\frac{2}{\sqrt{5}})\right]\right|$$

$$= \left|\frac{-18}{\sqrt{5}}\right| = \frac{18}{\sqrt{5}}.$$

Exercises:

1- Determine the value of λ so that $\overleftrightarrow{P_1P_2} \perp \overleftrightarrow{P_3P_4}$ such that:
$P_1(-\lambda,-1,2)$, $P_2(0,2,4)$, $P_3(1,\lambda,1)$, $P_4(\lambda+1,0,2)$.

2- Determine the length of the shortest distance between two straight lines; direction ratios of one of them are $2,-2,1$ and passes the point $(2,5,1)$, and direction ratios of the other are $6,3,-2$ and passes the point $(-2,2,6)$.

3- Determine the length of the shortest distance between the two straight lines $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{P_3P_4}$ such that:
$P_1(0,2,4)$, $P_2(3,4,5)$, $P_3(1,0,5)$, $P_4(4,6,3)$.

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Chapter 3

Analytical Solid Geometry

3.1 INTRODUCTION

In 1637, Rene Descartes* represented geometrical figures (configurations) by equations and vice versa. Analytical Geometry involves algebraic or analytic methods in geometry. Analytical geometry in three dimensions also known as Analytical solid** geometry or solid analytical geometry, studies geometrical objects in space involving three dimensions, which is an extension of coordinate geometry in plane (two dimensions).

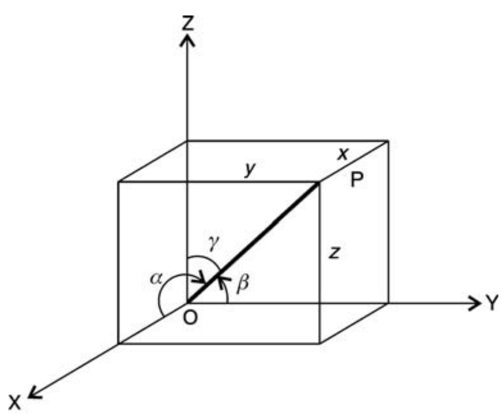


Fig. 3.1

* Rene Descartes (1596–1650) French philosopher and mathematician, latinized name for Renatus Cartesius.

** *Not* used in the sense of "non-hollowness". By a sphere or cylinder we mean a hollow sphere or cylinder.

Rectangular Cartesian Coordinates

The position (location) of a point in space can be determined in terms of its perpendicular distances (known as rectangular cartesian coordinates or simply *rectangular coordinates*) from three mutually perpendicular planes (known as **coordinate planes**). The lines of intersection of these three coordinate planes are known as *coordinate axes* and their point of intersection the *origin*.

The three axes called x-axis, y-axis and z-axis are marked positive on one side of the origin. The positive sides of axes *OX*, *OY*, *OZ* form a right handed system. The coordinate planes divide entire space into eight parts called *octants*. Thus a point P with coordinates x, y, z is denoted as $P(x, y, z)$. Here x, y, z are respectively the perpendicular distances of *P* from the *YZ*, *ZX* and *XY* planes. Note that a line perpendicular to a plane is perpendicular to every line in the plane.

Distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$. *Distance* from origin $O(0, 0, 0)$ is $\sqrt{x_2^2 + y_2^2 + z_2^2}$. Divisions of the line joining two points P_1, P_2: The coordinates of $Q(x, y, z)$, the point on P_1P_2 dividing the line segment P_1P_2 in the ratio $m : n$ are $\left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}, \frac{nz_1+mz_2}{m+n}\right)$ or putting k for $\frac{m}{n}$, $\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$; $k \neq -1$. Coordinates of mid point are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Direction of a line: A line in space is said to be directed if it is taken in a definite sense from one extreme (end) to the other (end).

Angle between Two Lines

Two straight lines in space may or may not intersect. If they intersect, they form a plane and are said to be coplanar. If they do not intersect, they are called *skew lines*.

Angle between two intersecting (coplanar) lines is the angle between their positive directions.

Angle between two non-intersecting (non-coplanar or skew) lines is the angle between two intersecting lines whose directions are same as those of given two lines.

3.2 DIRECTION COSINES AND DIRECTION RATIOS

Direction Cosines of a Line

Let L be a directed line OP from the origin $O(0, 0, 0)$ to a point $P(x, y, z)$ and of length r (Fig. 1.2). Suppose OP makes angles α, β, γ with the positive directions of the coordinate axes. Then α, β, γ are known as the *direction angles* of L. The cosines of these angles $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are known as the *direction cosines* of the line $L(OP)$ and are in general denoted by l, m, n respectively.

Thus

$$l = \cos\alpha = \frac{x}{r}, \quad m = \cos\beta = \frac{y}{r}, \quad n = \cos\gamma = \frac{z}{r}.$$

where $$r = \sqrt{x^2 + y^2 + z^2}.$$

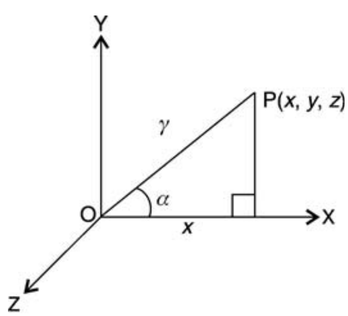


Fig. 3.2

Corollary 1: Lagrange's identity: $l^2+m^2+n^2 = 1$ i.e., sum of the squares of the direction cosines of any line is one, since $l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$.

Corollary 2: Direction cosines of the coordinate axes OX, OY, OZ are (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively.

Corollary 3: The coordinates of P are (lr, mr, nr) where l, m, n are the direction cosines of OP and r is the length of OP.

Note: Direction cosines is abbreviated as DC's.

Direction Ratios

(abbreviated as DR's:) of a line L are any set of three numbers a, b, c which are proportional to l, m, n the DC's of the line L. DR's are also known as direction numbers of L. Thus $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (proportionality constant) or $l = ak, m = bk, n = ck$. Since $l^2 + m^2 + n^2 = 1$ or $(ak)^2 + (bk)^2 + (ck)^2 = 1$ or $k = \frac{\pm 1}{\sqrt{a^2+b^2+c^2}}$. Then the actual direction cosines are $\cos\alpha = l = ak = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}$, $\cos\beta = m = bk = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$, $\cos\gamma = m = ck = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$ with $a^2 + b^2 + c^2 \neq 0$. Here +ve sign corresponds to positive direction and −ve sign to negative direction.

Note 1: Sum of the squares of DR's need not be one.

Note 2: Direction of line is $[a, b, c]$ where a, b, c are DR's.

Direction cosines of the line joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$l = \cos\alpha = \frac{PQ}{r} = \frac{LM}{r} = \frac{OM - OL}{r} = \frac{x_2 - x_1}{r}.$$

Similarly, $m = \cos\beta = \frac{y_2-y_1}{r}$ and $n = \cos\gamma = \frac{z^2-z_1}{r}$. Then the DR's of P_1P_2 are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$

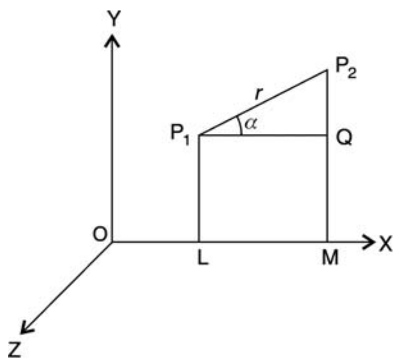


Fig. 3.3

Projections

Projection of a point P on line L is Q, the foot of the perpendicular from P to L.

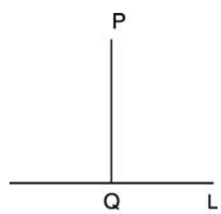


Fig. 3.4

Projection of line segment

P_1P_2 on a line L is the line segment MN where M and N are the feet of the perpendiculars from P and Q on to L. If θ is the angle between P_1P_2 and line L, then projection of P_1P_2 on $L = MN = PR = P_1P_2 \cos\theta$. Projection of line segment P_1P_2 on line L with (whose) DC's l, m, n is

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

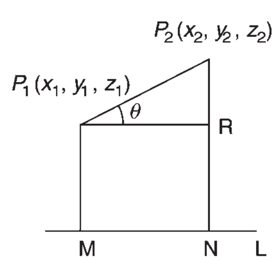


Fig. 3.5

Angle between Two Lines

Let θ be the angle between the two lines OP_1 and OP_2. Let $OP_1 = r_1$, $OP_2 = r_2$. Let l_1, m_1, n_1 be DC's of OP_1 and l_2, m_2, n_2 are DC's of OP_2. Then the coordinates of P_1 are l_1r_1, m_1r_1, n_1r_1 and of P_2 and l_2r_2, m_2r_2, n_2r_2.

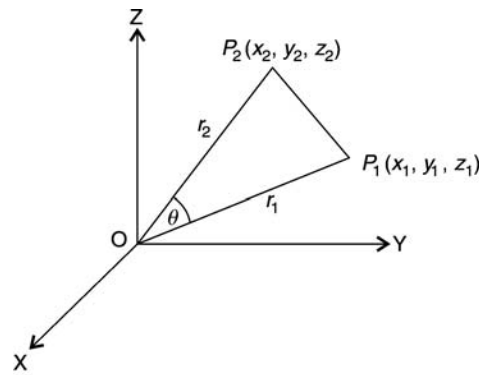


Fig. 3.6

From ΔOP_1P_2, we have

$$P_1P_2^2 = OP_1^2 + OP_2^2 - 2OP_1 \cdot OP_2 \cdot \cos\theta$$

$$(l_2r_2 - l_1r_1)^2 + (m_2r_2 - m_1r_1)^2 + (n_2r_2 - n_1r_1)^2$$
$$= \left[(l_1r_1)^2 + (m_1r_1)^2 + (n_1r_1)^2\right]$$
$$+ \left[(l_2r_2)^2 + (m_2r_2)^2 + (n_2r_2)^2\right] - 2 \cdot r_1r_2 \cos\theta.$$

Using $l_1^2 + m_1^2 + n_1^2 = 1$ and $l_2^2 + m_2^2 + n_2^2 = 1$,

$$r_1^2 + r_2^2 - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)$$
$$= r_1^2 + r_2^2 - 2r_1r_2 \cos\theta.$$

Then $\boxed{\cos\theta = l_1l_2 + m_1m_2 + n_1n_2}$

Corollary 1:

$$\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$$
$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2)$$
$$-(l_1l_2 + m_1m_2 + n_1n_2)^2$$
$$= (l_1m_2 - m_1l_2)^2 + (m_1n_2 - n_1m_2)^2$$
$$+(n_1l_2 - n_2l_1)^2$$

using the Lagrange's identity. Then

$$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2)$$
$$= (l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2.$$

Thus $\sin\theta = \sqrt{\sum(l_1m_2 - m_1l_2)^2}$

Corollary 2: $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\sum(l_1m_2-m_1l_2)^2}}{l_1l_2+m_1m_2+n_1n_2}$.

Corollary 3: If a_1, b_1, c_1 and a_2, b_2, c_2 are DR's of OP_1 and OP_2

Then $l_1 = \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$, $m_1 = \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$, $n_1 = \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}$ etc.

Then $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$,

$$\sin\theta = \frac{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Corollary: Condition for perpendicularity:
The two lines are perpendicular if $\theta = 90°$. Then

$$\cos\theta = \cos 90 = 0$$

Thus $$\boxed{l_1l_2 + m_1m_2 + n_1n_2 = 0}$$

or $$\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$

Corollary: Condition for parallelism:
If the two lines are parallel then $\theta = 0$. So $\sin\theta = 0$.

$$(l_1m_2 - m_1l_2)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 = 0$$

or $$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{l_1^2+m_1^2+n_1^2}}{\sqrt{l_2^2+m_2^2+n_2^2}} = \frac{1}{1}.$$

Thus $$\boxed{l_1 = l_2, \quad m_1 = m_2, \quad n_1 = n_2}$$

or $$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Worked Out Examples

Example 1: Find the angle between the lines $A(-3, 2, 4)$, $B(2, 5, -2)$ and $C(1, -2, 2)$, $D(4, 2, 3)$.

Solution: DR's of AB: $2-(-3)$, $5-2$, $-2-4$ $= 5, 3, -6$
DR's of CD: $3, 4, 1$. Then DC's of AB are $l_1 = \cos\alpha_1 = \frac{5}{\sqrt{5^2+3^2+6^2}} = \frac{5}{\sqrt{25+9+36}} = \frac{5}{\sqrt{70}}$ and $m_1 = \cos\beta_1 = \frac{3}{70}$, $n_1 = \cos\gamma_1 = \frac{-6}{\sqrt{70}}$. Similarly, $l_2 = \cos\alpha_2 = \frac{3}{\sqrt{3^2+4^2+1^2}} = \frac{3}{\sqrt{9+16+1}} = \frac{3}{\sqrt{26}}$, and $m_2 = \cos\beta_2 = \frac{4}{\sqrt{26}}$, $n_2 = \cos\gamma_2 = \frac{1}{\sqrt{26}}$. Now

$$\cos\theta = \cos\alpha_1\cdot\cos\alpha_2 + \cos\beta_1\cdot\cos\beta_2 + \cos\gamma_1\cdot\cos\gamma_2$$
$$= l_1l_2 + m_1m_2 + n_1n_2$$
$$\cos\theta = \frac{5}{\sqrt{70}}\cdot\frac{3}{\sqrt{26}} + \frac{3}{\sqrt{70}}\cdot\frac{4}{\sqrt{26}} - \frac{6}{\sqrt{70}}\cdot\frac{1}{\sqrt{26}}$$
$$= 0.49225$$
$$\therefore \quad \theta = \cos^{-1}(0.49225) = 60°30.7'$$

Example 2: Find the DC's of the line that is $\perp^r$ to each of the two lines whose directions are $[2, -1, 2]$ and $[3, 0, 1]$.

Solution: Let $[a, b, c]$ be the direction of the line. Since this line is $\perp^r$ to the line with direction $[2, -1, 2]$, by orthogonality

$$2a - b + 2c = 0$$

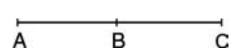
Similarly, direction $[a, b, c]$ is $\perp^r$ to direction $[3, 0, 1]$. So

$$3a + 0 + c = 0.$$

Solving $c = -3a$, $b = -4a$ or
direction $[a, b, c] = [a, -4a, -3a] = [1, -4, -3]$.
$\therefore$ DC's of the line: $\frac{1}{\sqrt{1^2+4^2+3^2}} = \frac{1}{\sqrt{26}}, \frac{-4}{\sqrt{26}}, \frac{-3}{\sqrt{26}}$.

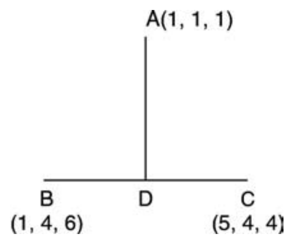
Example 3: Show that the points $A(1, 0, -2)$, $B(3, -1, 1)$ and $C(7, -3, 7)$ are collinear.

Solution: DR's of AB: $[2, -1, 3]$, DR's of AC: $[6, -3, 9]$, DR's of BC: $[4, -2, 6]$. Thus DR's of AB, AC, BC are same. Hence A, B, C are collinear.



Example 4: Find the coordinates of the foot of the perpendicular from $A(1, 1, 1)$ on the line joining $B(1, 4, 6)$ and $C(5, 4, 4)$.

Solution: Suppose D divides BC in the ratio $k : 1$. Then the coordinates of D are $\left(\frac{5k+1}{k+1}, \frac{4k+4}{k+1}, \frac{4k+6}{k+1}\right)$.
DR's of AD: $\frac{4k}{k+1}, 3, \frac{3K+5}{k+1}$, DR's of BC: $4, 0, -2$ AD is $\perp^r$ BC: $16k - 6k - 10 = 0$, or $k = 1$.



Coordinates of the foot of perpendicular are (3, 4, 5).

Example 5: Show that the points $A(1, 0, 2)$, $B(3, -1, 3)$, $C(2, 2, 2)$, $D(0, 3, 1)$ are the vertices of a parallelogram.

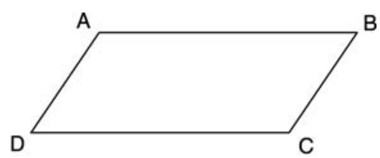


Fig. 3.7

Solution: DR's of *AB* are $[3-1, -1-0, 3-2] = [2, -1, 1]$. Similarly, DR's of *BC* are $[-1, 3, -1]$, of $CD[-2, 1, -1]$ of $DA[-1, 3, -1]$. Since DR's of *AB* and *CD* are same, they are parallel. Similarly *BC* and *DA* are parallel since DR's are same. Further *AB* is not $\perp^r$ to *AD* because

$$2(+1) + (-1)(-3) + 1(+1) = 6 \neq 0$$

Similarly, *AD* is not $\perp^r$ to *BC* because

$$2(-1) + (-1)3 + 1(-1) = -6 \neq 0.$$

Hence *ABCD* is a parallelogram.

EXERCISE

1. Show that the points $A(7, 0, 10)$, $B(6, -1, 6)$, $C(9, -4, 6)$ form an isoscales right angled triangle.

   Hint: $AB^2 = BC^2 = 18$, $CA^2 = 36$, $AB^2 + BC^2 = CA^2$

2. Prove that the points $A(3, -1, 1)$, $B(5, -4, 2)$, $C(11, -13, 5)$ are collinear.

   Hint 1: $AB^2 = 14$, $BC^2 = 126$, $CA^2 = 224$, $AB + BC = 4\sqrt{14} = CA$

   Hint 2: DR's of $AB = 2, -3, 1$; $BC: 6, -9, 3$; $AB \|^l$ to BC

3. Determine the internal angles of the triangle ABC where $A(2, 3, 5)$, $B(-1, 3, 2)$, $C(3, 5, -2)$.

   Hint: $AB^2 = 18$, $BC^2 = 36$, $AC^2 = 54$. DC's $AB: -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$; $BC: \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$; $AC: \frac{1}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}}$.

   Ans. $\cos A = \frac{1}{\sqrt{3}}$, $\cos B = 0$ i.e., $B = 90°$, $\cos C = \frac{\sqrt{6}}{3}$.

4. Show that the foot of the perpendicular from $A(0, 9, 6)$ on the line joining $B(1, 2, 3)$ and $C(7, -2, 5)$ is $D(-2, 4, 2)$.

   Hint: D divides BC in $k : 1$, $D\left(\frac{7k+1}{k+1}, \frac{-2k+2}{k+1}, \frac{5k+3}{k+1}\right)$. DR's AD: $(7k+1, -11k-7, -k-3)$, DR's BC: $6, -4, 2$. $AD \perp^r BC$: $k = -\frac{1}{3}$.

5. Find the condition that three lines with DC's l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are concurrent.

   Hint: Line with DC's l, m, n through point of concurrency will be $\perp^r$ to all three lines, $ll_i + mm_i + nn_i = 0$, $i = 1, 2, 3$.

   Ans.
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

6. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$ where $\alpha, \beta, \gamma, \delta$ are the angles which a line makes with the four diagonals of a cube.

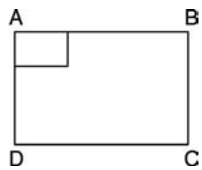
   Hint: DC's of four diagonals are (k, k, k), $(-k, k, k)$, $(k, -k, k)$, $(k, k, -k)$ where $k = \frac{1}{\sqrt{3}}$; l, m, n are DC's of line. $\cos\alpha = l.k. + mk + nk$, $\cos\beta = (-l + m + n)k$, $\cos\gamma = (l - m + n)k$, $\cos\delta = (l + m - n)k$.

7. Show that the points $A(-1, 1, 3)$, $B(1, -2, 4)$, $C(4, -1, 1)$ are vertices of a right triangle.

   Hint: DR's $AB : [2, -3, 1]$, $BC : [3, 1, -3]$, $CA : [5, -2, -2]$. AB is $\perp^r$ BC.

8. Prove that $A(3, 1, -2)$, $B(3, 0, 1)$, $C(5, 3, 2)$, $D(5, 4, -1)$ form a rectangle.

   Hint: DR's: AB: $[0, -1, 3]$; AC: $[2, 2, 4]$, $CD[0, 1, -3]$; $AD[2, 3, 1]$; $BC[2, 3, 1]$; $AB \| CD$, $AD \| BC$, $AD \perp AB$: $0 - 3 + 3 = 0$, $BC \perp DC$: $0 + 3 - 3 = 0$.



9. Find the interior angles of the triangle

   $A(3, -1, 4)$, $B(1, 2, -4)$, $C(-3, 2, 1)$.

   Hint: DC's of AB: $(-2, 3, -8)k_1$, BC: $(-4, 0, 5)k_2$, AC: $(-6, 3, -3)k_3$ where $k_1 = \frac{1}{\sqrt{77}}$, $k_2 = \frac{1}{\sqrt{41}}$, $k_3 = \frac{1}{\sqrt{54}}$.

Ans. $\cos A = \frac{15}{\sqrt{462}}$, $\cos B = \frac{32}{\sqrt{3157}}$, $\cos C = \frac{3}{\sqrt{246}}$.

10. Determine the DC's of a line $\perp^r$ to a triangle formed by $A(2, 3, 1)$, $B(6, -3, 2)$, $C(4, 0, 3)$.

Ans. $(3, 2, 0)k$ where $k = \frac{1}{\sqrt{13}}$.

   Hint: DR: AB: $[4, -6, 1]$, BC: $[-2, 3, 1]$, CA: $[2, -3, 2]$. $[a, b, c]$ of $\perp^r$ line: $4a - 6b + c = 0$, $-2a + 3b + c = 0$, $2a - 3b + 2c = 0$.

3.3 THE PLANE

Surface is the locus of a point moving in space satisfying a single condition.

Example: Surface of a sphere is the locus of a point that moves at a constant distance from a fixed point.

Surfaces are either plane or curved. Equation of the locus of a point is the analytical expression of the given condition(s) in terms of the coordinates of the point.

Exceptional cases: Equations may have locus other than a surface. Examples: (i) $x^2 + y^2 = 0$ is z-axis (ii) $x^2 + y^2 + z^2 = 0$ is origin (iii) $y^2 + 4 = 0$ has no locus.

Plane is a surface such that the straight line PQ, joining any two points P and Q on the plane, lies completely on the plane.

General equation of first degree in x, y, z is of the form

$$Ax + By + Cz + D = 0$$

Here A, B, C, D are given real numbers and A, B, C are not all zero (i.e., $A^2 + B^2 + C^2 \neq 0$)

Book Work: Show that every equation of the first degree in x, y, z represents a plane.

Proof: Let

$$Ax + By + Cz + D = 0 \tag{1}$$

be the equation of first degree in x, y, z with the condition that not all A, B, C are zero (i.e., $A^2 + B^2 + C^2 \neq 0$). Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the surface represented by (1). Then

$$Ax_1 + By_1 + Cz_1 + D_1 = 0 \tag{2}$$

$$Ax_2 + By_2 + Cz_2 + D_2 = 0 \tag{3}$$

Multiplying (3) by k and adding to (2), we get

$$A(x_1 + kx_2) + B(y_1 + ky_2) + C(z_1 + kz_2) + D(1 + k) = 0 \tag{4}$$

Assuming that $1 + k \neq 0$, divide (4) by $(1 + k)$.

$$A\left(\frac{x_1 + kx_2}{1 + k}\right) + B\left(\frac{y_1 + ky_2}{1 + k}\right) + C\left(\frac{z_1 + kz_2}{1 + k}\right) + D = 0$$

i.e., the point $R\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$ which is point dividing the line PQ in the ratio $k : 1$, also lies on the surface (1). Thus any point on the line joining P and Q lies on the surface i.e., line PQ completely lies on the surface. Therefore the surface by definition must be a plane.

General form of the equation of a plane is

$$Ax + By + Cz + D = 0$$

Special cases:

(i) Equation of plane passing through origin is

$$Ax + By + Cz = 0 \tag{5}$$

(ii) Equations of the coordinate planes XOY, YOZ and ZOX are respectively $z = 0$, $x = 0$ and $y=0$

(iii) $Ax + By + D = 0$ plane $\perp^r$ to xy-plane
$Ax + Cz + D = 0$ plane $\perp^r$ to xz-plane
$Ay + Cz + D = 0$ plane $\perp^r$ to yz-plane.

Similarly, $Ax + D = 0$ is $\|^l$ to yz-plane, $By + D = 0$ is $\|^l$ to zx-plane, $cz + D = 0$ is $\|^l$ to xy-plane.

One point form

Equation of a plane through a fixed point $P_1(x_1, y_1, z_1)$ and whose normal CD has DC's proportional to (A, B, C): For any point $P(x, y, z)$ on the given plane, the DR's of the line P_1P are $(x - x_1, y - y_1, z - z_1)$. Since a line perpendicular to a plane is perpendicular to every line in the plane, so ML is perpendicular to P_1, P. Thus

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \qquad (6)$$

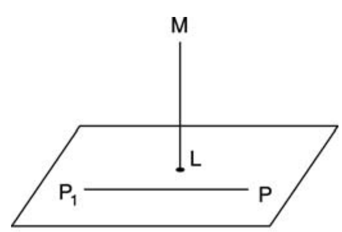


Fig. 3.8

Note 1: Rewriting (6), we get the general form of plane

$$Ax + By + Cz + D = 0 \qquad (1)$$

where $D = -ax_1 - by_1 - cz_1$

Note 2: The real numbers A, B, C which are the coefficients of x, y, z respectively in (1) are proportional to DC's of the normal ot the plane (1).

Note 3: Equation of a plane parallel to plane (1) is

$$Ax + By + Cz + D^* = 0 \qquad (7)$$

x-*intercept* of a plane is the point where the plane cuts the x-axis. This is obtained by putting $y = 0$, $z = 0$. Similarly, y-, z-intercepts. Traces of a plane are the lines of intersection of plane with coordinate axis.

Example: xy-trace is obtained by putting $z = 0$ in equation of plane.

Intercept form

Suppose $P(a, 0, 0)$, $Q(0, b, 0)$, $R(0, 0, c)$ are the x-, y-, z-intercepts of the plane. Then P, Q, R lies on the plane. From (1)

$$Aa + 0 + 0 + D = 0$$

or $$A = -\frac{D}{a}.$$

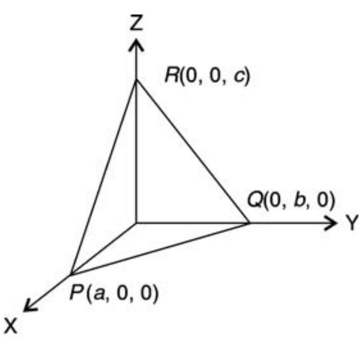


Fig. 3.9

similarly, $0 + bB + 0 + D = 0$ or $B = -\frac{D}{b}$ and $C = -\frac{D}{c}$.

Eliminating A, B, C the equation of the plane in the intercept form is

$$-\frac{D}{a}x - \frac{D}{b} - \frac{D}{c}z + D = 0$$

or $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (8)$$

Normal form

Let $P(x, y, z)$ be any point on the plane. Let ON be the perpendicular from origin O to the given plane. Let $ON = p$. (i.e., length of the perpendicular ON is p). Suppose l, m, n are the DC's of ON. Now ON is perpendicular to PN. Projection of OP on ON is ON itself i.e., p.

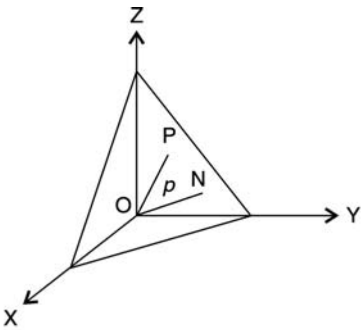


Fig. 3.10

Also the projection OP joining origin (0, 0, 0) to $P(x, y, z)$ on the line ON with DC's l, m, n is

$$l(x-0)+m(y-0)+n(z-0)$$

or $$lx+my+nz \tag{9}$$

Equating the two projection values from (8) & (9)

$$lx+my+nz=p \tag{10}$$

Note 1: p is always positive, since p is the perpendicular distance from origin to the plane.

Note 2: Reduction from general form.

Transpose constant term to R.H.S. and make it positive (if necessary by multiplying throughout by -1). Then divide throughout by $\pm\sqrt{A^2+B^2+C^2}$. Thus the general form $Ax+By+Cz+D=0$ takes the following normal form

$$\frac{Ax}{\pm\sqrt{A^2+B^2+C^2}}+\frac{By}{\pm\sqrt{A^2+B^2+C^2}}+\frac{Cz}{\pm\sqrt{A^2+B^2+C^2}}=\frac{-D}{\pm\sqrt{A^2+B^2+C^2}} \tag{11}$$

The sign before the radical is so chosen to make the R.H.S. in (11) positive.

Three point form

Equation of a plane passing through three given points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$:

Since the three points P_1, P_2, P_3 lie on the plane

$$Ax+By+Cz+D=0 \tag{1}$$

we have
$$Ax_1+By_1+Cz_1+D=0 \tag{12}$$
$$Ax_2+By_2+Cz_2+D=0 \tag{13}$$
$$Ax_3+By_3+Cz_3+D=0 \tag{14}$$

Eliminating A, B, C, D from (1), (12), (13), (14) (i.e., a non trivial solution A, B, C, D for the homogeneous system of 4 equations exist if the determinant coefficient is zero)

$$\begin{vmatrix} x & y & z & 1\\ x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0 \tag{15}$$

Equation (15) is the required equation of the plane through the 3 points P_1, P_2, P_3.

Corollary 1: Coplanarity of four given points: The four points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, $P_4(x_4, y_4, z_4)$ are coplanar (lie in a plane) if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1\\ x_2 & y_2 & z_2 & 1\\ x_3 & y_3 & z_3 & 1\\ x_4 & y_4 & z_4 & 1 \end{vmatrix}=0 \tag{16}$$

Angle between Two Given Planes

The angle between two planes

$$A_1x+B_1y+C_1z+D_1=0 \tag{17}$$
$$A_2x+B_2y+C_2z+D_2=0 \tag{18}$$

is the angle θ between their normals. Here A_1, B_1, C_1 and A_2, B_2, C_2 are the DR's of the normals respectively to the planes (17) and (18). Then

$$\cos\theta=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$$

Condition for perpendicularity

If $\theta=0$ then the two planes are $\perp^r$ to each other. Then

$$A_1A_2+B_1B_2+C_1C_2=0 \tag{19}$$

Condition for parallelism

If $\theta=0$, the two planes are $\|^l$ to each other. Then

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2} \tag{20}$$

Note: Thus parallel planes differ by a constant.

Although there are four constants A, B, C, D in the equation of plane, essentially three conditions are required to determine the three ratios of A, B, C, D, for example plane passing through:

a. three non-collinear points

b. two given points and $\perp^r$ to a given plane

c. a given point and $\perp^r$ to two given planes etc.

Coordinate of the Foot of the Perpendicular from a Point to a Given Plane

Let $Ax + By + Cz + D = 0$ be the given plane and $P(x_1, y_1, z_1)$ be a given point. Let PN be the perpendicular from P to the plane. Let the coordinates of the foot of the perpendicular PN be $N(\alpha, \beta, \gamma)$. Then DR's of $PN(x_1 - \alpha, y_1 - \beta, z_1 - \gamma)$ are proportional to the coefficients A, B, C i.e.,

$$x_1 - \alpha = kA, \quad y_1 - B = kB, \quad z_1 - \gamma = kC$$

or $$\alpha = x_1 - kA, \quad y_1 = \beta - kB, \quad z_1 = \gamma - kC$$

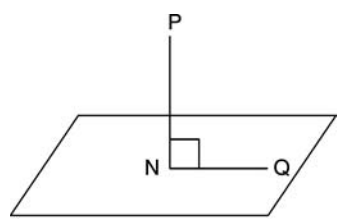


Fig. 3.11

Since N lies in the plane

$$A\alpha + B\beta + C\gamma + D = 0$$

Substituting α, β, γ,

$$A(x_1 - kA) + b(y_1 - kB) + c(z_1 - kC) + D = 0$$

Solving $$k = \frac{Ax_1 + By_1 + CZ_1 + D}{A^2 + B^2 + C^2}$$

Thus the coordinates of $N(\alpha, \beta, \gamma)$ the foot of the perpendicular from $P(x_1, y_1, z_1)$ to the plane are

$$\alpha = x_1 - \frac{A(Ax_1 + By_1 + Cz_1 + D)}{A^2 + B^2 + C^2},$$

$$\beta = y_1 - \frac{B(Ax_1 + By_1 + Cz_1 + D)}{A^2 + B^2 + C^2},$$

$$\gamma = z_1 - \frac{C(Ax_1 + By_1 + Cz_1 + D)}{A^2 + B^2 + C^2} \qquad (21)$$

Corollary 1: Length of the perpendicular from a given point to a given plane:

$$PN^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2$$
$$= (kA)^2 + (kB)^2 + (kC)^2$$
$$= k^2(A^2 + B^2 + C^2)$$
$$= \left[\frac{Ax_1 + By_1 + Cz_1 + D}{A^2 + B^2 + C^2}\right]^2 (A^2 + B^2 + C^2)$$
$$= \frac{(Ax_1 + By_1 + Cz_1 + D)^2}{A^2 + B^2 + C^2}$$

or $$PN = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

The sign before the radical is chosen as positive or negative according as D is positive or negative. Thus the numerical values of the length of the perpendicular PN is

$$PN = \left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right| \qquad (22)$$

Note: PN is obtained by substituting the coordinates (x_1, y_1, z_1) in the L.H.S. of the Equation (1) and dividing it by $\sqrt{A^2 + B^2 + C^2}$.

Equation of a plane passing through the line of intersection of two given planes $u \equiv A_1x + B_1y + C_1z + D_1 = 0$ and $v \equiv A_2x + B_2y + C_2z + D_2 = 0$ is $u + kv = 0$ where k is any constant.

Equations of the two planes bisecting the angles between two planes are

$$\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \pm\frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

Worked Out Examples

Example 1: Find the equation of the plane which passes through the point (2, 1, 4) and is

a. Parallel to plane $2x + 3y + 5z + 6 = 0$

b. Perpendicular to the line joining (3, 2, 5) and (1, 6, 4)

c. Perpendicular to the two planes $7x + y + 2z = 6$ and $3x + 5y - 6z = 8$

d. Find intercept points and traces of the plane in case c.

Solution:

a. Any plane parallel to the plane

$$2x + 3y + 5z + 6 = 0$$

is given by $2x+3y+5z+k=0$ (1) (differs by a constant). Since the point (2, 1, 4) lies on the plane (1), $2(2)+3(1)+5(4)+k=0$, $k=-27$. Required equation of plane is $2x+3y+5z-27=0$.

b. Any plane through the point (2, 1, 4) is (one point form)

$$A(x-2)+B(y-1)+C(z-4)=0 \qquad (2)$$

DC's of the line joining $M(3,2,5)$ and $N(1,6,4)$ are proportional to 2, −4, 1. Since MN is perpendicular to (2), A, B, C are proportional to 2, −4, 1. Then $2(x-2)-4(y-1)+1(z-4)=0$. The required equation of plane is $2x-4y+z-4=0$.

c. The plane through (2, 1, 4) is

$$A(x-2)+B(y-1)+C(z-4)=0. \qquad (2)$$

This plane (2) is perpendicular to the two planes $7x+y+2z=6$ and $3x+5y-6z=8$.
Using $A_1A_2+B_1B_2+C_1C_2=0$, we have

$$7a+b+2c=0$$

$$3a+5b-6c=0$$

Solving $\frac{a}{-6-10}=\frac{-b}{-42-8}=\frac{c}{35-3}$ or $\frac{a}{1}=\frac{b}{-3}=\frac{c}{-2}$.

Required equation of plane is

$$1(x-4)-3(y-1)-2(z-4)=0$$

or

$$x-3y-2z+7=0$$

d. x-intercept: Put $y=z=0$, $\therefore x=-7$ or (−7, 0, 0) is the x-intercept. Similarly, y-intercept is $(0, \frac{7}{3}, 0)$ and z-intercept is $\left(0,0,\frac{7}{2}\right)$. xy-trace is obtained by putting $z=0$. It is $x-3y+7=0$. Similarly, yz-trace is $3y+2z-7=0$ and zx-trace is $x-2z+7=0$.

Example 2: Find the equation of the plane containing the points $P(3,-1,-4)$, $Q(-2,2,1)$, $R(0,4,-1)$.

Solution: Equation of plane through the point $P(3,-1,-4)$ is

$$A(x+3)+B(y+1)+C(z+4)=0. \qquad (1)$$

DR's of PQ: −5, 3, 5; DR's of PR: −3, 5, 3. Since line PQ and PR completely lies in the plane (1), normal to (1) is perpendicular to PQ and PR. Then

$$-5A+3B+5C=0$$

$$-3A+5B+3C=0$$

Solving $A=C=1, \quad B=0$

$$(x-3)+0+(z+4)=0$$

Equation of the plane is

$$x+z+1=0$$

Aliter: Equation of the plane by 3-point form is

$$\begin{vmatrix} x & y & z & 1 \\ 3 & -1 & -4 & 1 \\ -2 & 2 & 1 & 1 \\ 0 & 4 & -1 & 1 \end{vmatrix}=0$$

Expanding $D_1x-D_2y+D_3z-1.D_4=0$ where

$$D_1=\begin{vmatrix} -1 & -4 & 1 \\ 2 & 1 & 1 \\ 4 & -1 & 1 \end{vmatrix}=-16, \quad D_2=\begin{vmatrix} 3 & -4 & 1 \\ -2 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}0=0$$

$$D_3=\begin{vmatrix} 3 & -1 & 1 \\ -2 & 2 & 1 \\ 0 & 4 & 1 \end{vmatrix}=-16, \quad D_4=\begin{vmatrix} 3 & -1 & -4 \\ -2 & 2 & 1 \\ 0 & 4 & -1 \end{vmatrix}=16$$

or required equation is $x+z+1=0$.

Example 3: Find the perpendicular distance between (a) The Point (3, 2, −1) and the plane $7x-6y+6z+8=0$ (b) between the parallel planes $x-2y+2z-8=0$ and $x-2y+2z+19=0$ (c) find the foot of the perpendicular in case (a).

Solution:
Perpendicular distance $=\left(\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right)$

a. Point (3, 2, −1), plane is $7x-6y+6z+8=0$. So perpendicular distance from (3, 2, −1) to plane is

$$=\frac{7(3)-6(2)+6(-1)+8}{\sqrt{7^2+6^2+6^2}}=\frac{11}{-11}=|-1|=1$$

b. x-intercept point of plane $x-2y+2z-8=0$ is (8, 0, 0) (obtained by putting $y=0, z=0$ in the equation). Then perpendicular distance from the point (8, 0, 0) to the second plane $x - 2y + 2z + 19 = 0$ is $\frac{1.8-2.0+2.0+19}{\sqrt{1^2+2^2+2^2}} = \frac{27}{3} = 9$

c. Let $N(\alpha, \beta, \gamma)$ be the foot of the perpendicular from $P(3, 2, -1)$. DR's of PN: $3 - \alpha, 2 - \beta, -1 - \gamma$. DR's of normal to plane are $7, -6, 6$. These are proportional. $\frac{3-\alpha}{7} = \frac{2-\beta}{-6} = \frac{-1-\gamma}{6}$ or $\alpha = 3 - 7k, \beta = 2 + 6k, \gamma = -1 - 6k$. Now (α, β, γ) lies on the plane. $7(3 - 7k) - 6(2 + 6k) + 6(-1 - 6k) + 8 = 0$ or $k = \frac{1}{11}$.
$\therefore$ the coordinates of the foot of perpendicular are $\left(\frac{26}{11}, \frac{28}{11}, \frac{-17}{11}\right)$.

Example 4: Are the points (2, 3, −5) and (3, 4, 7) on the same side of the plane $x + 2y - 2z = 9$?

Solution: Perpendicular distance of the point $(2, 3, -5)$ from the plane $x + 2y - 2z - 9 = 0$ or $-x - 2y + 2z + 9 = 0$ is $\frac{-1.2-2(3)-2(-5)+9}{\sqrt{1^2+2^2+2^2}} = -\frac{9}{3} = -3$.

$\perp^r$ distance of (3, 4, 7) is $\frac{-1.3-2.4+2.7+9}{\sqrt{1^2+2^2+2^2}} = \frac{12}{3} = 6$

$\perp^r$ distance from origin (0, 0, 0) is $\frac{0+0+0+9}{3} = 3$

So points (2, 3, −5) and (3, 4, 7) are on opposite sides of the given plane.

Example 5: Find the angle between the planes $4x - y + 8z = 9$ and $x + 3y + z = 4$.

Solution: DR's of the planes are [4, −1, 8] and [1, 3, 1]. Now

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$
$$= \frac{4.1 + 3\cdot(-1) + 1.8}{\sqrt{16 + 1 + 64}\sqrt{1 + 9 + 1}}$$
$$= \frac{9}{\sqrt{81}\sqrt{11}} = \frac{1}{\sqrt{11}} \quad \text{or} \quad \theta = \cos^{-1}\frac{1}{\sqrt{11}}.$$

Example 6: Find the equation of a plane passing through the line of intersection of the planes.

a. $3x + y - 5z + 7 = 0$ and $x - 2y + 4z - 3 = 0$ and passing through the point (−3, 2, −4)

b. $2x - 5y + z = 3$ and $x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$.

Solution:

a. Equation of plane is $u + kv = 0$ i.e.,

$$(3x + y - 5z + 7) + k(x - 2y + 4z - 3) = 0.$$

Since point (−3, 2, −4) lies on the intersection plane

$$[3(-3) + 1.(2) - 5(-4) + 7] + k[1(-3) - 2(2) + 4(-4) - 3] = 0.$$

So $k = \frac{10}{13}$. Then the required plane is

$$49x - 7y - 25z + 61 = 0.$$

b. Equation of plane is $u + kv = 0$ i.e.,

$$(2x - 5y + z - 3) + k(x + y + 4z - 5) = 0$$
or $$(2+k)x+(-5+k)y+(1+4k)z+(-3-5k) = 0.$$

Since this intersection plane is parallel to $x + 3y + 6z - 1 = 0$

$$\text{So} \quad \frac{2+k}{1} = \frac{-5+k}{3} = \frac{1+4k}{6} \quad \text{or} \quad k = -\frac{11}{2}.$$

Required equation of plane is $7x + 21y + 42z - 49 = 0$.

Example 7: Find the planes bisecting the angles between the planes $x + 2y + 2z = 9$ and $4x - 3y + 12z + 13 = 0$. Specify the angle θ between them.

Solution: Equations of the bisecting planes are

$$\frac{x + 2y + 2z - 9}{\sqrt{1 + 2^2 + 2^2}} = \pm\frac{4x - 3y + 12z + 13}{\sqrt{4^2 + 3^2 + 12^2}}$$
$$\frac{x + 2y + 2z - 9}{3} = \pm\frac{4x - 3y + 12z + 13}{13}$$

or $$25x + 17y + 62z - 78 = 0 \quad \text{and}$$
$$x + 35y - 10z - 156 = 0.$$

$$\cos\theta = \frac{25\cdot 1 + 17\cdot 35 - 62\times 10}{\sqrt{25^2 + 17^2 + 62^2}\sqrt{1 + 35^2 + 10^2}} = 0$$

$$\therefore \quad \theta = \frac{\pi}{2}$$

i.e, angle between the bisecting planes is $\frac{\pi}{2}$.

Example 8: Show that the planes

$$7x + 4y - 4z + 30 = 0 \quad (1)$$

$$36x - 51y + 12z + 17 = 0 \quad (2)$$

$$14x + 8y - 8z - 12 = 0 \quad (3)$$

$$12x - 17y + 4z - 3 = 0 \quad (4)$$

form four faces of a rectangular parallelopiped.

Solution: (1) and (3) are parallel since $\frac{7}{14} = \frac{4}{8} = \frac{-4}{-8} = \frac{1}{2}$. (2) and (4) are parallel since $\frac{36}{12} = \frac{-51}{-17} = \frac{12}{4} = 3$. Further (1) and (2) are $\perp^r$ since

$$7 \cdot 36 + 4(-51) - 4(12) = 252 - 204 - 48 = 0.$$

EXERCISE

1. Find the equation of the plane through $P(4, 3, 6)$ and perpendicular to the line joining $P(4, 3, 6)$ to the point $Q(2, 3, 1)$.

   Hint: DR's PQ: $[2, 0, 5]$, DR of plane through $(4, 3, 6) : x - 4, y - 3, z - 6; \perp^r$: $2(x - 4) + 0(y - 3) + 5(z - 6) = 0$

Ans. $2x + 5z - 38 = 0$

2. Find the equation of the plane through the point $P(1, 2, -1)$ and parallel to the plane $2x - 3y + 4z + 6 = 0$.

   Hint: Eq. $2x - 3y + 4z + k = 0, (1, 2, -1)$ lies, $k = 8$.

Ans. $2x - 3y + 4z + 8 = 0$

3. Find the equation of the plane that contains the three points $P(1, -2, 4)$, $Q(4, 1, 7)$, $R(-1, 5, 1)$.

   Hint: $A(x - 1) + B(y + 2) + C(z - 4) = 0$, DR: PQ: $[3, 3, 3]$, PR: $[-2, 7, -3]$. $\perp^r$ $3A + 3B + 3C = 0$, $-2A + 7B - 3C = 0$, $A = -10B, C = 9B$.

   Aliter: $\begin{vmatrix} x & y & z & 1 \\ 1 & -2 & 4 & 1 \\ 4 & 1 & 7 & 1 \\ -1 & 5 & 1 & 1 \end{vmatrix} = 0,$

   $$D_1x - D_2y + D_3z - D_4 = 0$$

   where $D_1 = \begin{vmatrix} -2 & 4 & 1 \\ 1 & 7 & 1 \\ 5 & 1 & 1 \end{vmatrix}$ etc.

Ans. $10x - y - 9z + 24 = 0$

4. Find the equation of the plane
   a. passing through $(1, -1, 2)$ and $\perp^r$ to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$
   b. passing through $(-1, 3, -5)$ and parallel to the plane $6x - 3y - 2z + 9 = 0$
   c. passing through $(2, 0, 1)$ and $(-1, 2, 0)$ and $\perp^r$ to the plane $2x - 4y - z = 7$.

Ans. **a.** $5x - 4y - z = 7$
b. $6x - 3y - 2z + 5 = 0$
c. $6x + 5y - 8z = 4$

5. Find the perpendicular distance between
   a. the point $(-2, 8, -3)$ and plane $9x - y - 4z = 0$
   b. the two planes $x - 2y + 2z = 6, 3x - 6y + 6z = 2$
   c. the point $(1, -2, 3)$ and plane $2x - 3y + 2z - 14 = 0$.

Ans. (a) $\sqrt{2}$ (b) $\frac{-16}{9}$ (c) 0 i.e., lies on the plane.

6. Find the angle between the two planes
   a. $x + 4y - z = 5, y + z = 2$
   b. $x - 2y + 3z + 4 = 0, 2x + y - 3z + 7=0$

Ans. (a) $\cos\theta = \frac{1}{2}, \theta = 60^\circ$ (b) $\cos\theta = \frac{-9}{14}$.

7. Prove that the planes $5x - 3y + 4z = 1$, $8x + 3y + 5z = 4$, $18x - 3y + 13z = 6$ contain a common line.

   Hint: $u + kv = 0$ substitute in $w = 0, k = \frac{1}{2}$

8. Find the coordinates of N, the foot of the perpendicular from the point $P(-3, 0, 1)$ on the plane $4x - 3y + 2z = 19$. Find the length of this perpendicular. Find also the image of P in the plane.

   Hint: $PN = NQ$ i.e., N is the mid point.

Ans. $N(1, -3, 3)$, $\sqrt{29}$, image of P is $Q(5, -6, 5)$

9. Find the equation of the plane through the line of intersection of the two planes $x - 3y + 5z - 7 = 0$ and $2x + y - 4z + 1 = 0$ and $\perp^r$ to the plane $x + y - 2z + 4 = 0$.

Ans. $3x - 2y + z - 6 = 0$

10. A variable plane passes through the fixed point (a, b, c) and meets the coordinate axes in P, Q, R. Prove that the locus of the point common to the planes through P, Q, R parallel to the coordinate plane is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$.

    Hint: $OP = x_1$, $OQ = y_1$, $OR = z_1$, $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$, (a, b, c) lies, $\frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1$.

3.4 THE STRAIGHT LINE

Two surfaces will in general intersect in a curve. In particular two planes, which are not parallel, intersect in a straight line.

Example: The coordinate planes ZOX and XOY, whose equations are $y = 0$ and $z = 0$ respectively, intersect in a line the x-axis.

Straight line

The locus of two simultaneous equations of first degree in x, y, z

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{aligned} \qquad (1)$$

is a straight line, provided $A_1 : B_1 : C_1 \neq A_2 : B_2 : C_2$ (i.e., not parallel). Equation (1) is known as the **general form** of the equation of a straight line. Thus the equation of a straight line or simply line is the pair of equations taken together i.e., equations of two planes together represent the equation of a line. However this representation is not unique, because many planes can pass through a given line. Thus a given line can be represented by different pairs of first degree equations.

Projecting planes

Of the many planes passing through a given line, those that are perpendicular to the coordinate planes are known as projecting planes and their traces give the **projections** of the line on the coordinate planes.

Symmetrical Form

The equation of line passing through a given point $P_1(x_1, y_1, z_1)$ and having direction cosines l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad (2)$$

since for any point $P(x, y, z)$ on the line, the DR's of PP_1: $x - x_1, y - y_1, z - z_1$ be proportional to l, m, n. Equation (2) represent two independent linear equations and are called the symmetrical (or symmetric) form of the equation of a line.

Corollary: Any point P on the line (2) is given by

$$x = x_1 + lr, \qquad y = y_1 + mr, \qquad z = z_1 + nr \qquad (3)$$

for different values of r, where $r = PP_1$.

Corollary: Lines perpendicular to one of the coordinate axes:

a. $x = x_1, \frac{y-y_1}{m} = \frac{z-z_1}{n}$, ($\perp^r$ to x-axis i.e., $\|^l$ to yz-plane)

b. $y = y_1, \frac{x-x_1}{l} = \frac{z-z_1}{n}$, ($\perp^r$ to y-axis i.e., $\|^l$ to xz-plane)

c. $z = z_1, \frac{x-x_1}{l} = \frac{y-y_1}{m}$, ($\perp^r$ to z-axis i.e., $\|^l$ to xy-plane)

Corollary: Lines perpendicular to two axes

a. $x = x_1, y = y_1$ ($\perp^r$ to x- & y-axis i.e., $\|^l$ to z-axis):

b. $x = x_1, z = z_1$ ($\perp^r$ to x- & z-axis i.e., $\|^l$ to y-axis)

c. $y = y_1, z = z_1$ ($\perp^r$ to y- & z-axis i.e., $\|^l$ to x-axis)

Corollary: Projecting planes: (containing the given line)

(a) $\frac{x-x_1}{l} = \frac{y-y_1}{m}$ (b) $\frac{x-x_1}{l} = \frac{z-z_1}{n}$ (c) $\frac{y-y_1}{m} = \frac{z-z_1}{n}$.

Note: When any of the constants l, m, n are zero, the Equation (2) are equivalent to equations

$$\frac{l}{x - x_1} = \frac{m}{y - y_1} = \frac{n}{z - z_1}.$$

Example: $\frac{x}{0} = \frac{y}{2} = \frac{z}{0}$ means $\frac{0}{x} = \frac{2}{y} = \frac{0}{z}$.

Corollary: If a, b, c are the DR's of the line, then (2) takes the form $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

Corollary: **Two point form** of a line passing through two given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad (4)$$

since the DR's of P_1P_2 are $x_2-x_1, y_2-y_1, z_2-z_1$.

Transformation of General Form to Symmetrical Form

The general form also known as unsymmetrical form of the equation of a line can be transformed to symmetrical form by determining

(a) one point on the line, by putting say $z = 0$ and solving the simultaneous equations in x and y.

(b) the DC's of the line from the fact that this line is $\perp^r$ to both normals of the given planes.

For example,

(a) by putting $z = 0$ in the general form

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{aligned} \quad (2)$$

and solving the resulting equations

$$A_1x + B_1y + D_1 = 0$$

$$A_2x + B_2y + D_2 = 0,$$

we get a point on the line as

$$\left(\frac{B_1D_2 - B_2D_1}{A_1B_2 - A_2B_1}, \frac{A_2D_1 - A_1D_2}{A_1B_2 - A_2B_1}, 0\right) \quad (5)$$

(b) Using the orthogonality of the line with the two normals of the two planes, we get

$$lA_1 + mB_1 + nC_1 = 0$$

$$lA_2 + mB_2 + nC_2 = 0$$

where $(l, m, n), (A_1, B_1, C_1)$ and (A_2, B_2, C_2) are DR's of the line, normal to first plane, normal to second plane respectively. Solving, we get the DR's l, m, n of the line as

$$\frac{l}{B_1C_2 - B_2C_1} = \frac{m}{C_1A_2 - C_2A_1} = \frac{n}{A_1B_2 - A_2B_1} \quad (6)$$

Using (5) and (6), thus the given general form (2) of the line reduces to the symmetrical form

$$\frac{x - \frac{(B_2D_1-B_1D_2)}{A_1B_2-A_2B_1}}{B_1C_2 - B_2C_1} = \frac{y - \frac{(A_2D_1-A_1D_2)}{A_1B_2-A_2B_1}}{C_1A_2 - C_2A_1} = \frac{z - 0}{A_1B_2 - A_2B_1} \quad (7)$$

Note 1: In finding a point on the line, one can put $x = 0$ or $y = 0$ instead of $z = 0$ and get similar results.

Note 2: General form (2) can also be reduced to the two point form (4) (special case of symmetric form) by determining two points on the line.

Angle between a Line and a Plane

Let π be the plane whose equation is

$$Ax + By + Cz + D = 0 \quad (8)$$

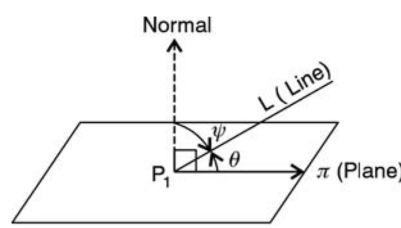


Fig. 3.12

and L be the straight line whose symmetrical form is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (2)$$

Let θ be the angle between the line L and the plane π. Let ψ be the angle between L and the normal to the plane π. Then

$$\cos\psi = \frac{lA + mB + nC}{\sqrt{l^2 + m^2 + n^2}\sqrt{A^2 + B^2 + C^2}} = \cos(90 - \theta) = \sin\theta \quad (9)$$

since $\psi = 90 - \theta$. The angle between a line L and plane π is the complement of the angle between the line L and the normal to the plane). Thus θ is determined from (9).

Corollary: Line is $\|^l$ to the plane if $\theta = 0$ then $\sin\theta = 0$ i.e.,

$$\boxed{lA + mB + nC = 0} \tag{10}$$

Corollary: Line is $\perp^r$ to the plane if $\theta = \frac{\pi}{2}$, then $\sin\theta = 1$ i.e.,

$$\boxed{\frac{l}{A} = \frac{m}{B} = \frac{n}{C}} \tag{11}$$

(i.e., DR's of normal and the line are same).

Conditions for a Line L to Lie in a Plane π

If every point of line L is a point of plane π, then line L lies in plane π. Substituting any point of the line $L : (x_1 + lr, y_1 + mr, z_1 + nr)$ in the equation of the plane (8), we get

$$A(x_1 + lr) + B(y_1 + mr) + C(z_1 + nr) + D = 0$$

$$\text{or} \quad (Al + Bm + Cn)r + (Ax_1 + By_1 + Cz_1 + D) = 0 \tag{12}$$

This Equation (12) is satisfied for all values of r if the coefficient of r and constant term in (12) are both zero i.e.,

$$\boxed{\begin{array}{c} Al + Bm + Cn = 0 \quad \text{and} \\ Ax_1 + By_1 + Cz_1 + D = 0 \end{array}} \tag{13}$$

Thus the two conditions for a line L to lie in a plane π are given by (13) which geometrically mean that (i) line L is $\perp^r$ to the nomal ot the plne and (ii) a (any one) point of line L lies on the plane.

Corollary: General equation of a plane containing line L (2) is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \tag{14}$$

subject to

$$Al + Bm + Cn = 0$$

Corollary: Equation of any plane through the line of intersection of the two planes

$$u \equiv A_1x + B_1y + C_1z + D_1 = 0 \quad \text{and}$$
$$v \equiv A_2x + B_2y + C_2z + D_2 = 0$$

is $u + kv = 0$ or $(A_1x + B_1y + C_1z + D_1) + k(A_2x + B_2y + C_2z + D_2) = 0$ where k is a constant.

Coplanar Lines

Consider two given straight lines L_1

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \tag{15}$$

and line L_2

$$\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2} \tag{16}$$

From (14), equation of any plane containing line L_1 is

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \tag{17}$$

subject to

$$Al_1 + Bm_1 + Cn_1 = 0 \tag{18}$$

If the plane (17) contains line L_2 also, then the point (x_2, y_2, z_2) of L_2 should also lie in the plane (17). Then

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \tag{19}$$

But the line L_2 is $\perp^r$ to the normal to the plane (17). Thus

$$Al_2 + Bm_2 + Cn_2 = 0 \tag{20}$$

Therefore the two lines L_1 and L_2 will lie in the same plane if (17), (18), (20) are simultaneously satisfied. Eliminating A, B, C from (19), (18), (20)(i.e., homogeneous system consistent if coefficient determinant is zero), we have

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \tag{21}$$

Thus (21) is the condition for coplanarity of the two lines L_1 and L_2. Now the equation of the plane containing lines L_1 and L_2 is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \tag{22}$$

which is obtained by eliminating A, B, C from (17), (18), (20).

Corollary: Condition for the two lines L_1

$$\begin{aligned} & u_1 \equiv A_1x + B_1y + C_1z + D_1 = 0, \\ & u_2 \equiv A_2x + B_2y + C_2z + D_2 = 0 \\ \text{and Line } L_2 \quad & u_3 \equiv A_3x + B_3y + C_3z + D_3 = 0, \\ & u_4 \equiv A_4x + B_4y + C_4z + D_4 = 0 \end{aligned} \tag{23}$$

to be coplanar is

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0 \tag{24}$$

If $P(\alpha, \beta, \gamma)$ is the point of intersection of the two lies, then P should satisfy the four Equations (23): $u_i|$ at $(\alpha, \beta, \gamma) = 0$ for $i = 1, 2, 3, 4$. Elimination of (α, β, γ) from these four equations leads to (24).

Corollary: The general form of equations of a line L_3 intersecting the lines L_1 and L_2 given by (23) are

$$u_1 + k_1u_2 = 0 \quad \text{and} \quad u_3 + k_2u_4 = 0 \tag{25}$$

where k_1 and k_2 are any two numbers.

Foot and length of the perpendicular from a point $P_1(\alpha, \beta, \gamma)$ to a given line L: $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

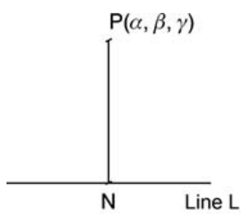


Fig. 3.13

Any point on the line L be $(x_1 + lr, y_1 + mr, z_1 + nr)$. The DR's of PN are $x_1 + lr - \alpha$, $y_1 + mr - \beta$, $z_1 + nr - \gamma$. Since PN is $\perp^r$ to line L, then

$$l(x_1 + lr - \alpha) + m(y_1 + mr - \beta) + n(z_1 + nr - \gamma) = 0.$$

Solving

$$r = \frac{l(\alpha - x_1) + m(\beta - y_1) + n(\gamma - z_1)}{l^2 + m^2 + n^2} \tag{26}$$

The coordinates of N, the foot of the perpendicular PN is $(x_1 + lr - \alpha, y_1 + mr - \beta, z_1 + nr - \gamma)$ where r is given by (26).

The length of the perpendicular PN is obtained by distance formula between P (given) and N (found).

Line of greatest slope in a plane

Let ML be the line of intersection of a horizontal plane I with slant plane II. Let P be any point on plane II. Draw $PN \perp^r$ to the line ML. Then the line of greatest slope in plane II is the line PN, because no other line in plane II through P is inclined to the horizontal plane I more steeply than PN.

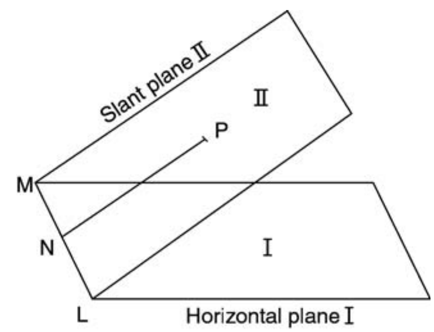


Fig. 3.14

Worked Out Examples

Example 1: Find the points where the line $x - y + 2z = 2$, $2x - 3y + 4z = 0$ pierces the coordinate planes.

Solution: Put $z = 0$ to find the point at which the line pierces the xy-plane: $x - y = 2$ and $2x - 3y = 0$ or $x = 6$, $y = 4$. ∴ (6, 4, 0).
Put $x = 0$, $-y + 2z = 2$, $-3y + 4z = 0$ or $y = 4$, $z = 3$ ∴ (0, 4, 3) is piercing point.
Put $y = 0$, $x + 2z = 2$, $2x + 4z = 0$ no unique solution.
Note that DR's of the line are [2, 0, −1]. So this line is $\perp^r$ to y-axis whose DR's are [0, 1, 0] (i.e., $2 \cdot 0 + 0 \cdot 1 + (-1) \cdot 0 = 0$). Hence the given line does not pierce the xz-plane.

Example 2: Transfer the general (unsymmetrical) form $x + 2y + 3z = 1$ and $x + y + 2z = 0$ to the symmetrical form.

Solution: Put $x = 0$, $2y + 3z = 1$, $y + 2z = 0$. Solving $z = -1$, $y = 2$. So $(0, 2, -1)$ is a point on the line. Let l, m, n be the DR's of the line. Since this line is $\perp^r$ to both normals of the given two planes, we have

$$1 \cdot l + 2 \cdot m + 3 \cdot n = 0$$

$$1 \cdot l + 1 \cdot m + 2 \cdot n = 0$$

Solving $\frac{l}{4-3} = -\frac{m}{2-3} = \frac{n}{1-2}$ or $\frac{l}{1} = \frac{m}{1} = -\frac{n}{1}$

Equation of the line passing through the point $(0, 2, -1)$ and having DR's $1, 1, -1$ is

$$\frac{x-0}{1} = \frac{y-2}{2} = \frac{z+1}{-1}$$

Aliter: Two point form.

Put $y = 0$, $x + 3z = 1$, $x + 27 = 0$. Solving $z = 1$, $x = -2$ or $(-2, 0, 1)$ is another point on the line. Now DR's of the line joining the two points $(0, 2, -1)$ and $(-2, 0, 1)$ are $-2, -2, 2$. Hence the equation of the line in the two point form is

$$\frac{x-0}{-2} = \frac{y-2}{-2} = \frac{z+1}{2} \quad \text{or} \quad \frac{x}{1} = \frac{y-2}{1} = \frac{z+1}{-1}.$$

Example 3: Find the acute angle between the lines $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x}{5} = \frac{y}{4} = \frac{z}{-3}$.

Solution: DR's are $[2, 2, 1]$ and $[5, 4, -3]$. If θ is the angle between the two lines, then

$$\cos\theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$= \frac{2 \cdot 5 + 2 \cdot 4 + 1 \cdot (-3)}{\sqrt{4+4+1}\sqrt{25+16+9}} = \frac{15}{3\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = 45^\circ$$

Example 4: Find the equation of the plane containing the line $x = y = z$ and passing through the point $(1, 2, 3)$.

Solution: General form of the given line is

$$x - y = 0 \quad \text{and} \quad x - z = 0.$$

Equation of a plane containing this line is

$$(x - y) + k(x - z) = 0$$

Since point $(1, 2, 3)$ lies on this line, it also lies on the above plane. Then

$$(1 - 2) + k(1 - 3) = 0 \quad \text{or} \quad k = -\frac{1}{2}$$

Equation of required plane is

$$(x - y) - \frac{1}{2}(x - z) = 0$$

or

$$x - 2y + z = 0.$$

Example 5: Show that the lines $\frac{x}{1} = \frac{y+3}{2} = \frac{z+1}{3}$ and $\frac{x-3}{2} = \frac{y}{1} = \frac{z-1}{-1}$ intersect. Find the point of intersection.

Solution: Rewriting the equation in general form, we have

$$2x - y = 3, \quad 3x - z = 1$$

and

$$x - 2y = 3, \quad x + 2z = 5$$

If these four equations have a common solution, then the given two lines intersect. Solving, $y = -1$, then $x = 1$, $z = 2$. So the point of intersection is $(1, -1, 2)$.

Example 6: Find the acute angle between the lines $\frac{x}{3} = \frac{y}{1} = \frac{z}{0}$ and the plane $x + 2y - 7 = 0$.

Solution: DR's of the line: $[3, 1, 0]$. DR's of normal to the plane is $[1, 2, 0]$. If ψ is the angle between the line and the normal, then

$$\cos\psi = \frac{3 \cdot 1 + 1 \cdot 2 + 0 \cdot 0}{\sqrt{3^2 + 1^2 + 0^2}\sqrt{1^2 + 2^2 + 0^2}}$$

$$= \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}} \quad \text{so} \quad \psi = 45^\circ.$$

Angle θ between the line and the plane is the complement of the angle ψ i.e., $\theta = 90 - \psi = 90 - 45 = 45^\circ$.

Example 7: Show that the lines $x + y - 3z = 0$, $2x + 3y - 8z = 1$ and $3x - y - z = 3$, $x + y - 3z = 5$ are parallel.

Solution: DR's of the first line are

l_1	m_1	n_1
1	1	−3
2	3	−8

or $\frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{1}$

Similarly, DR's of the second line are

$$\begin{matrix} l_2 & m_2 & n_2 \\ 3 & -1 & -1 \\ 1 & 1 & -3 \end{matrix} \quad \text{or} \quad \frac{l_2}{4}=\frac{m_2}{8}=\frac{n_2}{4} \quad \text{i.e.,} \quad \frac{l_2}{1}=\frac{m_2}{2}=\frac{n_2}{1}$$

Since the DR's of the two lines are same, they are parallel.

Example 8: Find the acute angle between the lines $2x - y + 3z - 4 = 0$, $3x + 2y - z + 7 = 0$ and $x + y - 2z + 3 = 0$, $4x - y + 3z + 7 = 0$.

Solution: The line represented by the two planes is perpendicular to both the normals of the two planes. If l_1, m_1, n_1 are the DR's of this line, then

$$\begin{matrix} l_1 & m_1 & n_1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{matrix} \quad \text{or} \quad \frac{l_1}{-5} = \frac{m_1}{11} = \frac{n_1}{7}$$

Similarly, DR's of the 2nd line are

$$\begin{matrix} l_2 & m_2 & n_2 \\ 1 & +1 & -2 \\ 4 & -1 & -3 \end{matrix} \quad \text{or} \quad \frac{l_2}{-1} = \frac{m_2}{11} = \frac{n_2}{5}$$

If θ is the angle between the lines, then

$$\cos\theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$= \frac{5 + 121 + 35}{\sqrt{195}\sqrt{147}} = \frac{23}{3\sqrt{65}}$$

$\therefore$ So $\theta = 180^\circ 1.4'$

Example 9: Prove that the line $\frac{x-4}{2} = \frac{y-2}{3} = \frac{z-3}{6}$ lies in the plane $3x - 4y + z = 7$.

Solution: The point of the line (4, 2, 3) should also lie in the plane. So $3 \cdot 4 - 4 \cdot 2 + 1 \cdot 3 = 7$ satisfied. The line and normal to the plane are perpendicular. So $2 \cdot 3 + 3 \cdot (-4) + 6 \cdot 1 = 6 - 12 + 6 = 0$. Thus the given line completely lies in the given plane.

Example 10: Show that the lines $\frac{x-2}{2} = \frac{y-3}{-1} = \frac{z+4}{3}$ and $\frac{x-3}{1} = \frac{y+1}{3} = \frac{z-1}{-2}$ are coplanar. Find their common point and determine the equation of the plane containing the two given lines.

Solution: Here first line passes through $(2, 3, -4)$ and has DR's $l_1, m_1, n_1 : 2, -1, 3$. The second line passes through $(3, -1, 1)$ and has DR's $l_2, m_2, n_2 : 1, 3, -2$. Condition for coplanarity:

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 3-2 & -1-3 & 1+4 \\ 2 & -1 & 3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= \begin{matrix} 7+28-35 = 0 \\ \text{satisfied.} \end{matrix}$$

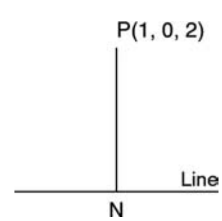
Point of intersection: Any point on the first line is $(2 + 2r_1, 3 - r_1 - 4 + 3r_1)$ and any point on the second line is $(3 + r_2, -1 + 3r_2, 1 - 2r_2)$. When the two lines intersect in a common point then co-ordinates on line (1) and line (2) must be equal, i.e., $2 + 2r_1 = 3 + r_2$, $3 - r_1 = -1 + 3r_2$ and $-4 + 3r_1 = 1 - 2r_2$. Solving $r_1 = r_2 = 1$. Therefore the point of intersection is $(2 + 2 \cdot 1, 3 - 1, -4 + 3 \cdot 1) = (4, 2, -1)$.
Equation of plane containing the two lines:

$$\begin{vmatrix} x-x_1 & y-y_1, & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} x-2 & y-3 & z+4 \\ 2 & -1 & 3 \\ 1 & 3 & -2 \end{vmatrix} = 0$$

Expanding $-7(x-2) - (-7)(y-3) + 7(z+4) = 0$ or $x - y - z + 3 = 0$.

Example 11: Find the coordinates of the foot of the perpendicular from $P(1, 0, 2)$ to the line $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$. Find the length of the perpendicular and its equation.

Solution: Any point N on the given line is $(3r - 1, 2 - 2r, -1 - r)$. DR's of PN are $(3r - 2, 2 - 2r, -3 - r)$. Now PN is normal to line if $3(3r - 2) + (-2)(2 - 2r) + (-1)(-3 - r) = 0$ or $r = \frac{1}{2}$. So the coordinates of N the foot of the perpendicular from P to the line are $\left(3 \cdot \frac{1}{2} - 1, 2 - 2 \cdot \frac{1}{2}, -1 - \frac{1}{2}\right)$ or $\left(\frac{1}{2}, 1, -\frac{3}{2}\right)$.



Length of the perpendicular

$$PN = \sqrt{\left(\frac{1}{2}-1\right)^2 + (1-0)^2 + \left(-\frac{3}{2}-2\right)^2}$$

$$= \sqrt{\frac{1}{4} + 1 + \frac{49}{4}} = \sqrt{\frac{54}{4}} = \frac{3}{2}\sqrt{6}.$$

DR's of PM with $r = \frac{1}{2}$ are $[3 \cdot \frac{1}{2} - 2, 2 - 2 \cdot \frac{1}{2}, -3 - \frac{1}{2}]$ i.e., DR's of PM are $\frac{1}{2}, -1, \frac{7}{2}$. And PM passes through $P(1, 0, 2)$. Therefore the equation of the perpendicular PM

$$\frac{x-1}{\frac{1}{2}} = \frac{y-0}{-1} = \frac{z-2}{\frac{7}{2}} \quad \text{or} \quad x - 1 = \frac{y}{-2} = \frac{z-2}{7}.$$

Example 12: Find the equation of the line of the greatest slope through the point (2, 1, 1) in the slant plane $2x + y - 5z = 0$ to the horizontal plane $4x - 3y + 7z = 0$.

Solution: Let l_1, m_1, n_1 be the DR's of the line of intersection ML of the two given planes. Since ML is $\perp^r$ to both normals,

$$2l_1 + m_1 - 5n_1 = 0, \qquad 4l_1 - 3m_1 + 7n_1 = 0.$$

Solving $\frac{l_1}{4} = \frac{m_1}{17} = \frac{n_1}{5}$. Let PN be the line of greatest slope and let l_2, m_2, n_2 be its DR's. Since PN and ML are perpendicular

$$4l_2 + 17m_2 + 5n_2 = 0$$

Also PN is perpendicular to normal of the slant plane $2x + y - 5z = 0$. So

$$2l_2 + m_2 - 5n_2 = 0$$

Solving $\frac{l_2}{3} = \frac{m_2}{-1} = \frac{n_2}{1}$.
Therefore the equation of the line of greatest slope PN having DR's $3, -1, 1$ and passing through $P(2, 1, 1)$ is

$$\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}.$$

EXERCISE

1. Find the points where the line $x + y + 4z = 6$, $2x - 3y - 2z = 2$ pierce the coordinate planes.

Ans. $(0, -2, 2), (4, 2, 0), (2, 0, 1)$

2. Transform the general form $3x + y - 2z = 7$, $6x - 5y - 4z = 7$ to symmetrical form and two point form.

Hint: $(0, 1, -3), (2, 1, 0)$ are two points on the line.

Ans. $\frac{x-2}{2} = \frac{y-1}{0} = \frac{z-0}{3}$

3. Show that the lines $x = y = z + 2$ and $\frac{x-1}{1} = \frac{y}{0} = \frac{z}{2}$ intersect and find the point of intersection.

Hint: Solve $x - y = 0$, $y - z = 2$, $y = 0$, $2x - z = 2$ simultaneously.

Ans. $(0, 0, -2)$

4. Find the equation plane containing the line $x = y = z$ and

a. Passing through the line $x + 1 = y + 1 = z$

b. Parallel to the line $\frac{x+1}{3} = \frac{y}{2} = \frac{z}{-1}$.

Ans. (a) $x - y = 0$; (b) $3x - 4y + z = 0$

5. Show that the line $\frac{x+1}{1} = \frac{y}{-1} = \frac{z-2}{2}$ is in the plane $2x + 4y + z = 0$.

Hint: $2(1) + 4(-1) + 1(2) = 0$,
$2(-1) + 4(0) + 2 = 0$

6. Find the equation of the plane containing line $\frac{x-1}{3} = \frac{y-1}{4} = \frac{z-2}{2}$ and parallel to the line $x - 2y + 3z = 4$, $2x - 3y + 4z = 5$.

Hint: Eq. of 2nd line $\frac{x-0}{\frac{1}{2}} = \frac{y-1}{1} = \frac{z-2}{\frac{1}{2}}$, contains 1st line: $3A + 4B + 2C = 0$. Parallel to 2nd line $A + 2B + C = 0$, $A = 0$, $B = -\frac{1}{2}C$, $D = -\frac{3}{2}C$.

Ans. $y - 2z + 3 = 0$

7. Show that the lines $x + 2y - z = 3$, $3x - y + 2z = 1$ and $2x - 2y + 3z = 2$, $x - y + z + 1 = 0$ are coplanar. Find the equation of the plane containing the two lines.

Hint: $\frac{x-0}{3} = \frac{y-\frac{7}{3}}{-5} = \frac{z-\frac{5}{3}}{-7}$, $\frac{x-0}{1} = \frac{y-5}{+1} = \frac{z-4}{0}$.

$$\begin{vmatrix} x-0 & y-5 & z-4 \\ 3 & -5 & -7 \\ 1 & 1 & 0 \end{vmatrix} = 0, \quad \text{Expand.}$$

Ans. $7x - 7y + 8z + 3 = 0$

8. Prove that the equation of the plane through the origin containing the line $\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{5}$ is $x - 5y + 3z = 0$.

Hint: $A(x-1) + B(y-2) + C(z-3) = 0$, $5A + 2B + 3C = 0$, $A + 2B + 3C = 0$,

Expand $\begin{vmatrix} x-1 & y-2 & z-3 \\ 5 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix} = 0$

9. Find the image of the point $P(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$.

Hint: Line through P and $\perp^r$ to plane: $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1}$. Image Q: $(2r+1, -r+3, r+4)$. Mid point L of PQ is $(r+1, -\frac{1}{2}r+3, \frac{1}{2}r+4)$. L lies on plane, $r = -2$.

Ans. $(-3, 5, 2)$

10. Determine the point of intersection of the lines

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}, \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$$

Hint: General points: $(r_1 + 4, -4r_1 - 3, 7r_1 - 1)$, $(2r_2 + 1, -3r_2 - 1, 8r_2 - 10)$, Equating $r_1 + 4 = 2r_2 + 1$, $-4r_1 - 3 = -3r_2 - 1$, solving $r_1 = 1, r_2 = 2$.

Ans. $(5, -7, 6)$

11. Show that the lines $\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$, $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are coplanar. Find the equation of the plane containing them.

Ans. $6x - 5y - z = 0$

12. Find the equation of the line which passes through the point $(2, -1, 1)$ and intersect the lines $2x + y = 4$, $y + 2z = 0$, and $x + 3z = 4$, $2x + 5z = 8$.

Ans. $x + y + z = 2, x + 2z = 4$

13. Find the coordinates of the foot of the perpendicular from $P(5, 9, 3)$ to the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Find the length of the perpendicular and its equations.

Ans. $(3, 5, 7)$, Length: 6, Equation $\frac{x-5}{-2} = \frac{y-9}{-4} = \frac{z-3}{4}$.

14. Find the equation of the line of greatest slope in the slant plane $2x + y - 5z = 12$ and passing through the point $(2, 3, -1)$ given that the line $\frac{x}{4} = \frac{y}{-3} = \frac{z}{7}$ is vertical.

Ans.

15. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $3x + y + z = 7$.

Hint: DR's of line: 2, 3, 6; DR's of normal to plane 3, 1, 1

$$\cos(90-\theta) = \sin\theta = \frac{2\cdot 3 + 3\cdot 1 + 6\cdot 1}{\sqrt{4+9+36}\sqrt{9+1+1}}.$$

Ans. $\sin\theta = \frac{15}{7\sqrt{11}}$

16. Find the angle between the line $x + y - z = 1$, $2x - 3y + z = 2$ and the plane $3x + y - z + 5 = 0$.

Hint: DR's of line 2, 3, 5, DR's of normal: 3, 1, −1

$$\cos(90-\theta) = \sin\theta = \frac{2\cdot 3 + 3\cdot 1 + 5\cdot(-1)}{\sqrt{4+9+25}\sqrt{9+1+1}}.$$

Ans. $\sin\theta = \frac{4}{\sqrt{38}\sqrt{11}}$

3.5 SHORTEST DISTANCE BETWEEN SKEW LINES

Skew lines: Any two straight lines which do not lie in the same plane are known as skew lines (or non-planar lines). Such lines neither intersect nor are parallel. **Shortest distance between two skew lines:**

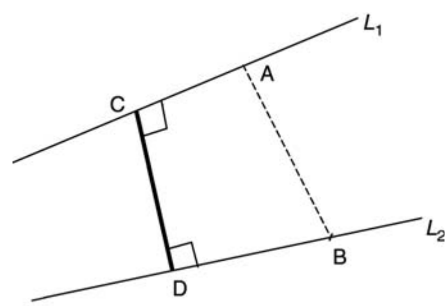


Fig. 3.15

Let L_1 and L_2 be two skew lines; L_1 passing through a given point A and L_2 through a given point

B. Shortest distance between the two skew lines L_1 and L_2 is the length of the line segment CD which is perpendicular to **both** L_1 and L_2. The equation of the shortest distance line CD can be uniquely determined since it intersects both lines L_1 and L_2 at right angles. Now CD = projection of AB on $CD = AB\cos\theta$ where θ is the angle between AB and CD. Since $\cos\theta < 1$, $CD < AB$, thus CD is the shortest distance between the lines L_1 and L_2.

Magnitude (length) and the equations of the line of shortest distance between two lines L_1 and L_2:

Suppose the equation of given line L_1 be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \tag{1}$$

and of line L_2 be

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \tag{2}$$

Assume the equation of shortest distance line CD as

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \tag{3}$$

where (α,β,γ) and (l,m,n) are to be determined. Since CD is perpendicular to both L_1 and L_2,

$$ll_1+mm_1+nn_1=0$$

$$ll_2+mm_2+nn_2=0$$

Solving

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$

$$=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

$$=\frac{1}{\sqrt{\sum(m_1n_2-m_2n_1)^2}}=\frac{1}{k}$$

where $k=\sqrt{\sum(m_1n_2-m_2n_1)^2}$

or $l=\dfrac{m_1n_2-m_2n_1}{k}$, $m=\dfrac{n_1l_2-n_2l_1}{k}$,

$$n=\frac{l_1m_2-l_2m_1}{k} \tag{4}$$

Thus the DC's l,m,n of the shortest distance line CD are determined by (4).

Magnitude of shortest distance CD = projection of AB on CD where $A(x_1,y_1,z_1)$ is a point on L_1 and $B(x_2,y_2,z_2)$ is a point on L_2.

$\therefore$ shortest distance CD =

$$=l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1) \tag{5}$$

In the determinant form,

$$\text{Shortest distance } CD=\frac{1}{k}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix} \tag{5$'$}$$

Note: If shortest distance is zero, then the two lines L_1 and L_2 are coplanar.

Equation of the line of shortest distance *CD*: Observe that CD is coplanar with both L_1 and L_2. Let P_1 be the plane containing L_1 and CD. Equation of plane P_1 containing coplanar lines L_1 and CD is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1\\ l_1 & m_1 & n_1\\ l & m & n\end{vmatrix}=0 \tag{6}$$

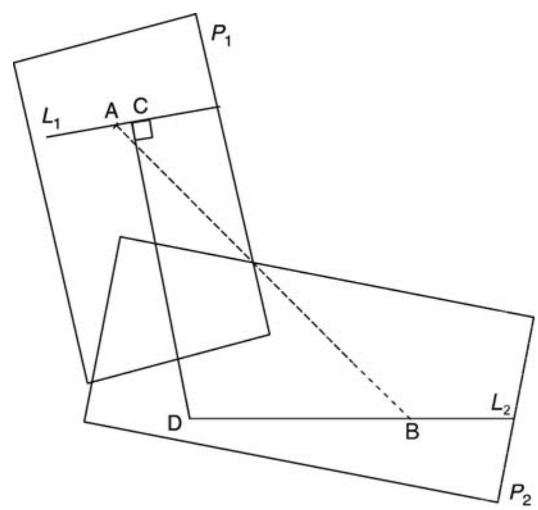


Fig. 3.16

Similarly, equation of plane P_2 containing L_2 and CD is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2\\ l_2 & m_2 & n_2\\ l & m & n\end{vmatrix}=0 \tag{7}$$

Equations (6) and (7) together give the equation of the line of shortest distance.

Points of intersection C and D with L_1 and L_2: Any general point C^* on L_1 is

$$(x_1 + l_1 r_1, \quad y_1 + m_1 r_1, \quad z_1 + n_1 r_1)$$

and any general point D^* on L_2 is

$$(x_2 + l_2 r_2, \quad y_2 + m_2 r_2, \quad z_2 + n_2 r_2)$$

DR's of C^*D^*: $(x_2 - x_1 + l_2 r_2 - l_1 r_1,\ y_2 - y_1 + m_2 r_2 - m_1 r_1,\ z_2 - z_1 + n_2 r_2 - n_1 r_1)$

If C^*D^* is $\perp^r$ to both L_1 and L_2, we get two equations for the two unknowns r_1 and r_2. Solving and knowing r_1 and r_2, the coordinates of C and D are determined. Then the magnitude of CD is obtained by length formula, and equation of CD by two point formula.

Parallel planes: Shortest distance CD = perpendicular distance from any point on L_1 to the plane parallel to L_1 and containing L_2.

Worked Out Examples

Example 1: Find the magnitude and equation of the line of shortest distance between the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4},$$
$$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}.$$

Solution: Point $A(x_1, y_1, z_1)$ on first line is (1, 2, 3) and $B(x_2, y_2, z_2)$ on second line is (2, 4, 5). Also (l_1, m_1, n_1) are (2, 3, 4) and $(l_2, m_2, n_2) = (3, 4, 5)$. Then

$$k^2 = (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2$$
$$= (15-16)^2 + (12-10)^2 + (8-9)^2$$
$$= 1 + 4 + 1 = 6 \quad \text{or} \quad k = \sqrt{6}.$$

So DR's is of line of shortest of distance: $-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$.

$$\text{Shortest distance} = \frac{1}{k}\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \frac{1}{\sqrt{6}}$$
$$= \frac{(15-16) - 2(10-12) + 2(8-9)}{\sqrt{6}}$$
$$= \frac{-1+4-2}{\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

Equation of shortest distance line:

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{vmatrix} = 0 \quad \text{or} \quad 11x + 2y - 7z + 6 = 0$$

and

$$\begin{vmatrix} x-1 & y-4 & z-5 \\ 2 & 3 & 4 \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{vmatrix} = 0 \quad \text{or} \quad 7x + y - 5z + 7 = 0.$$

Example 2: Determine the points of intersection of the line of shortest distance with the two lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}; \ \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}.$$

Also find the magnitude and equation of shortest distance.

Solution: Any general point C^* on first line is $(3 + 3r_1, 8 - r_1, 3 + r_1)$ and any general point D^* on the second line is $(-3 - 3r_2, -7 + 2r_2, 6 - 4r_2)$. DR's of C^*D^* are $(6 + 3r_1 + 3r_2, 15 - r_1 - 2r_2, -3 + r_1 - 4r_2)$. If C^*D^* is $\perp^r$ to both the given lines, then

$$3(6+3r_1+3r_2) - 1(15-r_1-2r_2) + 1(-3+r_1-4r_2) = 0$$
$$-3(6+3r_1+3r_2) + 2(15-r_1-2r_2) + 4(-3+r_1-4r_2) = 0$$

Solving for r_1 and r_2, $11r_1 - 7r_2 = 0, +7r_1 + 29r_2 = 0$ so $r_1 = r_2 = 0$. Then the points of intersection of shortest distance line CD with the given two lines are $C(3, 8, 3)$, $D(-3, -7, 6)$.

$$\text{Length of } CD = \sqrt{(-6)^2 + (-15)^2 + (3)^2}$$
$$= \sqrt{270} = 3\sqrt{30}$$

$$\text{Equation } CD: \frac{x-3}{-3-3} = \frac{y-8}{-7-8} = \frac{z-3}{6-3}$$

$$\text{i.e.,} \quad \frac{x-3}{-6} = \frac{y-8}{-15} = \frac{z-3}{3}.$$

Example 3: Calculate the length and equation of line of shortest distance between the lines

$$5x - y - z = 0, \quad x - 2y + z + 3 = 0 \qquad (1)$$

$$7x - 4y - 2z = 0, \quad x - y + z - 3 = 0 \qquad (2)$$

Solution: Any plane containing the second line (2) is

$$(7x - 4y - 2z) + \mu(x - y + z - 3) = 0$$

or $(7 + \mu)x + (-4 - \mu)y + (-2 + \mu)z - 3\mu = 0$ (3)

DR's of first line (1) are $(l, m, n) = (-3, -6, -9)$ obtained from:

l	m	n
5	−1	−1
l	−2	1

The plane (3) will be parallel to the line (1) with $l = -3, m = -6, n = -9$ if

$$-3(7 + \mu) + 6(4 + \mu) + 9(2 - \mu) = 0 \quad \text{or} \quad \mu = \frac{7}{2}$$

Substituting μ in (3), we get the equation of a plane containing line (2) and parallel to line (1) as

$$7x - 5y + z - 7 = 0 \qquad (4)$$

To find an arbitrary point on line (1), put $x = 0$. Then $-y - z = 0$ or $y = -z$ and $-2y + z + 3 = 0, z = -1, y = 1$. $\therefore$ $(0, 1, -1)$ is a point on line (1). Now the length of the shortest distance = perpendicular distance of $(0, 1, -1)$ to plane (4)

$$= \frac{0 - 5(1) + (-1) - 7}{\sqrt{49 + 25 + 1}} = \left|\frac{-13}{\sqrt{75}}\right| = \frac{13}{\sqrt{75}} \qquad (5)$$

Equation of any plane through line (1) is

$$5x - y - z + \lambda(x - 2y + z + 3) = 0$$

or $(5 + \lambda)x + (-y - 2\lambda)y + (-1 + \lambda)z + 3\lambda = 0$ (6)

DR's of line (2) are $(l, m, n) = (2, 3, 1)$ obtained from

l	m	n
7	−4	−2
1	−1	1

plane (6) will be parallel to line (2) if

$$2(5 + \lambda) + 3(-y - 2\lambda) + 1(-1 + \lambda) = 0 \quad \text{or} \quad \lambda = 2.$$

Thus the equation of plane containing line (1) and parallel to line (2) is

$$7x - 5y + z + 6 = 0 \qquad (7)$$

Hence equation of the line of shortest distance is given by (6) and (7) together.

Aliter: A point on line (2) is $(0, -1, 2)$ obtained by putting $x = 0$ and solving (2). Then the length of shortest distance = perpendicular distance of $(0, -1, 2)$ to the plane (7) $= \frac{0+5+2+6}{\sqrt{75}} = \frac{13}{\sqrt{75}}$

Note: By reducing (1) and (2) to symmetric forms

$$\frac{x - \frac{1}{3}}{1} = \frac{y - \frac{5}{3}}{2} = \frac{z}{3}$$
$$\frac{x + 4}{1} = \frac{y + 7}{\frac{3}{2}} = \frac{z}{\frac{1}{2}}$$

The problem can be solved as in above worked Example 1.

Example 4: Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar.

Solution: Shortest distance between the two lines is

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \begin{matrix} = (-1) - (-2) + (-1) \\ = 0 \end{matrix}$$

$\therefore$ Lines are coplanar.

Example 5: If a, b, c are the lengths of the edges of a rectangular parallelopiped, show that the shortest distance between a diagonal and an edge not meeting the diagonal is $\frac{bc}{\sqrt{b^2+c^2}} \left(\text{or} \frac{ca}{\sqrt{c^2+a^2}} \text{ or } \frac{ab}{\sqrt{a^2+b^2}} \right)$.

Solution: Choose coterminus edges OA, OB, OC along the X, Y, Z axes. Then the coordinates are $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $E(a, b, 0)$, $D(0, b, c)$, $G(a, 0, c) F(a, b, c)$ etc. so that $OA = a$, $OB = b$, $OC = c$.
To find the shortest distance between a diagonal OF and an edge GC. Here GC does not interest OF

Equation of the line OF: $\frac{x - 0}{a - 0} = \frac{y - 0}{b - 0} = \frac{z - 0}{c - 0}$

$$\text{or} \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \qquad (1)$$

Equation of the line GC: $\frac{x - 0}{a - 0} = \frac{y - 0}{b - 0} = \frac{z - c}{c - c}$

$$\text{or} \quad \frac{x}{1} = \frac{y}{0} = \frac{z - c}{0} \qquad (2)$$

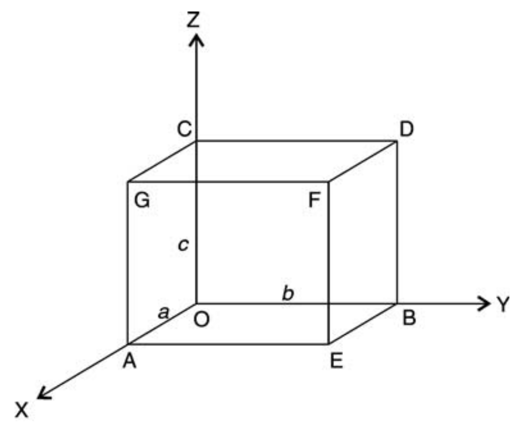


Fig. 3.17

Equation of a plane containing line (1) and parallel to (2) is

$$\begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 0 & 0 \end{vmatrix} = 0 \quad \text{or} \quad cy - bz = 0 \qquad (3)$$

Shortest distance = Length of perpendicular drawn from a point say $C(0, 0, c)$ to the plane (3)

$$= \frac{c \cdot 0 - b \cdot c}{\sqrt{0^2 + c^2 + b^2}} = \frac{bc}{\sqrt{c^2 + b^2}}.$$

In a similar manner, it can be proved that the shortest distance between the diagonal OF and non-intersecting edges AN and AM are respectively $\frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$.

EXERCISE

1. Determine the magnitude and equation of the line of shortest distance between the lines. Find the points of intersection of the shortest distance line, with the given lines
$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}, \quad \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Ans. 14, $117x + 4y - 41z - 490 = 0, 9x - 4y - z = 14$, points of intersection (5, 7, 3), (9, 13, 15).

2. Calculate the length, points of intersection, the equations of the line of shortest distance between the two lines
$\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4}, \quad \frac{x+1}{3} = \frac{y}{4} = \frac{z}{5}$.

Ans. $\frac{1}{\sqrt{6}}, \frac{x-\frac{5}{3}}{\frac{1}{6}} = \frac{y-3}{-\frac{1}{2}} = \frac{z-\frac{15}{2}}{\frac{1}{6}}, \left(\frac{5}{3}, 3, \frac{13}{3}\right),$ $\left(\frac{3}{2}, \frac{10}{3}, \frac{25}{6}\right)$.

3. Find the magnitude and equations of shortest distance between the two lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}, \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}.$$

Ans. $\frac{1}{\sqrt{6}}$, $11x + 2y - 7z + 6 = 0$, $7x + y - 5z + 7 = 0$.

4. Show that the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ is $\frac{1}{\sqrt{3}}$ and its equations are $4x + y - 5z = 0$, $7x + y - 8z = 31$.

5. Determine the points on the lines $\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1}$, $\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$ which are nearest to each other. Hence find the shortest distance between the lines and find its equations.

Ans. $(3, 8, 3)$, $(-3, -7, 6)$, $3\sqrt{30}$, $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$.

6. Prove that the shortest distance between the two lines $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$, $\frac{x+1}{2} = \frac{y-1}{-4} = \frac{z+2}{1}$ is $\frac{120}{\sqrt{341}}$

   Hint: Equation of a plane passing through the first lines nad parallel to the second line is $6x + 7y + 16z = 98$. A point on second line is $(-1, 1, -2)$. Perpendicular distance $= \frac{6(-1)+7(1)+16(-2)}{\sqrt{6^2+7^2+16^2}}$.

7. Find the length and equations of shortest distance between the lines $x - y + z = 0$, $2x - 3y + 4z = 0$; and $x + y + 2z - 3 = 0$, $2x + 3y + 3z - 4 = 0$.

   Hint: Equations of two lines in symmetric form are $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$, $\frac{x-5}{-3} = \frac{y+2}{1} = \frac{z}{1}$.

Ans. $\frac{13}{\sqrt{66}}$, $3x - y - z = 0$, $x + 2y + z - 1 = 0$.

8. Determine the magnitude and equations of the line of shortest distance between the lines $\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$ and $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$.

Ans. $4\sqrt{3}$, $-4x + y + 3z = 0$, $4x - 5y + z = 0$ (or $x = y = z$).

9. Obtain the coordinates of the points where the line of shortest distance between the lines $\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3}$ and $\frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$ meets them. Hence find the shortest distance between the two lines.

Ans. $(11, 11, 31)$, $(3, 5, 7)$, 26

10. Find the shortest distance between any two opposite edges of a tetrahedron formed by the planes $x + y = 0$, $y + z = 0$, $z + x = 0$, $x + y + z = a$. Also find the point of intersection of three lines of shortest distances.

    Hint: Vertices are $(0, 0, 0)$, $(a, -a, a)$, $(-a, a, a)$, $(a, a, -a)$.

Ans. $\frac{2a}{\sqrt{6}}$, $(-a, -a, -a)$.

11. Find the shortest distance between the lines PQ and RS where $P(2, 1, 3)$, $Q(1, 2, 1)$, $R(-1, -2, -2)$, $S(-1, 4, 0)$.

Ans. $3\sqrt{2}$

3.6 THE RIGHT CIRCULAR CONE

Cone

A **cone** is a surface generated by a straight line (known as **generating line or generator**) passing through a fixed point (known as **vertex**) and satisfying a condition, for example, it may intersect a given curve (known as **guiding curve**) or touches a given surface (say a sphere). Thus cone is a set of points on its generators. Only cones with second degree equations known as quadratic cones are considered here. In particular, quadratic cones with vertex at origin are homogeneous equations of second degree.

Equation of cone with vertex at (α, β, γ) and the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, $z = 0$ as the guiding curve:

The equation of any line through vertex (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \tag{1}$$

(1) will be generator of the cone if (1) intersects the given conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad z = 0 \tag{2}$$

Since (1) meets $z = 0$, put $z = 0$ in (1), then the point

$\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ will lie on the conic (2), if

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+$$
$$+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0 \quad (3)$$

From (1)

$$\frac{l}{n}=\frac{x-\alpha}{z-\gamma}, \qquad \frac{m}{n}=\frac{y-\beta}{z-\gamma} \quad (4)$$

Eliminate l, m, n from (3) using (4),

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2+$$
$$+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)+$$
$$+b\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)^2+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)+$$
$$+2f\left(\beta-\frac{\gamma-\beta}{z-\gamma}\cdot\gamma\right)+c=0$$

or

$$a(\alpha z-x\gamma)^2+2h(\alpha z-x\gamma)(\beta z-y\gamma)+$$
$$+b(\beta z-y\gamma)^2+2g(\alpha z-x\gamma)(z-\gamma)+$$
$$+2f(\beta z-y\gamma)(z-\gamma)+c(z-\gamma)^2=0$$

or

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2+$$
$$+2f(z-\gamma)(y-\beta)+2g(x-\alpha)(z-\gamma)+$$
$$+2h(x-\alpha)(y-\beta)=0 \quad (5)$$

Thus (5) is the equation of the quadratic cone with vertex at (α, β, γ) and guiding curve as the conic (2). **Special case: Vertex at origin (0, 0, 0)**. Put $\alpha = \beta = \gamma = 0$ in (5). Then (5) reduces to

$$ax^2+by^2+cz^2+2fzy+2gxz+2hxy=0 \quad (6)$$

Equation (6) which is a homogeneous and second degree in x, y, z is the equation of cone with vertex at origin.

Right circular cone

A right circular cone is a surface generated by a line (**generator**) through a fixed point (**vertex**) making a constant angle θ (**semi-vertical angle**) with the fixed line (**axis**) through the fixed point (**vertex**). Here the guiding curve is a circle with centre at c. Thus every section of a right circular cone by a plane perpendicular to its axis is a circle.

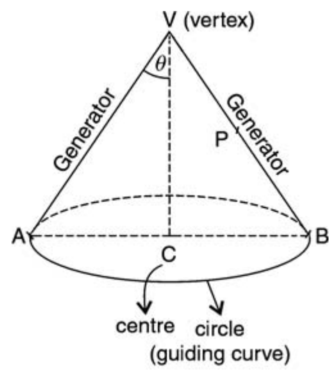


Fig. 3.18

Equation of a right circular cone: with vertex at (α, β, γ), semi vertical angle θ and equation of axis

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad (1)$$

Let $P(x, y, z)$ be any point on the generating line VB. Then the DC's of VB are proportional to $(x-\alpha, y-\beta, z-\gamma)$. Then

$$\cos\theta=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Rewriting, the required equation of cone is

$$\left[l(x-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2=$$
$$=(l^2+m^2+n^2)\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]\cos^2\theta \quad (2)$$

Case 1: If vertex is origin (0, 0, 0) then (2) reduces

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta \quad (3)$$

Case 2: If vertex is origin and axis of cone is z-axis (with $l = 0, m = 0, n = 1$) then (2) becomes

$$z^2=(x^2+y^2+z^2)\cos^2\theta \quad \text{or} \quad z^2\sec^2\theta=x^2+y^2+z^2$$
$$z^2(1+\tan^2\theta)=x^2+y^2+z^2$$

i.e.,
$$x^2+y^2=z^2\tan^2\theta \quad (4)$$

Similarly, with y-axis as the axis of cone

$$x^2 + z^2 = y^2 \tan^2 \theta$$

with x-axis as the axis of cone

$$y^2 + z^2 = x^2 \tan^2 \theta.$$

If the right circular cone admits sets of three mutually perpendicular generators then the semi-vertical angle $\theta = \tan^{-1}\sqrt{2}$ (since the sum of the coefficients of x^2, y^2, z^2 in the equation of such a cone must be zero i.e., $1 + 1 - \tan^2\theta = 0$ or $\tan\theta = \sqrt{2}$).

WORKED OUT EXAMPLES

Example 1: Find the equation of cone with base curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ and vertex (α, β, γ). Deduce the case when base curve is $\frac{x^2}{16} + \frac{y^2}{9} = 1, z = 0$ and vertex at (1, 1, 1).

Solution: The equation of any generating line through the vertex (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (1)$$

This generator (1) meets $z = 0$ in the point

$$\left(x = \alpha - \frac{l\gamma}{n}, \qquad y = \beta - \frac{m\gamma}{n}, \qquad z = 0\right) \qquad (2)$$

Point (2) lies on the generating curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (3)$$

Substituting (2) in (3)

$$\frac{\left(\alpha - \frac{l\gamma}{n}\right)^2}{a^2} + \frac{\left(\beta - \frac{m\gamma}{n}\right)^2}{b^2} = 1 \qquad (4)$$

Eliminating l, m, n from (4) using (1),

$$\frac{\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2}{a^2} + \frac{\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2}{b^2} = 1$$

$$b^2\Big[\alpha(z-\gamma) - \gamma(x-\alpha)\Big]^2 + a^2\Big[\beta(z-\gamma) - \gamma(y-\beta)\Big]^2 = a^2b^2(z-\gamma)^2$$

Deduction: When $a=4, b=3, \alpha=1, \beta=1, \gamma=1$,

$$9\Big[(z-1)-(x-1)\Big]^2 + 16\Big[(z-1)-(y-1)\Big]^2 = 144(z-1)^2$$

$$9x^2 + 16y^2 - 119z^2 - 18xz - 32yz + 288z - 144 = 0.$$

Example 2: Find the equation of the cone with vertex at (1, 0, 2) and passing through the circle $x^2 + y^2 + z^2 = 4, x + y - z = 1$.

Solution: Equation of generator is

$$\frac{x-1}{l} = \frac{y-0}{m} = \frac{z-2}{n} \qquad (1)$$

Any general point on the line (1) is

$$(1 + lr, \qquad mr, \qquad 2 + nr). \qquad (2)$$

Since generator (1) meets the plane

$$x + y - z = 1 \qquad (3)$$

substitute (2) in (3)

$$(1 + lr) + (mr) - (2 + nr) = 1$$

or
$$r = \frac{2}{l + m - n}. \qquad (4)$$

Since generator (1) meets the sphere

$$x^2 + y^2 + z^2 = 4 \qquad (5)$$

substitute (2) in (5)

$$(1 + lr)^2 + (mr)^2 + (2 + nr)^2 = 4$$

or
$$r^2(l^2 + m^2 + n^2) + 2r(l + 2n) + 1 = 0 \qquad (6)$$

Eliminate r from (6) using (4), then

$$\frac{4}{(l+m-n)^2}(l^2+m^2+n^2) + 2\frac{2}{(l+m-n)}(l+2n) + 1 = 0$$

$$9l^2 + 5m^2 - 3n^2 + 6lm + 2ln + 6nm = 0 \qquad (7)$$

Eliminate l, m, n from (7) using (1), then

$$9\left(\frac{x-1}{r}\right)^2 + 5\left(\frac{y}{r}\right)^2 - 3\left(\frac{z-2}{r}\right)^2 + 6\left(\frac{x-1}{r}\right)\left(\frac{y}{r}\right) + 2\left(\frac{x-1}{r}\right)\left(\frac{z-2}{r}\right) + 6\left(\frac{z-2}{r}\right)\left(\frac{y}{r}\right) = 0$$

or
$$9(x-1)^2 + 5y^2 - 3(z-2)^2 + 6y(x-1) + 2(x-1)(x-2) + 6(z-2)y = 0$$

Vertex (0, 0, 0):

Example 3: Determine the equation of a cone with vertex at origin and base curve given by

a. $ax^2 + by^2 = 2z, \quad lx + my + nz = p$

b. $ax^2 + by^2 + cz^2 = 1, \quad lx + my + nz = p$

c. $x^2 + y^2 + z^2 = 25, \quad x + 2y + 2z = 9$

Solution: We know that the equation of a quadratic cone with vertex at origin is a homogeneous equation of second degree in x, y, z. By eliminating the non-homogeneous terms in the base curve, we get the required equation of the cone.

a. $2z$ is the term of degree one and is non homogeneous. Solving

$$\frac{lx + my + nz}{p} = 1$$

rewrite the equation

$$ax^2 + by^2 = 2 \cdot z(1) = 2z\left(\frac{lx + my + nz}{p}\right)$$

$$apx^2 + bpy^2 - 2nz^2 - 2lxz - 2myz = 0$$

which is the equation of cone.

b. Except the R.H.S. term 1, all other terms are of degree 2 (and homogeneous). Rewriting, the required equation of cone as

$$ax^2 + by^2 + cz^2 = (1)^2 = \left(\frac{lx + my + nz}{p}\right)^2$$

$$(ap^2 - l^2)x^2 + (bp^2 - m^2)y^2 + (cp^2 - n^2)z^2 -$$

$$-2lmxy - 2mnyz - 2lnxz = 0$$

c. On similar lines

$$x^2 + y^2 + z^2 = 25 = 25(1)^2 = 25\left(\frac{x + 2y + 2z}{9}\right)^2$$

$$56x^2 - 19y^2 - 19z^2 - 100xy - 200yz - 100xz = 0$$

Right circular cone:

Example 4: Find the equation of a right circular cone with vertex at (2, 0, 0), semi-vertical angle $\theta = 30^\circ$ and axis is the line $\frac{x-2}{3} = \frac{y}{4} = \frac{z}{6}$.

Solution: Here $\alpha = 2, \beta = 0, \gamma = 0, l = 3, m = 4, n = 6$

$$\frac{\sqrt{3}}{2} = \cos 30 = \cos\theta$$

$$= \frac{l(x - \alpha) + m(y - \beta) + n(z - \gamma)}{\sqrt{(l^2 + m^2 + n^2)[(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2]}}$$

$$\frac{\sqrt{3}}{2} = \frac{3(x - 2) + 4y + 6z}{\sqrt{9 + 16 + 36}\sqrt{(x - 2)^2 + y^2 + z^2}}$$

$$183[(x - 2)^2 + y^2 + z^2] = 4[3(x - 2) + 4y + 6z]^2$$

$$147x^2 + 119y^2 + 39z^2 - 192yz - 144zx - 96xy -$$

$$-588x + 192y + 288z + 588 = 0$$

Vertex (0, 0, 0):

Example 5: Find the equation of the right circular cone which passes through the line $2x = 3y = -5z$ and has $x = y = z$ as its axis.

Solution: DC's of the generator $2x = 3y = -5z$ are $\frac{1}{2}, \frac{1}{3}, -\frac{1}{5}$. DC's of axis are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. Point of intersection of the generator and axis is (0, 0, 0). Now

$$\cos\theta = \frac{\frac{1}{2} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot \frac{1}{\sqrt{3}} - \frac{1}{5} \cdot \frac{1}{\sqrt{3}}}{\sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}}\sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{25}}} = \frac{\frac{19}{30}}{\sqrt{\frac{361}{900}}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Equation of cone with vertex at origin

$$\frac{1}{\sqrt{3}} = \cos\theta = \frac{\frac{1}{\sqrt{3}}(x + y + z)}{1\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 = (x + y + z)^2$$

$$xy + yz + zx = 0.$$

Example 6: Determine the equation of a right circular cone with vertex at origin and the guiding curve circle passing through the points (1, 2, 2), (1, −2, 2)(2, −1, −2).

Solution: Let l, m, n be the DC's of OL the axis of the cone. Let θ be the semi- vertical angle. Let $A(1, 2, 2)$, $B(1, -2, 2)$, $C(2, -1, -2)$ be the three points on the guiding circle. Then the lines OA, OB, OC make the same angle θ with the axis OL. The DC's of OA, OB, OC are proportional to

(1, 2, 2)(1, −2, 2)(2, −1, −2) respectively. Then

$$\cos\theta = \frac{l(1)+m(2)+n(2)}{\sqrt{1}\cdot\sqrt{1+4+4}} = \frac{l+2m+2n}{3} \quad (1)$$

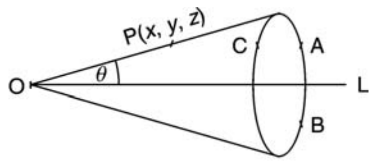


Fig. 3.19

Similarly,

$$\cos\theta = \frac{l(1)+m(-2)+n(2)}{\sqrt{1}\sqrt{1+4+4}} = \frac{l-2m+2n}{3} \quad (2)$$

$$\cos\theta = \frac{2l-m-2n}{3} \quad (3)$$

From (1) and (2), $4m = 0$ or $m = 0$.
From (2) and (3), $l + m - 4n = 0, l - 4n = 0$ or $l = 4n$.

DC's $\frac{l}{4} = \frac{m}{0} = \frac{n}{1}$ or $\frac{l}{\frac{4}{\sqrt{17}}} = \frac{m}{0} = \frac{n}{\frac{1}{\sqrt{17}}}$.

From (1) $\cos\theta = \dfrac{\frac{4}{\sqrt{17}} + 2\cdot 0 + 2\frac{1}{\sqrt{17}}}{3} = \dfrac{2}{\sqrt{17}}$.

Equation of right circular cone is

$$(l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta = (lx+my+nz)^2$$

$$\left(\frac{16}{17}+0+\frac{1}{17}\right)(x^2+y^2+z^2)\frac{4}{17} = \left(\frac{4}{\sqrt{17}}x+0+\frac{1}{\sqrt{17}}z\right)^2$$

$$4(x^2+y^2+z^2) = (4x+z)^2$$

$$12x^2 - 4y^2 - 3z^2 + 8xz = 0$$

is the required equation of the cone.

EXERCISE

1. Find the equation of the cone whose vertex is (3, 1, 2) and base circle is $2x^2 + 3y^2 = 1$, $z = 1$.

Ans. $2x^2 + 3y^2 + 20z^2 - 6yz - 12xz + 12x + 6y - 38z + 17 = 0$

2. Find the equation of the cone whose vertex is origin and guiding curve is $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1, x + y + z = 1$.

Ans. $27x^2 + 32y^2 + 72(xy + yz + zx) = 0$.

3. Determine the equation of the cone with vertex at origin and guiding curve $x^2 + y^2 + z^2 - x - 1 = 0, x^2 + y^2 + z^2 + y - z = 0$.

   Hint: Guiding curve is circle in plane $x + y = 1$. Rewrite $x^2 + y^2 + z^2 - x(x + y) - (x + y)^2 = 0$.

Ans. $x^2 + 3xy - z^2 = 0$

4. Show that the equation of cone with vertex at origin and base circle $x = a, y^2 + z^2 = b^2$ is $a^2(y^2 + z^2) = b^2x^2$. Further prove that the section of the cone by a plane parallel to the XY-plane is a hyperbola.

Ans. $b^2x^2 - a^2y^2 = a^2c^2, z = c$ (put $z = c$ in equation of cone)

5. Find the equation of a cone with vertex at origin and guiding curve is the circle passing through the X, Y, Z intercepts of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Ans. $a(b^2 + c^2)yz + b(c^2 + a^2)zx + c(a^2 + b^2)xy = 0$

6. Write the equation of the cone whose vertex is (1, 1, 0) and base is $y^2 + z^2 = 9, x = 0$.

   Hint: Substitute $\left(0, 1 - \frac{m}{l}, -\frac{n}{l}\right)$ in base curve and eliminate $\frac{m}{l} = \frac{y-1}{x-1}, \frac{n}{l} = \frac{z}{z-1}$.

Ans. $x^2 + y^2 + z^2 - 2xy = 0$

Right circular cone (R.C.C.)

7. Find the equation of R.C.C. with vertex at (2, 3, 1), axis parallel to the line $-x = \frac{y}{2} = z$ and one of its generators having DC's proportional to (1, −1, 1).

   Hint: $\cos\theta = \frac{-1-2+1}{\sqrt{6}\sqrt{3}}, l = -1, m = 2, n = 1, \alpha = 2, \beta = 3, \gamma = 1$.

Ans. $x^2 - 8y^2 + z^2 + 12xy - 12yz + 6zx - 46x + 36y + 22z - 19 = 0$

8. Determine the equation of R.C.C. with vertex at origin and passes through the point (1, 1, 2) and axis line $\frac{x}{2} = \frac{-y}{4} = \frac{z}{3}$.

Hint: $\cos\theta = \frac{2-4+6}{\sqrt{6}\sqrt{29}}$, DC's of generator: 1, 1, 2, axis: 2, −4, 3

Ans. $4x^2+40y^2+19z^2-48xy-72yz+36xz=0$

9. Find the equation of R.C.C. whose vertex is origin and whose axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and which has semi- vertical angle of 30°

   Hint: $\cos 30 = \frac{\sqrt{3}}{2} = \frac{x(1)+y(2)+z(3)}{\sqrt{(x^2+y^2+z^2)}\sqrt{1+4+9}}$

Ans. $19x^2+13y^2+3z^2-8xy-24yz-12zx=0$

10. Obtain the equation of R.C.C. generated when the straight line $2y+3z=6, x=0$ revolves about z-axis.

    Hint: Vertex (0, 0, 2), generator $\frac{x}{0}=\frac{y}{3}=\frac{z-2}{-2}$, $\cos\theta=-\frac{2}{\sqrt{13}}$.

Ans. $4x^2+4y^2-9z^2+36z-36=0$

11. Lines are drawn from the origin with DC's proportional to (1, 2, 2), (2, 3, 6), (3, 4, 12). Find the equation of R.C.C.

    Hint: $\cos\alpha=\frac{l+2m+2n}{3}=\frac{2l+3m+6n}{7}=\frac{3l+4m+12n}{13}$ $\frac{l}{-1}=\frac{m}{1}=\frac{n}{1}$, $\cos\alpha=\frac{1}{\sqrt{3}}$, DC's of axis: −1, 1, 1.

Ans. $xy-yz+zx=0$

12. Determine the equation of the R.C.C. generated by straight lines drawn from the origin to cut the circle through the three points (1, 2, 2), (2, 1, −2), and (2, −2, 1).

    Hint: $\cos\alpha=\frac{l+2m+2n}{3}=\frac{2l+m-2n}{3}=\frac{2l-2m+n}{3}$ $\frac{l}{5}=\frac{m}{1}=\frac{n}{1}$, $\cos\alpha=\frac{5+2+2}{3\sqrt{27}}=\frac{1}{\sqrt{3}}$.

Ans. $8x^2-4y^2-4z^2+5xy+5zx+yz=0$

3.7 THE RIGHT CIRCULAR CYLINDER

A cylinder is the surface generated by a straight line (known as **generator**) which is parallel to a fixed straight line (known as **axis**) and satisfies a condition; for example, it may intersect a fixed curve (known as the **guiding curve**) or touch a given surface. A **right circular cylinder** is a cylinder whose surface is generated by revolving the generator at a fixed distance (known as the **radius**) from the axis; i.e., the guiding curve in this case is a circle. In fact, the intersection of the right circular cylinder with any plane perpendicular to axis of the cylinder is a circle.

Equation of a cylinder with generators parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and guiding curve conic $ax^2+by^2+2hxy+2gx+2fy+c=0, z=0$.

Let $P(x_1, y_1, z_1)$ be any point on the cylinder. The equation of the generator through $P(x_1, y_1, z_1)$ which is parallel to the given line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \tag{1}$$

is
$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \tag{2}$$

Since (2) meets the plane $z=0$,

$$\therefore \quad \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{0-z_1}{n}$$

or
$$x=x_1-\frac{l}{n}z_1,\ y=y_1-\frac{m}{n}z_1 \tag{3}$$

Since this point (3) lies on the conic

$$ax^2+by^2+2hxy+2gx+2fy+c=0 \tag{4}$$

substitute (3) in (4). Then

$$a\left(x_1-\frac{l}{n}z_1\right)^2+b\left(y_1-\frac{m}{n}z_1\right)^2+$$
$$+2h\left(x_1-\frac{l}{n}z_1\right)\left(y_1-\frac{m}{n}z_1\right)+2g\left(x_1-\frac{l}{n}z_1\right)+$$
$$+2f\left(y_1-\frac{m}{n}z_1\right)+c=0.$$

The required equation of the cylinder is

$$a(nx-lz)^2+b(ny-mz)^2+2h(nx-lz)(ny-mz)+$$
$$+2ng(nx-lz)+2nf(ny-mz)+cn^2=0 \tag{5}$$

where the subscript 1 is droped because (x_1, y_1, z_1) is any general point on the cylinder.

Corollary 1: The equation of a cylinder with axis parallel to z-axis is obtained from (5) by putting $l=0, m=0, n=1$ which are the DC's of z-axis: i.e.,

$$ax^2+by^2+2hxy+2gx+2fy+c=0$$

which is **free** from z.

Thus the equation of a cylinder whose axis is paralle to x-axis (y-axis or z-axis) is obtained by eliminating the variable $x(y$ or $z)$ from the equation of the conic.

Equation of a right circular cylinder:

a. Standard form: with z-axis as axis and of radius a. Let $P(x, y, z)$ be any point on the cylinder. Then M the foot of the perpendicular PM has $(0, 0, z)$ and $PM = a$ (given). Then

$$a = PM = \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2}$$
$$x^2 + y^2 = a^2$$

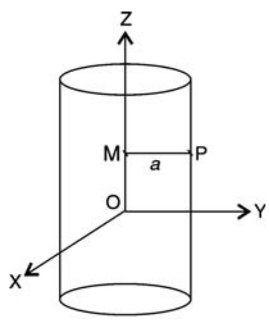


Fig. 3.20

Corollary 2: Similarly, equation of right circular cylinder with y-axis is $x^2 + z^2 = a^2$, with x-axis is $y^2 + z^2 = a^2$.

b. General form with the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ as axis and of radius a.

Axis AB passes through the point (α, β, γ) and has DR's l, m, n. Its DC's are $\frac{l}{k}, \frac{m}{k}, \frac{n}{k}$ where $k = \sqrt{l^2 + m^2 + n^2}$.

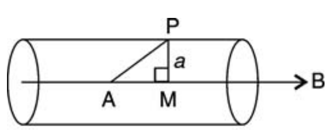


Fig. 3.21

From the right angled triangle APM

$$AP^2 = PM^2 + AM^2$$
$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2$$
$$= a^2 + \left[l(x-\alpha) + m(y-\beta) + n(z-\gamma)\right]^2$$

which is the required equation of the cylinder (Here AM is the projection of AP on the line AB is equal to $l(x-\alpha) + m(y-\beta) + n(z-\gamma)$).

Enveloping cylinder of a sphere is the locus of the tangent lines to the sphere which are parallel to a given line. Suppose

$$x^2 + y^2 + z^2 = a^2 \tag{1}$$

is the sphere and suppose that the generators are parallel to the given line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \tag{2}$$

Then for any point $P(x_1, y_1, z_1)$ on the cylinder, the equation of the generating line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \tag{3}$$

Any general point on (3) is

$$(x_1 + lr, \quad y_1 + mr, \quad z_1 + nr) \tag{4}$$

By substituting (4) in (1), we get the points of intersection of the sphere (1) and the generating line (3) i.e.,

$$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2$$

Rewriting as a quadratic in r, we have

$$(l^2 + m^2 + n^2)r^2 + 2(lx_1 + my_1 + nz_1)r + $$
$$+ (x_1^2 + y_1^2 + z_1^2 - a^2) = 0 \tag{5}$$

If the roots of (5) are equal, then the generating line (3) meets (touches) the sphere in a single point i.e., when the discriminant of the quadratic in r is zero.

or $$4(lx_1 + my_1 + nz_1)^2 - 4(l^2 + m^2 + n^2) \times$$
$$\times (x_1^2 + y_1^2 + z_1^2 - a^2) = 0$$

Thus the required equation of the enveloping cylinder is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2)$$

where the subscript 1 is droped to indicate that (x, y, z) is a general point on the cylinder.

Worked Out Examples

Example 1: Find the equation of the quadratic cylinder whose generators intersect the curve $ax^2 +$

$by^2 + cz^2 = k, lx + my + nz = p$ and parallel to the y-axis. Deduce the case for $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$ and parallel to y-axis

Solution: Eliminate y between

$$ax^2 + by^2 + cz^2 = k \quad (1)$$

and
$$lx + my + nz = p \quad (2)$$

Solving (2) for y, we get

$$y = \frac{p - lx - nz}{m} \quad (3)$$

Substitute (3) in (1), we have

$$ax^2 + b\left(\frac{p - lx - nz}{m}\right)^2 + cz^2 = k.$$

The required equation of the cylinder is

$$(am^2 + l^2)x^2 + (bn^2 + m^2c)z^2 - 2pblx$$
$$- 2npbz + 2blnxz + (bp^2 - m^2k) = 0.$$

Deduction: Put $a = 1, b = 1, c = 1, k = 1, l = m = n = p = 1$

$$2x^2 + 2z^2 - 2x - 2z + 2xz = 0$$

or
$$x^2 + z^2 + xz - x - z = 0.$$

Example 2: If l, m, n are the DC's of the generators and the circle $x^2 + y^2 = a^2$ in the XY-plane is the guiding curve, find the equation of the cylinder. Deduce the case when $a = 4, l = 1, m = 2, n = 3$.

Solution: For any point $P(x_1, y_1, z_1)$ on the cylinder, the equation of the generating line through P is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (1)$$

Since the line (1) meets the guiding curve $x^2 + y^2 = a^2, z = 0$,

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{0 - z_1}{n}$$

or
$$x = x_1 - \frac{lz_1}{n}, \qquad y = y_1 - \frac{mz_1}{n} \quad (2)$$

This point (2) lies on the circle $x^2 + y^2 = a^2$ also. Substituting (2) in the equation of circle, we have

$$\left(x_1 - \frac{lz_1}{n}\right)^2 + \left(y_1 - \frac{mz_1}{n}\right)^2 = a^2$$

or
$$(nx - lz)^2 + (ny - mz)^2 = n^2a^2$$

is the equation of the cylinder.

Deduction: Equation of cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and pass through the curve $x^2 + y^2 = 16, z = 0$. With $a = 4, l = 1, m = 2, n = 3$, the required equation of the cylinder is

$$(3x - z)^2 + (3y - 2z)^2 = 9(16) = 144$$

or
$$9x^2 + 9y^2 + 5z^2 - 6zx - 12yz - 144 = 0.$$

Example 3: Find the equation of the right circular cylinder of radius 3 and the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$ as axis.

Solution: Let $A(1, 3, 5)$ be the point on the axis and DR's of AB are $2, 2, -1$ or DC's of AB are $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$. Radius $PM = 3$ given. Since AM is the projection of AP on AB, we have

$$AM = \frac{2}{3}(x - 1) + \frac{2}{3}(y - 3) - \frac{1}{3}(z - 5)$$

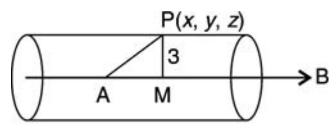


Fig. 3.22

From the right angled triangle APM

$$AP^2 = AM^2 + MP^2$$

$$(x - 1)^2 + (y - 3)^2 + (z - 5)^2$$
$$= \left[2\frac{(x - 1)}{3} + 2\frac{y - 3}{3} - 1\frac{(z - 5)}{3}\right] + 9$$

$$9[x^2 + 1 - 2x + y^2 + 9 - 6y + z^2 + 25 - 10z]$$
$$= [2x + 2y - z - 3]^2 + 81$$

$$9[x^2 + y^2 + z^2 - 2x - 6y - 10z + 35]$$
$$= [4x^2 + 4y^2 + z^2 + 9 + 8xy - 4xz - 12x$$
$$-4yz - 12y + 6z] + 81$$

is the required equation of the cylinder.

Example 4: Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$ having its generators parallel to the line $x = -2y = 2z$.

Solution: Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the equation of the generating line through P and parallel to the line $x = -2y = 2z$ or $\frac{x}{1} = \frac{y}{-\frac{1}{2}} = \frac{z}{\frac{1}{2}}$ is

$$\frac{x - x_1}{1} = \frac{y - y_1}{-\frac{1}{2}} = \frac{z - z_1}{\frac{1}{2}} \quad (1)$$

Any general point on (1) is

$$\left(x_1 + r, \quad y_1 - \frac{1}{2}r, \quad z_1 + \frac{1}{2}r\right) \quad (2)$$

The points of intersection of the line (1) and the sphere

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0 \quad (3)$$

are obtained by substituting (2) in (3).

$$(x_1 + r)^2 + \left(y_1 - \frac{1}{2}r\right)^2 + \left(z_1 + \frac{1}{2}r^2\right)^2 - 2\left(y_1 - \frac{1}{2}r\right) - 4\left(z_1 + \frac{1}{2}r\right) - 11 = 0$$

Rewriting this as a quadratic in r

$$\frac{3}{2}r^2 + (2x_1 - y_1 + z_1 - 1)r + (x_1^2 + y_1^2 + z_1^2 - 2y_1 - 4z_1 - 11) = 0 \quad (4)$$

The generator touches the sphere (3 if (4) has equal roots i.e., discriminant is zero or

$$(2x_1 - y_1 + z_1 - 1)^2 = 4 \cdot \frac{3}{2} \cdot (x_1^2 + y_1^2 + z_1^2 - 2y_1 - 4z_1 - 11).$$

The required equation of the cylinder is

$$2x^2 + 5y^2 + 5z^2 + 4xy - 4xz + 2yz + 4x - 14y - 22z - 67 = 0.$$

EXERCISE

1. Find the equation of the quadratic cylinder whose generators intersect the curve
   a. $ax^2 + by^2 = 2z, lx + my + nz = p$ and are parallel to z-axis.
   b. $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$ and are parallel to x-axis.

   Hint: Eliminate z

Ans. **a.** $n(ax^2 + by^2) + 2lx + 2my - 2p = 0$

Hint: Eliminate x.

Ans. **b.** $(bl^2 + am^2)y^2 + (cl^2 + an^2)z^2 + 2amnyz - 2ampy - 2anpz + (ap^2 - l^2) = 0$

2. If l, m, n are the DC's of the generating line and the circle $x^2 + z^2 = a^2$ in the zx-plane is the guiding curve, find the equation of the sphere.

Ans. $(mx - ly)^2 + (mz - ny)^2 = a^2m^2$

Find the equation of a right circular cylinder (4 to 9)

4. Whose axis is the line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$ and radius is 2 units.

Ans. $26x^2 + 29y^2 + 5z^2 + 4xy + 10yz - 20zx + 150y + 30z + 75 = 0$

5. Having for its base the circle $x^2 + y^2 + z^2 = 9, x - y + z = 3$.

Ans. $x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$

6. Whose axis passes through the point (1, 2, 3) and has DC's proportional to (2, −3, 6) and of radius 2.

Ans. $45x^2 + 40y^2 + 13z^2 + 36yz - 24zx + 12xy - 42x - 280y - 126z + 294 = 0.$

7. Whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$ and radius 2 units.

Ans. $5x^2 + 8y^2 + 5z^2 - 4yz - 8zx - 4xy + 22x - 16y - 14z - 10 = 0$

8. The guiding curve is the circle through the three points (1, 0, 0), (0, 1, 0)(0, 0, 1).

Ans. $x^2 + y^2 + z^2 - xy - yz - zx = 1$

9. The directing curve is $x^2 + z^2 - 4x - 2z + 4 = 0$, $y = 0$ and whose axis contains the point (0, 3, 0). Also find the area of the section of the cylinder by a plane parallel to xz-plane.

   Hint: Centre of circle (2, 0, 1) radius: 1

Ans. $9x^2 + 5y^2 + 9z^2 + 12xy + 6yz - 36x - 30y - 18z + 36 = 0, \pi$

10. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$, having its generators parallel to the line $x = y = z$.

Ans. $x^2 + y^2 + z^2 - xy - yz - zx - 2x + 7y + z - 2 = 0$.