Lecture in Linear Algebra

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1. Matrix

In 1848, G.G. Sylvester introduces the concept of matrices as the name of a group of numbers arranged in a rectangular in the form of rows and columns. In 1855, Arthur Cayley studied matrices from an algebraic perspective. In this study, he defined the process of multiplying matrices using linear transformations.

Definition 1.0.1 A matrix is a rectangular arrangement of numbers (real or complex) which may be represented as,

$$A = (a_{ij})_{m \times n} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix},$$

the general form of a matrix with m rows and n columns.

Ⓡ Capital letters $A, B, \ldots$ denote matrices, whereas lower case letters $a, b, \ldots$ denote elements.

■ **Example 1.1** Build a matrix $A = (a_{ij})_{2\times 3}$, where

$$a_{ij} = \begin{cases} i+j & if\ i<j \\ i & if\ i=j \\ i-j & if\ i>j \end{cases}$$

Solution:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix},$$

■

■ **Example 1.2** Build a matrix $B = (b_{ij})_{3\times 3}$;

$$b_{ij} = \begin{cases} i+j & if\ i<j \\ 0 & if\ i=j \\ i^2-j^2 & if\ i>j \end{cases}$$

Solution:

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

$$b_{11} = 0\ ,\ b_{12} = 1+2 = 3\ ,\ b_{13} = 1+3 = 4\ ,$$
$$b_{21} = 2^2 - 1^2 = 3\ ,\ b_{22} = 0\ ,\ b_{23} = 2+3 = 5,$$
$$b_{31} = 3^2 - 1^2 = 8\ ,\ b_{32} = 3^2 - 2^2 = 5\ ,\ b_{33} = 0,$$

$$\therefore B = \begin{pmatrix} 0 & 3 & 4 \\ 3 & 0 & 5 \\ 8 & 5 & 0 \end{pmatrix}.$$

■

Definition 1.0.2 Two matrices $A_{m\times n} = (a_{ij})$ and $B_{p\times q} = (b_{kl})$ are equal, if
1- $m = p$ and $n = q$.
2- $a_{ij} = b_{kl}\ \forall i, j, k, l$.

■ **Example 1.3** Given

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

disuss the possibility that

1. $A = B$.
2. $B = C$.
3. $A = C$.

Solution

1. $A = B$ is impossible because A and B are of different size.
2. Similarly, $B = C$ is impossible.
3. $A = C$ is possible.

■

Definition 1.0.3 A matrix whose elements are all zero is called a zero matrix and denoted by 0 or O.

Definition 1.0.4 A matrix with the same number of rows as columns is called a square matrix.
A square matrix with n rows and n columns is called a $n-$square matrix.

■ **Example 1.4** The matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -4 & -1 \\ 5 & 3 & 2 \end{pmatrix},$$

is a 3 square matrix. ■

Definition 1.0.5 The main diagonal or simply diagonal of a square matrix $A = (a_{ij})$ is the numbers $a_{11}, a_{22}, \ldots, a_{nn}$.

■ **Example 1.5** In the above Example 1.4, the numbers along the main diagonal are $1, -4, 2$. ■

Definition 1.0.6 The square matrix with $1s$ along the main diagonal and $0s$ elsewhere is called the unit matrix or the identity matrix and will be denoted by I.
For any square matrix A, $AI = IA = A$.

■ **Example 1.6** The matrix

$$I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

is a unit matrix of type 3×3. ■

1.1 Matrix Addition

Definition 1.1.1 The sum of the two matrices A and B, written $A+B$, is the matrix obtained by adding the corresponding element from A and B i.e.,

$$A + B = (a_{ij} + b_{ij}).$$

(R) $A+B$ have the same type as A and B.

(R) The sum of two matrices with different types is not defined.

■ **Example 1.7** Let A and B;

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ -1 & 4 & 1 \end{pmatrix},$$

be two matrices, then

$$A + B = \begin{pmatrix} 2 & 4 & 1 \\ 6 & 2 & 0 \\ -1 & 8 & 2 \end{pmatrix}.$$

■

■ **Example 1.8** Let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix},$$

$$B = \begin{pmatrix} 3 & 0 & -6 \\ 2 & -3 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 0 & 5 & -2 \\ 1 & -3 & -1 \end{pmatrix}.$$

Find $A+B$ and $C+D$.
Solution

$$A + B = \begin{pmatrix} 4 & -2 & -3 \\ 2 & 1 & 6 \end{pmatrix},$$

and the sum of $C+D$ is not defined. ■

Theorem 1.1.1 Let A, B and C be matrices with the same type, then
1. $(A+B)+C = A+(B+C)$.
2. $A+B = B+A$.
3. $A+O = O+A = A$.

Where O is a zero matrix with the same type of A.

Proof. Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}\, and\, C = (c_{ij})_{m\times n}$, then

$$\begin{aligned}(i)\ (A+B)+C &= [(a_{ij})+(b_{ij})]+(c_{ij})\\ &= (a_{ij}+b_{ij})+c_{ij}\\ &= (a_{ij}+b_{ij}+c_{ij})\\ &= (a_{ij})+(b_{ij}+c_{ij})\\ &= A+(B+C)\end{aligned}$$

$$\begin{aligned}(ii)\ A+B &= (a_{ij})+(b_{ij})\\ &= (a_{ij}+b_{ij})\\ &= (b_{ij}+a_{ij})\\ &= (b_{ij})+(a_{ij}).\end{aligned}$$

(iii) Trivial. ■

■ **Example 1.9** Solve

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} + X = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

where X is a matrix.

Solution:

To solve

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} + X = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},$$

simply subtract the matrix

$$\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix},$$

from both sides to get

$$X = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}.$$

■

1.2 Scalar Multiplication

Definition 1.2.1 The product of a scalar k and a matrix A, written kA is the matrix obtained by multiplying each element of A by k, i.e.,

$$kA = (ka_{ij})_{m\times n}.$$

■ **Example 1.10** $3\begin{pmatrix} 1 & -2 & 0 \\ 4 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -6 & 0 \\ 12 & 9 & -15 \end{pmatrix}.$ ■

■ **Example 1.11** If $kA = 0$, show that either $k = 0$ or $A = 0$.
Solution:
Write $A = (a_{ij})$, so that $kA = 0$, means $ka_{ij} = 0$, for all i and j. If $k = 0$, there is nothing to do. If $k \neq 0$, then $ka_{ij} = 0$ implies that $a_{ij} = 0$, for all i and j; that is, $A = 0$. ■

1.3 Matrix Multiplication

Definition 1.3.1 Le $A_{m\times n} = (a_{ij})$ and $B_{n\times q} = \left(b_{jk}\right)$, then

$$C_{m\times p} = AB = (\sum_{j=1}^{n} a_{ij}b_{jk}).$$

■ **Example 1.12** Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix},$$

and

$$D = \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix},$$

then find
(1) AB.

(2) AC.
(3) AD.
Solution
(1) $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ 23 & -3 \end{pmatrix}$.
(2) $AC = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -6 & 13 \\ 3 & 10 & 29 \end{pmatrix}$.
(3) $AD = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix}$ is not defined. ■

Theorem 1.3.1 Let A, B and C be matrices with the same type, then

(i) $(AB)C = A(BC)$
(ii) $A(B+C) = AB + AC$
(iii) $(B+C)A = BA + CA$
(iv) $k(AB) = (kA)B = A(kB)$ *where k is a scalar.*

Proof. Let $A = (a_{ij})_{m\times n}, B = (b_{jk})_{n\times p}\, and\, C = (c_{kl})_{p\times q}$, then

$$\begin{aligned}
(i)\ L.H.S &= (AB)C \\
&= (\textstyle\sum_{j=1}^{n} a_{ij}b_{jk}).(c_{kl}) \\
&= (\textstyle\sum_{k=1}^{p}[(\sum_{j=1}^{n} a_{ij}b_{jk}).c_{kl}]) \\
&= (\textstyle\sum_{k=1}^{p}\sum_{j=1}^{n} a_{ij}b_{jk}.c_{kl}). \\
R.H.S &= A(BC) \\
&= (a_{ij})(\textstyle\sum_{k=1}^{p} b_{jk}c_{kl}) \\
&= (\textstyle\sum_{j=1}^{n} a_{ij}[(\sum_{k=1}^{p} b_{jk}c_{kl})] \\
&= (\textstyle\sum_{j=1}^{n}\sum_{k=1}^{p} a_{ij}b_{jk}c_{kl}).
\end{aligned}$$

Assuming I have written these correctly, we can make two observations: first, the summands are equivalent, as multiplication is associative. Second, the order of the summations doesn't matter when we're summing a finite number of entries. Thus, $(AB)C = A(BC)$.

$$\begin{aligned}
&(ii)\ Let\, A = (a_{ij})_{m\times n},\ B = (b_{jk})_{n\times n}\, and\, C = (c_{jk})_{n\times n},\ then\\
&L.H.S = A\,(B+C)\\
&\qquad = (\textstyle\sum_{j=1}^{n} a_{ij}\left(b_{jk}+c_{jk}\right))\\
&\qquad = (\textstyle\sum_{j=1}^{n}(a_{ij}b_{jk}+a_{ij}c_{jk}))\\
&\qquad = (\textstyle\sum_{j=1}^{n} a_{ij}b_{jk}) + (\textstyle\sum_{j=1}^{n} a_{ij}c_{jk})\\
&\qquad = AB+AC.
\end{aligned}$$

(iii) In the same way.

(iv) Trivial.

■

R The matrix product is not commutative in general i.e.,

$$AB \neq BA.$$

■ **Example 1.13** Simplify the expression

$$A(BC-CD)+A(C-B)D-AB(C-D).$$

Solution

$A(BC-CD)+A(C-B)D-AB(C-D) = A(BC)-A(CD)+(AC-AB)D-(AB)C+(AB)D = ABC-ACD+ACD-ABD-ABC+ABC = 0.$ ■

■ **Example 1.14** Show that $AB = BA$ if and only if

$$(A-B)(A+B) = A^2-B^2.$$

Solution

In general the following hold

$$(A-B)(A+B) = A(A+B)-B(A+B) = A^2+AB-BA-B^2.$$

Hence if $AB = BA$, then $(A-B)(A+B) = A^2-B^2$. Conversely, if this last equation holds, then equation becomes

$$(A-B)(A+B) = A(A+B)-B(A+B) = A^2+AB-BA-B^2.$$

This gives $0 = AB-BA$, and then $AB = Bc$. ■

1.4 Transpose

Definition 1.4.1 The *transpose of a matrix*

$$A = (a_{ij})_{m\times n},$$

written by A^T is the matrix obtained by writing the rows of A, in order, as columns, i.e.,

$$A^T = (a_{ji})_{n\times m}.$$

Example 1.15 Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \end{pmatrix},$$

then

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & -5 \\ 3 & 6 \end{pmatrix}.$$

■

The transpose operation on a matrix satisfies the following properties:

Theorem 1.4.1 Let A and B be matrices with the same type, then

1. $(A+B)^T = A^T + B^T$.
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$, *for k a scalar.*
4. $(AB)^T = B^T A^T$.

Proof. Let $A = (a_{ij})_{m\times n}, B = (b_{jk})_{m\times n}$, *then*

1. $$\begin{aligned} L.H.S &= (A+B)^T \\ &= (a_{ij}+b_{ij})^T \\ &= (a_{ji}+b_{ji}) \\ &= (a_{ji})+(b_{ji}) \\ &= A^T + B^T \\ &= R.H.S. \end{aligned}$$
2. $$\begin{aligned} L.H.S &= \left(A^T\right)^T \\ &= \left((a_{ij})^T\right)^T \end{aligned}$$

$$= (a_{ji})^T$$
$$= (a_{ij})$$
$$= A = R.H.S.$$

3. Exercise.
4. Exercise.

■

Definition 1.4.2 A matrix A is called *symmetric* if

$$A = A^T.$$

Definition 1.4.3 A matrix A is called *skew-symmetric* if

$$A = -A^T.$$

R A symmetric matrix A is necessarily square.

■ **Example 1.16** If A and B are symmetric $n \times n$ matrices, show that $A + B$ is symmetric.
Solution:
Since $A = A^T$ and B^T, so, we have

$$(A+B)^T = A^T + B^T = A + B.$$

Hence $A + B$ is symmetric. ■

■ **Example 1.17** Let A be a square matrix satisfies,

$$A = 2A^T.$$

show that necessarily $A = 0$.
Solution:
If we iterate the given equation, gives

$$A = 2A^T.$$

$$= 2(2A^T)^T.$$

$$= 2((2A^T)^T).$$
$$= 4A.$$

This lead to $3A = O$ and hence $A = 0$. ■

■ **Example 1.18** If A and B are two skew symmetric matrices of same order, then AB issymmetric matrix if ....
Solution
$AB = BA$. ■

1.5 The inverse of a matrix

The inverse of a square $n \times n$ matrix A is another $n \times n$ matrix denoted by A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

where I is the $n \times n$ identity matrix. That is, multiplying a matrix by its inverse produces an identity matrix. Not all square matrices have an inverse matrix. If the determinant of the matrix is zero, then it will not have an inverse, and the matrix is said to be singular. Only non-singular matrices have inverses.

Definition 1.5.1 If A is a square matrix, a matrix B is called an inverse of A if and only if

$$AB = I \text{ and } BA = I.$$

■ **Example 1.19** Show that

$$B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

is an inverse of

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution:

Compute AB and BA.

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $AB = I = BA$, so B is indeed an inverse of A. ■

■ **Example 1.20** If

$$A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

show that $A^3 = I$ and so find A^{-1}.
Solution:
We have

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},$$

and so

$$A^3 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $A^3 = I$, as asserted. This can be written as

$$A^2A = AA^2 = I,$$

so it shows that A^2 is the inverse of A. That is,

$$A^{-1} = A^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

■

1.5.1 Adjoint of a square matrix

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n and let c_{ij} be the cofactor of a_{ij} in the determinant |A|, then the adjoint of A, denoted by adj (A), is defined as the transpose of the matrix, formed by the cofactors of the matrix.

Theorem 1.5.1 Given any non-singular matrix A, its inverse can be found from the formula

$$A^{-1} = \frac{adj\,A}{|A|}.$$

where $adj\,A$ is the adjoint matrix and $|A|$ is the determinant of A.

■ **Example 1.21** Find A^{-1} where

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

Solution:
We calculate the value of the determinant of the matrix

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 2 \\ 0 & -3 & -6 \\ 0 & -6 & -3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -3 & -6 \\ -6 & -3 \end{vmatrix}$$

$$= -27 \neq 0 \quad .$$

The cofactors of the matrix

$$\Delta_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = -3,$$

$$\Delta_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} = -6,$$

$$\Delta_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 2 & -2 \end{vmatrix} = -6,$$

$$\Delta_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} = -6,$$

$$\Delta_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3,$$

$$\Delta_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 6,$$

$$\Delta_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} = -6,$$

$$\Delta_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = 6,$$

$$\Delta_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3.$$

So,

$$\tilde{A} = (\Delta_{ij}) = \begin{pmatrix} -3 & -6 & -6 \\ -6 & -3 & 6 \\ -6 & 6 & -3 \end{pmatrix} = -3\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix},$$

and

$$ad\,jA = (\tilde{A})^t = -3\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix},$$

thus

$$A^{-1} = \frac{ad\,jA}{|A|}$$

$$= \frac{-3}{-27}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}.$$

■

Theorem 1.5.2 All the following matrices are square matrices of the same size.

1. I is invertible and $I^{-1} = I$.
2. If A is invertible, so is A^{-1}, and $\left(A^{-1}\right)^{-1} = A$.
3. If A and B are invertible, so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

.

4. If A is invertible, then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

1.6 Rank

Definition 1.6.1 A positive integer r is said to be the rank of a non-zero matrix A, if:

1. There exists at least one minor in A of order r which is not zero.
2. Every minor in A of order greater than r is zero, the rank of a matrix A is denoted by $\rho(A) = r$.
3. If A is an $m \times n$ matrix, then the rank of A is $0 \leq \rho \leq \min\{m, n\}$.

■ **Example 1.22** The rank of a null matrix is zero *i.e*, $\rho(O) = 0$. ■

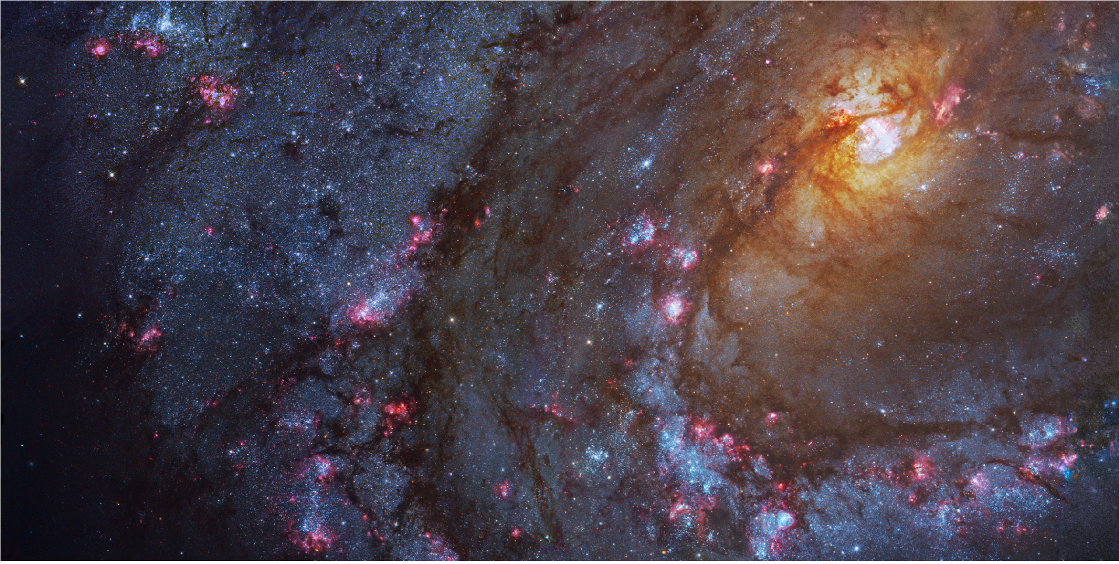
■ **Example 1.23** If I_n is an identity matrix of order n, then $\rho(I_n) = n$. ■

1.7 Exercises

1- Prove that
(i) $(kA)^T = kA^T, for\ k\ a\ scalar.$
(ii) $(AB)^T = B^T A^T$.
(iii) I is invertible and $I^{-1} = I$.
(iv) If A is invertible, so is A^{-1}, and $\left(A^{-1}\right)^{-1} = A$.
(v) If A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.
(vi) If A is invertible, then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.
2- Build matrices $A = (a_{ij})_{3\times 2}$, $B = (b_{ij})_{2\times 3}$;

$$a_{ij} = \begin{cases} i+j & if\ i<j \\ i & if\ i=j \\ i-j & if\ i>j \end{cases} \quad , \; b_{ij} = \begin{cases} 2i-1 & if\ i=j \\ i+j-2 & if\ i\neq j \end{cases}$$

3- If $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -4 & 0 & 1 \\ 2 & -1 & 3 & -1 \\ 4 & 0 & -2 & 0 \end{pmatrix}$. Compute AB.
4- If $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$, $B = \begin{pmatrix} -11 & -4 & 6 \\ 2 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix}$. Compute AB^t.
6- If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 & 1 \\ -4 & 3 & -2 \\ 3 & -2 & 1 \end{pmatrix}$. Compute AB^t.
7- Find the inverse of the matrices
(i) $\begin{pmatrix} -2 & 3 \\ -5 & -6 \end{pmatrix}$.
(ii) $\begin{pmatrix} 3 & 5 \\ 7 & 9 \end{pmatrix}$.
(iii) $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}$.



2. Systems of Linear Equati

In mathematics, the theory of linear systems is the basis and a fundamental part of linear algebra, a subject which is used in most parts of modern mathematics. In this chapter, we introduce and study the system of linear equation.

Definition 2.0.1 A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, ..., x_n$ of the form,

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2, \\ \;..... \;\; ..... \;\; ..... \;\; ..... \;\; ..... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m, \end{array}$$

where $a_{ij}, b_i \in \mathbf{R}$ or $\mathbf{C}$, for all $i = 1, 2, ..., m;\; j = 1, 2, ..., n >$

The two matrices:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} & b_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} & b_m \end{pmatrix},$$

are the *coefficient matrix*, the *augmented matrix* for the system respectively.

Definition 2.0.2 The solution set to a system with n unknowns $x_1, x_2, \ldots, x_n$ is a set of numbers $t_1, t_2, \ldots, t_n$ so that we set $x_1 = t_1$, $x_2 = t_2$, $\ldots$, $x_n = t_n$ then all of the equations in the system will be satisfied.

Definition 2.0.3 A system of equations

$$AX = B$$

is called a *homogeneous system* if $B = 0$ and if B $\neq 0$, then it is called *a non-homogeneous system* of equations.

Theorem 2.0.1 Given a system of m equations and n unknowns there will be one of three possibilities for solutions of the system:

1. There will be no solution.
2. There will be exactly one solution.
3. There will be infinitely many solutions.

Definition 2.0.4 If there is no solution to the system we call the system *inconsistent*
- If there is at least one solution to the system we call the system *consistent*.

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

in which the graphs of the equations are lines in the xy -plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities:

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.
2. The lines may intersect at only one point, in which case the system has exactly one solution.
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

■ **Example 2.1** The system of linear equations:

$$\begin{aligned} x + y &= 1 \\ x + 8y &= 1 \end{aligned}$$

consistent; since $x = 1$, $y = 0$. ■

■ **Example 2.2** The system of linear equations:

$$\begin{aligned} x + y + 2z \quad &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

consistent; since $x = 1$, $y = 2$, $z = 3$. ■

■ **Example 2.3** The system of linear equations:

$$\begin{aligned} 3x - 6y &= 1 \\ 2x - 4y &= 5 \end{aligned}$$

inconsistent. ■

Example 2.4 The system of linear equations:

$$\begin{aligned} x-2y+3z&=2\\ 2x+3y-2z&=5\\ 4x-y+4z&=1 \end{aligned}$$

inconsistent. ■

2.1 The Row Reduction Algorithm

Elementary operation on matrices:

1. Interchange of any two rows.
2. Multiplication of a row by a scalar.
3. Addition of a multiple of one row to another row.

A matrix (any matrix) is said to be in *reduced row-echelon form* if it satisfies all four of the following conditions:

1. If there are any rows of all zeros then they are at the bottom of the matrix.
2. If the row does not consist of all zeros then its first non-zero entry is a 1. This 1 is called a *leading 1*.
3. In any two successive rows, neither of which consists of all zeros, the leading 1 of the lower row is to the right of the leading 1 of the higher row.
4. If a column contains a leading 1 then all the other entries of that column are zero.

Example 2.5 The following matrices are all in the reduced row-echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 4\\ 0 & 1 & 0 & 7\\ 0 & 0 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & -7 & 10\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 & 0 & 1\\ 0 & 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

And the following matrices are not in the reduced row-echelon form:

$$\begin{pmatrix} 1 & 0 & 5\\ 0 & 1 & 3\\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 & 3 & 7\\ 0 & 1 & 6 & 2\\ 0 & 0 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 6 & 0\\ 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

■

■ **Example 2.6** Put the matrix $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & -5 & 5 \end{pmatrix}$ in reduced row-echelon form.

Solution:

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & -5 & 5 \end{pmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & -1 \\ 0 & -1 & -8 & -1 \end{pmatrix} \xrightarrow{-r_2}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -8 & -1 \end{pmatrix} \xrightarrow[r_2+r_3]{-r_2+r_1} \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -7 & 0 \end{pmatrix} \xrightarrow{(-1/7)r_3}$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{-r_2+r_1} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{-r_3+r_2}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

■

2.2 The solution of system of equations

2.2.1 Non-homogeneous system of equations

Let $AX = B$ be a system of n linear equations in n variables and the *augmented matrix* $D = (A : B)$.

(i) If $|A| \neq 0$ or $\rho(A) = \rho(D) = n$, then the system of equations is consistent and has a unique solution.

(ii) If $|A| = 0$ and $(adjA)B = O$ or $\rho(A) = \rho(D) < n$, then the system of equations is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and $(adjA)B \neq O$ or $\rho(A) \neq \rho(D)$, then the system of equations is inconsistent i.e., having no solution.

2.2.2 Homogeneous system of equations

Let $AX = B$ is a system of n linear equations in n variables.

(i) If $|A| \neq 0$ or $\rho(A) = n$, then it has only solution $X = 0$, is called the trivial solution.

(ii) If $|A|=0$ or $\rho(A)<n$, then the system has infinitely many solutions, called a non-trivial solution.

R The ***rank*** of a matrix is the number of Ones in the principal diagonal of the reduced matrix.

To find the solution of a system of linear equations:

Step I: Write the augmented matrix [A: B]

Step II: Reduce the augmented matrix to Echelon form using elementary row transformation.

Step III: Determine the rank of the coefficient matrix A and augmented matrix [A: B] by counting the number of non-zero rows in A and [A: B].

And write the final reduced matrix as a system of linear equations, then we can get the values of the unknowns (if the system is consistent), that is called the *Gauss-Jordan Elimination*.

Now, we show some examples:

Example 2.7 Use the row reduction algorithm to put the augmented matrix in reduced row-echelon form, then find the solution set for each of the following systems of linear equations:

(i) $\begin{aligned} x+y+2z&=9\\ 2x+4y-3z&=1\\ 3x+6y-5z&=0 \end{aligned}$

(ii) $\begin{aligned} x+y-z&=0\\ x-4y+2z&=0\\ 2x-3y+z&=0 \end{aligned}$

(iii) $\begin{aligned} x_1+5x_2+4x_3-13x_4&=3\\ 3x_1-x_2+2x_3+5x_4&=2\\ 2x_1+2x_2+3x_3-4x_4&=1 \end{aligned}$

Solution:

(i) We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 1 & 2 & 9\\ 2 & 4 & -3 & 1\\ 3 & 6 & -5 & 0 \end{pmatrix} \underset{-3r_1+r_3}{\overset{-2r_1+r_2}{\longrightarrow}} \begin{pmatrix} 1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ 0 & 3 & -11 & -27 \end{pmatrix} \overset{(1/2)r_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{pmatrix} \overset{-3r_2+r_3}{\rightarrow} \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{pmatrix} \overset{-2r_3}{\rightarrow}$$

$$\begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \overset{-r_2+r_1}{\rightarrow} \begin{pmatrix} 1 & 0 & 11/2 & 35/2 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \underset{(-11/2)r_3+r_1}{\overset{(7/2)r_3+r_2}{\rightarrow}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

So $\rho(A) = \rho(D) = 3$, then the system has a unique solution and this solution is

$$x = 1$$
$$y = 2$$
$$z = 3$$

(ii) We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & -4 & 2 & 0 \\ 2 & -3 & 1 & 0 \end{pmatrix} \xrightarrow[-r_1+r_2]{-2r_1+r_3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 0 \\ 0 & -5 & 3 & 0 \end{pmatrix} \xrightarrow{-r_2+r_3}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{(-1/5)r_2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-r_2+r_1}$$

$$\begin{pmatrix} 1 & 0 & -\frac{2}{5} & 0 \\ 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$x = \frac{2}{5}z,$$

$$y = \frac{3}{5}z,$$

also, $\rho(A) = \rho(D) < 3$. Hence the system has more than one solution.

(iii) We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 5 & 4 & -13 & 3 \\ 3 & -1 & 2 & 5 & 2 \\ 2 & 2 & 3 & -4 & 1 \end{pmatrix} \underset{-2r_1+r_3}{\overset{-3r_1+r_2}{\longrightarrow}} \begin{pmatrix} 1 & 5 & 4 & -13 & 3 \\ 0 & -16 & -10 & 44 & -7 \\ 0 & -8 & -5 & 22 & -5 \end{pmatrix} \overset{-2r_3+r_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 5 & 4 & -13 & 3 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & -8 & -5 & 22 & -5 \end{pmatrix}.$$

Since $\rho(A) \neq \rho(D)$, so the system has no solution. ■

■ **Example 2.8** Use the row reduction algorithm to put the augmented matrix in reduced row-echelon form, then find the solution set for the following system of linear equations:

$$\begin{aligned} x+2y-3z &= 0 \\ 3x-y+5z &= 0 \\ 4x+y-2z &= 0 \end{aligned}$$

Solution:

We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 3 & -1 & 5 & 0 \\ 4 & 1 & -2 & 0 \end{pmatrix} \underset{-4r_1+r_3}{\overset{-3r_1+r_2}{\longrightarrow}} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & -7 & 14 & 0 \\ 0 & -7 & 10 & 0 \end{pmatrix} \overset{(-1/7)r_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -7 & 10 & 0 \end{pmatrix} \overset{7r_2+r_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -4 & 0 \end{pmatrix} \overset{(-1/4)r_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \overset{-2r_2+r_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \underset{-r_3+r_1}{\overset{2r_3+r_2}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So $\rho(A) = \rho(D) = 2$, and the system has a unique solution i.e.,

$$\begin{aligned} x &= 0\,, \\ y &= 0\,, \\ z &= 0. \end{aligned}$$

■

■ **Example 2.9** Use the row reduction algorithm to put the augmented matrix in reduced row-echelon form, then find the solution set for the following system of linear equations:

$$\begin{aligned} x + 3y - 2z &= 0 \\ x - 8y + 8z &= 0 \\ 3x - 2y + 4z &= 0 \end{aligned}$$

Solution:
We reduce the augmented matrix as follows:

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 1 & -8 & 8 & 0 \\ 3 & -2 & 4 & 0 \end{pmatrix} \underset{-3r_1+r_3}{\overset{-r_1+r_2}{\longrightarrow}} \begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & -11 & 10 & 0 \\ 0 & -11 & 10 & 0 \end{pmatrix} \overset{-r_2+r_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & -11 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{(-1/11)r_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & -2 & 0 \\ 0 & 1 & -10/11 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{-3r_2+r_1}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 8/11 & 0 \\ 0 & 1 & -10/11 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So $\rho(A) = \rho(D) = 2 < 3$, then the system has more than one solution i.e.,

$$\begin{aligned} x + (8/11)z = 0 &\Rightarrow x = -(8/11)z \\ y - (10/11)z = 0 &\Rightarrow y = (10/11)z \end{aligned}$$

■

2.3 Exercises

Use the Row Reduction Algorithm to put the augmented matrix in reduced row-echelon form, then find the solution set for each of the following systems of linear equations:

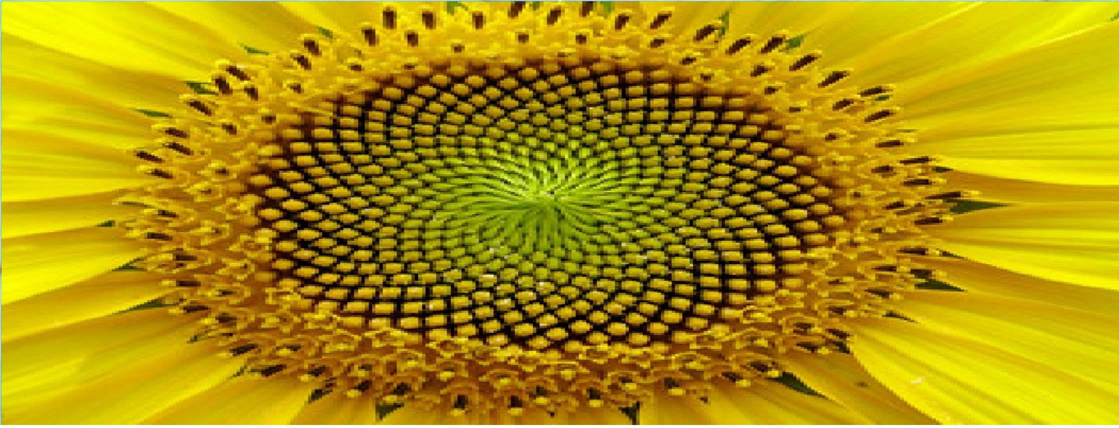
(i) $\begin{aligned} -2x_1 + x_2 - x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 13 \\ 3x_1 + x_3 &= -1 \end{aligned}$

(ii) $\begin{aligned} x + 3y - 2z &= 0 \\ x - 8y + 8z &= 0 \\ 3x - 2y + 4z &= 0 \end{aligned}$

(iii) $\begin{aligned} x_1 + 2x_2 + 6x_3 &= 4 \\ 2x_1 + 4x_2 + 4x_3 &= -1 \\ -x_1 - 2x_2 + 2x_3 &= 8 \end{aligned}$

(iv) $\begin{aligned} 2x - y + 3z &= 0 \\ 4x + 5y - z &= 0 \\ x + 3y - 2z &= 0 \end{aligned}$

(v) $\begin{aligned} -2x_1 + x_2 + x_3 &= 3 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 + 3x_2 + 3x_3 &= 0 \end{aligned}$



3. Vector Space

The notion of an "abstract vector space" evolved over many years and had many contributors. The idea crystallized with the work of the German mathematician H. G. Grassmann, who published a paper in 1862 in which he considered abstract systems of unspecified elements on which he defined formal operations of addition and scalar multiplication. In this chapter, we introduce vector (linear) space, subspace and give some examples. Also, the concepts of linear independence, dependence basis and dimension.

3.1 Linear space

Definition 3.1.1 Let V be an arbitrary nonempty vectors on which two operations are defined: addition, and multiplication by numbers called scalars. If the following axioms are satisfied by all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars k and m, then we call V a vector space and we call the vectors in V vectors.

1. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$.
4. There is an object $\mathbf{0}$ in V, called a zero vector for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a negative of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$.
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$.
9. $k(m\mathbf{u})=(km)(\mathbf{u})$.
10. $1\mathbf{u}=\mathbf{u}$.

■ **Example 3.1** Let $V=R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n -tuples; that is,

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\ldots,u_n)+(v_1,v_2,\ldots,v_n)=(u_1+v_1,u_2+v_2,\ldots,u_n+v_n)$$
$$k\mathbf{u}=(ku_1,ku_2,\ldots,ku_n)$$

The set $V=R^n$ is closed under addition and scalar multiplication and these operations satisfy Axioms 2 , 3,4,5,7,8,9, and 10. ■

■ **Example 3.2** Let V be the set of 2×2 matrices with real entries, and the operations on V define as follows,

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}\mu_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=\begin{bmatrix}u_{11}+v_{11} & u_{12}+v_{12}\\ u_{21}+v_{21} & u_{22}+v_{22}\end{bmatrix}$$
$$k\mathbf{u}=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\begin{bmatrix}ku_{11} & ku_{12}\\ ku_{21} & ku_{22}\end{bmatrix}$$

Show that $(V,+,.)$ be a vector space.

Solution:

The set V is closed under addition and scalar multiplication because the foregoing operations produce 2×2 matrices as the end result. Thus, it remains to confirm that Axioms 2,3,4,5,7,8,9, and 10 hold. Axiom 2 follows since

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}+\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\mathbf{v}+\mathbf{u}.$$

Similarly, Axioms $3,7,8$, and 9 are easy to verify. This leaves Axioms $4,5$, and 10 that remain to be verified.

To confirm that Axiom 4 is satisfied, we must find a 2×2 matrix 0 in V for which $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}$ for all 2×2 matrices in V. We can do this by taking

$$0=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With this definition,

$$\mathbf{0}+\mathbf{u}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\mathbf{u}$$

and similarly $\mathrm{u}+0=\mathrm{u}$. To verify that Axiom 5 holds we must show that each object $\mathbf{u}$ in V has a negative $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ and $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$. This can be done by defining the negative of u to be

$$-\mathbf{u}=\begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

With this definition,

$$\mathbf{u}+(-\mathbf{u})=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=0$$

and similarly $(-\mathbf{u})+\mathbf{u}=0$. Finally, Axiom 10 holds because

$$1\mathbf{u}=1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\mathbf{u}$$

■

■ **Example 3.3** Determine whether or not V it is a vector space? Give reasons for your assertion.

$V=\{(x,y):x,y\in R\}$
$(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$
$k(x,y)=(2kx,2ky)$.

Solution:

Since

$$1u=1(x,y)=(2x,2y)\neq u,$$

where $u=(x,y)$, then V is not a vector space. ■

■ **Example 3.4** Determine whether or not V it is a vector space? Give reasons for your assertion.

$V = \{(x,y) : x,y \in R\}$
$(x_1,y_1)+(x_2,y_2) = (x_1+y_2,y_1+x_2)$
$k(x,y) = (kx,ky)$.

Solution:

Since

$$(1,0)+(0,1) = (2,0),$$

and

$$(0,1)+(1,0) = (0,2),$$

this means $u+v \neq v+u$, then V is not a vector space. ■

■ **Example 3.5** Determine whether or not V it is a vector space? Give reasons for your assertion.

$V = \{(x,y) \in R^2 : x = 2\}$
$(x_1,y_1)+(x_2,y_2) = (x_1+x_2,y_1+y_2)$
$k(x,y,z) = (kx,y,z)$.

Solution:

Since

$$(2,0)+(2,1) = (4,1) \notin V,$$

thus V is not a vector space.

■

■ **Example 3.6** Determine whether or not V it is a vector space? Give reasons for your assertion.

$V = \{(x,y,z) : x,y,z \in R\}$
$(x_1,y_1,z_1)+(x_2,y_2,z_2) = (x_1+x_2,y_1+y_2,z_1+z_2)$
$k(x,y,z) = (kx,ky,z)$.

Solution: Since

$$\begin{aligned}(\lambda+\mu)u &= (\lambda+\mu)(x,y,z)\\ &= ((\lambda+\mu)x,y,z),\end{aligned}$$

and

$$\lambda u+\mu u=(\lambda x,y,z)+(\mu x,y,z)$$
$$=((\lambda+\mu)x,(\lambda+\mu)y,z)$$

i.e.,

$$(\lambda+\mu)u\neq\lambda u+\mu u.$$

Thus, V is not a vector space. ■

■ **Example 3.7** Determine whether or not V it is a vector space? Give reasons for your assertion.

$V=\{(x,y,z):x,y,z\in R\}$
$(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)$
$k(x,y,z)=(kx,1,kz)$.

Solution:

Since

$$1u=1(x,y,z)=(x,1,z)\neq u,$$

where $u=(x,y,z)$ and thus V is not a vector space. ■

■ **Example 3.8** Determine whether or not V it is a vector space? Give reasons for your assertion.

$V=\{(x,y,z):x,y,z\in R\}$
$(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_2,y_1+y_2,z_2)$
$k(x,y,z)=(kx,ky,kz)$.

Solution:

Since

$$(1,2,3)+(4,5,6)=(4,7,6)\ ,\ (4,5,6)+(1,2,3)=(1,7,3)$$

i.e.,

$$u+v\neq v+u.$$

and thus V is not a vector space. ■

■ **Example 3.9** Determine whether or not V it is a vector space? Give reasons for your assertion.

$V = \{(0,0,z) : z \in R\}$
$(0,0,z_1) + (0,0,z_2) = (0,0,z_1 + z_2)$
$k(0,0,z) = (0,0,kz)$.

Solution:

V is a vector space; all conditions and properties hold. ■

Theorem 3.1.1 Let V be a vector spaces and $u \in V$, $k \in K$, then

1. $0u = 0$.
2. $k0 = 0$.
3. $(-1)u = -u$.
4. $ku = 0 \Rightarrow u = 0 \vee k = 0$.

Proof. 1.

$$\begin{aligned} 0u &= (0+0)u \\ &= 0u + 0u \\ 0u + (-0u) &= [0u + 0u] + (-0u) \ (add\ -0u) \\ &= 0u + [0u + (-0u)] \\ 0 &= 0u + 0 = 0u. \end{aligned}$$

Another prove

$$\begin{aligned} 0 &= u + (-u) \\ &= 1u + (-1)u \\ &= [1 + (-1)]u \\ &= 0u. \end{aligned}$$

2.

$$\begin{aligned} k0 &= k(u + (-u)) \\ &= ku + k(-u) \\ &= ku + (-k)u \\ &= (k + (-k))u \\ &= 0u. \end{aligned}$$

3.

$$\begin{aligned} u+(-1)u &= 1u+(-1)u \\ &= [1+(-1)]u \\ &= 0u \\ &= 0. \end{aligned}$$

So $(-1)u = -u$.
4.let ku = 0 , k $\neq$0, then

$$\begin{aligned} 0 &= (1/k)0 \\ &= (1/k)(ku) \\ &= [(1/k)k]u \\ &= 1u \\ &= u. \end{aligned}$$

The second directions, let $ku = 0$, $u \neq 0$, then

$$ku = 0,\ 0u = 0 \Rightarrow k = 0.$$

■

3.2 Subspaces

Definition 3.2.1 Suppose that V is a vector space and W is a subset of V. If under the addition and scalar multiplication that is defined on V, W is also a vector space then we call W is a *subspace* of V.

Theorem 3.2.1 A nonempty subset W of a vector space V is a subspace of V if and only if the following two conditions:
(i) $u+v \in W$.
(ii) $ku \in W \forall u, v \in W, k \in K$.
are satisfied.

Proof. Suppose W be a subspace of V, the two conditions (i), (ii) are satisfied with the definition of a subspace.

Suppose the above two conditions (i) and (ii) are satisfied, we prove that W is a subspace of V as follow:

The properties (3),(4),(7),(8),(9),(10) are true simply based on the fact that W is a subset of V, we only need to verify (5),(6):

From the condition (ii) put $k = 0 \Rightarrow ou = 0 \in W \ \forall u \in W$ and put $k = -1 \Rightarrow (-1)u = -u \in W \ \forall u \in W$, therefore W is a subspace of V. ■

■ **Example 3.10** Let $W = \{(a,b,c) \in R^3 : b = 2a\}$. Is W a subspace of a vector space R^3?

Solution:

It is shown that $W \subset R^3$

(i) let $(a_1,b_1,c_1),(a_2,b_2,c_2) \in W \Rightarrow b_1 = 2a_1, b_2 = 2a_2$

$$\therefore (a_1,b_1,c_1)+(a_2,b_2,c_2) = (a_1+a_2, b_1+b_2, c_1+c_2) \in W;$$

since $b_1 + b_2 = 2(a_1 + a_2)$.

(ii) let $(a,b,c) \in W$, k scalar $\Rightarrow b = 2a$

$$\therefore k(a,b,c) = (ka,kb,kc) \in W;$$

since $kb = k(2a)$.

$\therefore$ W is a subspace of a vector space R^3. ■

■ **Example 3.11** Let $W = \{(a,b,c) \in R^3 : ab = 0\}$. Is W a subspace of a vector space R^3?

Solution:

W is not a subspace of a vector space R^3 for $(1,0,1),(0,1,1) \in W$ but

$$(1,0,1)+(0,1,1) = (1,1,2) \notin W.$$

■

Theorem 3.2.2 If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Proof. Let W be the intersection of the subspaces $W_1, W_2, \ldots, W_r$. This set is not empty because each of these subspaces contains the zero vector of V, and hence so does their intersection. Thus, it remains to show that W is closed under addition and scalar multiplication. Now, we assume that **u** and v are vectors in W. Then

$$\begin{aligned}
(1)\ u,v \in W &\Rightarrow u,v \in W_1 \wedge u,v \in W_2 \wedge \ldots \wedge u,v \in W_r \\
&\Rightarrow u+v \in W_1 \wedge u+v \in W_2 \wedge \ldots \wedge u+v \in W_r \\
&\Rightarrow u+v \in W. \\
(2)\ u \in W,\ k \in \mathbb{K} &\Rightarrow u \in W_1 \wedge u \in W_2 \wedge \ldots \wedge u \in W_r,\ k \in \mathbb{K} \\
&\Rightarrow ku \in W_1 \wedge ku \in W_2 \wedge \ldots \wedge ku \in W_r \\
&\Rightarrow ku \in W.
\end{aligned}$$

■

3.3 Exercises

1- Let $W = \{(a,b,c) \in R^3 : b = a^2\}$. Is W a subspace of a vector space R^3?
2- Let $W = \{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} : a,b \in R\}$. Is W a subspace of a vector space $M_{2\times2}(R)$?
3- Let $W = \{A \in M_{2\times2}(R) \ : \ |A| = 0\}$. Is W a subspace of a vector space $M_{2\times2}(R)$?
4- The set of all real numbers with the standard operations of addition and multiplication.
5- The set of all pairs of real numbers of the form $(x,0)$ with the standard operations on R^2.
6- The set of all pairs of real numbers of the form (x,y), where $x \geq 0$, with the standard operations on R^2.

3.4 Linear Combinations

Definition 3.4.1 If u is a vector in a vector space V, then u is said to be a *linear combination* of the vectors $v_1, v_2, \ldots, v_n$ in V if u can be expressed in the form

$$u = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars. These scalars are called the *coefficients* of the linear combination.

■ **Example 3.12** Verify that the vector $u = (9,2,7)$ is a linear combination of the vectors $v_1 = (1,2,-1)$, $v_2 = (6,4,2)$, but the vector $w = (4,-1,8)$ is not a linear combination of them.

Solution:

Let $u = c_1 v_1 + c_2 v_2$, then

$$(9,2,7) = c_1(1,2,-1) + c_2(6,4,2),$$

this mean

$$\begin{aligned} 9 &= c_1 + 6c_2 \\ 2 &= 2c_1 + 4c_2 \\ 7 &= -c_1 + 2c_2 \end{aligned}$$

We reduce the augmented matrix

$$\begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix},$$

as follows:

$$\begin{pmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{pmatrix} \underset{r_1+r_3}{\overset{-2r_1+r_2}{\longrightarrow}} \begin{pmatrix} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{pmatrix} \overset{(-1/8)r_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 8 & 16 \end{pmatrix} \underset{-6r_2+r_1}{\overset{-8r_2+r_3}{\longrightarrow}} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $c_1 = -3$ and $c_2 = 2$, therefor $u = -3v_1 + 2v_2$.

Similarly, let Let $w = c_1v_1 + c_2v_2$, then

$$(4,-1,8) = c_1(1,2,-1) + c_2(6,4,2),$$

this mean

$$\begin{aligned} 4 &= c_1 + 6c_2 \\ -1 &= 2c_1 + 4c_2 \\ 8 &= -c_1 + 2c_2 \end{aligned}$$

We reduce the augmented matrix

$$\begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{pmatrix},$$

as follows:

$$\begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 8 \end{pmatrix} \underset{r_1+r_3}{\overset{-2r_1+r_2}{\rightarrow}} \begin{pmatrix} 1 & 6 & 4 \\ 0 & -8 & -9 \\ 0 & 8 & 12 \end{pmatrix} \overset{-1r_2}{\rightarrow}$$

$$\begin{pmatrix} 1 & 6 & 4 \\ 0 & 8 & 9 \\ 0 & 8 & 12 \end{pmatrix},$$

this system inconsistent so w is not linear combination of the vectors v_1, v_2.

■

Theorem 3.4.1 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

(a) The set W of all possible linear combinations of the vectors in S is a subspace of V.

(b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

Proof. (a) Let W be the set of all possible linear combinations of the vectors in S. We must show that W is closed under addition and scalar multiplication. To prove closure under addition, let

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r \text{ and } \mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r$$

be two vectors in W. It follows that their sum can be written as

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\,\mathbf{w}_1 + (c_2 + k_2)\,\mathbf{w}_2 + \cdots + (c_r + k_r)\,\mathbf{w}_r$$

which is a linear combination of the vectors in S. Thus, W is closed under addition. We leave it for you to prove that W is also closed under scalar multiplication and hence is a subspace of V

(b) Let W' be any subspace of V that contains all of the vectors in S. Since W' is closed under addition and scalar multiplication, it contains all linear combinations of the vectors in S and hence contains W. ■

3.5 Exercises

1- write the vector $(1,-2,5)$as a linear combination of the vectors $v_1 = (1,1,1)$, $v_2 = (1,2,3)$, $v_3 = (2,-1,1)$

2- Verify that the vector $(0\,,\,-3\,,\,1)$is a linear combination of the rows vectors

of a matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix}$, but is not a linear combination of its columns vectors.

3.6 Spanning Sets

Definition 3.6.1 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a non-empty set of vectors in a vector space V, then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V generated by S, and we say that the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ are **Span** W. We denote this subspace as

$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S),$$

where $\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} = \{c_1 w_1 + c_2 w_2 + ... + c_r w_r : c_1, c_2, ..., c_r \in \mathbb{R}\}$.

■ **Example 3.13** Show that the standard unit vectors is Span R^n

Solution:

Recall that the standard unit vectors in R^n are

$$\mathbf{e}_1 = (1,0,0,\ldots,0), \quad \mathbf{e}_2 = (0,1,0,\ldots,0),\ldots, \quad \mathbf{e}_n = (0,0,0,\ldots,1)$$

These vectors span R^n since every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. ■

■ **Example 3.14** Show that the vectors

$$\mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0), \quad \mathbf{k} = (0,0,1)$$

span R^3?

Solution:

Since every vector $\mathbf{v} = (a,b,c)$ in this space can be expressed as

$$\mathbf{v} = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

■

■ **Example 3.15** Let $S = v_1, v_2, v_3$ be a set of vector in R^3. Show that S span R^3 where $v_1 = (1,1,2)$, $v_2 = (1,0,2)$, $v_3 = (1,1,0)$.

Soluation:

Let $v = (x,y,z) \in R^3$, then

$$v = c_1 v_l + c_2 v_2 + c_3 v_3,$$
$$(x,y,z) = c_1(l,1,2) + c_2(l,0,2) + c_3(l,l,0).$$

This lead to

$$x = c_1 + c_2 + c_3$$
$$y = c_1 + c_3$$
$$z = 2c_1 + 2c_2$$

And this system has a solution

$$c_l = x + y + z/2,$$
$$c_2 = x - y,$$
$$c_3 = x - z/2.$$

Thus, S span R^3. ■

■ **Example 3.16** Determent $S = \{v_l, v_2, v_3\}$ is span R^3 yeas or no, where $v_l = (l,l,2), v_2 = (l,0,l), v_3 = (2,l,3)$.

Solution: Let $u = (x,y,z)$ in R^3, then

$$u = c_l v_l + c_2 v_2 + c_3 v_3$$

i.e.,

$$(x,y,z) = c_l(l,l,2) + c_2(l,0,l) + c_3(2,l,3).$$

$$x = c_l + c_2 + 2c_3$$
$$y = c_l + c_3$$
$$z = 2c_l + c_2 + 3c_3$$

Since the coefficient matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ has

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} \equiv \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix} = 0.$$

So the above system has no solution and then S is not spanned R^3.

■

■ **Example 3.17** Consider the vectors $p_1 = 1 + x + 4x^2$ and $p_2 = 1 + 5x + x^2$ in $\mathbf{P}_2$. Determine whether p_1 and p_2 lie in span $\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Solution:

For p_1, we want to determine if s and t exist such that

$$p_1 = s\left(1 + 2x - x^2\right) + t\left(3 + 5x + 2x^2\right)$$

Equating coefficients of powers of x (where $x^0 = 1$) gives

$$1 = s + 3t, \quad 1 = 2s + 5t, \text{ and } \quad 4 = -s + 2t.$$

These equations have the solution $s = -2$ and $t = 1$, so p_1 is indeed in span $\{1 + 2x - x^2, 3 + 5x + 2x^2\}$.

Turning to $p_2 = 1 + 5x + x^2$, we are looking for s and t such that $p_2 = s\left(1 + 2x - x^2\right) + t(3 + 5x + 2x^2)$. Again equating coefficients of powers of x gives equations $1 = s + 3t, 5 = 2s + 5t$, and $1 = -s + 2t$. But in this case there is no solution, so p_2 is not in span $\{1 + 2x - x^2, 3 + 5x + 2x^2\}$. ■

Theorem 3.6.1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

■ **Example 3.18** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in a vector space V. Show that

$$\operatorname{span}\{\mathbf{u}, \mathbf{v}\} = \operatorname{span}\{\mathbf{u} + 2\mathbf{v}, \mathbf{u} - \mathbf{v}\}$$

Solution:

We have $\operatorname{span}\{\mathbf{u} + 2\mathbf{v}, \mathbf{u} - \mathbf{v}\} \subseteq \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$ because both $\mathbf{u} + 2\mathbf{v}$ and $\mathrm{u} - \mathrm{v}$ lie in $\operatorname{span}\{\mathrm{u}, \mathrm{v}\}$. On the other hand,

$$\mathbf{u} = \frac{1}{3}(\mathbf{u} + 2\mathbf{v}) + \frac{2}{3}(\mathbf{u} - \mathbf{v}) \quad \text{and} \quad \mathbf{v} = \frac{1}{3}(\mathbf{u} + 2\mathbf{v}) - \frac{1}{3}(\mathbf{u} - \mathbf{v})$$

■

3.7 Exercise

1. Show that $\mathbb{R}^3$ is spanned by

$$\{(1,0,1),(1,1,0),(0,1,1)\}.$$

2. Show that $\mathbf{P}_2$ is spanned by

$$\{1+2x^2, 3x, 1+x\}.$$

3. Show that $\mathbf{M}_{22}$ is spanned by

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right\}.$$

3.8 Linear independence & Linear dependence

Definition 3.8.1 Let Vbe a vector space over a field K. The vectors $v_1, v_2, ..., v_n \in V$ are said to be *linearly dependent* over K if there exist scalars $c_1, c_2, ..., c_n \in K$, not all of them 0, such that

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0.$$

Otherwise, the vectors are said to be *linearly independent* over$K$; i.e.,

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0 \Rightarrow c_1 = 0,\ c_2 = 0,\ ...\ ,\ c_n = 0.$$

A set $S = \{v_1, v_2, ..., v_n\}$of vectors is linearly dependent if the vectors $v_1, v_2, ..., v_n$ are linearly dependent, otherwise S is linearly independent.

The trivial linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the one with every coefficient zero:

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$$

This is obviously one way of expressing $\mathbf{0}$ as a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and they are linearly independent when it is the only way.

■ **Example 3.19** Show that the set $S_1 = \{v_1, v_2, v_3\}$; $v_1 = (2, -1, 0, 3)$, $v_2 = (1, 2, 5, -1)$, $v_3 = (7, -1, 5, 8)$ is linearly dependent inR^4.

Solution:

Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$,

$$\therefore c_1(2, -1, 0, 3) + c_2(1, 2, 5, -1) + c_3(7, -1, 5, 8) = 0$$

$$\therefore \begin{aligned} 2c_1 + c_2 + 7c_3 &= 0, \\ -c_1 + 2c_2 - c_3 &= 0, \\ 5c_2 + 5c_3 &= 0, \\ 3c_1 - c_2 + 8c_3 &= 0. \end{aligned}$$

We reduce the augmented matrix:

$$\begin{pmatrix} 2 & 1 & 7 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 5 & 5 & 0 \\ 3 & -1 & 8 & 0 \end{pmatrix} \overset{?}{\rightarrow} \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{(verify that?)}$$

$$\therefore c_1 = -3c_3 \ , \ c_2 = -\,c_3.$$

Hence the system of these equations has more than one solution; i.e.,

$$c_3 = 1 \Rightarrow c_1 = -3\,, c_2 = -1,$$

so S_1 is linearly dependent. ■

■ **Example 3.20** Show that the set $S_2 = \{v_1, v_2, v_3\}$; $v_1 = (1,0,1,2)$, $v_2 = (0,1,1,2)$, $v_3 = (1,1,1,3)$ is linearly independent inR^4.

Solution:

Let $c_1v_1 + c_2v_2 + c_3v_3 = 0$,

$c_1(1,0,1,2) + c_2(0,1,1,2) + c_3(1,1,1,3) = 0$

$$\therefore \begin{aligned} c_1 + c_3 &= 0\,, \\ c_2 + c_3 &= 0\,, \\ c_1 + c_2 + c_3 &= 0\,, \\ 2c_1 + 2c_2 + 3c_3 &= 0. \end{aligned}$$

We reduce the augmented matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 \end{pmatrix} \overset{?}{\rightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{(verify that?)}$$

$$\therefore c_1 = c_2 = c_3 = 0$$

So S_2 is linearly independent. ■

■ **Example 3.21** Show that $\{\sin x, \cos x\}$ is independent in the vector space $\mathbf{F}[0,2\pi]$ of functions defined on the interval $[0,2\pi]$.

Solution:

Suppose that a linear combination of these functions vanishes.

$$s_1(\sin x) + s_2(\cos x) = 0$$

This must hold for all values of x in $[0, 2\pi]$ (by the definition of equality in $\mathbf{F}[0, 2\pi]$). Taking $x = 0$ yields $s_2 = 0$ (because $\sin 0 = 0$ and $\cos 0 = 1$). Similarly, $s_1 = 0$ follows from taking $x = \frac{\pi}{2}$ (because $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$).

■

■ **Example 3.22** Suppose that $\{\mathbf{u}, \mathbf{v}\}$ is an independent set in a vector space V. Show that $\{\mathbf{u}+2\mathbf{v}, \mathbf{u}-3\mathbf{v}\}$ is also independent.

Solution:

Suppose a linear combination of $\mathbf{u}+2\mathbf{v}$ and $\mathbf{u}-3\mathbf{v}$ vanishes:

$$s(\mathbf{u}+2\mathbf{v})+t(\mathbf{u}-3\mathbf{v}) = \mathbf{0}$$

We must deduce that $s = t = 0$. Collecting terms involving $\mathbf{u}$ and $\mathbf{v}$ gives

$$(s+t)\mathbf{u}+(2s-3t)\mathbf{v} = \mathbf{0}$$

Because $\{\mathbf{u}, \mathbf{v}\}$ is independent, this yields linear equations $s+t = 0$ and $2s-3t = 0$. The only solution is $s = t = 0$. ■

■ **Example 3.23** Let V denote a vector space, then prove the following
1. If $\mathbf{v} \neq \mathbf{0}$ in V, then $\{\mathbf{v}\}$ is an independent set.
2. No independent set of vectors in V can contain the zero vector.

Solution:

1. Let $t\mathbf{v} = \mathbf{0}, t$ in $\mathbb{R}$. If $t \neq 0$, then $\mathbf{v} = 1\mathbf{v} = \frac{1}{t}(t\mathbf{v}) = \frac{1}{t}\mathbf{0} = \mathbf{0}$, contrary to assumption. So $t = 0$.
2. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is independent and (say) $\mathbf{v}_2 = \mathbf{0}$, then

$$0\mathbf{v}_1 + 1\mathbf{v}_2 + \ldots + 0\mathbf{v}_k = \mathbf{0},$$

is a nontrivial linear combination that vanishes, contrary to the independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. ■

Theorem 3.8.1 (a) *A* finite set that contains $\mathbf{0}$ is linearly dependent.
(b) A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.

(c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Proof. We will prove part (a) and leave the rest as exercises.
(a) For any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{0}\}$ is linearly depen- dent since the equation

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_r + 1(\mathbf{0}) = \mathbf{0}$$

expresses $\mathbf{0}$ as a linear combination of the vectors in S with coefficients that are not all zero.

■

3.9 Exercise

(1) In each part, determine whether the vectors are linearly independent or are linearly dependent in R^3.
(a) (-3,0,4),(5,-1,2),(1,1,3).
(b) (-2,0,1),(3,2,5),(6,-1,1),(7,0,-2).
(2) In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.
(a) (3,8,7,-3),(1,5,3,-1),(2,-1,2,6),(4,2,6,4).
(b) (3,0,-3,6),(0,2,3,1),(0,-2,-2,0),(-2,1,2,1).

3.10 Basis & Dimension

Definition 3.10.1 A set $S = \{v_1, v_2, ..., v_n\}$ of vectors is *a basis* of a vector space V if the following hold:
(i) S span V, and
(ii) S is linearly independent.

Definition 3.10.2 The number of the vectors in the basis is called the *dimension* of a vector space V, denoted by $\dim V$.

■ **Example 3.24** Verify that the set of vectors

$$S_1 = \{(1,2,1),(2,9,0),(3,3,4)\}$$

is a basis of R^3.
Solution:
The matrix which columns are $(1,2,1),(2,9,0),(3,3,4)$ respectively be:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{pmatrix},$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 9 & -5 \\ 1 & 0 & 0 \end{vmatrix} = -1 \neq 0.$$

Then, A is invertible, therefore, S_1 is a basis of R^3.

■

■ **Example 3.25** Is$S_2 = \{(1,1,2),(1,0,1),(2,1,3)\}$ it a basis of R^3?
Solution:
The matrix which columns are $(1,1,2),(1,0,1),(2,1,3)$ respectively be:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix},$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Then, A is not invertible, therefore, S_2 is not a basis of R^3. ■

■ **Example 3.26** Determine the basis and the dimension of the column space of a matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{pmatrix}.$$

Solution:
We reduce the transpose of a matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{pmatrix},$$

as follows:

$$A^T = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 1 & 5 & 4 \\ 1 & 1 & -4 \end{pmatrix} \underset{-r_1+r_4}{\overset{-r_1+r_3}{\longrightarrow}} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{pmatrix} \overset{(1/2)r_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{pmatrix} \underset{-3r_2+r_1}{\overset{-2r_2+r_3,\, 2r_2+r_4}{\longrightarrow}} \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $\{(1,0,-6),(0,1,2)\}$ is a basis of $C(A)$, and $\dim C(A) = 2$. ■

■ **Example 3.27** Determine a set of vectors in R^3 to be a basis of the null space of a matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 5 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & 1 & 5 & 0 \end{pmatrix} \underset{-3r_1+r_3}{\overset{-2r_1+r_2}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \overset{-r_2+r_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So

$$\therefore \begin{array}{l} x_1 + 2x_3 = 0, \\ x_2 - x_3 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -2x_3, \\ x_2 = x_3 \end{array}$$

$\therefore$ thus,

$$N(A) = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \},$$

let c_1, c_2 *scalars*

$$\therefore c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow c_1 = c_2 = 0.$$

Therefore, $\{(-1,1,0),(-1,0,1)\}$ be a basis of $N(A)$. ■

■ **Example 3.28** Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.
Solution:
We must show that the matrices are linearly independent and span M_{22}. To prove linear independence we must show that the equation

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \mathbf{0}$$

has only the trivial solution, where $\mathbf{0}$ is the 2×2 zero matrix; and to prove that the matrices span M_{22} we must show that every 2×2 matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

can be expressed as

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = B$$

The matrix forms of Equations (4) and (5) are

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

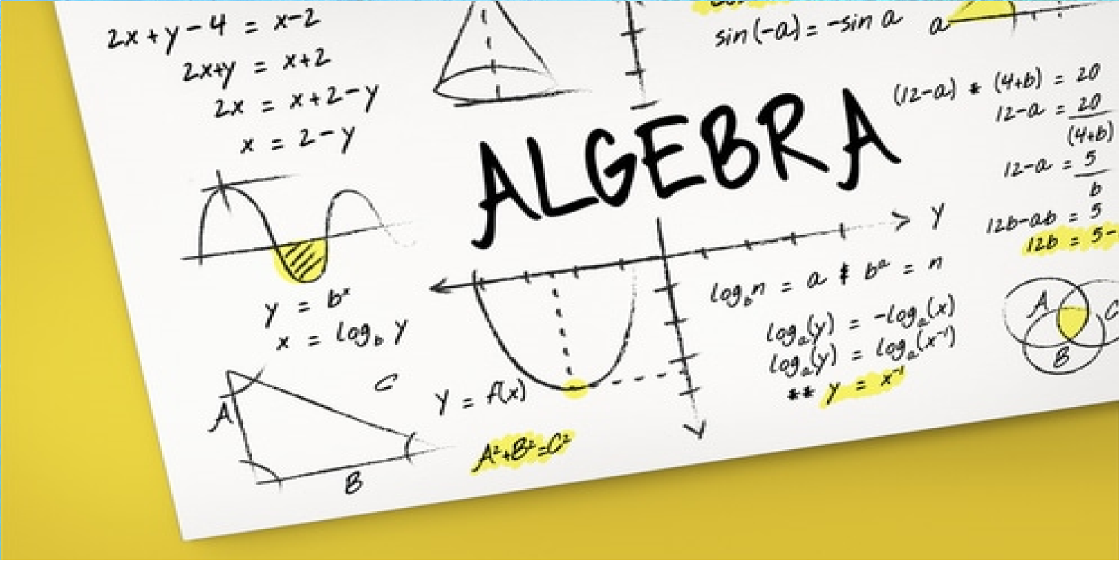
and

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

■

3.11 Exercise

1. Verify that $\{(1,2,1),(2,9,0),(3,3,4)\}$ is linearly independent inR^3.
2. Verify that $\{(1,3,-1),(2,0,1),(1,-1,1)\}$ is linearly dependent inR^3.
3. True or False (explain): If $\{v_1,v_2,v_3\}$ is linearly independent, then so is
$$\{v_1,v_1+v_2,v_1+v_2+v_3\}.$$
4. Let $S=\{v_1,v_2,v_3,...,v_n\}$ be a set of nonzero vectors such that $v_i\cdot v_j=0\ \forall\ i\neq j$. Verify that S is linearly independent.
5. Show that the vectors $v_1,v_2,...,v_n$ are linearly dependent iff one of them is a linear combination of the others.
6. Verify that the set of vectors $S=\{v_1,v_2,v_3,v_4\}$; $v_1=(1,0,1,0)$, $v_2=(0,1,-1,2)$, $v_3=(0,2,2,1)$, $v_4=(1,0,0,1)$ is a basis of R^4.
7. Determine a basis of the space of the solution of the system of linear equations:
$$\begin{array}{l} 2x_1+2x_2-x_3+x_5=0 \\ -x_1-x_2+2x_3-3x_4+x_5=0 \\ x_1+x_2-2x_3-x_5=0 \\ x_3+x_4+x_5=0 \end{array}.$$



4. Euclidean n-Space

Euclidean space is the fundamental space of classical geometry. Originally it was the three-dimensional space of Euclidean geometry, but in modern mathematics, there are Euclidean spaces of any non-negative integer dimension, including the three-dimensional space and the Euclidean plane (dimension two).

In this chapter, we define and study the notions norm, dot product, and distance in R^n.

4.1 Euclidean n-Space

Definition 4.1.1 An ordered n-tuple is an ordered sequence of n real numbers $(x_1, x_2, \ldots, x_n)$.
If n = 2 we have an ordered pair.
If n = 3 we have an ordered triple.

n-tuples can either represent points or vectors. We use the convention that

$x = (x_1, x_2, \ldots, x_n)$, etc.
The set of all possible n-tuples for a fixed n denoted R^ni.e.,

$$\mathrm{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathrm{R}\, for\, each\, i\}\,.$$

Definition 4.1.2 Let $u, v \in \mathrm{R}^n$ be two vectors, then the addition

$$\begin{aligned} u + v &= (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n). \end{aligned}$$

■ **Example 4.1** Let $u = (1,2), v = (0,3) \in \mathrm{R}^2$ be two vectors, then the addition

$$\begin{aligned} u + v &= (1,2) + (0,3) \\ &= (1,5)\,. \end{aligned}$$

■

Scalar multiplication

Definition 4.1.3 Let $u \in \mathrm{R}^n$ be a vector and $\alpha \in K$, then

$$\begin{aligned} \alpha u &= \alpha\,(u_1, u_2, \ldots, u_n) \\ &= (\alpha u_1, \alpha u_2, \ldots, \alpha u_n) \end{aligned}$$

Theorem 4.1.1 Given vectors u, v, w $\in \mathrm{R}^n$ and a scalars k, l$\in$ R, then:

1. $u + v = v + u$.
2. $(u + v) + w = u + (v + w)$.
3. $u + O = O + u = u$.
4. $u + (-u) = (-u) + u = O$.
5. $k(lu) = (kl)u$.
6. $k(u + v) = ku + kv$.
7. $(k + l)u = ku + lu$.

Proof. We will prove (1), (2) and leave the remaining results to be proven in the exercises.

1- Let $u, v \in \mathrm{R}^n$ be two vectors, then

$$\begin{aligned} u+v &= (u_1, u_2, ..., u_n) + (v_1, v_2, ..., v_n) \\ &= (u_1+v_1, u_2+v_2, ..., u_n+v_n) \\ &= (v_1+u_1, v_2+u_2, ..., v_n+u_n) \\ &= v+u \end{aligned}$$

2- Let $u, v, w \in \mathrm{R}^n$ be two vectors, then

$$\begin{aligned} (u+v)+w &= [(u_1, u_2, ..., u_n) + (v_1, v_2, ..., v_n)] + (w_1, w_2, ..., w_n) \\ &= (u_1+v_1, u_2+v_2, ..., u_n+v_n) + (w_1, w_2, ..., w_n) \\ &= (u_1+v_1+w_1, u_2+v_2+w_2, ..., u_n+v_n+w_n) \\ &= (u_1, u_2, ..., u_n) + (v_1+w_1, v_2+w_2, ..., v_n+w_n) \\ &= u+(v+w). \end{aligned}$$

■

4.2 Vector scalar product

Definition 4.2.1 Let $u, v \in \mathrm{R}^n$be two vectors, then the scalar product

$$\begin{aligned} u.v &= (u_1, u_2, \ldots, u_n) \,.\, (v_1, v_2, \ldots, v_n) \\ &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n. \end{aligned}$$

■ **Example 4.2** Let $u = (1, 2), v = (0, 3) \in \mathrm{R}^2$ be two vectors, then the addition

$$\begin{aligned} u.v &= (1, 2) \,.\, (0, 3) \\ &= 0 + 6 = 6. \end{aligned}$$

■

Now, we show properties of dot product

Theorem 4.2.1 Given vectors u, v, w $\in \mathrm{R}^n$ and a scalars k, l$\in$ R, then:

1. $u.v = v.u$.
2. $(u + v).w = u.w + v.w$.
3. $(ku).v = k(u.v)$.

Proof. We will prove (1), (2) and leave the remaining results to be proven in the exercises.

1- Let $u, v \in \mathrm{R}^n$ be two vectors, then

$$\begin{aligned} u.v &= (u_1, u_2, \ldots, u_n) \,.\, (v_1, v_2, \ldots, v_n) \\ &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \\ &= v_1 u_1 + v_2 u_2 + \ldots + v_n u_n \\ &= v.u. \end{aligned}$$

2- Let $u, v, w \in \mathrm{R}^n$ be two vectors, then

$$\begin{aligned} (u + v)\,.w &= [(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)].\,(w_1, w_2, \ldots, w_n) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \,.\, (w_1, w_2, \ldots, w_n) \\ &= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2, \ldots + u_n w_n + v_n w_n \\ &= (u_1, u_2, \ldots, u_n) \,.\, (w_1, w_2, \ldots, w_n) + (v_1, v_2, \ldots, v_n) \,.\, (w_1, w_2, \ldots, w_n) \\ &= u.w + v.w. \end{aligned}$$

■

Theorem 4.2.2 For any vector $v \in \mathbb{R}^n$ $v.v \geq 0$ and $v.v = 0$ if and only if $v = O$.

Proof. Let $v \in \mathrm{R}^n$ be two vectors, then

$$\begin{aligned} v.v &= (v_1, v_2, \ldots, v_n) \,.\, (v_1, v_2, \ldots, v_n) \\ &= v_1 v_1 + v_2 v_2 + \ldots + v_n v_n \\ &= {v^2}_1 + {v^2}_2 + \ldots + {v^2}_n \, (i.e., \; v.v \geq 0). \end{aligned}$$

Also if $v.v = 0 \Leftrightarrow v = 0$. ■

4.2.1 Length and the Distance between two Vectors

Definition 4.2.2 The dot product of a vector $u \in \mathrm{R}^n$with itself is the square of the length or magnitude or norm of u i.e.,

$$\| u \| = \sqrt{u.u}.$$

Theorem 4.2.3 Let $u, v \in \mathrm{R}^n$, then

$$\begin{aligned} &(i) \; \| u \| \geq 0. \\ &(ii) \; \| u \| = 0 \; if \, and \, only \, if \; u = 0. \\ &(iii) \; \| ku \| = k \| u \| \,. \\ &(iv) \; \| u + v \| \leq \| u \| + \| v \| \,. \text{"}Triangle \, Inequality\text{"} \end{aligned}$$

Proof. Let $u, v \in \mathrm{R}^n$ be two vectors, then

(i) $\| u \| = (u.u)^{\frac{1}{2}} \geq 0$.

(ii) If $\| u \| = 0$, then

$$\| u \| = (u.u)^{\frac{1}{2}} = 0 \Leftrightarrow u = 0.$$

(iii)

$$\begin{aligned} \| ku \| &= (ku.ku)^{\frac{1}{2}} \\ &= k \, (u.u)^{\frac{1}{2}} \\ &= k \| u \| \,. \end{aligned}$$

(iv)

$$\begin{aligned}\| u+v \|^2 &= (u+v).(u+v)\\ &= u.u+u.v+v.u+v.v\\ &= \| u \|^2 + u.v+v.u+ \| v \|^2\\ &= \| u \|^2 + 2u.v+ \| v \|^2\\ &\leq \| u \| + \| v \|^2 .\end{aligned}$$

Thus,

$$\| u+v \| \leq \| u \| + \| v \| .$$

■

■ **Example 4.3** Find the magnitude of the vector $u = (1,2,3)$.

Solution:

$$u.u = (1,2,3).(1,2,3) = 14.$$

Thus, $\| u \| = \sqrt{14}$. ■

Definition 4.2.3 Let $u, v \in \mathrm{R}^n$ be two vectors, then the distance between u and v define as follow

$$d(u,v) = \| u-v \| .$$

Theorem 4.2.4 Let $u, v \in \mathrm{R}^n$, then
$(i) d(u,v) \geq 0.$
$(ii) d(u,v) = 0 \; if \; and \; only \; if \; u = v.$
$(iii) d(u,v) = d(v,u).$
$(iv) d(u,w) \leq d(u,v) + d(v,w).$

Proof. Let $u, v \in \mathrm{R}^n$ be two vectors, then

(i)

$$\begin{aligned}d(u,v) &\Leftrightarrow \| u-v \| = 0\\ &= [(u-v).(u-v)]^{\frac{1}{2}}\\ &\geq 0.\end{aligned}$$

(ii)

$$\begin{aligned} d(u,v) = 0 = \parallel u - v \parallel &\Leftrightarrow [(u-v)\,.\,(u-v)]^{\frac{1}{2}} = 0 \\ &\Leftrightarrow (u-v)\,.\,(u-v) = 0 \\ &\Leftrightarrow (u-v) = 0 \\ &\Leftrightarrow u = v. \end{aligned}$$

(iii)

$$\begin{aligned} d(u,v) &= \parallel u - v \parallel \\ &= \parallel (-1)\,(v-u) \parallel \\ &= |(-1)| \parallel (v-u) \parallel \\ &= \parallel (v-u) \parallel \\ &= d(v,u). \end{aligned}$$

(iv)

$$\begin{aligned} d(u,w) &= \parallel u - w \parallel \\ &= \parallel u - v + v - w \parallel \\ &\leq \parallel u - v \parallel + \parallel v - w \parallel \\ &\leq d(u,v) + d(v,w). \end{aligned}$$

■

4.3 Orthogonality

Definition 4.3.1 Let $u, v \in \mathbb{R}^n$ be two vectors, then u, v are orthogonal if and only if $u.v = 0$.

■ **Example 4.4** Show that $u = (1,2,3)$ is orthogonal to $v = (3,0,-1)$.

Solution:

since

$$u.v = (1,2,3).(3,0,-1) = 0,$$

so u is orthogonal to v. ■

Theorem 4.3.1 Let $u, v \in \mathbb{R}^n$ be orthogonal, then

$$\| u+v \|^2 = \| u \|^2 + \| v \|^2 .$$

Proof. Let $u, v \in \mathbb{R}^n$ be orthogonal, then

$$\begin{aligned} \| u+v \|^2 &= (u+v).(u+v) \\ &= u.u + u.v + v.u + v.v \\ &= u.u + v.v \ \ (since\, u.v = v.u = 0) \\ &= \| u \|^2 + \| v \|^2 . \end{aligned}$$

■

4.3.1 The Angle Between Two Vectors

Theorem 4.3.2 Given two vectors u and v

$$u.v = \| u \| \| v \| \, cos\theta,$$

where θ is the angle between the two vectors.

■ **Example 4.5** Find the angle between $\mathbf{u} = (1,0,1)$ and $\mathbf{v} = (1,1,0)$.

Solution:

$$\mathbf{u}\cdot\mathbf{v} = (1,0,1)\cdot(1,1,0) = 1+0+0 = 1.$$
$$\|\mathbf{u}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}.$$
$$\|\mathbf{v}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}.$$
$$\therefore \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

So $\theta = \frac{\pi}{3}$. ∎

4.4 Exercises

A- In Exercises $1-6$, find the length of the vector.

1. $\mathbf{v}=(4,3)$.
2. $\mathbf{v}=(0,1)$.
3. $\mathbf{v}=(1,2,2)$.
4. $\mathbf{v}=(2,0,6)$.
5. $\mathbf{v}=(2,0,-5,5)$.
6. $\mathbf{v}=(2,-4,5,-1,1)$.

B-In Exercises $7-12$, find

$$(a)\|\mathbf{u}\|,(b)\|\mathbf{v}\|,\ and\ (\mathrm{c})\|\mathbf{u}+\mathbf{v}\|.$$

7. $\mathbf{u}=\left(-1,\frac{1}{4}\right),\quad \mathbf{v}=\left(4,-\frac{1}{8}\right)$
8. $\mathbf{u}=\left(1,\frac{1}{2}\right),\quad \mathbf{v}=\left(2,-\frac{1}{2}\right)$
9. $\mathbf{u}=(0,4,3),\quad \mathbf{v}=(1,-2,1)$
10. $\mathbf{u}=(1,2,1),\quad \mathbf{v}=(0,2,-2)$
11. $\mathbf{u}=(0,1,-1,2),\quad \mathbf{v}=(1,1,3,0)$
12. $\mathbf{u}=(1,0,0,0),\quad \mathbf{v}=(0,1,0,0)$

13- Prove that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

$$\mathbf{u}\cdot\mathbf{v}=\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$$

14- Prove that

$$\|\mathbf{u}+\mathbf{v}\|=\|\mathbf{u}\|+\|\mathbf{v}\|$$

if and only if $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

5. Row, Column, and Null S

Definition 5.0.1 Let A be an $m \times n$ matrix define as follow

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

then the vectors

$$\mathbf{r}_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$
$$\mathbf{r}_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$
$$\mathbf{r}_m = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

in R^n that are formed from the rows of A are called the row vectors of

A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the column vectors of A.

■ **Example 5.1** Row and Column Vectors of a 2×3 Matrix Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A are $\mathbf{r}_1 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$ and $\mathbf{r}_2 = \begin{bmatrix} 3 & -1 & 4 \end{bmatrix}$ tivate Windol and the column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

The following definition defines three important vector spaces associated with a matrix. ■

5.1 Row and column space

Definition 5.1.1 If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the row space of A, and the subspace of R^m spanned by the column vectors of A is called the column space of A.

Definition 5.1.2 Suppose A it is an $m \times n$ matrix. The *column space* A is the set $B = \{(b_1, b_2, ..., b_m)\} \subset R^m$ such that $AX = B$.
(we denote the column space of a matrix$A$ by $C(A)$)

■ **Example 5.2** Describe the column space of a matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 5 \end{pmatrix}.$$

Solution:

We reduce the augmented matrix

$$\begin{pmatrix} 1 & 1 & 2 & b_1 \\ 2 & 1 & 3 & b_2 \\ 3 & 2 & 5 & b_3 \end{pmatrix},$$

as follows:

$$\begin{pmatrix} 1 & 1 & 2 & b_1 \\ 2 & 1 & 3 & b_2 \\ 3 & 2 & 5 & b_3 \end{pmatrix} \underset{-3r_1+r_3}{\overset{-2r_1+r_2}{\longrightarrow}} \begin{pmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2-2b_1 \\ 0 & -1 & -1 & b_3-3b_1 \end{pmatrix} \overset{-r_2+r_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-b_1 \end{pmatrix}.$$

$$\therefore C(A) = \{(b_1,b_2,b_3) \in R^3 : b_3 = b_1+b_2\}.$$

■

■ **Example 5.3** Describe the column space of a matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -4 & -8 & 4 \end{pmatrix}.$$

Solution:

$$C(A) = \{(b_1,b_2,b_3) \in R^3 : b_2 = 2b_1\,,\ b_3 = -4b_1\}.$$

(verify that?) ■

R The row space of A equal $C(A^T)$

Theorem 5.1.1 A system of linear equations $Ax = b$ is consistent if and only if b is in the column space of A.

5.2 Null space

Definition 5.2.1 Suppose A it is an $m \times n$ matrix. The *null space* A is the set of all x in R^n such that $AX = 0$. (we denote the null space of a matrix$A$ by$N(A)$

■ **Example 5.4** Determine the null space of a matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution:
We reduce the augmented matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \overset{-r_1+r_3}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \overset{-r_2+r_3}{\rightarrow}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \overset{-r_3}{\rightarrow} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \underset{-r_3+r_1}{\overset{-r_3+r_2}{\rightarrow}}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\begin{aligned} x_1 &= 0, \\ \therefore \; x_2 &= 0, \\ x_3 &= 0 \end{aligned}$$

$$\therefore N(A) = \{(0,0,0)\}.$$

■

■ **Example 5.5** Determine the null space of a matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}.$$

Solution:
We reduce the augmented matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix},$$

as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \end{pmatrix} \underset{-3r_1+r_3}{\overset{-2r_1+r_2}{\rightarrow}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \overset{-r_2+r_3}{\rightarrow}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \begin{array}{l} x_1 + x_3 = 0, \\ x_2 - 2x_3 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = -x_3, \\ x_2 = 2x_3 \end{array}$$

$$\therefore N(A) = \{(x_1, x_2, x_3) = x_3(-1, 2, 1) : x_3 \in R\}.$$

■

■ **Example 5.6** For a matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix}$$

Solution:
$N(A) = \{(x_1, x_2, x_3) = x_2(-2, 1, 3) : x_2 \in R\}$ (verify that?). ■

Theorem 5.2.1 Elementary row operations do not change the null space of a matrix.

5.3 Exercises

A- For a matrix $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 2 & -5 & -3 & -2 \end{pmatrix}$.

(i) Determine the null space $N(A)$.
(ii) Describe the column space $C(A)$.
(iii) Describe the column space $R(A)$.

B- For a matrix $A = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 0 & 3 \\ 3 & 5 & -5 \end{pmatrix}$.

(i) Determine the null space $N(A)$.
(ii) Describe the column space $C(A)$.
(iii) Describe the column space $R(A)$.

Wish you all the best, Dr. A. Elrawy