Chapter 4 — The SIMPLE HARMONIC OSCILLATOR and the SIMPLE PENDULUM

THE SIMPLE HARMONIC OSCILLATOR

In Fig. 4-1(*a*) the mass m lies on a frictionless horizontal table indicated by the x axis. It is attached to one end of a spring of negligible mass and unstretched length l whose other end is fixed at E.

If m is given a displacement along the x axis [see Fig. 4-1(*b*)] and released, it will vibrate or oscillate back and forth about the *equilibrium position* O.

To determine the equation of motion, note that at any instant when the spring has length $l + x$ [Fig. 4-1(*b*)] there is a force tending to restore m to its equilibrium position. According to *Hooke's law* this force, called the *restoring force*, is proportional to the stretch x and is given by

$$\mathbf{F}_R = -\kappa x \mathbf{i} \tag{1}$$

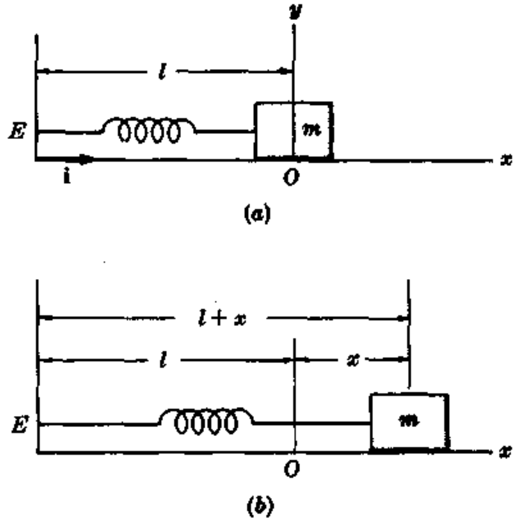

Fig. 4-1

where the subscript R stands for "restoring force" and where κ is the constant of proportionality often called the *spring constant, elastic constant, stiffness factor* or *modulus of elasticity* and $\mathbf{i}$ is the unit vector in the positive x direction. By Newton's second law we have

$$m\frac{d^2(x\mathbf{i})}{dt^2} = -\kappa x\mathbf{i} \quad \text{or} \quad m\ddot{x} + \kappa x = 0 \tag{2}$$

This vibrating system is called a *simple harmonic oscillator* or *linear harmonic oscillator*. This type of motion is often called *simple harmonic motion.*

AMPLITUDE, PERIOD AND FREQUENCY OF SIMPLE HARMONIC MOTION

If we solve the differential equation (*2*) subject to the initial conditions $x = A$ and $dx/dt = 0$ at $t = 0$, we find that

$$x = A\cos\omega t \qquad \text{where} \quad \omega = \sqrt{\kappa/m} \tag{3}$$

For the case where $A = 20$, $m = 2$ and $\kappa = 8$, see Problem 4.1.

Since $\cos\omega t$ varies between -1 and $+1$, the mass oscillates between $x = -A$ and $x = A$. A graph of x vs. t appears in Fig. 4-2.

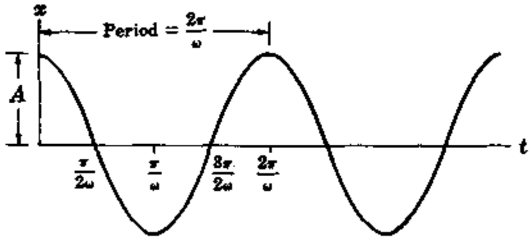

Fig. 4-2

The *amplitude* of the motion is the distance A and is the greatest distance from the equilibrium position.

The *period* of the motion is the time for one complete oscillation or vibration [sometimes called a *cycle*] such as, for example, from $x = A$ to $x = -A$ and then back to $x = A$ again. If P denotes the period, then

$$P = 2\pi/\omega = 2\pi\sqrt{m/\kappa} \tag{4}$$

The *frequency* of the motion, denoted by f, is the number of complete oscillations or cycles per unit time. We have

$$f = \frac{1}{P} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{m}} \tag{5}$$

In the general case, the solution of (2) is

$$x = A\cos\omega t + B\sin\omega t \qquad \text{where} \quad \omega = \sqrt{\kappa/m} \tag{6}$$

where A and B are determined from initial conditions. As seen in Problem 4.2, we can write (6) in the form

$$x = C\cos(\omega t - \phi) \qquad \text{where} \quad \omega = \sqrt{\kappa/m} \tag{7}$$

and where

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \phi = \tan^{-1}(B/A) \tag{8}$$

The amplitude in this case is C while the period and frequency remain the same as in (4) and (5), i.e. they are unaffected by change of initial conditions. The angle ϕ is called the *phase angle* or *epoch* chosen so that $0 \leqq \phi \leqq \pi$. If $\phi = 0$, (7) reduces to (3).

ENERGY OF A SIMPLE HARMONIC OSCILLATOR

If T is the kinetic energy, V the potential energy and $E = T + V$ the total energy of a simple harmonic oscillator, then we have

$$T = \tfrac{1}{2}mv^2, \quad V = \tfrac{1}{2}\kappa x^2 \tag{9}$$

and

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\kappa x^2 \tag{10}$$

See Problem 4.17.

THE DAMPED HARMONIC OSCILLATOR

In practice various forces may act on a harmonic oscillator, tending to reduce the magnitude of successive oscillations about the equilibrium position. Such forces are sometimes called *damping forces*. A useful approximate damping force is one which is proportional to the velocity and is given by

$$\mathbf{F}_D = -\beta\mathbf{v} = -\beta v\mathbf{i} = -\beta\frac{dx}{dt}\mathbf{i} \tag{11}$$

where the subscript D stands for "damping force" and where β is a positive constant called the *damping coefficient*. Note that $\mathbf{F}_D$ and $\mathbf{v}$ are in opposite directions.

If in addition to the restoring force we assume the damping force (*11*), the equation of motion of the harmonic oscillator, now called a *damped harmonic oscillator*, is given by

$$m\frac{d^2x}{dt^2} = -\kappa x - \beta\frac{dx}{dt} \quad \text{or} \quad m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \kappa x = 0 \tag{12}$$

on applying Newton's second law. Dividing by m and calling

$$\beta/m = 2\gamma, \quad \kappa/m = \omega^2 \tag{13}$$

this equation can be written

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \tag{14}$$

where the dots denote, as usual, differentiation with respect to t.

OVER-DAMPED, CRITICALLY DAMPED AND UNDER-DAMPED MOTION

Three cases arise in obtaining solutions to the differential equation (*14*).

***Case 1*, Over-damped motion,** $\gamma^2 > \omega^2$, i.e. $\beta^2 > 4\kappa m$

In this case (*14*) has the general solution

$$x = e^{-\gamma t}(Ae^{\alpha t} + Be^{-\alpha t}) \quad \text{where} \quad \alpha = \sqrt{\gamma^2 - \omega^2} \tag{15}$$

and where the arbitrary constants A and B can be found from the initial conditions.

***Case 2*, Critically damped motion,** $\gamma^2 = \omega^2$, i.e. $\beta^2 = 4\kappa m$

In this case (*14*) has the general solution

$$x = e^{-\gamma t}(A + Bt) \tag{16}$$

where A and B are found from initial conditions.

***Case 3*, Under-damped or damped oscillatory motion,** $\gamma^2 < \omega^2$, i.e. $\beta^2 < 4\kappa m$

In this case (*14*) has the general solution

$$\begin{aligned} x &= e^{-\gamma t}(A \sin \lambda t + B \cos \lambda t) \\ &= Ce^{-\gamma t} \cos(\lambda t - \phi) \quad \text{where} \quad \lambda = \sqrt{\omega^2 - \gamma^2} \end{aligned} \tag{17}$$

and where $C = \sqrt{A^2 + B^2}$, called the *amplitude* and ϕ, called the *phase angle* or *epoch*, are determined from the initial conditions.

In Cases 1 and 2 damping is so large that no oscillation takes place and the mass m simply returns gradually to the equilibrium position $x = 0$. This is indicated in Fig. 4-3 where we have assumed the initial conditions $x = x_0$, $dx/dt = 0$. Note that in the critically damped case, mass m returns to the equilibrium position faster than in the over-damped case.

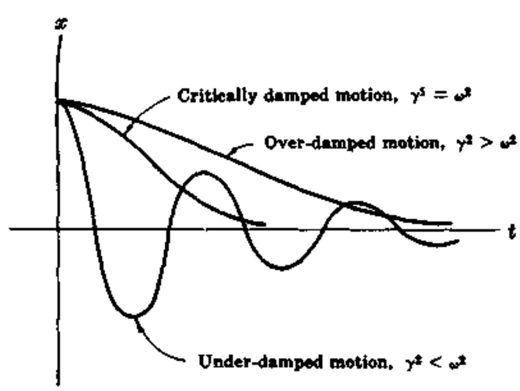

Fig. 4-3

In Case 3, damping has been reduced to such an extent that oscillations about the equilibrium position do take place, although the magnitude of these oscillations tend to decrease with time as indicated in Fig. 4-3. The difference in times between two successive maxima [or minima] in the under-damped [or damped oscillatory] motion of Fig. 4-3 is called the *period* of the motion and is given by

$$P = \frac{2\pi}{\lambda} = \frac{2\pi}{\sqrt{\omega^2 - \gamma^2}} = \frac{4\pi m}{\sqrt{4\kappa m - \beta^2}} \tag{18}$$

and the frequency, which is the reciprocal of the period, is given by

$$f = \frac{1}{P} = \frac{\lambda}{2\pi} = \frac{\sqrt{\omega^2 - \gamma^2}}{2\pi} = \frac{\sqrt{4\kappa m - \beta^2}}{4\pi m} \tag{19}$$

Note that if $\beta = 0$, *(18)* and *(19)* reduce to *(4)* and *(5)* respectively. The period and frequency corresponding to $\beta = 0$ are sometimes called the *natural period* and *natural frequency* respectively.

The period P given by *(18)* is also equal to two successive values of t for which $\cos(\lambda t - \phi) = 1$ [or $\cos(\lambda t - \phi) = -1$] as given in equation *(17)*. Suppose that the values of x corresponding to the two successive values t_n and $t_{n+1} = t_n + P$ are x_n and x_{n+1} respectively. Then

$$x_n/x_{n+1} = e^{-\gamma t_n}/e^{-\gamma(t_n+P)} = e^{\gamma P} \tag{20}$$

The quantity

$$\delta = \ln(x_n/x_{n+1}) = \gamma P \tag{21}$$

which is a constant, is called the *logarithmic decrement.*

FORCED VIBRATIONS

Suppose that in addition to the restoring force $-\kappa x\mathbf{i}$ and damping force $-\beta v\mathbf{i}$ we impress on the mass m a force $F(t)\mathbf{i}$ where

$$F(t) = F_0 \cos \alpha t \tag{22}$$

Then the differential equation of motion is

$$m\frac{d^2x}{dt^2} = -\kappa x - \beta\frac{dx}{dt} + F_0 \cos \alpha t \tag{23}$$

or

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = f_0 \cos \alpha t \tag{24}$$

where

$$\gamma = \beta/2m, \quad \omega^2 = \kappa/m, \quad f_0 = F_0/m \tag{25}$$

The general solution of *(24)* is found by adding the general solution of

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0 \tag{26}$$

[which has already been found and is given by *(15)*, *(16)* or *(17)*] to any particular solution of *(24)*. A particular solution of *(24)* is given by [see Problem 4.18]

$$x = \frac{f_0}{\sqrt{(\alpha^2 - \omega^2)^2 + 4\gamma^2\alpha^2}} \cos(\alpha t - \phi) \tag{27}$$

where

$$\tan\phi = \frac{2\gamma\alpha}{\alpha^2 - \omega^2} \qquad 0 \leqq \phi \leqq \pi \tag{28}$$

Now, as we have seen, the general solution of *(26)* approaches zero within a short time and we thus call this solution the *transient solution*. After this time has elapsed, the motion of the mass m is essentially given by *(27)* which is often called the *steady-state solution.* The vibrations or oscillations which take place, often called *forced vibrations* or *forced oscillations,* have a frequency which is equal to the frequency of the impressed force but lag behind by the phase angle ϕ.

RESONANCE

The amplitude of the steady-state oscillation (*27*) is given by

$$\mathcal{A} \quad = \quad \frac{f_0}{\sqrt{(\alpha^2 - \omega^2)^2 + 4\gamma^2\alpha^2}} \tag{29}$$

assuming $\gamma \neq 0$, i.e. $\beta \neq 0$, so that damping is assumed to be present. The maximum value of $\mathcal{A}$ in this case occurs where the frequency $\alpha/2\pi$ of the impressed force is such that

$$\alpha^2 \;=\; \alpha_R^2 \;=\; \omega^2 - 2\gamma^2 \tag{30}$$

assuming that $\gamma^2 < \frac{1}{2}\omega^2$ [see Problem 4.19]. Near this frequency very large oscillations may occur, sometimes causing damage to the system. The phenomenon is called *resonance* and the frequency $\alpha_R/2\pi$ is called the *frequency of resonance* or *resonant frequency.*

The value of the maximum amplitude at the resonant frequency is

$$\mathcal{A}_{\max} \quad = \quad \frac{f_0}{2\gamma\sqrt{\omega^2 - \gamma^2}} \tag{31}$$

The amplitude (*29*) can be written in terms of α_R as

$$\mathcal{A} \quad = \quad \frac{f_0}{\sqrt{(\alpha^2 - \alpha_R^2)^2 + 4\gamma^2(\omega^2 - \gamma^2)}} \tag{32}$$

A graph of $\mathcal{A}$ vs. α^2 is shown in Fig. 4-4. Note that the graph is symmetric around the resonant frequency and that the resonant frequency, frequency with damping and natural frequency (without damping) are all different. In case there is no damping, i.e. $\gamma = 0$ or $\beta = 0$, all of these frequencies are identical. In such case resonance occurs where the frequency of the impressed force equals the natural frequency of oscillation. The general solution for this case is

$$x \quad = \quad A\cos\omega t \;+\; B\sin\omega t \;+\; \frac{f_0 t}{2\omega}\sin\omega t \tag{33}$$

From the last term in (*33*) it is seen that the oscillations build up with time until finally the spring breaks. See Problem 4.20.

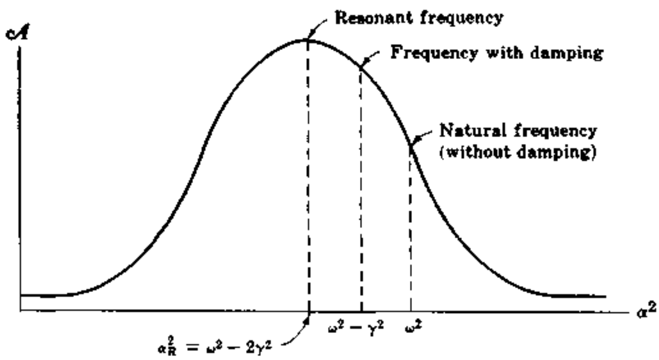

Fig. 4-4

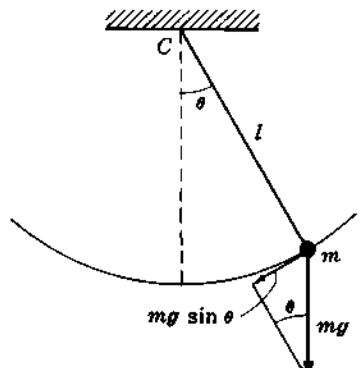

Fig. 4-5

THE SIMPLE PENDULUM

A *simple pendulum* consists of a mass m [Fig. 4-5] at the end of a massless string or rod of length l [which always remains straight, i.e. rigid]. If the mass m, sometimes called the *pendulum bob,* is pulled aside and released, the resulting motion will be oscillatory.

Calling θ the instantaneous angle which the string makes with the vertical, the differential equation of motion is [see Problem 4.23]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \tag{34}$$

assuming no damping forces or other external forces are present.

For small angles [e.g. less than 5° with the vertical], $\sin\theta$ is very nearly equal to θ, where θ is in radians, and equation (34) becomes, to a high degree of approximation,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \tag{35}$$

This equation has the general solution

$$\theta = A\cos\sqrt{g/l}\,t + B\sin\sqrt{g/l}\,t \tag{36}$$

where A and B are determined from initial conditions. For example, if $\theta = \theta_0$, $\dot{\theta} = 0$ at $t = 0$, then

$$\theta = \theta_0\cos\sqrt{g/l}\,t \tag{37}$$

In such case, the motion of the pendulum bob is that of simple harmonic motion. The period is given by

$$P = 2\pi\sqrt{l/g} \tag{38}$$

and the frequency is given by

$$f = \frac{1}{P} = \frac{1}{2\pi}\sqrt{g/l} \tag{39}$$

If the angles are not necessarily small, we can show [see Problems 4.29 and 4.30] that the period is equal to

$$\begin{aligned} P &= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= 2\pi\sqrt{\frac{l}{g}}\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2k^6+\cdots\right\} \end{aligned} \tag{40}$$

where $k = \sin(\theta_0/2)$. For small angles this reduces to (38).

For cases where damping and other external forces are considered, see Problems 4.25 and 4.114.

THE TWO AND THREE DIMENSIONAL HARMONIC OSCILLATOR

Suppose a particle of mass m moves in the xy plane under the influence of a force field $\mathbf{F}$ given by

$$\mathbf{F} = -\kappa_1 x\mathbf{i} - \kappa_2 y\mathbf{j} \tag{41}$$

where κ_1 and κ_2 are positive constants.

In this case the equations of motion of m are given by

$$m\frac{d^2x}{dt^2} = -\kappa_1 x, \quad m\frac{d^2y}{dt^2} = -\kappa_2 y \tag{42}$$

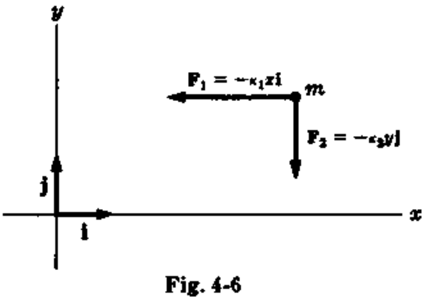

Fig. 4-6

and have solutions

$$x = A_1\cos\sqrt{\kappa_1/m}\,t + B_1\sin\sqrt{\kappa_1/m}\,t, \quad y = A_2\cos\sqrt{\kappa_2/m}\,t + B_2\sin\sqrt{\kappa_2/m}\,t \tag{43}$$

where A_1, B_1, A_2, B_2 are constants to be determined from the initial conditions. The mass m subjected to the force field (41) is often called a *two-dimensional harmonic oscillator*. The various curves which m describes in its motion are often called *Lissajous curves* or *figures*.

These ideas are easily extended to a three dimensional harmonic oscillator of mass m which is subject to a force field given by

$$\mathbf{F} = -\kappa_1 x\mathbf{i} - \kappa_2 y\mathbf{j} - \kappa_3 z\mathbf{k} \tag{44}$$

where $\kappa_1, \kappa_2, \kappa_3$ are positive constants.

Solved Problems

SIMPLE HARMONIC MOTION AND THE SIMPLE HARMONIC OSCILLATOR

4.1. A particle P of mass 2 moves along the x axis attracted toward origin O by a force whose magnitude is numerically equal to $8x$ [see Fig. 4-7]. If it is initially at rest at $x = 20$, find (*a*) the differential equation and initial conditions describing the motion, (*b*) the position of the particle at any time, (*c*) the speed and velocity of the particle at any time, and (*d*) the amplitude, period and frequency of the vibration.

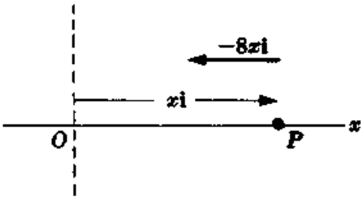

Fig. 4-7

(*a*) Let $\mathbf{r} = x\mathbf{i}$ be the position vector of P. The acceleration of P is $\frac{d^2}{dt^2}(x\mathbf{i}) = \frac{d^2x}{dt^2}\mathbf{i}$. The net force acting on P is $-8x\mathbf{i}$. Then by Newton's second law,

$$2\frac{d^2x}{dt^2}\mathbf{i} = -8x\mathbf{i} \quad \text{or} \quad \frac{d^2x}{dt^2} + 4x = 0 \tag{1}$$

which is the required differential equation of motion. The initial conditions are

$$x = 20, \; dx/dt = 0 \quad \text{at} \quad t = 0 \tag{2}$$

(*b*) The general solution of (*1*) is

$$x = A\cos 2t + B\sin 2t \tag{3}$$

When $t = 0$, $x = 20$ so that $A = 20$. Thus

$$x = 20\cos 2t + B\sin 2t \tag{4}$$

Then

$$dx/dt = -40\sin 2t + 2B\cos 2t \tag{5}$$

so that on putting $t = 0$, $dx/dt = 0$ we find $B = 0$. Thus (*3*) becomes

$$x = 20\cos 2t \tag{6}$$

which gives the position at any time.

(*c*) From (*6*) $dx/dt = -40\sin 2t$ which gives the speed at any time. The velocity is given by

$$\frac{dx}{dt}\mathbf{i} = -40\sin 2t\,\mathbf{i}$$

(*d*) Amplitude = 20. Period = $2\pi/2 = \pi$. Frequency = 1/period = $1/\pi$.

4.2. (*a*) Show that the function $A\cos\omega t + B\sin\omega t$ can be written as $C\cos(\omega t - \phi)$ where $C = \sqrt{A^2+B^2}$ and $\phi = \tan^{-1}(B/A)$. (*b*) Find the amplitude, period and frequency of the function in (*a*).

(*a*)
$$\begin{aligned} A\cos\omega t + B\sin\omega t &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right) \\ &= \sqrt{A^2+B^2}\,(\cos\phi\cos\omega t + \sin\phi\sin\omega t) \\ &= \sqrt{A^2+B^2}\cos(\omega t - \phi) = C\cos(\omega t - \phi) \end{aligned}$$

where $\cos\phi = A/\sqrt{A^2+B^2}$ and $\sin\phi = B/\sqrt{A^2+B^2}$, i.e. $\tan\phi = B/A$ or $\phi = \tan^{-1} B/A$, and $C = \sqrt{A^2+B^2}$. We generally choose that value of ϕ which lies between 0° and 180°, i.e. $0 \leqq \phi \leqq \pi$.

(*b*) Amplitude = maximum value = $C = \sqrt{A^2+B^2}$. Period = $2\pi/\omega$. Frequency = $\omega/2\pi$.

4.3. Work Problem 4.1 if P is initially at $x = 20$ but is moving (*a*) to the right with speed 30, (*b*) to the left with speed 30. Find the amplitude, period and frequency in each case.

(*a*) The only difference here is that the condition $dx/dt = 0$ at $t = 0$ of Problem 4.1 is replaced by $dx/dt = 30$ at $t = 0$. Then from (*5*) of Problem 4.1 we find $B = 15$, and (*3*) of Problem 4.1 becomes

$$x = 20\cos 2t + 15\sin 2t \tag{1}$$

which gives the position of P at any time. This may be written [see Problem 4.2] as

$$\begin{aligned} x &= \sqrt{(20)^2+(15)^2}\left\{\frac{20}{\sqrt{(20)^2+(15)^2}}\cos 2t + \frac{15}{\sqrt{(20)^2+(15)^2}}\sin 2t\right\} \\ &= 25(\tfrac{4}{5}\cos 2t + \tfrac{3}{5}\sin 2t) = 25\cos(2t-\phi) \end{aligned}$$

where

$$\cos\phi = \tfrac{4}{5}, \quad \sin\phi = \tfrac{3}{5} \tag{2}$$

The angle ϕ which can be found from (*2*) is often called the *phase angle* or *epoch*.

Since the cosine varies between -1 and $+1$, the amplitude $= 25$. The period and frequency are the same as before, i.e. period $= 2\pi/2 = \pi$ and frequency $= 2/2\pi = 1/\pi$.

(*b*) In this case the condition $dx/dt = 0$ at $t = 0$ of Problem 4.1 is replaced by $dx/dt = -30$ at $t = 0$. Then $B = -15$ and the position is given by

$$x = 20\cos 2t - 15\sin 2t$$

which as in part (*a*) can be written

$$\begin{aligned} x &= 25(\tfrac{4}{5}\cos 2t - \tfrac{3}{5}\sin 2t) \\ &= 25(\cos\psi\cos 2t + \sin\psi\sin 2t) = 25\cos(2t-\psi) \end{aligned}$$

where $\cos\psi = \frac{4}{5}$, $\sin\psi = -\frac{3}{5}$.

The amplitude, period and frequency are the same as in part (*a*). The only difference is in the phase angle. The relationship between ψ and ϕ is $\psi = \phi + \pi$. We often describe this by saying that the two motions are 180° *out of phase* with each other.

4.4. A spring of negligible mass, suspended vertically from one end, is stretched a distance of 20 cm when a 5 g mass is attached to the other end. The spring and mass are placed on a horizontal frictionless table as in Fig. 4-1(*a*), page 86, with the suspension point fixed at E. The mass is pulled away a distance 20 cm beyond the equilibrium position O and released. Find (*a*) the differential equation and initial conditions describing the motion, (*b*) the position at any time t, and (*c*) the amplitude, period and frequency of the vibrations.

(*a*) The gravitational force on a 5 g mass [i.e. the weight of a 5 g mass] is 5(980) dynes = 4900 dynes. Then since 4900 dynes stretches the spring 20 cm, the spring constant is $\kappa = 4900/20 = 245$ dynes/cm. Thus when the spring is stretched a distance x cm beyond the equilibrium position, the restoring force is $-245x\mathbf{i}$. Then by Newton's second law we have, if $\mathbf{r} = x\mathbf{i}$ is the position vector of the mass,

$$5\frac{d^2(x\mathbf{i})}{dt^2} = -245x\mathbf{i} \quad \text{or} \quad \frac{d^2x}{dt^2} + 49x = 0 \tag{1}$$

The initial conditions are

$$x = 20, \; dx/dt = 0 \quad \text{at } t = 0 \tag{2}$$

(*b*) The general solution of (*1*) is

$$x = A\cos 7t + B\sin 7t \tag{3}$$

Using the conditions (*2*) we find $A = 20$, $B = 0$ so that $x = 20\cos 7t$.

(*c*) From $x = 20\cos 7t$ we see that: amplitude = 20 cm; period = $2\pi/7$ s; frequency = $7/2\pi$ s^{-1} or $7/2\pi$ Hz.

4.5. A particle of mass m moves along the x axis, attracted toward a fixed point O on it by a force proportional to the distance from O. Initially the particle is at distance x_0 from O and is given a velocity v_0 away from O. Determine (a) the position at any time, (b) the velocity at any time, and (c) the amplitude, period, frequency, and maximum speed.

(a) The force of attraction toward O is $-\kappa x\mathbf{i}$ where κ is a positive constant of proportionality. Then by Newton's second law,

$$m\frac{d^2x}{dt^2}\mathbf{i} = -\kappa x\mathbf{i} \quad \text{or} \quad \ddot{x} + \frac{\kappa x}{m} = 0 \qquad (1)$$

Solving (1), we find

$$x = A\cos\sqrt{\kappa/m}\,t + B\sin\sqrt{\kappa/m}\,t \qquad (2)$$

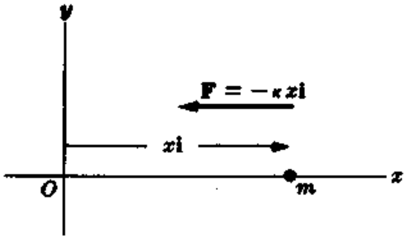

Fig. 4-8

We also have the initial conditions

$$x = x_0,\ dx/dt = v_0 \quad \text{at } t = 0 \qquad (3)$$

From $x = x_0$ at $t = 0$ we find, using (2), that $A = x_0$. Thus

$$x = x_0\cos\sqrt{\kappa/m}\,t + B\sin\sqrt{\kappa/m}\,t \qquad (4)$$

so that

$$dx/dt = -x_0\sqrt{\kappa/m}\sin\sqrt{\kappa/m}\,t + B\sqrt{\kappa/m}\cos\sqrt{\kappa/m}\,t \qquad (5)$$

From $dx/dt = v_0$ at $t = 0$ we find, using (5), that $B = v_0\sqrt{m/\kappa}$. Thus (4) becomes

$$x = x_0\cos\sqrt{\kappa/m}\,t + v_0\sqrt{m/\kappa}\sin\sqrt{\kappa/m}\,t \qquad (6)$$

Using Problem 4.2, this can be written

$$x = \sqrt{x_0^2 + mv_0^2/\kappa}\cos(\sqrt{\kappa/m}\,t - \phi) \qquad (7)$$

where

$$\phi = \tan^{-1}(v_0/x_0)\sqrt{m/\kappa} \qquad (8)$$

(b) The velocity is, using (6) or (7),

$$\begin{aligned}\mathbf{v} = \frac{dx}{dt}\mathbf{i} &= (-x_0\sqrt{\kappa/m}\sin\sqrt{\kappa/m}\,t + v_0\cos\sqrt{\kappa/m}\,t)\,\mathbf{i}\\ &= -\sqrt{\kappa/m}\sqrt{x_0^2 + mv_0^2/\kappa}\sin(\sqrt{\kappa/m}\,t - \phi)\,\mathbf{i}\\ &= -\sqrt{v_0^2 + \kappa x_0^2/m}\sin(\sqrt{\kappa/m}\,t - \phi)\,\mathbf{i} \qquad (9)\end{aligned}$$

(c) The amplitude is given from (7) by $\sqrt{x_0^2 + mv_0^2/\kappa}$.

From (7), the period is $P = 2\pi/\sqrt{k/m}$. The frequency is $f = 1/P = (2\pi\sqrt{m/k})^{-1}$.

From (9), the speed is a maximum when $\sin(\sqrt{\kappa/m}\,t - \phi) = \pm 1$; this speed is $\sqrt{v_0^2 + \kappa x_0^2/m}$.

4.6. An object of mass 20 kg moves with simple harmonic motion on the x axis. Initially ($t = 0$) it is located at the distance 4 meters away from the origin $x = 0$, and has velocity 15 m/s and acceleration 100 m/s² directed toward $x = 0$. Find (a) the position at any time, (b) the amplitude, period and frequency of the oscillations, and (c) the force on the object when $t = \pi/10$ s.

(a) If x denotes the position of the object at time t, then the initial conditions are

$$x = 4,\ dx/dt = -15,\ d^2x/dt^2 = -100 \quad \text{at } t = 0 \qquad (1)$$

Now for simple harmonic motion,

$$x = A\cos\omega t + B\sin\omega t \qquad (2)$$

Differentiating, we find

$$dx/dt = -A\omega\sin\omega t + B\omega\cos\omega t \qquad (3)$$

$$d^2x/dt^2 = -A\omega^2\cos\omega t - B\omega^2\sin\omega t \qquad (4)$$

Using conditions (1) in (2), (3) and (4), we find $4 = A$, $-15 = B\omega$, $-100 = -A\omega^2$. Solving simultaneously, we find $A = 4$, $\omega = 5$, $B = -3$ so that

$$x = 4\cos 5t - 3\sin 5t \tag{5}$$

which can be written

$$x = 5\cos(5t - \phi) \qquad \text{where} \quad \cos\phi = \tfrac{4}{5},\ \sin\phi = -\tfrac{3}{5} \tag{6}$$

(b) From (6) we see that: amplitude = 5 m, period = $2\pi/5$ s, frequency = $5/2\pi$ Hz.

(c) Magnitude of acceleration = $d^2x/dt^2 = -100\cos 5t + 75\sin 5t = 75\ \text{m/s}^2$ at $t = \pi/10$.

Force on object = (mass)(acceleration) = (20 kg)(75 m/s²) = 1500 newtons.

4.7. An object of 100 N weight suspended from the end of a vertical spring of negligible mass stretches it 0.16 m. (a) Determine the position of the object at any time if initially it is pulled down 0.05 m and then released. (b) Find the amplitude, period and frequency of the motion. (Use $g = 10\ \text{m/s}^2$).

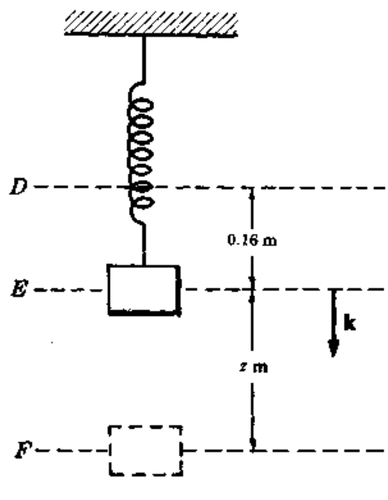

Fig. 4-9

(a) Let D and E [Fig. 4-9] represent the position of the end of the spring before and after the object is put on the spring. Position E is the equilibrium position of the object.

Choose a coordinate system as shown in Fig. 4-9 so that the positive z axis is downward with origin at the equilibrium position.

By Hooke's law, since 100 N stretches the spring 0.16 m, 200 N stretches it 0.32 m, then $200(0.16 + z)/0.32$ N stretches it $(0.16 + z)$m. Thus when the object is at position F, there is an upward force acting on it of magnitude $200(0.16 + z)/0.32$ and a downward force due to its weight of magnitude 100. By Newton's second law we thus have

$$\frac{100}{10}\frac{d^2z}{dt^2}\mathbf{k} = 100\mathbf{k} - 200(0.16 + z)/0.32\,\mathbf{k} \quad \text{or} \quad \frac{d^2z}{dt^2} + \frac{25}{4}z = 0$$

Solving,

$$z = A\cos\frac{5t}{2} + B\sin\frac{5t}{2} \tag{1}$$

Now at $t = 0$, $z = \frac{1}{20}$ and $dz/dt = 0$; thus $A = \frac{1}{20}$ $B = 0$ and

$$z = \frac{1}{20}\cos\frac{5t}{2} \tag{2}$$

(b) From (2): amplitude = 0.05 m, period = $\frac{4\pi}{5}$ s, frequency = $\frac{5}{4\pi}$ Hz

4.8. Work Problem 4.7 if initially the object is pulled down 0.08 m (instead of 0.05 m) and then given an initial velocity of 0.6 m/s downward.

In this case the solution (1) of Problem 4.7 still holds but the initial conditions are: at $t = 0$, $z = 0.08$ and $dz/dt = 0.6$. From these we find

$$A = 0.08 \text{ and } B = 0.24, \text{ so that } z = 0.08\cos 5t/2 + 0.24\sin 5t/2 = 0.253\cos\left(\frac{5}{2}t - 1.249\right)$$

Thus amplitude = 0.253 m, period = $4\pi/5$ s, frequency = $5/4\pi$ Hz. Note that the period and frequency are unaffected by changing the initial conditions.

4.9. A particle travels with uniform angular speed ω around a circle of radius b. Prove that its projection on a diameter oscillates with simple harmonic motion of period $2\pi/\omega$ about the center.

Choose the circle in the xy plane with center at the origin O as in Fig. 4-10 below. Let Q be the projection of particle P on diameter AB chosen along the x axis.

If the particle is initially at B, then in time t we will have $\angle BOP = \theta = \omega t$. Then the position of P at time t is

$$\mathbf{r} = b\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j} \qquad (1)$$

The projection Q of P on the x axis is at distance

$$\mathbf{r}\cdot\mathbf{i} = x = b\cos\omega t \qquad (2)$$

from O at any time t. From (2) we see that the projection Q oscillates with simple harmonic motion of period $2\pi/\omega$ about the center O.

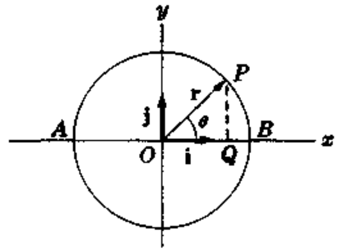

Fig. 4-10

THE DAMPED HARMONIC OSCILLATOR

4.10. Suppose that in Problem 4.1 the particle P has also a damping force whose magnitude is numerically equal to 8 times the instantaneous speed. Find (a) the position and (b) the velocity of the particle at any time. (c) Illustrate graphically the position of the particle as a function of time t.

(a) In this case the net force acting on P is [see Fig. 4-11] $-8x\mathbf{i} - 8\dfrac{dx}{dt}\mathbf{i}$. Then by Newton's second law,

$$2\frac{d^2x}{dt^2}\mathbf{i} = -8x\mathbf{i} - 8\frac{dx}{dt}\mathbf{i}$$

or

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0$$

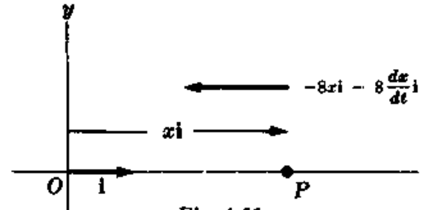

Fig. 4-11

This has the solution [see Appendix, page 352, Problem C.14]

$$x = e^{-2t}(A + Bt)$$

When $t = 0$, $x = 20$ and $dx/dt = 0$; thus $A = 20$, $B = 40$, and $x = 20e^{-2t}(1 + 2t)$ gives the position at any time t.

(b) The velocity is given by

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} = -80te^{-2t}\mathbf{i}$$

(c) The graph of x vs. t is shown in Fig. 4-12. It is seen that the motion is non-oscillatory. The particle approaches O slowly but never reaches it. This is an example where the motion is *critically damped*.

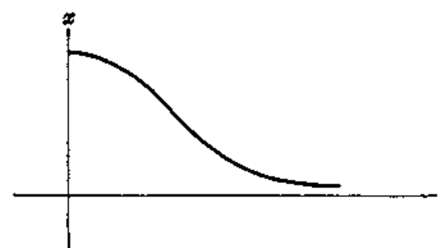

Fig. 4-12

4.11. A particle of mass 5 g moves along the x axis under the influence of two forces: (i) a force of attraction to origin O which in dynes is numerically equal to 40 times the instantaneous distance from O, and (ii) a damping force proportional to the instantaneous speed such that when the speed is 10 cm/s the damping force is 200 dynes. Assuming that the particle starts from rest at a distance 20 cm from O, (a) set up the differential equation and conditions describing the motion, (b) find the position of the particle at any time, (c) determine the amplitude, period and frequency of the damped oscillations, and (d) graph the motion.

(a) Let the position vector of the particle P be denoted by $\mathbf{r} = x\mathbf{i}$ as indicated in Fig. 4-13. Then the force of attraction (directed toward O) is

$$-40x\mathbf{i} \qquad (1)$$

The magnitude of the damping force f is proportional to the speed, so that $f = \beta\, dx/dt$ where β is constant. Then since $f = 200$ when $dx/dt = 10$, we have $\beta = 20$ and $f = 20\, dx/dt$. To get $\mathbf{f}$, note that when $dx/dt > 0$ and $x > 0$ the particle is on the positive x axis and moving to

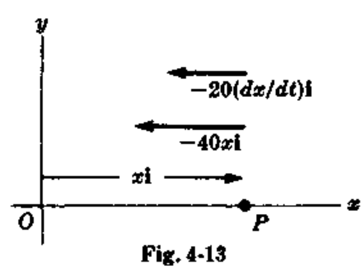

Fig. 4-13

the right. Thus the resistance force must be directed toward the left. This can only be accomplished if

$$\mathbf{f} = -20\frac{dx}{dt}\mathbf{i} \tag{2}$$

This same form for $\mathbf{f}$ is easily shown to be correct if $x > 0$, $dx/dt < 0$, $x < 0$, $dx/dt > 0$, $x < 0$, $dx/dt < 0$ [see Problem 4.45].

Hence by Newton's second law we have

$$5\frac{d^2x}{dt^2}\mathbf{i} = -20\frac{dx}{dt}\mathbf{i} - 40x\mathbf{i} \tag{3}$$

or

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = 0 \tag{4}$$

Since the particle starts from rest at 20 cm from O, we have

$$x = 20,\ dx/dt = 0 \quad \text{at } t = 0 \tag{5}$$

where we have assumed that the particle starts on the positive side of the x axis [we could just as well assume that the particle starts on the negative side, in which case $x = -20$].

(b) $x = e^{\alpha t}$ is a solution of (4) if

$$\alpha^2 + 4\alpha + 8 = 0 \quad \text{or} \quad \alpha = \tfrac{1}{2}(-4 \pm \sqrt{16-32}) = -2 \pm 2i$$

Then the general solution is

$$x = e^{-2t}(A\cos 2t + B\sin 2t) \tag{6}$$

Since $x = 20$ at $t = 0$, we find from (6) that $A = 20$, i.e.,

$$x = e^{-2t}(20\cos 2t + B\sin 2t) \tag{7}$$

Thus by differentiation,

$$dx/dt = (e^{-2t})(-40\sin 2t + 2B\cos 2t) + (-2e^{-2t})(20\cos 2t + B\sin 2t) \tag{8}$$

Since $dx/dt = 0$ at $t = 0$, we have from (8), $B = 20$. Thus from (7) we obtain

$$x = 20e^{-2t}(\cos 2t + \sin 2t) = 20\sqrt{2}\,e^{-2t}\cos(2t - \pi/4) \tag{9}$$

using Problem 4.2.

(c) From (9): amplitude $= 20\sqrt{2}\,e^{-2t}$ cm, period $= 2\pi/2 = \pi$ s, frequency $= 1/\pi$ Hz

(d) The graph is shown in Fig. 4-14. Note that the amplitudes of the oscillation decrease toward zero as t increases.

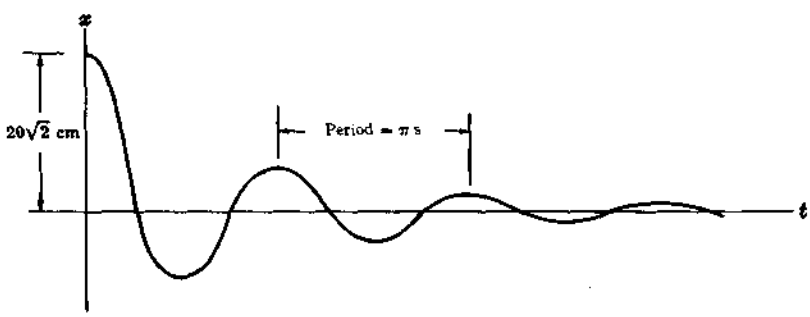

Fig. 4-14

4.12. Find the logarithmic decrement in Problem 4-11.

Method 1. The maxima (or minima) of x occur where $dx/dt = 0$. From (9) of Problem 4.11,

$$dx/dt = -80e^{-2t}\sin 2t = 0$$

when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi, 5\pi/2, \ldots$. The maxima occur when $t = 0, \pi, 2\pi, \ldots$; the minima occur when $t = \pi/2, 3\pi/2, 5\pi/2, \ldots$. The ratio of two successive maxima is $e^{-2(0)}/e^{-2(\pi)}$ or $e^{-2(\pi)}/e^{-2(2\pi)}$, etc., i.e. $e^{2\pi}$. Then the logarithmic decrement is $\delta = \ln(e^{2\pi}) = 2\pi$.

Method 2.

From (9) of Problem 4.11, the difference between two successive values of t, denoted by t_n and t_{n+1}, for which $\cos(2t - \pi/4) = 1$ (or -1) is π, which is the period. Then

$$\frac{x_n}{x_{n+1}} = \frac{20\sqrt{2}\,e^{-2t_n}}{20\sqrt{2}\,e^{-2t_{n+1}}} = e^{2\pi} \quad \text{and} \quad \delta = \ln(x_n/x_{n+1}) = 2\pi$$

Method 3. From (13), (18) and (21), pages 88 and 89, we have

$$\delta = \gamma P = \left(\frac{\beta}{2m}\right)\left(\frac{4\pi m}{\sqrt{4\kappa m - \beta^2}}\right) = \frac{2\pi\beta}{\sqrt{4\kappa m - \beta^2}}$$

Then since $m = 5$, $\beta = 20$, $\kappa = 40$ [Problem 4.11, equation (3)], $\delta = 2\pi$.

4.13. **Determine the natural period and frequency of the particle of Problem 4.11.**

The natural period is the period when there is no damping. In such case the motion is given by removing the term involving dx/dt in equation (3) or (4) of Problem 4.11. Thus

$$d^2x/dt^2 + 8x = 0 \quad \text{or} \quad x = A\cos 2\sqrt{2}\,t + B\sin 2\sqrt{2}\,t$$

Then natural period $= 2\pi/2\sqrt{2}$ s $= \pi/\sqrt{2}$ s; natural frequency $= \sqrt{2}/\pi$ Hz.

4.14. **For what range of values of the damping constant in Problem 4.11 will the motion be (a) overdamped, (b) underdamped or damped oscillatory, (c) critically damped?**

Denoting the damping constant by β, equation (3) of Problem 4.11 is replaced by

$$5\frac{d^2x}{dt^2}\mathbf{i} = -\beta\frac{dx}{dt}\mathbf{i} - 40x\mathbf{i} \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{\beta}{5}\frac{dx}{dt} + 8x = 0$$

Then the motion is:

(a) Overdamped if $(\beta/5)^2 > 32$, i.e. $\beta > 20\sqrt{2}$.

(b) Underdamped if $(\beta/5)^2 < 32$, i.e. $\beta < 20\sqrt{2}$.
[Note that this is the case for Problem 4.11 where $\beta = 20$.]

(c) Critically damped if $(\beta/5)^2 = 32$, i.e. $\beta = 20\sqrt{2}$.

4.15. **Solve Problem 4.7 taking into account an external damping force given numerically in newtons by βv where v is the instantaneous speed in m/s and (a) $\beta = 10$, (b) $\beta = 50$, (c) $\beta = 62.5$.**

The equation of motion is

$$\frac{100}{10}\frac{d^2z}{dt^2}\mathbf{k} = 100\mathbf{k} - 200(0.16+z)/0.32\mathbf{k} - \beta\frac{dz}{dt}\mathbf{k} \quad \text{or} \quad \frac{d^2z}{dt^2} + \frac{\beta}{10}\frac{dz}{dt} + \frac{25}{4}z = 0$$

(a) If $\beta = 10$, then $d^2z/dt^2 + 4\,dz/dt + 25/4\,z = 0$. The solution is

$$z = e^{-2t}(A\cos 3/2\,t + B\sin 3/2\,t)$$

Using the conditions $z = 0.05$, $dz/dt = 0$ at $t = 0$, we find $A = \frac{1}{20}$, $B = \frac{1}{15}$ so that

$$z = \frac{1}{5}e^{-2t}\left(\frac{1}{4}\cos\frac{3}{2}t + \frac{1}{3}\sin\frac{3}{2}t\right) = \frac{1}{25}e^{-2t}\cos\left(\frac{3}{2}t - 53°\,8'\right)$$

The motion is *damped oscillatory* with period $2\pi/4.8 = 5\pi/12$ s.

(b) If $\beta = 50$, then $d^2z/dt^2 + 5dz/dt + 25/4\,z = 0$. The solution is

$$z = e^{-5/2\,t}(A + Bt)$$

Solving subject to the initial conditions gives $A = \frac{1}{20}$, $B = \frac{1}{8}$; then $z = \frac{1}{4}e^{-5/2\,t}\left(\frac{1}{2} + \frac{1}{5}\right)$.

The motion is *critically damped* since any decrease in β would produce oscillatory motion.

(c) If $\beta = 62.5$ then $d^2z/dt^2 + \frac{25}{4}dz/dt + \frac{25}{4}z = 0$. The solution is

$$z = Ae^{-5/4\,t} + Be^{-5t}$$

Solving subject to initial conditions gives $A = \frac{1}{15}$, $B = -\frac{1}{60}$; then $z = \frac{1}{15}e^{-5/4\,t} - \frac{1}{60}e^{-5t}$.

The motion is *overdamped.*

ENERGY OF A SIMPLE HARMONIC OSCILLATOR

4.16. (a) Prove that the force $\mathbf{F} = -\kappa x\mathbf{i}$ acting on a simple harmonic oscillator is conservative. (b) Find the potential energy of a simple harmonic oscillator.

(a) We have
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\kappa x & 0 & 0 \end{vmatrix} = \mathbf{0}$$
so that $\mathbf{F}$ is conservative.

(b) The potential or potential energy is given by V where $\mathbf{F} = -\nabla V$ or

$$-\kappa x\mathbf{i} = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right)$$

Then $\partial V/\partial x = \kappa x$, $\partial V/\partial y = 0$, $\partial V/\partial z = 0$ from which $V = \frac{1}{2}\kappa x^2 + c$. Assuming $V = 0$ corresponding to $x = 0$, we find $c = 0$ so that $V = \frac{1}{2}\kappa x^2$.

4.17. Express in symbols the principle of conservation of energy for a simple harmonic oscillator.

By Problem 4.16(b), we have

$$\text{Kinetic energy} + \text{Potential energy} = \text{Total energy}$$

or
$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}\kappa x^2 = E$$

which can also be written, since $v = dx/dt$, as $\frac{1}{2}m(dx/dt)^2 + \frac{1}{2}\kappa x^2 = E$.

Another method. The differential equation for the motion of a simple harmonic oscillator is

$$m\,d^2x/dt^2 = -\kappa x$$

Since $dx/dt = v$, this can also be written as

$$m\frac{dv}{dt} = -\kappa x \quad\text{or}\quad m\frac{dv}{dx}\frac{dx}{dt} = -\kappa x, \qquad\text{i.e.}\quad mv\frac{dv}{dx} = -\kappa x$$

Integration yields $\frac{1}{2}mv^2 + \frac{1}{2}\kappa x^2 = E$.

FORCED VIBRATIONS AND RESONANCE

4.18. Derive the steady-state solution (27) corresponding to the differential equation (24) on page 89.

The differential equation is
$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = f_0 \cos \alpha t \tag{1}$$

Consider a particular solution having the form

$$x = c_1 \cos \alpha t + c_2 \sin \alpha t \tag{2}$$

where c_1 and c_2 are to be determined. Substituting (2) into (1), we find

$$(-\alpha^2 c_1 + 2\gamma\alpha c_2 + \omega^2 c_1)\cos\alpha t \;+\; (-\alpha^2 c_2 - 2\gamma\alpha c_1 + \omega^2 c_2)\sin\alpha t \;=\; f_0\cos\alpha t$$

from which $$-\alpha^2 c_1 + 2\gamma\alpha c_2 + \omega^2 c_1 \;=\; f_0, \qquad -\alpha^2 c_2 - 2\gamma\alpha c_1 + \omega^2 c_2 \;=\; 0 \tag{3}$$

or $$(\alpha^2-\omega^2)c_1 - 2\gamma\alpha c_2 \;=\; -f_0, \qquad 2\gamma\alpha c_1 + (\alpha^2-\omega^2)c_2 \;=\; 0 \tag{4}$$

Solving these simultaneously, we find

$$c_1 \;=\; \frac{f_0(\omega^2-\alpha^2)}{(\alpha^2-\omega^2)^2+4\gamma^2\alpha^2}, \qquad c_2 \;=\; \frac{2f_0\gamma\omega}{(\alpha^2-\omega^2)^2+4\gamma^2\alpha^2} \tag{5}$$

Thus (2) becomes

$$x \;=\; \frac{f_0[(\omega^2-\alpha^2)\cos\alpha t \;+\; 2\gamma\alpha\sin\alpha t]}{(\alpha^2-\omega^2)^2+4\gamma^2\alpha^2} \tag{6}$$

Now by Problem 4.2, page 92,

$$(\omega^2-\alpha^2)\cos\alpha t \;+\; 2\gamma\alpha\sin\alpha t \;=\; \sqrt{(\omega^2-\alpha^2)^2+4\gamma^2\alpha^2}\,\cos(\alpha t-\phi) \tag{7}$$

where $\tan\phi = 2\gamma\alpha/(\alpha^2-\omega^2)$, $0 \leqq \phi \leqq \pi$. Using (7) in (6), we find as required

$$x \;=\; \frac{f_0}{\sqrt{(\alpha^2-\omega^2)^2+4\gamma^2\alpha^2}}\cos(\alpha t-\phi)$$

4.19. Prove (*a*) that the amplitude in Problem 4.18 is a maximum where the resonant frequency is determined from $\alpha = \sqrt{\omega^2-2\gamma^2}$ and (*b*) that the value of this maximum amplitude is $f_0/(2\gamma\sqrt{\omega^2-\gamma^2})$.

Method 1. The amplitude in Problem 4.18 is

$$f_0/\sqrt{(\alpha^2-\omega^2)^2+4\gamma^2\alpha^2} \tag{1}$$

It is a maximum when the denominator [or the square of the denominator] is a minimum. To find this minimum, write

$$\begin{aligned}(\alpha^2-\omega^2)^2+4\gamma^2\alpha^2 \;&=\; \alpha^4 - 2(\omega^2-2\gamma^2)\alpha^2 + \omega^4\\ &=\; \alpha^4 - 2(\omega^2-2\gamma^2)\alpha^2 + (\omega^2-2\gamma^2)^2 + \omega^4 - (\omega^2-2\gamma^2)^2\\ &=\; [\alpha^2-(\omega^2-2\gamma^2)]^2 + 4\gamma^2(\omega^2-\gamma^2)\end{aligned}$$

This is a minimum where the first term on the last line is zero, i.e. when $\alpha^2 = \omega^2-2\gamma^2$, and the value is then $4\gamma^2(\omega^2-\gamma^2)$. Thus the value of the maximum amplitude is given from (1) by $f_0/(2\gamma\sqrt{\omega^2-\gamma^2})$.

Method 2. The function $U \;=\; (\alpha^2-\omega^2)^2+4\gamma^2\alpha^2$ has a minimum or maximum when

$$\frac{dU}{d\alpha} \;=\; 2(\alpha^2-\omega^2)2\alpha + 8\gamma^2\alpha \;=\; 0 \qquad\text{or}\qquad \alpha(\alpha^2-\omega^2+2\gamma^2) \;=\; 0$$

i.e. $\alpha = 0$, $\alpha = \sqrt{\omega^2-2\gamma^2}$ where $\gamma^2 < \frac{1}{2}\omega^2$. Now

$$d^2U/d\alpha^2 \;=\; 12\alpha^2 - 4\omega^2 + 8\gamma^2$$

For $\alpha = 0$, $d^2U/d\alpha^2 = -4(\omega^2-2\gamma^2) < 0$. For $\alpha = \sqrt{\omega^2-2\gamma^2}$, $d^2U/d\alpha^2 = 8(\omega^2-2\gamma^2) > 0$. Thus $\alpha = \sqrt{\omega^2-2\gamma^2}$ gives the minimum value.

4.20. (*a*) Obtain the solution (33), page 90, for the case where there is no damping and the impressed frequency is equal to the natural frequency of the oscillation. (*b*) Give a physical interpretation.

(*a*) The case to be considered is obtained by putting $\gamma = 0$ or $\beta = 0$ and $\alpha = \omega$ in equations (23) or (24), page 89. We thus must solve the equation

$$\ddot{x} + \omega^2 x \;=\; f_0\cos\omega t \tag{1}$$

To find the general solution of this equation we add the general solution of

$$\ddot{x} + \omega^2 x = 0$$

to a particular solution of *(1)*.

Now the general solution of *(2)* is

$$x = A\cos\omega t + B\sin\omega t$$

To find a particular solution of *(1)* it would do no good to assume a particular solution of the fo

$$x = c_1\cos\omega t + c_2\sin\omega t$$

since when we substitute *(4)* [which is identical in form to *(3)*] into the left side of *(1)*, we wo get zero. We must therefore modify the form of the assumed particular solution *(4)*. As s in Appendix C, the assumed particular solution has the form

$$x = t(c_1\cos\omega t + c_2\sin\omega t)$$

To see that this yields the required particular solution, let us differentiate *(5)* to obtain

$$\dot{x} = t(-\omega c_1\sin\omega t + \omega c_2\cos\omega t) + (c_1\cos\omega t + c_2\sin\omega t)$$

$$\ddot{x} = t(-\omega^2 c_1\cos\omega t - \omega^2 c_2\sin\omega t) + 2(-\omega c_1\sin\omega t + \omega c_2\cos\omega t)$$

Substituting *(5)*, *(6)* and *(7)* into *(1)*, we find after simplifying

$$-2\omega c_1\sin\omega t + 2\omega c_2\cos\omega t = f_0\cos\omega t$$

from which $c_1 = 0$ and $c_2 = f_0/2\omega$. Thus the required particular solution *(5)* is $x = (f_0/2\omega)t\sin\omega t$. The general solution of *(1)* is therefore

$$x = A\cos\omega t + B\sin\omega t + (f_0/2\omega)t\sin\omega t \quad (8)$$

(*b*) The constants A and B in *(8)* are determined from the initial conditions. Unlike the case with damping, the terms involving A and B do not become small with time. However, the last term involving t increases with time to such an extent that the spring will finally break. A graph of the last term shown in Fig. 4-15 indicates how the oscillations build up in magnitude.

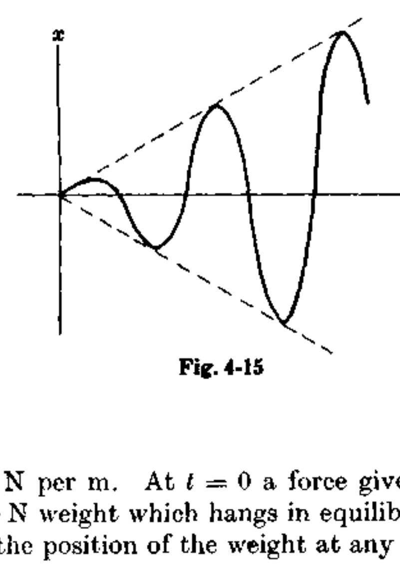

Fig. 4-15

4.21. A vertical spring has a stiffness factor equal to 48 N per m. At $t = 0$ a force give newtons by $F(t) = 51\sin 4t$, $t \geqq 0$ is applied to a 30 N weight which hangs in equilib at the end of the spring. Neglecting damping, find the position of the weight at any time t.

Using the method of Problem 4.7, we have by Newton's second law,

$$\frac{30}{10}\frac{d^2z}{dt^2} = -48z + 51\sin 4t$$

or

$$\frac{d^2z}{dt^2} + 16z = 17\sin 4t$$

Solving,

$$z = A\cos 4t + B\sin 4t - 17/8\cos 4t$$

When $t = 0$, $z = 0$ and $dz/dt = 0$; then $A = 0$, $B = 17/32$ and

$$z = 17/32\sin 4t - 17/8\, t\cos 4t$$

As t gets larger the term $-17/8\, t\cos 4t$ increases numerically without bound, and physically the will ultimately break. The example illustrates the phenomenon of resonance. Note that the r frequency of the spring ($4/2\pi = 2/\pi$) equals the frequency of the impressed force.

4.22. Work Problem 4.21 if $F(t) = 120 \cos 6t, t \geqq 0$.

In this case the equation (*1*) of Problem 4.21 becomes

$$d^2z/dt^2 + 16z = 40 \cos 6t \tag{1}$$

and the initial conditions are

$$z = 0, \ dz/dt = 0 \quad \text{at} \quad t = 0 \tag{2}$$

The general solution of (*1*) is

$$z = A \cos 4t + B \sin 4t - 2 \cos 6t \tag{3}$$

Using conditions (*2*) in (*3*), we find $A = 2$, $B = 0$, and

$$z = 2(\cos 4t - \cos 6t) = 2\{\cos(5t - t) - \cos(5t + t)\} = 4 \sin t \sin 5t$$

The graph of z vs. t is shown by the heavy curve of Fig. 4-16. The dashed curves are the curves $z = \pm 4 \sin t$ obtained by placing $\sin 5t = \pm 1$. If we consider that $4 \sin t$ is the amplitude of $\sin 5t$, we see that the amplitude varies sinusoidally. The phenomenon is known as amplitude modulation and is of practical importance in communications and electronics.

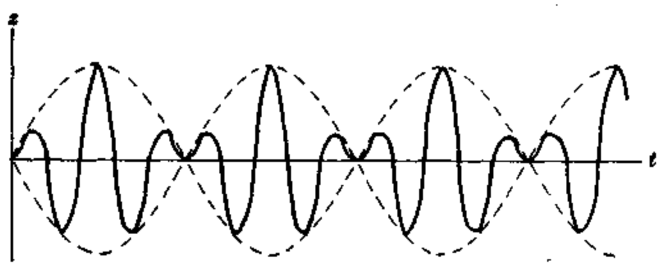

Fig. 4-16

THE SIMPLE PENDULUM

4.23. Determine the motion of a simple pendulum of length l and mass m assuming small vibrations and no resisting forces.

Let the position of m at any time be determined by s, the arclength measured from the equilibrium position O [see Fig. 4-17]. Let θ be the angle made by the pendulum string with the vertical.

If $\mathbf{T}$ is a unit tangent vector to the circular path of the pendulum bob m, then by Newton's second law

$$m \frac{d^2 s}{dt^2} \mathbf{T} = -mg \sin \theta \, \mathbf{T} \tag{1}$$

or, since $s = l\theta$,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin \theta \tag{2}$$

For small vibrations we can replace $\sin \theta$ by θ so that to a high degree of accuracy equation (*2*) can be replaced by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{3}$$

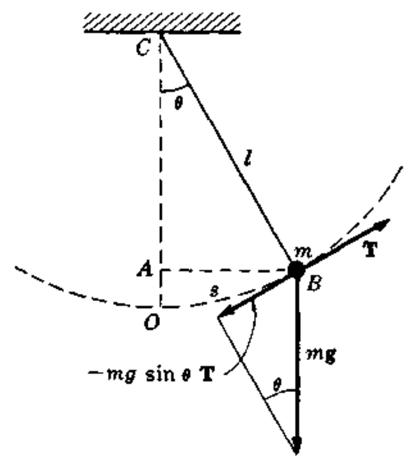

Fig. 4-17

which has solution

$$\theta = A \cos \sqrt{g/l}\, t + B \sin \sqrt{g/l}\, t$$

Taking as initial conditions $\theta = \theta_0$, $d\theta/dt = 0$ at $t = 0$, we find $A = \theta_0$, $B = 0$ and so

$$\theta = \theta_0 \cos \sqrt{g/l}\, t$$

From this we see that the period of the pendulum is $2\pi\sqrt{l/g}$.

4.24. Show how to obtain the equation (2) for the pendulum of Problem 4.23 by using the principle of conservation of energy.

We see from Fig. 4-17 that $OA = OC - AC = l - l\cos\theta = l(1-\cos\theta)$. Then by the conservation of energy [taking the reference level for the potential energy as a horizontal plane through the lowest point O] we have

$$\text{Potential energy at } B + \text{Kinetic energy at } B = \text{Total energy} = E = \text{constant}$$

$$mgl(1-\cos\theta) + \tfrac{1}{2}m(ds/dt)^2 = E \qquad (1)$$

Since $s = l\theta$, this becomes

$$mgl(1-\cos\theta) + \tfrac{1}{2}ml^2(d\theta/dt)^2 = E \qquad (2)$$

Differentiating both sides of (2) with respect to t, we find

$$mgl\sin\theta\,\dot{\theta} + ml^2\dot{\theta}\,\ddot{\theta} = 0 \quad \text{or} \quad \ddot{\theta} + (g/l)\sin\theta = 0$$

in agreement with equation (2) of Problem 4.23.

4.25. Work Problem 4.23 if a damping force proportional to the instantaneous velocity is taken into account.

In this case the equation of motion (1) of Problem 4.23 is replaced by

$$m\frac{d^2s}{dt^2}\mathbf{T} = -mg\sin\theta\,\mathbf{T} - \beta\frac{ds}{dt}\mathbf{T} \quad \text{or} \quad \frac{d^2s}{dt^2} = -g\sin\theta - \frac{\beta}{m}\frac{ds}{dt}$$

Using $s = l\theta$ and replacing $\sin\theta$ by θ for small vibrations, this becomes

$$\frac{d^2\theta}{dt^2} + \frac{\beta}{m}\frac{d\theta}{dt} + \frac{g}{l}\theta = 0$$

Three cases arise:

Case 1. $\beta^2/4m^2 < g/l$

$$\theta = e^{-\beta t/2m}(A\cos\omega t + B\sin\omega t) \qquad \text{where } \omega = \sqrt{g/l - \beta^2/4m^2}$$

This is the case of *damped oscillations* or *underdamped motion.*

Case 2. $\beta^2/4m^2 = g/l$

$$\theta = e^{-\beta t/2m}(A + Bt)$$

This is the case of *critically damped motion.*

Case 3. $\beta^2/4m^2 > g/l$

$$\theta = e^{-\beta t/2m}(Ae^{\lambda t} + Be^{-\lambda t}) \qquad \text{where } \lambda = \sqrt{\beta^2/4m^2 - g/l}$$

This is the case of *overdamped motion.*

In each case the constants A and B can be determined from the initial conditions. In Case 1 there are continually decreasing oscillations. In Cases 2 and 3 the pendulum bob gradually returns to the equilibrium position without oscillation.

THE TWO AND THREE DIMENSIONAL HARMONIC OSCILLATOR

4.26. Find the potential energy for (*a*) the two dimensional and (*b*) the three dimensional harmonic oscillator.

(*a*) In this case the force is given by

$$\mathbf{F} = -\kappa_1 x\mathbf{i} - \kappa_2 y\mathbf{j}$$

Since $\nabla \times \mathbf{F} = \mathbf{0}$, the force field is conservative. Thus a potential does exist, i.e. there exists a function V such that $\mathbf{F} = -\nabla V$. We thus have

$$\mathbf{F} = -\kappa_1 x\mathbf{i} - \kappa_2 y\mathbf{j} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}$$

from which $\partial V/\partial x = \kappa_1 x,\ \partial V/\partial y = \kappa_2 y,\ \partial V/\partial z = 0$ or

$$V = \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{2}\kappa_2 y^2$$

choosing the arbitrary additive constant to be zero. This is the required potential energy.

(b) In this case we have $\mathbf{F} = -\kappa_1 x\mathbf{i} - \kappa_2 y\mathbf{j} - \kappa_3 z\mathbf{k}$ which is also conservative since $\nabla \times \mathbf{F} = \mathbf{0}$. *We then find as in part (a),* $\partial V/\partial x = \kappa_1 x$, $\partial V/\partial y = \kappa_2 y$, $\partial V/\partial z = \kappa_3 z$ *from which the required potential energy is*

$$V = \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{2}\kappa_2 y^2 + \tfrac{1}{2}\kappa_3 z^2$$

4.27. A particle moves in the xy plane in a force field given by $\mathbf{F} = -\kappa x\mathbf{i} - \kappa y\mathbf{j}$. Prove that in general it will move in an elliptical path.

If the particle has mass m, its equation of motion is

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} = -\kappa x\mathbf{i} - \kappa y\mathbf{j} \tag{1}$$

or, since $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$,

$$m\frac{d^2x}{dt^2}\mathbf{i} + m\frac{d^2y}{dt^2}\mathbf{j} = -\kappa x\mathbf{i} - \kappa y\mathbf{j}$$

Then

$$m\frac{d^2x}{dt^2} = -\kappa x, \qquad m\frac{d^2y}{dt^2} = -\kappa y \tag{2}$$

These equations have solutions given respectively by

$$x = A_1\cos\sqrt{\kappa/m}\,t + A_2\sin\sqrt{\kappa/m}\,t, \qquad y = B_1\cos\sqrt{\kappa/m}\,t + B_2\sin\sqrt{\kappa/m}\,t \tag{3}$$

Let us suppose that at $t = 0$ the particle is located at the point whose position vector is $\mathbf{r} = a\mathbf{i} + b\mathbf{j}$ and moving with velocity $d\mathbf{r}/dt = v_1\mathbf{i} + v_2\mathbf{j}$. Using these conditions, we find $A_1 = a$, $B_1 = b$, $A_2 = v_1\sqrt{m/\kappa}$, $B_2 = v_2\sqrt{m/\kappa}$ and so

$$x = a\cos\omega t + c\sin\omega t, \qquad y = b\cos\omega t + d\sin\omega t \tag{4}$$

where $c = v_1\sqrt{m/\kappa}$, $d = v_2\sqrt{m/\kappa}$. Solving for $\sin\omega t$ and $\cos\omega t$ in (4) we find, if $ad \neq bc$,

$$\cos\omega t = \frac{dx - cy}{ad - bc}, \qquad \sin\omega t = \frac{ay - bx}{ad - bc}$$

Squaring and adding, using the fact that $\cos^2\omega t + \sin^2\omega t = 1$, we find

$$(dx - cy)^2 + (ay - bx)^2 = (ad - bc)^2$$

or

$$(b^2 + d^2)x^2 - 2(cd + ab)xy + (a^2 + c^2)y^2 = (ad - bc)^2 \tag{5}$$

Now the equation

$$Ax^2 + Bxy + Cy^2 = D \qquad \text{where } A > 0,\ C > 0,\ D > 0$$

is an ellipse if $B^2 - 4AC < 0$, a parabola if $B^2 - 4AC = 0$, and a hyperbola if $B^2 - 4AC > 0$. To determine what (5) is, we see that $A = b^2 + d^2$, $B = -2(cd + ab)$, $C = a^2 + c^2$ so that

$$B^2 - 4AC = 4(cd + ab)^2 - 4(b^2 + d^2)(a^2 + c^2) = -4(ad - bc)^2 < 0$$

provided $ad \neq bc$. Thus in general the path is an ellipse, and if $A = C$ it is a circle. If $ad = bc$ the ellipse reduces to the straight line $ay = bx$.

MISCELLANEOUS PROBLEMS

4.28. A cylinder having axis vertical floats in a liquid of density σ. It is pushed down slightly and released. Find the period of the oscillation if the cylinder has weight W and cross sectional area A.

Let RS, the equilibrium position of the cylinder, be distant z from the liquid surface PQ at any time t. By *Archimedes' principle*, the buoyant force on the cylinder is $(Az)\sigma g$. Then by Newton's second law,

$$\frac{W}{g}\frac{d^2z}{dt^2} = -Az\sigma g$$

or

$$\frac{d^2z}{dt^2} + \frac{g^2A\sigma}{W}z = 0$$

Solving,

$$z = c_1 \cos\sqrt{g^2A\sigma/W}\,t + c_2 \sin\sqrt{g^2A\sigma/W}\,t$$

and the period of the oscillation is $2\pi\sqrt{W/g^2A\sigma}$.

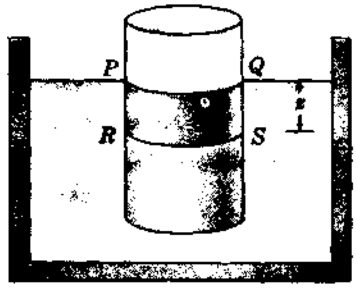

Fig. 4-18

4.29. Show that if the assumption of small vibrations is not made, then the period of a simple pendulum is

$$4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \qquad \text{where} \quad k = \sin(\theta_0/2)$$

The equation of motion for a simple pendulum if small vibrations are not assumed is [equation (34), page 91]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \tag{1}$$

Let $d\theta/dt = u$. Then

$$\frac{d^2\theta}{dt^2} = \frac{du}{dt} = \frac{du}{d\theta}\frac{d\theta}{dt} = u\frac{du}{d\theta}$$

and (1) becomes

$$u\frac{du}{d\theta} = -\frac{g}{l}\sin\theta \tag{2}$$

Integrating (2) we obtain

$$\frac{u^2}{2} = \frac{g}{l}\cos\theta + c \tag{3}$$

Now when $\theta = \theta_0$, $u = 0$ so that $c = -(g/l)\cos\theta_0$. Thus (3) can be written

$$u^2 = (2g/l)(\cos\theta - \cos\theta_0) \qquad \text{or} \qquad d\theta/dt = \pm\sqrt{(2g/l)(\cos\theta - \cos\theta_0)} \tag{4}$$

If we restrict ourselves to that part of the motion where the bob goes from $\theta = \theta_0$ to $\theta = 0$, which represents a time equal to one fourth of the period, then we must use the minus sign in (4) so that it becomes

$$d\theta/dt = -\sqrt{(2g/l)(\cos\theta - \cos\theta_0)}$$

Separating the variables and integrating, we have

$$t = -\sqrt{\frac{l}{2g}}\int\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

Since $t = 0$ at $\theta = \theta_0$ and $t = P/4$ at $\theta = 0$, where P is the period,

$$P = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{5}$$

Making use of the trigonometric identity $\cos\theta = 2\sin^2(\theta/2) - 1$, with a similar one replacing θ by θ_0, (5) can be written

$$P = 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} \tag{6}$$

Now let

$$\sin(\theta/2) = \sin(\theta_0/2)\sin\phi \tag{7}$$

Then taking the differential of both sides,

$$\tfrac{1}{2}\cos(\theta/2)\,d\theta \quad = \quad \sin(\theta_0/2)\cos\phi\,d\phi$$

or calling $k = \sin(\theta_0/2)$,

$$d\theta \quad = \quad \frac{2\sin(\theta_0/2)\cos\phi\,d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

Also from *(7)* we see that when $\theta = 0,\ \phi = 0$; and when $\theta = \theta_0,\ \phi = \pi/2$. Hence *(6)* becomes, as required,

$$P \quad = \quad 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{8}$$

Note that if we have small vibrations, i.e. if k is equal to zero very nearly, then the period *(8)* becomes

$$P \quad = \quad 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}d\phi \quad = \quad 2\pi\sqrt{\frac{l}{g}} \tag{9}$$

as we have already seen.

The integral in *(8)* is called an *elliptic integral* and cannot be evaluated exactly in terms of elementary functions. The equation of motion of the pendulum can be solved for θ in terms of *elliptic functions* which are generalizations of the trigonometric functions.

4.30. Show that period given in Problem 4.29 can be written as

$$P \quad = \quad 2\pi\sqrt{l/g}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right\}$$

The *binomial theorem* states that if $|x| < 1$, then

$$(1+x)^p \quad = \quad 1 + px + \frac{p(p-1)}{2\cdot 1}x^2 + \frac{p(p-1)(p-2)}{3\cdot 2\cdot 1}x^3 + \cdots$$

If $p = -\frac{1}{2}$, this can be written

$$(1+x)^{-1/2} \quad = \quad 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$$

Letting $x = -k^2\sin^2\phi$ and integrating from 0 to $\pi/2$, we find

$$\begin{aligned} P \quad &= \quad 4\sqrt{l/g}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \\ &= \quad 4\sqrt{l/g}\int_0^{\pi/2}\left\{1 + \frac{1}{2}k^2\sin^2\phi + \frac{1\cdot 3}{2\cdot 4}k^4\sin^4\phi + \cdots\right\}d\phi \\ &= \quad 2\pi\sqrt{l/g}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right\} \end{aligned}$$

where we have made use of the integration formula

$$\int_0^{\pi/2}\sin^{2n}\phi\,d\phi \quad = \quad \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,\frac{\pi}{2}$$

The term by term integration is possible since $|k| < 1$.

4.31. A bead of mass m is constrained to move on a frictionless wire in the shape of a cycloid [Fig. 4-19 below] whose parametric equations are

$$x \;=\; a(\phi - \sin\phi), \qquad y \;=\; a(1-\cos\phi) \tag{1}$$

which lies in a vertical plane. If the bead starts from rest at point O, *(a)* find the speed at the bottom of the path and *(b)* show that the bead performs oscillations with period equivalent to that of a simple pendulum of length $4a$.

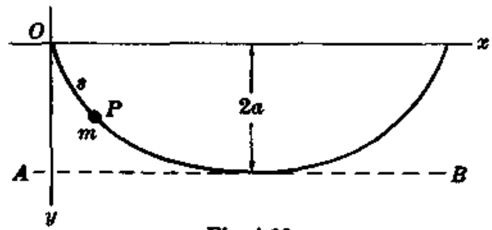

Fig. 4-19

(a) Let P be the position of the bead at any time t and let s be the arclength along the cycloid measured from point O.

By the conservation of energy, measuring potential energy relative to line AB through the minimum point of the cycloid, we have

P.E. at P + K.E. at P = P.E. at O + K.E. at O

$$mg(2a - y) + \tfrac{1}{2}m(ds/dt)^2 = mg(2a) + 0 \quad (2)$$

Thus $$v^2 = (ds/dt)^2 = 2gy \quad \text{or} \quad v = ds/dt = \sqrt{2gy} \quad (3)$$

At the lowest point $y = 2a$ the speed is $v = \sqrt{2g(2a)} = 2\sqrt{ga}$.

(b) From part (a), $(ds/dt)^2 = 2gy$. But

$$(ds/dt)^2 = (dx/dt)^2 + (dy/dt)^2 = a^2(1 - \cos\phi)^2\dot{\phi}^2 + a^2\sin^2\phi\,\dot{\phi}^2 = 2a^2(1 - \cos\phi)\dot{\phi}^2$$

Then $2a^2(1 - \cos\phi)\dot{\phi}^2 = 2ga(1 - \cos\phi)$ or $\dot{\phi}^2 = g/a$. Thus

$$d\phi/dt = \sqrt{g/a} \quad \text{and} \quad \phi = \sqrt{g/a}\,t + c_1 \quad (4)$$

When $\phi = 0$, $t = 0$; when $\phi = 2\pi$, $t = P/2$ where P is the period. Hence from the second equation of (4),

$$P = 4\pi\sqrt{a/g} = 2\pi\sqrt{4a/g}$$

and the period is the same as that of a simple pendulum of length $l = 4a$.

For some interesting applications see Problems 4.86-4.88.

4.32. A particle of mass m is placed on the inside of a smooth paraboloid of revolution having equation $cz = x^2 + y^2$ at a point P which is at height H above the horizontal [assumed as the xy plane]. Assuming that the particle starts from rest, (a) find the speed with which it reaches the vertex O, (b) find the time τ taken, and (c) find the period for small vibrations.

It is convenient to choose the point P in the yz plane so that $x = 0$ and $cz = y^2$. By the principle of conservation of energy we have if Q is any point on the path PQO,

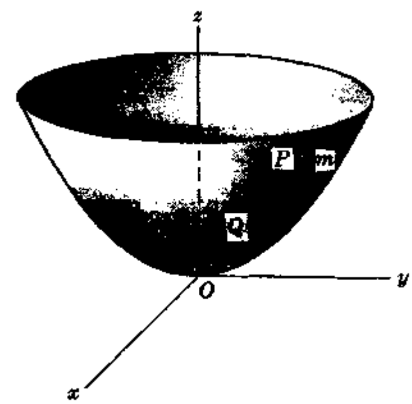

Fig. 4-20

$$\text{P.E. at } P + \text{K.E. at } P = \text{P.E. at } Q + \text{K.E. at } Q$$

$$mgH + \tfrac{1}{2}m(0)^2 = mgz + \tfrac{1}{2}m(ds/dt)^2$$

where s is the arclength along OPQ measured from O. Thus

$$(ds/dt)^2 = 2g(H - z) \quad (1)$$

or

$$ds/dt = -\sqrt{2g(H - z)} \quad (2)$$

using the negative sign since s is decreasing with t.

(a) Putting $z = 0$, we see that the speed is $\sqrt{2gH}$ at the vertex.

(b) We have, since $x = 0$ and $cz = y^2$,

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{dy}{dt}\right)^2 + \frac{4y^2}{c^2}\left(\frac{dy}{dt}\right)^2 = \left(1 + \frac{4y^2}{c^2}\right)\left(\frac{dy}{dt}\right)^2$$

Thus (1) can be written $(1 + 4y^2/c^2)(dy/dt)^2 = 2g(H - y^2/c)$. Then

$$\frac{dy}{dt} = -\sqrt{2gc}\,\frac{\sqrt{cH - y^2}}{\sqrt{c^2 + 4y^2}} \quad \text{or} \quad -\sqrt{2gc}\,dt = \frac{\sqrt{c^2 + 4y^2}}{\sqrt{cH - y^2}}\,dy$$

Integrating, using the fact that $z = H$ and thus $y = \sqrt{cH}$ at $t = 0$ while at $t = \tau$, $y = 0$, we have

$$\int_0^\tau -\sqrt{2gc}\,dt \quad = \quad \int_{\sqrt{cH}}^0 \frac{\sqrt{c^2 + 4y^2}}{\sqrt{cH - y^2}}\,dy \qquad \text{or} \qquad \tau \quad = \quad \frac{1}{\sqrt{2gc}} \int_0^{\sqrt{cH}} \frac{\sqrt{c^2 + 4y^2}}{\sqrt{cH - y^2}}\,dy$$

Letting $y = \sqrt{cH}\cos\theta$, the integral can be written

$$\tau \quad = \quad \frac{1}{\sqrt{2gc}} \int_0^{\pi/2} \sqrt{c^2 + 4cH\cos^2\theta}\,d\theta \quad = \quad \frac{1}{\sqrt{2gc}} \int_0^{\pi/2} \sqrt{c^2 + 4cH - 4cH\sin^2\theta}\,d\theta$$

and this can be written

$$\tau \quad = \quad \sqrt{\frac{c + 4H}{2g}} \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\,d\theta \tag{3}$$

where

$$k \quad = \quad \sqrt{4H/(c + 4H)} \quad < \quad 1 \tag{4}$$

The integral in (3) is an *elliptic integral* and cannot be evaluated in terms of elementary functions. It can, however, be evaluated in terms of series [see Problem 4.119].

(c) The particle oscillates back and forth on the inside of the paraboloid with period given by

$$P \quad = \quad 4\tau \quad = \quad 4\sqrt{\frac{c + 4H}{2g}} \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\theta}\,d\theta \tag{5}$$

For small vibrations the value of k given by (4) can be assumed so small so as to be zero for practical purposes. Hence (5) becomes

$$P \quad = \quad 2\pi\sqrt{(c + 4H)/2g}$$

The length of the equivalent simple pendulum is $l = \frac{1}{2}(c + 4H)$.

Supplementary Problems

SIMPLE HARMONIC MOTION AND THE SIMPLE HARMONIC OSCILLATOR

4.33. A particle of mass 12 g moves along the x axis attracted toward the point O on it by a force in dynes which is numerically equal to 60 times its instantaneous distance x cm from O. If the particle starts from rest at $x = 10$, find the (a) amplitude, (b) period and (c) frequency of the motion.

Ans. (a) 10 cm, (b) $2\pi/\sqrt{5}$ s, (c) $\sqrt{5}/2\pi$ Hz

4.34. (a) If the particle of Problem 4.33 starts at $x = 10$ with a speed toward O of 20 cm/sec, determine its amplitude, period and frequency. (b) Determine when the particle reaches O for the first time.

Ans. (a) Amplitude $= 6\sqrt{5}$ cm, period $= 2\pi/\sqrt{5}$ s, frequency $= \sqrt{5}/2\pi$ Hz; (b) 0.33 s

4.35. A particle moves on the x axis attracted toward the origin O on it with a force proportional to its instantaneous distance from O. If it starts from rest at $x = 5$ cm and reaches $x = 2.5$ cm for the first time after 2 s, find (a) the position at any time t after it starts, (b) the speed at $x = 0$, (c) the amplitude, period and frequency of the vibration, (d) the maximum acceleration, (e) the maximum speed.

Ans. (a) $x = 5\cos(\pi t/6)$; (b) $5\pi/6$ cm/s; (c) 5 cm, 12 s, 1/12 Hz; (d) $5\pi^2/36$ cm/s²; (e) $5\pi/6$ cm/s

4.36. If a particle moves with simple harmonic motion along the x axis, prove that (a) the acceleration is numerically greatest at the ends of the path, (b) the velocity is numerically greatest in the middle of the path, (c) the acceleration is zero in the middle of the path, (d) the velocity is zero at the ends of the path.

4.37. A particle moves with simple harmonic motion in a straight line. Its maximum speed is 6 m/s and its maximum acceleration is 24 m/s. Find the period and frequency of the motion.

Ans. $\pi/2$ s, $2/\pi$ Hz

4.38. A particle moves with simple harmonic motion. If its acceleration at distance D from the equilibrium position is A, prove that the period of the motion is $2\pi\sqrt{D/A}$.

4.39. A particle moving with simple harmonic motion has speeds of 3 cm/s and 4 cm/s at distances 8 cm and 6 cm respectively from the equilibrium position. Find the period of the motion. *Ans.* 4π s

4.40. An 8 kg weight placed on a vertical spring stretches it 20 cm. The weight is then pulled down a distance of 40 cm and released. (*a*) Find the amplitude, period and frequency of the oscillations. (*b*) What is the position and speed at any time?

Ans. (*a*) 40 cm, $2\pi/7$ s, $7/2\pi$ Hz
(*b*) $x = 40 \cos 7t$ cm, $v = -280 \sin 7t$ cm/s

4.41 A mass of 200 g placed at the lower end of a vertical spring stretches it 20 cm. When it is in equilibrium the mass is hit and due to this goes up a distance of 8 cm before coming down again. Find (*a*) the magnitude of the velocity imparted to the mass when it is hit and (*b*) the period of the motion. the motion. *Ans.* (*a*) 56 cm/s, (*b*) $2\pi/7$ s

4.42. A 5 kg mass at the end of a spring moves with simple harmonic motion along a horizontal straight line with period 3 s and amplitude 2 meters. (*a*) Determine the spring constant. (*b*) What is the maximum force exerted on the spring?

Ans. (*a*) 1140 dynes/cm or 1.14 newtons/meter
(*b*) 2.28×10^5 dynes or 2.28 newtons

4.43. When a mass M hanging from the lower end of a vertical spring is set into motion, it oscillates with period P. Prove that the period when mass m is added is $P\sqrt{1+m/M}$.

THE DAMPED HARMONIC OSCILLATOR

4.44. (*a*) Solve the equation $d^2x/dt^2 + 2\,dx/dt + 5x = 0$ subject to the conditions $x = 5$, $dx/dt = -3$ at $t = 0$ and (*b*) give a physical interpretation of the results.

Ans. (*a*) $x = e^{-t}(5 \cos 2t + \sin 2t)$

4.45. Verify that the damping force given by equation (*2*) of Problem 4.11 is correct regardless of the position and velocity of the particle.

4.46. A 1.5 kg weight hung on a vertical spring stretches it 0.4 m. The weight is then pulled down 1 m and released. (*a*) Find the position of the weight at any time if a damping force numerically equal to 15 times the instantaneous speed is acting. (*b*) Is the motion oscillatory damped, overdamped or critically damped? (Use $g = 10$ m/s²). *Ans.* (*a*) $x = e^{-5t}(5t+1)$, (*b*) critically damped

4.47. Work Problem 4.46 if the damping force is numerically 18.75 times the instantaneous speed. *Ans.* (*a*) $x = \frac{1}{3}(4e^{-5/2t} - e^{-10t})$, (*b*) overdamped

4.48. In Problem 4.46, suppose that the damping force is numerically 7.5 times the instantaneous speed. (*a*) Prove that the motion is damped oscillatory. (*b*) Find the amplitude, period and frequency of the oscillations. (*c*) Find the logarithmic decrement.

Ans. (*b*) Amplitude $= \frac{2}{\sqrt{3}} e^{-5t/2}$ m, period $= 4\pi/5\sqrt{3}$ s, frequency $= 5\sqrt{3}/4\pi$ Hz; (*c*) $2\pi/\sqrt{3}$

4.49. Prove that the logarithmic decrement is the time required for the maximum amplitude during an oscillation to reduce to $1/e$ of this value.

4.50. The natural frequency of a mass vibrating on a spring is 20 Hz, while its frequency with damping is 16 Hz. Find the logarithmic decrement. *Ans.* $(3/4)2\pi$

4.51. Prove that the difference in times corresponding to the successive maximum displacements of a damped harmonic oscillator with equation given by (*12*) of page 88 is constant and equal to $4\pi m/\sqrt{4\kappa m - \beta^2}$.

4.52. Is the difference in times between successive minimum displacements of a damped harmonic oscillator the same as in Problem 4.51? Justify your answer.

FORCED VIBRATIONS AND RESONANCE

4.53. The position of a particle moving along the x axis is determined by the equation $d^2x/dt^2 + 4dx/dt + 8x = 20\cos 2t$. If the particle starts from rest at $x = 0$, find (a) x as a function of t, (b) the amplitude, period and frequency of the oscillation after a long time has elapsed.

Ans. (a) $x = \cos 2t + 2\sin 2t - e^{-2t}(\cos 2t + 3\sin 2t)$

(b) Amplitude $= \sqrt{5}$, period $= \pi$, frequency $= 1/\pi$

4.54. (a) Give a physical interpretation to Problem 4.53 involving a mass at the end of a vertical spring. (b) What is the natural frequency of such a vibrating spring? (c) What is the frequency of the impressed force? *Ans.* (b) $\sqrt{2}/\pi$, (c) $1/\pi$

4.55. The weight on a vertical spring undergoes forced vibrations according to the equation $d^2x/dt^2 + 4x = 8\sin\omega t$ where x is the displacement from the equilibrium position and $\omega > 0$ is a constant. If at $t = 0$, $x = 0$ and $dx/dt = 0$, find (a) x as a function of t, (b) the period of the external force for which resonance occurs.

Ans. (a) $x = (8\sin\omega t - 4\omega\sin 2t)/(4 - \omega^2)$ if $\omega \neq 2$; $x = \sin 2t - 2t\cos 2t$ if $\omega = 2$

(b) $\omega = 2$ or period $= \pi$

4.56. A vertical spring having constant 272 N/m has a 16 kg weight suspended from it. An external force given as a function of time t by $F(t) = 240\sin 4t$, $t \geq 0$ is applied. A damping force given numerically in newtons by $32\,v$, where v is the instantaneous speed of the object in m/s, is assumed to act. Initially the weight is at rest at the equilibrium position. (a) Determine the position of the weight at any time. (b) Indicate the transient and steady-state solutions, giving physical interpretations of each. (c) Find the amplitude, period and frequency of the steady-state solution. (Use $g = 10$ m/s².)

Ans. (a) $x = \frac{3}{13}e^{-t}(8\cos 4t + \sin 4t) + \frac{3}{13}\sin 4t - \frac{24}{13}\cos 4t$

(b) Transient, $\frac{3}{13}e^{-t}(8\cos 4t + \sin 4t)$; Steady state, $\frac{3}{13}\sin 4t - \frac{24}{13}\cos 4t$

(c) Amplitude $= 3\sqrt{\frac{5}{13}}$ m, period $= \pi/2$ s, frequency $= 2/\pi$ Hz

4.57. A spring is stretched 5 cm by a force of 50 dynes. A mass of 10 g is placed on the lower end of the spring. After equilibrium has been reached, the upper end of the spring is moved up and down so that the external force acting on the mass is given by $F(t) = 20\cos\omega t$, $t \geq 0$. (a) Find the position of the mass at any time, measured from its equilibrium position. (b) Find the value of ω for which resonance occurs. *Ans.* (a) $x = 2(\cos\omega t - \cos t)/(1 - \omega^2)$, (b) $\omega = 1$

4.58. A periodic external force acts on a 6 kg mass suspended from the lower end of a vertical spring having constant 150 newtons/meter. The damping force is proportional to the instantaneous speed of the mass and is 80 newtons when the speed is 2 meters/sec. Find the frequency at which resonance occurs. *Ans.* $5/6\pi$ Hz

THE SIMPLE PENDULUM

4.59. Find the length of a simple pendulum whose period is 1 second. Such a pendulum which registers seconds is called a *seconds pendulum*. *Ans.* 99.3 cm

4.60. Will a pendulum which registers seconds at one location lose or gain time when it is moved to another location where the acceleration due to gravity is greater? Explain.

Ans. Gain time

4.61. A simple pendulum whose length is 2 meters has its bob drawn to one side until the string makes an angle of 30° with the vertical. The bob is then released. (a) What is the speed of the bob as it passes through its lowest point? (b) What is the angular speed at the lowest point? (c) What is the maximum acceleration and where does it occur?

Ans. (a) 2.93 m/s, (b) 1.46 rad/s, (c) 2 m/s²

4.62. Prove that the tension in the string of a vertical simple pendulum of length l and mass m is given by $mg \cos\theta$ where θ is the instantaneous angle made by the string with the vertical.

4.63. A seconds pendulum which gives correct time at a certain location is taken to another location where it is found to lose T seconds per day. Determine the gravitational acceleration at the second location. *Ans.* $g(1 - T/86{,}400)^2$ where g is the gravitational acceleration at the first location

4.64. What is the length of a seconds pendulum on the surface of the moon where the acceleration due to gravity is approximately 1/6 that on the earth? *Ans.* 16.5 cm

4.65. A simple pendulum of length l and mass m hangs vertically from a fixed point O. The bob is given an initial horizontal velocity of magnitude v_0. Prove that the arc through which the bob swings in one period has a length given by $4l\cos^{-1}(1 - v_0^2/2gl)$

4.66. Find the minimum value of v_0 in Problem 4.65 in order that the bob will make a complete vertical circle with center at O. *Ans.* $2\sqrt{gl}$

THE TWO AND THREE DIMENSIONAL HARMONIC OSCILLATOR

4.67. A particle of mass 2 moves in the xy plane attracted to the origin with a force given by $\mathbf{F} = -18x\mathbf{i} - 50y\mathbf{j}$. At $t = 0$ the particle is placed at the point $(3, 4)$ and given a velocity of magnitude 10 in a direction perpendicular to the x axis. (*a*) Find the position and velocity of the particle at any time. (*b*) What curve does the particle describe?

Ans. (*a*) $\mathbf{r} = 3\cos 3t\,\mathbf{i} + [4\cos 5t + 2\sin 5t]\mathbf{j}$, $\mathbf{v} = -9\sin 3t\,\mathbf{i} + [10\cos 5t - 20\sin 5t]\mathbf{j}$

4.68. Find the total energy of the particle of Problem 4.67. *Ans.* 581

4.69. A two dimensional harmonic oscillator of mass 2 has potential energy given by $V = 8(x^2 + 4y^2)$. If the position vector and velocity of the oscillator at time $t = 0$ are given respectively by $\mathbf{r}_0 = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{v}_0 = 4\mathbf{i} + 8\mathbf{j}$, (*a*) find its position and velocity at any time $t > 0$ and (*b*) determine the period of the motion.

Ans. (*a*) $\mathbf{r} = (2\cos 4t + \sin 4t)\mathbf{i} + (\sin 8t - \cos 8t)\mathbf{j}$, $\mathbf{v} = (4\cos 4t - 8\sin 4t)\mathbf{i} + (8\cos 8t + 8\sin 8t)\mathbf{j}$
(*b*) $\pi/8$

4.70. Work Problem 4.69 if $V = 8(x^2 + 2y^2)$. Is there a period defined for the motion in this case? Explain.

4.71. A particle of mass m moves in a 3 dimensional force field whose potential is given by $V = \frac{1}{2}\kappa(x^2 + 4y^2 + 16z^2)$. (*a*) Prove that if the particle is placed at an arbitrary point in space other than the origin, then it will return to the point after some period of time. Determine this time. (*b*) Is the velocity on returning to the starting point the same as the initial velocity? Explain.

4.72. Suppose that in Problem 4.71 the potential is $V = \frac{1}{2}\kappa(x^2 + 2y^2 + 5z^2)$. Will the particle return to the starting point? Explain.

MISCELLANEOUS PROBLEMS

4.73. A vertical spring of constant κ having natural length l is supported at a fixed point A. A mass m is placed at the lower end of the spring, lifted to a height h below A and dropped. Prove that the lowest point reached will be at a distance below A given by $l + mg/\kappa + \sqrt{m^2g^2/\kappa^2 + 2mgh/\kappa}$.

4.74. Work Problem 4.73 if damping proportional to the instantaneous velocity is taken into account.

4.75 Given the equation $m\ddot{x} + \beta\dot{x} + \kappa x = 0$ for damped oscillations of a harmonic oscillator. Prove that if $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\kappa x^2$, then $\dot{E} = -\beta\dot{x}^2$. Thus show that if there is damping the total energy E decreases with time. What happens to the energy lost? Explain.

4.76. (a) Prove that $A_1 \cos(\omega t - \phi_1) + A_2 \cos(\omega t - \phi_2) = A \cos(\omega t - \phi)$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2)}$, $\phi = \tan^{-1}\left(\dfrac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}\right)$.

(b) Use (a) to demonstrate that the sum of two simple harmonic motions of the same frequency and in the same straight line is simple harmonic of the same frequency.

4.77. Give a vector interpretation to the results of Problem 4.76.

4.78. Discuss Problem 4.76 in case the frequencies of the two simple harmonic motions are not equal. Is the resultant motion simple harmonic? Justify your answer.

4.79. A particle oscillates in a plane so that its distances x and y from two mutually perpendicular axes are given as functions of time t by

$$x = A\cos(\omega t + \phi_1), \qquad y = B\cos(\omega t + \phi_2)$$

(a) Prove that the particle moves in an ellipse inscribed in the rectangle defined by $x = \pm A$, $y = \pm B$. (b) Prove that the period of the particle in the elliptical path is $2\pi/\omega$.

4.80. Suppose that the particle of Problem 4.79 moves so that

$$x = A\cos(\omega t + \phi_1), \qquad y = B\cos(\omega t + \epsilon t + \phi_2)$$

where ϵ is assumed to be a positive constant which is assumed to be much smaller than ω. Prove that the particle oscillates in slowly rotating ellipses inscribed in the rectangle $x = \pm A$, $y = \pm B$.

4.81. Illustrate Problem 4.80 by graphing the motion of a particle which moves in the path

$$x = 3\cos(2t + \pi/4), \qquad y = 4\cos(2.4t)$$

4.82. In Fig. 4-21 a mass m which is on a frictionless table is connected to fixed points A and B by two springs of equal natural length, of negligible mass and spring constants κ_1 and κ_2 respectively. The mass m is displaced horizontally and then released. Prove that the period of oscillation is given by $P = 2\pi\sqrt{m/(\kappa_1 + \kappa_2)}$.

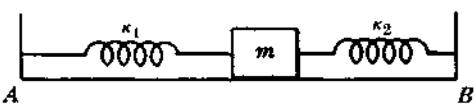

Fig. 4-21

4.83. A spring having constant κ and negligible mass has one end fixed at point A on an inclined plane of angle α and a mass m at the other end, as indicated in Fig. 4-22. If the mass m is pulled down a distance x_0 below the equilibrium position and released, find the displacement from the equilibrium position at any time if (a) the incline is frictionless, (b) the incline has coefficient of friction μ.

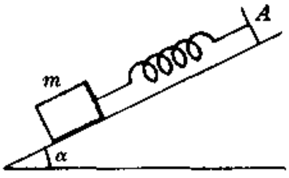

Fig. 4-22

4.84. A particle moves with simple harmonic motion along the x axis. At times t_0, $2t_0$ and $3t_0$ it is located at $x = a, b$ and c respectively. Prove that the period of oscillation is $\dfrac{4\pi t_0}{\cos^{-1}(a+c)/2b}$.

4.85. A seconds pendulum giving the correct time at one location is taken to another location where it loses 5 minutes per day. By how much must the pendulum rod be lengthened or shortened in order to give the correct time?

4.86. A vertical pendulum having a bob of mass m is suspended from the fixed point O. As it oscillates, the string winds up on the constraint curves ODA [or OC] as indicated in Fig. 4-23. Prove that if curve ABC is a cycloid, then the period of oscillation will be the same regardless of the amplitude of the oscillations. The pendulum in this case is called a *cycloidal pendulum*. The curves ODA and OC are constructed to be *evolutes* of the cycloid. [*Hint.* Use Problem 4.31.]

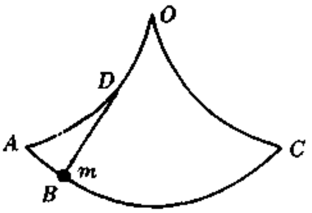

Fig. 4-23

4.87. A bead slides down a frictionless wire located in a vertical plane. It is desired to find the shape of the wire so that regardless of where the bead is placed on the wire it will slide under the influence of gravity to the bottom of the wire in the same time. This is often called the *tautochrone* problem. Prove that the wire must have the shape of a cycloid.
[*Hint.* Use Problem 4.31.]

4.88. Prove that the curves ODA and OC of Problem 4.86 are cycloids having the same shape as the cycloid ABC.

4.89. A simple pendulum of length l has its point of support moving back and forth on a horizontal line so that its distance from a fixed point on the line is $A \sin \omega t$, $t \geqq 0$. Find the position of the pendulum bob at any time t assuming that it is at rest at the equilibrium position at $t = 0$.

4.90. Work Problem 4.89 if the point of support moves vertically instead of horizontally and if at $t = 0$ the rod of the pendulum makes an angle θ_0 with the vertical.

4.91. A particle of mass m moves in a plane under the influence of forces of attraction toward fixed points which are directly proportional to its instantaneous distance from these points. Prove that in general the particle will describe an ellipse.

4.92. A vertical elastic spring of negligible weight and having its upper end fixed, carries a weight W at its lower end. The weight is lifted so that the tension in the spring is zero, and then it is released. Prove that the tension in the spring will not exceed $2W$.

4.93. A vertical spring having constant κ has a pan on top of it with a weight W on it [see Fig. 4-24]. Determine the largest frequency with which the spring can vibrate so that the weight will remain in the pan.

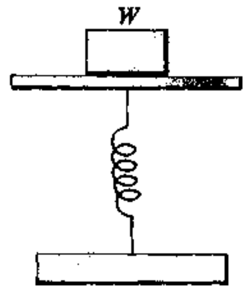

Fig. 4-24

4.94. A spring has a natural length of 50 cm and a force of 100 dynes is required to stretch it 25 cm. Find the work done in stretching the spring from 75 cm to 100 cm, assuming that the elastic limit is not exceeded so that the spring characteristics do not change.
Ans. 3750 ergs

4.95. A particle moves in the xy plane so that its position is given by $x = A \cos \omega t$, $y = B \cos 2\omega t$. Prove that it describes an arc of a parabola.

4.96. A particle moves in the xy plane so that its position is given by $x = A \cos(\omega_1 t + \phi_1)$, $y = B \cos(\omega_2 t + \phi_2)$. Prove that the particle describes a closed curve or not, according as ω_1/ω_2 is rational or not. In which cases is the motion periodic?

4.97. The position of a particle moving in the xy plane is described by the equations $d^2x/dt^2 = -4y$, $d^2y/dt^2 = -4x$. At time $t = 0$ the particle is at rest at the point $(6, 3)$. Find (*a*) its position and (*b*) its velocity at any later time t.

4.98. Find the period of a simple pendulum of length 1 meter if the maximum angle which the rod makes with the vertical is (*a*) 30°, (*b*) 60°, (*c*) 90°.

4.99. A simple pendulum of length 0.9 m is suspended vertically from a fixed point. At $t = 0$ the bob is given a horizontal velocity of 2.4 m/s. Find (*a*) the maximum angle which the pendulum rod makes with the vertical, (*b*) the period of the oscillations.
Ans. (*a*) 47° 38′, (*b*) 1.98 s

4.100. Prove that the time averages over a period of the potential energy and kinetic energy of a simple harmonic oscillator are equal to $2\pi^2A^2/P^2$ where A is the amplitude and P is the period of the motion.

4.101. A cylinder of radius 3 m with its axis vertical oscillates vertically in water of density 10^3 kg m^{-3} with a period of 5 seconds. How much does it weigh? *Ans.* 1.72×10^6 N

4.102. A particle moves in the xy plane in a force field whose potential is given by $V = x^2 + xy + y^2$. If the particle is initially at the point $(3, 4)$ and is given a velocity of magnitude 10 in a direction parallel to the positive x axis, (*a*) find the position at any time and (*b*) determine the period of the motion if one exists.

4.103. In Problem 4.96 suppose that ω_1/ω_2 is irrational and that at $t = 0$ the particle is at the particular point (x_0, y_0) inside the rectangle defined by $x = \pm A$, $y = \pm B$. Prove that the point (x_0, y_0) will never be reached again but that in the course of its motion the particle will come arbitrarily close to the point.

4.104. A particle oscillates on a vertical frictionless cycloid with its vertex downward. Prove that the projection of the particle on a vertical axis oscillates with simple harmonic motion.

4.105. A mass of 5 kg at the lower end of a vertical spring which has an elastic constant equal to 20 newtons/meter oscillates with a period of 10 seconds. Find (*a*) the damping constant, (*b*) the natural period and (*c*) the logarithmic decrement. *Ans.* (*a*) 19 N s/m, (*b*) 3.14 s

4.106. A mass of 100 g is supported in equilibrium by two identical springs of negligible mass having elastic constant equal to 50 dynes/cm. In the equilibrium position shown in Fig. 4-25 the springs make an angle of 30° with the horizontal and are 100 cm in length. If the mass is pulled down a distance of 2 cm and released, find the period of the resulting oscillation.

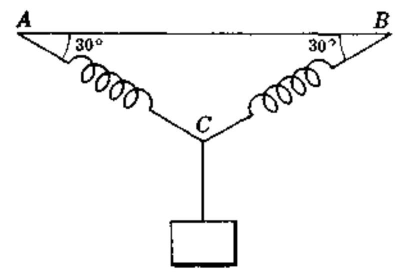

Fig. 4-25

4.107. A thin hollow circular cylinder of inner radius 10 cm is fixed so that its axis is horizontal. A particle is placed on the inner frictionless surface of the cylinder so that its vertical distance above the lowest point of the inner surface is 2 cm. Find (*a*) the time for the particle to reach the lowest point and (*b*) the period of the oscillations which take place.

4.108. A cubical box of side a and weight W vibrates vertically in water of density σ. Prove that the period of vibration is $(2\pi/a)\sqrt{\dfrac{W}{\sigma g^2}}$

4.109. A spring vibrates so that its equation of motion is

$$m\, d^2x/dt^2 + \kappa x = F(t)$$

If $x = 0$, $dx/dt = 0$ at $t = 0$, find x as a function of time t.

Ans. $x = \dfrac{1}{\sqrt{m\kappa}} \displaystyle\int_0^t F(u) \sin \sqrt{\kappa/m}\,(t-u)\, du$

4.110. Work Problem 4.109 if damping proportional to dx/dt is taken into account.

4.111. A spring vibrates so that its equation of motion is

$$m\, d^2x/dt^2 + \kappa x = 5 \cos \omega t + 2 \cos 3\omega t$$

If $x = 0$, $\dot{x} = v_0$ at $t = 0$, (*a*) find x at any time t and (*b*) determine for what values of ω resonance will occur.

4.112. A vertical spring having elastic constant κ carries a mass m at its lower end. At $t = 0$ the spring is in equilibrium and its upper end is suddenly made to move vertically so that its distance from the original point of support is given by $A \sin \omega t$, $t \geqq 0$. Find (*a*) the position of the mass m at any time and (*b*) the values of ω for which resonance occurs.

4.113. (*a*) Solve $d^2x/dt^2 + x = t \sin t + \cos t$ where $x = 0$, $dx/dt = 0$ at $t = 0$, and (*b*) give a physical interpretation.

4.114. Discuss the motion of a simple pendulum for the case where damping and external forces are present.

4.115. **Find the period of small vertical oscillations of a cylinder of radius a and height h floating with its axis horizontal in water of density σ.**

4.116. A vertical spring having elastic constant 2 newtons per meter has a 50 g weight suspended from it. A force in newtons which is given as a function of time t by $F(t) = 6\cos^4 t$, $t \geq 0$ is applied. Assuming that the weight, initially at the equilibrium position, is given an upward velocity of 4 m/s and that damping is negligible, determine the (*a*) position and (*b*) velocity of the weight at any time.

4.117. **In Problem 4.55, can the answer for $\omega = 2$ be deduced from the answer for $\omega \neq 2$ by taking the limit as $\omega \to 2$? Justify your answer.**

4.118. **An oscillator has a restoring force acting on it whose magnitude is $-\kappa x - \epsilon x^2$ where ϵ is small compared with κ. Prove that the displacement of the oscillator [in this case often called an *anharmonic oscillator*] from the equilibrium position is given approximately by**

$$x = A\cos(\omega t - \phi) + \frac{A\epsilon^2}{6\kappa}\{\cos 2(\omega t - \phi) - 3\}$$

where A and ϕ are determined from the initial conditions.

4.119. **Prove that if the oscillations in Problem 4.32 are not necessarily small, then the period is given by**

$$P = 2\pi\sqrt{\frac{c + 4H}{2g}}\left\{1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{k^6}{5} - \cdots\right\}$$

Chapter 5

CENTRAL FORCES and PLANETARY MOTION

CENTRAL FORCES

Suppose that a force acting on a particle of mass m is such that [see Fig. 5-1]:

(a) it is always directed from m toward or away from a fixed point O,

(b) its magnitude depends only on the distance r from O.

Then we call the force a *central force* or *central force field* with O as the *center of force*. In symbols $\mathbf{F}$ is a central force if and only if

$$\mathbf{F} = f(r)\,\mathbf{r}_1 = f(r)\,\mathbf{r}/r \qquad (1)$$

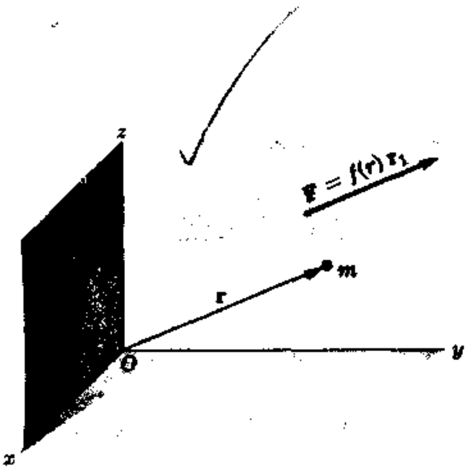

Fig. 5-1

where $\mathbf{r}_1 = \mathbf{r}/r$ is a unit vector in the direction of $\mathbf{r}$.

The central force is one of *attraction* toward O or *repulsion* from O according as $f(r) < 0$ or $f(r) > 0$ respectively.

SOME IMPORTANT PROPERTIES OF CENTRAL FORCE FIELDS

If a particle moves in a central force field, then the following properties are valid.

1. The path or *orbit* of the particle must be a plane curve, i.e. the particle moves in a plane. This plane is often taken to be the xy plane. See Problem 5.1.
2. The angular momentum of the particle is conserved, i.e. is constant. See Problem 5.2.
3. The particle moves in such a way that the position vector or radius vector drawn from O to the particle sweeps out equal areas in equal times. In other words, the time rate of change in area is constant. This is sometimes called the *law of areas*. See Problem 5.6.

EQUATIONS OF MOTION FOR A PARTICLE IN A CENTRAL FIELD

By Property 1, the motion of a particle in a central force field takes place in a plane. Choosing this plane as the xy plane and the coordinates of the particle as polar coordinates (r, θ), the equations of motion are found to be [see Problem 5.3]

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad (2)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad (3)$$

where dots denote differentiations with respect to time t.

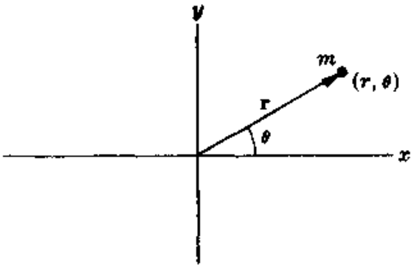

Fig. 5-2

From equation (3) we find

$$r^2\dot{\theta} = \text{constant} = h \tag{4}$$

This is related to Properties 2 and 3 above.

IMPORTANT EQUATIONS DEDUCED FROM THE EQUATIONS OF MOTION

The following equations deduced from the fundamental equations (2) and (3) often prove to be useful.

1.
$$\ddot{r} - \frac{h^2}{r^3} = \frac{f(r)}{m} \tag{5}$$

2.
$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}f(1/u) \tag{6}$$

where $u = 1/r$.

3.
$$\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = \frac{r^4 f(r)}{mh^2} \tag{7}$$

POTENTIAL ENERGY OF A PARTICLE IN A CENTRAL FIELD

A central force field is a conservative field, hence it can be derived from a potential. This potential which depends only on r is, apart from an arbitrary additive constant, given by

$$V(r) = -\int f(r)\,dr \tag{8}$$

This is also the potential energy of a particle in the central force field. The arbitrary additive constant can be obtained by assuming, for example, $V = 0$ at $r = 0$ or $V \to 0$ as $r \to \infty$.

CONSERVATION OF ENERGY

By using (8) and the fact that in polar coordinates the kinetic energy of a particle is $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, the equation for conservation of energy can be written

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E \tag{9}$$

or

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E \tag{10}$$

where E is the total energy and is constant. Using (4), equation (10) can also be written as

$$\frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] - \int f(r)\,dr = E \tag{11}$$

and also as

$$\frac{m}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) - \int f(r)\,dr = E \tag{12}$$

In terms of $u = 1/r$, we can also write equation (9) as

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{mh^2} \tag{13}$$

DETERMINATION OF THE ORBIT FROM THE CENTRAL FORCE

If the central force field is prescribed, i.e. if $f(r)$ is given, it is possible to determine the orbit or path of the particle. This orbit can be obtained in the form

$$r = r(\theta) \tag{14}$$

i.e. r as a function of θ, or in the form

$$r = r(t), \quad \theta = \theta(t) \tag{15}$$

which are parametric equations in terms of the time parameter t.

To determine the orbit in the form (*14*) it is convenient to employ equations (*6*), (*7*) or (*11*). To obtain equations in the form (*15*), it is sometimes convenient to use (*12*) together with (*4*) or to use equations (*4*) and (*5*).

DETERMINATION OF THE CENTRAL FORCE FROM THE ORBIT

Conversely if we know the orbit or path of the particle, then we can find the corresponding central force. If the orbit is given by $r = r(\theta)$ or $u = u(\theta)$ where $u = 1/r$, the central force can be found from

$$f(r) = \frac{mh^2}{r^4}\left\{\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r\right\} \tag{16}$$

or

$$f(1/u) = -mh^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\} \tag{17}$$

which are obtained from equations (*6*) and (*7*) on page 117. The law of force can also be obtained from other equations, as for example equations (*9*)-(*13*).

It is important to note that given an orbit there may be infinitely many force fields for which the orbit is possible. However, if a central force field exists it is unique, i.e. it is the only one.

CONIC SECTIONS, ELLIPSE, PARABOLA AND HYPERBOLA

Consider a fixed point O and a fixed line AB distant D from O, as shown in Fig. 5-3. Suppose that a point P in the plane of O and AB moves so that the ratio of its distance from point O to its distance from line AB is always equal to the positive constant ϵ.

Then the curve described by P is given in polar coordinates (r, θ) by

$$r = \frac{p}{1 + \epsilon\cos\theta} \tag{18}$$

See Problem 5.16.

The point O is called a *focus*, the line AB is called a *directrix* and the ratio ϵ is called the *eccentricity*. The curve is often called a *conic section* since it can be obtained by intersecting a plane and a cone at different angles. Three possible types of curves exist, depending on the value of the eccentricity.

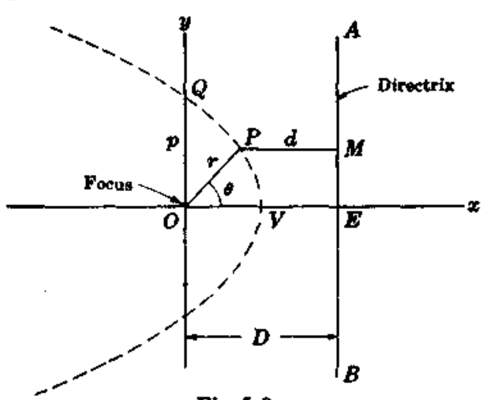

Fig. 5-3

1. **Ellipse:** $\epsilon < 1$ [See Fig. 5-4 below.]

   If C is the *center* of the ellipse and $CV = CU = a$ is the length of the *semi-major axis*, then the equation of the ellipse can be written as

$$r = \frac{a(1-\epsilon^2)}{1 + \epsilon\cos\theta} \tag{19}$$

   Note that the *major axis* is the line joining the *vertices* V and U of the ellipse and has length $2a$.

If b is the length of the *semi-minor axis* [CW or CS in Fig. 5-4] and c is the distance CO from center to focus, then we have the important result

$$c = \sqrt{a^2 - b^2} = a\epsilon \qquad (20)$$

A circle can be considered as a special case of an ellipse with eccentricity equal to zero.

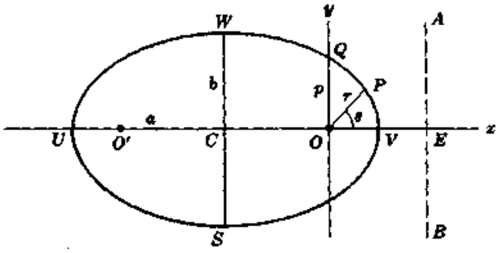

Fig. 5-4

2. **Parabola:** $\epsilon = 1$ [See Fig. 5-5.]

The equation of the parabola is

$$r = \frac{p}{1 + \cos\theta} \qquad (21)$$

We can consider a parabola to be a limiting case of the ellipse (*19*) where $\epsilon \to 1$, which means that $a \to \infty$ [i.e. the major axis becomes infinite] in such a way that $a(1 - \epsilon^2) = p$.

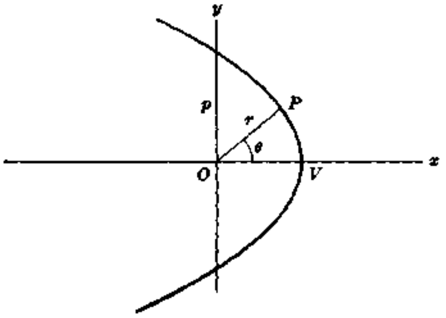

Fig. 5-5

3. **Hyperbola:** $\epsilon > 1$ [See Fig. 5-6.]

The hyperbola consists of two branches as indicated in Fig. 5-6. The branch on the left is the important one for our purposes. The hyperbola is asymptotic to the dashed lines of Fig. 5-6 which are called its *asymptotes*. The intersection C of the asymptotes is called the *center*. The distance $CV = a$ from the center C to vertex V is called the *semi-major axis* [the major axis being the distance between vertices V and U by analogy with the ellipse]. The equation of the hyperbola can be written as

$$r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon\cos\theta} \qquad (22)$$

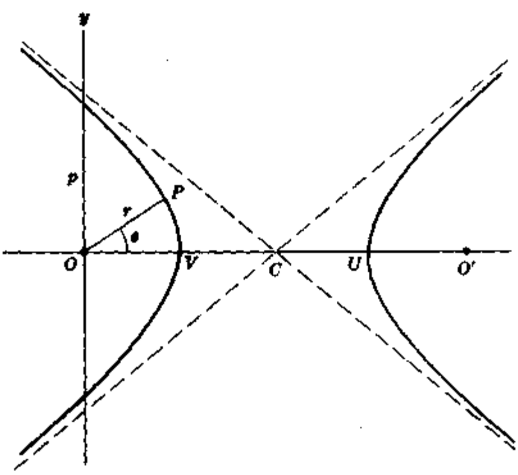

Fig. 5-6

Various other alternative definitions for conic sections may be given. For example, an ellipse can be defined as the *locus* or *path* of all points the sum of whose distances from two fixed points is a constant. Similarly, a hyperbola can be defined as the locus of all points the difference of whose distances from two fixed points is a constant. In both these cases the two fixed points are the foci and the constant is equal in magnitude to the length of the major axis.

SOME DEFINITIONS IN ASTRONOMY

A *solar system* is composed of a *star* [such as our sun] and objects called *planets* which revolve around it. The star is an object which emits its own light, while the planets do not emit light but can reflect it. In addition there may be objects revolving about the planets. These are called *satellites*.

In our solar system, for example, the *moon* is a satellite of the earth which in turn is a planet revolving about our sun. In addition there are *artificial* or *man-made satellites* which can revolve about the planets or their moons.

The path of a planet or satellite is called its *orbit*. The largest and smallest distances of a planet from the sun about which it revolves are called the *aphelion* and *perihelion* respectively. The largest and smallest distances of a satellite around a planet about which it revolves are called the *apogee* and *perigee* respectively.

The time for one complete revolution of a body in an orbit is called its *period*. This is sometimes called a *sidereal period* to distinguish it from other periods such as the period of earth's motion about its axis, etc.

KEPLER'S LAWS OF PLANETARY MOTION

Before Newton had enunciated his famous laws of motion, Kepler, using voluminous data accumulated by Tycho Brahe formulated his three laws concerning the motion of planets around the sun [see Fig. 5-7].

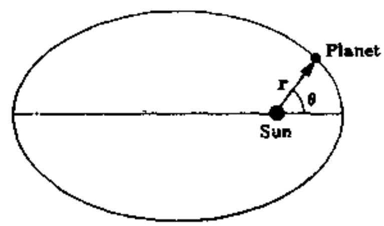

Fig. 5-7

1. Every planet moves in an orbit which is an ellipse with the sun at one focus.
2. The radius vector drawn from the sun to any planet sweeps out equal areas in equal times (the *law of areas*, as on page 116).
3. The squares of the periods of revolution of the planets are proportional to the cubes of the semi-major axes of their orbits.

NEWTON'S UNIVERSAL LAW OF GRAVITATION

By using Kepler's first law and equations *(16)* or *(17)*, Newton was able to deduce his famous law of gravitation between the sun and planets, which he postulated as valid for any objects in the universe [see Problem 5.21].

Newton's Law of Gravitation. Any two particles of mass m_1 and m_2 respectively and distance r apart are attracted toward each other with a force

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\mathbf{r}_1 \tag{23}$$

where G is a universal constant called the *gravitational constant*.

By using Newton's law of gravitation we can, conversely, deduce Kepler's laws [see Problems 5.13 and 5.23]. The value of G is shown in the table on page 342.

ATTRACTION OF SPHERES AND OTHER OBJECTS

By using Newton's law of gravitation, the forces of attraction between large objects such as spheres can be determined. To do this, we use the fact that each large object is composed of particles. We then apply the law of gravitation to find the forces between particles and sum over these forces, usually by methods of integration, to find the resultant force of attraction. An important application of this is given in the following

Theorem 5.1. Two solid or hollow uniform spheres of masses m_1 and m_2 respectively which do not intersect are attracted to each other as if they were particles of the same mass situated at their respective geometric centers.

Since the potential corresponding to

$$\mathbf{F} = -\frac{Gm_1m_2}{r^2}\mathbf{r}_1 \tag{24}$$

is
$$V = -\frac{Gm_1m_2}{r} \tag{25}$$

it is also possible to find the attraction between objects by first finding the potential and then using $\mathbf{F} = -\nabla V$. See Problems 5.26-5.33.

MOTION IN AN INVERSE SQUARE FORCE FIELD

As we have seen, the planets revolve in elliptical orbits about the sun which is at one focus of the ellipse. In a similar manner, satellites (natural or man-made) may revolve around planets in elliptical orbits. However, the motion of an object in an inverse square field of attraction need not always be elliptical but may be parabolic or hyperbolic. In such cases the object, such as a *comet* or *meteorite*, would enter the solar system and then leave but never return again.

The following simple condition in terms of the total energy E determines the path of an object.

(i) if $E < 0$ the path is an ellipse

(ii) if $E = 0$ the path is a parabola

(iii) if $E > 0$ the path is a hyperbola

Other conditions in terms of the speed of the object are also available. See Problem 5.37.

In this chapter we assume the sun to be fixed and the planets do not affect each other. Similarly in the motion of satellites around a planet such as the earth, for example, we assume the planet fixed and that the sun and all other planets have no effect.

Although such assumption is correct as a first approximation, the influence of other planets may have to be taken into account for more accurate purposes. The problems of dealing with the motions of two, three, etc., objects under their mutual attractions are often called the *two body problem, three body problem,* etc.

Solved Problems

CENTRAL FORCES AND IMPORTANT PROPERTIES

5.1. Prove that if a particle moves in a central force field, then its path must be a plane curve.

Let $\mathbf{F} = f(r)\,\mathbf{r}_1$ be the central force field. Then

$$\mathbf{r}\times\mathbf{F} = f(r)\,\mathbf{r}\times\mathbf{r}_1 = \mathbf{0} \tag{1}$$

since $\mathbf{r}_1$ is a unit vector in the direction of the position vector $\mathbf{r}$. Since $\mathbf{F} = m\,d\mathbf{v}/dt$, this can be written

$$\mathbf{r}\times d\mathbf{v}/dt = \mathbf{0} \tag{2}$$

or
$$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \mathbf{0} \tag{3}$$

Integrating, we find
$$\mathbf{r}\times\mathbf{v} = \mathbf{h} \tag{4}$$

where $\mathbf{h}$ is a constant vector. Multiplying both sides of (4) by $\mathbf{r}\cdot$,

$$\mathbf{r}\cdot\mathbf{h} = 0 \tag{5}$$

using the fact that $\mathbf{r}\cdot(\mathbf{r}\times\mathbf{v}) = (\mathbf{r}\times\mathbf{r})\cdot\mathbf{v} = 0$. Thus $\mathbf{r}$ is perpendicular to the constant vector $\mathbf{h}$, and so the motion takes place in a plane. We shall assume that this plane is taken to be the xy plane whose origin is at the center of force.

5.2. Prove that for a particle moving in a central force field the angular momentum is conserved.

From equation (4) of Problem 5.1, we have

$$\mathbf{r} \times \mathbf{v} = \mathbf{h}$$

where $\mathbf{h}$ is a constant vector. Then multiplying by mass m,

$$m(\mathbf{r} \times \mathbf{v}) = m\mathbf{h} \tag{1}$$

Since the left side of (1) is the angular momentum, it follows that the angular momentum is conserved, i.e. is always constant in magnitude and direction.

EQUATIONS OF MOTION FOR A PARTICLE IN A CENTRAL FIELD

5.3. Write the equations of motion for a particle in a central field.

By Problem 5.1 the motion of the particle takes place in a plane. Choose this plane to be the xy plane and the coordinates describing the position of the particle at any time t to be polar coordinates (r, θ). Using Problem 1.49, page 27, we have

$$\text{(mass)(acceleration)} = \text{net force}$$

$$m\{(\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1\} = f(r)\,\mathbf{r}_1 \tag{1}$$

Thus the required equations of motion are given by

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \tag{2}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{3}$$

5.4. Show that $r^2\dot{\theta} = h$, a constant.

Method 1. Equation (3) of Problem 5.3 can be written

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{m}{r}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

Thus $\dfrac{d}{dt}(r^2\dot{\theta}) = 0$ and so

$$r^2\dot{\theta} = h \tag{1}$$

where h is a constant.

Method 2. By Problem 1.49, page 27, the velocity in polar coordinates is

$$\mathbf{v} = \dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1$$

Then from equation (4) of Problem 5.1

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \dot{r}(\mathbf{r} \times \mathbf{r}_1) + r\dot{\theta}(\mathbf{r} \times \boldsymbol{\theta}_1) = r^2\dot{\theta}\mathbf{k} \tag{2}$$

since $\mathbf{r} \times \mathbf{r}_1 = \mathbf{0}$ and $\mathbf{r} \times \boldsymbol{\theta}_1 = r\mathbf{k}$ where $\mathbf{k}$ is the unit vector in a direction perpendicular to the plane of motion [the xy plane], i.e. in the direction $\mathbf{r} \times \mathbf{v}$. Using $\mathbf{h} = h\mathbf{k}$ in (2), we see that $r^2\dot{\theta} = h$.

5.5. Prove that $r^2\dot{\theta} = 2\dot{A}$ where $\dot{A}$ is the time rate at which area is swept out by the position vector $\mathbf{r}$.

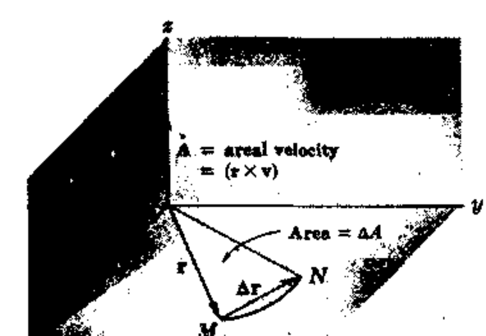

Fig. 5-8

Suppose that in time Δt the particle moves from M to N [see Fig. 5-8]. The area ΔA swept out by the position vector in this time is approximately half the area of a parallelogram with sides r and Δr or [see Problem 1.18, page 15]

$$\Delta A = \tfrac{1}{2}|\mathbf{r} \times \Delta\mathbf{r}|$$

Dividing by Δt and letting $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2}\left|\mathbf{r} \times \frac{\Delta\mathbf{r}}{\Delta t}\right| = \frac{1}{2}|\mathbf{r} \times \mathbf{v}|$$

i.e.,

$$\dot{A} = \tfrac{1}{2}|\mathbf{r}\times\mathbf{v}| = \tfrac{1}{2}r^2\dot{\theta}$$

using the result in Problem 5.4. Thus $r^2\dot{\theta} = 2\dot{A}$, as required. The vector quantity

$$\dot{\mathbf{A}} = \dot{A}\mathbf{k} = \tfrac{1}{2}(\mathbf{r}\times\mathbf{v}) = \tfrac{1}{2}(r^2\dot{\theta})\mathbf{k}$$

is often called the *areal velocity*.

5.6. Prove that for a particle moving in a central force field the areal velocity is constant.

By Problem 5.4, $r^2\dot{\theta} = h =$ a constant. Then the areal velocity is

$$\dot{\mathbf{A}} = \tfrac{1}{2}r^2\dot{\theta}\mathbf{k} = \tfrac{1}{2}h\mathbf{k} = \tfrac{1}{2}\mathbf{h}, \text{ a constant vector}$$

The result is often stated as follows: *If a particle moves in a central force field with O as center, then the radius vector drawn from O to the particle sweeps out equal areas in equal times.* This result is sometimes called the *law of areas*.

5.7. Show by means of the substitution $r = 1/u$ that the differential equation for the path of the particle in a central field is

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{mh^2u^2}$$

From Problem 5.4 or equation (3) of Problem 5.3, we have

$$r^2\dot{\theta} = h \quad \text{or} \quad \dot{\theta} = h/r^2 = hu^2 \tag{1}$$

Substituting into equation (2) of Problem 5.3, we find

$$m(\ddot{r} - h^2/r^3) = f(r) \tag{2}$$

Now if $r = 1/u$, we have

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{du}{d\theta} \tag{3}$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2} \tag{4}$$

From this we see that (2) can be written

$$m(-h^2u^2\,d^2u/d\theta^2 - h^2u^3) = f(1/u) \tag{5}$$

or, as required,

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{mh^2u^2} \tag{6}$$

POTENTIAL ENERGY AND CONSERVATION OF ENERGY FOR CENTRAL FORCE FIELDS

5.8. (*a*) Prove that a central force field is conservative and (*b*) find the corresponding potential energy of a particle in this field.

Method 1.

If we can find the potential or potential energy, then we will have also incidentally proved that the field is conservative. Now if the potential V exists, it must be such that

$$\mathbf{F}\cdot d\mathbf{r} = -dV \tag{1}$$

where $\mathbf{F} = f(r)\,\mathbf{r}_1$ is the central force. We have

$$\mathbf{F}\cdot d\mathbf{r} = f(r)\,\mathbf{r}_1\cdot d\mathbf{r} = f(r)\frac{\mathbf{r}}{r}\cdot d\mathbf{r} = f(r)\,dr$$

since $\mathbf{r}\cdot d\mathbf{r} = r\,dr$.

Since we can determine V such that

$$-dV = f(r)\,dr$$

for example,
$$V = -\int f(r)\,dr \tag{2}$$

it follows that the field is conservative and that (2) represents the potential or potential energy.

Method 2.

We can show that $\nabla \times \mathbf{F} = \mathbf{0}$ directly, but this method is tedious although straightforward.

5.9. Write the conservation of energy for a particle of mass m in a central force field.

Method 1. The velocity of a particle expressed in polar coordinates is [Problem 1.49, page 27]

$$\mathbf{v} = \dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1 \qquad \text{so that} \qquad v^2 = \mathbf{v}\cdot\mathbf{v} = \dot{r}^2 + r^2\dot{\theta}^2$$

Then the principle of conservation of energy can be expressed as

$$\tfrac{1}{2}mv^2 + V = E \qquad \text{or} \qquad \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E$$

where E is a constant.

Method 2. The equations of motion for a particle in a central field are, by Problem 5.3,

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \tag{1}$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \tag{2}$$

Multiply equation (1) by $\dot{r}$, equation (2) by $r\dot{\theta}$ and add to obtain

$$m(\dot{r}\ddot{r} + r^2\dot{\theta}\ddot{\theta} + r\dot{r}\dot{\theta}^2) = f(r)\dot{r} \tag{3}$$

This can be written
$$\tfrac{1}{2}m\frac{d}{dt}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{d}{dt}\int f(r)\,dr \tag{4}$$

Then integrating both sides, we obtain

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E \tag{5}$$

5.10. Show that the differential equation describing the motion of a particle in a central field can be written as

$$\frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] - \int f(r)\,dr = E$$

From Problem 5.9 we have by the conservation of energy,

$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E \tag{1}$$

We also have
$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} \tag{2}$$

Substituting (2) into (1), we find

$$\tfrac{1}{2}m\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\dot{\theta}^2 - \int f(r)\,dr = E \qquad \text{or} \qquad \frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] - \int f(r)\,dr = E$$

since $\dot{\theta} = h/r^2$.

5.11. (a) If $u = 1/r$, prove that $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2\{(du/d\theta)^2 + u^2\}$.

(b) Use (a) to prove that the conservation of energy equation becomes

$$(du/d\theta)^2 + u^2 = 2(E - V)/mh^2$$

(a) From equations (1) and (3) of Problem 5.7 we have $\dot{\theta} = hu^2$, $\dot{r} = -h\,du/d\theta$. Thus

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2(du/d\theta)^2 + (1/u^2)(hu^2)^2 = h^2\{(du/d\theta)^2 + u^2\}$$

(b) From the conservation of energy [Problem 5.9] and part (a),

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{r}^2 + r\dot{\theta}^2) = E - V \qquad \text{or} \qquad (du/d\theta)^2 + u^2 = 2(E-V)/mh^2$$

DETERMINATION OF ORBIT FROM CENTRAL FORCE, OR CENTRAL FORCE FROM ORBIT

5.12. Show that the position of the particle as a function of time t can be determined from the equations

$$t = \int [G(r)]^{-1/2}\,dr, \qquad t = \frac{1}{h}\int r^2\,d\theta$$

where

$$G(r) = \frac{2E}{m} + \frac{2}{m}\int f(r)\,dr - \frac{h^2}{2m^2r^2}$$

Placing $\dot{\theta} = h/r^2$ in the equation for conservation of energy of Problem 5.9,

$$\tfrac{1}{2}m(\dot{r}^2 + h^2/r^2) - \int f(r)\,dr = E$$

or

$$\dot{r}^2 = \frac{2E}{m} + \frac{2}{m}\int f(r)\,dr - \frac{h^2}{r^2} = G(r)$$

Then assuming the positive square root, we have

$$dr/dt = \sqrt{G(r)}$$

and so separating the variables and integrating, we find

$$t = \int [G(r)]^{-1/2}\,dr$$

The second equation follows by writing $\dot{\theta} = h/r^2$ as $dt = r^2\,d\theta/h$ and integrating.

5.13. Show that if the law of central force is defined by

$$f(r) = -K/r^2, \qquad K > 0$$

i.e. an inverse square law of attraction, then the path of the particle is a conic.

Method 1.

In this case $f(1/u) = -Ku^2$. Substituting into the differential equation of motion in Problem 5.7, we find

$$d^2u/d\theta^2 + u = K/mh^2 \tag{1}$$

This equation has the general solution

$$u = A\cos\theta + B\sin\theta + K/mh^2 \tag{2}$$

or using Problem 4.2, page 92,

$$u = K/mh^2 + C\cos(\theta - \phi) \tag{3}$$

i.e.,

$$r = \frac{1}{K/mh^2 + C\cos(\theta - \phi)} \tag{4}$$

It is always possible to choose the axes so that $\phi = 0$, in which case we have

$$r = \frac{1}{K/mh^2 + C\cos\theta} \tag{5}$$

This has the general form of the conic [see Problem 5.16]

$$r = \frac{p}{1 + \epsilon \cos\theta} = \frac{1}{1/p + (\epsilon/p)\cos\theta} \tag{6}$$

Then comparing (5) and (6) we see that

$$1/p = K/mh^2, \qquad \epsilon/p = C \tag{7}$$

or

$$p = mh^2/K, \qquad \epsilon = mh^2C/K \tag{8}$$

Method 2. Since $f(r) = -K/r^2$, we have

$$V = -\int f(r)\,dr = -K/r + c_1 \tag{9}$$

where c_1 is a constant. If we assume that $V \to 0$ as $r \to \infty$, then $c_1 = 0$ and so

$$V = -K/r \tag{10}$$

Using Problem 5.10, page 124, we find

$$\frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] = E + \frac{K}{r} \tag{11}$$

from which

$$\frac{dr}{d\theta} = \pm r\sqrt{\frac{2Er^2}{mh^2} + \frac{2Kr}{mh^2} - 1} \tag{12}$$

By separating variables and integrating [see Problem 5.66] we find the solution (5) where C is expressed in terms of the energy E.

5.14. (*a*) Obtain the constant C of Problem 5.13 in terms of the total energy E and (*b*) thus show that the conic is an ellipse, parabola or hyperbola according as $E < 0$, $E = 0$, $E > 0$ respectively.

Method 1.

(*a*) The potential energy is

$$V = -\int f(r)\,dr = \int (K/r^2)\,dr = -K/r = -Ku \tag{1}$$

where we use $u = 1/r$ and choose the constant of integration so that $\lim_{r\to\infty} V = 0$. Now from equation (5) of Problem 5.13,

$$u = 1/r = K/mh^2 + C\cos\theta \tag{2}$$

Thus from Problem 5.11(*b*) together with (1), we have

$$(C\sin\theta)^2 + \left(\frac{K}{mh^2} + C\cos\theta\right)^2 = \frac{2E}{mh^2} + \frac{2K}{mh^2}\left(\frac{K}{mh^2} + C\cos\theta\right)$$

or

$$C^2 = \frac{K^2}{m^2h^4} + \frac{2E}{mh^2} \qquad \text{or} \qquad C = \sqrt{\frac{K^2}{m^2h^4} + \frac{2E}{mh^2}} \tag{3}$$

assuming $C > 0$.

(*b*) Using the value of C in part (*a*), the equation of the conic becomes

$$u = \frac{1}{r} = \frac{K}{mh^2}\left\{1 + \sqrt{1 + \frac{2Emh^2}{K^2}}\cos\theta\right\}$$

Comparing this with (4) of Problem 5.16, we see that the eccentricity is

$$\epsilon = \sqrt{1 + \frac{2Emh^2}{K^2}} \tag{4}$$

From this we see that the conic is an ellipse if $E < 0$ [but greater than $-K^2/2mh^2$], a parabola if $E = 0$ and a hyperbola if $E > 0$, since in such cases $\epsilon < 1$, $\epsilon = 1$ and $\epsilon > 1$ respectively.

Method 2. The value of C can also be obtained as in the second method of Problem 5.13.

5.15. Under the influence of a central force at point O, a particle moves in a circular orbit which passes through O. Find the law of force.

Method 1.

In polar coordinates the equation of a circle of radius a passing through O is [see Fig. 5-9]

$$r = 2a\cos\theta$$

Then since $u = 1/r = (\sec\theta)/2a$, we have

$$\frac{du}{d\theta} = \frac{\sec\theta\tan\theta}{2a}$$

$$\frac{d^2u}{d\theta^2} = \frac{(\sec\theta)(\sec^2\theta) + (\sec\theta\tan\theta)(\tan\theta)}{2a} = \frac{\sec^3\theta + \sec\theta\tan^2\theta}{2a}$$

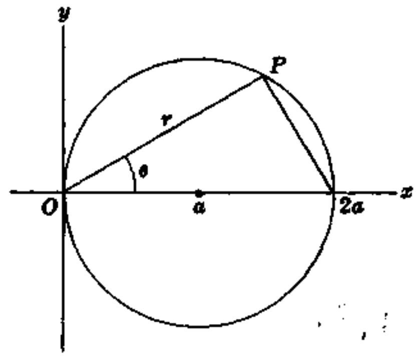

Fig. 5-9

Thus by Problem 5.7,

$$f(1/u) = -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = -mh^2u^2\left(\frac{\sec^3\theta + \sec\theta\tan^2\theta + \sec\theta}{2a}\right)$$
$$= -\frac{mh^2u^2}{2a}\{\sec^3\theta + \sec\theta(\tan^2\theta + 1)\} = -\frac{mh^2u^2}{2a}\cdot 2\sec^3\theta$$
$$= -8mh^2a^2u^5$$

or

$$f(r) = -\frac{8mh^2a^2}{r^5}$$

Thus the force is one of attraction varying inversely as the fifth power of the distance from O.

Method 2. Using $r = 2a\cos\theta$ in equation *(16)*, page 118, we have

$$f(r) = \frac{mh^2}{r^4}\left\{-2a\cos\theta - \frac{2}{2a\cos\theta}(-2a\sin\theta)^2 - 2a\cos\theta\right\}$$
$$= -\frac{4amh^2}{r^4\cos\theta} = -\frac{8a^2mh^2}{r^5}$$

CONIC SECTIONS. ELLIPSE, PARABOLA AND HYPERBOLA

5.16. Derive equation *(18)*, page 118, for a conic section.

Referring to Fig. 5-3, page 118, by definition of a conic section we have for any point P on it,

$$r/d = \epsilon \qquad \text{or} \qquad d = r/\epsilon \tag{1}$$

Corresponding to the particular point Q, we have

$$p/D = \epsilon \qquad \text{or} \qquad p = \epsilon D \tag{2}$$

But

$$D = d + r\cos\theta = \frac{r}{\epsilon} + r\cos\theta = \frac{r}{\epsilon}(1 + \epsilon\cos\theta) \tag{3}$$

Then from *(2)* and *(3)*, we have on eliminating D,

$$p = r(1 + \epsilon\cos\theta) \qquad \text{or} \qquad r = \frac{p}{1 + \epsilon\cos\theta} \tag{4}$$

The equation is a circle if $\epsilon = 0$, an ellipse if $0 < \epsilon < 1$, a parabola if $\epsilon = 1$ and a hyperbola if $\epsilon > 1$.

5.17. Derive equation *(19)*, page 118, for an ellipse.

Referring to Fig. 5-4, page 119, we see that when $\theta = 0$, $r = OV$ and when $\theta = \pi$, $r = OU$. Thus using equation *(4)* of Problem 5.16,

$$OV = p/(1+\epsilon), \qquad OU = p/(1-\epsilon) \tag{1}$$

But since $2a$ is the length of the major axis,

$$OV + OU = 2a \qquad \text{or} \qquad p/(1+\epsilon) + p/(1-\epsilon) = 2a \tag{2}$$

from which

$$p = a(1-\epsilon^2) \tag{3}$$

Thus the equation of the ellipse is

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} \tag{4}$$

5.18. Prove that in Fig. 5-4, page 119, (a) $OV = a(1-\epsilon)$, (b) $OU = a(1+\epsilon)$.

(a) From Problem 5.17, equation (3) and the first equation of (1),

$$OV = \frac{p}{1+\epsilon} = \frac{a(1-\epsilon^2)}{1+\epsilon} = a(1-\epsilon) \tag{1}$$

(b) From Problem 5.17, equation (3) and the second equation of (1),

$$OU = \frac{p}{1-\epsilon} = \frac{a(1-\epsilon^2)}{1-\epsilon} = a(1+\epsilon) \tag{2}$$

5.19. Prove that $c = a\epsilon$ where c is the distance from the center to the focus of the ellipse. a is the length of the semi-major axis and ϵ is the eccentricity.

From Fig. 5-4, page 119, we have $c = CO = CV - OV = a - a(1-\epsilon) = a\epsilon$.

An analogous result holds for the hyperbola [see Problem 5.73(c), page 139].

5.20. If a and c are as in Problem 5.19 and b is the length of the semi-minor axis, prove that (a) $c = \sqrt{a^2-b^2}$, (b) $b = a\sqrt{1-\epsilon^2}$.

(a) From Fig. 5-4, page 119, and the definition of an ellipse, we have

$$\epsilon = \frac{OV}{VE} = \frac{CV - CO}{VE} = \frac{a-c}{VE} \qquad \text{or} \qquad VE = \frac{a-c}{\epsilon} \tag{1}$$

Also since the eccentricity is the distance from O to W divided by the distance from W to the directrix AB [which is equal to CE], we have

$$OW/CE = \epsilon$$

or, using (1) and the result of Problem 5.19,

$$OW = \epsilon CE = \epsilon(CV + VE) = \epsilon[a + (a-c)/\epsilon] = \epsilon a + a - c = a$$

Then $(OW)^2 = (OC)^2 + (CW)^2$ or $a^2 = b^2 + c^2$, i.e. $c = \sqrt{a^2-b^2}$.

(b) From Problem 5.19 and part (a), $a^2 = b^2 + a^2\epsilon^2$ or $b = a\sqrt{1-\epsilon^2}$.

KEPLER'S LAWS OF PLANETARY MOTION AND NEWTON'S UNIVERSAL LAW OF GRAVITATION

5.21. Prove that if a planet is to revolve around the sun in an elliptical path with the sun at a focus [Kepler's first law], then the central force necessary varies inversely as the square of the distance of the planet from the sun.

If the path is an ellipse with the sun at a focus, then calling r the distance from the sun, we have by Problem 5.16,

$$r = \frac{p}{1 + \epsilon \cos\theta} \qquad \text{or} \qquad u = \frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p}\cos\theta \tag{1}$$

where $\epsilon < 1$. Then the central force is given as in Problem 5.7 by

$$f(1/u) = -mh^2u^2(d^2u/d\theta^2 + u) = -mh^2u^2/p \tag{2}$$

on substituting the value of u in (1). From (2) we have on replacing u by $1/r$,

$$f(r) = -mh^2/pr^2 = -K/r^2 \tag{3}$$

5.22. Discuss the connection of Newton's universal law of gravitation with Problem 5.21.

Historically, Newton arrived at the inverse square law of force for planets by using Kepler's first law and the method of Problem 5.21. He was then led to the idea that perhaps all objects of the universe were attracted to each other with a force which was inversely proportional to the square of the distance r between them and directly proportional to the product of their masses. This led to the fundamental postulate

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{r}_1 \tag{1}$$

where G is the universal gravitational constant. Equivalently, the law of force (3) of Problem 5.21 is the same as (1) where

$$K = GMm \tag{2}$$

5.23. Prove Kepler's third law: The squares of the periods of the various planets are proportional to the cubes of their corresponding semi-major axes.

If a and b are the lengths of the semi-major and semi-minor axes, then the area of the ellipse is πab. Since the areal velocity has magnitude $h/2$ [Problem 5.6], the time taken to sweep over area πab, i.e. the period, is

$$P = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h} \tag{1}$$

Now by Problem 5.17 equation (3), Problem 5.20(b), and Problem 5.13 equation (3), we have

$$b = a\sqrt{1-\epsilon^2}, \qquad p = a(1-\epsilon^2) = mh^2/K \tag{2}$$

Then from (1) and (2) we find

$$P = 2\pi m^{1/2}a^{3/2}/K^{1/2} \qquad \text{or} \qquad P^2 = 4\pi^2 ma^3/K$$

Thus the squares of the periods are proportional to the cubes of the semi-major axes.

5.24. Prove that $GM = gR^2$.

On the earth's surface, i.e. $r = R$ where R is the radius, the force of attraction of the earth on an object of mass m is equal to the weight mg of the object. Thus if M is the mass of the earth,

$$GMm/R^2 = mg \qquad \text{or} \qquad GM = gR^2$$

5.25. Calculate the mass of the earth.

From Problem 5.24, $GM = gR^2$ or $M = gR^2/G$. Taking the radius of the earth as 6.38×10^6 m, $g = 9.80$ m/s^2 and $G = 6.67 \times 10^{-11}$ SI units, we find $M = 5.98 \times 10^{24}$ kg.

ATTRACTION OF OBJECTS

5.26. Find the force of attraction of a thin uniform rod of length $2a$ on a particle of mass m placed at a distance b from its midpoint.

Choose the x axis along the rod and the y axis perpendicular to the rod and passing through its center O, as shown in Fig. 5-10. Let σ be the mass per unit length of the rod. The force of attraction $d\mathbf{F}$ between an element of mass $\sigma\,dx$ of the rod and m is, by Newton's universal law of gravitation,

$$d\mathbf{F} = \frac{Gm\sigma\,dx}{x^2+b^2}(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})$$

$$= \frac{Gm\sigma x\,dx}{(x^2+b^2)^{3/2}}\mathbf{i} - \frac{Gm\sigma b\,dx}{(x^2+b^2)^{3/2}}\mathbf{j}$$

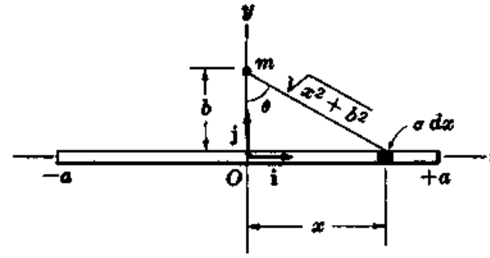

Fig. 5-10

since from Fig. 5-10, $\sin\theta = x/\sqrt{x^2+b^2}$, $\cos\theta = b/\sqrt{x^2+b^2}$. Then the total force of attraction is

$$\mathbf{F} = \mathbf{i}\int_{x=-a}^{a}\frac{Gm\sigma x\,dx}{(x^2+b^2)^{3/2}} - \mathbf{j}\int_{x=-a}^{a}\frac{Gm\sigma b\,dx}{(x^2+b^2)^{3/2}}$$

$$= 0 - 2\mathbf{j}\int_0^a\frac{Gm\sigma b\,dx}{(x^2+b^2)^{3/2}} = -2Gm\sigma b\,\mathbf{j}\int_0^a\frac{dx}{(x^2+b^2)^{3/2}}$$

Let $x = b\tan\theta$ in this integral. Then when $x=0$, $\theta = 0$; and when $x = a$, $\theta = \tan^{-1}(a/b)$. Thus the integral becomes

$$\mathbf{F} = -2Gm\sigma b\,\mathbf{j}\int_0^{\tan^{-1}(a/b)}\frac{b\sec^2\theta\,d\theta}{(b^2\sec^2\theta)^{3/2}} = -\frac{2Gm\sigma a}{b\sqrt{a^2+b^2}}\mathbf{j}$$

Since the mass of the rod is $M = 2a\sigma$, this can also be written as

$$\mathbf{F} = -\frac{GMm}{b\sqrt{a^2+b^2}}\mathbf{j}$$

Thus we see that the force of attraction is directed from m to the center of the rod and of magnitude $2Gm\sigma a/b\sqrt{a^2+b^2}$ or $GMm/b\sqrt{a^2+b^2}$.

5.27. A mass m lies on the perpendicular through the center of a uniform thin circular plate of radius a and at distance b from the center. Find the force of attraction between the plate and the mass m.

Method 1.

Let $\mathbf{n}$ be a unit vector drawn from point P where m is located to the center O of the plate. Subdivide the circular plate into circular rings [such as ABC in Fig. 5-11] of radius r and thickness dr. If σ is the mass per unit area, then the mass of the ring is $\sigma(2\pi r\,dr)$. Since all points of the ring are at the same distance $\sqrt{r^2+b^2}$ from P, the force of attraction of the ring on m will be

$$d\mathbf{F} = \frac{G\sigma(2\pi r\,dr)m}{r^2+b^2}\cos\phi\,\mathbf{n}$$

$$= \frac{G\sigma\,2\pi r\,dr\,mb}{(r^2+b^2)^{3/2}}\mathbf{n} \qquad (1)$$

where we have used the fact that due to symmetry the resultant force of attraction is in the direction $\mathbf{n}$. By integrating over all rings from $r=0$ to $r=a$, we find that the total attraction is

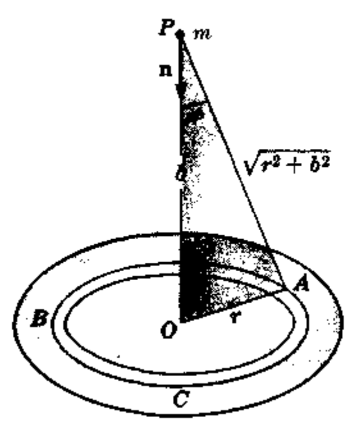

Fig. 5-11

$$\mathbf{F} = 2\pi G\sigma mb\,\mathbf{n}\int_0^a\frac{r\,dr}{(r^2+b^2)^{3/2}} \qquad (2)$$

To evaluate the integral, let $r^2+b^2 = u^2$ so that $r\,dr = u\,du$. Then since $u = b$ when $r = 0$ and $u = \sqrt{a^2+b^2}$ when $r = a$, the result is

$$\mathbf{F} = 2\pi G\sigma mb\,\mathbf{n}\int_b^{\sqrt{a^2+b^2}} \frac{u\,du}{u^3} = 2\pi G\sigma m\,\mathbf{n}\left(1 - \frac{b}{\sqrt{a^2+b^2}}\right)$$

If we let α be the value of ϕ when $r = a$, this can be written

$$\mathbf{F} = 2\pi G\sigma m\,\mathbf{n}\,(1 - \cos\alpha) \qquad (3)$$

Thus the force is directed from m to the center O of the plate and has magnitude $2\pi G\sigma m(1 - \cos\alpha)$.

Method 2.

The method of double integration can also be used. In such case the element of area at A is $r\,dr\,d\theta$ where θ is the angle measured from a line [taken as the x axis] in the plane of the circular plate and passing through the center O. Then we have as in equation *(1)*,

$$d\mathbf{F} = \frac{G\sigma(r\,dr\,d\theta)mb}{(r^2+b^2)^{3/2}}\,\mathbf{n}$$

and by integrating over the circular plate

$$\mathbf{F} = G\sigma mb\,\mathbf{n}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{r\,dr\,d\theta}{(r^2+b^2)^{3/2}} = G\sigma mb\,\mathbf{n}\int_{r=0}^{a} \frac{2\pi r\,dr}{(r^2+b^2)^{3/2}} = 2\pi G\sigma m\,\mathbf{n}\,(1-\cos\alpha)$$

5.28. A uniform plate has its boundary consisting of two concentric half circles of inner and outer radii a and b respectively, as shown in Fig. 5-12. Find the force of attraction of the plate on a mass m located at the center O.

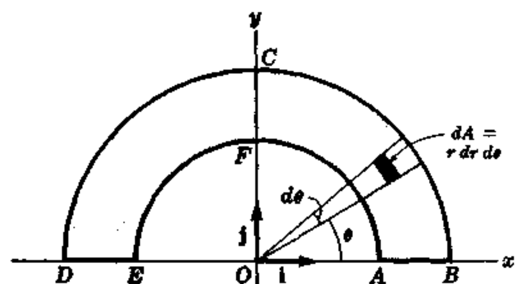

Fig. 5-12

It is convenient to use polar coordinates (r, θ). The element of area of the plate [shaded in Fig. 5-12] is $dA = r\,dr\,d\theta$, and the mass is $\sigma r\,dr\,d\theta$. Then the force of attraction between dA and O is

$$d\mathbf{F} = \frac{G(\sigma r\,dr\,d\theta)m}{r^2}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

Thus the total force of attraction is

$$\mathbf{F} = \int_{\theta=0}^{\pi}\int_{r=a}^{b} \frac{G(\sigma r\,dr\,d\theta)m}{r^2}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

$$= G\sigma m\ln\left(\frac{b}{a}\right)\int_{\theta=0}^{\pi}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})\,d\theta = 2G\sigma m\ln\left(\frac{b}{a}\right)\mathbf{j}$$

Since $M = \sigma(\frac{1}{2}\pi b^2 - \frac{1}{2}\pi a^2)$, we have $\sigma = 2M/\pi(b^2-a^2)$ and the force can be written

$$\mathbf{F} = \frac{4GMm}{\pi(b^2-a^2)}\ln\left(\frac{b}{a}\right)\mathbf{j}$$

The method of single integration can also be used by dividing the region between $r = a$ and $r = b$ into circular rings as in Problem 5.27.

5.29. Find the force of attraction of a thin spherical shell of radius a on a particle P of mass m at a distance $r > a$ from its center.

Let O be the center of the sphere. Subdivide the surface of the sphere into circular elements such as $ABCDA$ of Fig. 5-13 below by using parallel planes perpendicular to OP.

The area of the surface element $ABCDA$ as seen from Fig. 5-13 is

$$2\pi(a\sin\theta)(a\,d\theta) = 2\pi a^2\sin\theta\,d\theta$$

since the radius is $a\sin\theta$ [so that the perimeter is $2\pi(a\sin\theta)$] and the thickness is $a\,d\theta$. Then if σ is the mass per unit area, the mass of $ABCDA$ is $2\pi a^2\sigma\sin\theta\,d\theta$.

Since all points of $ABCDA$ are at the same distance $w = AP$ from P, the force of attraction of the element $ABCDA$ on m is

$$d\mathbf{F} = \frac{G(2\pi a^2\sigma \sin\theta\, d\theta)m}{w^2}\cos\phi\,\mathbf{n} \qquad (1)$$

where we have used the fact that from symmetry the net force will be in the direction of the unit vector $\mathbf{n}$ from P toward O. Now from Fig. 5-13,

$$\cos\phi = \frac{PE}{AP} = \frac{PO - EO}{AP} = \frac{r - a\cos\theta}{w} \qquad (2)$$

Using (2) in (1) together with the fact that by the cosine law

$$w^2 = a^2 + r^2 - 2ar\cos\theta \qquad (3)$$

we find

$$d\mathbf{F} = \frac{G(2\pi a^2\sigma\sin\theta\, d\theta)m(r - a\cos\theta)}{(a^2 + r^2 - 2ar\cos\theta)^{3/2}}\,\mathbf{n}$$

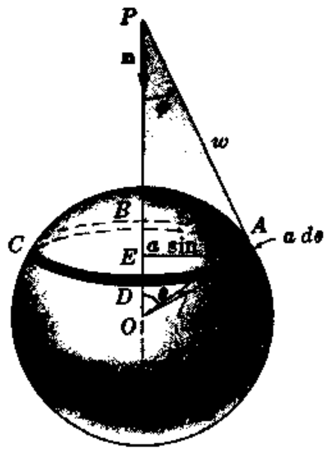

Fig. 5-13

Then the total force is

$$\mathbf{F} = 2\pi G a^2\sigma m\,\mathbf{n}\int_{\theta=0}^{\pi}\frac{(r - a\cos\theta)\sin\theta}{(a^2 + r^2 - 2ar\cos\theta)^{3/2}}\,d\theta \qquad (4)$$

We can evaluate the integral by using the variable w given by (3) in place of θ. When $\theta = 0$, $w^2 = a^2 - 2ar + r^2 = (r-a)^2$ so that $w = r - a$ if $r > a$. Also when $\theta = \pi$, $w^2 = a^2 + 2ar + r^2 = (r+a)^2$ so that $w = r + a$. In addition, we have

$$2w\,dw = 2ar\sin\theta\,d\theta$$

$$r - a\cos\theta = r - a\left(\frac{a^2 + r^2 - w^2}{2ar}\right) = \frac{w^2 - a^2 + r^2}{2r}$$

Then (4) becomes

$$\mathbf{F} = \frac{\pi G a\sigma m\,\mathbf{n}}{r^2}\int_{r-a}^{r+a}\left(1 + \frac{r^2 - a^2}{w^2}\right)dw = \frac{4\pi G a^2\sigma m\,\mathbf{n}}{r^2}$$

5.30. Work Problem 5.29 if $r < a$.

In this case the force is also given by (4) of Problem 5.29. However, in evaluating the integral we note that on making the substitution (3) of Problem 5.29 that $\theta = 0$ yields $w^2 = (a-r)^2$ or $w = a - r$ if $r < a$. Then the result (4) of Problem 5.29 becomes

$$\mathbf{F} = \frac{\pi G a\sigma m\,\mathbf{n}}{r^2}\int_{a-r}^{a+r}\left(1 - \frac{a^2 - r^2}{w^2}\right)dw = \mathbf{0}$$

Thus there will be no force of attraction of a spherical shell on any mass placed inside. This means that in such case a particle will be in equilibrium inside of the shell.

5.31. Prove that the force of attraction in Problem 5.29 is the same as if all the mass of the spherical shell were concentrated at its center.

The mass of the shell is $M = 4\pi a^2\sigma$. Thus the force is $\mathbf{F} = (GMm/r^2)\mathbf{n}$, which proves the required result.

5.32. (a) Find the force of attraction of a solid uniform sphere on a mass m placed outside of it and (b) prove that the force is the same as if all the mass were concentrated at its center.

(a) We can subdivide the solid sphere into thin concentric spherical shells. If ρ is the distance of any of these shells from the center and $d\rho$ is the thickness, then by Problem 5.29 the force of attraction of this shell on the mass m is

$$d\mathbf{F} = \frac{G\sigma(4\pi\rho^2\,d\rho)m}{r^2}\,\mathbf{n} \tag{1}$$

where σ is the mass per unit volume. Then the total force obtained by integrating from $r=0$ to $r=a$ is

$$\mathbf{F} = \frac{4\pi G\sigma m\,\mathbf{n}}{r^2}\int_0^a \rho^2\,d\rho = \frac{G(\frac{4}{3}\pi a^3)\sigma m\,\mathbf{n}}{r^2} \tag{2}$$

(b) Since the mass of the sphere is $M = \frac{4}{3}\pi a^3\sigma$, (2) can be written as $\mathbf{F} = (GMm/r^2)\mathbf{n}$, which shows that the force of attraction is the same as if all the mass were concentrated at the center.

We can also use triple integration to obtain this result [see Problem 5.130].

5.33. Derive the result of Problems 5.29 and 5.30 by first finding the potential due to the mass distribution.

The potential dV due to the element $ABCDA$ is

$$dV = -\frac{G(2\pi a^2\sigma\sin\theta\,d\theta)m}{w} = -\frac{G(2\pi a^2\sigma\sin\theta\,d\theta)m}{\sqrt{a^2+r^2-2ar\cos\theta}}$$

Then the total potential is

$$V = -2\pi Ga^2\sigma m\int_0^\pi \frac{\sin\theta\,d\theta}{\sqrt{a^2+r^2-2ar\cos\theta}} = -\frac{2\pi Ga\sigma m}{r}\{\sqrt{(a+r)^2}-\sqrt{(a-r)^2}\}$$

If $r > a$ this yields $\qquad V = -\dfrac{4\pi Ga^2\sigma m}{r} = -\dfrac{GMm}{r}$

If $r < a$ it yields $\qquad V = -4\pi Ga\sigma m$

Then if $r > a$ the force is

$$\mathbf{F} = -\nabla V = -\nabla\left(-\frac{GMm}{r}\right) = -\frac{GMm}{r^2}\mathbf{r}_1$$

and if $r < a$ the force is

$$\mathbf{F} = -\nabla V = -\nabla(-4\pi Ga\sigma m) = \mathbf{0}$$

in agreement with Problems 5.29 and 5.30.

MISCELLANEOUS PROBLEMS

5.34. An object is projected vertically upward from the earth's surface with init[ial] speed v_0. Neglecting air resistance, (a) find the speed at a distance H above t[he] earth's surface and (b) the smallest velocity of projection needed in order that t[he] object never return.

(a) Let r denote the radial distance of the object at time t from the center of the earth, which we assume is fixed [see Fig. 5-14]. If M is the mass of the earth and R is its radius, then by Newton's universal law of gravitation and Problem 5.29, the force between m and M is

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{r}_1 \tag{1}$$

where $\mathbf{r}_1$ is a unit vector directed radially outward from the earth's center in the direction of motion of the object.

If v is the speed at time t, we have by Newton's second law,

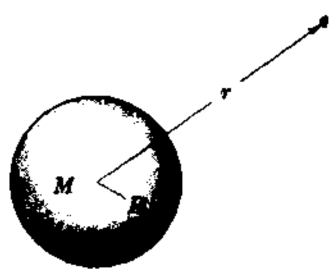

Fig. 5-14

$$m\frac{dv}{dt}\mathbf{r}_1 = -\frac{GMm}{r^2}\mathbf{r}_1 \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{GM}{r^2} \tag{2}$$

This can be written as

$$\frac{dv}{dr}\frac{dr}{dt} = -\frac{GM}{r^2} \qquad \text{or} \qquad v\frac{dv}{dr} = -\frac{GM}{r^2} \tag{3}$$

Then by integrating, we find $\quad v^2/2 = GM/r + c_1 \quad (4)$

Since the object starts from the earth's surface with speed v_0, we have $v = v_0$ when $r = R$ so that $c_1 = v_0^2/2 - GM/R$. Then (4) becomes

$$v^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) + v_0^2 \tag{5}$$

Thus when the object is at height H above the earth's surface, i.e. $r = R + H$,

$$v^2 = 2GM\left(\frac{1}{R+H} - \frac{1}{R}\right) + v_0^2 = v_0^2 - \frac{2GMH}{R(R+H)}$$

i.e.,

$$v = \sqrt{v_0^2 - \frac{2GMH}{R(R+H)}}$$

Using Problem 5.24, this can be written

$$v = \sqrt{v_0^2 - \frac{2gRH}{R+H}} \tag{6}$$

(b) As $H \to \infty$, the limiting speed (6) becomes

$$\sqrt{v_0^2 - 2GM/R} \qquad \text{or} \qquad \sqrt{v_0^2 - 2gR} \tag{7}$$

since $\lim_{H\to\infty} \dfrac{H}{(R+H)} = 1$. The minimum initial speed occurs where (7) is zero or where

$$v_0 = \sqrt{2GM/R} = \sqrt{2gR} \tag{8}$$

This minimum speed is called the *escape speed* and the corresponding velocity is called the *escape velocity* from the earth's surface.

5.35. Show that the magnitude of the escape velocity of an object from the earth's surface is about 11 km/s.

From equation (8) of Problem 5.34, $v_0 = \sqrt{2gR}$. Taking $g = 9.80\ \text{m/s}^2$ and $R = 6.38 \times 10^6$ m, we find $v_0 = 11.2$ km/s

5.36. Prove, by using vector methods primarily, that the path of a planet around the sun is an ellipse with the sun at one focus.

Since the force $\mathbf{F}$ between the planet and sun is

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = -\frac{GMm}{r^2}\mathbf{r}_1 \tag{1}$$

we have

$$\frac{d\mathbf{v}}{dt} = -\frac{GM}{r^2}\mathbf{r}_1 \tag{2}$$

Also, by Problem 5.1, equation (4), we have

$$\mathbf{r} \times \mathbf{v} = \mathbf{h} \tag{3}$$

Now since $\mathbf{r} = r\mathbf{r}_1$, $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = r\dfrac{d\mathbf{r}_1}{dt} + \dfrac{dr}{dt}\mathbf{r}_1$. Thus from (3),

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = r\mathbf{r}_1 \times \left(r\frac{d\mathbf{r}_1}{dt} + \frac{dr}{dt}\mathbf{r}_1\right) = r^2\mathbf{r}_1 \times \frac{d\mathbf{r}_1}{dt} \tag{4}$$

From (2),

$$\frac{d\mathbf{v}}{dt}\times\mathbf{h} = -\frac{GM}{r^2}\mathbf{r}_1\times\mathbf{h} = -GM\,\mathbf{r}_1\times\left(\mathbf{r}_1\times\frac{d\mathbf{r}_1}{dt}\right)$$

$$= -GM\left\{\left(\mathbf{r}_1\cdot\frac{d\mathbf{r}_1}{dt}\right)\mathbf{r}_1 - (\mathbf{r}_1\cdot\mathbf{r}_1)\frac{d\mathbf{r}_1}{dt}\right\} = GM\frac{d\mathbf{r}_1}{dt}$$

using equation (4) above and equation (7), page 5.

But since $\mathbf{h}$ is a constant vector, $\frac{d\mathbf{v}}{dt}\times\mathbf{h} = \frac{d}{dt}(\mathbf{v}\times\mathbf{h})$ so that

$$\frac{d}{dt}(\mathbf{v}\times\mathbf{h}) = GM\frac{d\mathbf{r}_1}{dt}$$

Integrating,

$$\mathbf{v}\times\mathbf{h} = GM\,\mathbf{r}_1 + \mathbf{c}$$

from which

$$\mathbf{r}\cdot(\mathbf{v}\times\mathbf{h}) = GM\,\mathbf{r}\cdot\mathbf{r}_1 + \mathbf{r}\cdot\mathbf{c} = GMr + r\mathbf{r}_1\cdot\mathbf{c} = GMr + rc\cos\theta$$

where $\mathbf{c}$ is an arbitrary constant vector having magnitude c, and θ is the angle between $\mathbf{c}$ and $\mathbf{r}_1$.

Since $\mathbf{r}\cdot(\mathbf{v}\times\mathbf{h}) = (\mathbf{r}\times\mathbf{v})\cdot\mathbf{h} = \mathbf{h}\cdot\mathbf{h} = h^2$ [see Problem 1.72(*a*), page 27],

$$h^2 = GMr + rc\cos\theta$$

and so

$$r = \frac{h^2}{GM + c\cos\theta} = \frac{h^2/GM}{1 + (c/GM)\cos\theta}$$

which is the equation of a conic. Since the only conic which is a closed curve is an ellipse, the required result is proved.

5.37. Prove that the speed v of a particle moving in an elliptical path in an inverse square field is given by

$$v^2 = \frac{K}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$$

where a is the semi-major axis.

By (8) of Problem 5.13, (4) of Problem 5.14 and (3) of Problem 5.17, we have

$$p = \frac{mh^2}{K} = a(1-\epsilon^2) = a\left(-\frac{2Emh^2}{K^2}\right) \tag{1}$$

from which

$$E = -K/2a \tag{2}$$

Thus by the conservation of energy we have, using $V = -K/r$,

$$\tfrac{1}{2}mv^2 = E - V = -\frac{K}{2a} + \frac{K}{r}$$

or

$$v^2 = \frac{K}{m}\left(\frac{2}{r} - \frac{1}{a}\right) \tag{3}$$

We can similarly show that for a hyperbola,

$$v^2 = \frac{K}{m}\left(\frac{2}{r} + \frac{1}{a}\right) \tag{4}$$

while for a parabola [which corresponds to letting $a \to \infty$ in either (3) or (4)],

$$v^2 = 2K/mr$$

5.38. An artificial (man-made) satellite revolves about the earth at height H above the surface. Determine the (*a*) orbital speed and (*b*) orbital period so that a man in the satellite will be in a state of weightlessness.

(*a*) Assume that the earth is spherical and has radius R. Weightlessness will result when the centrifugal force [equal and opposite to the centripetal force, i.e. the force due to the centripetal acceleration] acting on the man due to rotation of the satellite just balances his attraction to the earth. Then if v_0 is the orbital speed,

$$\frac{mv_0^2}{R+H} = \frac{GMm}{(R+H)^2} = \frac{gR^2m}{(R+H)^2} \qquad \text{or} \qquad v_0 = \frac{R}{R+H}\sqrt{(R+H)g}$$

If H is small compared with R, this is $\sqrt{Rg}$ approximately.

(b) $$\text{Orbital speed} = \frac{\text{distance traveled in one revolution}}{\text{time for one revolution, or period}}$$

Thus $$v_0 = \frac{2\pi(R+H)}{P}$$

Then from part (a)

$$P = \frac{2\pi(R+H)}{v_0} = 2\pi\left(\frac{R+H}{R}\right)\sqrt{\frac{R+H}{g}}$$

If H is small compared with R, this is $2\pi\sqrt{R/g}$ approximately.

5.39. Calculate the (a) orbital speed and (b) period in Problem 5.38 assuming that the height H above the earth's surface is small compared with the earth's radius.

Taking the earth's radius as 6380 km and $g = 9.80$ m/s², we find (a) $v_0 = \sqrt{Rg} = 7.92$ km/s and (b) $P = 2\pi\sqrt{R/g} = 1.42$ h $= 85$ minutes, approximately.

5.40. Find the force of attraction of a solid sphere of radius a on a particle of mass m at a distance $b < a$ from its center.

By Problem 5.30 the force of attraction of any spherical shell containing m in its interior [such as the spherical shell shown dashed in Fig. 5-15] is zero.

Thus the force of attraction on m is the force due to a sphere of radius $b < a$ with center at O. If σ is the mass per unit volume, the force of attraction is

$$G(\tfrac{4}{3}\pi b^3)\sigma m/b^2 = (\tfrac{4}{3}\pi G\sigma m)b$$

Thus the force varies as the distance b from the mass to the center.

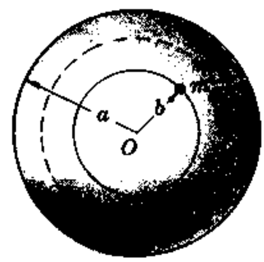

Fig. 5-15

Supplementary Problems

CENTRAL FORCES AND EQUATIONS OF MOTION

5.41. Indicate which of the following central force fields are attractive toward origin O and which are repulsive from O. (a) $\mathbf{F} = -4r^3\mathbf{r}_1$; (b) $\mathbf{F} = K\mathbf{r}_1/\sqrt{r}$, $K > 0$; (c) $\mathbf{F} = r(r-1)\mathbf{r}_1/(r^2+1)$; (d) $\mathbf{F} = \sin \pi r\, \mathbf{r}_1$.

Ans. (a) attractive; (b) repulsive; (c) attractive if $0 < r < 1$, repulsive if $r > 1$; (d) repulsive for $2n < r < 2n+1$, attractive for $2n+1 < r < 2n+2$ where $n = 0, 1, 2, 3, \ldots$.

5.42. Prove that in rectangular coordinates the magnitude of the areal velocity is $\frac{1}{2}(x\dot{y} - y\dot{x})$.

5.43. Give an example of a force field directed toward a fixed point which is not a central force field.

5.44. Derive equation (7), page 117.

5.45. If a particle moves in a circular orbit under the influence of a central force at its center, prove that its speed around the orbit must be constant.

5.46. A particle of mass m moves in a force field defined by $\mathbf{F} = -K\mathbf{r}_1/r^3$. If it starts on the positive x axis at distance a away from the origin and moves with speed v_0 in direction making angle α with the positive x axis, prove that the differential equation for the radial position r of the particle at any time t is

$$\frac{d^2r}{dt^2} = -\frac{(K - ma^2v_0^2 \sin^2 \alpha)}{mr^3}$$

5.47. (a) Show that the differential equation for the orbit in Problem 5.46 is given in terms of $u = 1/r$ by

$$\frac{d^2u}{d\theta^2} + (1-\gamma)u = 0 \qquad \text{where} \quad \gamma = \frac{K}{ma^2v_0^2 \sin^2 \alpha}$$

(b) Solve the differential equation in (a) and interpret physically.

5.48. A particle is to move under the influence of a central force field so that its orbital speed is always constant and equal to v_0. Determine all possible orbits.

POTENTIAL ENERGY AND CONSERVATION OF ENERGY

5.49. Find the potential energy or potential corresponding to the central force fields defined by (a) $\mathbf{F} = -K\mathbf{r}_1/r^3$, (b) $\mathbf{F} = (\alpha/r^2 + \beta/r^3)\mathbf{r}_1$, (c) $\mathbf{F} = Kr\mathbf{r}_1$, (d) $\mathbf{F} = \mathbf{r}_1/\sqrt{r}$, (e) $\mathbf{F} = \sin \pi r\, \mathbf{r}_1$.
Ans. (a) $-K/2r^2$, (b) $\alpha/r + \beta/2r^2$, (c) $\frac{1}{2}Kr^2$, (d) $2\sqrt{r}$, (e) $(\cos \pi r)/\pi$

5.50. (a) Find the potential energy for a particle which moves in the force field $\mathbf{F} = -K\mathbf{r}_1/r^2$. (b) How much work is done by the force field in (a) in moving the particle from a point on the circle $r = a > 0$ to another point on the circle $r = b > 0$? Does the work depend on the path? Explain.
Ans. (a) $-K/r$, (b) $K(a-b)/ab$

5.51. Work Problem 5.50 for the force field $\mathbf{F} = -K\mathbf{r}_1/r$. *Ans.* (a) $-K \ln r$, (b) $-K \ln (a/b)$

5.52. A particle of mass m moves in a central force field defined by $\mathbf{F} = -K\mathbf{r}_1/r^3$. (a) Write an equation for the conservation of energy. (b) Prove that if E is the total energy supplied to the particle, then its speed is given by $v = \sqrt{K/mr^2 + 2E/m}$.

5.53. A particle moves in a central force field defined by $\mathbf{F} = -Kr^2\mathbf{r}_1$. It starts from rest at a point on the circle $r = a$. (a) Prove that when it reaches the circle $r = b$ its speed will be $\sqrt{2K(a^3 - b^3)/3m}$ and that (b) the speed will be independent of the path.

5.54. A particle of mass m moves in a central force field $\mathbf{F} = K\mathbf{r}_1/r^n$ where K and n are constants. It starts from rest at $r = a$ and arrives at $r = 0$ with finite speed v_0. (a) Prove that we must have $n < 1$ and $K > 0$. (b) Prove that $v_0 = \sqrt{2Ka^{1-n}/m(n-1)}$. (c) Discuss the physical significance of the results in (a).

5.55. By differentiating both sides of equation (13), page 117, obtain equation (6).

DETERMINATION OF ORBIT FROM CENTRAL FORCE OR CENTRAL FORCE FROM ORBIT

5.56. A particle of mass m moves in a central force field given in magnitude by $f(r) = -Kr$ where K is a positive constant. If the particle starts at $r = a$, $\theta = 0$ with a speed v_0 in a direction perpendicular to the x axis, determine its orbit. What type of curve is described?

5.57. (a) Work Problem 5.56 if the speed is v_0 in a direction making angle α with the positive x axis. (b) Discuss the cases $\alpha = 0$, $\alpha = \pi$ and give the physical significance.

5.58. A particle moving in a central force field located at $r = 0$ describes the spiral $r = e^{-\theta}$. Prove that the magnitude of the force is inversely proportional to r^3.

5.59. Find the central force necessary to make a particle describe the lemniscate $r^2 = a^2 \cos 2\theta$ [see Fig. 5-16]. *Ans.* A force proportional to r^{-7}.

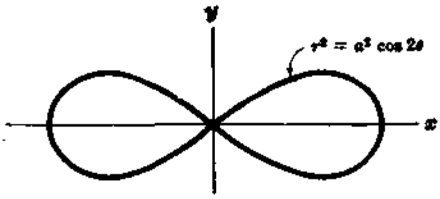

Fig. 5-16

5.60. Obtain the orbit for the particle of Problem 5.46 and describe physically.

5.61. Prove that the orbits $r = e^{-\theta}$ and $r = 1/\theta$ are both possible for the case of an inverse cube field of force. Explain physically how this is possible.

5.62. (a) Show that if the law of force is given by

$$\mathbf{F} = \frac{A\mathbf{r}_1}{r^4 \cos\theta} \qquad \text{or} \qquad \mathbf{F} = \frac{B\mathbf{r}_1}{r^2 \cos^3\theta}$$

then a particle can move in the circular orbit $r = 2a\cos\theta$. (b) What can you conclude about the uniqueness of forces when the orbit is specified? (c) Answer part (b) when the forces are central forces.

5.63. (a) What central force at the origin O is needed to make a particle move around O with a speed which is inversely proportional to the distance from O. (b) What types of orbits are possible in such case? *Ans.* (a) Inverse cube force.

5.64. Discuss the motion of a particle moving in a central force field given by $\mathbf{F} = (\alpha/r^2 + \beta/r^3)\mathbf{r}_1$.

5.65. Prove that there is no central force which will enable a particle to move in a straight line.

5.66. Complete the integration of equation (*12*) of Problem 5.13, page 125 and thus arrive at equation (*5*) of the same problem. [*Hint.* Let $r = 1/u$.]

5.67. Suppose that the orbit of a particle moving in a central force field is given by $\theta = \theta(r)$. Prove that the law of force is $-\dfrac{mh^2[2\theta' + r\theta'' + r^2(\theta')^3]}{r^5(\theta')^3}$ where primes denote differentiations with respect to r.

5.68. (a) Use Problem 5.67 to show that if $\theta = 1/r$, the central force is one of attraction and varies inversely as r^3. (b) Graph the orbit in (a) and explain physically.

CONIC SECTIONS, ELLIPSE, PARABOLA AND HYPERBOLA

5.69. The equation of a conic is $r = \dfrac{12}{3 + \cos\theta}$. Graph the conic, finding (a) the foci, (b) the vertices, (c) the length of the major axis, (d) the length of the minor axis, (e) the distance from the center to the directrix.

5.70. Work Problem 5.69 for the conic $r = \dfrac{24}{3 + 5\cos\theta}$.

5.71. Show that the equation of a parabola can be written as $r = p\sec^2(\theta/2)$.

5.72. Find an equation for an ellipse which has one focus at the origin, its center at the point $(-4, 0)$, and its major axis of length 10. *Ans.* $r = 9/(5 + 4\cos\theta)$

5.73. In Fig. 5-17, SR or TN is called the *minor axis* of the hyperbola and its length is generally denoted by $2b$. The length of the *major axis* VU is $2a$, while the distance between the foci O and O' is $2c$ [i.e. the distance from the center C to a focus O or O' is C].

(a) Prove that $c^2 = a^2 + b^2$.

(b) Prove that $b = a\sqrt{\epsilon^2 - 1}$ where ϵ is the eccentricity.

(c) Prove that $c = a\epsilon$. Compare with results for the ellipse.

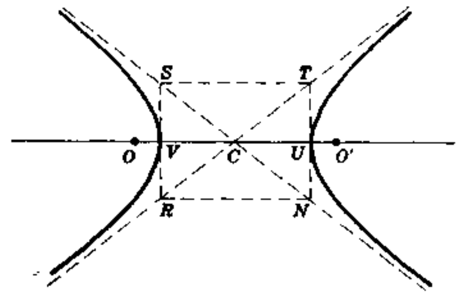

Fig. 5-17

5.74. Derive equation (*22*), page 119, for a hyperbola.

5.75. In rectangular coordinates the equations for an ellipse and hyperbola in standard form are given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

respectively, where a and b are the lengths of the semi-major and semi-minor axes. Graph these equations, locating vertices, foci and directrices, and explain the relation of these equations to equations (*19*), page 118, and (*22*), page 119.

5.76. Using the alternative definitions for an ellipse and hyperbola given on pages 118-119, obtain the equations (*19*) and (*22*).

5.77. Prove that the angle between the asymptotes of a hyperbola is $2\cos^{-1}(1/\epsilon)$.

KEPLER'S LAWS AND NEWTON'S LAW OF GRAVITATION

5.78. Assuming that the planet Mars has a period about the sun equal to 687 earth days approximately, find the mean distance of Mars from the sun. Take the distance of the earth from the sun as 150 million km. *Ans.* 225 million km

5.79. Work Problem 5.78 for (a) Jupiter and (b) Venus which have periods of 4333 earth days and 225 earth days respectively. *Ans.* (a) 778 million km, (b) 108 million km

5.80. Suppose that a small spherical planet has a radius of 10 km and a mean density of 5 g/cm³.
(a) What would be the acceleration due to gravity at its surface? (b) What would a man weigh on this planet if he weighed 80 kgf on earth?

5.81. If the acceleration due to gravity on the surface of a spherically shaped planet P is g_P while its mean density and radius are given by σ_P and R_P respectively, prove that $g_P = \frac{4}{3}\pi G R_P \sigma_P$ where G is the universal gravitational constant.

5.82. If L, M, T represent the dimensions of length, mass and time, find the dimensions of the universal gravitational constant. *Ans.* $L^3M^{-1}T^{-2}$

5.83. Calculate the mass of the sun using the fact that the earth is approximately 150×10^6 kilometers from it and makes one complete revolution about it in approximately 365 days. *Ans.* 2×10^{30} kg

5.84. Calculate the force between the sun and the earth if the distance between the earth and the sun is taken as 150×10^6 kilometers and the masses of the earth and sun are 6×10^{24} kg and 2×10^{30} kg respectively. *Ans.* 1.16×10^{24} newtons

ATTRACTION OF OBJECTS

5.85. Find the force of attraction of a thin uniform rod of length a on a mass m outside the rod but on the same line as the rod and distance b from an end. *Ans.* $GMm/b(a+b)$

5.86. In Problem 5.85 determine where the mass of the rod should be concentrated so as to give the same force of attraction. *Ans.* At a point in the rod a distance $\sqrt{b(a+b)} - b$ from the end

5.87. Find the force of attraction of an infinitely long thin uniform rod on a mass m at distance b from it. *Ans.* Magnitude is $2Gm\sigma/b$

5.88. A uniform wire is in the form of an arc of a circle of radius b and central angle ψ. Prove that the force of attraction of the wire on a mass m placed at the center of the circle is given in magnitude by

$$\frac{2GMm\sin(\psi/2)}{b^2\psi} \quad \text{or} \quad \frac{2G\sigma m\sin(\psi/2)}{b}$$

where M is the mass of the wire and σ is the mass per unit length. Discuss the cases $\psi = \pi/2$ and $\psi = \pi$.

5.89. In Fig. 5-18, AB is a thin rod of length $2a$ and m is a mass located at point C a distance b from the rod. Prove that the force of attraction of the rod on m has magnitude

$$\frac{GMm}{ab}\sin\tfrac{1}{2}(\alpha+\beta)$$

in a direction making an angle with the rod given by

$$\tan^{-1}\left(\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}\right)$$

Discuss the case $\alpha = \beta$ and compare with Problem 5.26.

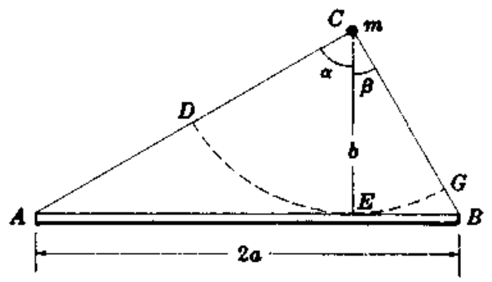

Fig. 5-18

5.90. By comparing Problem 5.89 with Problem 5.88, prove that the rod of Problem 5.89 can be replaced by a wire in the form of circular arc DEG [shown dashed in Fig. 5-18] which has its center at C and is tangent to the rod at E. Prove that the direction of the attraction is toward the midpoint of this arc.

5.91. A hemisphere of mass M and radius a has a particle of mass m located at its center. Find the force of attraction if (*a*) the hemisphere is a thin shell, (*b*) the hemisphere is solid.
Ans. (*a*) $GMm/2a^2$, (*b*) $3GMm/2a^2$

5.92. Work Problem 5.91 if the hemisphere is a shell having outer radius a and inner radius b.

5.93. Deduce from Kepler's laws that if the force of attraction between sun and planets is given in magnitude by $\gamma m/r^2$, then γ must be independent of the particular planet.

5.94. A cone has height H and radius a. Prove that the force of attraction on a particle of mass m placed at its vertex has magnitude $\dfrac{6GMm}{a^2}\left(1-\dfrac{H}{\sqrt{a^2+H^2}}\right)$.

5.95. Find the force of attraction between two non-intersecting spheres.

5.96. A particle of mass m is placed outside of a uniform solid hemisphere of radius a at a distance a on a line perpendicular to the base through its center. Prove that the force of attraction is given in magnitude by $GMm(\sqrt{2}-1)/a^2$.

5.97. Work (*a*) Problem 5.26, (*b*) Problem 5.27, and (*c*) Problem 5.94 by first finding the potential.

MISCELLANEOUS PROBLEMS

5.98. A particle is projected vertically upward from the earth's surface with initial speed v_0.

(*a*) Prove that the maximum height H reached above the earth's surface is $H = v_0^2R/(2gR - v_0^2)$.

(*b*) Discuss the significance of the case where $v_0^2 = 2gR$.

(*c*) Prove that if H is small, then it is equal to $v_0^2/2g$ very nearly.

5.99. (a) Prove that the time taken to reach the maximum height of Problem 5.98 is

$$\sqrt{\frac{R+H}{2g}}\left\{\sqrt{\frac{H}{R}} + \frac{R+H}{2R}\cos^{-1}\left(\frac{R-H}{R+H}\right)\right\}$$

(b) Prove that if H is very small compared with R, then the time in (a) is very nearly $\sqrt{2H/g}$.

5.100. (a) Prove that if an object is dropped to the earth's surface from a height H, then if air resistance is negligible it will hit the earth with a speed $v = \sqrt{2gRH/(R+H)}$ where R is the radius of the earth.

(b) Calculate the speed in part (a) for the cases where $H = 100$ km and $H = 10{,}000$ km respectively. Take the radius of the earth as 6380 km.

5.101. Find the time taken for the object of Problem 5.100 to reach the earth's surface in each of the two cases.

5.102. What must be the law of force if the speed of a particle in a central force field is to be proportional to r^{-n} where n is a constant?

5.103. What velocity must a space ship have in order to keep it in an orbit around the earth at a distance of (a) 200 km, (b) 2000 km above the earth's surface?

5.104. An object is thrown upward from the earth's surface with velocity v_0. Assuming that it returns to earth and that air resistance is negligible, find its velocity on returning.

5.105. (a) What is the work done by a space ship of mass m in moving from a distance a above the earth's surface to a distance b?

(b) Does the work depend on the path? Explain. *Ans.* (a) $GmM(a-b)/ab$

5.106. (a) Prove that it is possible for a particle to move in a circle of radius a in any central force field whose law of force is $f(r)$.

(b) Suppose the particle of part (a) is displaced slightly from its circular orbit. Prove that it will return to the orbit, i.e. the motion is *stable*, if

$$a\,f'(a) + 3\,f(a) > 0$$

but is unstable otherwise.

(c) Illustrate the result in (b) by considering $f(r) = 1/r^n$ and deciding for which values of n stability can occur. *Ans.* (c) For $n < 3$ there is stability.

5.107. If the moon were suddenly stopped in its orbit, how long would it take to fall to the earth assuming that the earth remained at rest? *Ans.* About 4 days 18 hours

5.108. If the earth were suddenly stopped in its orbit, how long would it take for it to fall into the sun? *Ans.* About 65 days

5.109. Work Problem 5.34, page 133, by using energy methods.

5.110. Find the velocity of escape for an object on the surface of the moon. Use the fact that the acceleration due to gravity on the moon's surface is approximately 1/6 that on the earth and that the radius of the moon is approximately 1/4 of the earth's radius. *Ans.* 2.29 km/s

5.111. An object is dropped through a hole bored through the center of the earth. Assuming that the resistance to motion is negligible, show that the speed of the particle as it passes through the center of the earth is slightly less than 8 km/s.
[*Hint.* Use Problem 5.40, page 136.]

5.112. In Problem 5.111 show that the time taken for the object to return is about 85 minutes.

5.113. Work Problems 5.111 and 5.112 if the hole is straight but does not pass through the center of the earth.

5.114. Discuss the relationship between the results of Problems 5.111 and 5.112 and that of Problem 5.39.

5.115. How would you explain the fact that the earth has an atmosphere while the moon has none?

5.116. Prove Theorem 5.1, page 120.

5.117. Discuss Theorem 5.1 if the spheres intersect.

5.118. Explain how you could use the result of Problem 5.27 to find the force of attraction of a solid sphere on a particle.

5.119. Find the force of attraction between a uniform circular ring of outer radius a and inner radius b and a mass m located on its axis at a distance b from its center.

5.120. Two space ships move about the earth on the same elliptical path of eccentricity ϵ. If they are separated by a small distance D at perigee, prove that at apogee they will be separated by the distance $D(1-\epsilon)/(1+\epsilon)$.

5.121. (*a*) Explain how you could calculate the velocity of escape from a planet. (*b*) Use your method to calculate the velocity of escape from Mars. *Ans.* (*b*) 5 km/s

5.122. Work Problem 5.121 for (*a*) Jupiter, (*b*) Venus. *Ans.* (*a*) about 61 km/s, (*b*) about 10 km/s

5.123. Three infinitely long thin uniform rods having the same mass per unit length lie in the same plane and form a triangle. Prove that force of attraction on a particle will be zero if and only if the particle is located at the intersection of the medians of the triangle.

5.124. Find the force of attraction between a uniform rod of length a and a sphere of radius b if they do not intersect and the line of the rod passes through the center.

5.125. Work Problem 5.124 if the rod is situated so that a line drawn from the center perpendicular to the line of the rod bisects the rod.

5.126. A satellite of radius a revolves in a circular orbit about a planet of radius b with period P. If the shortest distance between their surfaces is c, prove that the mass of the planet is $4\pi^2(a+b+c)^3/GP^2$.

5.127. Given that the moon is approximately 386,000 km from the earth and makes one complete revolution about the earth in $27\frac{1}{3}$ days approximately, find the mass of the earth.
Ans. 6×10^{24} kg

5.128. Discuss the relationship of Problem 5.126 with Kepler's third law.

5.129. Prove that the only central force field $\mathbf{F}$ whose divergence is zero is an inverse square force field.

5.130. Work Problem 5.32, page 132, by using triple integration.

5.131. A uniform solid right circular cylinder has radius a and height H. A particle of mass m is placed on the extended axis of the cylinder so that it is at a distance D from one end. Prove that the force of attraction is directed along the axis and given in magnitude by

$$\frac{2GMm}{a^2H}\{H + \sqrt{a^2+D^2} - \sqrt{a^2+(D+H)^2}\}$$

5.132. Suppose that the cylinder of Problem 5.131 has a given volume. Prove that the force of attraction when the particle is at the center of one end of the cylinder is a maximum when $a/H = \frac{1}{8}(9-\sqrt{17})$.

5.133. Work (*a*) Problem 5.26 and (*b*) Problem 5.27 assuming an inverse cube law of attraction.

5.134. Do the results of Problems 5.29 and 5.30 apply if there is an inverse cube law of attraction? Explain.

5.135. What would be the velocity of escape from the small planet of Problem 5.80?

5.136. A spherical shell of inner radius a and outer radius b has constant density σ. Prove that the gravitational potential $V(r)$ at distance r from the center is given by

$$V(r) = \begin{cases} 2\pi\sigma(b^2 - a^2) & r < a \\ 2\pi\sigma(b^2 - \frac{1}{3}r^2) - 4\pi\sigma a^3/3r & a < r < b \\ 4\pi\sigma(b^3 - a^3)/3r & r > b \end{cases}$$

5.137. If *Einstein's theory of relativity* is taken into account, the differential equation for the orbit of a planet becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{K}{mh^2} + \gamma u^2$$

where $\gamma = 3K/mc^2$, c being the speed of light. (a) Prove that if axes are suitably chosen, then the position r of the planet can be determined approximately from

$$r = \frac{mh^2/K}{1 + \epsilon \cos \alpha\theta} \qquad \text{where} \quad \alpha = 1 - \gamma K/mh^2$$

(b) Use (a) to show that a planet actually moves in an elliptical path but that this ellipse slowly rotates in space, the rate of angular rotation being $2\pi\gamma K/mh^2$. (c) Show that in the case of Mercury this rotation amounts to 43 seconds of arc per century. This was actually observed, thus offering experimental proof of the validity of the theory of relativity.

5.138. Find the position of a planet in its orbit around the sun as a function of time t measured from where it is furthest from the sun.

5.139. At apogee of 300 km from the earth's surface, two space ships in the same elliptical path are 150 m apart. How far apart will they be at perigee 250 km assuming that they drift without altering their path in any way?

5.140. A particle of mass m is located on a perpendicular line through the center of a rectangular plate of sides $2a$ and $2b$ at a distance D from this center. Prove that the force of attraction of the plate on the particle is given in magnitude by

$$\frac{GMm}{ab} \sin^{-1}\left(\frac{ab}{\sqrt{(a^2 + D^2)(b^2 + D^2)}}\right)$$

5.141. Find the force of attraction of a uniform infinite plate of negligible thickness and density σ on a particle at distance D from it. *Ans.* $2\pi\sigma Gm$

5.142. Points where $\dot{r} = 0$ are called *apsides* [singular, *apsis*]. (a) Prove that apsides for a central force field with potential $V(r)$ and total energy E are roots of the equation $V(r) + mh^2/2r^2 = E$. (b) Find the apsides corresponding to an inverse square field of force, showing that there are two, one or none according as the orbit is an ellipse, hyperbola or parabola.

5.143. A particle moving in a central force field travels in a path which is the cycloid $r = a(1 - \cos\theta)$. Find the law of force. *Ans.* Inverse fourth power of r.

5.144. Set up equations for the motion of a particle in a central force field if it takes place in a medium where the resistance is proportional to the instantaneous speed of the particle.

5.145. A satellite has its largest and smallest orbital speeds given by v_{max} and v_{min} respectively. Prove that the eccentricity of the orbit in which the satellite moves is equal to $\dfrac{v_{max} - v_{min}}{v_{max} + v_{min}}$.

5.146. Prove that if the satellite of Problem 5.145 has a period equal to τ, then it moves in an elliptic path having major axis whose length is $\frac{\tau}{2\pi}\sqrt{v_{max}\, v_{min}}$.

Chapter 6

MOVING COORDINATE SYSTEMS

NON-INERTIAL COORDINATE SYSTEMS

In preceding chapters the coordinate systems used to describe the motions of particles were assumed to be inertial [see page 33]. In many instances of practical importance, however, this assumption is not warranted. For example, a coordinate system fixed in the earth is not an inertial system since the earth itself is rotating in space. Consequently if we use this coordinate system to describe the motion of a particle relative to the earth we obtain results which may be in error. We are led therefore to consider the motion of particles relative to moving coordinate systems.

ROTATING COORDINATE SYSTEMS

In Fig. 6-1 let XYZ denote an inertial coordinate system with origin O which we shall consider fixed in space. Let the coordinate system xyz having the same origin O be rotating with respect to the XYZ system.

Consider a vector $\mathbf{A}$ which is changing with time. To an observer fixed relative to the xyz system the time rate of change of $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ is found to be

$$\left.\frac{d\mathbf{A}}{dt}\right|_M = \frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k} \qquad (1)$$

where subscript M indicates the derivative in the moving (xyz) system.

However, the time rate of change of $\mathbf{A}$ relative to the fixed XYZ system symbolized by the subscript F is found to be [see Problem 6.1]

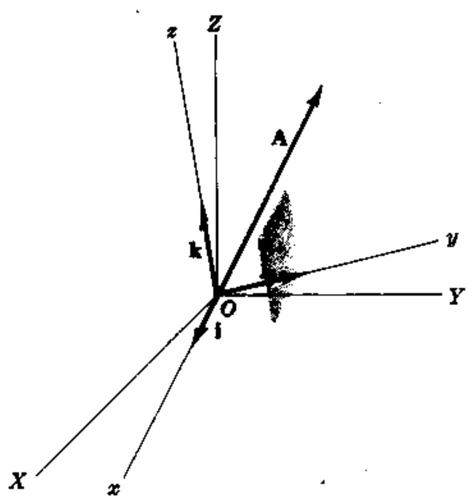

Fig. 6-1

$$\left.\frac{d\mathbf{A}}{dt}\right|_F = \left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{A} \qquad (2)$$

where $\boldsymbol{\omega}$ is called the *angular velocity* of the xyz system with respect to the XYZ system.

DERIVATIVE OPERATORS

Let D_F and D_M represent time derivative operators in the fixed and moving systems. Then we can write the operator equivalence

$$D_F \equiv D_M + \boldsymbol{\omega}\times \qquad (3)$$

This result is useful in relating higher order time derivatives in the fixed and moving systems. See Problem 6.6.

VELOCITY IN A MOVING SYSTEM

If, in particular, vector **A** is the position vector **r** of a particle, then (2) gives

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{r} \tag{4}$$

or

$$D_F\mathbf{r} = D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r} \tag{5}$$

Let us write

$\mathbf{v}_{P|F} = d\mathbf{r}/dt\,|_F = D_F\mathbf{r}$ = velocity of particle P relative to fixed system

$\mathbf{v}_{P|M} = d\mathbf{r}/dt\,|_M = D_M\mathbf{r}$ = velocity of particle P relative to moving system

$\mathbf{v}_{M|F} = \boldsymbol{\omega}\times\mathbf{r}$ = velocity of moving system relative to fixed system.

Then (4) or (5) can be written

$$\mathbf{v}_{P|F} = \mathbf{v}_{P|M} + \mathbf{v}_{M|F} \tag{6}$$

ACCELERATION IN A MOVING SYSTEM

If $D_F^2 = d^2/dt^2\,|_F$ and $D_M^2 = d^2/dt^2\,|_M$ are second derivative operators with respect to t in the fixed and moving systems, then application of (3) yields [see Problem 6.6]

$$D_F^2\mathbf{r} = D_M^2\mathbf{r} + (D_M\boldsymbol{\omega})\times\mathbf{r} + 2\boldsymbol{\omega}\times D_M\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{7}$$

Let us write

$\mathbf{a}_{P|F} = d^2\mathbf{r}/dt^2\,|_F = D_F^2\mathbf{r}$ = acceleration of particle P relative to fixed system

$\mathbf{a}_{P|M} = d^2\mathbf{r}/dt^2\,|_M = D_M^2\mathbf{r}$ = acceleration of particle P relative to moving system

$\mathbf{a}_{M|F} = (D_M\boldsymbol{\omega})\times\mathbf{r} + 2\boldsymbol{\omega}\times D_M\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$
= acceleration of moving system relative to fixed system

Then (7) can be written

$$\mathbf{a}_{P|F} = \mathbf{a}_{P|M} + \mathbf{a}_{M|F} \tag{8}$$

CORIOLIS AND CENTRIPETAL ACCELERATION

The last two terms on the right of (7) are called the *Coriolis acceleration* and *centripetal acceleration* respectively, i.e.,

$$\text{Coriolis acceleration} = 2\boldsymbol{\omega}\times D_M\mathbf{r} = 2\boldsymbol{\omega}\times\mathbf{v}_M \tag{9}$$

$$\text{Centripetal acceleration} = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{10}$$

The second term on the right of (7) is sometimes called the *linear acceleration*, i.e.,

$$\text{Linear acceleration} = (D_M\boldsymbol{\omega})\times\mathbf{r} = \left(\left.\frac{d\boldsymbol{\omega}}{dt}\right|_M\right)\times\mathbf{r} \tag{11}$$

and $D_M\boldsymbol{\omega}$ is called the *angular acceleration*. For many cases of practical importance [e.g. in the rotation of the earth] $\boldsymbol{\omega}$ is constant and $D_M\boldsymbol{\omega} = 0$.

The quantity $-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is often called the *centrifugal acceleration.*

MOTION OF A PARTICLE RELATIVE TO THE EARTH

Newton's second law is strictly applicable only to inertial systems. However, by using (7) we obtain a result valid for non-inertial systems. This has the form

$$mD_M^2\mathbf{r} = \mathbf{F} - m(D_M\boldsymbol{\omega})\times\mathbf{r} - 2m(\boldsymbol{\omega}\times D_M\mathbf{r}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{12}$$

where **F** is the resultant of all forces acting on the particle as seen by the observer in the fixed or inertial system.

In practice we are interested in expressing the equations of motion in terms of quantities as determined by an observer fixed on the earth [or other moving system]. In such case we may omit the subscript M and write (*12*) as

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m(\dot{\boldsymbol{\omega}} \times \mathbf{r}) - 2m(\boldsymbol{\omega} \times \mathbf{v}) - m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] \tag{13}$$

For the case of the earth rotating with constant angular $\boldsymbol{\omega}$ about its axis, $\dot{\boldsymbol{\omega}} = 0$ and (*13*) becomes

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}) - m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] \tag{14}$$

CORIOLIS AND CENTRIPETAL FORCE

Referring to equations (*13*) or (*14*) we often use the following terminology

$$\text{Coriolis force} = 2m(\boldsymbol{\omega} \times \dot{\mathbf{r}}) = 2m(\boldsymbol{\omega} \times \mathbf{v})$$

$$\text{Centripetal force} = m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$$

$$\text{Centrifugal force} = -m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]$$

MOVING COORDINATE SYSTEMS IN GENERAL

In the above results we assumed that the coordinate systems xyz and XYZ [see Fig. 6-1] have common origin O. In case they do not have a common origin, results are easily obtained from those already considered.

Suppose that $\mathbf{R}$ is the position vector of origin Q relative to origin O [see Fig. 6-2]. Then if $\dot{\mathbf{R}}$ and $\ddot{\mathbf{R}}$ denote the velocity and acceleration of Q relative to O, equations (*5*) and (*7*) are replaced respectively by

$$D_F\mathbf{r} = \dot{\mathbf{R}} + D_M\mathbf{r} + \boldsymbol{\omega} \times \mathbf{r}$$

$$= \dot{\mathbf{R}} + \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \tag{15}$$

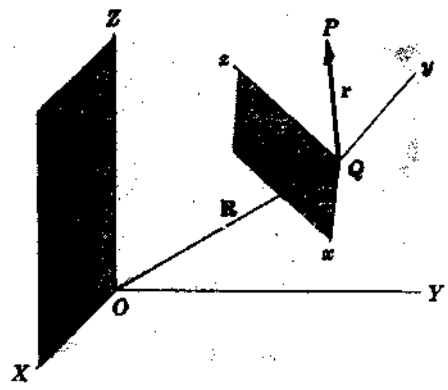

Fig. 6-2

and

$$D_F^2\mathbf{r} = \ddot{\mathbf{R}} + D_M^2\mathbf{r} + (D_M\boldsymbol{\omega}) \times \mathbf{r} + 2\boldsymbol{\omega} \times D_M\mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \ddot{\mathbf{R}} + \frac{d^2\mathbf{r}}{dt^2} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{16}$$

Similarly equation (*14*) is replaced by

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - 2m(\boldsymbol{\omega} \times \mathbf{v}) - m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] - m\ddot{\mathbf{R}} \tag{17}$$

THE FOUCAULT PENDULUM

Consider a simple pendulum consisting of a long string and heavy bob suspended vertically from a frictionless support. Suppose that the bob is displaced from its equilibrium position and is free to rotate in any vertical plane. Then due to the rotation of the earth, the plane in which the pendulum swings will gradually precess about a vertical axis. In the northern hemisphere this precession is in the clockwise direction if we look down at the earth's surface. In the southern hemisphere the precession would be in the counterclockwise direction.

Such a pendulum used for detecting the earth's rotation was first employed by *Foucault* in 1851 and is called *Foucault's pendulum*.

Solved Problems

ROTATING COORDINATE SYSTEMS

6.1. An observer stationed at a point which is fixed relative to an xyz coordinate system with origin O [see Fig. 6-1, page 144] observes a vector $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and calculates its time derivative to be $\frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}$. Later, he finds that he and his coordinate system are actually rotating with respect to an XYZ coordinate system taken as fixed in space and having origin also at O. He asks, "What would be the time derivative of $\mathbf{A}$ for an observer who is fixed relative to the XYZ coordinate system?"

If $\left.\frac{d\mathbf{A}}{dt}\right|_F$ and $\left.\frac{d\mathbf{A}}{dt}\right|_M$ denote respectively the time derivatives of $\mathbf{A}$ relative to the fixed and moving systems, show that there exists a vector quantity $\boldsymbol{\omega}$ such that

$$\left.\frac{d\mathbf{A}}{dt}\right|_F = \left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega} \times \mathbf{A}$$

To the fixed observer the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ actually change with time. Hence such an observer would compute the time derivative as

$$\frac{d\mathbf{A}}{dt} = \frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k} + A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt} \tag{1}$$

i.e.,

$$\left.\frac{d\mathbf{A}}{dt}\right|_F = \left.\frac{d\mathbf{A}}{dt}\right|_M + A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt} \tag{2}$$

Since $\mathbf{i}$ is a unit vector, $d\mathbf{i}/dt$ is perpendicular to $\mathbf{i}$ and must therefore lie in the plane of $\mathbf{j}$ and $\mathbf{k}$. Then

$$d\mathbf{i}/dt = \alpha_1\mathbf{j} + \alpha_2\mathbf{k} \tag{3}$$

Similarly,

$$d\mathbf{j}/dt = \alpha_3\mathbf{k} + \alpha_4\mathbf{i} \tag{4}$$

$$d\mathbf{k}/dt = \alpha_5\mathbf{i} + \alpha_6\mathbf{j} \tag{5}$$

From $\mathbf{i}\cdot\mathbf{j} = 0$, differentiation yields $\mathbf{i}\cdot\frac{d\mathbf{j}}{dt} + \frac{d\mathbf{i}}{dt}\cdot\mathbf{j} = 0$. But $\mathbf{i}\cdot\frac{d\mathbf{j}}{dt} = \alpha_4$ from (4) and $\frac{d\mathbf{i}}{dt}\cdot\mathbf{j} = \alpha_1$ from (3). Thus $\alpha_4 = -\alpha_1$.

Similarly from $\mathbf{i}\cdot\mathbf{k} = 0$, $\mathbf{i}\cdot\frac{d\mathbf{k}}{dt} + \frac{d\mathbf{i}}{dt}\cdot\mathbf{k} = 0$ and $\alpha_5 = -\alpha_2$; from $\mathbf{j}\cdot\mathbf{k} = 0$, $\mathbf{j}\cdot\frac{d\mathbf{k}}{dt} + \frac{d\mathbf{j}}{dt}\cdot\mathbf{k} = 0$ and $\alpha_6 = -\alpha_3$. Then

$$d\mathbf{i}/dt = \alpha_1\mathbf{j} + \alpha_2\mathbf{k}, \quad d\mathbf{j}/dt = \alpha_3\mathbf{k} - \alpha_1\mathbf{i}, \quad d\mathbf{k}/dt = -\alpha_2\mathbf{i} - \alpha_3\mathbf{j}$$

It follows that

$$A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt} = (-\alpha_1A_2 - \alpha_2A_3)\mathbf{i} + (\alpha_1A_1 - \alpha_3A_3)\mathbf{j} + (\alpha_2A_1 + \alpha_3A_2)\mathbf{k} \tag{6}$$

which can be written as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha_3 & -\alpha_2 & \alpha_1 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Then if we choose $\alpha_3 = \omega_1$, $-\alpha_2 = \omega_2$, $\alpha_1 = \omega_3$ this determinant becomes

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_1 & \omega_2 & \omega_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \boldsymbol{\omega} \times \mathbf{A}$$

where $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$.

From (2) and (6) we find, as required,

$$\left.\frac{d\mathbf{A}}{dt}\right|_F = \left.\frac{d\mathbf{A}}{dt}\right|_M + \boldsymbol{\omega} \times \mathbf{A}$$

The vector quantity $\boldsymbol{\omega}$ is the *angular velocity* of the moving system relative to the fixed system.

6.2. Let D_F and D_M be symbolic time derivative operators in the fixed and moving systems respectively. Demonstrate the operator equivalence

$$D_F \equiv D_M + \boldsymbol{\omega}\times$$

By definition
$$D_F\mathbf{A} = \left.\frac{d\mathbf{A}}{dt}\right|_F = \text{derivative in fixed system}$$
$$D_M\mathbf{A} = \left.\frac{d\mathbf{A}}{dt}\right|_M = \text{derivative in moving system}$$

Then from Problem 6.1,

$$D_F\mathbf{A} = D_M\mathbf{A} + \boldsymbol{\omega}\times\mathbf{A} = (D_M + \boldsymbol{\omega}\times)\mathbf{A}$$

which shows the equivalence of the operators $D_F \equiv D_M + \boldsymbol{\omega}\times$.

6.3. Prove that the angular acceleration is the same in both XYZ and xyz coordinate systems.

Let $\mathbf{A} = \boldsymbol{\omega}$ in Problem 6.1. Then

$$\left.\frac{d\boldsymbol{\omega}}{dt}\right|_F = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M + \boldsymbol{\omega}\times\boldsymbol{\omega} = \left.\frac{d\boldsymbol{\omega}}{dt}\right|_M$$

Since $d\boldsymbol{\omega}/dt$ is the angular acceleration, the required statement is proved.

'ELOCITY AND ACCELERATION IN MOVING SYSTEMS

4. Determine the velocity of a moving particle as seen by the two observers in Problem 6.1.

Replacing $\mathbf{A}$ by the position vector $\mathbf{r}$ of the particle, we have

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{r} \tag{1}$$

If $\mathbf{r}$ is expressed in terms of the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the moving coordinate system, then the velocity of the particle relative to this system is, on dropping the subscript M,

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \tag{2}$$

and the velocity of the particle relative to the fixed system is from (1)

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times\mathbf{r} \tag{3}$$

The velocity (3) is sometimes called the *true velocity*, while (2) is the *apparent velocity*.

An xyz coordinate system is rotating with respect to an XYZ coordinate system having the same origin and assumed to be fixed in space [i.e. it is an inertial system]. The angular velocity of the xyz system relative to the XYZ system is given by $\boldsymbol{\omega} = 2t\mathbf{i} - t^2\mathbf{j} + (2t+4)\mathbf{k}$ where t is the time. The position vector of a particle at time t as observed in the xyz system is given by $\mathbf{r} = (t^2+1)\mathbf{i} - 6t\mathbf{j} + 4t^3\mathbf{k}$. Find (a) the apparent velocity and (b) the true velocity at time $t = 1$.

(a) The apparent velocity at any time t is

$$d\mathbf{r}/dt = 2t\mathbf{i} - 6\mathbf{j} + 12t^2\mathbf{k}$$

At time $t = 1$ this is $2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}$.

(b) The true velocity at any time t is

$$d\mathbf{r}/dt + \boldsymbol{\omega}\times\mathbf{r} = (2t\mathbf{i} - 6\mathbf{j} + 12t^2\mathbf{k}) + [2t\mathbf{i} - t^2\mathbf{j} + (2t+4)\mathbf{k}] \times [(t^2+1)\mathbf{i} - 6t\mathbf{j} + 4t^3\mathbf{k}]$$

At time $t = 1$ this is

$$2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix} = 34\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

6.6. Determine the acceleration of a moving particle as seen by the two observers in Problem 6.1.

The acceleration of the particle as seen by the observer in the fixed XYZ system is $D_F^2\mathbf{r} = D_F(D_F\mathbf{r})$. Using the operator equivalence established in Problem 6.2, we have

$$\begin{aligned} D_F(D_F\mathbf{r}) &= D_F(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) \\ &= (D_M + \boldsymbol{\omega}\times\ )(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) \\ &= D_M(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times(D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r}) \\ &= D_M^2\mathbf{r} + D_M(\boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times D_M\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \end{aligned}$$

or since $D_M(\boldsymbol{\omega}\times\mathbf{r}) = (D_M\boldsymbol{\omega})\times\mathbf{r} + \boldsymbol{\omega}\times(D_M\mathbf{r})$,

$$D_F^2\mathbf{r} = D_M^2\mathbf{r} + (D_M\boldsymbol{\omega})\times\mathbf{r} + 2\boldsymbol{\omega}\times(D_M\mathbf{r}) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{1}$$

If $\mathbf{r}$ is the position vector expressed in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the moving coordinate system, then the acceleration of the particle relative to this system is, on dropping the subscript M,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \tag{2}$$

The acceleration of the particle relative to the fixed system is given from (1) as

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_F = \frac{d^2\mathbf{r}}{dt^2} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{3}$$

The acceleration (3) is sometimes called the *true acceleration*, while (2) is the *apparent acceleration*.

6.7. Find (a) the apparent acceleration and (b) the true acceleration of the particle in Problem 6.5.

(a) The apparent acceleration at any time t is

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}(2t\mathbf{i} - 6\mathbf{j} + 12t^2\mathbf{k}) = 2\mathbf{i} + 24t\mathbf{k}$$

At time $t = 1$ this is $2\mathbf{i} + 24\mathbf{k}$.

(b) The true acceleration at any time t is

$$\frac{d^2\mathbf{r}}{dt^2} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

At time $t = 1$ this equals

$$\begin{aligned} &2\mathbf{i} + 24\mathbf{k} + (4\mathbf{i} - 2\mathbf{j} + 12\mathbf{k})\times(2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) \\ &\qquad + (2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})\times(2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}) \\ &\qquad\qquad + (2\mathbf{i} - \mathbf{j} + 6\mathbf{k})\times\{(2\mathbf{i} - \mathbf{j} + 6\mathbf{k})\times(2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k})\} \\ &= 2\mathbf{i} + 24\mathbf{k} + (48\mathbf{i} - 24\mathbf{j} - 20\mathbf{k}) + (4\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}) + (-14\mathbf{i} + 212\mathbf{j} + 40\mathbf{k}) \\ &= 40\mathbf{i} + 184\mathbf{j} + 36\mathbf{k} \end{aligned}$$

CORIOLIS AND CENTRIPETAL ACCELERATION

6.8. Referring to Problem 6.5, find (*a*) the Coriolis acceleration, (*b*) the centripetal acceleration and (*c*) their magnitudes at time $t = 1$.

(*a*) From Problem 6.5 we have,

$$\text{Coriolis acceleration} = 2\boldsymbol{\omega} \times d\mathbf{r}/dt = (4\mathbf{i} - 2\mathbf{j} + 12\mathbf{k}) \times (2\mathbf{i} - 6\mathbf{j} + 12\mathbf{k}) = 48\mathbf{i} - 24\mathbf{j} - 20\mathbf{k}$$

(*b*) From Problem 6.5 we have,

$$\text{Centripetal acceleration} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = (2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) \times (32\mathbf{i} + 4\mathbf{j} - 10\mathbf{k}) = -14\mathbf{i} + 212\mathbf{j} + 40\mathbf{k}$$

(*c*) From parts (*a*) and (*b*) we have

$$\text{Magnitude of Coriolis acceleration} = \sqrt{(48)^2 + (-24)^2 + (-20)^2} = 4\sqrt{205}$$

$$\text{Magnitude of centripetal acceleration} = \sqrt{(-14)^2 + (212)^2 + (40)^2} = 2\sqrt{11{,}685}$$

MOTION OF A PARTICLE RELATIVE TO THE EARTH

6.9. (*a*) Express Newton's second law for the motion of a particle relative to an XYZ coordinate system fixed in space (inertial system). (*b*) Use (*a*) to find an equation of motion for the particle relative to an xyz system having the same origin as the XYZ system but rotating with respect to it.

(*a*) If m is the mass of the particle (assumed constant), $d^2\mathbf{r}/dt^2|_F$ its acceleration in the fixed system and $\mathbf{F}$ the resultant of all forces acting on the particle as viewed in the fixed system, then Newton's second law states that

$$m\left.\frac{d^2\mathbf{r}}{dt^2}\right|_F = \mathbf{F} \tag{1}$$

(*b*) Using subscript M to denote quantities as viewed in the moving system, we have from Problem 6.6,

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_F = \left.\frac{d^2\mathbf{r}}{dt^2}\right|_M + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{2}$$

Substituting this into (*1*), we find the required equation

$$m\left.\frac{d^2\mathbf{r}}{dt^2}\right|_M = \mathbf{F} - m(\dot{\boldsymbol{\omega}} \times \mathbf{r}) - 2m\left(\boldsymbol{\omega} \times \left.\frac{d\mathbf{r}}{dt}\right|_M\right) - m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] \tag{3}$$

We can drop the subscript M provided it is clear that all quantities except $\mathbf{F}$ are as determined by an observer in the moving system. The quantity $\mathbf{F}$, it must be emphasized, is the resultant force as observed in the fixed or inertial system. If we do remove the subscript M and write $d\mathbf{r}/dt = \mathbf{v}$, then (*3*) can be written

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m(\dot{\boldsymbol{\omega}} \times \mathbf{r}) - 2m(\boldsymbol{\omega} \times \mathbf{v}) - m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] \tag{4}$$

6.10. Calculate the angular speed of the earth about its axis.

Since the earth makes one revolution [2π radians] about its axis in approximately 24 hours = 86,400 s, the angular speed is

$$\omega = \frac{2\pi}{86{,}400} = 7.27 \times 10^{-5} \text{ rad/s}$$

The actual time for one revolution is closer to 86,164 s and the angular speed 7.29×10^{-5} rad/s.

MOVING COORDINATE SYSTEMS IN GENERAL

6.11. Work Problem 6.4 if the origins of the XYZ and xyz systems do not coincide.

Let $\mathbf{R}$ be the position vector of origin Q of the xyz system relative to origin O of the fixed (or inertial) XYZ system [see Fig. 6-3]. The velocity of the particle P relative to the moving system is, as before,

$$\left.\frac{d\mathbf{r}}{dt}\right|_M = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \qquad (1)$$

Now the position vector of P relative to O is $\boldsymbol{\rho} = \mathbf{R} + \mathbf{r}$ and thus the velocity of P as viewed in the XYZ system is

$$\frac{d\boldsymbol{\rho}}{dt} = \left.\frac{d}{dt}(\mathbf{R}+\mathbf{r})\right|_F = \left.\frac{d\mathbf{R}}{dt}\right|_F + \left.\frac{d\mathbf{r}}{dt}\right|_F$$

$$= \dot{\mathbf{R}} + \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times\mathbf{r} \qquad (2)$$

using equation (3) of Problem 6.4. Note that $\dot{\mathbf{R}}$ is the velocity of Q with respect to O. If $\mathbf{R} = \mathbf{0}$ this reduces to the result of Problem 6.4.

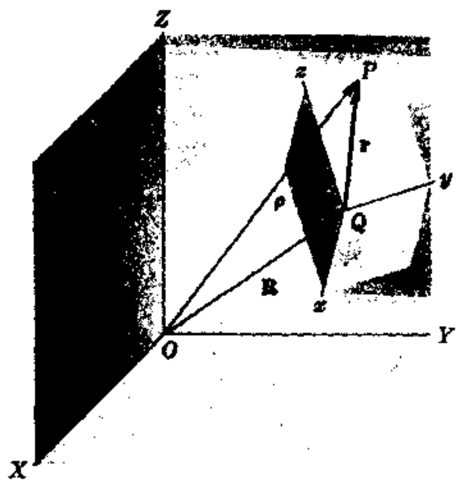

Fig. 6-3

6.12. Work Problem 6.6 if the origins of the XYZ and xyz systems do not coincide.

Referring to Fig. 6-3, the acceleration of the particle P relative to the moving system is, as before,

$$\left.\frac{d^2\mathbf{r}}{dt^2}\right|_M = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \qquad (1)$$

Since the position vector of P relative to O is $\boldsymbol{\rho} = \mathbf{R} + \mathbf{r}$, the acceleration of P as viewed in the XYZ system is

$$\left.\frac{d^2\boldsymbol{\rho}}{dt^2}\right|_F = \left.\frac{d^2}{dt^2}(\mathbf{R}+\mathbf{r})\right|_F = \left.\frac{d^2\mathbf{R}}{dt^2}\right|_F + \left.\frac{d^2\mathbf{r}}{dt^2}\right|_F$$

$$= \ddot{\mathbf{R}} + \frac{d^2\mathbf{r}}{dt^2} + \frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \qquad (2)$$

using equation (3) of Problem 6.6. Note that $\ddot{\mathbf{R}}$ is the acceleration of Q with respect to O. If $\mathbf{R} = \mathbf{0}$ this reduces to the result of Problem 6.6.

6.13. Work Problem 6.9 if the origins of the XYZ and xyz systems do not coincide.

(a) The position vector of the particle relative to the fixed (XYZ) system is $\boldsymbol{\rho}$. Then th required equation of motion is

$$m\left.\frac{d^2\boldsymbol{\rho}}{dt^2}\right|_F = \mathbf{F} \qquad (1$$

(b) Using the result (2) of Problem 6.12 in (1), we obtain

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m\ddot{\mathbf{R}} - m(\dot{\boldsymbol{\omega}}\times\mathbf{r}) - 2m(\boldsymbol{\omega}\times\mathbf{v}) - m[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})] \qquad ($$

where $\mathbf{F}$ is the force acting on m as viewed in the inertial system and where $\mathbf{v} = \dot{\mathbf{r}}$.

6.14. Find the equation of motion of a particle relative to an observer on the eartl surface.

We assume the earth to be a sphere with center at O [Fig. 6-4] rotating about the Z axis with angular velocity $\boldsymbol{\omega} = \omega\mathbf{K}$. We also use the fact that the effect of the earth's rotation around the sun is negligible, so that the XYZ system can be taken as an inertial system.

Then we can use equation (2) of Problem 6.12. For the case of the earth, we have

$$\dot{\boldsymbol{\omega}} = \mathbf{0} \qquad (1)$$

$$\ddot{\mathbf{R}} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) \qquad (2)$$

$$\mathbf{F} = -\frac{GMm}{\rho^3}\boldsymbol{\rho} \qquad (3)$$

the first equation arising from the fact that the rotation of the earth about its axis proceeds with constant angular velocity, the second arising from the fact that the acceleration of origin Q relative to O is the centripetal acceleration, and the third arising from Newton's law of gravitation. Using these in (2) of Problem 6.12 yields the required equation,

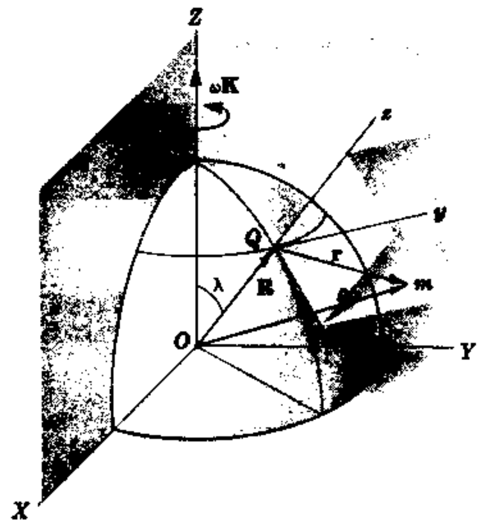

Fig. 6-4

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{\rho^3}\boldsymbol{\rho} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) - 2(\boldsymbol{\omega} \times \mathbf{v}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (4)$$

assuming that other forces acting on m [such as air resistance, etc.] are neglected.

We can define

$$\mathbf{g} = -\frac{GM}{\rho^3}\boldsymbol{\rho} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) \qquad (5)$$

as the acceleration due to gravity, so that (4) becomes

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{g} - 2(\boldsymbol{\omega} \times \mathbf{v}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (6)$$

Near the earth's surface the last term in (6) can be neglected, so that to a high degree of approximation,

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{g} - 2(\boldsymbol{\omega} \times \mathbf{v}) \qquad (7)$$

In practice we choose $\mathbf{g}$ as constant in magnitude although it varies slightly over the earth's surface. If other external forces act, we must add them to the right side of equations (6) or (7).

6.15. Show that if the particle of Problem 6.14 moves near the earth's surface, then the equations of motion are given by

$$\ddot{x} = 2\omega\cos\lambda\,\dot{y}$$
$$\ddot{y} = -2(\omega\cos\lambda\,\dot{x} + \omega\sin\lambda\,\dot{z})$$
$$\ddot{z} = -g + 2\omega\sin\lambda\,\dot{y}$$

where the angle λ is the *colatitude* [see Fig. 6-4] and $90° - \lambda$ is the *latitude*.

From Fig. 6-4 we have

$$\mathbf{K} = (\mathbf{K}\cdot\mathbf{i})\mathbf{i} + (\mathbf{K}\cdot\mathbf{j})\mathbf{j} + (\mathbf{K}\cdot\mathbf{k})\mathbf{k}$$
$$= (-\sin\lambda)\mathbf{i} + 0\mathbf{j} + (\cos\lambda)\mathbf{k} = -\sin\lambda\,\mathbf{i} + \cos\lambda\,\mathbf{k}$$

and so $\qquad \boldsymbol{\omega} = \omega\mathbf{K} = -\omega\sin\lambda\,\mathbf{i} + \omega\cos\lambda\,\mathbf{k}$

Then
$$\begin{aligned}\boldsymbol{\omega}\times\mathbf{v} &= \boldsymbol{\omega}\times(\dot{x}\mathbf{i}+\dot{y}\mathbf{j}+\dot{z}\mathbf{k})\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\omega\sin\lambda & 0 & \omega\cos\lambda\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix}\\ &= (-\omega\cos\lambda\,\dot{y})\mathbf{i} + (\omega\cos\lambda\,\dot{x}+\omega\sin\lambda\,\dot{z})\mathbf{j} - (\omega\sin\lambda\,\dot{y})\mathbf{k}\end{aligned}$$

Thus from equation (7) of Problem 6.14 we have
$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \mathbf{g} - 2(\boldsymbol{\omega}\times\mathbf{v})\\ &= -g\mathbf{k} + 2\omega\cos\lambda\,\dot{y}\,\mathbf{i} - 2(\omega\cos\lambda\,\dot{x}+\omega\sin\lambda\,\dot{z})\mathbf{j} + 2\omega\sin\lambda\,\dot{y}\,\mathbf{k}\end{aligned}$$

Equating corresponding coefficients of $\mathbf{i},\mathbf{j},\mathbf{k}$ on both sides of this equation, we find, as required,
$$\ddot{x} = 2\omega\cos\lambda\,\dot{y} \tag{1}$$
$$\ddot{y} = -2(\omega\cos\lambda\,\dot{x}+\omega\sin\lambda\,\dot{z}) \tag{2}$$
$$\ddot{z} = -g + 2\omega\sin\lambda\,\dot{y} \tag{3}$$

6.16. An object of mass m initially at rest is dropped to the earth's surface from a height which is small compared with the earth's radius. Assuming that the angular speed of the earth about its axis is a constant ω, prove that after time t the object is deflected east of the vertical by the amount $\frac{1}{3}\omega g t^3 \sin\lambda$.

Method 1.

We assume that the object is located on the z axis at $x=0$, $y=0$, $z=h$ [see Fig. 6-4]. From equations (1) and (2) of Problem 6.15 we have on integrating,
$$\dot{x} = 2\omega\cos\lambda\,y + c_1, \qquad \dot{y} = -2(\omega\cos\lambda\,x + \omega\sin\lambda\,z) + c_2$$

Since at $t=0$, $\dot{x}=0$, $\dot{y}=0$, $x=0$, $y=0$, $z=h$ we have $c_1=0$, $c_2 = 2\omega\sin\lambda\,h$. Thus
$$\dot{x} = 2\omega\cos\lambda\,y, \qquad \dot{y} = -2(\omega\cos\lambda\,x + \omega\sin\lambda\,z) + 2\omega\sin\lambda\,h \tag{1}$$

Then (3) of Problem 6.15 becomes
$$\ddot{z} = -g + 2\omega\sin\lambda\,\dot{y} = -g - 4\omega^2\sin\lambda\,[\cos\lambda\,x + \sin\lambda\,(z-h)]$$

But since the terms on the right involving ω^2 are very small compared with $-g$ we can neglect them and write $\ddot{z} = -g$. Integration yields $\dot{z} = -gt + c_3$. Since $\dot{z}=0$ at $t=0$, we have $c_3 = 0$ or
$$\dot{z} = -gt \tag{2}$$

Using equation (2) and the first equation of (1) in equation (2) of Problem 6.15 we find
$$\begin{aligned}\ddot{y} &= (-2\omega\cos\lambda)(2\omega\cos\lambda\,y) + (-2\omega\sin\lambda)(-gt)\\ &= -4\omega^2\cos^2\lambda\,y + 2\omega\sin\lambda\,gt\end{aligned}$$

Then neglecting the first term, we have $\ddot{y} = 2\omega\sin\lambda\,gt$. Integrating,
$$\dot{y} = \omega g\sin\lambda\,t^2 + c_4$$

Since $\dot{y}=0$ at $t=0$, we have $c_4=0$ and $\dot{y} = \omega g\sin\lambda\,t^2$. Integrating again,
$$y = \tfrac{1}{3}\omega g\sin\lambda\,t^3 + c_5$$

Then since $y=0$ at $t=0$, $c_5=0$ so that, as required.
$$y = \tfrac{1}{3}\omega g\sin\lambda\,t^3 \tag{3}$$

Method 2.

Integrating equations (1), (2) and (3) of Problem 6.15, we have
$$\begin{aligned}\dot{x} &= 2\omega\cos\lambda\,y + c_1\\ \dot{y} &= -2(\omega\cos\lambda\,x + \omega\sin\lambda\,z) + c_2\\ \dot{z} &= -gt + 2\omega\sin\lambda\,y + c_3\end{aligned}$$

Using the fact that at $t = 0$, $\dot{x} = \dot{y} = \dot{z} = 0$ and $x = 0$; $y = 0$, $z = h$, we have $c_1 = 0$, $c_2 = 2\omega h \sin\lambda$, $c_3 = 0$. Thus

$$\begin{aligned} \dot{x} &= 2\omega\cos\lambda\, y \\ \dot{y} &= -2(\omega\cos\lambda\, x + \omega\sin\lambda\, z) + 2\omega h\sin\lambda \\ \dot{z} &= -gt + 2\omega\sin\lambda\, y \end{aligned}$$

Integrating these we find, using the above conditions,

$$x = 2\omega\cos\lambda\int_0^t y\,du \tag{4}$$

$$y = 2\omega ht\sin\lambda - 2\omega\cos\lambda\int_0^t x\,du - 2\omega\sin\lambda\int_0^t z\,du \tag{5}$$

$$z = h - \tfrac{1}{2}gt^2 + 2\omega\sin\lambda\int_0^t y\,du \tag{6}$$

Since the unknowns are under the integral sign, these equations are called *integral equations*. We shall use a method called the *method of successive approximations* or *method of iteration* to obtain a solution to any desired accuracy. The method consists of using a first guess for x, y, z under the integral signs in (4), (5) and (6) to obtain a better guess. As a first guess we can try $x = 0$, $y = 0$, $z = 0$ under the integral signs. Then we find as a second guess

$$x = 0, \qquad y = 2\omega ht\sin\lambda, \qquad z = h - \tfrac{1}{2}gt^2$$

Substituting these in (4), (5) and (6) and neglecting terms involving ω^2, we find the third guess

$$x = 0, \qquad y = 2\omega ht\sin\lambda - 2\omega\sin\lambda(ht - \tfrac{1}{6}gt^3) = \tfrac{1}{3}\omega gt^3\sin\lambda, \qquad z = h - \tfrac{1}{2}gt^2$$

Using these in (4), (5) and (6) and again neglecting terms involving ω^2, we find the fourth guess

$$x = 0, \qquad y = \tfrac{1}{3}\omega gt^3\sin\lambda, \qquad z = h - \tfrac{1}{2}gt^2$$

Since this fourth guess is identical with the third guess, these results are accurate up to terms involving ω^2, and no further guesses need be taken. It is thus seen that the deflection is $y = \frac{1}{3}\omega gt^3\sin\lambda$, as required.

6.17. Referring to Problem 6.16, show that an object dropped from height h above the earth's surface hits the earth at a point east of the vertical at a distance $\frac{2}{3}\omega h\sin\lambda\sqrt{2h/g}$.

From (2) of Problem 6.16 we have on integrating, $\dot{z} = -\frac{1}{2}gt^2 + c$. Since $z = h$ at $t = 0$, $c = h$ and $z = h - \frac{1}{2}gt^2$. Then at $z = 0$, $h = \frac{1}{2}gt^2$ or $t = \sqrt{2h/g}$. Substituting this value of t into (3) of Problem 6.16, we find the required distance.

THE FOUCAULT PENDULUM

6.18. Derive an equation of motion for a simple pendulum, taking into account the earth's rotation about its axis.

Choose the xyz coordinate system of Fig. 6-5. Suppose that the origin O is the equilibrium position of the bob B, A is the point of suspension and the length of string AB is l. If the tension in the string is $\mathbf{T}$, then we have

$$\begin{aligned} \mathbf{T} &= (\mathbf{T}\cdot\mathbf{i})\mathbf{i} + (\mathbf{T}\cdot\mathbf{j})\mathbf{j} + (\mathbf{T}\cdot\mathbf{k})\mathbf{k} \\ &= T\cos\alpha\,\mathbf{i} + T\cos\beta\,\mathbf{j} + T\cos\gamma\,\mathbf{k} \\ &= -T\left(\frac{x}{l}\right)\mathbf{i} - T\left(\frac{y}{l}\right)\mathbf{j} + T\left(\frac{l-z}{l}\right)\mathbf{k} \end{aligned} \tag{1}$$

Since the net force acting on B is $\mathbf{T} + m\mathbf{g}$, the equation of motion of B is given by [see Problem 6.14]

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{T} + m\mathbf{g} - 2m(\boldsymbol{\omega}\times\mathbf{v}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{2}$$

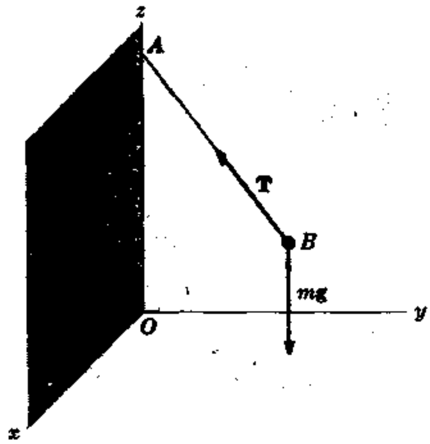

Fig. 6-5

If we neglect the last term in (2), put $\mathbf{g} = -g\mathbf{k}$ and use (1), then (2) can be written in component form as

$$m\ddot{x} = -T(x/l) + 2m\omega\dot{y}\cos\lambda \tag{3}$$

$$m\ddot{y} = -T(y/l) - 2m\omega(\dot{x}\cos\lambda + \dot{z}\sin\lambda) \tag{4}$$

$$m\ddot{z} = T(l-z)/l - mg + 2m\omega\dot{y}\sin\lambda \tag{5}$$

6.19. By assuming that the bob of the simple pendulum in Problem 6.18 undergoes small oscillations about the equilibrium position so that its motion can be assumed to take place in a horizontal plane, simplify the equations of motion.

Making the assumption that the motion of the bob takes place in a horizontal plane amounts to assuming that $\ddot{z}$ and $\dot{z}$ are zero. For small vibrations $(l-z)/l$ is very nearly equal to one. Then equation (5) of Problem 6.18 yields

$$0 = T - mg + 2m\omega\dot{y}\sin\lambda$$

or

$$T = mg - 2m\omega\dot{y}\sin\lambda \tag{1}$$

Substituting (1) into equations (3) and (4) of Problem 6.18 and simplifying, we obtain

$$\ddot{x} = -\frac{gx}{l} + \frac{2\omega x\dot{y}\sin\lambda}{l} + 2\omega\dot{y}\cos\lambda \tag{2}$$

$$\ddot{y} = -\frac{gy}{l} + \frac{2\omega y\dot{y}\sin\lambda}{l} - 2\omega\dot{x}\cos\lambda \tag{3}$$

These differential equations are non-linear because of the presence of the terms involving $x\dot{y}$ and $y\dot{y}$. However, these terms are negligible compared with the others since ω, x and y are small. Upon neglecting them we obtain the linear differential equations

$$\ddot{x} = -gx/l + 2\omega\dot{y}\cos\lambda \tag{4}$$

$$\ddot{y} = -gy/l - 2\omega\dot{x}\cos\lambda \tag{5}$$

6.20. Solve the equations of motion of the pendulum obtained in Problem 6.19, assuming suitable initial conditions.

Suppose that initially the bob is in the yz plane and is given a displacement from the z axis of magnitude $A > 0$, after which it is released. Then the initial conditions are

$$x = 0,\ \dot{x} = 0,\ y = A,\ \dot{y} = 0 \quad \text{at } t = 0 \tag{1}$$

To find the solution of equations (4) and (5) of Problem 6.19, it is convenient to place

$$K^2 = g/l, \quad \alpha = \omega\cos\lambda \tag{2}$$

so that they become

$$\ddot{x} = -K^2x + 2\alpha\dot{y} \tag{3}$$

$$\ddot{y} = -K^2y - 2\alpha\dot{x} \tag{4}$$

It is also convenient to use complex numbers. Multiplying equation (4) by i and adding to (3), we find

$$\ddot{x} + i\ddot{y} = -K^2(x+iy) + 2\alpha(\dot{y} - i\dot{x}) = -K^2(x+iy) - 2i\alpha(\dot{x} + i\dot{y})$$

Then calling $u = x + iy$, this can be written

$$\ddot{u} = -K^2u - 2i\alpha\dot{u} \quad \text{or} \quad \ddot{u} + 2i\alpha\dot{u} + K^2u = 0 \tag{5}$$

If $u = Ce^{\gamma t}$ where C and γ are constants, this becomes

$$\gamma^2 + 2i\alpha\gamma + K^2 = 0$$

so that

$$\gamma = (-2i\alpha \pm \sqrt{-4\alpha^2 - 4K^2})/2 = -i\alpha \pm i\sqrt{\alpha^2 + K^2} \tag{6}$$

Now since $\alpha^2 = \omega^2\cos^2\lambda$ is small compared to $K^2 = g/l$, we can write

$$\gamma = -i\alpha \pm iK \tag{7}$$

Then solutions of the equation are (allowing for complex coefficients)

$$(C_1 + iC_2)e^{-i(\alpha-K)t} \quad \text{and} \quad (C_3 + iC_4)e^{-i(\alpha+K)t}$$

and the general solution is

$$u = (C_1 + iC_2)e^{-i(\alpha-K)t} + (C_3 + iC_4)e^{-i(\alpha+K)t} \tag{8}$$

where C_1, C_2, C_3, C_4 are assumed real. Using Euler's formulas

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta \tag{9}$$

and the fact that $u = x + iy$, *(8)* can be written

$$x + iy = (C_1 + iC_2)\{\cos(\alpha - K)t - i\sin(\alpha - K)t\} + (C_3 + iC_4)\{\cos(\alpha + K)t - i\sin(\alpha + K)t\}$$

Equating real and imaginary parts, we find

$$x = C_1\cos(\alpha - K)t + C_2\sin(\alpha - K)t + C_3\cos(\alpha + K)t + C_4\sin(\alpha + K)t \tag{10}$$

$$y = -C_1\sin(\alpha - K)t + C_2\cos(\alpha - K)t - C_3\sin(\alpha + K)t + C_4\cos(\alpha + K)t \tag{11}$$

Using the initial condition $x = 0$ at $t = 0$, we find from *(10)* that $C_1 + C_3 = 0$ or $C_3 = -C_1$. Similarly, using $\dot{x} = 0$ at $t = 0$, we find from *(10)* that

$$C_4 = C_2\left(\frac{K - \alpha}{K + \alpha}\right) = C_2\left(\frac{\sqrt{g/l} - \omega\cos\lambda}{\sqrt{g/l} + \omega\cos\lambda}\right)$$

Now since $\omega\cos\lambda$ is small compared with $\sqrt{g/l}$, we have, to a high degree of approximation, $C_4 = C_2$.

Thus equations *(10)* and *(11)* become

$$x = C_1\cos(\alpha - K)t + C_2\sin(\alpha - K)t - C_1\cos(\alpha + K)t + C_2\sin(\alpha + K)t \tag{12}$$

$$y = -C_1\sin(\alpha - K)t + C_2\cos(\alpha - K)t + C_1\sin(\alpha + K)t + C_2\cos(\alpha + K)t \tag{13}$$

Using the initial condition $\dot{y} = 0$, *(13)* yields $C_1 = 0$. Similarly using $y = A$ at $t = 0$, we find $C_2 = \frac{1}{2}A$. Thus *(12)* and *(13)* become

$$x = \tfrac{1}{2}A\sin(\alpha - K)t + \tfrac{1}{2}A\sin(\alpha + K)t$$
$$y = \tfrac{1}{2}A\cos(\alpha - K)t + \tfrac{1}{2}A\cos(\alpha + K)t$$

or

$$\left.\begin{aligned} x &= A\cos Kt\sin\alpha t \\ y &= A\cos Kt\cos\alpha t \end{aligned}\right\} \tag{14}$$

i.e.,

$$\left.\begin{aligned} x &= A\cos\sqrt{g/l}\,t\,\sin(\omega\cos\lambda\; t) \\ y &= A\cos\sqrt{g/l}\,t\,\cos(\omega\cos\lambda\; t) \end{aligned}\right\} \tag{15}$$

6.21. Give a physical interpretation to the solution *(15)* of Problem 6.20.

In vector form, *(15)* can be written

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = A\cos\sqrt{g/l}\,t\,\mathbf{n}$$

where

$$\mathbf{n} = \mathbf{i}\sin(\omega\cos\lambda)t + \mathbf{j}\cos(\omega\cos\lambda)t$$

is a unit vector.

The period of $\cos\sqrt{g/l}\,t$ [namely, $2\pi\sqrt{l/g}$] is very small compared with the period of $\mathbf{n}$ [namely, $2\pi/(\omega\cos\lambda)$]. It follows that $\mathbf{n}$ is a very slowly turning vector. Thus physically the pendulum oscillates in a plane through the z axis which is slowly rotating (or *precessing*) about the z axis.

Now at $t = 0$, $\mathbf{n} = \mathbf{j}$ and the bob is at $y = A$. After a time $t = 2\pi/(4\omega\cos\lambda)$, for example, $\mathbf{n} = \frac{1}{2}\sqrt{2}\,\mathbf{i} + \frac{1}{2}\sqrt{2}\,\mathbf{j}$ so that the rotation of the plane is proceeding in the clockwise direction as viewed from above the earth's surface in the northern hemisphere [where $\cos\lambda > 0$]. In the southern hemisphere the rotation of the plane is counterclockwise.

The rotation of the plane was observed by Foucault in 1851 and served to provide laboratory evidence of the rotation of the earth about its axis.

MISCELLANEOUS PROBLEMS

6.22. The vertical rod AB of Fig. 6-6 is rotating with constant angular velocity ω. A light inextensible string of length l has one end attached at point O of the rod while the other end P of the string has a mass m attached. Find (*a*) the tension in the string and (*b*) the angle which string OP makes with the vertical when equilibrium conditions prevail.

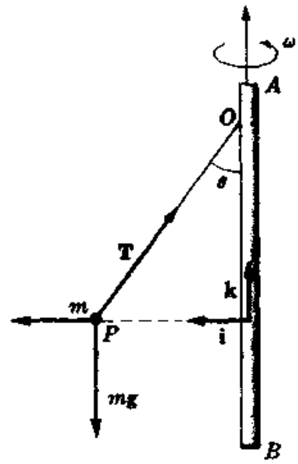

Fig. 6-6

Choose unit vectors **i** and **k** perpendicular and parallel respectively to the rod and rotating with it. The unit vector **j** can be chosen perpendicular to the plane of **i** and **k**. Let

$$\mathbf{r} = l \sin\theta\, \mathbf{i} - l\cos\theta\, \mathbf{k}$$

be the position vector of m with respect to O.

Three forces act on particle m

(i) The weight, $m\mathbf{g} = -mg\mathbf{k}$

(ii) The centrifugal force,

$$-m\{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\} = -m\{[\omega\mathbf{k}] \times ([\omega\mathbf{k}] \times [l \sin\theta\, \mathbf{i} - l\cos\theta\, \mathbf{k}])\}$$
$$= -m\{[\omega\mathbf{k}] \times (\omega l \sin\theta\, \mathbf{j})\} = m\omega^2 l \sin\theta\, \mathbf{i}$$

(iii) The tension, $\mathbf{T} = -T\sin\theta\, \mathbf{i} + T\cos\theta\, \mathbf{k}$

When the particle is in equilibrium, the resultant of all these forces is zero. Then

$$-mg\mathbf{k} + m\omega^2 l \sin\theta\, \mathbf{i} - T\sin\theta\, \mathbf{i} + T\cos\theta\, \mathbf{k} = \mathbf{0}$$

i.e.,
$$(m\omega^2 l \sin\theta - T\sin\theta)\mathbf{i} + (T\cos\theta - mg)\mathbf{k} = \mathbf{0}$$

or
$$m\omega^2 l \sin\theta - T\sin\theta = 0 \qquad (1)$$
$$T\cos\theta - mg = 0 \qquad (2)$$

Solving (*1*) and (*2*) simultaneously, we find (*a*) $T = m\omega^2 l$, (*b*) $\theta = \cos^{-1}(g/\omega^2 l)$.

Since the string OP with mass m at P describe the surface of a cone the system is sometimes called a *conical pendulum*.

6.23. A rod AOB [Fig. 6-7] rotates in a vertical plane [the yz plane] about a horizontal axis through O perpendicular to this plane [the x axis] with constant angular velocity ω. Assuming no frictional forces, determine the motion of a particle P of mass m which is constrained to move along the rod. An equivalent problem exists when the rod AOB is replaced by a thin hollow tube inside which the particle can move.

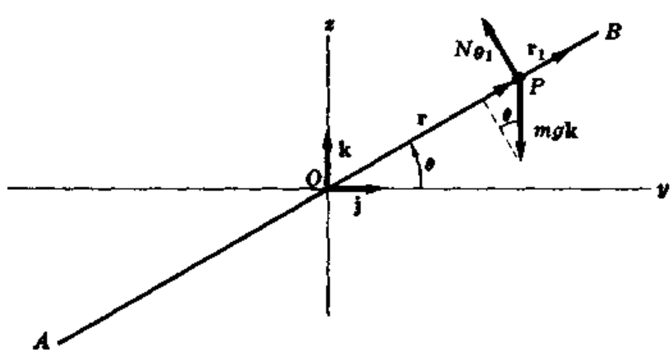

Fig. 6-7

At time t let $\mathbf{r}$ be the position vector of the particle and θ the angle made by the rod with the y axis. Choose unit vectors **j** and **k** in the y and z directions respectively and unit vector $\mathbf{i} = \mathbf{j} \times \mathbf{k}$. Let $\mathbf{r}_1$ be a unit vector in the direction $\mathbf{r}$ and $\boldsymbol{\theta}_1$ a unit vector in the direction of increasing θ.

There are three forces acting on P:

(i) The weight, $m\mathbf{g} = -mg\mathbf{k} = -mg \sin\theta\, \mathbf{r}_1 - mg\cos\theta\, \boldsymbol{\theta}_1$

(ii) The centrifugal force,

$$\begin{aligned} -m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] &= -m[\omega\mathbf{i} \times (\omega\mathbf{i} \times r\mathbf{r}_1)] \\ &= -m[\omega\mathbf{i}(\omega\mathbf{i} \cdot r\mathbf{r}_1) - r\mathbf{r}_1(\omega\mathbf{i} \cdot \omega\mathbf{i})] \\ &= -m[0 - \omega^2 r\mathbf{r}_1] \;=\; m\omega^2 r\mathbf{r}_1 \end{aligned}$$

(iii) The reaction force $\mathbf{N} = N\boldsymbol{\theta}_1$ of the rod which is perpendicular to the rod since there are no frictional or resistance forces.

Then by Newton's second law,

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{k} + m\omega^2 r\mathbf{r}_1 + N\boldsymbol{\theta}_1$$

or

$$\begin{aligned} m\frac{d^2r}{dt^2}\mathbf{r}_1 &= -mg\sin\theta\,\mathbf{r}_1 - mg\cos\theta\,\boldsymbol{\theta}_1 + m\omega^2 r\mathbf{r}_1 + N\boldsymbol{\theta}_1 \\ &= (m\omega^2 r - mg\sin\theta)\mathbf{r}_1 + (N - mg\cos\theta)\boldsymbol{\theta}_1 \end{aligned}$$

It follows that $N = mg\cos\theta$ and

$$d^2r/dt^2 = \omega^2 r - g\sin\theta \tag{1}$$

Since $\dot{\theta} = \omega$, a constant, we have $\theta = \omega t$ if we assume $\theta = 0$ at $t = 0$. Then (1) becomes

$$d^2r/dt^2 - \omega^2 r = -g\sin\omega t \tag{2}$$

If we assume that at $t = 0$, $r = r_0$, $dr/dt = v_0$, we find

$$r = \left(\frac{r_0}{2} + \frac{v_0}{2\omega} - \frac{g}{4\omega^2}\right)e^{\omega t} + \left(\frac{r_0}{2} - \frac{v_0}{2\omega} + \frac{g}{4\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t \tag{3}$$

or in terms of hyperbolic functions,

$$r = r_0\cosh\omega t + \left(\frac{v_0}{\omega} - \frac{g}{2\omega^2}\right)\sinh\omega t + \frac{g}{2\omega^2}\sin\omega t \tag{4}$$

6.24. (*a*) Show that under suitable conditions the particle of Problem 6.23 can oscillate along the rod with simple harmonic motion and find these conditions. (*b*) What happens to the particle if the conditions of (*a*) are not satisfied?

(*a*) The particle will oscillate with simple harmonic motion along the rod if and only if $r_0 = 0$ and $v_0 = g/2\omega$. In this case, $r = (g/2\omega^2)\sin\omega t$. Thus the amplitude and period of the simple harmonic motion in such case are given by $g/2\omega^2$ and $2\pi/\omega$ respectively.

(*b*) If $v_0 = (g/2\omega) - \omega r_0$ then $r = r_0 e^{-\omega t} + (g/2\omega^2)\sin\omega t$ and the motion is approximately simple harmonic after some time. Otherwise the mass will ultimately fly off the rod if it is finite.

6.25. A projectile located at colatitude λ is fired with velocity v_0 in a southward direction at an angle α with the horizontal. (*a*) Find the position of the projectile after time t. (*b*) Prove that after time t the projectile is deflected toward the east of the original vertical plane of motion by the amount

$$\tfrac{1}{3}\omega g\sin\lambda\, t^3 - \omega v_0\cos(\alpha - \lambda)\, t^2$$

(*a*) We use the equations of Problem 6.15. Assuming the projectile starts at the origin, we have

$$x = 0,\; y = 0,\; z = 0 \quad \text{at } t = 0 \tag{1}$$

Also, the initial velocity is $\mathbf{v}_0 = v_0\cos\alpha\,\mathbf{i} + v_0\sin\alpha\,\mathbf{k}$ so that

$$\dot{x} = v_0\cos\alpha,\; \dot{y} = 0,\; \dot{z} = v_0\sin\alpha \quad \text{at } t = 0 \tag{2}$$

Integrating equations (1), (2) and (3) of Problem 6.15, we obtain on using conditions (2),

$$\dot{x} = 2\omega\cos\lambda\, y + v_0\cos\alpha \qquad (3)$$

$$\dot{y} = -2(\omega\cos\lambda\, x + \omega\sin\lambda\, z) \qquad (4)$$

$$\dot{z} = -gt + 2\omega\sin\lambda\, y + v_0\sin\alpha \qquad (5)$$

Instead of attempting to solve these equations directly we shall use the *method of iteration* or *successive approximations* as in Method 2 of Problem 6.16. Thus by integrating and using conditions (1), we find

$$x = 2\omega\cos\lambda\int_0^t y\,du + (v_0\cos\alpha)t \qquad (6)$$

$$y = -2\omega\cos\lambda\int_0^t x\,du - 2\omega\sin\lambda\int_0^t z\,du \qquad (7)$$

$$z = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 + 2\omega\sin\lambda\int_0^t y\,du \qquad (8)$$

As a first guess we use $x = 0$, $y = 0$, $z = 0$ under the integral signs. Then (6), (7) and (8) become, neglecting terms involving ω^2,

$$x = (v_0\cos\alpha)t \qquad (9)$$

$$y = 0 \qquad (10)$$

$$z = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 \qquad (11)$$

To obtain a better guess we now use (9), (10) and (11) under the integral signs in (6), (7) and (8), thus arriving at

$$x = (v_0\cos\alpha)t \qquad (12)$$

$$y = -\omega v_0\cos(\alpha-\lambda)\,t^2 + \tfrac{1}{3}\omega g t^3\sin\lambda \qquad (13)$$

$$z = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 \qquad (14)$$

where we have again neglected terms involving ω^2. Further guesses again produce equations (12), (13) and (14), so that these equations are accurate up to terms involving ω^2.

(b) From equation (13) we see that the projectile is deflected toward the east of the xz plane by the amount $\tfrac{1}{3}\omega g t^3\sin\lambda - \omega v_0\cos(\alpha-\lambda)\,t^2$. If $v_0 = 0$ this agrees with Problem 6.16.

6.26. Prove that when the projectile of Problem 6.25 returns to the horizontal, it will be at the distance

$$\frac{\omega v_0^3\sin^2\alpha}{3g^2}(3\cos\alpha\cos\lambda + \sin\alpha\sin\lambda)$$

to the west of that point where it would have landed assuming no axial rotation of the earth.

The projectile will return to the horizontal when $z = 0$, i.e.,

$$(v_0\sin\alpha)t - \tfrac{1}{2}gt^2 = 0 \qquad \text{or} \qquad t = (2v_0\sin\alpha)/g$$

Using this value of t in equation (13) of Problem 6.25, we find the required result.

Supplementary Problems

ROTATING COORDINATE SYSTEMS. VELOCITY AND ACCELERATION

6.27. An xyz coordinate system moves with angular velocity $\boldsymbol{\omega} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ relative to a fixed or inertial XYZ coordinate system having the same origin. If a vector relative to the xyz system is given as a function of time t by $\mathbf{A} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + e^{-t}\mathbf{k}$, find (a) $d\mathbf{A}/dt$ relative to the fixed system, (b) $d\mathbf{A}/dt$ relative to the moving system.

Ans. (a) $(6\cos t - 3e^{-t})\mathbf{i} + (6\sin t - 2e^{-t})\mathbf{j} + (3\sin t - 2\cos t - e^{-t})\mathbf{k}$

(b) $\cos t\,\mathbf{i} + \sin t\,\mathbf{j} - e^{-t}\mathbf{k}$

6.28. Find $d^2\mathbf{A}/dt^2$ for the vector $\mathbf{A}$ of Problem 6.27 relative to (a) the fixed system and (b) the moving system.

Ans. (a) $(6\cos t - 45\sin t + 16e^{-t})\mathbf{i} + (40\cos t - 6\sin t - 11e^{-t})\mathbf{j}$
$+ (10\sin t - 23\cos t + 16e^{-t})\mathbf{k}$

(b) $-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + e^{-t}\mathbf{k}$

6.29. An xyz coordinate system is rotating with angular velocity $\boldsymbol{\omega} = 5\mathbf{i} - 4\mathbf{j} - 10\mathbf{k}$ relative to a fixed XYZ coordinate system having the same origin. Find the velocity of a particle fixed in the xyz system at the point $(3, 1, -2)$ as seen by an observer fixed in the XYZ system.
Ans. $18\mathbf{i} - 20\mathbf{j} + 17\mathbf{k}$

6.30. Discuss the physical interpretation of replacing $\boldsymbol{\omega}$ by $-\boldsymbol{\omega}$ in (a) Problem 6.4, page 148, and (b) Problem 6.6, page 149.

6.31. Explain from a physical point of view why you would expect the result of Problem 6.3, page 148, to be correct.

6.32. An xyz coordinate system rotates with angular velocity $\boldsymbol{\omega} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$ with respect to a fixed XYZ coordinate system having the same origin. If the position vector of a particle is given by $\mathbf{r} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t\mathbf{k}$, find (a) the apparent velocity and (b) the true velocity at any time t. *Ans.* (a) $\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}$ (b) $(t\sin t + 2\cos t)\mathbf{i} + (2\sin t - t\cos t)\mathbf{j}$

6.33. Determine (a) the apparent acceleration and (b) the true acceleration of the particle of Problem 6.32.
Ans. (a) $-\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$ (b) $(2t\cos t - 3\sin t)\mathbf{i} + (3\cos t + 2t\sin t)\mathbf{j} + (1 - t)\mathbf{k}$

CORIOLIS AND CENTRIPETAL ACCELERATIONS AND FORCES

6.34. A ball is thrown horizontally in the northern hemisphere. (a) Would the path of the ball, if the Coriolis force is taken into account, be to the right or to the left of the path when it is not taken into account as viewed by the person throwing the ball? (b) What would be your answer to (a) if the ball were thrown in the southern hemisphere? *Ans.* (a) to the right, (b) to the left

6.35. What would be your answer to Problem 6.34 if the ball were thrown at the north or south poles?

6.36. Explain why water running out of a vertical drain will swirl counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. What happens at the equator?

6.37. Prove that the centrifugal force acting on a particle of mass m on the earth's surface is a vector (a) directed away from the earth and perpendicular to the angular velocity vector $\boldsymbol{\omega}$ and (b) of magnitude $m\omega^2 R \sin\lambda$ where λ is the colatitude.

6.38. In Problem 6.37, where would the centrifugal force be (a) a maximum, (b) a minimum?
Ans. (a) at the equator, (b) at the north and south poles.

6.39. Find the centrifugal force acting on a train of mass 100,000 kg at (a) the equator (b) colatitude 30°.
Ans. (a) 343 N, (b) 171.5 N

6.40. (a) A river of width D flows northward with a speed v_0 at colatitude λ. Prove that the left bank of the river will be higher than the right bank by an amount equal to

$$(2D\omega v_0 \cos\lambda)(g^2 + 4\omega^2 v^2 \cos^2\lambda)^{-1/2}$$

where ω is the angular speed of the earth about its axis.

(b) Prove that the result in part (a) is for all practical purposes equal to $(2D\omega v_0 \cos\lambda)/g$.

6.41. If the river of Problem 6.40 is 2 km wide and flows at a speed of 5 km/h at colatitude 45°, how much higher will the left bank be than the right bank? *Ans.* 2.9 cm

6.42. An automobile rounds a curve whose radius of curvature is ρ. If the coefficient of friction is μ, prove that the greatest speed with which it can travel so as not to slip on the road is $\sqrt{\mu\rho g}$.

6.43. Determine whether the automobile of Problem 6.42 will slip if the speed is 100 km/h, $\mu = .05$ and (*a*) $\rho = 150$ m, (*b*) $\rho = 15$ m. Discuss the results physically.

MOTION OF A PARTICLE RELATIVE TO THE EARTH

6.44. An object is dropped at the equator from a height of 400 meters. If air resistance is neglected, how far will the point where it hits the earth's surface be from the point vertically below the initial position? *Ans.* 17.6 cm toward the east

6.45. Work Problem 6.44 if the object is dropped (*a*) at colatitude 60° and (*b*) at the north pole. *Ans.* (*a*) 15.2 cm toward the east

6.46. An object is thrown vertically upward at colatitude λ with speed v_0. Prove that when it returns it will be at a distance westward from its starting point equal to $(4\omega v_0^3 \sin\lambda)/3g^2$.

6.47. An object at the equator is thrown vertically upward with a speed of 100 km/h. How far from its initial position will it land? *Ans.* 2.17 cm

6.48. With what speed must the object of Problem 6.47 be thrown in order that it return to a point on the earth which is 6 m from its original position? *Ans.* 651.6 km/h

6.49. An object is thrown downward with initial speed v_0. Prove that after time t the object is deflected east of the vertical by the amount

$$\omega v_0 \sin\lambda\, t^2 + \tfrac{1}{3}\omega g \sin\lambda\, t^3$$

6.50. Prove that if the object of Problem 6.49 is thrown downward from height h above the earth's surface, then it will hit the earth at a point east of the vertical at a distance

$$\frac{\omega \sin\lambda}{3g^2}(\sqrt{v_0^2 + 2gh} - v_0)^2(\sqrt{v_0^2+2gh} + 2v_0)$$

6.51. Suppose that the mass m of a conical pendulum of length l moves in a horizontal circle of radius a. Prove that (*a*) the speed is $a\sqrt{g}/\sqrt[4]{l^2 - a^2}$ and (*b*) the tension in the string is $mgl/\sqrt{l^2 - a^2}$.

6.52. If an object is dropped to the earth's surface prove that its path is a semicubical parabola.

THE FOUCAULT PENDULUM

6.53. Explain physically why the plane of oscillation of a Foucault pendulum should rotate clockwise when viewed from above the earth's surface in the northern hemisphere but counterclockwise in the southern hemisphere.

6.54. How long would it take the plane of oscillation of a Foucault pendulum to make one complete revolution if the pendulum is located at (*a*) the north pole, (*b*) colatitude 45°, (*c*) colatitude 85°? *Ans.* (*a*) 23.94 h, (*b*) 33.86 h, (*c*) 92.50 h

6.55. Explain physically why a Foucault pendulum situated at the equator would not detect the rotation of the earth about its axis. Is this physical result supported mathematically? Explain.

MOVING COORDINATE SYSTEMS IN GENERAL

6.56. An xyz coordinate system rotates about the z axis with angular velocity $\boldsymbol{\omega} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ relative to a fixed XYZ coordinate system where t is the time. The origin of the xyz system has position vector $\mathbf{R} = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$ with respect to the XYZ system. If the position vector of a particle is given by $\mathbf{r} = (3t+1)\mathbf{i} - 2t\mathbf{j} + 5\mathbf{k}$ relative to the moving system, find the (*a*) apparent velocity and (*b*) true velocity at any time.

6.57. Determine (*a*) the apparent acceleration and (*b*) the true acceleration of the particle in Problem 6.56.

6.58. Work (*a*) Problem 6.5, page 148, and (*b*) Problem 6.7, page 149, if the position vector of the xyz system relative to the origin of the fixed XYZ system is $\mathbf{R} = t^2\mathbf{i} - 2t\mathbf{j} + 5\mathbf{k}$.

MISCELLANEOUS PROBLEMS

6.59. Prove that due to the rotation of the earth about its axis the apparent weight of an object of mass m at colatitude λ is $m\sqrt{(g - \omega^2 R \sin^2\lambda)^2 + (\omega^2 R \sin\lambda \cos\lambda)^2}$ where R is the radius of the earth.

6.60. Prove that the angle β which the apparent vertical at colatitude λ makes with the true vertical is given by $\tan\beta = \dfrac{\omega^2 R \sin\lambda \cos\lambda}{g - \omega^2 R \sin^2\lambda}$.

6.61. Explain physically why the true vertical and apparent vertical would coincide at the equator and also the north and south poles.

6.62. A stone is twirled in a vertical circle by a string of length 3 m. Prove that it must have a speed of at least 11 m/s at the bottom of its path in order to complete the circle.

6.63. A car C [Fig. 6-8] is to go completely around the vertical circular loop of radius a without leaving the track. Assuming the track is frictionless, determine the height H at which it must start.

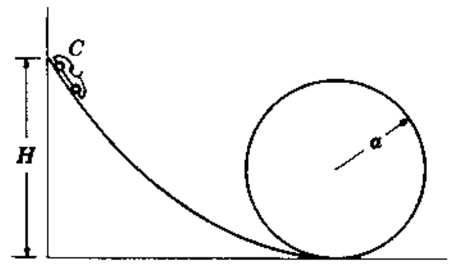

Fig. 6-8

6.64. A particle of mass m is constrained to move on a frictionless vertical circle of radius a which rotates about a fixed diameter with constant angular speed ω. Prove that the particle will make small oscillations about its equilibrium position with a frequency given by $2\pi a\omega/\sqrt{a^2\omega^4 - g^2}$.

6.65. Discuss what happens in Problem 6.64 if $\omega = \sqrt{g/a}$.

6.66. A hollow cylindrical tube AOB of length $2a$ [Fig. 6-9] rotates with constant angular speed • about a vertical axis through the center O. A particle is initially at rest in the tube at a distance b from O. Assuming no frictional forces, find (*a*) the position and (*b*) the speed of the particle at any time.

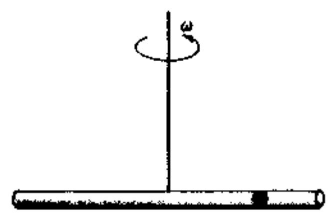

Fig. 6-9

6.67. (*a*) How long will it take the particle of Problem 6.66 to come out of the tube and (*b*) what will be its speed as it leaves? *Ans.* (*a*) $\dfrac{1}{\omega}\ln\left(\dfrac{a + \sqrt{a^2 - b^2}}{b}\right)$

6.68. Find the force on the particle of Problem 6.66 at any position in the tube.

6.69. A mass, attached to a string which is suspended from a fixed point, moves in a horizontal circle having center vertically below the fixed point with a speed of 20 revolutions per minute. Find the distance of the center of the circle below the fixed point. *Ans.* 2.23 meters

6.70. A particle on a frictionless horizontal plane at colatitude λ is given an initial speed v_0 in a northward direction. Prove that it describes a circle of radius $v_0/(2\omega\cos\lambda)$ with period $\pi/(\omega\cos\lambda)$.

6.71. The pendulum bob of a conical pendulum describes a horizontal circle of radius a. If the length of the pendulum is l, prove that the period is given by $4\pi^2\sqrt{l^2 - a^2}/g$.

6.72. A particle constrained to move on a circular wire of radius a and coefficient μ is given an initial velocity v_0. Assuming no other forces act, how long will it take for the particle to come to rest?

6.73. (*a*) Prove that if the earth were to rotate at an angular speed given by $\sqrt{2g/R}$ where R is its radius and g the acceleration due to gravity, then the weight of a particle of mass m would be the same at all latitudes. (*b*) What is the numerical value of this angular speed?
Ans. (*b*) 1.74×10^{-3} rad/s

6.74. A cylindrical tank containing water rotates about its axis with constant angular speed ω so that no water spills out. Prove that the shape of the water surface is a paraboloid of revolution.

6.75. Work (a) Problem 6.16 and (b) Problem 6.17, accurate to terms involving ω^2.

6.76. Prove that due to the earth's rotation about its axis, winds in the northern hemisphere traveling from a high pressure area to a low pressure area are rotated in a counterclockwise sense when viewed above the earth's surface. What happens to winds in the southern hemisphere?

6.77. (a) Prove that in the northern hemisphere winds from the north, east, south and west are deflected respectively toward the west, north, east and south as indicated in Fig. 6-10. (b) Use this to explain the origin of *cyclones*.

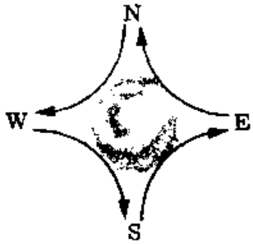

Fig. 6-10

6.78. Find the condition on the angular speed so that a particle will describe a horizontal circle inside of a frictionless vertical cone of angle α.

6.79. Work Problem 6.78 for a hemisphere.

6.80. The period of a simple pendulum is given by P. Prove that its period when it is suspended from the ceiling of a train moving with speed v_0 around a circular track of radius ρ is given by $P\sqrt{\rho g}/\sqrt[4]{v_0^4 + \rho^2 g^2}$.

6.81. Work Problem 6.25 accurate to terms involving ω^2.

6.82. A thin hollow cylindrical tube OA inclined at angle α with the horizontal rotates about the vertical with constant angular speed ω [see Fig. 6-11]. If a particle constrained to move in this tube is initially at rest at a distance a from the intersection O of the tube and the vertical axis of rotation, prove that its distance r from O at any time t is $r = a\cosh(\omega t\sin\alpha) - (g\cos\alpha)t^2$.

6.83. Work Problem 6.82 if the rod has coefficient of friction μ.

6.84. Prove that the particle of Problem 6.82 is in stable equilibrium between the distances from O given by

$$\frac{g\sin\alpha}{\omega^2}\left(\frac{1-\mu\tan\alpha}{\tan\alpha+\mu}\right) \quad \text{and} \quad \frac{g\sin\alpha}{\omega^2}\left(\frac{1+\mu\tan\alpha}{\tan\alpha-\mu}\right)$$

assuming $\tan\alpha < 1/\mu$.

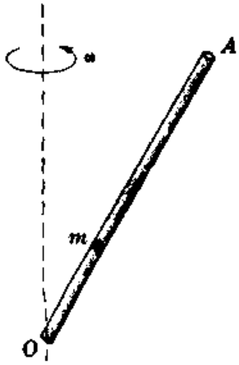

Fig. 6-11

6.85. A train having a maximum speed equal to v_0 is to round a curve with radius of curvature ρ. Prove that if there is to be no lateral thrust on the outer track, then this track should be at a height above the inner track given by $av_0^2/\sqrt{v_0^4 + \rho^2 g^2}$ where a is the distance between tracks.

6.86. A projectile is fired at colatitude λ with velocity v_0 directed toward the west and at angle α with the horizontal. Prove that if terms involving ω^2 are neglected, then the time taken to reach the maximum height is

$$\frac{v_0\sin\alpha}{g} - \frac{2\omega v_0^2\sin\lambda\sin\alpha\cos\alpha}{g^2}$$

Compare with the case where $\omega = 0$, i.e. that the earth does not rotate about its axis.

6.87. In Problem 6.86, prove that the maximum height reached is

$$\frac{v_0^2\sin^2\alpha}{2g} - \frac{2\omega v_0^3\sin\lambda\sin^2\alpha\cos\alpha}{g^2}$$

Compare with the case where $\omega = 0$.

6.88. Prove that the range of the projectile of Problem 6.86 is

$$\frac{v_0^2 \sin 2\alpha}{g} + \frac{\omega v_0^3 \sin\alpha \sin\lambda\,(8\sin^2\alpha - 6)}{3g^2}$$

Thus show that if terms involving ω^2 and higher are neglected, the range will be larger, smaller or the same as the case where $\omega = 0$, according as $\alpha > 60°$, $\alpha < 60°$ or $\alpha = 60°$ respectively.

6.89. If a projectile is fired with initial velocity $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ from the origin of a coordinate system fixed relative to the earth's surface at colatitude λ, prove that its position at any later time t will be given by

$$x = v_1 t + \omega v_2 t^2 \cos\lambda$$
$$y = v_2 t - \omega t^2(v_1\cos\lambda + v_3\sin\lambda) + \tfrac{1}{3}\omega g t^3 \sin\lambda$$
$$z = v_3 t - \tfrac{1}{2}gt^2 + \omega v_2 t^2 \sin\lambda$$

neglecting terms involving ω^2.

6.90. Work Problem 6.89 so as to include terms involving ω^2 but exclude terms involving ω^3.

6.91. An object of mass m initially at rest is dropped from height h to the earth's surface at colatitude λ. Assuming that air resistance proportional to the instantaneous speed of the object is taken into account as well as the rotation of the earth about its axis, prove that after time t the object is deflected east of the vertical by the amount

$$\frac{2\omega\sin\lambda}{\beta^3}\left[(g - 2h\beta^2)(1 - e^{-\beta t}) + \beta^3 h t e^{-\beta t} - \beta g t + \tfrac{1}{2}g\beta^2 t^2\right]$$

neglecting terms of order ω^2 and higher.

6.92. Work Problem 6.91, obtaining accuracy up to and including terms of order ω^2.

6.93. A frictionless inclined plane of length l and angle α located at colatitude λ is so situated that a particle placed on it would slide under the influence of gravity from north to south. If the particle starts from rest at the top, prove that it will reach the bottom in a time given by

$$\sqrt{\frac{2l}{g\sin\alpha}} + \frac{2\omega l \sin\lambda\cos\alpha}{3g}$$

and that its speed at the bottom is

$$\sqrt{2gl\sin\alpha} - \tfrac{4}{3}\omega l \sin\alpha\cos\alpha\sin\lambda$$

neglecting terms of order ω^2.

6.94. (*a*) Prove that by the time the particle of Problem 6.93 reaches the bottom it will have undergone a deflection of magnitude

$$\frac{2l\omega}{3}\sqrt{\frac{2l}{g\sin\alpha}}\cos(\alpha + \lambda)$$

to the east or west respectively according as $\cos(\alpha + \lambda)$ is greater than or less than zero. (*b*) Discuss the case where $\cos(\alpha + \lambda) = 0$. (*c*) Use the result of (*a*) to arrive at the result of Problem 6.17.

6.95. Work Problems 6.93 and 6.94 if the inclined plane has coefficient of friction μ.