APPLIED MATHEMATICS 4

PROF. MAHDY

Second year

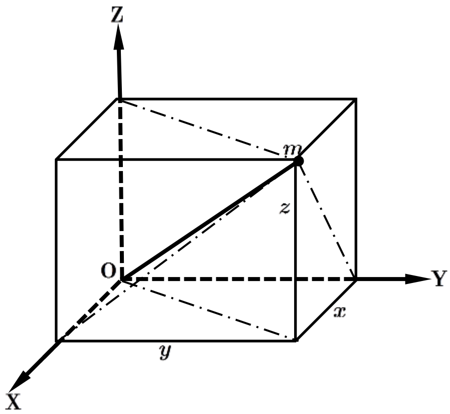
SVU

Chapters

MOMENTS OF INERTIA
(SECOND MOMENT)

In attempting to solve the equations of motion of a Rigid Body in a manner similar to that employed for a single particle, it will be found that certain new quantities appear, which depend on the extent and shape of the body, on its density, and on the way in which it may be moving in respect of some particular line or system of coordinate axes. These quantities are called Moments of Inertia and Products of Inertia or second moments. A moment of inertia of a body about any line is denned to be the sum of the products of all the material elements of the body by the squares of their perpendicular distances from this line. It may be denoted in general by the letter I . When rectangular coordinate axes are used, the moments of inertia of a point of mass m about the axes are defined by

$$I_X = (z^2 + y^2)m$$
$$I_Y = (x^2 + z^2)m$$
$$I_Z = (x^2 + y^2)m \qquad \text{....(1)}$$



and about the planes are denoted by

$$I_{X=0} = x^2 m$$
$$I_{Y=0} = y^2 m$$
$$I_{Z=0} = z^2 m \qquad \text{....(2)}$$

and about a point $\mathbf{O}$ is given by

$$I_{\mathrm{O}} = (x^2 + y^2 + z^2)m \qquad \text{....(3)}$$

♦ It is clear that we can deduce from these equations that

1- The moment of inertia of a body about three perpendicular axes equals the moment of inertia about the point of their intersection , that is $I_X + I_Y + I_Z = 2I_O$

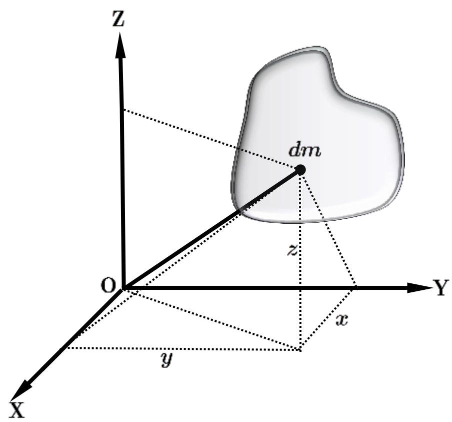
2- The moment of inertia of a body about two perpendicular planes equals the moment of inertia about the axis of their intersection , that is $I_{X=0} + I_{Y=0} = I_Z$

3- The moment of inertia of a body about three perpendicular planes equals the moment of inertia about the point of their intersection, that is $I_{X=0} + I_{Y=0} + I_{Z=0} = I_O$

♦ General case

Now to obtain the moment of inertia of a rigid body we divide the rigid body to an infinite number of differential elements each of them of mass dm therefore, the previous relations of moment of inertia become About the axes

$$I_X = \int (z^2 + y^2)dm$$
$$I_Y = \int (x^2 + z^2)dm$$
$$I_Z = \int (x^2 + y^2)dm \qquad ....(1')$$



about the planes are denoted by

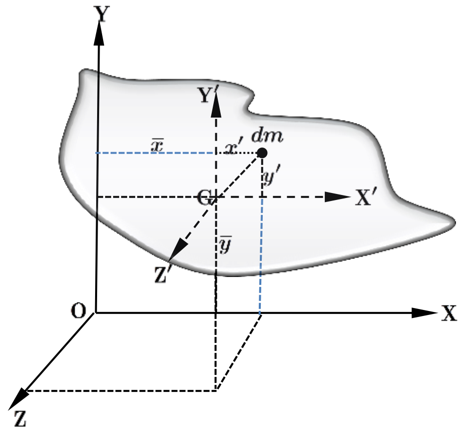
$$I_{X=0} = \int x^2 dm$$
$$I_{Y=0} = \int y^2 dm$$
$$I_{Z=0} = \int z^2 dm \qquad ....(2')$$

and about a point $\mathbf{O}$ is given by

$$I_O = \int (x^2 + y^2 + z^2)dm \qquad ....(3')$$

■ Theorem of Parallel Axes

There is a simple relationship between the moments of inertia about two parallel axes, provided that one of the axes passes through the centroid of the body. Referring to the shown figure, let be the centroid of the body, therefore the moment of inertia of the body about an axis ($\mathbf{Z}$ say) is equal to its moment of inertia about a parallel axis through its center of gravity $(\mathbf{Z'})$, together with the product of the whole mass and the square of the distance between the axes.



Let the given axis be taken as axis of $\mathbf{Z}$. Let (x, y, z) be the coordinates of a differential element of mass dm, $(\bar{x}, \bar{y}, \bar{z})$ the coordinates of the center of gravity of the body G, and let $x = \bar{x} + x'$, $\quad y = \bar{y} + y'$, $\quad z = \bar{z} + z'$.

Where (x', y', z') represents the coordinates of the differential element with respect to $\mathbf{GX'Y'Z'}$ passing through the center of gravity. The moment of inertia about $\mathbf{OZ}$

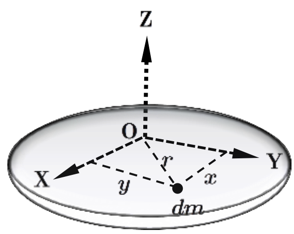
$$\begin{aligned} I_Z &= \int (x^2 + y^2) dm \\ &= \int \ (\bar{x} + x')^2 + (\bar{y} + y')^2 \ dm \\ &= \int (x'^2 + y'^2) dm + \ \bar{x}^2 + \bar{y}^2 \ \int dm + 2\bar{x} \int x' dm + 2\bar{y} \int y' dm \end{aligned}$$

But $\int x' dm = 0$ and $\int y' dm = 0$, therefore the moment of inertia about $\mathbf{OZ}$

$$\begin{aligned} I_Z &= \int (x'^2 + y'^2) dm + M \ \bar{x}^2 + \bar{y}^2 \\ &= I_{Z'} + Md^2 \qquad\qquad \int dm = M \end{aligned}$$

where the first sum is the moment of inertia about an axis through $\mathbf{G}$ parallel to $\mathbf{OZ}$ $(\mathbf{Z'})$, and the remaining terms are the product of the whole mass and the square of the distance between the two axes $\bar{x}^2 + \bar{y}^2 = d^2$.

■ Plane Lamina



The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moments of inertia about any two perpendicular axes in the plane that intersect on the first axis. Take the plane of the lamina as the plane of $\mathbf{XY}$ and the perpendicular axis as $\mathbf{OZ}$. Then, since $\mathbf{Z} = \mathbf{0}$ at all points on the lamina, we have

$$I_X = \int y^2 dm, \qquad I_Y = \int x^2 dm, \qquad I_Z = \int r^2 dm$$

But $r^2 = x^2 + y^2$ hence we get

$$\Rightarrow I_Z = \int (x^2 + y^2) dm$$
$$= \underbrace{\int x^2 dm}_{I_Y} + \underbrace{\int y^2 dm}_{I_X} = I_X + I_Y$$

■ Radius of Gyration of an Area

In some structural engineering applications it is common practice to introduce the radius of gyration of area. The radii of gyration of an area about the axes $\mathbf{X}$, $\mathbf{Y}$ and point $\mathbf{O}$ respectively, are defined as

$$K_x = \sqrt{\frac{I_X}{A}}, \qquad K_y = \sqrt{\frac{I_Y}{A}}, \qquad K_0 = \sqrt{\frac{I_0}{A}}$$

The radius of gyration of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. However, the radii of gyration are not a distance that has a clear-cut physical meaning, nor can be determined by direct measurement; its value can be determined only using the previous formulas. The radii of gyration are related by the equation $K_0^2 = K_x^2 + K_y^2$ which can be obtained from the previous relation.

■ Moments of Inertia for an Area about Inclined Axes

In structural and mechanical design, it is sometimes necessary to calculate the moments and product of inertia $I_{X'}$, $I_{Y'}$, and $I_{X'Y'}$ for an area with respect to a set of inclined $\mathbf{X'}$ and $\mathbf{Y'}$ axes when the values for θ, I_X, I_Y, and I_{XY} are known. To do this we will use transformation equations which relate the $\mathbf{X}, \mathbf{Y}$ and $\mathbf{X'}, \mathbf{Y'}$ coordinates. From the shown figure, these equations are

$$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

With these equations, the moments and product of inertia of dA about the X' and Y' axes become

$$\begin{aligned} dI_{X'} &= y'^2 dA = (y\cos\theta - x\sin\theta)^2 dA \\ dI_{Y'} &= x'^2 dA = (x\cos\theta + y\sin\theta)^2 dA \\ dI_{X'Y'} &= x'y'dA = (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dA \end{aligned}$$

Expanding each expression and integrating, realizing that

$I_X = \int y^2 dA$, $I_Y = \int x^2 dA$ and $I_{XY} = \int xy dA$, we obtain

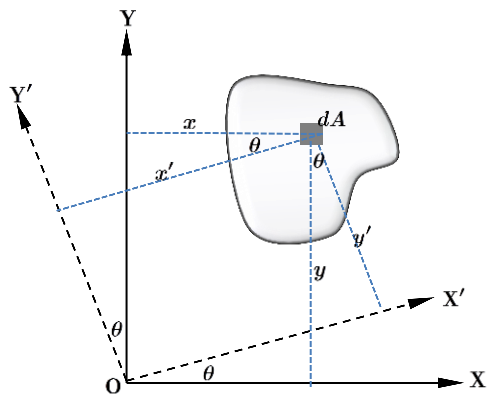
$$\begin{aligned} I_{X'} &= I_X\cos^2\theta + I_Y\sin^2\theta - 2I_{XY}\sin\theta\cos\theta \\ I_{Y'} &= I_X\sin^2\theta + I_Y\cos^2\theta + 2I_{XY}\sin\theta\cos\theta \\ I_{X'Y'} &= I_{XY}(\cos^2\theta - \sin^2\theta) + (I_X - I_Y)\sin\theta\cos\theta \end{aligned}$$

Using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we can simplify the above expressions, in which case

$$\begin{aligned} I_{X'} &= I_X\cos^2\theta + I_Y\sin^2\theta - I_{XY}\sin 2\theta \\ I_{Y'} &= I_X\sin^2\theta + I_Y\cos^2\theta + I_{XY}\sin 2\theta \\ I_{X'Y'} &= I_{XY}\cos 2\theta + \frac{1}{2}(I_X - I_Y)\sin 2\theta \end{aligned}$$

If $I_{X'Y'} = 0$ then $\tan 2\theta = \dfrac{2I_{XY}}{I_Y - I_X}$

It is evident that when the law of m is known and the shape of the body is given, the finding of a moment or of a product of inertia involves integration; and the following examples will serve to show how the process of integration may be used for this purpose. Further on, several propositions will be given by which the method may be usually much simplified.



■Illustrative Examples■

❒ EXAMPLE 1

Determine the moment of inertia of a uniform rod of length L about an axis normal to it at one of its ends.

❒ SOLUTION

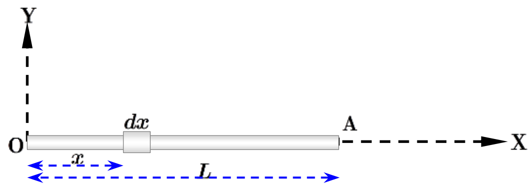
Consider the differential element of the rod as shown in the figure It is located on the rod at the arbitrary point x from the one end of the rod say $\mathbf{O}$ with length dx then the mass of the element is $dm = \rho dx$ where ρ is the density (mass per unit length) and therefore the moment of inertia of this element about $\mathbf{Y}$ axis is $dI_Y = x^2 dm$ therefore the moment of inertia of the whole rod is obtained by integration, so

$$dI_Y = x^2 dm = x^2 \rho dx$$

$$\Rightarrow I_Y = \int x^2 dm$$

$$= \rho \int_0^L x^2 dx = \rho \left(\frac{1}{3} x^3 \right)_0^L$$

$$= \frac{1}{3} L^3 \rho = \frac{1}{3} \underbrace{\rho L}_{M} L^2 = \frac{1}{3} M L^2$$



Remember $\quad M = \int dm = \int_0^L \rho dx = \rho L$

❒ EXAMPLE 2

Determine the moment of inertia of the area of a rectangular with lengths a, b about an axis coincident with side a.

❒ SOLUTION

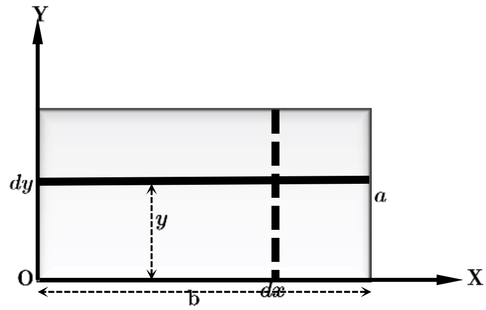
Consider a strip (dashed line) having a thickness dx; length a, and located in an arbitrary position and has a distance y from $\mathbf{X}$ axis. The mass of the element is $dm = \rho a dx$, ρ is the mass per unit area.

$$\because dI_X = y^2 dm = y^2 \ \rho b dy$$

$$\Rightarrow I_X = \int y^2 dm$$

$$= \rho b \int_0^a y^2 dy = \rho \left(\frac{1}{3} y^3 \right)_0^a$$

$$= \frac{1}{3}\rho b a^3 = \frac{1}{3} \underbrace{\rho a b}_{M} a^2 = \frac{1}{3} M a^2$$



Note that $M = \int dm = \int_0^a \rho b dy = \rho b a$

❐ EXAMPLE 3

Determine the moment of inertia of the area of triangle about an axis coincident with side h as shown.

❐ SOLUTION

Consider a thin rectangular element (dashed line) having a thickness dx; length x, and located in an arbitrary position so it has a distance x from $\mathbf{Y}$ axis. The mass of the element is $dm = \rho y dx$ where $y = h - \frac{h}{a} x$

Thus the moment of inertia of the triangle about axis $\mathbf{Y}$ is given by

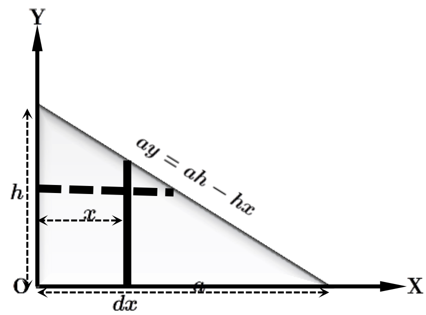
$$\because dI_Y = x^2 dm$$

$$\Rightarrow I_Y = \int x^2 dm$$

$$= \rho \int_0^a x^2 y dx = \rho \int_0^a x^2 \left(h - \frac{h}{a} x \right) dx$$

$$= \rho \left(\frac{h}{3} x^3 - \frac{h}{4a} x^4 \right)_0^a = \frac{1}{12} \rho h a^3$$

$$= \frac{1}{6} \underbrace{\left(\frac{1}{2} \rho h a \right)}_{M} a^2 = \frac{1}{6} M a^2$$



Where $M = \int dm = \int_0^a \rho y dx$

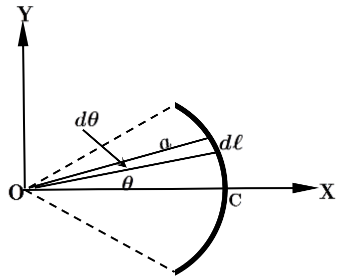
$$= \rho \int_0^a \left(h - \frac{h}{a} x \right) dx = \rho \left(hx - \frac{h}{2a} x^2 \right)_0^a = \frac{1}{2} \rho h a$$

❐ EXAMPLE 4

Determine the moment of inertia of a circular arc of radius a which subtends an angle 2α at its center $\mathbf{O}$ about an axis normal to the its plane at $\mathbf{O}$.

❒ SOLUTION

Take the axis of $\mathbf{OX}$, with C is the middle point of the arc, and a perpendicular axis $\mathbf{OY}$. Let $d\ell$ be the length of any differential element on the arc as shown where the mass of this element is given by $dm = \rho d\ell, \quad d\ell = ad\theta$ hence



$$\Rightarrow dI_{\mathrm{O}} = R^2 dm = R^2(\rho R d\theta)$$

$$\Rightarrow I_{\mathrm{O}} = \int R^2 dm = R^3\rho \int_{-\alpha}^{\alpha} d\theta$$

$$\Rightarrow I_{\mathrm{O}} = R^3\rho\,\theta\Big|_{-\alpha}^{\alpha}$$
$$= 2R^3\rho\alpha = \underbrace{(2R\rho\alpha)}_{M} R^2 = MR^2$$

♦ Now for special case, semicircle where $2\alpha = \pi$

$$\Rightarrow I_{\mathrm{O}} = R^3\rho\,\theta\Big|_{-\pi/2}^{\pi/2} = \underbrace{(R\rho\pi)}_{M} R^2 = MR^2 \qquad (M = \rho\pi R)$$

♦ Now for special case, a circle (cord) where $2\alpha = 2\pi$

$$\Rightarrow I_{\mathrm{O}} = R^3\rho\,\theta\Big|_{-\pi}^{\pi} = \underbrace{(2R\rho\pi)}_{M} R^2 = MR^2 \qquad (M = 2\pi R\rho)$$

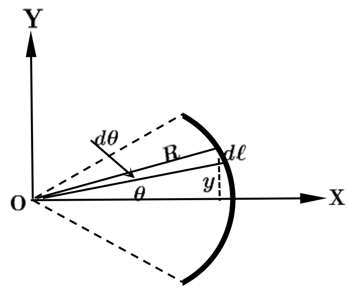
That is the moment of a circle of radius R about an axis normal to it and passing through the center is MR^2

❒ EXAMPLE 5

Determine the moment of inertia of a circular arc of radius a which subtends an angle 2α at its center $\mathbf{O}$ about a symmetric axis $\mathbf{X}$ as shown.

❒ SOLUTION

Take the axis $\mathbf{OX}$, with C is the middle point of the arc, and a perpendicular axis $\mathbf{OY}$. Let $d\ell$ be the length of any differential element on the arc as shown where the mass of this element is given by $dm = \rho d\ell, \quad d\ell = ad\theta$ hence



$$dI_X = y^2 dm = R^2 \sin^2\theta(\rho R d\theta)$$

$$\Rightarrow I_X = \rho R^3 \int_{-\alpha}^{\alpha} \sin^2\theta d\theta$$

$$= \rho R^3 \int_{-\alpha}^{\alpha} \frac{(1-\cos 2\theta)}{2} d\theta$$

$$I_X = \frac{1}{2}\rho R^3 \left(\theta - \frac{\sin 2\theta}{2}\right)_{-\alpha}^{\alpha}$$

$$= \frac{1}{2}\rho R^3 \;\; 2\alpha - \sin 2\alpha \;\; = \rho R^3 \left(\alpha - \frac{\sin 2\alpha}{2}\right) = \underbrace{(2\alpha\rho R)}_{M}\left(\frac{1}{2} - \frac{\sin 2\alpha}{4\alpha}\right)R^2$$

$$\Rightarrow I_X = \frac{1}{2}M\left(1 - \frac{\sin 2\alpha}{2\alpha}\right)R^2$$

♦ Now for special case, semicircle where $2\alpha = \pi$

$$\Rightarrow I_X = \frac{1}{2}M\left(1 - \frac{\sin 2\alpha}{2\alpha}\right)R^2 = \frac{1}{2}MR^2 \qquad (M = \rho\pi R)$$

♦ Now for special case, a circle (cord) where $2\alpha = 2\pi$

$$\Rightarrow I_X = \frac{1}{2}M\left(1 - \frac{\sin 2\alpha}{2\alpha}\right)R^2 = \frac{1}{2}MR^2 \qquad (M = 2\pi R\rho)$$

In the same manner the moment of inertia about $\mathbf{Y}$ axis is given by

$$I_Y = \int x^2 dm = \int_{-\alpha}^{\alpha} R^2\cos^2\theta(\rho R d\theta)$$

Now we can deduce the moment of inertia about a normal axis to the plane of the circular arc passing through its center (as obtained before) by applying perpendicular axes theorem

$$I_O = I_X + I_Y$$

$$= \rho R^3 \left(\int_{-\alpha}^{\alpha} \cos^2\theta d\theta + \int_{-\alpha}^{\alpha} \sin^2\theta d\theta\right)$$

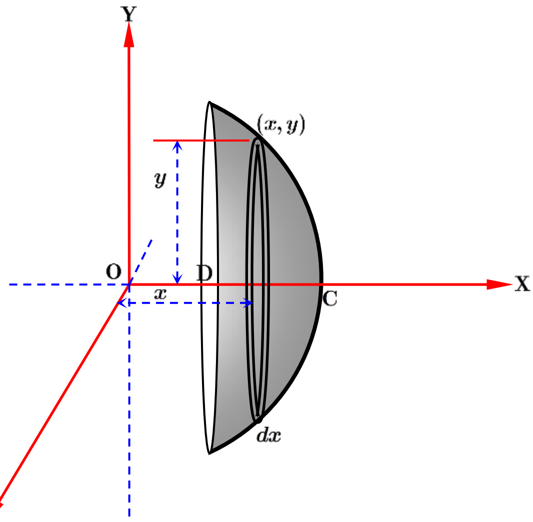
$$= \rho R^3 \left(\int_{-\alpha}^{\alpha} \;\; \cos^2\theta + \sin^2\theta \;\; d\theta\right)$$

$$\therefore \; I_O = \rho R^3 \int_{-\alpha}^{\alpha} d\theta = 2\rho\alpha R^3 = MR^2$$

❒ EXAMPLE 6

Find the moment of inertia of a solid spherical segment of height h of a sphere of radius a about $\mathbf{X}$-axis as shown

❒ SOLUTION



Let $\mathbf{O}$ be the center of the sphere and $\mathbf{OX}$ an axis at right angles to the base of the segment meeting it in $\mathbf{D}$ and the curved surface in $\mathbf{C}$, so that $\mathbf{OD} = a - h$ and $\mathbf{DC} = h$. Taking slices of the segment (thin disks) parallel to its base of radius y, then $dm = \rho\pi y^2 dx$ But x, y are co-ordinates of a point on a circle of radius a, viz. the section of the sphere by the plane of the paper; so that $y^2 = a^2 - x^2$, the moment of inertia of the element is $dI_X = \frac{1}{2} y^2 dm$

$$I_X = \frac{1}{2}\int y^2 dm = \frac{1}{2}\int_{a-h}^{a} \pi y^4 \rho dx \qquad (x^2 + y^2 = a^2)$$

$$= \frac{1}{2}\pi\rho \int_{a-h}^{a} (a^2 - x^2)^2 dx$$

$$= \frac{1}{2}\pi\rho \int_{a-h}^{a} (a^4 - 2a^2x^2 + x^4) dx$$

$$= \frac{1}{2}\pi\rho \left(a^4 x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5 \right)_{a-h}^{a} =$$

♦ Now for special case, solid hemisphere where $h = a$ then

$$I_X = \frac{1}{2}\pi\rho \left(a^4 x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5 \right)_{0}^{a}$$

$$= \frac{4}{15}\pi\rho a^5 = \frac{2}{5}\underbrace{\left(\frac{2}{3}\pi\rho a^3\right)}_{M} a^2 = \frac{2}{5} M a^2$$

❒ EXAMPLE 7

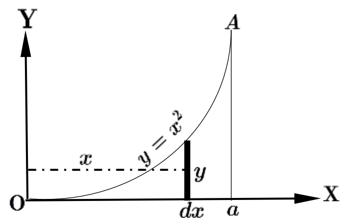
Determine the moment of inertia for the area under the curve $y = x^2$ and bounded by the **X, Y** axes and the line $x = a$

❒ SOLUTION

A differential element is chosen to be a thin rectangular (strip) parallel to **Y** axis of thickness dx as shown in the figure. The element intersects the curve at the arbitrary point (x, y), and so it has a height y.

The mass of the element is $dm = \rho y dx$, then we obtain

$$
\begin{aligned}
dI_Y &= \int x^2 dm = \int_0^a x^2 (\rho y dx) \\
&= \rho \int_0^a x^4 dx = \frac{1}{5}\rho x^5 \Big|_0^a = \frac{1}{5}\rho a^5 \\
&= \frac{3}{5}\underbrace{\left(\frac{1}{3}\rho a^3\right)}_{M} a^2 = \frac{3}{5}Ma^2
\end{aligned}
$$



$$M = \int dm = \int \rho y dx = \int_0^a \rho x^2 dx = \frac{1}{3}\rho a^3$$

◈ Obtain the moment of inertia of this case about **X** axis

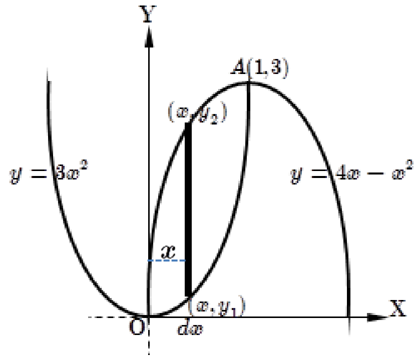
❒ EXAMPLE 8

Find the moment of inertia for the area bounded by the two parabola $y = 3x^2$ and $y = 4x - x^2$ about **Y** axis as shown.

❒ SOLUTION

The differential element is chosen to be a thin rectangular parallel to **Y** axis of thickness dx as shown in the figure. The element intersects the curves at the arbitrary point (x, y), and so it has a height $(y_2 - y_1)$. The mass of the element is determined by

$$dm = \rho\ \ y_2 - y_1\ \ dx$$
$$= \rho\ \ 4x - x^2 - 3x^2\ \ \ dx$$
$$= 4\rho\ \ \ x - x^2\ \ \ dx$$



$$\because\ dI_Y = x^2 dm$$

$$\Rightarrow I_Y = \int x^2 dm = 4\rho\int_0^1 x^2(x - x^2)dx$$
$$= 4\rho\int_0^1 (x^3 - x^4)dx$$
$$= 4\rho\left(\frac{1}{4}x^4 - \frac{1}{5}x^5\right)_0^1 = \frac{1}{5}\rho$$

$$M = \int dm = 4\rho\int_0^1 (x - x^2)dx = \frac{2}{3}\rho$$

$$\Rightarrow I_Y = \frac{1}{5}\rho = \frac{3}{10}\underbrace{\left(\frac{2}{3}\rho\right)}_{M} = \frac{3}{10}M$$

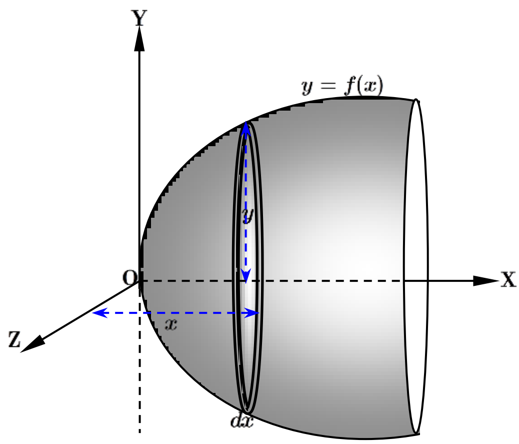
◈ Obtain the moment of inertia about $\mathbf{X}$ axis

❒ EXAMPLE 9

Obtain the moment of inertia of the volume of revolution that generated by revolving a given curve $y = f(x)$ round the $\mathbf{X}$ axis from $x = 0$ and $x = a$.

❒ SOLUTION

Consider a differential element (thin disk of radius y and thickness dx) is selected to be parallel to $\mathbf{Y}$ axis with thickness dx it intersects the generating curve at the arbitrary point $(x, y, 0)$ and so its radius is y. The mass of the element is $dm = \rho\pi y^2 dx = \rho\pi f^2(x)dx$, therefore



$$\Rightarrow I_X = \frac{1}{2}\int y^2 dm = \frac{1}{2}\pi\rho\int y^4 dx$$

$$\Rightarrow I_X = \frac{1}{2}\pi\rho\int_0^a f^4(x)dx$$

♦ Again for a special case as $y = x$ we get a solid right circular cone and then

$$\Rightarrow I_X = \frac{1}{2}\int y^2 dm = \frac{1}{2}\pi\rho\int y^4 dx$$

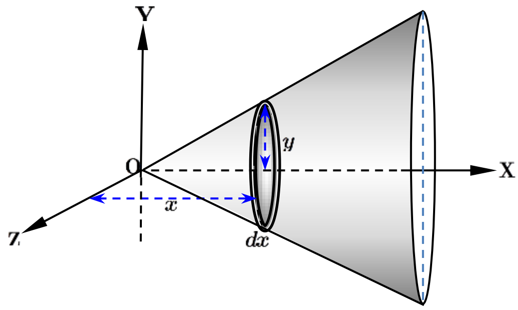
$$= \frac{1}{2}\pi\rho\int_0^h \left(\frac{a}{h}x\right)^4 dx = \frac{1}{2}\pi\rho\frac{a^4}{h^4}\left(\frac{x^5}{5}\right)_0^h = \frac{1}{10}\pi\rho a^4 h$$

Since we have

$$M = \int dm = \pi\rho\int_0^h \left(\frac{a}{h}x\right)^2 dx = \frac{1}{3}\pi\rho a^2 h$$

$$\Rightarrow I_X = \frac{1}{10}\pi\rho a^4 h = \frac{3}{10}\underbrace{\left(\frac{1}{3}\pi\rho a^2 h\right)}_{M}a^2$$

$$= \frac{3}{10}Ma^2$$

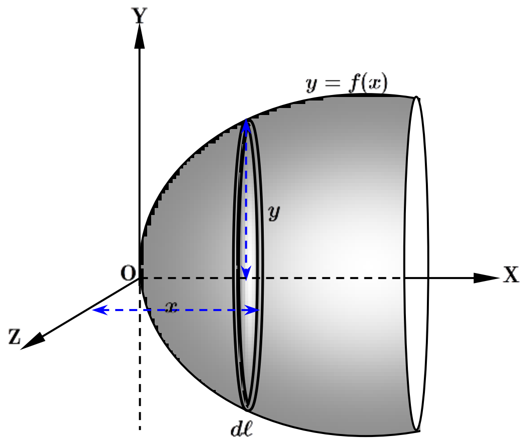


where h is the height of the circular cone.

❒ EXAMPLE 10

Determine the moment of inertia for the surface of revolution that generated by revolving a given curve $y = f(x)$ round the $\mathbf{X}$ axis from $x = 0$ and $x = a$.

❒ SOLUTION



Let us consider a differential element (thin ring of radius y and thickness $d\ell$) is selected to be parallel to $\mathbf{Y}$ axis with thickness $d\ell$ it intersects the generating curve at the arbitrary point $(x, y, 0)$ and so its radius is y. The mass of the selected element is $dm = 2\rho\pi y d\ell$ $= 2\rho\pi f(x)d\ell$, by differentiation the equation $y = f(x)$ with respect to x we get $\dfrac{dy}{dx} = f'(x)$ then $\therefore\ d\ell = \sqrt{1 + f'^2(x)}\, dx.$ Since we have $dI_X = y^2 dm$

$$\Rightarrow I_X = \int y^2 dm = 2\pi\rho \int f^3(x) d\ell$$

$$= 2\pi\rho \int_0^a f^3(x)\sqrt{1 + f'^2(x)}\, dx$$

◈ Obtain moment of inertia for a hollow right circular cone .

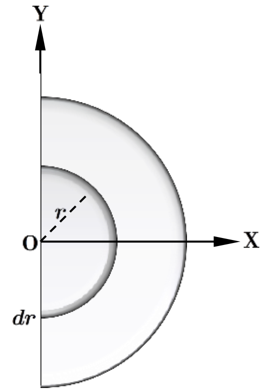
❒ EXAMPLE 11

Determine the moment of inertia for area of semicircle of radius a when the density is proportional to the distance from **O** .

❒ SOLUTION

Let us consider a differential element of semicircle of radius r and thickness dr and let the density of this element is $\rho \propto r$ **Or** $\rho = \mu r$

Then the mass of the element is $dm = \pi r \rho dr = \pi \mu r^2 dr$



The moment of the element about **O** is dI_O

$$dI_O = r^2 dm \qquad \Rightarrow dI_O = r^2 \ \pi \mu r^2 dr$$

$$\Rightarrow I_O = \int r^2 dm$$

$$= \pi\mu \int_0^a r^4 dr$$

$$= \frac{\pi\mu}{5} r^5 \Big|_0^a = \frac{\pi\mu}{5} a^5$$

Note $M = \displaystyle\int dm = \int_0^a \pi r \rho dr$

$$= \pi\mu \int_0^a r^2 dr = \frac{\pi\mu}{3} r^3 \Big|_0^a = \frac{\pi\mu}{3} a^3$$

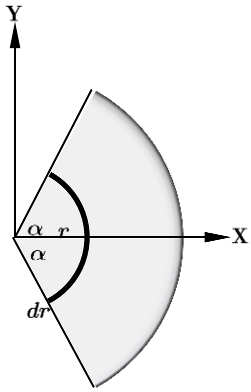
$$\Rightarrow I_O = \frac{\pi\mu}{5} a^5 = \frac{3}{5} \underbrace{\left(\frac{\pi\mu}{3} a^3 \right)}_{M} a^2 = \frac{3}{5} M a^2$$

❒ EXAMPLE 12

Find the moment of inertia of the area of Circular Sector of radius a and which subtends an angle $\mathbf{X}$ at its center **O** when the density is proportional to the distance from **O** .

❐ SOLUTION

Let the differential element be a circular arc of radius r which subtends an angle $\mathbf{X}$ at its center and has thickness dr which its moment of inertia is $\frac{1}{2}\left(1-\frac{\sin 2\alpha}{2\alpha}\right)r^2dm$ (Ex. 5) mass $dm = 2\alpha\rho r dr$ -where $\rho = \mathrm{k}r,\ \mathrm{k}$ is a constant, then $dm = 2\alpha\mathrm{k}r^2dr$



Therefore the moment of inertia about $\mathbf{X}$ axis is dI_X.

$$dI_X = \frac{1}{2}\left(1-\frac{\sin 2\alpha}{2\alpha}\right)r^2dm$$

$$\Rightarrow dI_X = \frac{1}{2}\left(1-\frac{\sin 2\alpha}{2\alpha}\right)r^2\ \ 2\mathrm{k}\alpha r^2dr$$

$$\Rightarrow I_X = \int r^2dm$$

$$\Rightarrow I_X = \mathrm{k}\alpha\left(1-\frac{\sin 2\alpha}{2\alpha}\right)\int_0^a r^4dr = \frac{1}{5}\mathrm{k}\alpha\left(1-\frac{\sin 2\alpha}{2\alpha}\right)a^5$$

$$\because M = \int dm = \int_0^a 2\mathrm{k}\alpha r^2dr$$

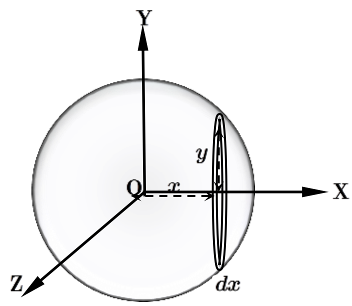
$$= 2\mathrm{k}\alpha\int_0^a r^2dr = \frac{2}{3}\mathrm{k}\alpha r^3\Big|_0^a = \frac{2}{3}\mathrm{k}\alpha a^3$$

$$\Rightarrow I_X = \frac{1}{5}\mathrm{k}\alpha\left(1-\frac{\sin 2\alpha}{2\alpha}\right)a^5$$

$$= \frac{3}{10}\underbrace{\left(\frac{2}{3}\mathrm{k}\alpha a^3\right)}_{M}\left(1-\frac{\sin 2\alpha}{2\alpha}\right)a^2 = \frac{3}{10}\left(1-\frac{\sin 2\alpha}{2\alpha}\right)Ma^2$$

❐ EXAMPLE 13

Find the moment of inertia of a solid sphere of radius a about $\mathbf{X}$ axis, as shown.



❐ SOLUTION

Consider a differential element (thin disk of radius y) is selected to be parallel to $\mathbf{Y}$ axis with thickness dx it intersects the generating curve at the arbitrary point

$(x, y, 0)$ and so its radius is y. The mass of the element is $dm = \rho\pi y^2 dx$, therefore $dI_X = \frac{1}{2}y^2 dm$

$$\Rightarrow dI_X = \frac{1}{2}y^2 dm = \frac{1}{2}y^2 \quad \rho\pi y^2 dx \quad = \frac{1}{2}\rho\pi y^4 dx$$

$$\Rightarrow \quad I_X = \int_{-a}^{a} \frac{1}{2}\rho\pi y^4 dx = \int_{-a}^{a} \frac{1}{2}\rho\pi(a^2 - x^2)^2 dx$$

$$= \frac{1}{2}\rho\pi \int_{-a}^{a} (a^4 - 2a^2x^2 + x^4)dx \quad \textbf{Even Function}$$

$$= \rho\pi \int_{0}^{a} (a^4 - 2a^2x^2 + x^4)dx = \rho\pi \left(a^4x - \frac{2}{3}a^2x^3 + \frac{1}{5}x^5 \right)_0^a$$

$$\Rightarrow I_X = \frac{8}{15}\rho\pi a^5$$

But we have

$$M = \int dm = \rho\pi \int_{-a}^{a} (a^2 - x^2)dx$$

$$= 2\rho\pi \int_{0}^{a} (a^2 - x^2)dx$$

$$= 2\rho\pi \left(a^2x - \frac{1}{3}x^3 \right)_0^a = \frac{4}{3}\rho\pi a^3$$

$$\Rightarrow I_X = \frac{8}{15}\rho\pi a^5 = \frac{2}{5}\underbrace{\left(\frac{4}{3}\rho\pi a^3\right)}_{M} a^2 = \frac{2}{5}Ma^2$$

Note according to symmetry we get $I_X = I_Y = I_z$

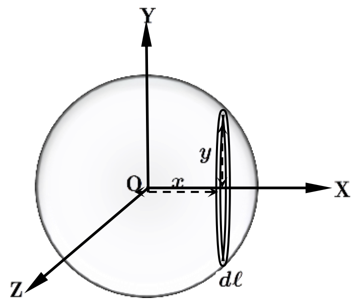
♦ This problem can be resolved by considering the differential element a hollow sphere of radius r and thickness dr

❒ EXAMPLE 14

Find the moment of inertia of a hollow sphere of radius a about $\mathbf{X}$ axis, as shown.

❐ SOLUTION

Consider a differential element (thin ring or cord of radius y) is selected to be parallel to $\mathbf{Y}$ axis with thickness dx it intersects the generating curve at the arbitrary point $(x, y, 0)$ and so its radius is y. The mass of the element is $dm = 2\rho\pi y d\ell$, therefore the moment of inertia of the element about $\mathbf{X}$ axis is



$$dI_X = y^2 dm$$

$$\Rightarrow dI_X = y^2 dm = y^2 \ 2\rho\pi y d\ell \ = 2\rho\pi y^3 d\ell$$

$$\Rightarrow \ I_X = 2\int_{-a}^{a} \rho\pi y^3 d\ell$$

$$d\ell = \sqrt{1 + \left\{\frac{dy}{dx}\right\}^2} dx, \qquad y^2 = a^2 - x^2$$

$$\Rightarrow 2ydy = -2xdx \quad \text{Or} \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow d\ell = \sqrt{1 + \frac{x^2}{y^2}} dx, \qquad \Rightarrow d\ell = \frac{a}{y} dx,$$

$$\Rightarrow \ I_X = 2\int_{-a}^{a} \rho\pi y^3 d\ell = 2\int_{-a}^{a} \rho\pi y^3 \frac{a}{y} dx = 2a\rho\pi \int_{-a}^{a} y^2 dx$$

$$= 2a\rho\pi \int_{-a}^{a} (a^2 - x^2) dx = 4a\rho\pi \int_{0}^{a} (a^2 - x^2) dx$$

$$= 4a\rho\pi \left(a^2 x - \frac{1}{3} x^3 \right)_0^a = \frac{8}{3} a^4 \rho\pi$$

but we have

$$M = \int dm = 2\rho\pi \int_{-a}^{a} y d\ell$$

$$= 2\rho\pi \int_{-a}^{a} y \frac{a}{y} dx = 2\rho\pi a(2a) = 4\rho\pi a^2$$

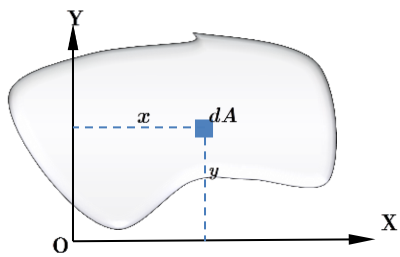
$$\Rightarrow I_X = \frac{8}{3} a^4 \rho\pi = \frac{1}{3} \underbrace{4\rho\pi a^2}_{M} a^2 = \frac{2}{3} M a^2$$

■ Product of Inertia for an Area

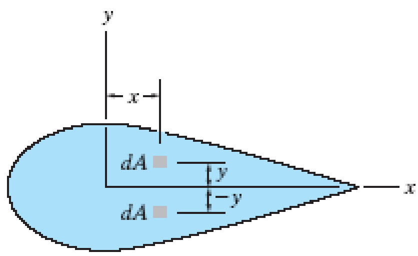
It will be shown here that the property of n area, called the product of inertia, is required in order to determine the maximum and minimum moments of inertia for the area. These maximum and minimum values are important properties needed for designing structural and mechanical members such as beams, columns, and shafts. The product of inertia of the area in the figure with respect to the $\mathbf{X}$ and $\mathbf{Y}$ axes is defined as

$$I_{XY} = \int xydm$$

If the chosen element of area has a differential size in two directions, as shown in the figure, a double integration must be performed to evaluate I_{XY}. Most often, however, it is easier to choose an element having a differential size or thickness in only one direction in which case the evaluation requires only a single integration.



Like the moment of inertia, the product of inertia has units of length raised to the fourth power, e.g., m^4, mm^4 or ft^4, in^4. However, since x or y may be negative, the product of inertia may be either positive, negative, or zero, depending on the location and orientation of the coordinate axes. For example, the product of inertia I_{XY} for an area will be zero if either the $\mathbf{X}$ or $\mathbf{Y}$ axis is an axis of symmetry for the area, as shown. Here every element dA located at point (x, y) has a corresponding element dA located at $(x, -y)$. Since the products of inertia for these elements are, respectively, $xydA$ and $-xydA$, the algebraic sum or integration of all the elements that are chosen in this way will cancel each other. Consequently, the product of inertia



for the total area becomes zero. It also follows from the definition of I_{XY} that the “sign” of this quantity depends on the quadrant where the area is located.

■ Moments of Inertia for Composite Areas, Volumes

A composite area consists of a series of connected “simpler” parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the algebraic sum of the moments of inertia of all its parts. The following examples will serve to show how this process can be used to determine the moment of inertia of the composite body.

■Illustrative Examples■

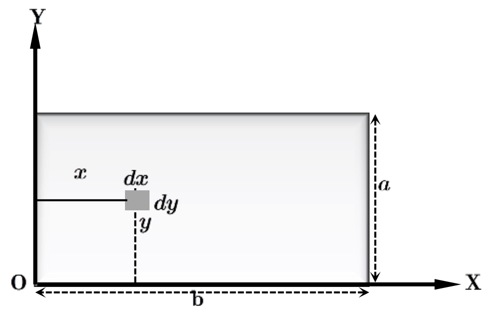
❒ EXAMPLE 1

Determine the product of inertia I_{XY} of the area of a rectangular with lengths a, b about axes coincident with its sides.

❒ SOLUTION

Consider an area $dA = dxdy$ differential element, and located in an arbitrary position and has a distance y from $\mathbf{X}$ axis and x from Y axis. The mass of the element is $dm = \rho dA = \rho dxdy$, ρ is the mass per unit area.

$$\because dI_{XY} = xydm = xy \;\; \rho dxdy$$



$$\Rightarrow I_{XY} = \int xydm$$

$$= \rho \int_0^b \int_0^a xydxdy$$

$$= \frac{1}{2}\rho \int_0^b \; yx^2 \Big|_0^a dy$$

$$= \frac{1}{2}\rho a^2 \int_0^b ydy$$

$$= \frac{1}{4}\rho b^2 a^2 = \frac{1}{4} \underbrace{\rho ab}_{M} \; ab = \frac{1}{3} Mab$$

❒ EXAMPLE 2

Determine the product of inertia of the area of the triangle about axes coincident with its sides as shown.

❒ SOLUTION

Consider a thin rectangular area having a thickness dx and dy; and located in an arbitrary position so it has a distance y from $\mathbf{X}$ axis and x from $\mathbf{Y}$ axis.

Again, the mass of the element is $dm = \rho dxdy$ where $y = h - \frac{h}{a}x$

Thus the product of inertia of the triangle is given by

$$\because\ dI_{XY} = xydm$$

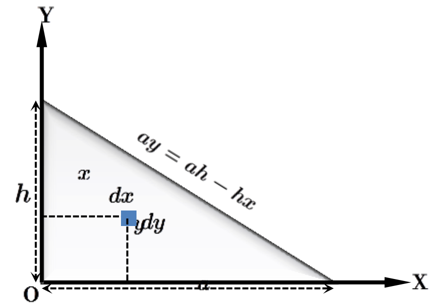
$$\Rightarrow\ I_{XY} = \int xydm$$

$$= \rho\int_0^h \int_0^{\frac{a}{h}(h-y)} xydxdy = \frac{1}{2}\rho\int_0^h y\, x^2\Big|_0^{\frac{a}{h}(h-y)} dy$$

$$= \frac{1}{2}\rho\left(\frac{a}{h}\right)^2 \int_0^h y\,(h-y)^2\Big|dy$$

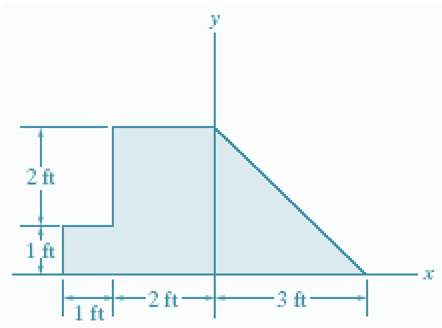
$$= \frac{1}{2}\rho\left(\frac{a}{h}\right)^2 \left(\frac{1}{2}h^2y^2 - \frac{2}{3}hy^3 + \frac{1}{4}y^4\right)\Bigg|_0^h$$

$$= \frac{1}{24}\rho\left(\frac{a}{h}\right)^2 h^4 = \frac{1}{12}\underbrace{\left(\frac{1}{2}\rho ha\right)}_{M} ah = \frac{1}{12}Mah$$



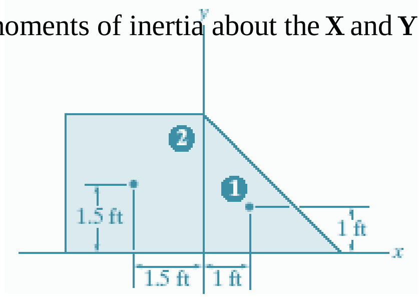
❒ Example 3

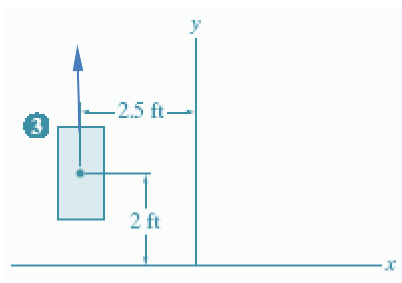
Determine the moment of inertia of the shaded area about the **X** and **Y** axes.



❒ Solution

The plate is divided into three segments as shown in the figures below. Here triangle area ❶ and the area of the small rectangle ❸ is considered “negative” since it must be subtracted from the larger one ❷ . The following table involves the moments of inertia about the **X** and **Y** axes.





No.	A_i ft^2	I_X ft^4	I_Y ft^4
❶	(1/2)(3)(3)=4.5	$(1/6)(4.5)(3)^2$	$(1/6)(4.5)(3)^2$
❷	(3)(3)=9	$(1/3)(9)(3)^2$	$(1/3)(9)(3)^2$
❸	(2)(1)=2	$(1/12)(2)(2)^2+2(2)^2$	$(1/12)(2)(1)^2+2(2.5)^2$

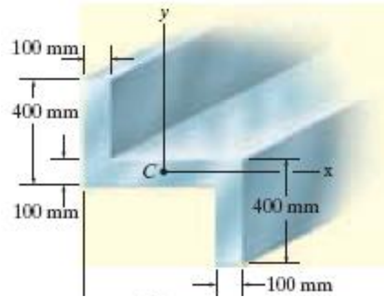
Therefore the moment of inertia of the shaded area is

$$\Rightarrow (I_X)_1 + (I_X)_2 - (I_X)_3 = 6.75 + 27 - 8.6667 \simeq 25.08 \quad \text{ft}^4$$
$$\Rightarrow (I_Y)_1 + (I_Y)_2 - (I_Y)_3 = 6.75 + 27 - 12.6667 \simeq 21.08 \quad \text{ft}^4$$

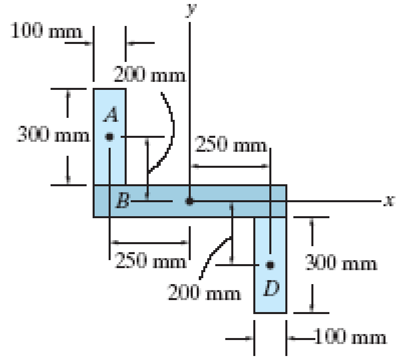
Note that the theorem of parallel axes is applied for the shape ❸

❒ EXAMPLE 4

Determine the moments of inertia for the cross-sectional area of the member shown about the **X** and **Y** centroid axes.



❒ SOLUTION



Composite Parts, the cross section can be subdivided into the three rectangular areas A, B, and D as shown. For the calculation, the centroid of each of these rectangles is located in the figure.

Parallel-axis theorem, the moment of inertia of a rectangle about its centroid axis is $\frac{1}{12}ba^3$. Hence, using the parallel-axis theorem for rectangles A and D, the calculations are as follows:

$$I_X = \frac{1}{12}(100)(300)^3 + 100(300)(200)^2 = 1.425(10^9) \quad \text{mm}^4$$
$$I_Y = \frac{1}{12}(300)(100)^3 + 100(300)(250)^2 = 1.90(10^9) \quad \text{mm}^4$$

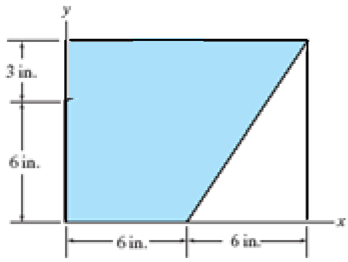
Rectangle B

$$I_X = \frac{1}{12}(600)(100)^3 = 0.05(10^9) \quad \text{mm}^4$$
$$I_Y = \frac{1}{12}(100)(600)^3 = 1.80(10^9) \quad \text{mm}^4$$

Then the moments of inertia for the entire cross section are thus

$$\Rightarrow I_X = 2(1.425(10^9)) + 0.05(10^9) = 2.90(10^9) \text{ mm}^4$$
$$\Rightarrow I_Y = 2(1.9(10^9)) + 1.80(10^9) = 5.60(10^9) \text{ mm}^4$$

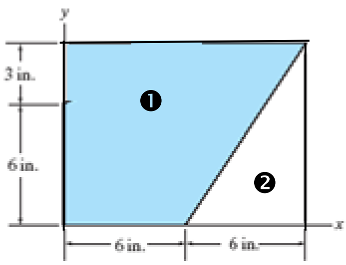
❒ EXAMPLE 5



Determine the moment of inertia of the shaded area about the $\mathbf{X}$ and $\mathbf{Y}$ axes.

❒ SOLUTION

Composite Parts, the cross section can be subdivided into the two areas, i.e. rectangular❶ and a triangle ❷ as shown. Parallel-axis theorem, the moment of inertia of a rectangle about $\mathbf{X}$ axis is $\frac{1}{3}M(9)^2$. using the parallel-axis theorem for triangle ❷, the calculations are as follows:



For the whole rectangular❶

$$I_X = \frac{1}{3}(108)(9)^2 = 2916 \quad \text{in}^4$$
$$I_Y = \frac{1}{3}(108)(12)^2 = 5148 \quad \text{in}^4$$

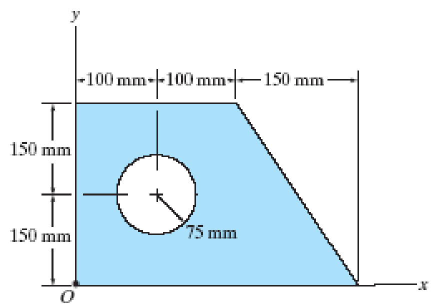
for the triangle ❷

$$I_X = \frac{1}{6}(27)(9)^2 = 364.5 \quad \text{in}^4$$
$$I_Y = \frac{1}{6}(27)(6)^2 - (27)(2)^2 + (27)(10)^2 = 2754 \ \text{in}^4$$

therefore the moment of inertia for shaded area is

$$\Rightarrow I_X = 2916 - 364.5 = 2550.5 \quad \text{in}^4$$
$$\Rightarrow I_Y = 5148 - 2754 = 2394 \quad \text{in}^4$$

❒ EXAMPLE 6

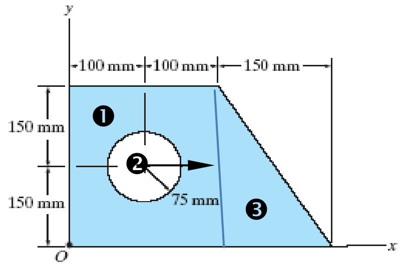


Determine the moment of inertia I_X and I_Y of the shaded area.

❒ SOLUTION

The plate is divided into three segments as shown in the figures below. Here the area of the rectangle ❶ and the triangle area ❸ and a circle area ❷ which is considered “negative” since it must be subtracted from the larger one ❶ .

For the whole rectangular❶



$$I_X = \frac{1}{3}(200)(300)^3 = 18(10^8) \quad mm^4$$
$$I_Y = \frac{1}{3}(300)(200)^3 = 8(10^8) \quad mm^4$$

for the triangle ❸

$$I_X = \frac{1}{6}(22500)(300)^2 \simeq 3.375(10^8) \quad mm^4$$
$$I_Y = \frac{1}{6}(22500)(150)^2 - (22500)(50)^2 + (22500)(250)^2$$
$$\simeq 14.343(10^8) \quad mm^4$$

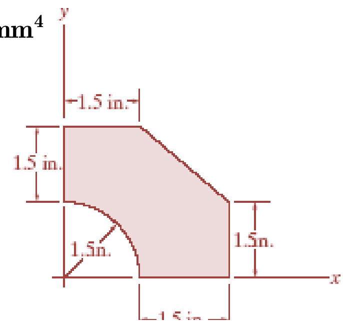
for the circle area ❷

$$I_X = \frac{1}{4}\left(\frac{22}{7}\right)(75)^4 + \left(\frac{22}{7}\right)(75)^2(150)^2 \simeq 4.226(10^8) \; mm^4$$
$$I_Y = \frac{1}{4}\left(\frac{22}{7}\right)(75)^4 + \left(\frac{22}{7}\right)(75)^2(100)^2 \simeq 2.016(10^8) \; mm^4$$

therefore the moment of inertia for shaded area is

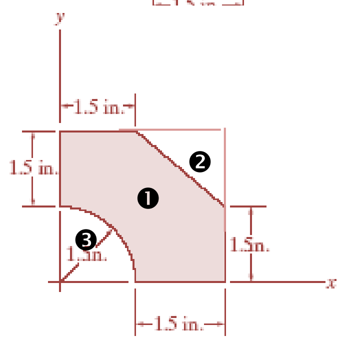
$$I_X = 18(10^8) + 3.375(10^8) - 4.226(10^8) = 17.149(10^8) \; mm^4$$
$$I_Y = 8(10^8) + 14.343(10^8) - 2.016(10^8) = 20.327(10^8) \; mm^4$$

❐ EXAMPLE 7



Evaluate the moment of inertia of the shaded area about Y axis.

❐ SOLUTION



The plate is divided into three segments as shown in the figure. Here the areas of the quarter circle ❸ and triangle ❷ are considered "negative" since it must be subtracted from the larger square ❶ .

for segment ❶ the moment of inertia about Y axis is

$$I_Y = \frac{1}{3}(9)(3)^2 = 27 \, in^4$$

for segment ❷ the moment of inertia about Y axis is

$$I_Y = \frac{1}{6}\left(\frac{1}{2}(1.5)(1.5)\right)(1.5)^2 - \left(\frac{1}{2}(1.5)(1.5)\right)(0.5)^2 + \left(\frac{1}{2}(1.5)(1.5)\right)(2.5)^2 \simeq 7.17 \, in^4$$

for segment ❸ the moment of inertia about Y axis is

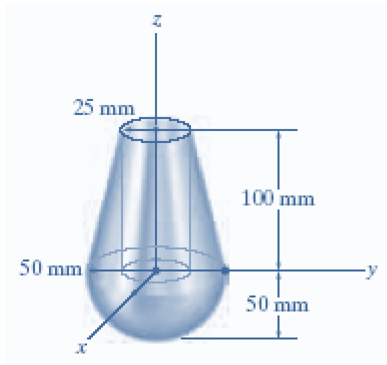
$$I_Y = \frac{1}{4}\left(\frac{1}{4}\pi(1.5)^2\right)(1.5)^2 = 0.994\,\text{in}^4$$

therefore the moment of inertia for shaded area is

$$I_Y = 27 - 7.17 - 0.994 \simeq 18.83\ \text{in}^4$$

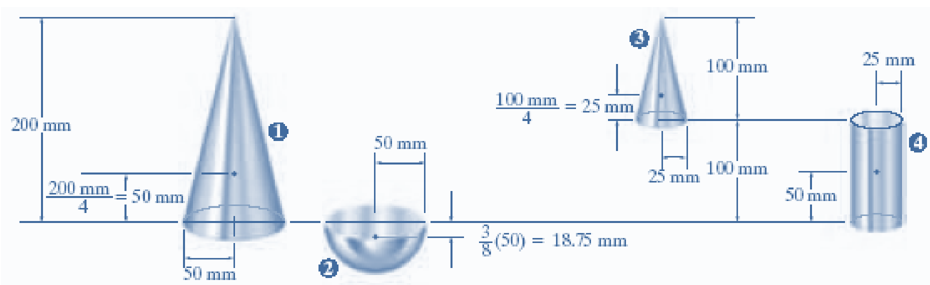
❒ EXAMPLE 8

Determine the moment of inertia of the assembly given about Z axis. The conical frustum has a density of $\rho = 8\,\text{Mg}\,\text{m}^{-3}$, and the hemisphere has a density of $\rho' = 4\,\text{Mg}\,\text{m}^{-3}$. There is a 25-mm-radius cylindrical hole in the center of the frustum.



❒ SOLUTION

The assembly can be thought of as consisting of four segments as shown in Figures below. For the calculations, ❸ and ❹ must be considered as "negative" segments in order that the four segments, when added together, yield the total composite shape. Using the following table, the computations for the moment of inertia of each piece are shown. The mass of each piece can be computed from $m = \rho V$ and used for the calculations. Also, $1\ \text{Mg}\ \text{m}^{-3} = 10^{-6}\ \text{kg}\,\text{mm}^{-3}$



so that we get for segment ❶ the moment of inertia about Z axis is

$$I_Z = \frac{3}{10}M(50)^2 = 1575\,\text{kg}\,\text{mm}^2 \qquad M = (8(10)^{-6})\frac{1}{3}\pi(50)^2(100) = 2.1\,\text{kg}$$

for segment ❷ the moment of inertia about **Z** axis is

$$I_Z = \frac{2}{5}M(50)^2 = 1048\ \text{kg mm}^2 \qquad M = (4(10)^{-6})\frac{2}{3}\pi(50)^3 \simeq 1.048\,\text{kg}$$

for segment ❸ the moment of inertia about **Z** axis is

$$I_Z = \frac{3}{10}M(25)^2 \simeq 98.25\ \text{kg}\,\text{mm}^2 \qquad M = (8(10)^{-6})\frac{1}{3}\pi(25)^2(100) = 0.524\,\text{kg}$$

for segment ❹ the moment of inertia about **Z** axis is

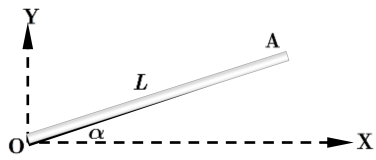
$$I_Z = \frac{1}{2}M(25)^2 = 490.625\ \text{kg}\,\text{mm}^2 \qquad M = (8(10)^{-6})\pi(25)^2(100) \simeq 1.57\ \text{kg}$$

hence the moment of inertia of the a the assembly

$$I_Z = 1575 + 1048 - 98.25 - 490.625 = 2034.125\ \text{kg}\,\text{mm}^2$$

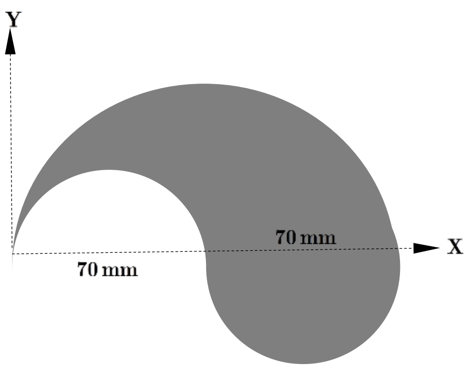
Problems

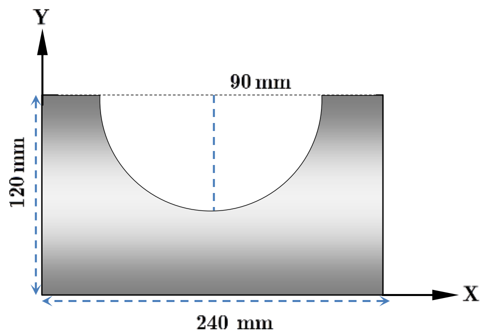
❒ Determine the moment of inertia of a uniform rod of length L about $\mathbf{X}$ and $\mathbf{Y}$ axes as shown.



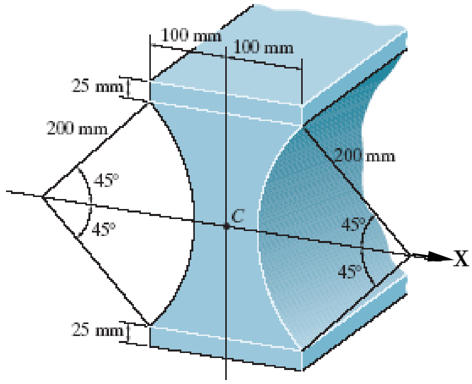
❒ Find the moment of inertia of an elliptic plate, of small thickness and uniform density.

❒ Find the moment of inertia of shaded area about $\mathbf{X}$ and $\mathbf{Y}$ axes as shown

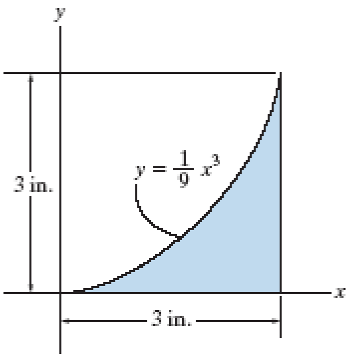


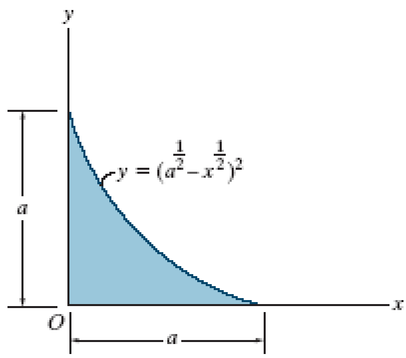


❒ Determine the moment of inertia for the beam's cross-sectional area about the **X** axis passing through the centroid *C* of the cross section.

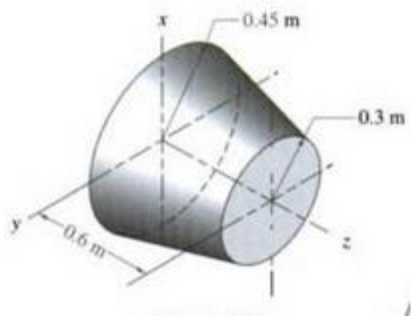


❒ Determine the product of inertia of the shaded area with respect to the **X** and **Y** axes.





❒ Determine the moment of inertia of truncated cone about the **Z-** axis.



KINEMATICS OF A RIGID BODY

When dealing with the kinematics of a rigid body, we are concerned with the geometric relationships that exist among displacements, velocities, and accelerations of various particles in a body in motion without regard to the forces causing the motion or caused by it. A rigid body is said to be in plane motion if all of its particles move in parallel planes. Three types of plane motion are identified in this chapter. The first type is known as translation which could be either rectilinear or curvilinear. The second type deals with rotation about a fixed axis. The third type is referred to as general plane motion which is a combination of a translation and a rotation.

Outer space provides one of many areas where kinematics plays a primary role. While on Earth, a force is required to maintain a body at a constant velocity because of friction; in outer space, no such force is required. However, basic kinematics applies equally will here on Earth and in space. The implication is that, irrespective of your engineering discipline, the knowledge of basic kinematics of rigid bodies will provide you with the basic foundations upon which dynamic principles are based. In a society where technology is pervasive, there is a need for understanding basic concepts and training of the individuals who design and organize the production of much of that technology. This chapter should enhance your enthusiasm for and the pleasure you derive from problem solving and reducing concepts to solid expression. The case study accompanying this chapter provides a clear illustration of the joys and frustrations of being an engineer.

■ Rectilinear and Curvilinear Translations

The kinematics of rigid bodies will be considered. We will investigate the relations existing between the time, the positions, the velocities, and the accelerations of the various particles forming a rigid body. The various types of rigid-body motion can be conveniently grouped as follows:

1. Translation. A rigid body is said to be in translation if any straight line inside the body keeps the same direction during the motion. It can also be observed that in a translation all the particles forming the body move along parallel paths. If these paths are straight lines Figure 1a, the motion is said to be a rectilinear translation if the paths are curved lines, the motion is a curvilinear translation Figure 1b. In both cases, note that a straight line, such as A_1B_1, remains parallel to itself throughout the entire motion.

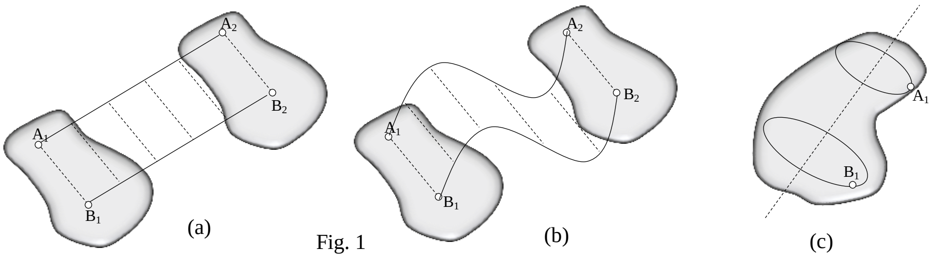


Fig. 1

2. Rotation about a Fixed Axis. In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis Figure 1c. If this axis, called the axis of rotation, intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration. Rotation should not be confused with certain types of curvilinear translation. For example, the plate shown in Figure 2a is in curvilinear translation, with all its particles moving along parallel circles, while the plate shown in Figure 2b is in rotation, with all its particles moving along concentric circles. In the first case, any given straight line drawn on the plate will maintain the same direction, whereas in the second case, point O remains fixed. Because each particle moves in a given plane, the rotation of a body about a fixed axis is said to be a plane motion.

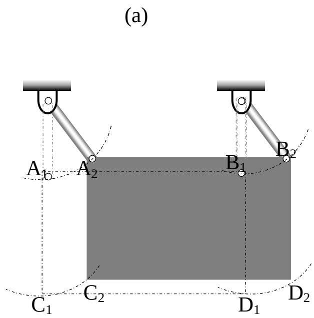
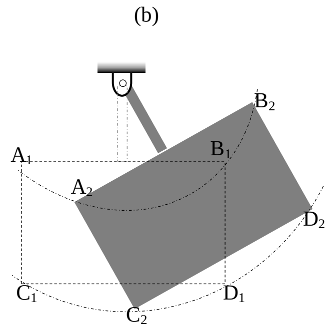
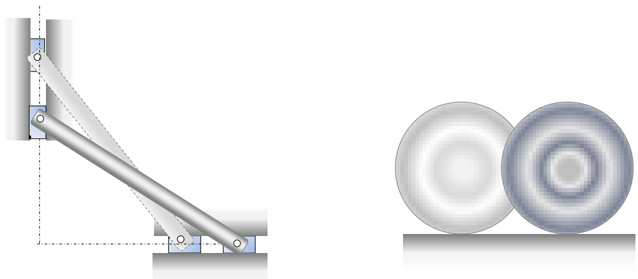


Fig. 2



3. General Plane Motion. There are many other types of plane motion, i.e., motions in which all the particles of the body move in parallel planes. Any plane motion which is neither a rotation nor a translation is referred to as a general plane motion. Two examples of general plane motion are given in Figure 3.

Fig. 3



4. Motion about a Fixed Point. The three-dimensional motion of a rigid body attached at a fixed point O, e.g., the motion of a top on a rough floor Figure 4a, is known as motion about a fixed point.

5. General Motion. Any motion of a rigid body which does not fall in any of the categories above is referred to as a general motion as shown in Figure 4b.

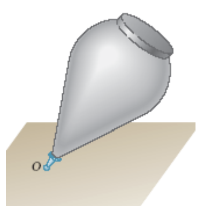


Fig. 4a

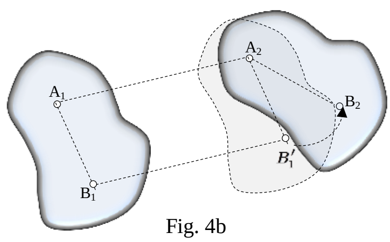


Fig. 4b

♦ **Translation Motion** Consider a rigid body in plane motion, as shown in Figure 5, which is assumed to be executing curvilinear translation parallel to the **X-Y** plane. The position vectors to particles A and B on this rigid body are $\underline{\mathbf{r}}_A$ and $\underline{\mathbf{r}}_B$ respectively, and the position vector of particle B relative to particle A is $\underline{\mathbf{r}}_{B|A}$. Therefore, the positions of any two particles A and B on a rigid body may be related by the following vector equation:

$$\underline{\mathbf{r}}_B = \underline{\mathbf{r}}_A + \underline{\mathbf{r}}_{B|A} \qquad \textbf{....(1)}$$

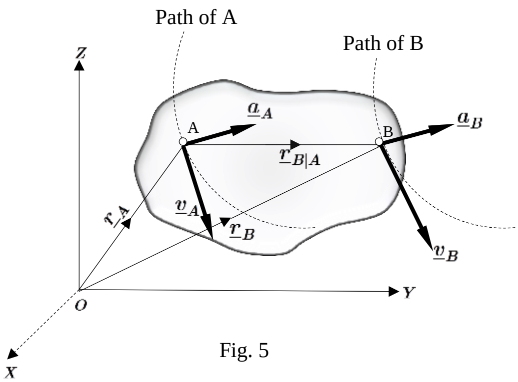


Fig. 5

Thus, if the position of any one particle on a rigid body, such as A, is known, the position of any other particle, such as B, may be determined by previous Eq. (1) if its position relative to particle A is known. If we differentiate Eq. (1) with respect to time, we have

$$\frac{d\underline{\mathbf{r}}_B}{dt} = \frac{d\underline{\mathbf{r}}_A}{dt} + \frac{d\underline{\mathbf{r}}_{B|A}}{dt} \qquad \textbf{....(2)}$$

By definition, because a rigid body implies that the positions of the particles relative to each other do not change, it follows that the term $\frac{d\underline{\mathbf{r}}_{B|A}}{dt}$ must vanish, and Eq. (2) reduces to $\frac{d\underline{\mathbf{r}}_B}{dt} = \frac{d\underline{\mathbf{r}}_A}{dt}$ Because $\frac{d\underline{\mathbf{r}}_B}{dt} = \underline{v}_B$ and $\frac{d\underline{\mathbf{r}}_A}{dt} = \underline{v}_A$ it follows that

$$\underline{v}_A = \underline{v}_B \qquad \text{....(3)}$$

where $\underline{v}_A$ and $\underline{v}_B$ represent the velocities of points A and B, respectively. Thus, according to Eq. (3), if the velocity of one particle, such as A, in a rigid body is known, the velocity of any other particle, such as B, may be found. In other words the velocities of all particles in a translating rigid body are the same at a given instant of time. As known, these velocity vectors must be tangent to the paths of motion, as shown in Figure 5. If we now differentiate once more Eq. (3) with respect to time and replace the time derivatives of the velocities by accelerations,

$$\underline{a}_A = \underline{a}_B \qquad \text{....(4)}$$

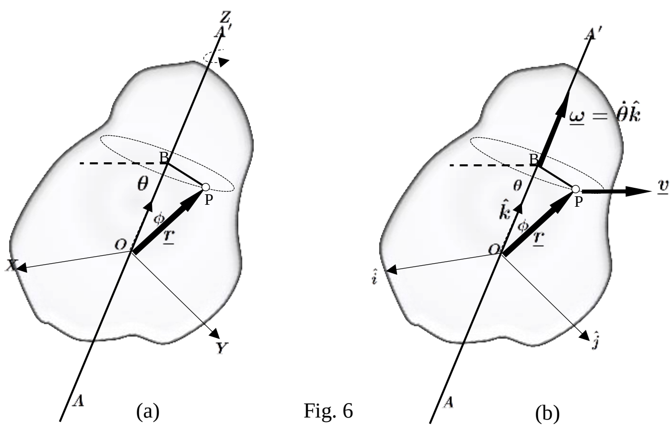
where $\underline{a}_A$ and $\underline{a}_B$ represent the acceleration vectors of particles A and B, respectively. Again, thus, if the acceleration of one particle, such as A, is known, the acceleration of any other particle, such as B, may be determined. In other words, the accelerations of all particles in a translating rigid body are all the same at a given instant of time. The accelerations of the two particles A and B are sketched in Figure 5. Note that these accelerations could have any direction depending upon the conditions specified in a given problem. It is obvious from the above discussion that the translating motion of a rigid body is completely defined if the motion of one of the particles is fully specified. Therefore, the equations developed for the motion of a particle may be used equally well to analyze the translating motion of a rigid body. The concepts discussed above are illustrated in the following examples.

That is, when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant (Figure 5). In the case of curvilinear translation, the velocity and acceleration change in direction as well as in magnitude at every instant. In the case of rectilinear translation, all particles of the body move along parallel straight lines, and their velocity and acceleration keep the same direction during the entire motion.

♦ Rotation about a Fixed Axis

Consider a rigid body which rotates about a fixed axis AA'. Let P be a point of the body and $\underline{r}$ its position vector with respect to a fixed frame of reference. For convenience, let us assume that the frame is centered at point O on AA' and that the $\mathbf{Z}$ axis coincides with AA' (Fig. 6a). Let B be the projection of P on AA'; since P must remain at a constant distance from B, it will describe a circle of center B and of radius $r\sin\phi$, where ϕ denotes the angle formed by $\underline{r}$ and AA'. The position of P and of the entire body is completely defined by the angle θ the line BP forms with the $\mathbf{ZX}$ plane. The angle θ is known as the angular coordinate of the body and is defined as positive when viewed as counterclockwise from A'. The angular coordinate will be expressed in radians (rad) or, occasionally, in degrees, or revolutions (rev). We recall that

$$1 \ \ \text{rev} = 2\pi \ \ \text{rad} = 360^{\circ}$$



(a) Fig. 6 (b)

We know that the velocity $\underline{v} = d\underline{r}/dt$ of a particle P is a vector tangent to the path of P and of magnitude $v = dS/dt$. Observing that the length $\triangle S$ of the arc described by P when the body rotates through $\triangle\theta$ is

$$\triangle S = (\mathrm{BP})\triangle\theta = (r\sin\phi)\triangle\theta$$

and dividing both members by Δt, we obtain at the limit, as Δt approaches zero,

$$v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{dS}{dt} = r\dot{\theta}\sin\phi \qquad \text{....(*)}$$

where $\dot{\theta}$ denotes the time derivative of θ. (Note that the angle θ depends on the position of P within the body, but the rate of change $\dot{\theta}$ is itself independent of P.) We conclude that the velocity $\underline{v}$ of P is a vector perpendicular to the plane containing AA' and $\underline{r}$, and of magnitude v defined by Eq. ($*$). But this is precisely the result we would obtain if we drew along AA' a vector $\underline{\omega} = \dot{\theta}\hat{k}$ and formed the vector product $\underline{\omega} \wedge \underline{r}$ (Figure 6b). We thus write

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{\omega} \wedge \underline{r} \qquad \text{....(5)}$$

The vector $\underline{\omega}$ is directed along the axis of rotation, is called the angular velocity of the body and is equal in magnitude to the rate of change $\dot{\theta}$ of the angular coordinate; its sense may be obtained by the right hand rule from the sense of rotation of the body. The acceleration $\underline{a}$ of the particle P will now be determined. Differentiating Eq. (5) and recalling the rule for the differentiation of a vector product, we write

$$\begin{aligned} \underline{a} &= \frac{d\underline{v}}{dt} = \frac{d}{dt}(\underline{\omega} \wedge \underline{r}) \\ &= \frac{d\underline{\omega}}{dt} \wedge \underline{r} + \underline{\omega} \wedge \frac{d\underline{r}}{dt} \\ &= \frac{d\underline{\omega}}{dt} \wedge \underline{r} + \underline{\omega} \wedge \underline{v} \qquad \text{....(6)} \end{aligned}$$

The vector $d\underline{\omega} / dt$ is denoted by $\underline{\alpha}$ and is called the angular acceleration of the body. Substituting also for $\underline{v}$ from Eq. (5), we have

$$\underline{a} = \frac{d\underline{v}}{dt} = \underline{\alpha} \wedge \underline{r} + \underline{\omega} \wedge (\underline{\omega} \wedge \underline{r}) \qquad \text{....(7)}$$

Differentiating $\underline{\omega} = \dot{\theta}\hat{k}$ and recalling that $\hat{k}$ is constant in magnitude and direction, we have

$$\underline{\alpha} = \frac{d\underline{\omega}}{dt} = \dot{\omega}\hat{k} = \ddot{\theta}\hat{k} \qquad \text{....(8)}$$

Thus, the angular acceleration of a body rotating about a fixed axis is a vector directed along the axis of rotation, and is equal in magnitude to the rate of change $\dot{\omega}$ of the angular velocity. Returning to Eq. (7), we note that the acceleration of P is the sum of two vectors. The first vector is equal to the vector product $\underline{\alpha} \wedge \underline{r}$; it is tangent to the circle described by P and therefore represents the tangential component of the acceleration. The second vector is equal to the vector triple product $\underline{\omega} \wedge (\underline{\omega} \wedge \underline{r})$ obtained by forming the vector product of $\underline{\omega}$ and $(\underline{\omega} \wedge \underline{r})$; since $(\underline{\omega} \wedge \underline{r})$ is tangent to the circle described by P, the vector triple product is directed toward the center B of the circle and therefore represents the normal component of the acceleration.

♦ Rotation of a Representative Slab

The rotation of a rigid body about a fixed axis can be defined by the motion of a representative slab in a reference plane perpendicular to the axis of rotation. Let us choose the XY plane as the reference plane and assume that it coincides with the plane of the figure, with the Z axis pointing out of the paper (Fig. 6c). Recalling from $\underline{\omega} = \omega\hat{k}$ we note that a positive value of the scalar ω corresponds to a counterclockwise rotation of the representative slab, and a negative value to a clockwise rotation. Substituting $\omega\hat{k}$ for ω into Eq. (5), we express the velocity of any given point P of the slab as $\underline{v} = \omega\hat{k} \wedge \underline{r}$

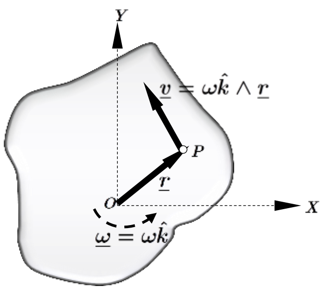


Fig. 6c

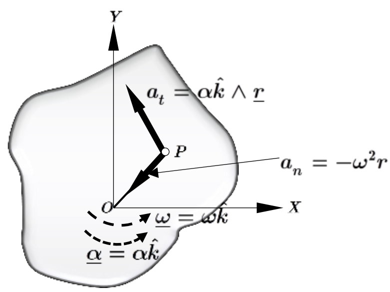


Fig. 6d

Since the vectors $\hat{k}$ and $\underline{r}$ are mutually perpendicular, the magnitude of the velocity $\underline{v}$ is $v = r\omega$ and its direction can be obtained by rotating $\underline{r}$ through 90° in the sense of rotation of the slab.

Substituting $\underline{\omega} = \omega\hat{k}$ and $\underline{\alpha} = \alpha\hat{k}$ into Eq. (7), and observing that cross-multiplying $\underline{r}$ twice by $\hat{k}$ results in a 180° rotation of the vector $\underline{r}$, we express the acceleration of point *P* as

$$\underline{a} = \alpha\hat{k} \wedge \underline{r} - \omega^2\underline{r} \qquad ....$$

Resolving $\underline{a}$ into tangential and normal components (Fig. 6d), we Write

$$a_t = \alpha\hat{k} \wedge \underline{r} \qquad a_t = r\alpha$$
$$a_n = -\omega^2\underline{r} \qquad a_n = r\omega^2 \qquad ....$$

The tangential component a_t points in the counterclockwise direction if the scalar α is positive, and in the clockwise direction if α is negative. The normal component a_n always points in the direction opposite to that of $\underline{r}$, that is, toward *O*.

♦ Equations Defining the Rotation of a Rigid Body about a Fixed Axis

The motion of a rigid body rotating about a fixed axis AA' is said to be known when its angular coordinate θ can be expressed as a known function of t. In practice, however, the rotation of a rigid body is seldom defined by a relation between θ and t. More often, the conditions of motion will be specified by the type of angular acceleration that the body possesses. For example, a may be given as a function of t, as a function of θ, or as a function of ω. Recalling the relation $\underline{\omega} = \dot{\theta}\hat{k}$ and Equation (8), we write

$$\omega = \frac{d\theta}{dt} \qquad \textbf{and} \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad ...(9)$$

From the chain rule we get,

$$\alpha = \frac{d\omega}{d\theta}\underset{\omega}{\frac{d\theta}{dt}} = \omega\frac{d\omega}{d\theta} \qquad \textbf{...(10)}$$

These equations are similar to those obtained before for the rectilinear motion of a particle, their integration can be performed by following the procedure outlined before. Two particular cases of rotation are frequently encountered:

1. Uniform Rotation. This case is characterized by the fact that the angular acceleration is zero. The angular velocity is thus constant, and the angular coordinate is given by the formula

$$\because\ d\theta = \omega dt \qquad \Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t \omega dt$$

$$\Rightarrow \theta - \theta_0 = \omega t$$

$$\textbf{Or} \qquad \theta = \theta_0 + \omega t \qquad \textbf{...(11)}$$

2. Uniformly Accelerated Rotation. In this case, the angular acceleration is constant. The following formulas relating angular velocity, angular coordinate, and time can then be derived in a manner similar to that described previous. The similarity between the formulas derived here and those obtained for the rectilinear uniformly accelerated motion of a particle is apparent.

$$\omega = \omega_0 + \alpha t \qquad \textbf{...(12)}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \qquad \textbf{...(13)}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad \textbf{...(14)}$$

It should be emphasized that formula (11) can be used only when $\alpha = \mathbf{0}$, and formulas (12-14) can be used only when $\alpha =$constant. In any other case, the general formulas (9) and (10) should be used.

♦ General Plane Motion

As indicated before, we understand by general plane motion a plane motion which is neither a translation nor a rotation. As we will presently see, however, a general plane motion can always be considered as the sum of a translation and a rotation.

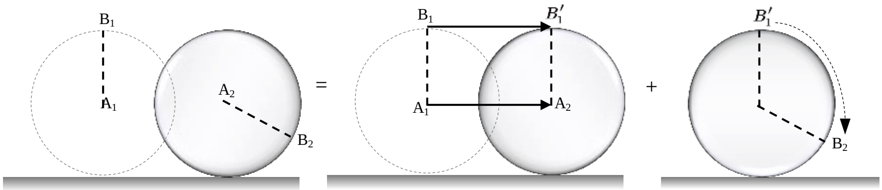


Fig. 7 Plane Motion = Translation with A + Rotation about A

Consider, for example, a wheel rolling on a straight track (Fig. 7). Over a certain interval of time, two given points A and B will have moved, respectively, from A_1 to A_2 and from B_1 to B_2. The same result could be obtained through a translation which would bring A and B into A_2 and B_1' (the line AB remaining vertical), followed by a rotation about A bringing B into B_2. Although the original rolling motion differs from the combination of translation and rotation when these motions are taken in succession, the original motion can be exactly duplicated by a combination of simultaneous translation and rotation. Another example of plane motion is given in Fig. 8, which represents a rod whose extremities slide along a horizontal and a vertical track, respectively.

This motion can be replaced by a translation in a horizontal direction and a rotation about A (Fig. 8a) or by a translation in a vertical direction and a rotation about B (Fig. 8b). In the general case of plane motion, we will consider a small displacement which brings two particles A and B of a representative slab, respectively, from A_1 and B_1 into A_2 and B_2 (Fig. 9). This displacement can be divided into two parts: in one, the particles move into A_2 and B_1' while the line AB maintains the same direction; in the other, B_1' moves into B_2 while A remains fixed. The first part of the motion is clearly a translation and the second is a rotation about A.

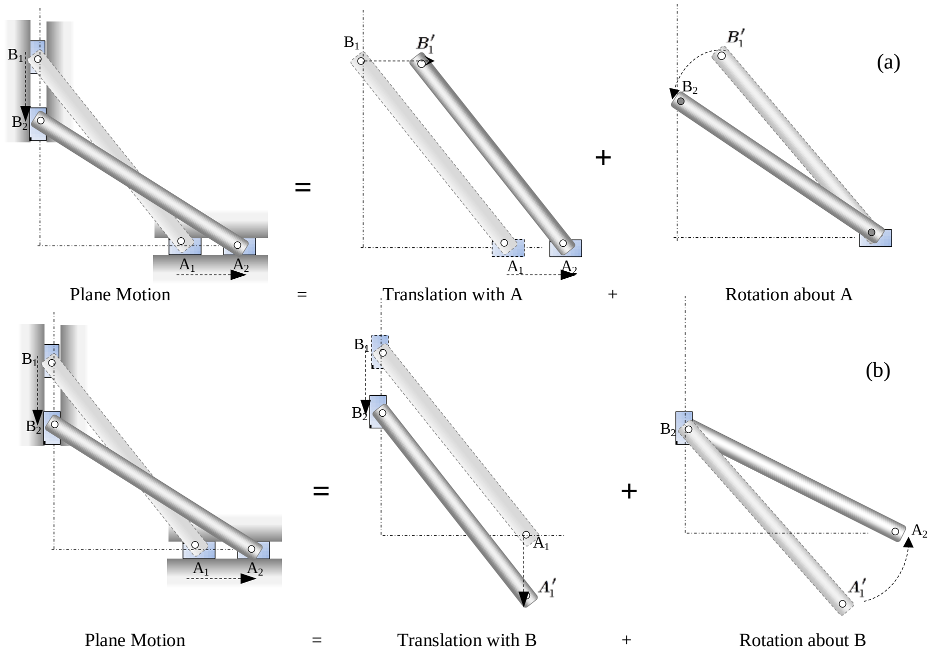


Fig. 8

Recalling, the definition of the relative motion of a particle with respect to a moving frame of reference—as opposed to its absolute motion with respect to a fixed frame of reference—we can restate as follows the result obtained above: Given two particles *A* and *B* of a rigid slab in plane motion, the relative motion of *B* with respect to a frame attached to *A* and of fixed orientation is a rotation. To an observer moving with *A* but not rotating, particle *B* will appear to describe an arc of circle centered at *A*.

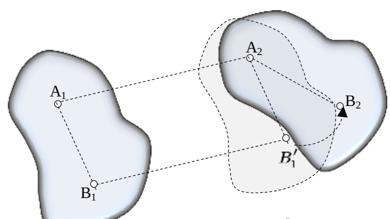
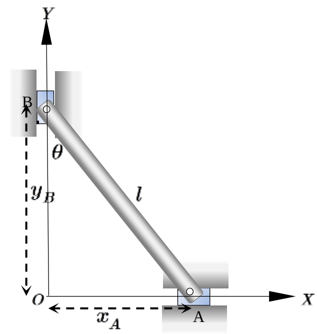


Fig. 9 Repeated

■ Analysis of Plane Motion in Terms of a Parameter



In the case of certain mechanisms, it is possible to express the coordinates *x* and *y* of all the significant points of the mechanism by means of simple analytic expressions containing a single parameter. It is sometimes advantageous in such a case to determine the absolute velocity and the absolute acceleration of the various points of the mechanism directly, since the components of the velocity and of the acceleration of a given point can be obtained by differentiating the coordinates *x* and *y* of that point. Let us consider again the rod *AB* whose extremities slide, respectively, in a horizontal and a vertical track (see figure besides). The coordinates x_A and y_B of the extremities of the rod can be expressed in terms of the angle θ the rod forms with the vertical:

$$x_A = l\sin\theta, \qquad y_B = l\cos\theta$$

Differentiating this equation twice with respect to time, we write

$$\begin{aligned} v_A &= \dot{x}_A = l\dot{\theta}\cos\theta, \qquad & a_A &= \ddot{x}_A = l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta \\ v_B &= \dot{y}_B = -l\dot{\theta}\sin\theta, \qquad & a_B &= \ddot{y}_B = -l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta \end{aligned}$$

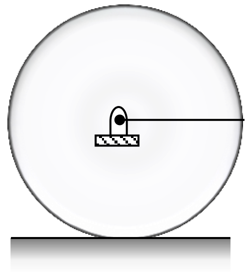
Recalling that $\dot{\theta} = \omega, \quad \ddot{\theta} = \alpha$, we obtain

$$\begin{aligned} v_A &= l\omega\cos\theta, \qquad & v_B &= -l\omega\sin\theta \qquad & ....(1) \\ a_A &= l\alpha\cos\theta - l\omega^2\sin\theta, \qquad & a_B &= -l\alpha\sin\theta - l\omega^2\cos\theta \qquad & ....(2) \end{aligned}$$

We note that a positive sign for v_A or a_A indicates that the velocity v_A or the acceleration a_A is directed to the right; a positive sign for v_B or a_B indicates that v_B or a_B is directed upward. Equations (1) can be used, for example to determine v_B and ω when v_A and θ are known. Substituting for ω in (2), we can then determine a_B and α if a_A is known

■Illustrative Examples■

❒ EXAMPLE 1



The motion of a disk rotating is defined by the relation $\theta = 0.4(1 - e^{-t/4})$, where θ is expressed in radians and t in seconds. Determine the angular coordinate, velocity, and acceleration of the disk when (i) $t = 0$, (ii) $t = 3$ s, (iii) $t = \infty$

❒ SOLUTION

Angular displacement of the disk

$$\begin{aligned} \theta = 0.4(1 - e^{-t/4}) \quad &\Rightarrow \theta\big|_{t=0} = 0 \text{ rad} \\ &\Rightarrow \theta\big|_{t=3} = 0.4(1 - e^{-0.75}) \simeq 0.211 \text{ rad} \\ &\Rightarrow \theta\big|_{t\to\infty} = 0.4(1 - e^{-\infty}) \simeq 0.4 \text{ rad} \end{aligned}$$

Angular velocity of rotating disk

$$\begin{aligned} \dot{\theta} = 0.1e^{-t/4} \quad &\Rightarrow \dot{\theta}\big|_{t=0} = 0.1 \text{ rad s}^{-1} \\ &\Rightarrow \dot{\theta}\big|_{t=3} = 0.1e^{-0.75} \simeq 0.00472 \text{ rad s}^{-1} \\ &\Rightarrow \dot{\theta}\big|_{t\to\infty} = 0.1e^{-\infty} \simeq 0 \text{ rad s}^{-1} \end{aligned}$$

Angular acceleration of the disk

$$\begin{aligned} \ddot{\theta} = -\frac{1}{40}e^{-t/4} \quad &\Rightarrow \ddot{\theta}\big|_{t=0} = -\frac{1}{40} \text{ rad s}^{-2} \\ &\Rightarrow \ddot{\theta}\big|_{t=3} = -\frac{1}{40}e^{-0.75} \simeq -0.012 \text{ rad s}^{-2} \\ &\Rightarrow \ddot{\theta}\big|_{t\to\infty} = -\frac{1}{40}e^{-\infty} \simeq 0 \text{ rad s}^{-2} \end{aligned}$$

❒ EXAMPLE 2

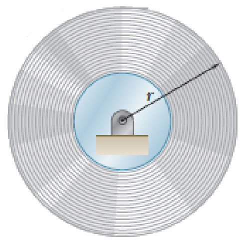
The angular acceleration of an oscillating disk is defined by the relation $\alpha = -\mu\theta$. Determine (i) the value of μ for which $\omega = 8$ rad s^{-1} when $\theta = 0$ and $\theta = 4$ rad when $\omega = 0$, (ii) the angular velocity of the disk when $\theta = 3$ rad.

❒ SOLUTION

Since the angular acceleration $\alpha = \ddot{\theta} = \dfrac{d\dot{\theta}}{dt} = \dfrac{d\dot{\theta}}{d\theta}\underbrace{\dfrac{d\theta}{dt}}_{\dot{\theta}} = \dot{\theta}\dfrac{d\dot{\theta}}{d\theta}$ therefore,

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = -\mu\theta \qquad \Rightarrow \dot{\theta}d\dot{\theta} = -\mu\theta d\theta$$

Integrating we get



$$\frac{1}{2}\dot{\theta}^2 = -\frac{1}{2}\mu\theta^2 + c$$

But as given $\omega = 8$ rad s^{-1} when $\theta = 0$ thus , $(\dot{\theta} = \omega)$

$$\frac{1}{2}(8)^2 = -\frac{1}{2}\mu(0)^2 + c \qquad \Rightarrow c = 32$$

Once again as $\theta = 4$ rad when $\omega = 0$ we get

$$\frac{1}{2}(0)^2 = -\frac{1}{2}\mu(4)^2 + 32 \qquad \Rightarrow \mu = 4$$

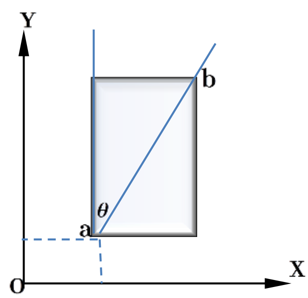
That is the angular coordinate is given by $\dot{\theta}^2 = 64 - 4\theta^2$ the angular velocity can be obtained by differentiation $\dot{\theta}^2 = 64 - 4\theta^2$ with respect to time, we get

$$2\dot{\theta}\ddot{\theta} = -8\theta\dot{\theta} \qquad \Rightarrow \ddot{\theta} = -4\theta$$

Then the angular acceleration when $\theta = 3$ rad is

$$\alpha = \ddot{\theta}\Big|_{t=3} = -4(3) = -12 \text{ rad s}^{-2}$$

❒ EXAMPLE 3



The figure shows the position of a rectangular lamina at $t = 0$, the displacement of a point a is given by $x = 3.9 + 4.8t$, $y = 2.2 - 9.6t + 5t^2$, where x, y is expressed in meters and t in seconds. In addition, the angular position of the line ab is given by $\theta = 1.9t^2 + 0.3$ where θ is expressed in radians. Determine the position after 2 s and calculate the velocity of the point a when $t = 1$ s .

❒ SOLUTION

$x = 3.9 + 4.8t \qquad \Rightarrow x\big|_{t=2} = 3.9 + 4.8(2) = 13.5 \text{ m}$

$y = 2.2 - 9.6t + 5t^2 \qquad \Rightarrow y\big|_{t=2} = 2.2 - 9.6(2) + 5(2)^2 = 3 \text{ m}$

$\theta = 1.9t^2 + 0.3 \qquad \Rightarrow \theta\big|_{t=2} = 1.9(2)^2 + 0.3 = 7.9 \text{ rad} = 452.45^o = 92.45^o$

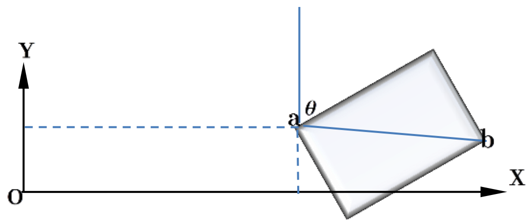
The components of velocity are

$\dot{x} = 4.8 \qquad \Rightarrow \dot{x}\big|_{t=1} = 4.8 \text{ m s}^{-1}$

$\dot{y} = -9.6 + 10t \qquad \Rightarrow \dot{y}\big|_{t=1} = -9.6 + 10(1) = 0.4 \text{ m s}^{-1}$

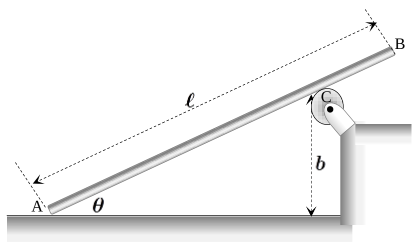
$\dot{\theta} = 3.8t \qquad \Rightarrow \dot{\theta}\big|_{t=1} = 3.8(1) = 3.8 \text{ rad s}^{-1}$

Therefore the translation velocity is $\sqrt{(4.8)^2 + (0.4)^2} \simeq 4.82 \text{ m s}^{-1}$ whereas the angular velocity is 3.8 rad s^{-1}



❒ EXAMPLE 4

Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity v_A. Derive expressions for the angular velocity and angular acceleration of the rod.



❒ SOLUTION

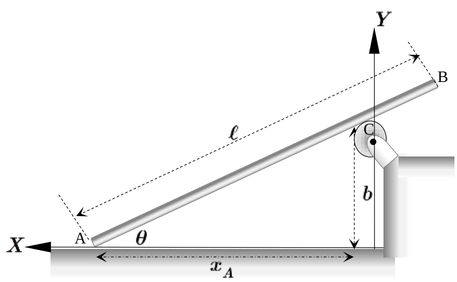
From the geometry of the figure we have

$$x_A = b\cot\theta$$

Thus by differentiation we get

$$v_A = \dot{x}_A = -b\dot{\theta}\csc^2\theta = -b\omega\csc^2\theta \qquad \ldots\ldots$$

where $\omega = \dot{\theta}$ is the angular velocity of the rigid rod. Therefore,



$$\omega = -\frac{v_A}{b \csc^2 \theta} \ \ \mathbf{rad\,s^{-1}} \qquad \ldots\ldots$$

Also $\quad a_A = \ddot{x}_A$

Because v_A is constant, $a_c = 0$ and it follows that

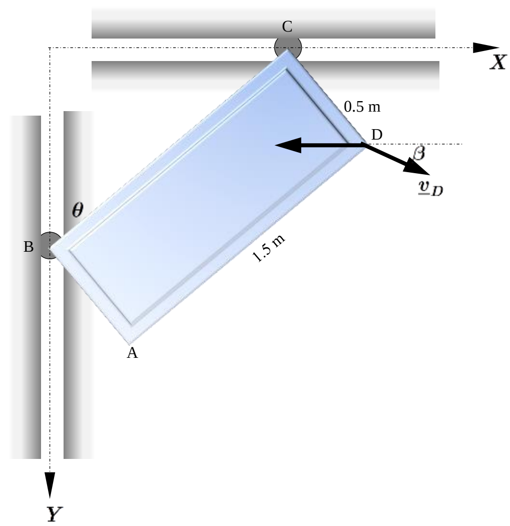
$$\begin{aligned} a_A &= 2b\dot{\theta}^2 \csc^2\theta \cot\theta - b\ddot{\theta}\csc^2\theta \\ &= 2b\omega^2 \csc^2\theta\cot\theta - b\alpha\csc^2\theta = 0 \end{aligned}$$

where $\alpha = \ddot{\theta}$ is the angular acceleration of the rigid plate. Therefore,

$$\alpha = 2\omega^2 \cot\theta \quad \mathbf{rad\,s^{-2}} \qquad \ldots..$$

❒ EXAMPLE 5

The rectangular plate ABCD, shown in the Figure, is constrained to move so that corner B slides in a vertical track and corner C in a horizontal track. If corner C moves to the right with a constant velocity $v_c = \mathbf{0.75\ m\,s^{-1}}$, determine the angular velocity ω and angular acceleration α of the plate. Express your answers in terms of the angular position θ measured clockwise from the vertical track.



❒ SOLUTION

From the geometry of the figure we have

$$x_c = a\sin\theta$$

Thus by differentiation we get $\qquad v_c = \dot{x}_c = a\dot{\theta}\cos\theta = a\omega\cos\theta \quad \ldots\ldots$

where $\omega = \dot{\theta}$ is the angular velocity of the rigid plate. Therefore,

$$\omega = \frac{v_c}{a\cos\theta} = \frac{0.5}{\cos\theta}\mathbf{rad\,s^{-1}} \qquad \ldots\ldots$$

Also $\quad a_c = \ddot{x}_c$

Because v_c is constant, $a_c = 0$ and it follows that

$$\Rightarrow a_c = -a\dot{\theta}^2 \sin\theta + a\ddot{\theta}\cos\theta = 0$$

Thus $$a_c = -a\omega^2 \sin\theta + a\alpha\cos\theta = 0$$

where $\alpha = \ddot{\theta}$ is the angular acceleration of the rigid plate. Therefore,

$$\alpha = \omega^2 \tan\theta$$

Or $$\alpha = \left(\frac{0.5}{\cos\theta}\right)^2 \tan\theta = \frac{0.25}{\cos^2\theta}\tan\theta \quad \text{rad s}^{-1} \quad .....$$

❒ EXAMPLE 6

Refer to previous example, determine the velocity v_D and the acceleration a_D of corner D. Express your answers in terms of the angular position *0* measured clockwise from the vertical track.

❒ SOLUTION

Once more again and according to the geometry of Figure in previous example,

$$x_D = x_c + b\cos\theta = a\sin\theta + b\cos\theta$$
$$y_D = b\sin\theta$$

Therefore

$$(v_D)_x = \dot{x}_D = a\omega\cos\theta - b\omega\sin\theta$$
$$\text{and } (v_D)_y = \dot{y}_D = b\omega\cos\theta$$

Using the results given in previous Example for v_C and ω, respectively,

$$(v_D)_x = a\omega\cos\theta - b\omega\sin\theta = v_v - v_c\left(\frac{b}{a}\right)\tan\theta = v_v\left(1 - \frac{b}{a}\tan\theta\right)$$
$$\text{and } (v_D)_y = b\omega\cos\theta = \frac{b}{a}v_c$$

Since $v_D = \sqrt{(v_D)_x^2 + (v_D)_y^2}$, it follows that

$$v_D = v_v\sqrt{\left(1 - \frac{b}{a}\tan\theta\right)^2 + \left(\frac{b}{a}\right)^2}$$
$$= 0.75\sqrt{\left(\frac{1}{9}\right) + \left(1 - \frac{1}{3}\tan\theta\right)^2} \quad \text{m s}^{-1} \quad ....$$

The direction of v_D is defined by the angle β (see Figure) where

$$\tan\beta = \frac{(v_D)_y}{(v_D)_x} = \frac{b / a}{\left(1 - \frac{b}{a}\tan\theta\right)} = \frac{b}{a - b\tan\theta}$$

$$\Rightarrow \beta = \tan^{-1}\left(\frac{1}{3 - \tan\theta}\right) \qquad \ldots\ldots$$

To obtain the components of acceleration of point D $(\ddot{x}_D, \ddot{y}_D)$ differentiation the components of velocity $(\dot{x}_D, \dot{y}_D)$ so that

$$(a_D)_x = \ddot{x}_D = -a\omega^2\sin\theta + a\alpha\cos\theta - b\omega^2\cos\theta - b\alpha\sin\theta$$

$$= -\omega^2(a\sin\theta + b\cos\theta) + \alpha(a\cos\theta - b\sin\theta) \qquad \ldots\ldots$$

$$\text{and } (a_D)_y = \ddot{y}_D = -b\omega^2\sin\theta + b\alpha\cos\theta \qquad \ldots\ldots$$

Or by substituting for ω and α we get

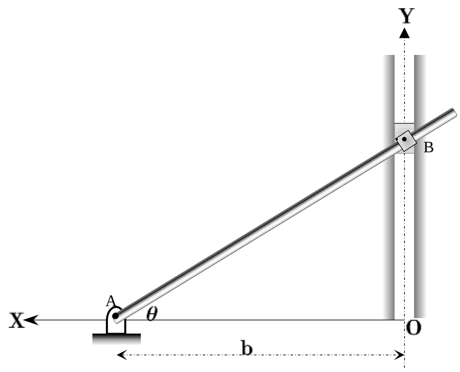
$$(a_D)_x = -\frac{bv_c^2}{a^2\cos^3\theta} \qquad \text{and} \qquad (a_D)_y = \frac{bv_c^2}{a^2\cos^3\theta}(-\sin\theta + \sin\theta) = 0$$

Since $a_D = \sqrt{(a_D)_x^2 + (a_D)_y^2}$, it follows that

$$a_D = (a_D)_x = -\frac{bv_c^2}{a^2\cos^3\theta} = -\frac{0.125}{\cos^3\theta}\ \mathrm{m\,s^{-2}} \qquad \ldots\ldots$$

where the negative sign indicates a sense for an which is opposite to the positive direction of the $\mathbf{X}$ axis. Thus, a_D is pointed to the left as shown in the Figure, for the case when the velocity of corner C is constant and pointed to the right (i.e., in the positive $\mathbf{X}$ direction).

❒ EXAMPLE 7



Member AB in the figure shown starts from rest, when $\theta = 0$, and rotates in counterclockwise direction at a constant angular acceleration α. In terms of b, θ and α, develop expressions for the velocity v_B and acceleration a_B of the collar-slider unit B which is constrained to move along a smooth vertical track. If $b = 0.75\,\mathrm{m}$ and $\alpha = 5.0\ \mathrm{rad\,s^{-2}}$, determine v_B and a_B for $\theta = 30^{\circ}$

❒ SOLUTION

From the geometry of the figure we have $\quad y_B = b\tan\theta$

Thus by differentiation we get $\quad v_B = \dot{y}_B = b\dot{\theta}\sec^2\theta = b\omega\sec^2\theta \quad .....$

where $\omega = \dot{\theta}$ is the angular velocity of the rigid member.

Differentiation again to evaluate the acceleration of point B we get,

$$\begin{aligned} a_B &= \ddot{y}_B = b\ddot{\theta}\sec^2\theta + 2b\omega\dot{\theta}\sec^2\theta\tan\theta \\ &= \sec^2\theta(b\alpha + 2b\omega^2\tan\theta) \qquad .... \end{aligned}$$

where $\alpha = \ddot{\theta}$ is the angular acceleration of the rigid member

since $\alpha = 5.0\ \mathrm{rad\,s^{-2}} \quad \Rightarrow \omega\dfrac{d\omega}{d\theta} = 5 \quad$ or $\quad \omega d\omega = 5d\theta$

by integrating we have, $\omega^2 = 10\theta + c$ since the member starts from rest $\omega = 0,$ when $\theta = 0$, so $c = 0$ and then $\omega^2 = 10\theta$

when $b = 0.75\,\mathrm{m}$ and $\alpha = 5.0\ \mathrm{rad\,s^{-2}}$ and $\theta = 30^{\circ}$ Note $\omega\big|_{\theta=30} = \sqrt{10\pi / 6}$

$$v_B = b\omega\sec^2\theta = (0.75)\ \sqrt{10\pi / 6}\ \sec^2 30 \simeq 2.288\ \mathrm{m\,s^{-1}}$$

$$\begin{aligned} a_B &= \sec^2\theta(b\alpha + 2b\omega^2\tan\theta) \\ &= \sec^2 30\left((0.75)(5) + 2(0.75)\left(\frac{10\pi}{6}\right)\tan 30\right) \simeq 11.09\,\mathrm{m\,s^{-2}} \end{aligned}$$

♦ Instantaneous Center of Rotation in Plane Motion

Consider the general plane motion of a slab. We propose to show that at any given instant the velocities of the various particles of the slab are the same as if the slab were rotating about a certain axis perpendicular to the plane of the slab, called the instantaneous axis of rotation. This axis intersects the plane of the slab at a point C, called the instantaneous center of rotation of the slab.

We first recall that the plane motion of a slab can always be replaced by a translation defined by the motion of an arbitrary reference point A and by a rotation about A. As far as the velocities are concerned, the translation is characterized by the velocity $\underline{v}_A$ of the reference point A and the rotation is characterized by the angular velocity ω of the slab (which is independent of the choice of A). Thus, the velocity $\underline{v}_A$ of point A and the angular velocity ω of the slab define completely the velocities of all the other particles of the slab (Figure 10a). Now let us assume that $\underline{v}_A$ and ω are known and that they are both different from zero. (If $\underline{v}_A = \mathbf{0}$, point A is itself the instantaneous center of rotation, and if $\omega = \mathbf{0}$, all the particles have the same velocity $\underline{v}_A$.) These velocities could be obtained by letting the slab rotate with the angular velocity ω about a point C located on the perpendicular to $\underline{v}_A$ at a distance $r = v_A / \omega$ from A as shown in Figure 10b. We check that the velocity of A would be perpendicular to AC and that its magnitude would be $r\omega = (v_A / \omega)\omega = v_A$.

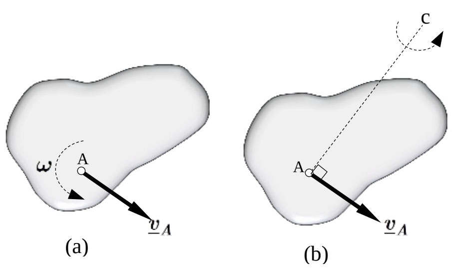


Fig. 10

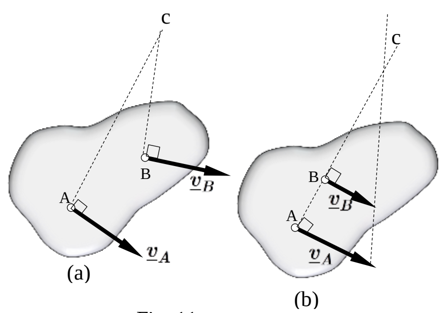
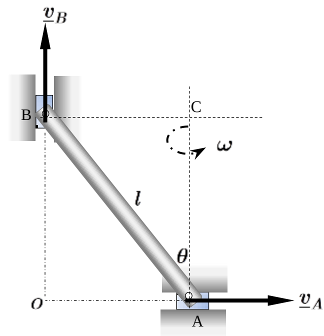


Fig. 11



Thus the velocities of all the other particles of the slab would be the same as originally defined. Therefore, as far as the velocities are concerned, the slab seems to rotate about the instantaneous center C at the instant considered. The position of the instantaneous center can be defined in two other ways. If the directions of the velocities of two particles A and B of the slab are known and if they are different, the instantaneous center C is obtained by drawing the perpendicular to $\underline{v}_A$ through A and the perpendicular to $\underline{v}_B$ through B and determining the point in which these two lines intersect (Figure 11a). If the velocities $\underline{v}_A$ and $\underline{v}_B$ of two particles A and B are perpendicular to the line AB and if their magnitudes are known, the instantaneous center can be found by intersecting the line AB with the line joining the extremities of the vectors $\underline{v}_A$ and $\underline{v}_B$ (Figure 11b). Note that if $\underline{v}_A$ and $\underline{v}_B$ were parallel in Figure 11a or if $\underline{v}_A$ and $\underline{v}_B$ had the same magnitude in Figure 11b, the instantaneous center C would be at an infinite distance and ω would be zero; all points of the slab would have the same velocity. To see how the concept of instantaneous center of rotation can be put to use, let us consider again the rod as shown. Drawing the perpendicular to $\underline{v}_A$ through A and the perpendicular to $\underline{v}_B$ through B (Figure 12), we obtain the instantaneous center C. At the instant considered, the velocities of all the particles of the rod are thus the same as if the rod rotated about C. Now, if the magnitude $\underline{v}_A$ of the velocity of A is known, the magnitude ω of the angular velocity of the rod can be obtained by writing

$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta}$$

The magnitude of the velocity of B can then be obtained by writing

$$v_B = (BC)\omega = l\sin\theta\,\frac{v_A}{l\cos\theta} = v_A\tan\theta$$

Note that only absolute velocities are involved in the computation.

The instantaneous center of a slab in plane motion can be located either on the slab or outside the slab. If it is located on the slab, the particle C coinciding with the instantaneous center at a given instant t must have zero velocity at that instant. However, it should be noted that the instantaneous center of rotation is valid only at a given instant. Thus, the particle C of the slab which coincides with the instantaneous center at time t will generally not coincide with the instantaneous center at time $t+\Delta t$; while its velocity is zero at time t, it will probably be different from zero at time $t+\Delta t$. This means that, in general, the particle C does not have zero acceleration and, therefore, that the accelerations of the various particles of the slab cannot be determined as if the slab was rotating about C. As the motion of the slab proceeds, the instantaneous center moves in space. But it was just pointed out that the position of the instantaneous center on the slab keeps changing. Thus, the instantaneous center describes one curve in space, called the space centrode, and another curve on the slab, called the body centrode (Figure 13). It can be shown that at any instant, these two curves are tangent at C and that as the slab moves; the body centrode appears to roll on the space centrode.

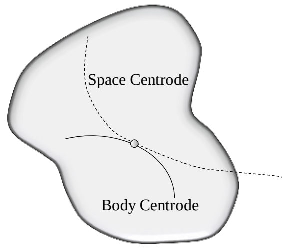


Fig. 13

♦ Rolling Motion without Slipping

A special type of motion of a rigid body is that which occurs in the case of bodies capable of rolling, such as cylinders, spheres, hoops, and wheels in general. Figure 14a shows a circular disk of radius R that is rolling on a horizontal surface with angular velocity ω and angular acceleration α, both clockwise. Observe that the path of the center O is a straight line parallel to the surface. Rolling without slipping occurs if the contact point C on the disk has no velocity; i.e., the disk does not slide along the surface. This case deserves special attention because it occurs in many engineering applications.

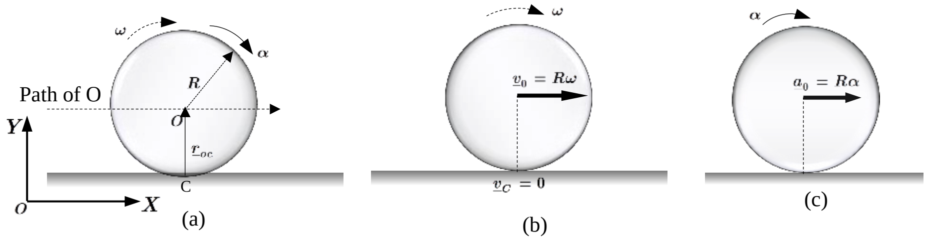


Fig. 14

Applying the relative velocity equation, to point C and O, where O is the center of the disk, we have $\underline{v}_O = \underline{v}_C + \underline{\omega} \wedge \underline{r}_{OC}$

Substituting $\underline{v}_C = \mathbf{0}$, $\underline{\omega} = -\omega\hat{k}$ and $\underline{r}_{OC} = R\hat{j}$

$$\underline{v}_O = -\omega\hat{k} \wedge (R\hat{j}) = R\omega\hat{i}$$

As expected, this result shows that the velocity of the center O is parallel to the surface on which the disk rolls, its magnitude being $v_O = R\omega$ As shown in Figure 14b. It is convenient here to derive the acceleration of O. The acceleration of O can be obtained by differentiation of the equation $\underline{v}_O = -\omega\hat{k} \wedge (R\hat{j}) = R\omega\hat{i}$. Noting that R and $\hat{i}$ are constants, we get

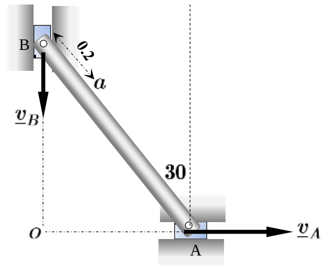
$$\underline{a}_O = R\frac{d\omega}{dt} = R\alpha\hat{i}$$

Thus the acceleration of O is also parallel to the surface of rolling, and its magnitude is $a_O = R\alpha$ as shown in Figure 14c. It should be noted that although the velocity of C is zero, its acceleration is not zero.

■Illustrative Examples■

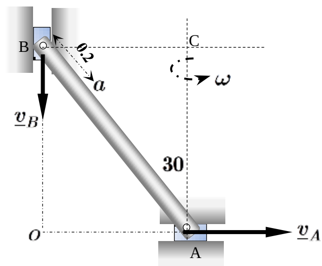
❒ EXAMPLE 1

A rod AB of length $0.8\ \mathrm{m}$ executes general plane motion such that end A is constrained to move along a horizontal track while end B moves along a vertical track, as shown in the figure. If the end A moves to the right with a constant velocity $v_A = 4\ \mathrm{m\,s^{-1}}$, determine the angular velocity ω and linear velocity of end B. Express your answers in terms of the angular position θ measured counterclockwise from the vertical track. Determine the velocity of point a.



❒ SOLUTION

By drawing the perpendicular to $\underline{v}_A$ through A and the perpendicular to $\underline{v}_B$ through B and determining the point in which these two lines intersect (Figure).



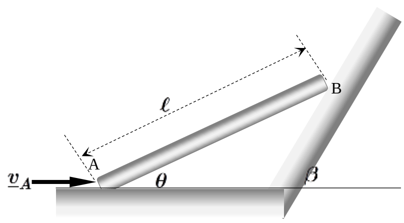
$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta} = \frac{4}{0.8\cos 30} = 5.774\ \mathrm{rad\,s^{-1}}$$

Therefore the linear velocity of end B is

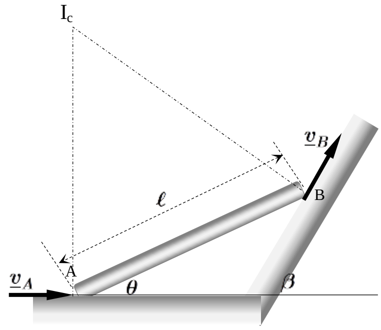
$$\underline{v}_B = \omega(BC) = 5.774(0.8\sin 30) = 2.3094\ \mathrm{m\,s^{-1}}$$

❒ EXAMPLE 2

Rod AB of length ℓ can slide freely along the floor and the inclined plane. Denoting by v_A the velocity of point A, derive an expression for (i) the angular velocity of the rod, (ii) the velocity of end B.



❒ SOLUTION



By drawing the perpendicular to $\underline{v}_A$ through A and the perpendicular to $\underline{v}_B$ through B and determining the point in which these two lines intersect (see Figure).

$$\omega = \frac{v_A}{AI_c} = \frac{v_B}{BI_c}$$

From the sin law (triangle ABI_c) we have

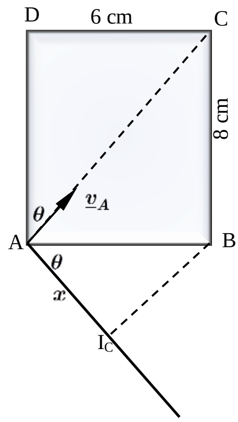
$$\frac{\ell}{\sin\beta} = \frac{AI_c}{\sin(90+\theta-\beta)} = \frac{BI_c}{\sin(90-\theta)} \Rightarrow AI_c = \frac{\cos(\theta-\beta)}{\sin\beta}\ell$$

hence

$$\omega = \frac{v_A}{AI_c} = \frac{v_A \sin\beta}{\ell\cos(\theta-\beta)} \text{ and } v_B = \omega BI_c = \frac{v_A\cos\theta}{\cos(\theta-\beta)}$$

❒ EXAMPLE 3

A uniform rectangular lamina ABCD is moving so that the velocity of A is in the direction of the diagonal Ad, and the magnitude of the velocity of point B is . Determine the angular velocity of the rectangular and the velocity of point D with respect to point B



❒ SOLUTION

Instantaneous center of rotation can be determined by drawing the perpendicular to $\underline{v}_A$ through A and the perpendicular to $\underline{v}_B$ through B and determining the point in which these two lines intersect in I_c as shown. Let I_c lies a distance x from A (since the direction of velocity of B is unknown), since $\underline{v}_B = 4\sqrt{13}$

$$(I_cB)^2 = x^2 + (6)^2 - 2(x)(6)\cos\theta = x^2 + 36 - 9.6x \qquad ......(1)$$

$$\underline{v}_A = 20 = \omega x, \qquad ....(2)$$

$$\text{and} \qquad \underline{v}_B = 4\sqrt{13} = \omega(I_cB) \qquad ....(3)$$

Dividing these two equations (2) and (3)

$$\frac{20}{4\sqrt{13}} = \frac{x}{I_c B} \qquad \text{by squaring}$$

$$\Rightarrow x^2 - 20x + 75 = 0 \qquad \text{Or} \qquad (x-5)(x-15) = 0 \ \text{ i.e. } \ x = 5, \ x = 15$$

The problem has two solutions

$$\omega_1 = \frac{20}{x} = \frac{20}{5} = 4 \ \text{rad s}^{-2} \quad ....(4) \ \text{ and } \quad \omega_2 = \frac{20}{15} = \frac{4}{3} \ \text{rad s}^{-2} \quad .....(5)$$

Therefore the corner D has two solutions

$$(I_c D)^2 = x^2 + (8)^2 + 2(x)(6)\cos\left(\theta + \frac{\pi}{2}\right) = x^2 + 64 + 9.6x$$

So the first solution is

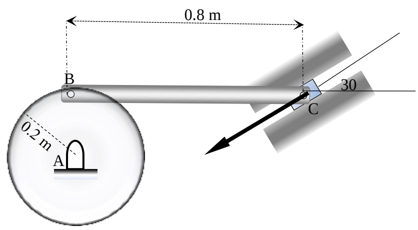
$$\underline{v}_D = \omega_1 (I_c D)\Big|_{x=5} = 4\sqrt{(5)^2 + 64 + 9.6(5)} = 46.8 \ \text{cm s}^{-1}$$

The second one is

$$\underline{v}_D = \omega_2 (I_c D)\Big|_{x=15} = \frac{4}{3}\sqrt{(15)^2 + 64 + 9.6(15)} = 27.74 \ \text{cm s}^{-1}$$

❒ EXAMPLE 4

Determine the angular velocity of link BC and the velocity of the piston C at the instant shown,



❒ SOLUTION

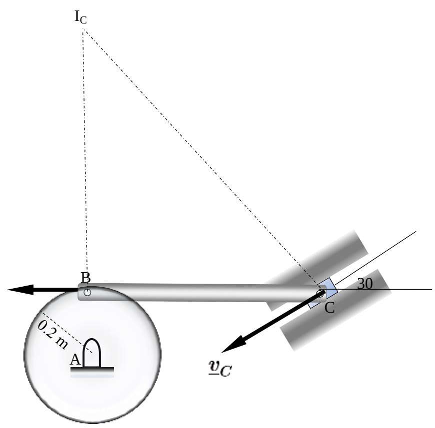
Since $$\omega_{BC} = \frac{v_B}{I_C B} = \frac{v_C}{I_C C} \quad ...(1)$$

and $v_B = \omega r = 6(0.2) = 1.2 \ \text{m s}^{-1}$

From sin law, we have

$$\frac{BI_C}{\sin 60} = \frac{CI_C}{\sin 90} = \frac{0.8}{\sin 30}$$

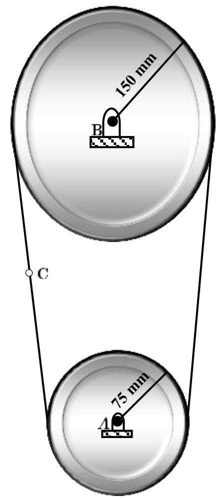
$$\Rightarrow CI_C = 1.6 \ \text{m}, \quad BI_C = 1.38 \ \text{m}$$



Therefore, from equation (1) $v_C (I_C C) = 1.2(1.38)$

$$\Rightarrow v_C = 1.39 \ \text{m s}^{-1} \quad \text{and} \ \omega_{BC} = 0.86 \ \text{rad s}^{-1}$$

❒ EXAMPLE 5



Pulley B is being driven by the motorized pulley A that is rotating at $\omega = 20 \text{ rad s}^{-1}$. At time $t = 0$, the current in the motor is cu off, and friction in the bearings causes the pulleys to coast to a stop. The angular acceleration of a during the deceleration is $\alpha = -2.5t \text{ rad s}^{-1}$, where is in seconds. Assuming that the derive belt does not slip on the pulleys, determine (i) the angular velocity of B as a function of time ; (ii) the angular displacement of B during the period of coasting; and (iii) the acceleration of point C on the straight portion of the belt as a function of time.

❒ SOLUTION

Because the belt does not slip, every point on the belt that is in contact with a pulley has the same velocity as the adjacent point on the pulley. Therefore, the speed of any point on the belt is

$$v = R_A\omega_A = R_B\omega_B \quad ...(1)$$

So that

$$\omega_B = \frac{R_A}{R_B}\omega_A = \frac{75}{150}\omega_A = 0.5\omega_A$$

Differentiating with respect to time, we obtain for the angular acceleration of pulley B $\alpha_B = 0.5\alpha_A = 0.5(-2.5t) = -1.25t \text{ rad s}^{-1}$

Because $\alpha_B = d\omega_B / dt$, we have $d\omega_B = \alpha_B dt$ or

$$\omega_B = \int \omega_B dt = -1.25\int t dt = -0.625t^2 + C_1$$

The initial condition, $\omega = 20 \text{ rad s}^{-1}$ when $t = 0$, yields $C_1 = 20 \text{ rad s}^{-1}$

hence the angular velocity of pulley B is $\omega_B = 20 - 0.625t^2$ ....

We let the angular position of a line in B measured from a fixed reference line. Recalling that $\omega_B = d\theta_B / dt$ we integrate $d\theta_B = \omega_B dt$ to obtain

$$\theta_B = \int \omega_B dt = \int (20 - 0.625t^2)dt = -0.2083t^3 + 20t + C_2$$

Letting $\theta_B = 0$ when $t = 0$, we have $C_2 = 0$, which gives

$$\theta_B = -0.2083t^3 + 20t$$

The pulley comes to rest when $\omega_B = 20 - 0.625t^2 = 0$, which yields $t = 5.627\,\text{s}$. The corresponding angular position of the line in B is

$$\theta_B\big|_{t=5.627} = -0.2083(5.627)^3 + 20(5.627) = 112 \text{ rad}$$

Therefore, the angular displacement of pulley B as it coasts to a stop is

$$\triangle\theta_B = \theta_B\big|_{t=5.627} - \theta_B\big|_{t=0} = 112 - 0 = 112 \text{ rad} \quad .....$$

Because the direction of rotation does not change, the total angle turned through by the pulley B during the deceleration is also 112 rad

Substituting $R_B = 150$ mm and $\omega_B = 20 - 0.625t^2$ into Eq. (1), the speed of point C (which is the same for all points on the belt) is

$$v_C = 150(20 - 0.625t^2) = -93.75t^2 + 3000 \text{ mm s}^{-1}$$

Because the path of point C on the belt is a straight line, the acceleration of C is

$$a_C = \dot{v}_C = -187.5t \text{ mm s}^{-1} \quad .....$$

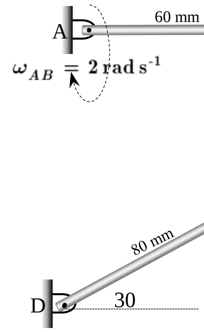
We could obtain the same result by observing that a_C is equal to the tangential component of acceleration of a point on the rim of pulley b (pulley A could also be used). Thus

$$a_C = R_B\alpha_B = 150(-1.25t) = -187.5t \text{ mm s}^{-1}$$

❒ EXAMPLE 6

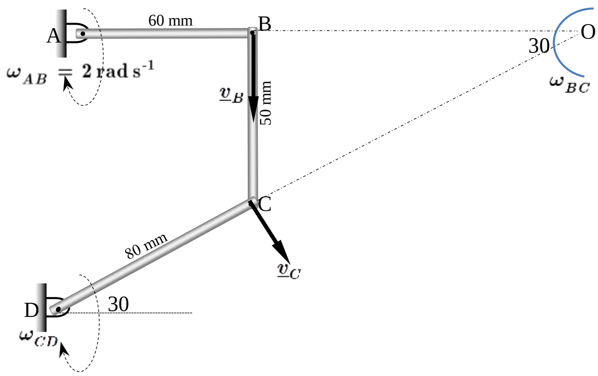
When the linkage in Figure is in the position shown, the angular velocity of bar AB is $\omega_{AB} = 2 \text{ rad s}^{-1}$ clockwise.

For this position, determine the angular velocities of bars BC and CD and the velocity of C using the instant centers for velocities



❒ SOLUTION

Because A and D are fixed points, they are the instant centers for bars AB and CD respectively. The instant center for the bar BC, labeled O in the figure, is located at the point of intersection of the lines that are perpendicular to the velocity vectors of B and C.



Because $\underline{v}_B$ and $\underline{v}_C$ are perpendicular to AB and CD, respectively, the instant center is at the intersection of these two lines

The distnace to B and C from O, found from the triangle OBC, are

$$r_{BO} = \frac{50}{\tan 30} = 86.6 \text{ mm}$$

$$r_{CO} = \frac{50}{\sin 30} = 100 \text{ mm}$$

The instant centers, A, O and D, can now be used to compute the required angular velocities directly from the figure. Considering the motion of AB (the instant center is at A), we find that $v_B = r_{BA}\omega_{AB} = 60(2) = 120 \text{ mm s}^{-1}$, directed as shown in figure. Analyzing the motion of BC (the instant center is at O) yields

$$\omega_{BC} = \frac{v_B}{r_{BO}} = \frac{120}{86.6} = 1.386 \text{ rad s}^{-1} \quad .....$$

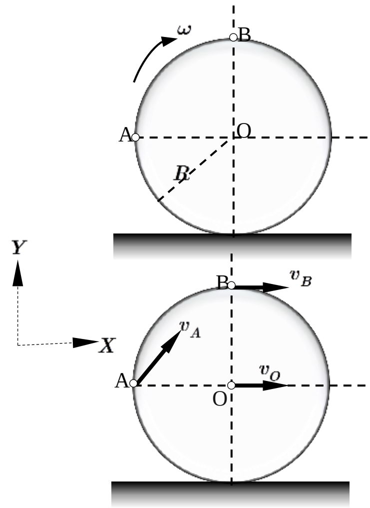
And $$v_C = r_{CO}\omega_{BC} = 100(1.386) = 138.6\,\mathrm{mm\,s^{-1}} \quad .....$$

❒ EXAMPLE 7

Figure beside shows a wheel of radius R that is rolling without slipping with the clockwise angular velocity ω. For the position shown, determine the velocity vectors for (i) point A; and (ii) point B

❒ SOLUTION

We choose the point O (the center of the wheel) as the reference point, because its velocity is known ($\underline{v}_O = Rw\,\hat{i}$)



The velocity $\underline{v}_{AO}$ is computed by assuming that the point O is fixed. Therefore the equation becomes

$$\underline{v}_A = \underline{v}_O + \underline{v}_{AO} \Rightarrow \underline{v}_A = R\omega\,\hat{i} + R\omega\,\hat{j}$$

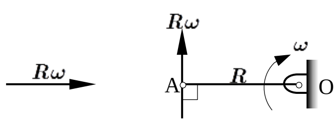
from which the velocity of A is found to be

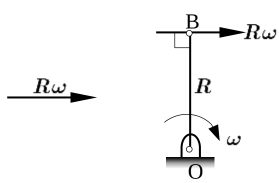
$$v_A = \sqrt{2}R\omega$$

Again for point B

$$\underline{v}_B = \underline{v}_O + \underline{v}_{BO} \Rightarrow \underline{v}_B = R\omega\,\hat{i} + R\omega\,\hat{i}$$

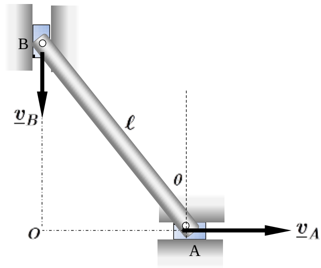
from which the velocity of A is found to be $\quad v_A = 2R\omega$



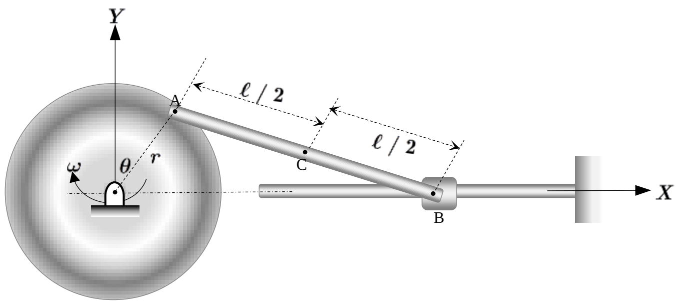


PROBLEMS

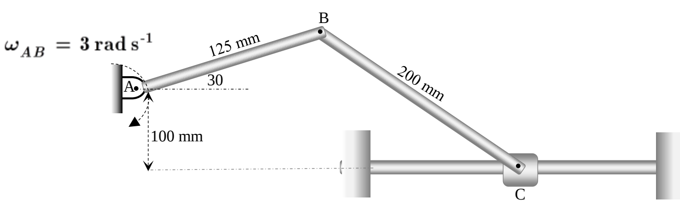
❒ Member AB of length ℓ executes general plane motion such that end A is constrained to move along a horizontal track while end B moves along a vertical track, as shown in the figure. If member AB has a constant counterclockwise angular velocity ω, determine in terms of ℓ, ω and θ, the velocity v_A and the acceleration a_A of end A. If $\omega = 4\ \mathbf{rad\,s^{-1}}$, $\theta = 45^{\circ}$, and $\ell = 15\ \mathbf{in}$, determine v_A and a_A.



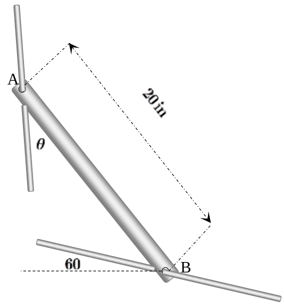
❒ If the wheel rotates clockwise at a constant angular velocity ω, determine the velocity v_B and acceleration a_B of point B in terms of θ, ω, and r. If $r = 1.5\ \mathbf{ft}$ $\omega = 500\,\mathbf{rpm}\ \ (\mathbf{rev\,m^{-1}})$, and $\ell = 5\ \mathbf{ft}$, determine v_c and a_c for $\theta = 30^{\circ}$



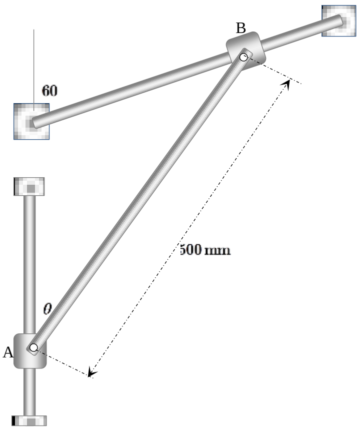
❒ The angualr velocity of bar AB in figure is 3 rad s^{-1} clockwise in the position shown. Determine the angular velocity of bar BC and the velocity of the slider C in this position.



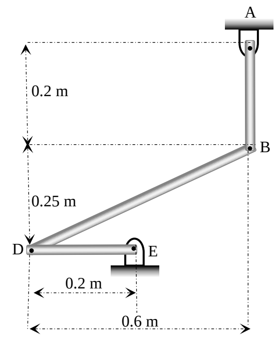
❒ The motion of rod AB is guided by pins attached at A and B which slide in the slots shown. At the instant shown, $\boldsymbol{\theta = 45^{o}}$ and the pin at B moves upward to the left with a constant velocity of $\mathbf{6\ in\,s^{-1}}$. Determine (a) the angular velocity of the rod, (b) the velocity of the pin at end A.



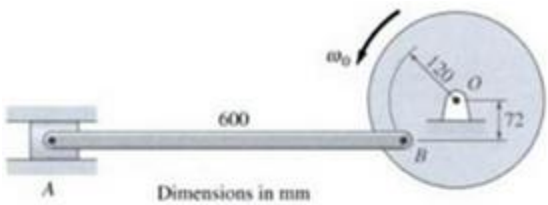
❒ Collar A moves upward with a constant velocity of $\mathbf{1.2\ m\,s^{-1}}$. At the instant shown when $\boldsymbol{\theta = 30^{o}}$, determine (a) the angular velocity of rod AB, (b) the velocity of collar B.



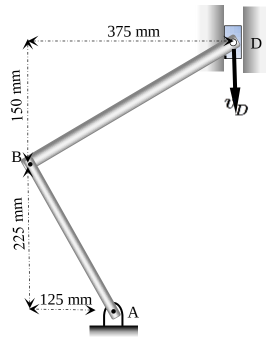
❒ Knowing that at the instant shown the angular velocity of rod AB is 15 rad/s clockwise, determine (a) the angular velocity of rod BD, (b) the velocity of the midpoint of rod BD.



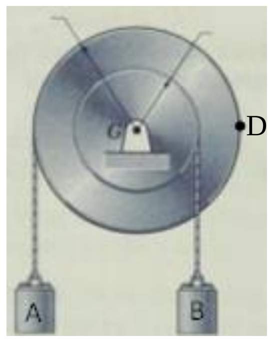
❒ The flywheel rotates counterclockwise about O with the constant angular velocity ω_0. When $\boldsymbol{\theta = 0}$, the speed of the piston A is determine ω_0.



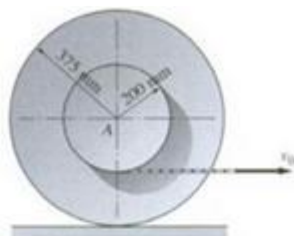
❒ When the mechanism is in the position shown, the velocity of slider D is $v_D = \mathbf{1.25\ m\,s^{-1}}$. Determine the angular velocities of bars AB and BD at this instant



❒ A pulley and two loads are connected by inextensible cords as shown. Load A has a constant acceleration of $\mathbf{300\ mm\,s^{-2}}$ and an initial velocity of $\mathbf{240\ mm\,s^{-1}}$, both directed upward. Determine (i) the number of revolutions executed by the pulley in $\mathbf{3\ s}$, (ii) the velocity and position of load B after $\mathbf{3\ s}$, (iii) the acceleration of point D on the rim of the pulley at $t = \mathbf{0}$.

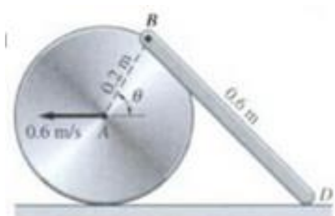


❒ The end of the cord that is wrapped around the hub of the wheel is pulled to the right with the velocity $\mathbf{700\ mm\,s^{-1}}$. Find the angular velocity of the wheel, assuming no slipping.



Dimensions in mm

❒ The wheel is rolling without slipping. Its center has a constant velocity of $\mathbf{0.6\ m\,s^{-1}}$ to the left. Compute the angular velocity of bar BD and the velocity of point D when $\theta = \mathbf{0}$.



KINETICS OF A RIGID BODY

In this chapter we will study the kinetics of rigid bodies, i.e., the relations existing between the forces acting on a rigid body, the shape and mass of the body, and the motion produced. We studied similar relations, assuming then that the body could be considered as a particle, i.e., that its mass could be concentrated in one point and that all forces acted at that point. The shape of the body, as well as the exact location of the points of application of the forces, will now be taken into account.

We will also be concerned not only with the motion of the body as a whole but also with the motion of the body about its mass center. Our approach will be to consider rigid bodies as made of large numbers of particles.

♦ Reduction a System of Forces

When a number of forces and couple moments are acting on a body, it is easier to understand their overall effect on the body if they are combined into a single force and couple moment having the same external effect. The two force and couple systems are called equivalent systems since they have the same external effect on the body Figure 1.

Suppose a system of forces $\underline{F}_1, \underline{F}_2, ..., \underline{F}_i, ..., \underline{F}_n$ is reduced at a chosen point $\mathbf{O}$ to a single force $\underline{F}$ and a single couple $\underline{M}$ viz. the obtaining result is $\underline{M}_o, \underline{F}$ where

$$\underline{M}_o = \sum_{i=1}^{n} \underline{r}_i \wedge \underline{F}_i, \quad \underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

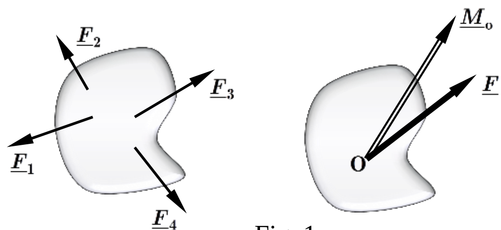


Fig. 1

♦ Angular Momentum of a Rigid Body

Consider a rigid slab in plane motion. Assuming that the slab is made of a large number n of particles P_i of mass Δm_i, we note that the angular momentum $\mathbf{H}_G$ of the slab about its mass center G can be computed by taking the moments about G of the momenta of the particles of the slab in their motion with respect to either of the frames Oxy or $Gx'y'$ (Figure 2).

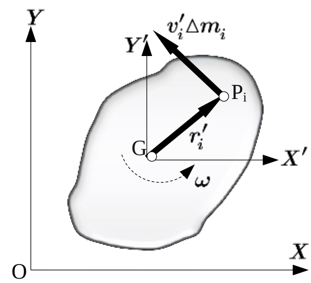


Fig. 2

Choosing the latter course, we write

$$H_G = \sum_{i=1}^{n} (\underline{r}_i' \wedge \underline{v}_i' \Delta m_i)$$

where $\underline{r}_i'$ and $\underline{v}_i' \Delta m_i$ denote, respectively, the position vector and the linear momentum of the particle P_i relative to the centroidal frame of reference $Gx'y'$. But since the particle belongs to the slab, we have $\underline{v}_i' = \underline{\omega} \wedge \underline{r}_i'$, where $\underline{\omega}$ is the angular velocity of the slab at the instant considered. We write

$$H_G = \sum_{i=1}^{n} (\underline{r}_i' \wedge (\underline{\omega} \wedge \underline{r}_i') \Delta m_i)$$

Referring to Figure 6, we easily verify that the expression obtained represents a vector of the same direction as $\underline{\omega}$ (that is, perpendicular to the slab) and of magnitude equal to $\omega \sum \underline{r}_i'^2 \Delta m_i$. Recalling that the sum $\sum \underline{r}_i'^2 \Delta m_i$ represents the moment of inertia I of the slab about a centroidal axis perpendicular to the slab, we conclude that the angular momentum $\mathbf{H}_G$ of the slab about its mass center is

$$H_G = I\underline{\omega}$$

Differentiating both members of previous equation we obtain

$$\dot{H}_G = I\dot{\underline{\omega}} = I\underline{\alpha}$$

Thus the rate of change of the angular momentum of the slab is represented by a vector of the same direction as $\underline{\alpha}$ (that is, perpendicular to the slab) and of magnitude $I\alpha$.

It should be kept in mind that the results obtained in this section have been derived for a rigid slab in plane motion. They remain valid in the case of the plane motion of rigid bodies which are symmetrical with respect to the reference plane. However, they do not apply in the case of nonsymmetrical bodies or in the case of three-dimensional motion.

♦ Plane Motion of a Rigid Body D'Alembert's Principle

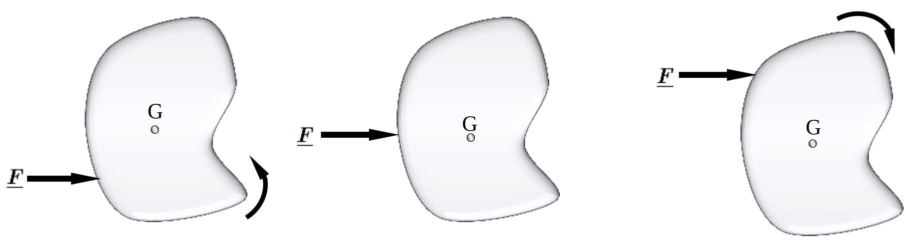
Consider a rigid slab of mass m moving under the action of several external forces $\underline{F}_1, \underline{F}_2, \underline{F}_3, ...$ contained in the plane of the slab (Figure 1). Substituting for H_G from equation $\dot{H}_G = I\underline{\alpha}$ into equation $\dot{H}_G = I\dot{\underline{\omega}} = I\underline{\alpha}$ and writing the fundamental equations of motion in scalar form, we have

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z, \quad \sum M_g = I\alpha$$

These equations show that the acceleration of the mass center G of the slab and its angular acceleration $\underline{\alpha}$ are easily obtained once the resultant of the external forces acting on the slab and their moment resultant about G have been determined. Note that the equations

$$\sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z$$

do not give any information about the rotation of the body, as shown in the figure, although the same force acts on the body but the body has different effect.



■ Constrained Plane

Most engineering applications deal with rigid bodies which are moving under given constraints. *For example*, cranks must rotate about a fixed axis, wheels must roll without sliding, and connecting rods must describe certain prescribed motions. In all such cases, definite relations exist between the components of the acceleration $\underline{a}$ of the mass center G of the body considered and its angular acceleration $\underline{\alpha}$; the corresponding motion is said to be a constrained motion.

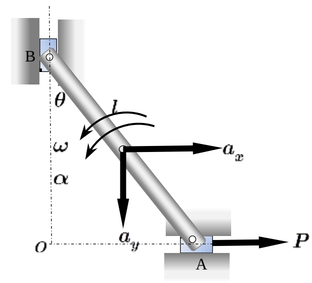


Fig. 3

The solution of a problem involving a constrained plane motion calls first for a kinematic analysis of the problem. Consider, for example, a slender rod AB of length l and mass m whose extremities are connected to blocks of negligible mass which slide along horizontal and vertical frictionless tracks. The rod is pulled by a force P applied at A as shown in Figure 3. We know that the acceleration a of the mass center G of the rod can be determined at any given instant from the position of the rod, its angular velocity, and its angular acceleration at that instant. Suppose, for example, that the values of θ, ω and α are known at a given instant and that we wish to determine the corresponding value of the force P, as well as the reactions at A and B. We should first determine the components a_x and a_y of the acceleration of the mass center G. We next apply D'Alembert's principle as plotted in Figure 4a, using the expressions obtained for a_x and a_y. The unknown forces P, N_A, and N_B can then be determined by writing and solving the appropriate equations.

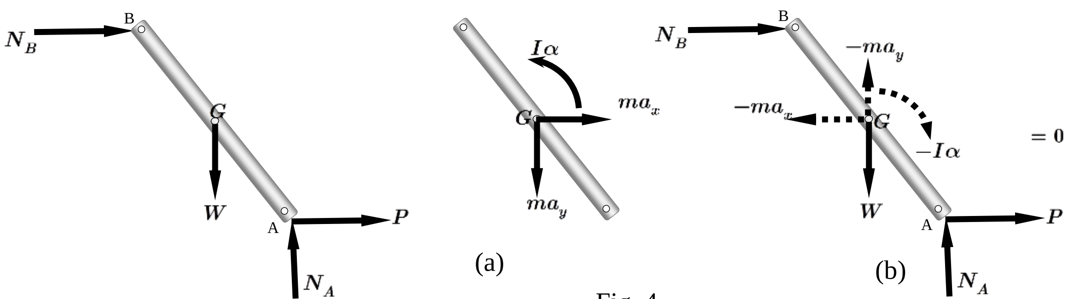


Fig. 4

Suppose now that the applied force P, the angle θ, and the angular velocity v of the rod are known at a given instant and that we wish to find the angular acceleration α of the rod and the components a_x and a_y of the acceleration of its mass center at that instant, as well as the reactions at A and B. The preliminary kinematic study of the problem will have for its object to express the components a_x and a_y of the acceleration of G in terms of the angular acceleration α of the rod. This will be done by first expressing the acceleration of a suitable reference point such as A in terms of the angular acceleration α. The components a_x and a_y of the acceleration of G can then be determined in terms of a, and the expressions obtained carried into Figure 4a. Three equations can then be derived in terms of α, N_A, and N_B and solved for the three unknowns. Note that the method of dynamic equilibrium can also be used to carry out the solution of the two types of problems we have considered (Figure 4b). When a mechanism consists of several moving parts, the approach just described can be used with each part of the mechanism. The procedure required to determine the various unknowns is then similar to the procedure followed in the case of the equilibrium of a system of connected rigid bodies. Earlier, we analyzed two particular cases of constrained plane motion: the translation of a rigid body, in which the angular acceleration of the body is constrained to be zero, and the centroidal rotation, in which the acceleration $\underline{a}$ of the mass center of the body is constrained to be zero. Two other particular cases of constrained plane motion are of special interest: the noncentroidal rotation of a rigid body and the rolling motion of a disk or wheel. These two cases can be analyzed by one of the general methods described above. However, in view of the range of their applications, they deserve a few special comments.

◇ **Non-Centroidal Rotation**, The motion of a rigid body constrained to rotate about a fixed axis which does not pass through its mass center is called noncentroidal rotation. The mass center G of the body moves along a circle of radius r centered at the point O, where the axis of rotation intersects the plane of reference as illustrated in Figure 5. Denoting, respectively, by $\underset{\sim}{\omega}$ and $\underset{\sim}{\alpha}$ the angular velocity and the angular acceleration of the line OG, we obtain the following expressions for the tangential and normal components of the acceleration of G:

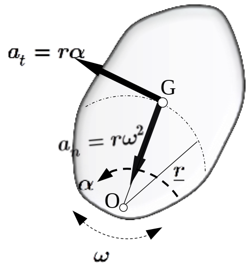
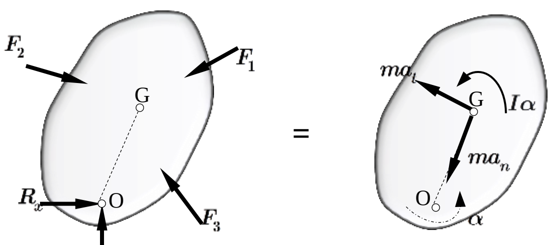


Fig. 5

$$a_t = r\alpha, \qquad \textbf{and} \qquad a_n = r\omega^2 \qquad \textbf{....(1)}$$

Since line OG belongs to the body, its angular velocity ω and its angular acceleration α also represent the angular velocity and the angular acceleration of the body in its motion relative to G. Equations (1) define, therefore, the kinematic relation existing between the motion of the mass center G and the motion of the body about G. They should be used to eliminate a_t and a_n from the equations obtained by applying d'Alembert's principle (Figure 6) or the method of dynamic equilibrium as shown in Figure 7.

An interesting relation is obtained by equating the moments about the fixed point O of the forces and vectors shown, respectively, in parts *a* and *b* of Figure 6. We write $\sum M_o = I\alpha + (Mr\alpha)r = (I + Mr^2)\alpha$



(a) Fig. 6 (b)

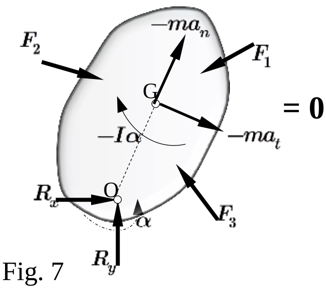


Fig. 7

But according to the parallel-axis theorem, we have, $I + Mr^2 = I_o$ where I_o denotes the moment of inertia of the rigid body about the fixed axis. We therefore write

$$\sum M_o = I_o\alpha \qquad .....(2)$$

Although formula (2) expresses an important relation between the sum of the moments of the external forces about the fixed point O and the product $I_o\alpha$, it should be clearly understood that this formula does not mean that the system of the external forces is equivalent to a couple of moment $I_o\alpha$. The system of the effective forces, and thus the system of the external forces, reduces to a couple only when O coincides with G—that is, only when the rotation is centroidal. In the more general case of noncentroidal rotation, the system of the external forces does not reduce to a couple.

A particular case of noncentroidal rotation is of special interest the case of uniform rotation, in which the angular velocity $\underline{\omega}$ is constant. Since $\underline{\alpha}$ is zero, the inertia couple in Figure (7) vanishes and the inertia vector reduces to its normal component. This component (also called centrifugal force) represents the tendency of the rigid body to break away from the axis of rotation.

■ Remember That

Plane motion of a rigid body: The problems that you will be asked to solve will fall into one of the following categories.

i. Rigid body in translation. For a body in translation, the angular acceleration is zero. The effective forces reduce to the vector $m\underline{a}$ applied at the mass center.

ii. Rigid body in centroidal rotation. For a body in centroidal rotation, the acceleration of the mass center is zero. The effective forces reduce to the couple $I\underline{\alpha}$.

iii. Rigid body in general plane motion. You can consider the general plane motion of a rigid body as the sum of a translation and a centroidal rotation. The effective forces are equivalent to the vector $m\underline{a}$ and the couple $I\underline{\alpha}$.

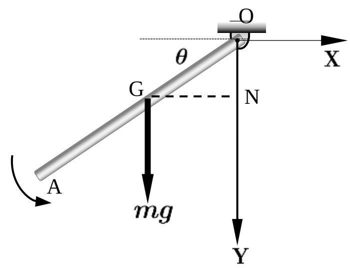
■Illustrative Examples■

❒ EXAMPLE 1

A thin uniform rod of length L and mass *m* hangs freely from a hinge at A. If it is allowed to fall with initial velocity equals zero from a horizontal position. Determine the maximum value of angular acceleration of its center of gravity and find the associated tangential acceleration a_t of the end of the rod.

❒ SOLUTION

Let the rod OA at any instant t makes an angle θ with the initial horizontal position OX. Let G be the center of gravity and GN perpendicular to OY.



The angular formula of Newton's second law we get

$$M_o = I_o \alpha$$

Taking the momentum about the point O

$$M_o = mg\left(\frac{1}{2}L\cos\theta\right)$$

Since the moment of inertia of the rod about any of its ends is $I_o = \frac{1}{3}mL^2$

therefore,

$$mg\left(\frac{1}{2}L\cos\theta\right) = \frac{1}{3}mL^2\alpha \qquad \Rightarrow \alpha = \frac{3g}{2L}\cos\theta$$

That is the angular acceleration is a function of θ hence the maximum value is obtained when $\theta = 0$ and its value is $\alpha_{\max} = \dfrac{3g}{2L}$

Therefore the associated tangential acceleration a_t of the end of the rod is

$$a_t = L\alpha_{\max} = \frac{3}{2}g$$

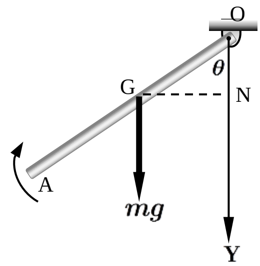
❒ EXAMPLE 2

A straight uniform rod of length *a* and mass *m* can turn freely about one end O, hangs from O vertically. Find the least angular velocity with which it must begin to move so that it may perform complete revolution in a vertical plane.

❒ SOLUTION

Let the rod OA at any instant t makes an angle θ with the initial vertical position OY. Let G be the center of gravity and GN perpendicular to OY. The equation of angular motion of the rod is

$$\frac{1}{3}ma^2\ddot{\theta} = -mg\left(\frac{a}{2}\right)\sin\theta$$



Since the moment of effective forces about O = $\frac{1}{3}ma^2\ddot{\theta}$

And moment of external forces about O = $-mg\left(\frac{a}{2}\right)\sin\theta$

Thus $$2a\ddot{\theta} = -3g\sin\theta$$

Multiply the above Equation by $\dot{\theta}$ and integrating we get

$a\dot{\theta}^2 = 3g\cos\theta + c$, c is integration constant.

Let $\dot{\theta} = \omega$ when $\theta = 0$ we have, $c = a\omega^2 - 3g$

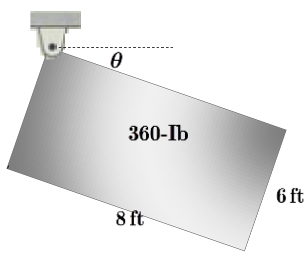
Therefore, $a\dot{\theta}^2 = a\omega^2 - 3g(1-\cos\theta)$

We require that $\dot{\theta} = 0$ when $\theta = \pi$ to complete revolution, hence

$$0 = a\omega^2 - 3g(1-\cos\pi) \quad \Rightarrow \omega = \sqrt{6g/a}$$

❒ EXAMPLE 3

The 360-Ib uniform plate shown in the figure rotates in the vertical plane about a smooth pin at A. The plate is released from rest when $\theta = 0$. (i) Show that the differential equation of motion for the plate is $\ddot{\theta} = 0.996(4\cos\theta - 3\sin\theta)$ rad s^{-2}



(ii) Integrate the differential equation of motion analytically to obtain the angular velocity of the plate as a function of θ

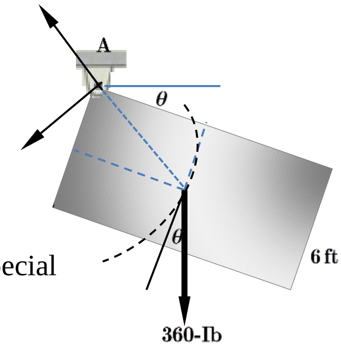
(iii) Find the maximum value of θ.

❒ SOLUTION

The mass of the plate is $m = 360/32.2 = 11.18$ slugs and the moment of inertia about its mass center G is

$$I = \frac{1}{12}(11.18)(8^2 + 6^2) = 93.17 \text{ slug ft}^2$$

The figure behind shows the free body diagram (FBD) and mass acceleration diagram for an arbitrary position of the plate. The FBD contains A_n and A_t, the unknown components of the pin reaction at A. Since the path of G is a circle of radius $r = 5$ ft centered at A, the normal components of $\underline{a}$ is $a_n = r\dot{\theta}^2 = 5\dot{\theta}^2$, and its tangential component is $a_t = r\ddot{\theta} = 5\ddot{\theta}$. Observe that the angular acceleration $\ddot{\theta}$ is assumed to be clockwise. Now, by taking the momentum about A we have



$$(360\cos\theta)4 - (360\sin\theta)(3) = 93.17\ddot{\theta} + 11.18(5\ddot{\theta})5$$

which reduces to

$$\ddot{\theta} = 0.966(4\cos\theta - 3\sin\theta) \text{ rad s}^{-2}$$

The identical result could be obtained by using the special case of momentum equation

$$\sum M_A = I_A\ddot{\theta}$$

$$(360\cos\theta)4 - (360\sin\theta)(3) = (93.17 + 11.18(5)^2)\ddot{\theta}$$

To find the angular velocity as a function of θ, we use the formula $\ddot{\theta} = \dot{\theta}\dfrac{d\dot{\theta}}{d\theta}$ hence

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = 0.966(4\cos\theta - 3\sin\theta) \text{ rad s}^{-2}$$
$$\Rightarrow \dot{\theta}d\dot{\theta} = 0.966(4\cos\theta - 3\sin\theta)\,d\theta$$

The result of integrating this equation analytically is

$$\frac{1}{2}\dot{\theta}^2 = 0.966(4\sin\theta + 3\cos\theta) + c$$

The constant of integration c is evaluated by applying the initial condition $\dot{\theta} = 0$ when $\theta = 0$, which gives $c = -3(0.966)\text{ rad}^2\text{ s}^{-2}$. Therefore, the last equation becomes

$$\frac{1}{2}\dot{\theta}^2 = 0.966(4\sin\theta + 3\cos\theta - 3)$$

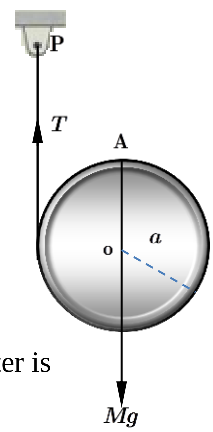
From which the angular velocity is found to be

❒ EXAMPLE 4

One end of a light string is fixed to a point of the rim of a uniform circular disk of radius a and mass M and the string is wounded several times around the rim. The free end is attached to a fixed point and the disk is held so that the part of the string not in contact with it is vertical. If the disk be let go, find the acceleration and tension of the string.

❒ SOLUTION

Let the free end be attached to the fixed point P and let A be the initial position of the center of gravity G. let T be the tension of the string, there being no horizontal force the center of gravity will move vertically downwards. Let y be the distance moved by G in time and during this period, θ be the angle turned through some radius.



$$\because y = a\theta \quad \Rightarrow \dot{y} = a\dot{\theta}, \quad \ddot{y} = a\ddot{\theta}$$

The equation of motion of the center of gravity of circular disk is

$$M\ddot{y} = Mg - T \qquad ....(1)$$

The equation of angular motion about the center of the disk is

$$M_o = I_o\ddot{\theta}$$

The moment of inertia of the disk about an axis passing through its center is

$$I_o = \frac{1}{2}Ma^2$$

Taking the momentum about the center of the disk

$$M_o = Ta$$

Therefore, from the last three equations we have

$$\frac{1}{2}Ma^2\ddot{\theta} = Ta \qquad \Rightarrow T = \frac{1}{2}M\underbrace{a\ddot{\theta}}_{\ddot{y}} = \frac{1}{2}M\ddot{y}$$

Substituting this value of tension in Equation (1) we get

$$M\ddot{y} = Mg - \frac{1}{2}M\ddot{y} \qquad \Rightarrow \ddot{y} = \frac{2}{3}g$$

In which gives the vertical acceleration of the center of the disk and to obtain the tension, substituting this value in Eq. (1)

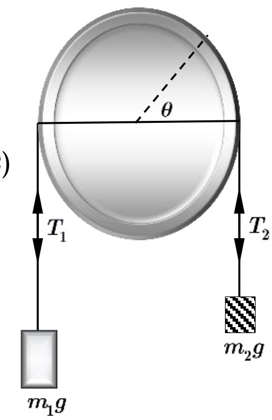
$$T = \frac{1}{2}M\ddot{y} = \frac{1}{3}Mg$$

❒ EXAMPLE 5

Two equal masses m_1 and m_2 $(m_1 > m_2)$ are suspended by a light string over a circular pulley of mass M and radius b. There is no slipping and the friction of the axis can be neglected. If a be the acceleration; show that this is constant, and if k^2 be the radius of gyration of the pulley about the axle, show that $k^2 = \dfrac{b^2}{ma}((g-a)m_2 - (g+a)m_2)$

❒ SOLUTION

Let in time t, m_1 moves a distance y downwards and m_2 moves distance upwards. Let θ be the angle through which the pulley has rotated in time t. Since $y = b\theta \Rightarrow \ddot{y} = b\ddot{\theta}$



Equations of motion of m_1 and m_2 are

$$m_1\ddot{y} = m_1g - T_1 \quad ...(1) \qquad \text{and} \qquad m_2\ddot{y} = T_2 - m_2g \quad ....(2)$$

Equation of motion of the pulley is

$$M_o = I_o\ddot{\theta} \quad \Rightarrow \frac{1}{2}Mb^2\ddot{\theta} = T_1b - T_2b$$

$$\Rightarrow \frac{1}{2}Mb^2\frac{\ddot{x}}{b^2} = T_1 - T_2 \quad \Rightarrow \frac{1}{2}M\ddot{x} = T_1 - T_2 \quad .....(3)$$

Adding three Equations (1), (2) and (3) , we get

$$\left(m_1 + m_2 + \frac{1}{2}M\right)\ddot{x} = (m_1 - m_2)g \quad \Rightarrow \ddot{x} = a = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{1}{2}M\right)}$$

which is constant. From above we get $\left(m_1 + m_2 + \frac{1}{2}M\right)a = (m_1 - m_2)g$

Since the radius of gyration of a circular disk (pulley) about its center is

$$k^2 = \frac{I_o}{A} = \frac{1}{2}\frac{Ab^2}{A} = \frac{1}{2}b^2$$

$$\left(m_1 + m_2 + \frac{k^2}{b^2}M\right)a = (m_1 - m_2)g$$

$$\Rightarrow k^2 = \frac{b^2}{Ma} \ \ (g-a)m_1 - (g+a)m_2$$

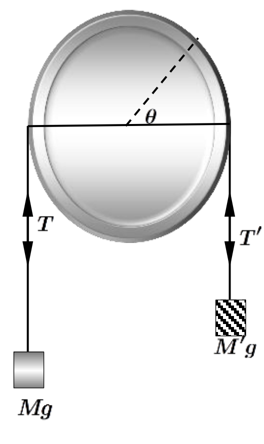
Again by subtracting Equation (1) from Equation (2), we get

$$\begin{aligned} m_2 - m_1 \ \ \ddot{x} &= T_2 + T_1 - (m_1 + m_2)g \\ \Rightarrow T_2 + T_1 &= \ m_2 - m_1 \ \ \ddot{x} + (m_1 + m_2)g \\ &= \ m_2 - m_1 \ \ a + (m_1 + m_2)g \end{aligned}$$

❒ EXAMPLE 6

A fine sting has two masses M and M' tied to its ends and passes over a rough pulley; of mass m and radius a whose center is fixed. If the string does not slip over the pulley, show that M will descend with acceleration $\ M - M' \ \ g/ \ \ 2M + 2M' + m \ $. If the pulley be not sufficiently rough to prevent sliding, and be the descending mass, show that its acceleration is $\ M - M'e^{\mu\pi} \ \ / \ \ M + M'e^{\mu\pi} \ $ and find the angular acceleration of the pulley.

❒ SOLUTION



First part, when the pulley is rough enough to prevent sliding. Proceeding like previous examples the equations of motion of masses and pulley are

$$M\ddot{x} = Mg - T \qquad \text{...(1)}$$

and $$M'\ddot{x} = T' - M'g \qquad \text{....(2)}$$

And taking moment of effective forces about the center of pulley, we get

$$\frac{1}{2}ma^2\ddot{\theta} = (T - T')a \qquad ...(3)$$

Again $x = a\theta$, $\Rightarrow \ddot{x} = a\ddot{\theta}$ therefore Equation (3) turns into

$$\frac{1}{2}m\ddot{x} = T - T' \qquad ...(4)$$

Adding Equations (1), (2) and (4), we have

$$\ddot{x}\left(M + M' + \frac{1}{2}m\right) = (M - M')g$$
$$\Rightarrow \ddot{x} = \frac{2(M - M')g}{2M + 2M' + m}$$

Second part, when the pulley is not sufficiently rough to prevent sliding, then we cannot take $x = a\theta$. In this case, from Statics, we have

$$T = T'e^{\mu\pi} \qquad ....(5)$$

Solving Equations (1), (2) and (5), we have

$$T = \frac{2MM'ge^{\mu\pi}}{M + M'e^{\mu\pi}}, \quad T' = \frac{2MM'g}{M + M'e^{\mu\pi}}, \quad \text{and} \quad \ddot{x} = \frac{M - M'ge^{\mu\pi}}{M + M'e^{\mu\pi}}g$$

Further putting above values of T and T' in Equation (3), we get

$$\ddot{\theta} = \frac{4ag \;\; e^{\mu\pi} - 1}{ma^2} \cdot \frac{MM'}{M + M'e^{\mu\pi}}$$

❒ EXAMPLE 7

Two unequal masses, M and M' rest on two rough planes inclined at an angles α and β to the horizon; they are connected by a fine string passing over a small pulley, of mass m and radius a, which is placed at the common vertex of the two planes; determine the acceleration of either mass. Where μ and μ' are the coefficients of friction, M is the mass which moves downwards.

❒ SOLUTION

Suppose that in time t, the mass M moves a distance x downwards. Also M' moves a distance x upwards. Let the pulley turn through an angle θ, in the same time t.

$$\therefore x = a\theta \Rightarrow \dot{x} = a\dot{\theta}, \quad \ddot{x} = a\ddot{\theta}$$

The equation of motion of the masses and are

$$M\ddot{x} = Mg\sin\alpha - Mg\mu\cos\alpha - T \qquad \text{...(1)}$$

$$M'\ddot{x} = T' - M'g\sin\beta - M'g\mu'\cos\beta \qquad \text{...(2)}$$

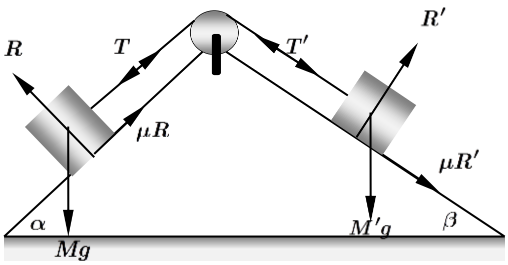
Equation of motion of pulley is

$$\frac{1}{3}ma^2\ddot{\theta} = (T - T')a \qquad \Rightarrow \frac{1}{3}m\ddot{x} = T - T' \qquad \text{...(3)}$$

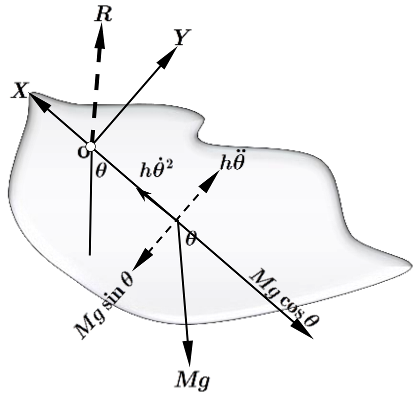
By adding the three Equations (1), (2) and (3), we have

$$\left(\frac{1}{3}m + M + M'\right)\ddot{x} = g\ \ M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta - \mu'\cos\beta)$$

$$\Rightarrow \ddot{x} = \frac{3g\ \ M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta - \mu'\cos\beta)}{m + 3M + 3M'}$$



♦ Reactions of the Axis Rotation



A body moves about a fixed axis under the action of forces and both the body and the forces are symmetrical with respect to the plane through the center of gravity perpendicular to the axis, find the reaction of the axis of rotation .

Let O be the point where the plane through G perpendicular to the axis of rotation meets this axis. By symmetry the actions on the axis reduce to a single force at O, the center of suspension. Let the components of this single force be X and Y along and perpendicular to GO respectively. Now G describes a circle around O as center, its acceleration along and perpendicular to GO are $h\dot{\theta}^2$ and $h\ddot{\theta}$. Equations of motion of center of gravity are

$$Mh\dot{\theta}^2 = X - Mg\cos\theta \qquad \text{....}(1)$$

$$Mh\ddot{\theta} = Y - Mg\sin\theta \qquad \text{....}(2)$$

By taking moments about O, $I_o\ddot{\theta} = Mk^2\ddot{\theta} = -Mgh \quad \text{....}(3)$ where k is the radius of gyration about the axis. Y is obtained by eliminating $\ddot{\theta}$ from Equations (2) and (3) by integrating Equation (3) and determining the constant from the initial conditions, and then from equation (1) we can find X. Resultant reaction $R = \sqrt{X^2 + Y^2}$ and $\tan\varphi = (Y/X)$ where φ is the angle which the direction of R makes with GO. Note that on resolving X and Y horizontally and vertically.

The horizontal reaction = $X\sin\theta - Y\cos\theta$

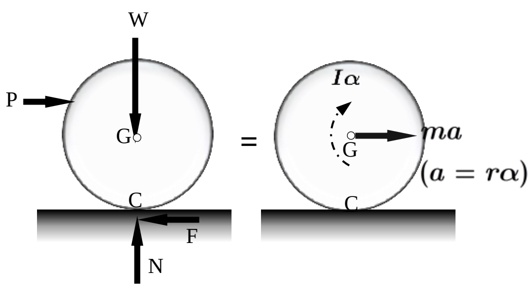
and vertical reaction = $X\cos\theta + Y\sin\theta$

♦ Rolling Motion

Another important case of plane motion is the motion of a disk or wheel rolling on a plane surface. If the disk is constrained to roll without sliding, the acceleration $\underline{a}$ of its mass center G and its angular acceleration $\ddot{\theta}$ are not independent. Assuming that the disk is balanced, so that its mass center and its geometric center coincide, we first write that the distance x traveled by G during a rotation θ of the disk is $x = r\theta$, where r represents the radius of the disk. Differentiating this relation twice, we write $\ddot{x} = r\ddot{\theta}(r\alpha)$

Recalling that the system of the effective forces in plane motion reduces to a vector $m\underline{a}$ and a couple $I\ddot{\theta}$, we find that in the particular case of the rolling motion of a balanced disk, the effective forces reduce to a vector of magnitude $mr\ddot{\theta}$ attached at G and to a couple of magnitude $I\ddot{\theta}$. We may thus express that the external forces are equivalent to the vector and couple shown in Figure.

When a disk rolls without sliding, there is no relative motion between the point of the disk in contact with the ground and the ground itself. Thus, as far as the computation of the friction force F is concerned, a rolling disk can be compared with a block at rest on surface. The magnitude F of the friction force can have any value, as long as this value does not exceed the maximum value $F = \mu N$, where μ represents the coefficient of static friction and N is the magnitude of the normal reaction force. In the case of a rolling disk, the magnitude F of the friction force should therefore be determined independently of N by solving the equation obtained from Figure beside.

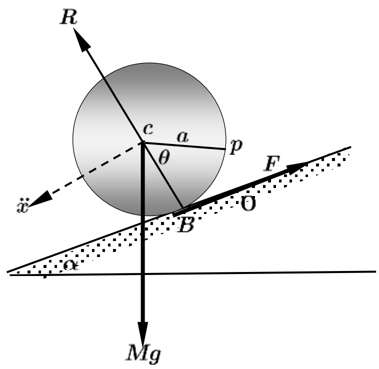


When sliding is impending, the friction force reaches its maximum value $F = \mu N$ and can be obtained from N. These two different cases can be summarized as follows:

Rolling, no sliding: $F \le \mu N, \quad \underline{a} = r\ddot{\theta}$

Rolling, no sliding: $F = \mu N, \quad \underline{a} = r\ddot{\theta}$

◇ A uniform sphere rolls down an inclined plane whose inclination to horizon be α, rough enough to prevent any sliding; discuss the motion.



Initially, the sphere was rest with its point P in contact with O. During the motion, after any time t, let the center C of the sphere describes a distance x on the inclined plane and θ is the angle through which the sphere turns. Thus CP a line fixed in the body, makes an angle θ with the normal to the plane, a line fixed in space. Let F be the frictional force and R the normal reaction force at the point of contact, then the equations of motion of the center of gravity of the sphere are

$$M\ddot{x} = Mg\sin\alpha - F \qquad \text{...(1)}$$

Since there is no motion perpendicular to the plane, we have

$$M\ddot{y} = Mg\cos\alpha - R = 0 \quad \text{Or} \quad R = Mg\cos\alpha \qquad \text{...(2)}$$

Also Equation of motion about the center of gravity is

$$\frac{2}{5}a^2M\ddot{\theta} = Fa \qquad \text{...(3)}$$

According to there is no sliding, so we have $OB = PB$

$$\Rightarrow x = a\theta, \quad \ddot{x} = a\ddot{\theta}$$

From Equation (3) $\Rightarrow \frac{2}{5}M\ddot{x} = F$

Substituting this value of F in Equation (1), we readily get

$$\Rightarrow \frac{7}{5}M\ddot{x} = Mg\sin\alpha \qquad \text{or} \qquad \ddot{x} = \frac{5}{7}g\sin\alpha \qquad \text{....(4)}$$

i.e. the sphere rolls down with a constant acceleration therefore by integrating

$$\frac{dx}{dt} = \left(\frac{5}{7}g\sin\alpha\right)t + c$$

The constant of integration c vanishes as t and $\dot{x}$ vanish together. Integrating again, we get,

$$x = \left(\frac{5}{14}g\sin\alpha\right)t^2$$

Because the constant of integration again vanishes, as t and x vanish simultaneously. For the case of hollow sphere, the acceleration is.

$$\ddot{x} = \frac{3}{5}g\sin\alpha$$

Pure rolling: Eliminating $\ddot{x}$ from Equations (1) and (4), we get

$$\begin{aligned} F &= mg\sin\alpha - \frac{5}{7}Mg\sin\alpha \\ &= \frac{2}{7}Mg\sin\alpha \end{aligned}$$

Also from Equation (2) $R = Mg\cos\alpha$. In order there may be no sliding F/R must be less than μ i.e. for pure rolling $F < \mu R$

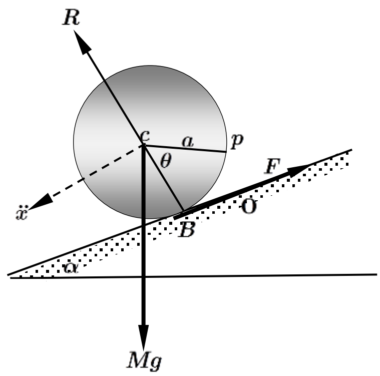
$$\text{thus,} \quad \mu > \frac{F}{R} \qquad \Rightarrow \mu = \frac{2}{7}\tan\alpha$$

■Illustrative Examples■

❒ EXAMPLE 1

A uniform solid cylinder is placed with its axis horizontal on a plane, whose inclination to the horizon is α, Prove that the least coefficient of friction between it and the plane, so that it may roll and not slide, is $\tan\alpha / 3$. If the cylinder be hollow, and of small thickness, the least value is $\tan\alpha / 2$

❒ SOLUTION



At any time t, Let the axis of the cylinder describe a distance x and θ be the angle turned, since there is no sliding so we have $x = a\theta$. Also the equations of motion of a center of gravity are given by

$$M\ddot{x} = Mg\sin\alpha - F \qquad ....(1)$$

$$\text{and} \qquad 0 = Mg\cos\alpha - R \qquad ...(2)$$

Again taking moments about the axis through G, the center of gravity of the solid cylinder, we have

$$\frac{1}{2}Ma^2\ddot{\theta} = Fa \qquad \Rightarrow \frac{1}{2}M\ddot{x} = F \qquad .....(3)$$

hence elimination of $M\ddot{x}$ in between Equations (1) and (3), we get

$$F = \frac{1}{3}Mg\sin\alpha \qquad ....(4) \qquad \text{but} \qquad R = Mg\cos\alpha \qquad ....(5)$$

For pure rolling, $\mu > \dfrac{F}{R} = \dfrac{1}{3}\tan\alpha$

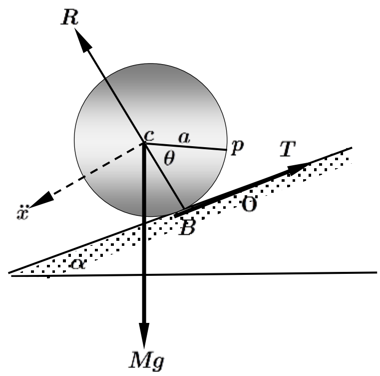
In the same manner for hollow cylinder and we can obtain the least value of coefficient of friction to prevent sliding is $\mu > \dfrac{F}{R} = \dfrac{1}{2}\tan\alpha$

❒ EXAMPLE 2

A cylinder rolls down a smooth plane whose inclination to the horizontal is α, unwrapping, as it goes, a fine string fixed to the highest point of the plane, find its acceleration and the tension of the string.

❒ SOLUTION

When the cylinder has rolled down a distance x along the plane, Let T be the tension in the string and in this time (say t), let θ be the angle turned by the cylinder, then as the string is tight, the motion of the



pure rolling i.e. $BP = OB \Rightarrow \ x = a\theta$ which gives

$\dot{x} = a\dot{\theta}, \ \ \ddot{x} = a\ddot{\theta}$

Now equations of motion of the center of gravity of the cylinder are

$$M\ddot{x} = Mg\sin\alpha - T \qquad ....(1)$$

and $$M\ddot{y} = 0 = Mg\cos\alpha - R \qquad ...(2)$$

Now taking moments about the center, we get

$$\frac{1}{2}Ma^2\ddot{\theta} = Ta \qquad \Rightarrow \frac{1}{2}M\ddot{x} = T \quad (\ddot{x} = a\ddot{\theta}) \ .....(3)$$

Equations (1) and (3) give

$$\frac{3}{2}M\ddot{x} = Mg\sin\alpha \qquad \Rightarrow \ddot{x} = \frac{2}{3}g\sin\alpha$$

and then $$T = \frac{1}{3}Mg\sin\alpha$$

❒ EXAMPLE 3

A rough uniform rod of length $2a$ is placed on a rough table at right angles to its edge; if its center of gravity be initially at distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle $\mu a^2(a^2 + 9b^2)^{-1}$ where μ represents the coefficient of friction.

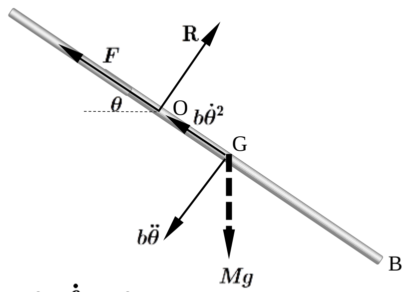
❒ SOLUTION

Initially the rod was at right angles to the edge of the rough table, now it has turned through an angle θ. Let there be no sliding when the rod has turned through this angle. Let F and R be the force of friction and normal reaction on the rod. Acceleration of G along and perpendicular to GO are respectively $b\dot{\theta}^2$ and $b\ddot{\theta}$. Equations of motion of center of gravity G are

$$M b\ddot{\theta} = Mg\cos\theta - R \qquad ....(1)$$

$$\text{and} \quad M b\dot{\theta}^2 = F - Mg\sin\theta \qquad ....(2)$$

Taking moments about O, the point of contact of the rod and table, we have



$$I_o\ddot{\theta} = Mgb\cos\theta, \qquad I_o = M\left(b^2 + \frac{1}{3}a^2\right)$$

$$\Rightarrow \ddot{\theta} = \frac{3gb}{a^2 + 3b^2}\cos\theta \qquad ....(3)$$

Multiply Eq. (3) by $2\dot{\theta}$ and integrate, we get

$$\dot{\theta}^2 = \frac{6gb}{a^2 + 3b^2}\sin\theta \quad ...(4)$$

The constant of integration vanishes as initially when $\theta = 0,\ \dot{\theta} = 0$.

Putting the values of $\ddot{\theta}$ and $\dot{\theta}^2$ in Equations (1) and (2) from (3) and (4), we get

$$R = -Mb.\frac{3gb}{a^2 + 3b^2}\cos\theta + Mg\cos\theta = \frac{Mga^2}{a^2 + 3b^2}\cos\theta$$

$$\text{and} \quad F = Mg\sin\theta + Mb.\frac{6gb}{a^2 + 3b^2}\sin\theta = \frac{a^2 + 9b^2}{a^2 + 3b^2}Mg\sin\theta$$

Now, the sliding commences when $F = \mu R$ i.e.

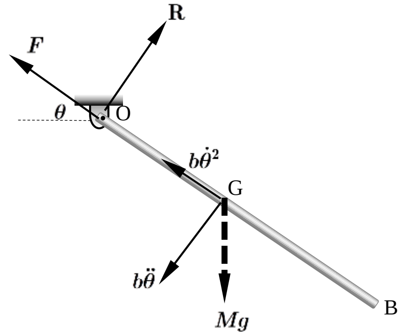
$$\frac{a^2 + 9b^2}{a^2 + 3b^2}Mg\sin\theta = \frac{Mga^2\mu}{a^2 + 3b^2}\cos\theta \quad \Rightarrow \tan\theta = \frac{\mu a^2}{a^2 + 9b^2}$$

❒ EXAMPLE 4

A thin uniform rod of mass m and length $2a$ has one end attached to a smooth hinge and is allowed to fall from a horizontal position. Show that the horizontal strain on the hinge is greatest when the rod is inclined at angle 45^o to the vertical, and that the vertical strain is then 1.375 times the weight of the rod.

❐ SOLUTION

Let $OA = 2a$ and let the rod makes an angle θ with the horizontal after time t. The figure behind shows the free body diagram (FBD) and mass acceleration diagram for an arbitrary position of the rod. The FBD contains X and Y, the unknown components of the pin reaction at O. Since the path of G is a circle of radius $r = a$ centered at O, the normal components of $\underline{a}$ is $a_n = r\dot{\theta}^2 = a\dot{\theta}^2$, and its tangential component is $a_t = r\ddot{\theta} = a\ddot{\theta}$. Observe that the angular acceleration $\ddot{\theta}$ is assumed to be clockwise. Equations of motion of G along and perpendicular to GO are



$$Ma\ddot{\theta} = -Y\cos\theta + X\sin\theta + Mg\cos\theta \qquad ....(1)$$

and $$Ma\dot{\theta}^2 = Y\sin\theta + X\cos\theta - Mg\sin\theta \qquad ....(2)$$

Again the moment equation about O is

$$\frac{4}{3}Ma^2\ddot{\theta} = Mga\cos\theta \qquad \Rightarrow \ddot{\theta} = \frac{3g}{4a}\cos\theta \qquad ....(3)$$

Integrating Equation (3) we get $\dot{\theta}^2 = \dfrac{3g}{2a}\sin\theta + C$

From initial condition $\dot{\theta} = 0$ when $\theta = 0$ therefore, $C = 0 \Rightarrow \dot{\theta}^2 = \dfrac{3g}{2a}\sin\theta$

Putting this value of $\dot{\theta}^2$ in Equation (2), we get

$$\frac{3}{2}Mg\sin\theta = Y\sin\theta + X\cos\theta - Mg\sin\theta$$

$$\Rightarrow Y\sin\theta + X\cos\theta = \frac{5}{2}Mg\sin\theta \qquad ....(4)$$

With the help of Equation (3), the Equation (1) becomes

$$\frac{3}{4}Mg\cos\theta = -Y\cos\theta + X\sin\theta + Mg\cos\theta$$

$$\Rightarrow Y\cos\theta - X\sin\theta = \frac{1}{4}Mg\cos\theta \qquad ...(5)$$

Multiply Equation (4) by $\cos\theta$ and Equation (5) by $\sin\theta$ and adding, we have

$$X = \left(\frac{5}{2} - \frac{1}{4}\right)Mg\sin\theta\cos\theta = \frac{9}{8}Mg\sin 2\theta$$

Similarly, we have

$$Y = Mg\left(\frac{5}{2}\sin^2\theta + \frac{1}{4}\cos^2\theta\right)$$

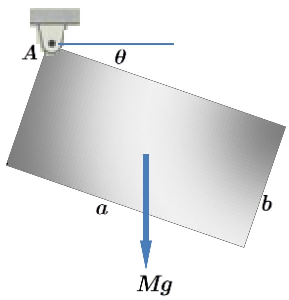
We observe that X is maximum when $\sin 2\theta = 1$ i.e. when $2\theta = \pi/2$ or $\theta = \pi/4$, so when $\theta = \pi/4$ we have

$$Y = Mg\left(\frac{5}{2}.\frac{1}{2} + \frac{1}{4}.\frac{1}{2}\right) = 1.375Mg$$

i.e. Y is 1.375 times the weight of the rod.

❒ EXAMPLE 5

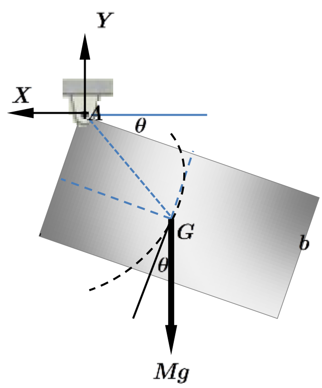
The 360-Ib uniform plate shown in the figure rotates in the vertical plane about a smooth pin at A. The plate is released from rest when $\theta = 0$. Determine the components of reaction at pin A.



❒ SOLUTION

The mass of the plate is $m = 360/32.2 = 11.18$ slugs and the moment of
inertia about its mass center G is

$$I = \frac{1}{12}M(a^2 + b^2)$$



The figure behind shows the free body diagram (FBD) and mass acceleration diagram for an arbitrary position of the plate. The FBD contains X and Y, the unknown components of the pin reaction at A. Since the path of G is a circle of radius $r = \sqrt{(0.5a)^2 + (0.5b)^2} = \ell$ (say) centered at A, the normal components of $\underline{a}$ is $a_n = \ell\dot{\theta}^2$, and its tangential component is $a_t = r\ddot{\theta} = \ell\ddot{\theta}$. Observe that the angular acceleration $\ddot{\theta}$ is assumed to be clockwise.

Equations of motion of G along and perpendicular to GA are

$$M\ell\ddot{\theta} = -Y\cos\theta + X\sin\theta + Mg\cos\theta \qquad ....(1)$$

$$\text{and} \quad M\ell\dot{\theta}^2 = Y\sin\theta + X\cos\theta - Mg\sin\theta \qquad ....(2)$$

Again the moment equation about A is

$$\frac{1}{3}M(a^2 + b^2)\ddot{\theta} = Mg\ell\cos\theta \qquad \Rightarrow \ddot{\theta} = \frac{3g\ell}{a^2 + b^2}\cos\theta \qquad ....(3)$$

Integrating Equation (3) we get $\dot{\theta}^2 = \dfrac{6g\ell}{a^2 + b^2}\sin\theta + C$

From initial condition $\dot{\theta} = 0$ when $\theta = 0$ therefore, $C = 0$

$$\Rightarrow \dot{\theta}^2 = \frac{6g\ell}{a^2 + b^2}\sin\theta$$

Putting this value of $\dot{\theta}^2$ in equation (2), we have

$$\ell^2 M\frac{6g}{a^2 + b^2}\sin\theta = Y\sin\theta + X\cos\theta - Mg\sin\theta$$

$$\Rightarrow Y\sin\theta + X\cos\theta = \frac{5}{2}Mg\sin\theta \qquad ....(4)$$

Via the help of Equation (3), the Equation (1) turns into

$$\ell^2 M\frac{3g}{a^2 + b^2}\cos\theta = -Y\cos\theta + X\sin\theta + Mg\cos\theta$$

$$\Rightarrow Y\cos\theta - X\sin\theta = \frac{1}{4}Mg\cos\theta \qquad ...(5)$$

From equations (4) and (5) the components of reaction at A can be found.

❒ EXAMPLE 6

A right cone of angle 2α can turn freely about an axis passing through the center of its base and perpendicular to the axis; if the cone starts from rest with its axis horizontal, show that when the axis is vertical, the thrust on the fixed axis is to the weight of the cone as $\dfrac{6+3\cos^2\alpha}{4+6\cos^2\alpha}$.

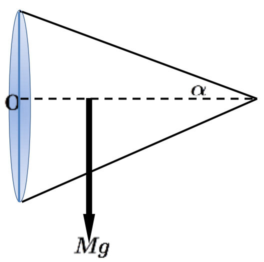
❒ SOLUTION

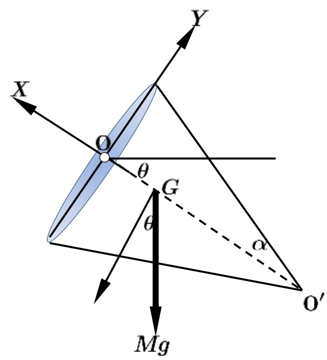
Let initially the cone be as shown in the figure. After any time t, let the cone take the position as shown. If the height of the cone is $OO' = h$, then $OG = \frac{1}{4}h$ where G denotes the center of gravity of the cone. Now since the C.G. of the cone i.e. point G is describing a circle of radius $(h/4)$, the equations of motion of G are

$$M\left(\frac{1}{4}h\right)\dot{\theta}^2 = X - Mg\sin\theta \qquad \text{....(1)}$$

$$\text{and} \quad M\left(\frac{1}{4}h\right)\ddot{\theta} = Mg\cos\theta - Y \qquad \text{....(2)}$$

Where denote the components of reaction at O along and perpendicular to OX. Taking moments about O, we get





$$M\frac{h^2}{20}(2+3\tan^2\alpha)\ddot{\theta} = Mg\left(\frac{1}{4}h\right)\cos\theta$$

$$\Rightarrow h\ddot{\theta} = \frac{5}{2+3\tan^2\alpha}g\cos\theta \qquad \text{....(3)}$$

Multiplying both sides of Equation (3) by $2\dot{\theta}$ and integrating we get

$$h\dot{\theta}^2 = \frac{10}{2+3\tan^2\alpha}g\sin\theta + C$$

Initially $\dot{\theta} = 0$ when $\theta = 0$ giving thereby the constant, $C = 0$

$$h\dot{\theta}^2 = \frac{10}{2 + 3\tan^2\alpha} g\sin\theta \qquad \text{....(4)}$$

Substituting Equation (4) in (1) we have

$$M\left(\frac{1}{4}h\right)\frac{10}{(2 + 3\tan^2\alpha)h} g\sin\theta = X - Mg\sin\theta h\dot{\theta}^2$$

$$\Rightarrow X = Mg\sin\theta\left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha}\right)$$

Also using Equation (3) in (2) we get

$$Y = \left(\frac{3 + 6\tan^2\alpha}{8 + 12\tan^2\alpha}\right) Mg\cos\theta$$

When the axis is vertical i.e. when $\theta = (\pi/2)$ we have

$$X = Mg\left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha}\right), \quad Y = 0$$

Then resultant pressure is $R = \sqrt{X^2 + Y^2} = X$

$$R = Mg\left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha}\right)$$

$$= Mg\left(\frac{9\cos^2\alpha + 6\sin^2\alpha}{4\cos^2\alpha + 6\sin^2\alpha}\right)$$

$$\Rightarrow \frac{X}{Mg} = \frac{6 + 3\cos^2\alpha}{4 + 6\cos^2\alpha}$$

■ The Compound Pendulum

In order to determine the motion of a body acted on by the force of gravity only and moving about a fixed horizontal axis.

Let us take plane of the paper as the plane through the center of gravity G of the body and perpendicular to the fixed axis. Let the plane meet the axis in C and let be the angle between a plane fixed in space and a plane in the body.

Let $CG = h$. The forces on the body are

(i) its weight Mg acting downward through G.

(ii) the reaction at C of the fixed axis.

We take moments about the fixed axis to eliminate this reaction.

The equation of motion is

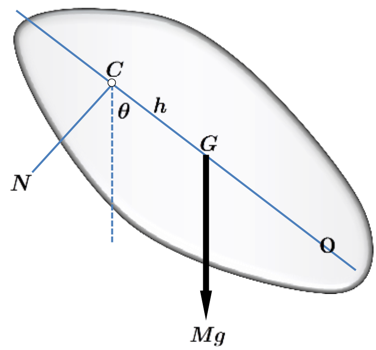
$$Mk^2\ddot{\theta} = -Mgh\sin\theta \qquad (Mk^2 = I_C)$$

$$\Rightarrow \ddot{\theta} = -\frac{gh}{k^2}\sin\theta = -\frac{gh}{k^2}\theta \qquad (\theta \text{ being small}) \qquad ...(1)$$

Equation (1) shows that the motion is Simple Harmonic Motion. Hence the time of complete oscillation of compound pendulum is $\tau = 2\pi\sqrt{k^2/(gh)}$

♦ Simple Equivalent Pendulum,

We know that equation of motion of a particle of any mass suspended by a string of length L is



$$\ddot{\theta} = -\frac{g}{L}\sin\theta = -\frac{g}{L}\theta \qquad (\theta \text{ being small})$$

The time of complete oscillation is $\tau = 2\pi\sqrt{L/g}$

If $2\pi\sqrt{L/g} = 2\pi\sqrt{k^2/(gh)}$ then $L = k^2/h$

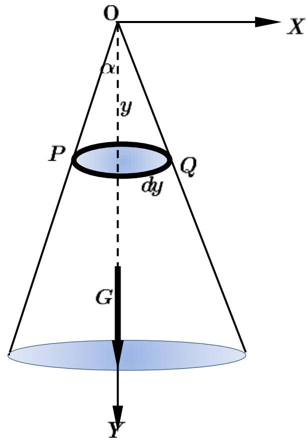
This length (k^2 / h) in the case of a compound pendulum is called the length of the simple equivalent pendulum.

If O is the point on CG produced such that (the length of the simple equivalent pendulum) then the point O is called the ***center of oscillation***.

Illustrative Examples

❐ EXAMPLE 6

A solid homogeneous cone of height h and vertical angle 2α oscillates about a horizontal axis through its vertex, show that the length of the simple equivalent pendulum is $\frac{1}{5}h(4+\tan^2\alpha)$



❐ SOLUTION

Let OX be the vertical axis through the vertex O. Let us take a circular disk PQ of thickness dy at distance y from O. Moment of inertia of disk about OX = $(\rho\pi y^2\tan^2\alpha\, dy)\left(\frac{1}{4}y^2\tan^2\alpha + y^2\right)$

Therefore Moment of inertia of whole cone about OX is

$$I_X = \rho\pi\tan^2\alpha\left(1+\frac{1}{4}\tan^2\alpha\right)\int_0^h y^4\,dy$$
$$= \frac{1}{5}\rho\pi\tan^2\alpha\left(1+\frac{1}{4}\tan^2\alpha\right)h^5$$
$$= \frac{1}{20}\rho\pi\tan^2\alpha\ \ 4+\tan^2\alpha\ \ h^5$$
$$= \frac{3}{20}M\ \ 4+\tan^2\alpha\ \ h^2 \qquad \left(M = \frac{1}{3}\rho\pi h^3\tan^2\alpha\right)$$

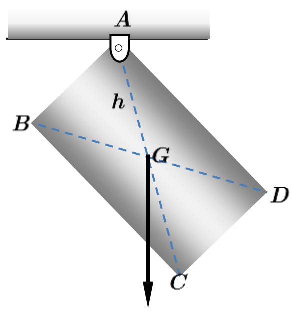
Since $k^2 = \frac{3}{20}\ \ 4+\tan^2\alpha\ \ h^2$ and $OG = \frac{3}{4}h$

Therefore the length of the simple equivalent pendulum is

$$l = \frac{k^2}{OG} = \frac{1}{5}\ \ \tan^2\alpha + 4\ \ h$$

❐ EXAMPLE 7

A rectangular plate swings in a vertical plane about one of its corners. If its period is one second, find the length of the diagonal.



❒ SOLUTION

Let k be the radius of gyration of the plate about the axis, through A and perpendicular to its plane, then we have

$$Mk^2 = \frac{1}{12}M(a^2+b^2) + Mh^2 \quad \text{(parallel axis theorem)}$$
$$= \frac{4}{3}Mh^2 \quad \Rightarrow k^2 = \frac{4}{3}h^2, \qquad h = \frac{1}{2}\sqrt{a^2+b^2}$$

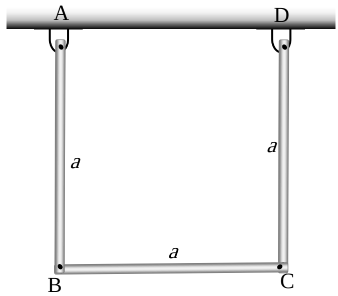
BG=GD, further distance of center of gravity from A = AG= h

Since the period = $2\pi\sqrt{\dfrac{k^2}{hg}} = 2\pi\sqrt{\dfrac{4h^2}{3hg}} = 4\pi\sqrt{\dfrac{h}{3g}}$

But as given the period =1 therefore, $4\pi\sqrt{h/(3g)} = 1 \quad \Rightarrow h = 3g/(16\pi^2)$

Hence the length of the diagonal is $2h = (3g\,/\,8\pi^2)$

❒ EXAMPLE 8



Three uniform rods AB, BC, CD each of length *a*, are freely jointed at B and C and suspended from the points A and D which are in the same horizontal line and a distance *a* apart. Prove that when the rods move in a vertical plane, the length of simple equivalent pendulum is 5*a*/6

❒ SOLUTION

The system forms a compound pendulum swinging about the horizontal AD. The figure is self-explanatory. Let *m* be the mass of each rod.

Let h be the depth of C.G of the system from AD and k be the radius of gyration of the system about the horizontal axis AD, then we easily obtain = sum of the moments of inertia of the three rods about AD, i.e.

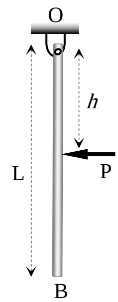
$$3mk^2 = m\ a^2/3 + m\ a^2/3 + ma^2 \quad \Rightarrow k^2 = 5a^2/9$$

And $$h = m\ a/2 + m\ a/2 + ma\ /\,(3m) = 2a/3$$

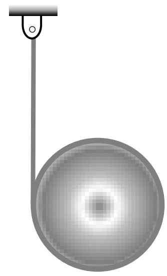
Therefore, the length of simple equivalent pendulum is $(k^2/h) = (5a/6)$

PROBLEMS

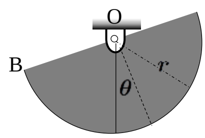
❒ A uniform slender rod of length L = 36 in. and weight W = 4 lb hangs freely from a hinge at A. If a force **P** of magnitude 1.5 lb is applied horizontally as shown to the left (h=24 in), determine (a) the angular acceleration of the rod, (b) the components of the reaction at A.



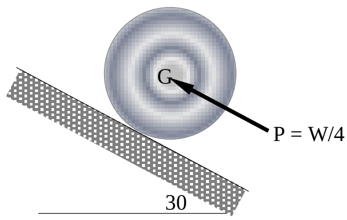
❒ A rope is wrapped around a cylinder of radius r and mass *m* as shown. Knowing that the cylinder is released from rest, determine the velocity of the center of the cylinder after it has moved downward a distance L.



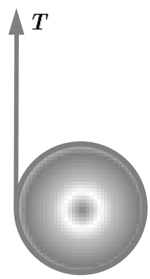
❒ A thin, homogeneous, semicircular plate of mass m and radius *r* as shown. It is released from rest when $\boldsymbol{\theta = 90^{o}}$. At this instant, determine the linear acceleration of point B expressing it in terms of the acceleration of gravity g.



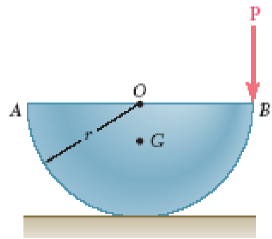
❒ The homogeneous thin hoop of weight *Wand* radius *r* shown is released from rest and rolls without sliding down the inclined plane under the action of its weight and the applied force *P*. Determine (a) the angular acceleration of the hoop, (b) the frictional force, (c) the acceleration of the mass center G, and (d) the minimum coefficient of friction required to assure that the thin hoop rolls without sliding. Express answers in terms of *W* and *r*.



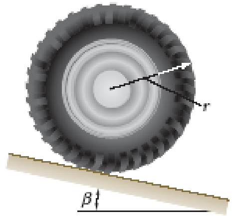
❒ The homogeneous cylinder of weight W = 100 N and radius r = 0.25 m is released from rest as shown in Figure. Determine the tension in the inextensible cord, the angular acceleration of the cylinder and the acceleration of its mass center G. Assume that the cylinder does not slip on the cord.



❒ A half section of a uniform cylinder of mass m is at rest when a force P is applied as shown. Assuming that the section rolls without sliding. Determine (a) its angular acceleration, (b) the minimum value of μ compatible with the motion.



❒ A wheel of radius r and centroidal radius of gyration k is released from rest on the incline and rolls without sliding. Derive an expression for the acceleration of the center of the wheel in terms of r, k, β , and g.



❒ A sphere of radius r and weight W is released with no initial velocity on the incline and rolls without slipping. Determine (a) the minimum value of the coefficient of static friction compatible with the rolling motion, (b) the velocity of the center G of the sphere after the sphere has rolled 10 ft, (c) the velocity of G if the sphere were to move 10 ft down a frictionless 30° incline.

❒ Find the length of the equivalent simple pendulum in the following cases, the axis being horizontal

(i) Circular disk; axis a tangent to it Ans.($5a/4$)

(ii) Hemisphere; axis a diameter of the base Ans. ($16a/15$)

(iii) An elliptic lamina when the axis is a latus rectum.