South valley University

Faculty of science-Qena
Mathematics Department

الكلية: التــربيــة بقنــــا

المقرر: رياضيات (4) – جزء الديناميكا

الفرقة: الثانية فيزياء وكيمياء - عام

البرنامج: اللغة الإنجليزية (English Programme of Study)

الفصل الدراسي: الثـــــــاني

المحاضر: أ. د. جمال عبدالله أحمد حشودي

Contents

Chapter: 1
Motion in a Resisting Medium

In studying the motion of a body in a resisting medium, we assume that the resistive force on a body, and hence its deceleration, is some function of its speed. Such resistive forces are not generally conservative, and kinetic energy is usually dissipated as heat. For simple theoretical studies one can assume a simple force law, such as the resistive force is proportional to the speed, or to the square of the speed, or to some function that we can conveniently handle mathematically. For slow, laminar, non-turbulent motion through a viscous fluid, the resistance is indeed simply proportional to the speed, as can be shown at least by dimensional arguments. One thinks, for example, of Stokes's Law for the motion of a sphere through a viscous fluid. For faster motion, when laminar flow breaks up and the flow becomes turbulent, a resistive force that is proportional to the square of the speed may represent the actual physical situation better.

1-Horizontal motion in a straight line

2-Vertical motion in a straight line

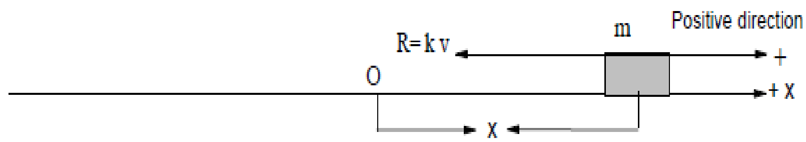
(i) - downwards (falling)

(ii)- upwards

1-Horizontal motion in a straight line

Example:1 A particle moves (travels) in a straight line is subject to a resistance $k\mathrm{v}$, where v is the velocity and k is the constant. Show that its velocity and position at time t are given, respectively, $\mathrm{v}=\mathrm{v}_0 e^{-\lambda t}$, $x=\frac{\mathrm{v}_0}{\lambda}\left(1-e^{-\lambda t}\right)$. If the particle starts its motion from the original point by initial velocity v_0 and determine maximum distance that travels the particle?

Solution



The motion of Equation in our case is

$$m\vec{a}=-R \qquad (1)$$

But, $R \;\alpha\; \mathrm{v} \;\rightarrow\; R=k\mathrm{v}$, and if we put $k=m\lambda$. Then Eq. (1) becomes

$$a=-\lambda\mathrm{v}=\frac{dv}{dt}$$

Separating variables and integrating yields:

$$\int\frac{dv}{\mathrm{v}}=-\lambda\int dt$$

$$\ln(\mathrm{v})=-\lambda t+c_1 \qquad (2)$$

From the initial condition at $t=0 \;\; \mathrm{v}=0$, this tends to $\ln(\mathrm{v}_0)=c_1$.

Substitute into Eq. (2) we have

$$\ln(\mathrm{v})=-\lambda t+\ln(\mathrm{v}_0)$$

Then $$\left\{\ln(\mathrm{v})-\ln(\mathrm{v}_0)\right\}=-\lambda t$$

$$\ln\left\{\frac{\mathrm{v}}{\mathrm{v}_0}\right\}=-\lambda t \;\rightarrow\; \frac{\mathrm{v}}{\mathrm{v}_0}=e^{-\lambda t}$$

$$\therefore\; \mathrm{v}=\mathrm{v}_0 e^{-\lambda t} \qquad \text{(Answer…..Ans.)} \qquad (3)$$

The displacement (distance) from Eq. (3) given by

$$\mathrm{v}=\frac{dx}{dt}=\mathrm{v}_0 e^{-\lambda t} \;\rightarrow \int dx=\int \mathrm{v}_0 e^{-\lambda t}\,dt \;\rightarrow x=-\frac{\mathrm{v}_0}{\lambda}e^{-\lambda t}+c_2 \qquad (4)$$

From the initial condition at $t=0 \;\; x=0$, this tends to

$$0=-\frac{\mathrm{v}_0}{\lambda}e^{-\lambda(0)}+c_2 \;\rightarrow\; c_2=\frac{\mathrm{v}_0}{\lambda}$$

Then In Eq. (4) we have

$$x=-\frac{\mathrm{v}_0}{\lambda}e^{-\lambda t}+\frac{\mathrm{v}_0}{\lambda}$$

$$\therefore\; x=\frac{\mathrm{v}_0}{\lambda}\left(1-e^{-\lambda t}\right) \qquad (5)$$

When the particle travels to maximum distance, then the velocity equal zero and from Eq. (3) we find that

$$\therefore\; \mathrm{v}=\mathrm{v}_0 e^{-\lambda t}=0 \;\Rightarrow\; t=\infty\,.$$

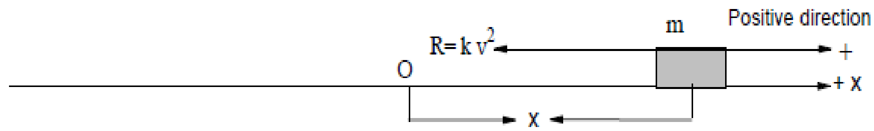
Substituting this value for time into the position function (Eq. 5) gives the maximum distance the particle travels as.

$$\therefore\; x=\frac{\mathrm{v}_0}{\lambda}\left(1-e^{-\lambda t}\right)=\frac{\mathrm{v}_0}{\lambda}\left(1-e^{-\lambda(\infty)}\right)=\frac{\mathrm{v}_0}{\lambda}(1-0)=\frac{\mathrm{v}_0}{\lambda}$$

Exercise: A particle moves (travels) in a straight line is subject to a resistance $k\text{v}$, where v is the velocity and k is the constant. Find both the velocity and position at time t, if the particle starts at $x=x_0$ by initial velocity v_0. Determine maximum distance that travels the particle?

Example:2- A particle moving in a straight line is subject to a resistance $k\text{v}^2$, where v is the velocity and k is the constant. Study the motion if the particle starts its motion from the original point by initial velocity v_0?

Solution



From the Newton's second law of motion

$m\vec{a}=-R$ (1)

But, $R \;\alpha\; \text{v} \rightarrow R=k\text{v}^2$, and if we put $k=m\lambda$. Then Eq. (1) becomes

$$a=-\lambda \text{v}^2=\frac{d\text{v}}{dt}=\rightarrow \quad \int\frac{d\text{v}}{\text{v}^2}=-\lambda\int dt$$

$$\therefore \frac{-1}{\text{v}}=-\lambda t+c_1 \qquad (2)$$

From the initial condition at $t=0$ $\text{v}=\text{v}_0$, this tends to $c_1=\frac{-1}{\text{v}_0}$.

Substitute into Eq. (2) we have

$\frac{-1}{\text{v}}=-\lambda t+\frac{-1}{\text{v}_0} \Rightarrow \frac{1}{\text{v}}=\frac{\lambda \text{v}_0 t+1}{\text{v}_0}$, then

$$\therefore \quad \text{v}=\frac{\text{v}_0}{\lambda \text{v}_0 t+1} \qquad (3)$$

The displacement from Eq. (3) given by

$$\text{v}=\frac{dx}{dt}=\frac{\text{v}_0}{\lambda \text{v}_0 t+1} \rightarrow \int dx=\frac{1}{\lambda}\int\frac{\text{v}_0\lambda}{\lambda \text{v}_0 t+1}dt \rightarrow x=\frac{1}{\lambda}\ln(\lambda \text{v}_0 t+1)+c_2 \qquad (4)$$

From the initial condition at $t=0$ $x=0$, this tends to

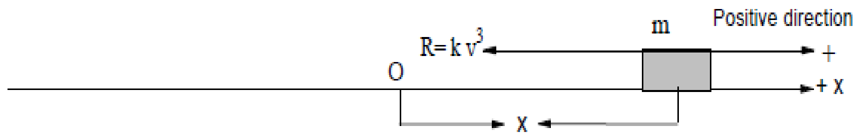
$0=\frac{1}{\lambda}\ln(\lambda(0)+1)+c_2 \rightarrow \quad c_2=\frac{1}{\lambda}\ln(1)=0$ and again in Eq. (4) we have

$$\therefore \qquad x=\frac{1}{\lambda}\ln(\lambda \text{v}_0 t+1) \qquad (5)$$

Example:3- A particle moves in a straight line is subject to a resistance $k\text{v}^3$, where v is the velocity and k is the constant. If the particle starts its motion from the original point by initial velocity v_0 prove that $t=\frac{x}{\text{v}_0}+\frac{1}{2}\lambda x^2$,

$\text{v}=\frac{\text{v}_0}{\text{v}_0\lambda x+1}$?

Solution



From the Newton's second law of motion

$$m\vec{a}=-R \qquad (1)$$

But, $R=k\text{v}^3$. Then in Eq. (1) becomes

$$ma=-k\text{v}^3,\ \lambda=km \quad \rightarrow \quad a=-\lambda\text{v}^3$$

$$\text{v}\frac{d\text{v}}{dx}=-\lambda\text{v}^3 \rightarrow \frac{d\text{v}}{dx}=-\lambda\text{v}^2 \rightarrow \frac{d\text{v}}{\text{v}^2}=-\lambda\,dx$$

$$\int\frac{d\text{v}}{\text{v}^2}=-\lambda\int dx \rightarrow \frac{-1}{\text{v}}=-\lambda x+c_1 \qquad (2)$$

From the initial condition at $t=0,\ x=0,\ \text{v}=\text{v}_0$, this tends to

$$\frac{-1}{\text{v}_0}=-\lambda(0)+c_1 \rightarrow c_1=\frac{-1}{\text{v}_0}$$

Substitute into Eq. (2) we have $\frac{-1}{\text{v}}=-\lambda x-\frac{1}{\text{v}_0} \rightarrow \frac{1}{\text{v}}=\lambda x+\frac{1}{\text{v}_0}=\frac{\text{v}_0\lambda x+1}{\text{v}_0}$

$$\text{v}=\frac{\text{v}_0}{\text{v}_0\lambda x+1} \qquad (3)$$

The displacement (distance) from Eq. (3) given by

$$\text{v}=\frac{dx}{dt}=\frac{\text{v}_0}{\text{v}_0\lambda x+1} \rightarrow \int\text{v}_0 dt=\int(\text{v}_0\lambda x+1)dx \rightarrow \text{v}_0 t=\left(\frac{1}{2}\text{v}_0\lambda x^2+x\right)+c_2 \qquad (4)$$

From the initial condition at $t=0,\ x=0\ \ \text{v}=\text{v}_0$, this tends to

$$\text{v}_0(0)=\left(\frac{1}{2}\text{v}_0\lambda(0)^2+(0)\right)+c_2 \rightarrow c_2=0$$

Substitute into Eq. (4) we have

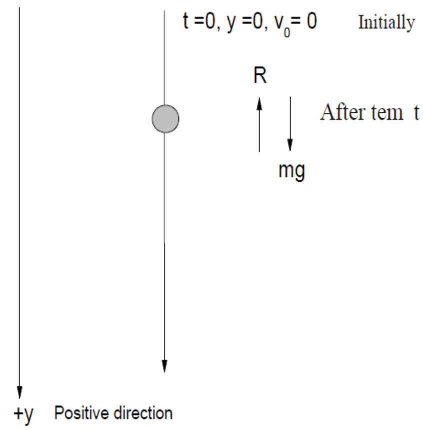
$$\text{v}_0 t=\left(\frac{1}{2}\text{v}_0\lambda x^2+x\right)+0 \rightarrow t=\frac{1}{2\text{v}_0}\text{v}_0\lambda x^2+\frac{x}{\text{v}_0}=\frac{x}{\text{v}_0}+\frac{1}{2}\lambda x^2 \rightarrow t=\frac{x}{\text{v}_0}+\frac{1}{2}\lambda x^2$$

2-Vertical motion under gravity with linear resistance

(i)- downwards (falling)

Example:**4 -** Determine the motion of a body (particle) falling under gravity and the resistance of air being assumed proportional to the velocity.

Solution



Taking the downward direction as positive, then the equation of motion will be

$$m\vec{a} = m\vec{g} - R \qquad (1)$$

Whereecomes b 1). Then Eq. ($k = m\lambda$ and if we put $R \,\alpha\, v \rightarrow R = kv$

$$m\,a = mg - m\lambda v = m\frac{dv}{dt} = \rightarrow \quad \int \frac{dv}{g - \lambda v} = \int dt \quad \rightarrow \quad -\frac{1}{\lambda}\int \frac{-\lambda\, dv}{g - \lambda v} = \int dt$$

$$\therefore -\frac{1}{\lambda}\ln(g - \lambda v) = t + c_1 \qquad (2)$$

From the initial condition at $t = 0,\ y = 0\ \ v = 0$, this tends to

$-\frac{1}{\lambda}\ln(g) = c_1$ and substituting in Eq. (2) we have $-\frac{1}{\lambda}\ln(g - \lambda v) = t - \frac{1}{\lambda}\ln(g)$

$$\left\{\ln(g - \lambda v) - \ln(g)\right\} = -\lambda t$$

$$\ln\left\{\frac{g - \lambda v}{g}\right\} = -\lambda t \quad \rightarrow \quad \frac{g - \lambda v}{g} = e^{-\lambda t} \quad \rightarrow \quad g - \lambda v = g\,e^{-\lambda t}$$

$$\therefore\ v = \frac{g}{\lambda}(1 - e^{-\lambda t}) \qquad (3)$$

This equation gives the velocity at any instant (moment). If the time increases to infinity, then $v = \frac{g}{\lambda}$ that is called the terminal velocity.

The displacement (distance) from Eq. (3) given by

$$v = \frac{dy}{dt} = \frac{g}{\lambda}(1 - e^{-\lambda t}) \ \rightarrow \int dy = \int \frac{g}{\lambda}(1 - e^{-\lambda t})dt \ \rightarrow y = \frac{g}{\lambda}(t + \frac{1}{\lambda}e^{-\lambda t}) + c_2 \qquad (4)$$

From the initial condition at $t=0,\ y=0$, this tends to

$$0=\frac{g}{\lambda}(0+\frac{1}{\lambda}e^{-0})+c_2 \ \rightarrow \ \frac{g}{\lambda}(0+\frac{1}{\lambda})+c_2 \ \rightarrow \ c_2=-\frac{g}{\lambda^2}$$ and into Eq. (4)

$$y=\frac{g}{\lambda}(t+\frac{1}{\lambda^2}e^{-\lambda t})-\frac{g}{\lambda^2}$$, then $$\therefore y=\frac{gt}{\lambda}-\frac{g}{\lambda^2}(1-e^{-\lambda t}) \qquad (5)$$

Example:5- Determine the motion of a body falling under gravity and the resistance of air being assumed proportional to the square of the velocity.

Solution

Taking the downward direction as positive, then the equation of motion will be

$$m\vec{a}=m\vec{g}-R \qquad (1)$$

Where. Then Eq. (1) becomes $k=m\lambda$ and if we put $R\ \alpha\ \text{v} \rightarrow R=k\text{v}^2$

$$ma=mg-m\lambda\text{v}^2 \quad Or \quad a=g-\lambda\text{v}^2$$

Where $a=\text{v}\frac{d\text{v}}{dy}$, then

$$\text{v}\frac{d\text{v}}{dy}=g-\lambda\text{v}^2 \ \rightarrow \ \int\frac{\text{v}\,d\text{v}}{g-\lambda\text{v}^2}=\int dy \ \rightarrow \ \frac{-1}{2\lambda}\int\frac{-2\lambda\text{v}\,d\text{v}}{g-\lambda\text{v}^2}=\int dy$$

$$\therefore \ln\left(g-\lambda\text{v}^2\right)=-2\lambda y+c_1 \qquad (2)$$

From the initial condition at $t=0,\ y=0\ \ \text{v}=0$, this tends to $\ln(g)=c_1$ and into Eq. (2), we have

$$\ln\left(g-\lambda\text{v}^2\right)=-2\lambda y+\ln(g)$$

$$y=\frac{1}{2\lambda}\ln\left(\frac{g}{g-\lambda\text{v}^2}\right) \qquad (3)$$

To determine the time from Eq. (2) and where, then $a=\frac{d\text{v}}{dt}$

$$\frac{d\text{v}}{dt}=g-\lambda\text{v}^2=g\left(1-\left\{\sqrt{\frac{\lambda}{g}}\text{v}\right\}^2\right)\rightarrow\sqrt{\frac{g}{\lambda}}\int\frac{d\left(\sqrt{\frac{\lambda}{g}}\text{v}\right)}{1-\left\{\sqrt{\frac{\lambda}{g}}\text{v}\right\}^2}=g\int dt$$

$$\sqrt{\frac{g}{\lambda}}\tanh^{-1}\left\{\sqrt{\frac{\lambda}{g}}\text{v}\right\}=gt+c_2 \qquad (4)$$

From the initial condition at $t=0,\ y=0\ \ \text{v}=0$, this tends to

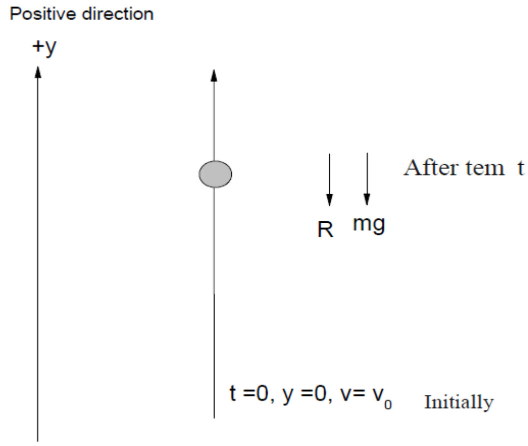
$$\sqrt{\frac{g}{\lambda}}\tanh^{-1}\left\{\sqrt{\frac{\lambda}{g}}(0)\right\}=c_2\Rightarrow c_2=0$$ and again in Eq. (4) we have

$$\sqrt{\frac{g}{\lambda}}\tanh^{-1}\left\{\sqrt{\frac{\lambda}{g}}\text{v}\right\}=gt$$, then $$t=\frac{1}{g}\sqrt{\frac{g}{\lambda}}\tan^{-1}\left\{\sqrt{\frac{\lambda}{g}}\text{v}\right\} \qquad (5)$$

(ii) Upwards

Example: 6- A body is projected vertically upwards with v_0 speed in a medium that exerts a drag-force kv, where k is a positive constant and v is the velocity of the body. Find the maximum height achieved by the body, the time taken to reach that height?

Solution



On including the linear resistance force, the scalar equation of motion becomes

$$m\vec{a} = m\vec{g} - R$$

Where $R = kv$, then the Equation of motions becomes

$$ma = -mg - kv = m\frac{dv}{dt} \to a = \frac{dv}{dt} = -g - \frac{k}{m}v \to \frac{dv}{dt} = -(g + \lambda v), \quad \lambda = \frac{k}{m}$$

$$\int \frac{dv}{dt} = -\int (g + \lambda v) \to \int \frac{dv}{g + \lambda v} = -\int dt \to$$

$$\therefore \frac{1}{\lambda}\ln(g + \lambda v) = -t + c_1 \qquad (1)$$

From the initial condition at $t = 0$, $v = v_0$, then

$$-\frac{1}{\lambda}\ln(g + kv_0) = c_1 \qquad (2)$$

Substitute from (2) into one we find that

$$-\frac{1}{\lambda}\ln(g + \lambda v) = t - \frac{1}{\lambda}\ln(g + \lambda v_0)$$

$$\left\{\ln(g + \lambda v) - \ln(g + \lambda v_0)\right\} = -\lambda t$$

$$\ln\left\{\frac{g + \lambda v}{g + \lambda v_0}\right\} = -\lambda t \to \frac{g + \lambda v}{g + \lambda v_0} = e^{-\lambda t} \to g + \lambda v = (g + \lambda v_0)e^{-\lambda t}$$

$$\therefore v = -\frac{g}{\lambda} + \frac{1}{\lambda}(g + \lambda v_0)e^{-\lambda t} = v_0 e^{-\lambda t} - \frac{g}{\lambda}\left(1 - e^{-\lambda t}\right) \qquad (3)$$

For the displacement from (3) we have

$$v = \frac{dy}{dt} = v_0 e^{-\lambda t} - \frac{g}{\lambda}\left(1 - e^{-\lambda t}\right) \to \int dy = \int \left[v_0 e^{-\lambda t} - \frac{g}{\lambda}\left(1 - e^{-\lambda t}\right)\right] dt$$

$$y=-\frac{v_0}{\lambda}e^{-\lambda t}-\frac{g}{\lambda}\left(t+\frac{1}{\lambda}e^{-\lambda t}\right)+c_2 \qquad (4)$$

Again, From the initial condition at $t=0,\ y=0$, then

$$0=-\frac{v_0}{\lambda}e^{-(0)}-\frac{g}{\lambda}\left((0)+\frac{1}{\lambda}e^{-(0)}\right)+c_2 \rightarrow -\frac{v_0}{\lambda}-\frac{g}{\lambda}(0+\frac{1}{\lambda})+c_2 \rightarrow c_2=\frac{v_0}{\lambda}+\frac{g}{\lambda^2}$$

$$y=-\frac{v_0}{\lambda}e^{-\lambda t}-\frac{g}{\lambda}\left(t+\frac{1}{\lambda}e^{-\lambda t}\right)+\frac{v_0}{\lambda}+\frac{g}{\lambda^2}$$

$$\therefore y=\frac{1}{\lambda^2}(g+\lambda v_0)\left(1-e^{-\lambda t}\right)-\frac{g}{\lambda}t \qquad (5)$$

when the body is stop to the move vertically upwards, in this case $v=0$

$$v=\frac{1}{\lambda}\left\{(g+\lambda v_0)e^{-\lambda t}-g\right\}=0\rightarrow \quad (g+\lambda v_0)e^{-\lambda t}-g=0$$

$$e^{-\lambda t}=\frac{g}{g+\lambda v_0} \quad Or \quad e^{\lambda t}=\frac{g+\lambda v_0}{g} \rightarrow \lambda t=\ln\left(\frac{g+\lambda v_0}{g}\right)$$

Then, the arrival time of maximum height

$$t_{max}=\frac{1}{\lambda}\ln\left(\frac{g+\lambda v_0}{g}\right) \qquad (6)$$

Substitution from Eq. (6) into Eq. (5), then the maximum height achieved by the body is given by

$$y_{\max}=\left\{\frac{1}{\lambda^2}(g+\lambda v_0)\left(1-e^{-\lambda t}\right)-\frac{g}{\lambda}t\right\}=\left\{\frac{1}{\lambda^2}(g+\lambda v_0)\left(1-\frac{g}{g+\lambda v_0}\right)-\frac{g}{\lambda}\frac{1}{\lambda}\ln\left(\frac{g+\lambda v_0}{g}\right)\right\}$$

$$y_{\max}=\frac{1}{\lambda^2}(g+\lambda v_0)\left(1-\frac{g}{g+\lambda v_0}\right)-\frac{g}{\lambda^2}\ln\left(\frac{g+\lambda v_0}{g}\right)=\frac{1}{\lambda^2}(g+\lambda v_0-g)-\frac{g}{\lambda^2}\ln\left(\frac{g+\lambda v_0}{g}\right)$$

$$\therefore y_{\max}=\frac{v_0}{\lambda}-\frac{g}{\lambda^2}\ln\left(\frac{g+\lambda v_0}{g}\right)$$

Chapter: 2
Projectiles

<u>Definition</u> Any object released into the air is called a projectile. Or a projectile is an object upon which the only force acting is gravity.

There are a variety of examples of projectiles. An object dropped from rest is a projectile (provided that the influence of air resistance is negligible). An object that is thrown vertically upward is also a projectile (provided that the influence of air resistance is negligible). And an object which is thrown upward at an angle to the horizontal is also a projectile (provided that the influence of air resistance is negligible). A projectile is any object that once projected or dropped continues in motion by its own inertia and is influenced only by the downward force of gravity.

Other examples:

(1) A ball after it has been thrown or hit

(2) A human body when jumping or diving

All projectiles have a "parabolic" flight path.

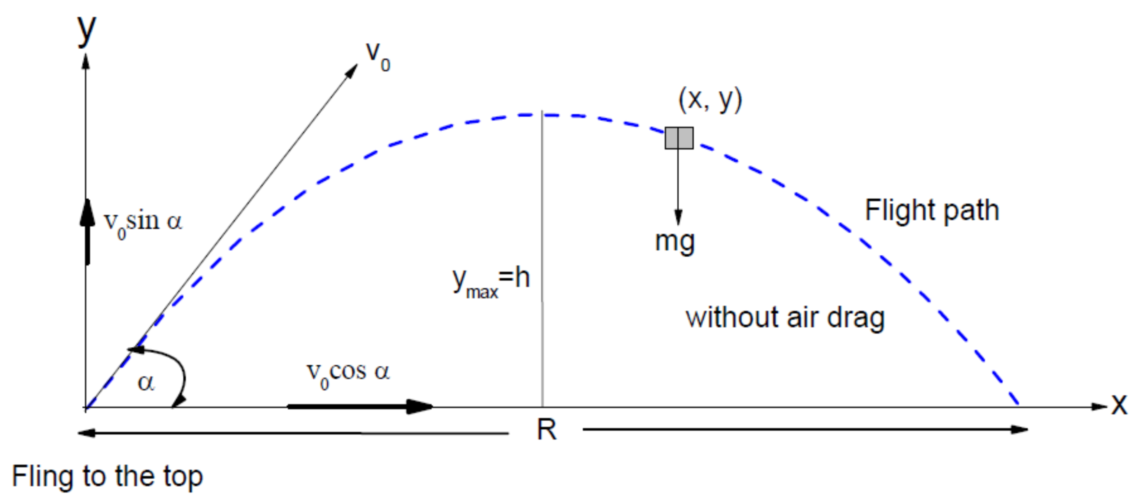
Trajectory = the flight path of a projectile.

The trajectory of a projectile consists of a vertical and horizontal component.

In the projectile motion the object moves in a bilaterally symmetrical, parabolic path. The path that the object follows is called its trajectory. Projectile motion only occurs when there is one force applied at the beginning on the trajectory, after which the only interference is from gravity. In a previous atom we discussed what the various components of an object in projectile motion are. In this atom we will discuss the basic equations that go along with them in the special case in which the projectile initial positions are null .

<u>How do we handle 2D projectile motion mathematically?</u>

One of the easiest ways to deal with 2D projectile motion is to just analyze the motion in each direction separately. In other words, we will use one set of equations to describe the horizontal motion of the lime, and another set of equations to describe the vertical motion of the lime. This turns a single difficult 2D problem into two simpler 1D problems. We're able to do this since the change in the vertical velocity of the lime does not affect the horizontal velocity of the lime. Similarly, throwing the lime with a large horizontal velocity does not affect the vertical acceleration of the lime. In other words, if you fire a bullet horizontally and drop a bullet at the same time, they will hit the ground at the same time.



When a particle is projected obliquely near the earth's surface, it moves simultaneously in the direction of horizontal and vertical. The motion of such a particle is called Projectile Motion. In the above diagram, where a particle is projected at an angle α , with an initial velocity v_0. For this particular case, we will study the motion in two direction (horizontal and vertical)

For the particular, we will calculate the following:

1- The velocity at any time "t " during the motion.
2- Time of reach maximum height
3-The maximum height reached during the motion.
4-Time of flight or total time
5-The horizontal distance (Range)

Horizontal direction:

There's no acceleration in the horizontal direction since gravity does not pull projectiles sideways, only downward. Air resistance would cause a horizontal acceleration, slowing the horizontal motion, but since we're going to only consider cases where air resistance is negligible we can assume that the horizontal velocity is constant for a projectile. So for the horizontal direction we can use the following equation

$m\ddot{x} = 0 \quad Or \quad \ddot{x} = 0$ (1)

Integration Eq. (1), we have $\ddot{x} = \frac{d\dot{x}}{dt} = 0 \rightarrow \dot{x} = c_1$

From the initial condition at $t = 0 \rightarrow \dot{x}|_{t=0} = v_0 \cos\alpha \rightarrow c_1 = v_0 \cos\alpha$. Then

$\dot{x} = v_0 \cos\alpha$ (2)

Integration Eq. (2), we have $x = v_0 t \cos\alpha + c_2$

From the initial condition at $t = 0$, $x = 0 \rightarrow$, then $c_2 = 0$

$x = v_0 t \cos\alpha$ (3)

Vertical direction:

Two-dimensional projectiles experience a constant downward acceleration due to gravity. Since the vertical acceleration is constant, we can solve for a vertical variable with one of the four kinematic formulas which are shown below.

$m\ddot{y} = -mg \quad Or \quad \ddot{y} = -g$ (4)

Integration Eq. (4), we have $\ddot{y} = \frac{d\dot{y}}{dt} = -g \rightarrow \dot{y} = -gt + c_3$

From the initial condition at $t = 0$, $\dot{y}|_{t=0} = v_0 \sin\alpha \rightarrow c_2 = v_0 \sin\alpha$. Then

$\dot{y} = v_0 \sin\alpha - gt$ (5)

Integration Eq. (5), we have $y = v_0 t \sin\alpha - \frac{1}{2} g t^2 + c_4$

From the initial condition at $t = 0$, $y = 0$, $\rightarrow c_4 = 0$. Then

$y = v_0 t \sin\alpha - \frac{1}{2} g t^2$ (6)

The properties of projectile

(1) Parametric Equation

From Eq. (3), we find $t = \dfrac{x}{v_0 \cos\alpha}$ and substitute in Eq.(6), we get

$$y = v_0 t \sin\alpha - \frac{1}{2} g t^2 = v_0 \left(\frac{x}{v_0 \cos\alpha}\right) \sin\alpha - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha}\right)^2 = \frac{\sin\alpha}{\cos\alpha} x - \frac{g x^2}{2 v_0^2 \cos^2\alpha}$$

$$y = x \tan\alpha - \frac{g x^2}{2 v_0^2} \sec^2\alpha \qquad (7)$$

2- Time of reach maximum height

The time is taken to reach the maximum point is called the time of reach maximum height. The maximum height is reached when $(\dot{y} = 0)$. Using this we can rearrange the velocity equation to find the time it will take for the object to reach maximum height. At the maximum point, the vertical velocity will vanish, i.e.

$\dot{y} = v_0 \sin\alpha - g t = 0$. Then the time of reach maximum height given by

$$t_{y = y_{\max}} = t = \frac{v_0}{g} \sin\alpha \qquad (8)$$

3-The maximum height (Greatest Height)

The maximum vertical distance to which the particle reaches during the motion is called the maximum height.

Substitute from Eq. (8) into Eq. (6), we have

$y = v_0 \left(\dfrac{v_0 \sin\alpha}{g}\right) \sin\alpha - \dfrac{1}{2} g \left(\dfrac{v_0 \sin\alpha}{g}\right)^2$, then

$$y_{\max.} = h = \frac{v_0^2}{2g} \sin^2\alpha \qquad \text{(Maximum Height Formula)} \qquad (9)$$

Note, at the maximum height $\left(x_{t_h}, y_{\max.}\right) = \left(\dfrac{v_0^2}{2g} \sin 2\alpha,\ \dfrac{v_0^2}{2g} \sin^2\alpha\right)$.

4-Time of flight (Total time of the whole journey)

The time of flight of a projectile motion is the time from when the object is projected to the time it reaches the surface. Or, the total time for which the projectile remains in the air is called the time of flight. In this case the projectile is fall on the $x-$ axis, i. e.

$(y=0)$, $y = v_0 t \sin\alpha - \frac{1}{2} g t^2 = 0 \quad \rightarrow \quad t = 0$, or

$t = 2\frac{v_0}{g}\sin\alpha$.

So the total time is given by

$$T = 2\frac{v_0}{g}\sin\alpha \qquad (10)$$

5-Range

The range of the motion is fixed by the condition ($y=0$). Using this we can rearrange the parabolic motion equation to find the range of the motion

$$R = \frac{2v_0^2 \sin\alpha \cos\alpha}{g} = \frac{v_0^2}{g}\sin 2\alpha \qquad (11)$$

6. The same property Range

If we throw (fling) an object by the same initial velocity and with the two angles α and $\frac{\pi}{2}-\alpha$. From Eq. (11), we find that

$$R_1 = \frac{v_0^2}{g}\sin 2\alpha \qquad (12)$$

Again in Eq. (11) $R_2 = \frac{v_0^2}{g}\sin 2(\frac{\pi}{2}-\alpha) = \frac{v_0^2}{g}\sin(\pi - 2\alpha) = \frac{v_0^2}{g}\left\{\underbrace{\sin\pi}_{0}\cos 2\alpha - \underbrace{\cos\pi}_{=-1}\sin 2\alpha\right\}$.Then

$$R_2 = \frac{v_0^2}{g}\sin 2\alpha \qquad (13)$$

From Eqs. (12) and (13), some note that, the same range happens at α and $\frac{\pi}{2}-\alpha$. For example, at $\alpha = 30^o$, $\alpha = 60^o$, we have the same range $R_2 = R_1 = \frac{\sqrt{3}}{4}\frac{v_0^2}{g}$

6. The maximum Range

From Eq. (11) the maximum Range happens at $\sin 2\alpha = 1$. In this case

$$R_{\max} = \frac{v_0^2}{2g} \text{ and } \alpha = \frac{\pi}{4}.$$

7. The velocity at any point

From Eq. (2) and Eq. (5), we have, respectively.

$$\dot{x}^2 = v_0^2 \cos^2 \alpha,$$

$$\dot{y}^2 = (v_0 \sin\alpha - gt)^2 = v_0^2 \sin^2 \alpha - 2g\,t\,v_0 \sin\alpha + (gt)^2$$

$$= v_0^2 \sin^2 \alpha - 2g\left(v_0\, t \sin\alpha - \frac{1}{2} g t^2 \right) = v_0^2 \sin^2 \alpha - 2g\,y,$$

Then $v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{v_0^2 \cos^2 \alpha + v_0^2 \sin^2 \alpha - 2g\,y} = \sqrt{v_0^2 (\cos^2 \alpha + \sin^2 \alpha) - 2g\,y}$

$$v = \sqrt{v_0^2 - 2g\,y} \qquad (14)$$

The direction of velocity is given by

$$\tan\theta = \frac{\dot{x}}{\dot{y}} = \frac{\dot{x}}{\sqrt{\dot{y}^2}} = \frac{v_0 \cos\alpha}{\sqrt{(v_0 \sin\alpha - gt)^2}} = \frac{v_0 \cos\alpha}{\sqrt{v_0^2 \sin^2 \alpha - 2g\,y}}$$

$$\tan\theta = \pm \frac{v_0 \cos\alpha}{\sqrt{v_0^2 \sin^2 \alpha - 2g\,y}} \qquad (15)$$

Also, it take the formula

$$\cos\theta = \pm \frac{v_0 \cos\alpha}{\sqrt{v_0^2 - 2g\,y}} \qquad (16)$$

Example:1 Determine the angle of projection for which maximum height is equal to the range of the projectile?

Solution

The maximum height $y_{\max.} = h = \dfrac{v_0^2 \sin^2 \alpha}{2g}$

The range $R = \dfrac{2 v_0^2 \sin\alpha \cos\alpha}{g} = \dfrac{v_0^2 \sin 2\alpha}{g}$

$$\frac{v_0^2 \sin^2 \alpha}{2g} = \frac{2 v_0^2 \sin\alpha \cos\alpha}{g} \rightarrow \frac{\sin\alpha}{2} = 2\cos\alpha \rightarrow \tan\alpha = 4 \rightarrow \alpha = \tan^{-1}(4)$$

Example 2: Prove that the maximum range of the projectile is given by $\frac{v_0^2}{g}$, what is the angle of projection for projectile to have maximum range?

Solution

The range $R = \frac{2v_0^2 \sin\alpha \cos\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g}$

Where g and v_0^2 are constants, so the maximum range verified when $\sin 2\alpha$ is the greatest possible, that it will be at $\sin 2\alpha = 1 \ \rightarrow \ 2\alpha = \frac{\pi}{2} \rightarrow \ \alpha = \frac{\pi}{4} = 45^O$

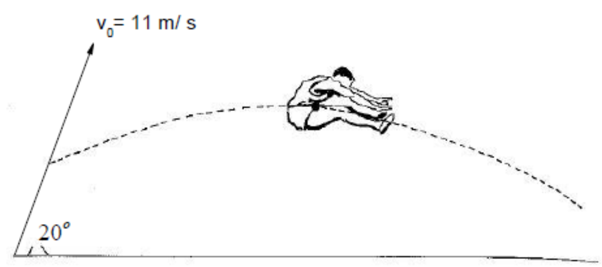
So, the maximum range of the projectile given by $\frac{v_0^2}{g}$

Example 3: A Jumper leaves the ground at an angle of 20^O above the horizontal and at a speed of $v_0 = 11 m/\sec$

(A) What is the maximum height reached?

(B) How far does he jump in the horizontal direction and the necessary time for that?

Solution



The maximum height (Greatest Height or Maximum Height Formula) of the jumper given by

$$y_{max.} = h = \frac{v_0^2}{2g}\sin^2\alpha = \frac{(11)^2}{2(9.8)}(\sin 20^O)^2 = \frac{(11)^2}{19.6}(0.342)^2 = \frac{(11)^2}{19.6}(0.116964) = \frac{14.152644}{19.6}$$

$$\therefore \ y_{max.} = 0.722m$$

The range of the jumper given by :

$$R = \frac{v_0^2}{g}\sin 2\alpha = \frac{(11)^2}{9.8}\sin 40^O = \frac{(11)^2}{9.8}(0.64279) = \frac{77.777}{9.8} = 7.94m$$

Time of Flight $T = 2\frac{v_0}{g}\sin\alpha = \frac{2(11)}{9.8}\sin 20^O = \frac{2(11)}{9.8}(0.342) = \frac{2(11)(0.342)}{9.8} = 0.7677\sec$

Example 4: A place kicker must kick a football from a point 36 m from the goal. Half the crowd hopes the ball will clear the crossbar, which is 3.05 meters high. When kicked, the ball leaves the ground with a speed of 20 m/s and an angle of 53 degrees above the horizontal.

(a) By how much does the ball clear or fall short of the crossbar?

(b) Does the ball approach the crossbar while still rising or while falling?

Solution

Assume air resistance is negligible, Assume no rotation of the ball.

The horizontal distance is $x = 36m$, The initial velocity is $v_0 = 20\ m/\sec$

The angle of initial velocity speed is $\alpha = 53^O\ m$

(a) Then the vertical distance is at $x = 36\ m$

$$y = x\tan\alpha - \frac{g\,x^2}{2v_0^2\cos^2\alpha} \quad Or \quad y = x\tan\alpha - \frac{g\,x^2}{2v_0^2}\sec^2\alpha$$

Then $y = x\tan\alpha - \dfrac{g\,x^2}{2v_0^2\cos^2\alpha}$, $\qquad y = 36\tan 53^O - \dfrac{9.8\,(36)^2}{2(20)^2\,(\cos 53)^2}$

$$y = (36)\,(1.327) - \frac{9.8\,(36)^2}{2(20)^2\,(0.6012)^2} = 47.77 - \frac{9.8\,(1296)}{2(400)(0.3622)}$$

$$= 47.77 - \frac{12700.8}{289.76} = 47.77 - 43.8321 = 3.93m$$

$\therefore\ y = 3.93m$

Then the football will pass over the crossbar with a distance of $3.93 - 3.05 = 88cm$

Then will falling at $R = \dfrac{v_0^2}{g}\sin 2\alpha = \dfrac{400}{9.8}\sin(106) = \dfrac{400}{9.8}(0.96126) = \dfrac{384.5046}{9.8} = 39.2351\,m$

(b) In order to the football barely makes it over the bar or descending

$$y = v_0\,t\sin\alpha - \frac{1}{2}g\,t^2 \ \rightarrow\ 3.05 = (20)t\sin(53) - \frac{1}{2}(9.8)t^2 \ \rightarrow\ 3.05 = (20)t\,(0.7986) - \frac{1}{2}(9.8)t^2$$

$$4.9t^2 - 15.9727t + 3.05 = 0 \ \rightarrow\ t = \frac{1}{2(4.9)}\left[15.9727 \pm \sqrt{(15.9727)^2 - 4(4.9)(3.05)}\right]$$

$$t = \frac{1}{2(4.9)}\left[15.9727 \pm \sqrt{255.1274 - 59.78}\right] = \frac{1}{2(4.9)}\left[15.9727 \pm \sqrt{195.3474}\right]$$

$$t = \frac{1}{9.8}\left[15.9727 \pm 13.97667\right] \rightarrow t = \frac{29.9493}{9.8} = 3.056\sec \ \ or \ \ \frac{1.99603}{9.8} = 0.2036\sec$$

Then the football takes $t = 3.056\sec$ to arrive the

The corresponding horizontal distance given by

$$x = v_0\,t\,\cos\alpha \ \rightarrow\ x = (20)(3.05)\cos(53) = (61)(0.60181) \rightarrow\ x = 36.71\ m$$

Example 5: A place kicker must kick a football from a point 33.8 m from a goal. As a result of the kick, the ball must clear the crossbar, which is 3.05 m high. When kicked the ball leaves the ground with a speed of 21.6 m/s at an angle of 53° to the horizontal.

(a) By how much does the ball clear or fall short of clearing the crossbar?

(b) Does the ball approach the crossbar while still rising, or falling? Prove your answer mathematically.

Solution

Assume air resistance is negligible, Assume no rotation of the ball.

The horizontal distance is $x = 33.8\ m$

The initial velocity is $v_0 = 21.6\ m/\sec$

The angle of initial velocity speed is $\alpha = 53^O\ m$

Then the vertical distance is at $x = 33.8\ m$

$$y = x\tan\alpha - \frac{g\,x^2}{2v_0^2\cos^2\alpha} \quad Or \quad y = x\tan\alpha - \frac{g\,x^2}{2v_0^2}\sec^2\alpha$$

Then $y = x\tan\alpha - \dfrac{g\,x^2}{2v_0^2\cos^2\alpha} \quad \rightarrow \quad y = 33.8\tan 53^O - \dfrac{9.8\,(33.8)^2}{2(21.6)^2(\cos 53)^2}$

$$y = (33.8)\,(1.33) - \frac{9.8\,(33.8)^2}{2(21.6)^2\,(0.6012)^2} = 44.954 - \frac{9.8\,(1142.44)}{2(466.56)(0.3622)} \quad \therefore\ y = 11.124m$$

$$= 47.77 - \frac{11195.912}{337.976} = 44.954 - 33.83 = 11.124m$$

The corresponding time given by $x = v_0\,t\cos\alpha$

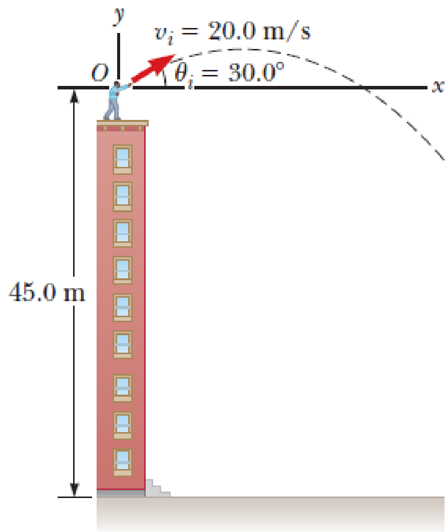
Then $t = \dfrac{x}{v_0\,\cos\alpha} = \dfrac{33.8}{(21.6)\cos 53^O} = \dfrac{33.8}{(21.6)(0.6012)} = \dfrac{33.8}{12.9859} = 2.6\ \sec$

Example 6: A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s. The height from which the stone is thrown is 45.0 m above the ground.

(A) How long does it take the stone to reach the ground and the horizontal distance from the building ?

(B) What is the speed of the stone just before it strikes the ground?

Solution



Analyze: We have the information $y = -45m$, $v_0 = 20m/\sec$, $\alpha = 30^o$

The answer is required

(i) The horizontal distance until the stone falls to the ground

(ii) Time taken until the stone falls to the ground

(iii) The speed of the stone just before it strikes the ground

(i) We know that, $y = x\tan\alpha - \dfrac{g\,x^2}{2v_0^2\cos^2\alpha} = x\tan\alpha - \dfrac{g\,x^2}{2v_0^2}\sec^2\alpha$, then

$$-45 = x\tan(30^O) - \frac{(10)x^2}{2(20)^2(\cos(30^O))^2} = x(0.577) - \frac{(10)x^2}{2(20)^2(0.866)^2} = (0.577)x - \frac{10\,x^2}{2(400)(0.75)}$$

$$-45 = (0.577)x - \frac{x^2}{(80)(0.75)} \rightarrow -(45) = (0.577)\,x - \frac{x^2}{60} \rightarrow -(45)(60) = (0.577)(60)\,x - x^2$$

$$x^2 - 34.62x - 2700 = 0$$

$$x = \frac{(34.62) \pm \sqrt{(34.62)^2 + (4)(2700)}}{2} = \frac{(34.62) \pm \sqrt{1198.5444 + 10800}}{2}$$

$$= \frac{(34.62) \pm \sqrt{11998.5444}}{2} = \frac{(34.62) \pm 109.5378}{2} = \frac{144.1578}{2}$$

$$x = 72.078m$$

(ii) The time given by $x = v_0\,t\cos\alpha$

$$t = \frac{x}{v_0\cos\alpha} = \frac{72.078}{20\cos 30^O} = \frac{72.078}{20(0.866025)} = \frac{72.078}{17.32} = 4.161\sec$$

Another Solution

We know that $y = v_0 t \sin\alpha - \frac{1}{2} g t^2 \Rightarrow -45 = (20)t \sin 30^O - \frac{1}{2}(10)t^2$

$-45 = (20)t \frac{1}{2} - \frac{1}{2}(10)t^2 \Rightarrow -450 = 10t - 5t^2 \Rightarrow -90 = 2t - t^2$

$t^2 - 2t - 9 = 0$

$t = \frac{2 \pm \sqrt{(2)^2 + (4)(9)}}{2} = \frac{2 \pm \sqrt{4 + (4)(9)}}{2} = \frac{2 \pm 2\sqrt{1+9}}{2} = 1 \pm \sqrt{10} = 1 \pm 3.16$

$t = 4.16 \sec$

(iii) $\left(\dot{x}, \dot{y}\right) = \left(v_0 \cos\alpha, v_0 \sin\alpha - gt\right),$

Before it strikes the ground the time is $t = 4.1 \sec$, then

$v = \sqrt{(v_0 \cos\alpha)^2 + (v_0 \sin\alpha - gt)^2} = \sqrt{(20\cos 30^O)^2 + (20\sin 30^O - 10t)^2}$

$= \sqrt{(20(0.866))^2 + (20\sin(0.5) - 10(4.1))^2} = \sqrt{(17.32)^2 + (10-41)^2}$

$v = \sqrt{(17.32)^2 + (-31)^2} = \sqrt{300 + 961} = \sqrt{1261} = 35.51 m/\sec$

Example 7: A fire fighter aims a fire hose upward, toward a fire in a skyscraper. The water leaving the hose has a velocity of 32.0 *m/s*. If the fire fighter holds the hose at an angle of 78.5°, what is the maximum height of the water stream?

Solution

Analyze: We have the information, $v_0 = 32 m/\sec, \ \alpha = 78.5^o$

The answer is required

(i) The maximum height of the water stream

(ii) Time taken until the stone falls to the ground

(iii) The speed of the stone just before it strikes the ground

$y_{\max.} = h = \frac{v_0^2 \sin^2\alpha}{2g} = \frac{(32)^2(\sin 78.5^O)^2}{2(9.8)} = \frac{(32)^2(0.9799)^2}{19.6} = \frac{(1024)(0.96025)}{19.6} = \frac{983.296}{19.6}$ The

$\therefore \ y_{\max.} = 50.16 m$

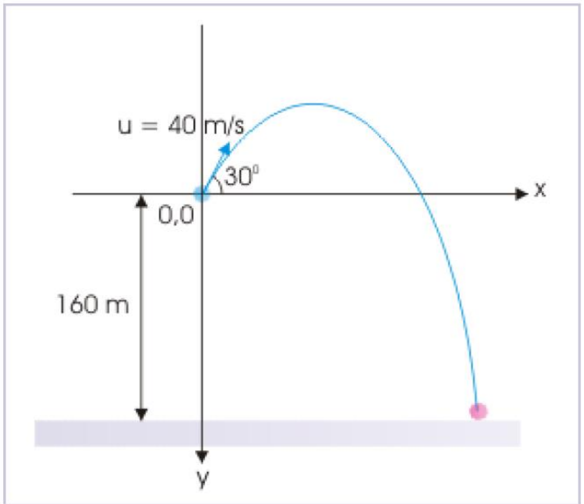
maximum height of the water from the hose is 50.2 *m*.

Example:8 A projectile is thrown from the top of a building $160\,m$ m high, at an angle of 30^{O} with the horizontal at a speed of $40\,m/s$ 40, find

(i) Time of flight, (ii) Horizontal distance covered at the end of journey

(iii) The maximum height of the projectile above the ground.

Solution



(i) We know that, $y = x\tan\alpha - \dfrac{g\,x^2}{2\mathrm{v}_0^2\cos^2\alpha} = x\tan\alpha - \dfrac{g\,x^2}{2\mathrm{v}_0^2}\sec^2\alpha$, then

$$-160 = x\tan(30^{O}) - \frac{(10)\,x^2}{2(40)^2(\cos(30^{O}))^2} = x(0.577) - \frac{(10)\,x^2}{(3200)(0.866)^2} = (0.577)x - \frac{10\,x^2}{(3200)(0.75)}$$

$$-160 = (0.577)x - \frac{x^2}{(320)(0.75)} \rightarrow -160 = (0.577)\,x - \frac{x^2}{240} \rightarrow -160(240) = (0.577)(240)\,x - x^2$$

$$x^2 - 138.48x - 2700 = 0$$

$$x = \frac{(34.62) \pm \sqrt{(34.62)^2 + (4)(2700)}}{2} = \frac{(34.62) \pm \sqrt{1198.5444 + 10800}}{2}$$

$$= \frac{(34.62) \pm \sqrt{11998.5444}}{2} = \frac{(34.62) \pm 109.5378}{2} = \frac{144.1578}{2}$$

$$x = 72.078m$$

(ii) The time given by $x = \mathrm{v}_0\,t\cos\alpha$

$$t = \frac{x}{\mathrm{v}_0\cos\alpha} = \frac{72.078}{20\cos 30^{O}} = \frac{72.078}{20(0.866025)} = \frac{72.078}{17.32} = 4.161\sec$$

Another Solution

We know that $y = \mathrm{v}_0\,t\,\sin\alpha - \dfrac{1}{2}g\,t^2$

$$-45 = (20)t\ \sin 30^{O} - \frac{1}{2}(10)t^2$$

$$-45 = (20)t\ \frac{1}{2} - \frac{1}{2}(10)t^2 \ \Rightarrow -450 = 10t - 5t^2 \ \Rightarrow -90 = 2t - t^2$$

$$t^2 - 2t - 9 = 0$$

$$t = \frac{2 \pm \sqrt{(2)^2 + (4)(9)}}{2} = \frac{2 \pm \sqrt{4 + (4)(9)}}{2} = \frac{2 \pm 2\sqrt{1+9}}{2} = 1 \pm \sqrt{10} = 1 \pm 3.16$$

$$t = 4.16\text{sec}$$

Chapter: 3 Mechanics of Rigid body

Definition of the Rigid body

In physics, a rigid body is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it. A rigid body is usually considered as a continuous distribution of mass.

Definition of moment of inertia

Physical; A measure of the resistance of a body to angular acceleration about a given axis

Mathematic; The Moment of Inertia is equal to the sum of the products of each element of mass in the body and the square of the element's distance from the axis. It is defined as the sum of second moment of area of individual section about an axis

(1) The basic shapes

(2) Systems of particles

(3) Composite bodies (shapes)

(4) Uninform shapes

The Moment of Inertia of masses

The mass moment of inertia about a fixed axis is the property of a body that measures the body's resilience to rotational acceleration. The greater its value, the greater the moment required to provide a given acceleration about a fixed pivot. The moment of inertia must be specified with respect to a chosen axis of rotation.

(1)- For a single mass, the moment of inertia can is expressed as

For the element dm that is located a distance a from the L-axis, the Moment of inertia referenced to L-axis is given as

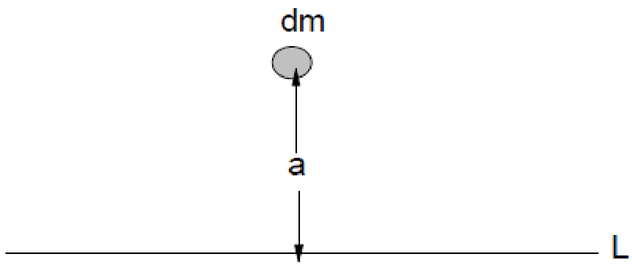


Fig. 1

$I_{LL} = dm\,a^2$

(2)- If a system consists of $n-$bodies, then the moment of inertia can be given as
For the $n-$elements, they have the mass $dm_1,\ dm_2,\ dm_3, ..., dm_n$ that is located a distance a from the L-axis, the moment of inertia referenced to L-axis is given as

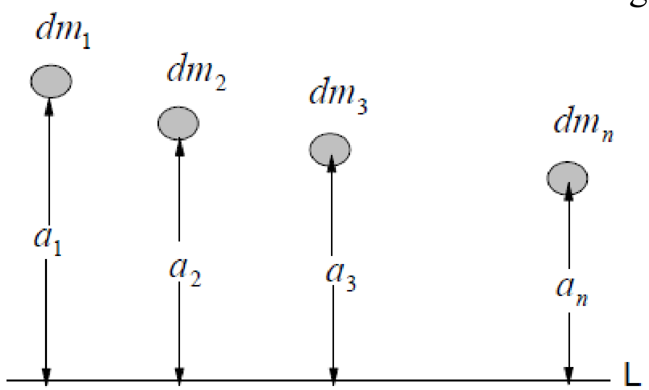


Fig. 2

$$I_{LL} = dm_1\,a_1^2 + dm_2\,a_2^2 + dm_3\,a_3^2 + ... + dm_n\,a_n^2 = \sum_{i=1}^{n} dm_i\,a_i^2$$

(3)- The Moment of Inertia in the plane

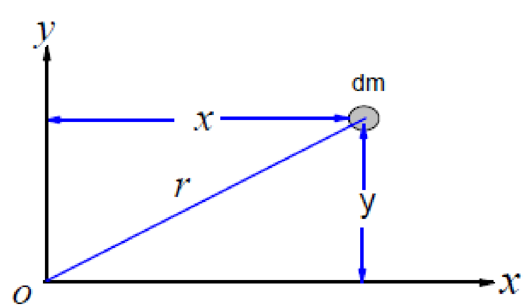


Fig. 3

Referenced to x-axis is given by $I_{xx} = dm\,y^2$,

Referenced to y-axis is given by $I_{yy} = dm\,x^2$,

Referenced to the original point (O) is given by

$$I_O = dmr^2 = m(x^2 + y^2) = I_{xx} + I_{yy}$$

I_O is called Polar moment inertial

(4)- The Moment of Inertia in the plane for number of elements

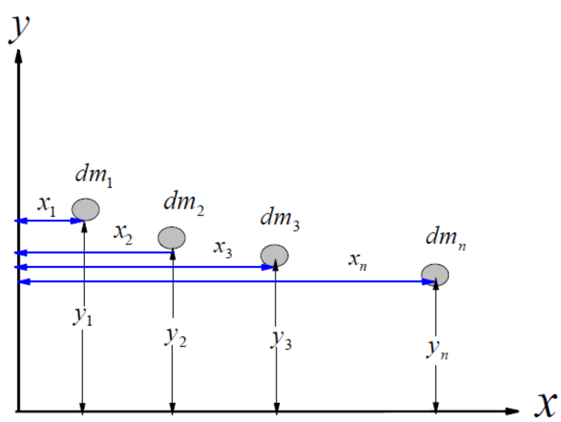


Fig. 4

Referenced to x-axis is given by $\qquad I_{xx} = \sum_{i=1}^{n} dm_i\, y_i^2$

Referenced to y-axis is given by $\qquad I_{yy} = \sum_{i=1}^{n} dm_i\, x_i^2$

(4)- The Moment of Inertia in space

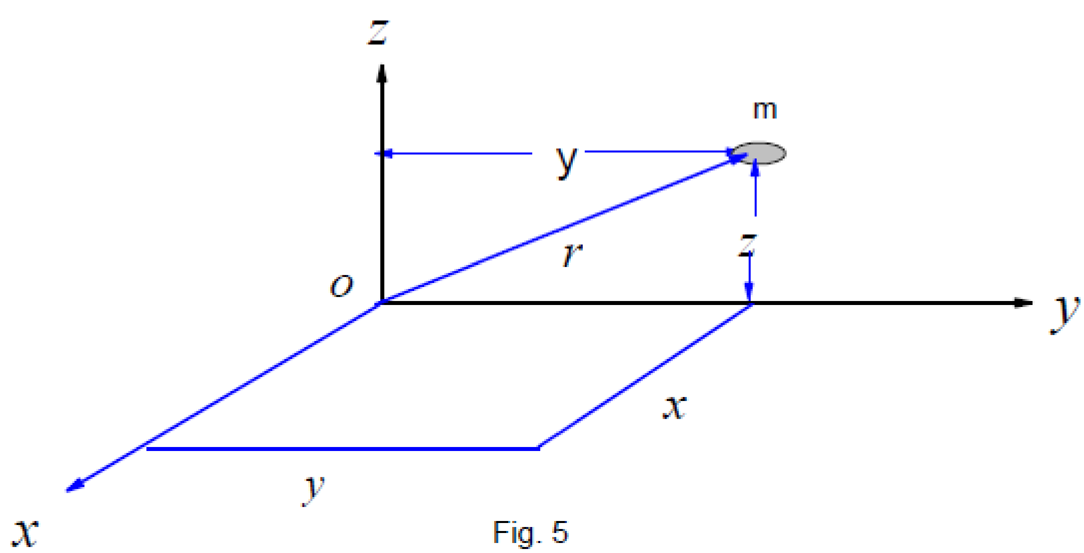


Fig. 5

Referenced to the original point (O) is given by

$$I_O = mr^2 = m(x^2 + y^2 + z^2) \qquad (1)$$

Referenced to x-axis is given by $\qquad I_{xx} = m(y^2 + z^2),$

Referenced to y -axis is given by $I_{yy} = m(x^2 + z^2)$,

Referenced to z -axis is given by $I_x = m(x^2 + y^2)$,

Referenced to the plane $-x = 0$ is given by $I_{xx} = m(y^2 + z^2)$,

Referenced to the plane $I_y = m(x^2 + z^2)$, is given by $-y=0$

Referenced to the plane $z=0$ is given by $I_z = m(x^2 + y^2)$,

From previous relation, we have

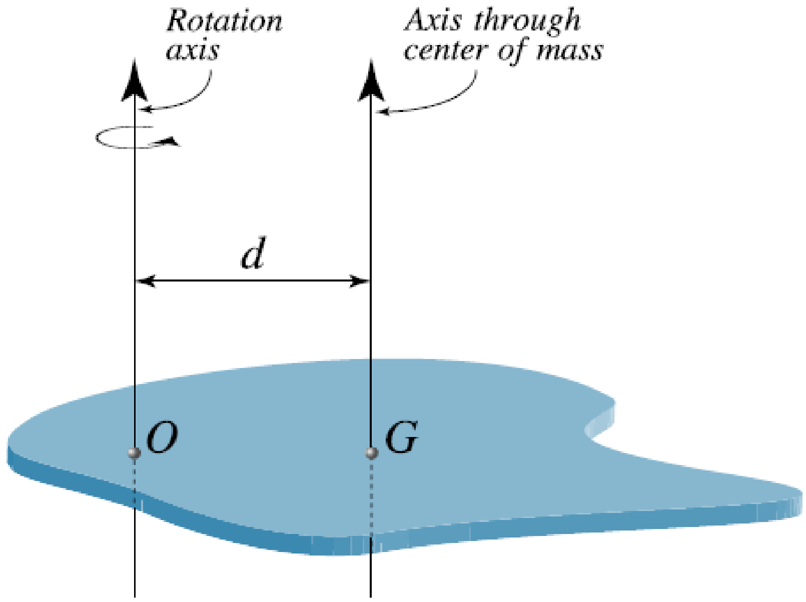
$$I_o = mr^2 = m(x^2 + y^2 + z^2) = I_{xoy} + I_{xoz} + I_{yoz}$$

$$2I_o = I_{xx} + I_{yy} + I_{zz} \quad \text{or} \quad I_o = mr^2 = m(x^2 + y^2 + z^2) = \frac{1}{2}(I_{xx} + I_{yy} + I_{zz})$$

$$I_{xx} = m(y^2 + z^2) = I_{xoy} + I_{xoz}$$
$$I_{yy} = m(x^2 + z^2) = I_{xoy} + I_{yoz}$$
$$I_{zz} = m(x^2 + y^2) = I_{xoz} + I_{yoz}$$

Parallel axis theorem

Parallel axis theorem is applicable to bodies of any shape. The theorem of parallel axis states that the moment of inertia of a body about an axis parallel to an axis passing through the centre of mass is equal to the sum of the moment of inertia of body about an axis passing through centre of mass and product of mass and square of the distance between the two axes. The parallel axis theorem is much easier to understand in equation form than in words. Here it is:



In physics, the parallel axis theorem can be used to determine the moment of inertia of a rigid object about any axis, given the moment of inertia of the object about the parallel axis through the object's center of mass and the perpendicular distance between the axes. We consider an element (m) and its center is (x_{cm}, y_{cm})(see below Figure)

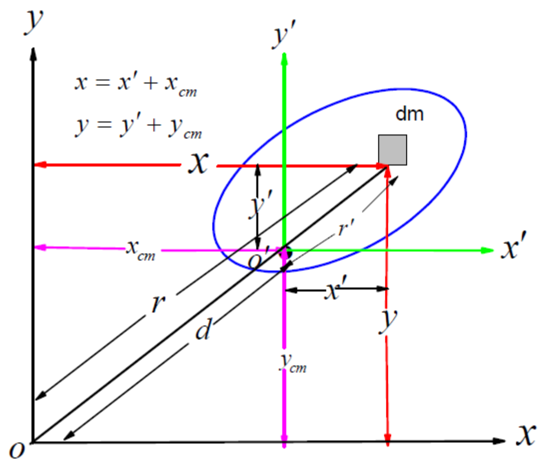


Fig. 7

$dI_{xx} = dm\,y^2$, the moment of inertial with respect to $x-$ axis

$dI_{yy} = dm\,x^2$, the moment of inertial with respect to $y-$ axis

$dI_O = dm\,r^2 = I_{xx} + I_{yy} = dm(x^2 + y^2)$, the moment of inertial with respect to the point(o)

$$I_O = \int r^2 dm = \int (x^2 + y^2) dm \qquad (1)$$

$$I_{cm} = \int r'^2 dm = \int (x'^2 + y'^2) dm \qquad (2)$$

$x = x' + x_{cm}, \quad y = y' + y_{cm}$

$$I_O = \int r^2 dm = \int \left\{ \left(x' + x_{cm} \right)^2 + \left(y' + y_{cm} \right)^2 \right\} dm$$

$$= \int \left\{ x'^2 + x_{cm}^2 + 2x'\,x_{cm} + y'^2 + y_{cm}^2 + 2y'\,y_{cm} \right\} dm$$

$$I_O = \underbrace{\int \left(x'^2 + y'^2 \right) dm}_{I_{cm}} + \int \underbrace{\left(x_{cm}^2 + y_{cm}^2 \right)}_{=d^2} dm + 2x_{cm} \int x'\,dm + 2y_{cm} \int y'\,dm$$

$$I_O = I_{cm} + \int d^2\,dm + 2x_{cm} \int x'\,dm + 2y_{cm} \int y'\,dm$$

$$I_O = I_{cm} + d^2 \int dm + 2x_{cm} \int x'\,dm + 2y_{cm} \int y'\,dm$$

$$I_O = I_{cm} + d^2\, m + 2x_{cm}\int x'\, dm + 2y_{cm}\int y'\, dm \qquad (3)$$

$$\bar{x} = \frac{\int x'\, dm}{\int dm} \rightarrow \int x'\, dm = \bar{x}\int dm, \qquad \bar{y} = \frac{\int y'\, dm}{\int dm} \rightarrow \int y'\, dm = \bar{y}\int dm \qquad (4)$$

$$I_O = I_{cm} + d^2\, m + 2x_{cm}\left\{\bar{x}\int dm\right\} + 2y_{cm}\left\{\bar{y}\int dm\right\}$$

$$I_O = I_{cm} + d^2\, m + 2x_{cm}\,\bar{x}\, m + 2y_{cm}\,\bar{y}\, m \qquad (5)$$

$$I_O = I_{cm} + m d^2 \qquad (6)$$

Question: Let I_A and I_B be moments of inertia of a body about two axes A and B respectively. The axis A passes through the centre of mass of the body but B does not, So.

(A) $I_A < I_B$ (B) $I_A > I_B$ (C) If the axes are parallel $I_A < I_B$

(D) If the axes are parallel $I_A > I_B$ (E) If the axes are not parallel $I_A > I_B$

The moment of inertia is always less for an axis passing through the center of mass than any other parallel axis. We cannot say anything of the moment of inertia about a non parallel axis. Thus C is correct.

Perpendicular Axis Theorem

This theorem is applicable only to the planar bodies. Bodies which are flat with very less or negligible thickness. This theorem states that the moment of inertia of a planar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with the perpendicular axis and lying in the plane of the body.

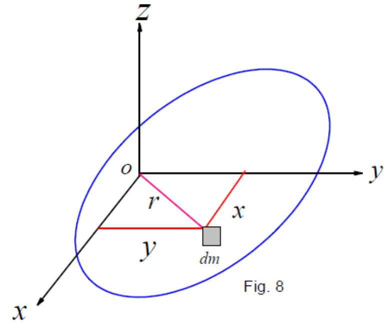


Fig. 8

$dI_{xx} = dm\, y^2$, the moment of inertial with respect to $x-$ axis

$dI_{yy} = dm\, x^2$, the moment of inertial with respect to $y-$ axis

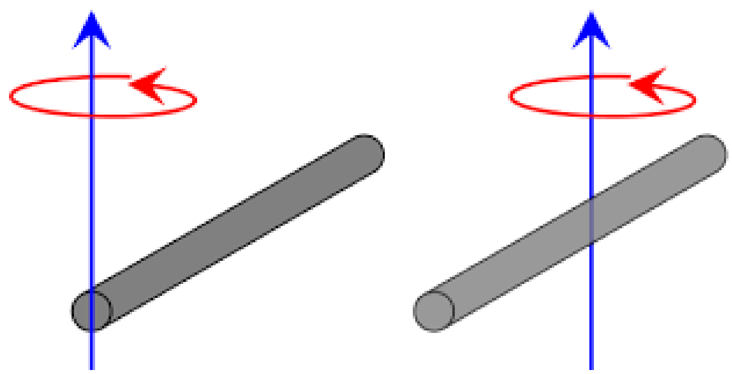
$dI_O = dm\, r^2 = I_{xx} + I_{yy} = dm(x^2 + y^2)$,, the moment of inertial with respect to the point (o)

$$I_O = \int (x^2 + y^2)\, dm = \int r^2 dm = r^2 \int dm = r^2 m \qquad (1)$$

$$I_{zz} = I_{xx} + I_{yy} \qquad (2)$$

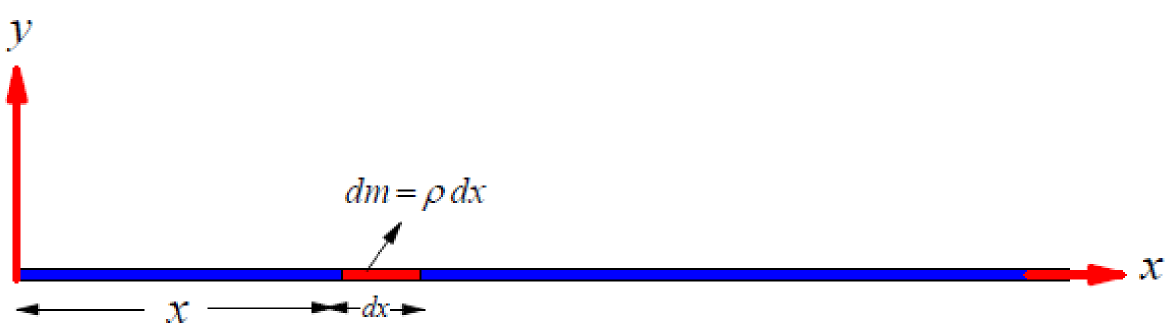
Example:1 Find the Mass moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through one of its ends. Also, about an axis perpendicular to its length and passing through its center?

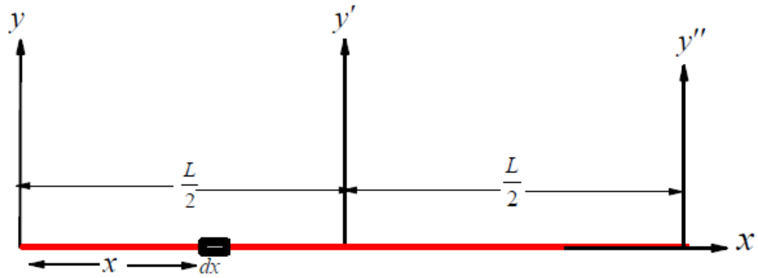
Solution



We consider L be the length of the Rod, M be the mass of the Rod and is the density ρ. We divided the Rod into many small elements. We select one of them, that has length dx, mass dm and has the distance x from the left end of the Rod

For the small element $dm = \rho\, dx \rightarrow m = \int_0^L \rho\, dx = \rho \int_0^L dx = \rho\, x\Big|_0^L \rightarrow m = \rho L$





The moment of inertia about its end is given by

$$I_{yy} = \int x^2 dm = \int_0^L x^2(\rho\, dx) = \frac{1}{3}\rho L^3 = \frac{1}{3}\rho L^3 \frac{m}{\rho L^3} = \frac{1}{3} m L^2 \qquad \therefore\ I_{yy} = \frac{1}{3} m L^2$$

This the moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through one of its ends.

The moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through its center. From the Parallel axis theorem

$$I_{yy} = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \rightarrow \frac{1}{3} m L^2 = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \rightarrow I_{y'y'} = \frac{1}{3} m L^2 - \frac{1}{4} mL^2 = \left(\frac{4-3}{12}\right) mL^2 = \frac{1}{12} mL^2$$

$$\therefore\ I_{y'y'} = \frac{1}{12} mL^2$$

The moment of inertia about its other end

$$I_{y''y''} = \frac{1}{12} m L^2 + m\left(\frac{1}{2}L\right)^2 = \frac{1}{12} m L^2 + \frac{1}{4} mL^2 = \left(\frac{1+3}{12}\right) mL^2 = \frac{4}{12} mL^2 \quad I_{y''y''} = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \rightarrow$$

$$\therefore\ I_{y''y''} = \frac{1}{3} mL^2$$

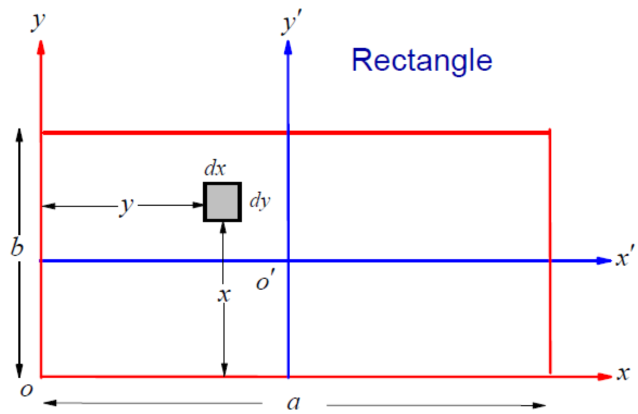
Note: The moment of inertia for a thin uniform Rod that rotates about the axis perpendicular to the rod and passing through one end is $\frac{1}{3} mL^2$. If the axis of rotation passes through the center of the Rod, then the moment of inertia is $\frac{1}{12} mL^2$.

Problem: Determine the Mass moment of inertia for a uniform rod with negligible thickness about its end if the Rod makes angle with the axis rotation?

Example 2: Find the Mass moment of inertia of a thin uniform rectangular plate about its base and its one of edges axes?

Solution

We consider a uniform strip with the length (dx) and thickness (dy) as shown in below Figure, where the density is ρ.



$dm = \rho\, dx\, dy \rightarrow m = \rho \int_0^b \int_0^a dx\, dy \rightarrow \; m = \rho\, a\, b$. The moment of inertia about its corner is given

by $dI_{yy} = x^2 dm = \rho\, x^2\, dx\, dy \quad \rightarrow I_{yy} = \rho \int_0^b \int_0^a x^2\, dx\, dy = \rho \left[\frac{x^3}{3}\right]_0^b [y]_0^b = \frac{b\,a^3}{3}\rho = \frac{b\,a^3}{3}\rho\frac{m}{\rho\, a\, b}$

$\therefore \; I_{yy} = \frac{1}{3} m\, a^2$

If we select a vertical strip (sector, section), we have

$dI_{yy} = x^2 dm = \rho\, x^2\, (b\, dx) \quad \rightarrow I_{yy} = \rho\, b \int_0^a x^2\; dx = \rho\, b\left[\frac{x^3}{3}\right]_0^a = \frac{b\,a^3}{3}\rho = \frac{b\,a^3}{3}\rho\frac{m}{\rho\, a\, b}$

$\therefore \; I_{yy} = \frac{1}{3} m\, a^2 \qquad\qquad I_{yy} = I_{y'y'} + m\left(\frac{1}{2}a\right)^2 \rightarrow$

$\frac{1}{3} m\, a^2 = I_{y'y'} + m\left(\frac{1}{2}a\right)^2 \rightarrow I_{y'y'} = \frac{1}{3} m\, a^2 - \frac{1}{4} m a^2 = \left(\frac{4-3}{12}\right) m a^2 = \frac{1}{12} m a^2$

$\therefore \; I_{y'y'} = \frac{1}{12} m a^2$ Similarly, if we select a horizontal strip, we can prove that:

$I_{xx} = \frac{1}{3} m\, b^2, \qquad I_{x'x'} = \frac{1}{12} m\, b^2$

For axis is perpendicular ox, oy $I_{zz} = I_{xx} + I_{yy} = \frac{1}{3} m\, b^2 + \frac{1}{3} m\, a^2 = \frac{1}{3} m\,(a^2 + b^2)$

For axis is perpendicular ox', oy' : $I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{1}{12} m\, b^2 + \frac{1}{12} m\, a^2 = \frac{1}{12} m\,(a^2 + b^2)$

The moment of inertia about its corner is given by (Mass *moment of inertia)*

$I_{xx} = \frac{1}{3} m b^2 = \frac{1}{3}(ab)b^2 = \frac{1}{3} a b^3,$ $\quad I_{yy} = \frac{1}{3} b a^3$ $\quad I_o = I_{xx} + I_{yy} = \frac{1}{3} ab(a^2 + b^2)$

$I_{x'x'} = \frac{1}{12} a b^3,$ $\quad I_{y'y'} = \frac{1}{12} b a^3$ $\quad I_{o'} = I_{x'x'} + I_{y'y'} = \frac{1}{12} a b(a^2 + b^2)$

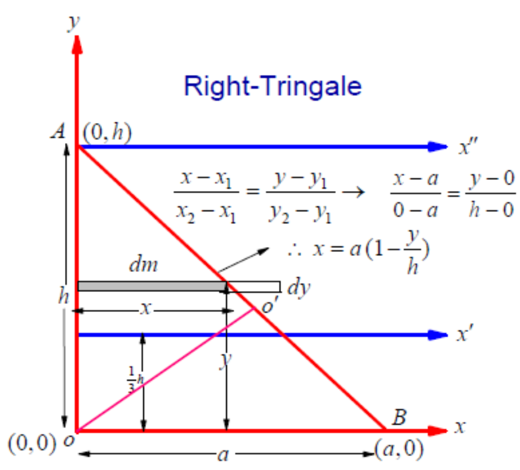
Uniform rectangular plate (a,b)	*Axis coincides with one of its sides*	*Axis passing through its centroid*	*Axis coincides to other side*
With respect toaxis I_{yy} −	$I_{yy} = \frac{1}{3} ma^2$	$I_{y'y'} = \frac{1}{12} ma^2$	$I_{y''y''} = \frac{1}{3} ma^2$
With respect toaxis I_{xx} −	$I_{xx} = \frac{1}{3} mb^2$	$I_{x'x'} = \frac{1}{12} mb^2$	$I_{x''x''} = \frac{1}{3} mb^2$
With respect to axis perpendicular to the plane oxy	$I_{zz} = \frac{1}{3} m(a^2 + b^2)$	$I_{z'z'} = \frac{1}{12} m(a^2 + b^2)$	$I_{z''z''} = \frac{1}{3} m(a^2 + b^2)$

Example 3: Determine the mass moment of inertia for right Triangular Plate (Right-angled triangle)?

Solution

We consider a uniform strip with the length (x) and thickness (dy), such that it is parallel to x - axis, as shown in below Figure. Then

$$dm = \rho x\, dy \rightarrow m = \rho \int_0^h x\, dy = \rho \int_0^h a(1 - \frac{y}{h})\, dy = a\rho \left[y - \frac{y^2}{2h} \right]_0^h == a\rho \left[h - \frac{h^2}{2h} \right] \rightarrow m = \frac{1}{2} a h \rho$$



Then moment of Inertia with respect to $x-$ axis:

$$dI_{xx} = y^2 dm = \rho\, x\, y^2\, dy \quad \rightarrow I_{xx} = \rho\int_0^h x\, y^2\, dy \text{ , but } \frac{x}{a}+\frac{y}{h}=1 \rightarrow x = a(1-\frac{y}{h})$$

$$I_{xx} = \rho\int_0^h a(1-\frac{y}{h})\, y^2\, dy = \rho\, a\int_0^h (y^2-\frac{y^3}{h})\, dy = \rho\, a\left[\frac{y^3}{3}-\frac{y^4}{4h}\right]_0^h$$

$$I_{xx} = \rho\, a\left[\frac{h^3}{3}-\frac{h^4}{4h}\right] = \frac{1}{12}\rho\, a\, h^3(4-3) = \frac{1}{12}\rho\, a\, h^3\frac{m}{\frac{1}{2}a\, h\rho} = \frac{1}{6}m\, h^2$$

$$\therefore\ I_{xx} = \frac{1}{6}m\, h^2$$

Then moment of Inertia with respect to $x'-$ axis:

$$I_{xx} = I_{x'x'} + m\left(\frac{1}{3}h\right)^2 \rightarrow \quad I_{x'x'} = \frac{1}{6}m\, h^2 - \frac{1}{9}m\, h^2 = \frac{1}{18}m\, h^2(3-2) = \frac{1}{18}m\, h^2 \qquad I_{x'x'} = \frac{1}{18}m\, h^2$$

Then moment of Inertia with respect to $x''-$ axis:

$$I_{x''x''} = I_{x'x'} + m\left(\frac{2}{3}h\right)^2 = \frac{1}{18}m\, h^2 + \frac{4}{9}m\, h^2 = \frac{1}{18}m\, h^2(1+8) = \frac{9}{18}m\, h^2 \qquad I_{x''x''} = \frac{1}{2}m\, h^2$$

Also, $I_{yy} = \frac{1}{6}m\, a^2$, $\qquad I_{y'y'} = \frac{1}{18}m\, a^2$, $\qquad I_{y''y''} = \frac{1}{2}m\, a^2$.

$$I_{zz} = I_{xx} + I_{yy} = \frac{1}{6}m\, a^2 + \frac{1}{6}m\, h^2 = \frac{1}{6}m(a^2+h^2) \quad I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{1}{18}m\, a^2 + \frac{1}{18}m\, h^2 = \frac{1}{18}m(a^2+h^2)$$

Again, $I_{AB} = \frac{1}{6}m(oo')^2$

where $\frac{1}{2}(oo')\, AB$, $\quad AB = \sqrt{(0-a)^2+(h-0)^2} = \sqrt{a^2+h^2}$

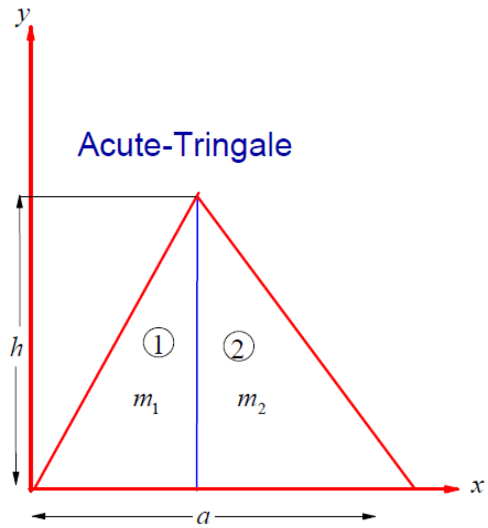
$$I_{AB} = \frac{1}{6}m(oo')^2 = \frac{a^2\, h^2}{6(a^2+h^2)}m \text{ , Also } \frac{1}{2}a\, h = \frac{1}{2}(oo')\, AB = \frac{1}{2}(oo')\sqrt{a^2+h^2} \rightarrow oo' = \frac{a\, h}{\sqrt{a^2+h^2}}$$

Right Triangular Plate of height h and bass a	About its corner	*About its center of mass*	*About its* vertex
About its base	$I_{xx} = \frac{1}{6}mh^2$	$I_{x'x'} = \frac{1}{18}mh^2$	$I_{x''x''} = \frac{1}{2}mh^2$
About its height	$I_{yy} = \frac{1}{6}ma^2$	$I_{y'y'} = \frac{1}{18}ma^2$	$I_{y''y''} = \frac{1}{2}ma^2$
About vertical axis	$I_{zz} = \frac{1}{6}m(a^2+h^2)$	$I_{z'z'} = \frac{1}{18}m(a^2+h^2)$	$I_{z''z''} = \frac{1}{6}m(3a^2+h^2)$, $I_{z''z''} = \frac{1}{6}m(a^2+3h^2)$

Example 4: The Mass Moment of inertia of acute triangular plate?

Solution

We divide the *acute triangular plate to two right triangular plate as is shown in Figure*



The Moment of inertia of about $x-$ axis for the two right triangular plate is given as

$$\left(I_{xx}\right)_1 = \frac{1}{6} m_1 h^2, \qquad \left(I_{xx}\right)_2 = \frac{1}{6} m_2 h^2,$$

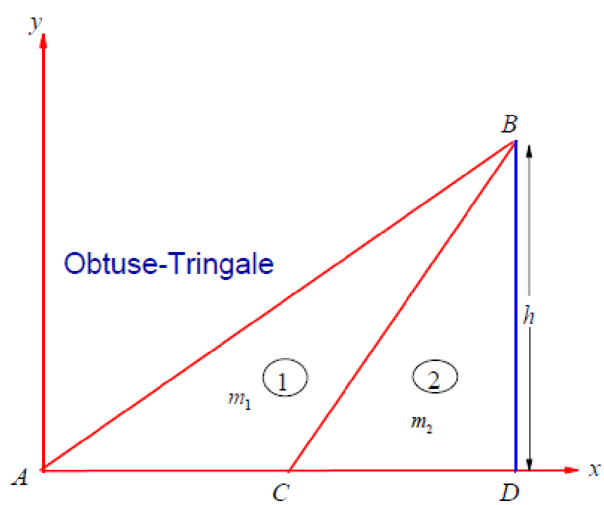
For the acute triangular plate

$$I_{xx} = \left(I_{xx}\right)_1 + \left(I_{xx}\right)_2 = \frac{1}{6} m_1 h^2 + \frac{1}{6} m_2 h^2 = \frac{1}{6}(m_1 + m_2) h^2 = \frac{1}{6} m h^2$$

Example 5: The Mass Moment of inertia of Obtuse *triangular plate?*

Solution

We divided the *obtuse triangular plate to two right- triangular plate as is shown below Figure*



The Moment of inertia of about $x-$ axis for the two right triangular plate is given as

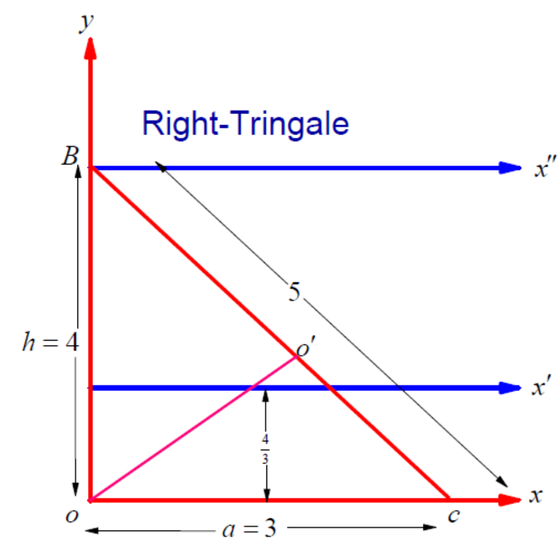
$$(I_{xx})_{ABD} = \frac{1}{6}(m_1 + m_2)h^2, \quad (I_{xx})_{CBD} = \frac{1}{6}m_2 h^2$$

For the acute triangular plate

$$(I_{xx})_{ABC} = (I_{xx})_{ABD} + (I_{xx})_{CBD} = \frac{1}{6}(m_1 + m_2)h^2 - \frac{1}{6}m_2 h^2 = \frac{1}{6}m_1 h^2$$

Example 6 : Find the Mass Moment of inertia of right- *triangular plate as is shown in figure about all different axes?*

Solution



From the Figure it is clear that $I_{xx} = \frac{1}{6}m h^2, \quad I_{yy} = \frac{1}{6}m a^2, \; I_{BC} = \frac{a^2 h^2}{6(a^2 + h^2)}m$

$$I_{xx} = \frac{1}{6}m h^2 = \frac{1}{6}m(4)^2 = \frac{16}{6}m = \frac{8}{3}m, \qquad I_{yy} = \frac{1}{6}m a^2 = \frac{1}{6}m(3)^2 = \frac{9}{6}m = \frac{3}{2}m$$

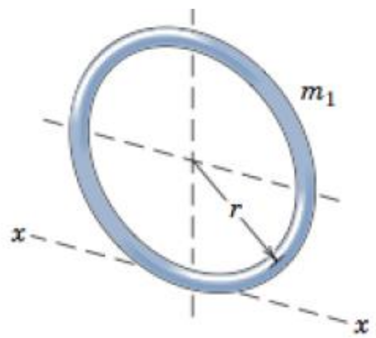
$$I_{BC} = \frac{a^2 h^2}{6(a^2 + h^2)}m = \frac{(3)^2(4)^2}{6((3)^2 + (4)^2)}m = \frac{(9)(16)}{6(9+16)}m = \frac{(9)(16)}{6(25)}m = \frac{24}{25}m$$

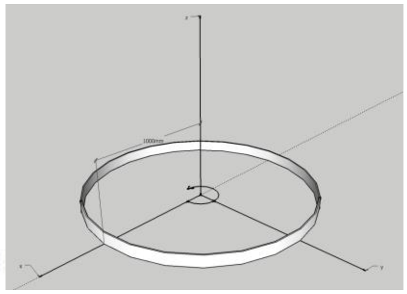
Note that $3 < 4 < 5, \quad I_{xx} = \frac{8}{3}m > I_{yy} = \frac{3}{2}m > I_{BC} = \frac{24}{25}m$

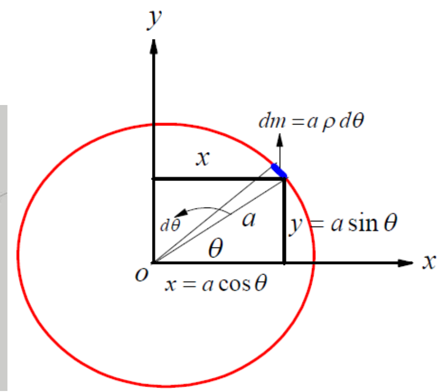
Example 7: The Mass Moment of inertia of Circular Ring?

Solution

We select a small element has the mass dm at any point located at distance (x, y) from the origin point







The Moment of inertia about $z-$axis (The axis is passing through the center (z-axis) and is perpendicular to the Ring) is given as

$$dI_{zz} = a^2 dm \, \ldots\ldots\ldots \quad I_{zz} = \int a^2\, dm = a^2 \int_0^m dm \rightarrow \quad I_{zz} = a^2\, m$$

From the Perpendicular axis theorem (Here, the distance between the tangent and the diameter is a) $I_{zz} = I_{xx} + I_{yy}$. So $\quad I_{xx} + I_{yy} = m\, a^2$

But I_{xx} and I_{yy} are symmetric, so $I_{xx} = I_{yy}$, Then

$I_{xx} = I_{yy} = \frac{1}{2} m\, a^2$ (The moment of inertia of a ring about of its diameter or the axis passes through the diameter)

From the parallel axis theorem $I_{y'y'} = I_{yy} + m\,a^2 \rightarrow I_{y'y'} = \frac{1}{2} m\,a^2 + m\,a^2 \rightarrow I_{y'y'} = \frac{1}{2} m\,a^2$

$$I_{x'x'} = I_{yy} + m\,a^2 \rightarrow I_{x'x'} = \frac{1}{2} m\,a^2 + m\,a^2 \rightarrow I_{x'x'} = \frac{1}{2} m\,a^2$$

Moment of inertia about an axis is passing through the edge of Ring and perpendicular to its plane and parallel an axis is passing through the center (z-axis) and is perpendicular to the Ring

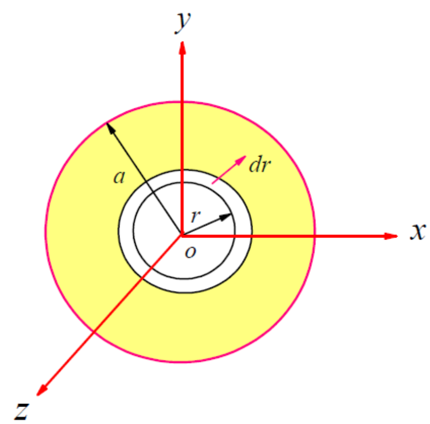
$$I_{z'z'} = I_{zz} + m\,a^2 \rightarrow I_{z'z'} = m\,a^2 + m\,a^2 \rightarrow I_{z'z'} = 2m\,a^2$$

Circular Ring	For Vertical axis	About axis in the plane of Circular Ring and passes in the its center *The moment of inertia of the ring about of its diameter*
Axis of rotation	$I_{zz} = ma^2$	$I_{xx} = I_{yy} = \frac{1}{2} ma^2$
Axis of rotation	$I_{z'z'} = 2ma^2$	$I_{x'x'} = I_{y'y'} = \frac{3}{2} ma^2$

Example 8: Find the Mass Moment of inertia of Circular area *?*

Solution

We divided the Circular area to the *small Circular Rings, we selected one of them has mass* (dm)*,thickness* (dr) *and raids* (r).



So, $dm = 2\pi r \rho\, dr \rightarrow m = 2\pi \rho \int_0^a r\, dr \rightarrow m = 2\pi \rho \left. \frac{r^2}{2} \right|_0^a = \pi a^2 \rho$

$$I_{zz} = \int r^2\, dm = \int r^2 (2\pi r \rho\, dr) = 2\pi\rho \int_0^a r^3\, dr = \left. \frac{2\pi\rho r^4}{4} \right|_0^a = \frac{\pi\rho a^4}{2} = \frac{\pi\rho a^4}{2} \frac{m}{m} = \frac{\pi\rho a^4}{2} \frac{m}{\pi a^2 \rho}$$

$$I_{zz} = \frac{1}{2} ma^2$$

From the Perpendicular axis theorem

$I_{zz} = I_{xx} + I_{yy}$. So $I_{xx} + I_{yy} = \frac{1}{2} m a^2$.

But I_{xx}, I_{yy} are symmetric, so $I_{xx} = I_{yy}$. Then $I_{xx} = I_{yy} = \frac{1}{4} m a^2$

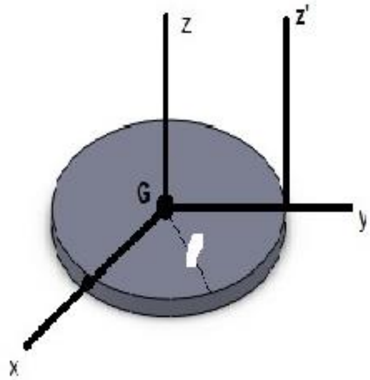
Circular area	For Vertical axis	About axis in the plane of Circular Ring and passes in the its center
Axis of rotation	$I_{zz} = \frac{1}{2} ma^2$	$I_{xx} = I_{yy} = \frac{1}{4} ma^2$

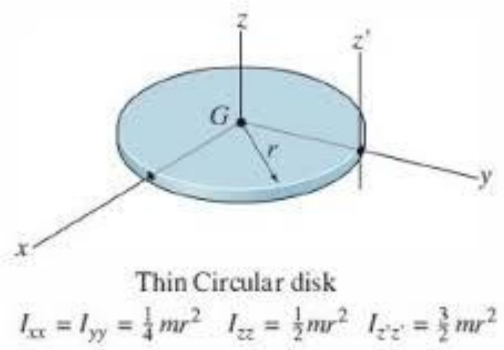
Axis of rotation	$I_{z'z'} = \frac{3}{2} ma^2$	$I_{x'x'} = I_{y'y'} = \frac{5}{4} ma^2$

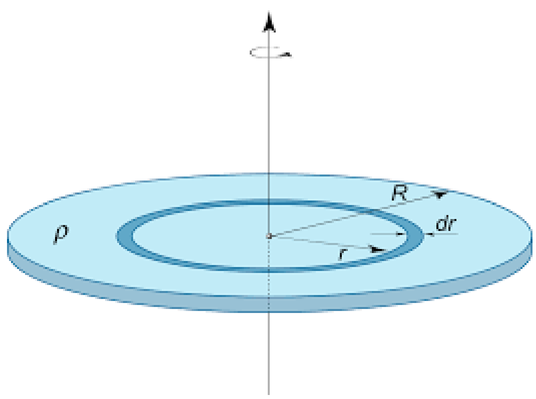
Example 9: Find the Moment of inertia of Thin Disc?

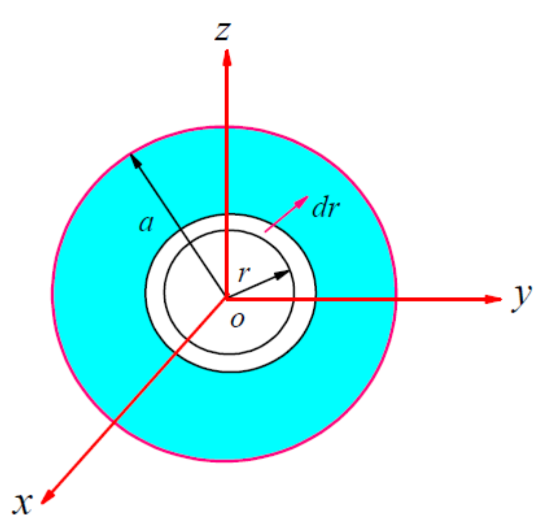
Solution

We divide the solid Disc to the *small Circular Rings, we selected one of them has mass* (dm)*,thickness* (dr),distraction thickness (Δz) *and raids* (r).









$$dm = 2\pi r\rho \Delta z\, dr \rightarrow \; m = 2\pi \rho \Delta z \int_0^a r\, dr \; \rightarrow m = 2\pi \rho \Delta z \left.\frac{r^2}{2}\right|_0^a = \pi a^2 \rho \Delta z$$

So, the Moment of inertia of thin Disc is

$$I_{zz} = \int r^2\, dm = \int r^2 (2\pi r \rho \Delta z\, dr) = 2\pi\rho \Delta z \int_0^a r^3\, dr = 2\pi\rho \Delta z \left.\frac{r^4}{4}\right|_0^a = \pi\rho \Delta z \frac{a^4}{2}$$

$$I_{zz} = \frac{\pi\rho a^4}{2}\frac{m}{m} = \frac{\pi\rho\Delta z\, a^4}{2}\frac{m}{\pi a^2 \rho \Delta z} \quad \rightarrow \quad I_{zz} = \frac{1}{2} m a^2$$

From the Parallel axis theorem $\quad I_{z'z'} = I_{zz} + m a^2 \rightarrow \quad I_{z'z'} = \frac{3}{2} m a^2$

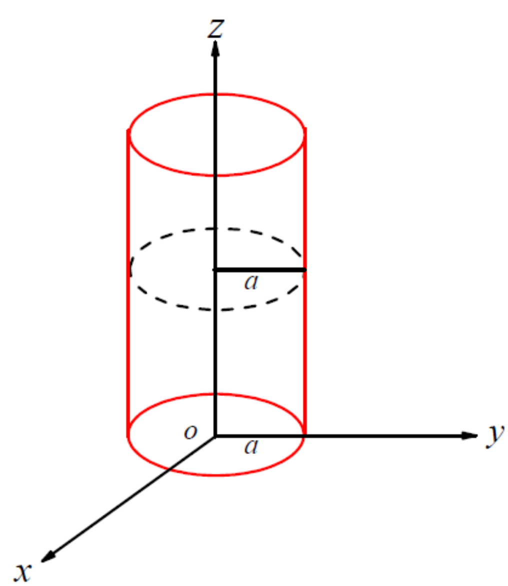
From the Perpendicular axis theorem $I_{zz} = I_{xx} + I_{yy}$. So $\quad I_{xx} + I_{yy} = \frac{1}{2} m a^2$.

But I_{xx}, I_{yy} are symmetric, so $I_{xx} = I_{yy}$. Then $\quad I_{xx} = I_{yy} = \frac{1}{4} m a^2$

Example: 10: Derive the *Mass* moment of inertia of Hollow Cylinder?

Solution

Take the hollow cylinder as the corresponding shape, divide it into an infinite number of regular circular rings and take one of these rings with the mass *(dm)* and the radius (a).



Then the moment of inertia of this ring is given as $\quad dI_{zz} = a^2 dm$.

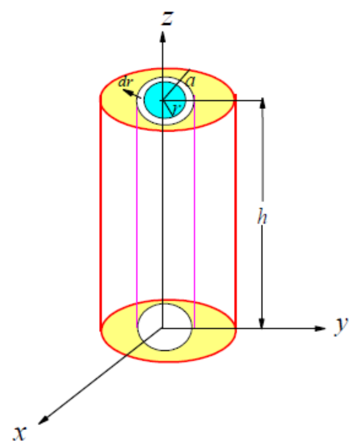
Then, the total moment of Hollow Cylinder

$$I_{zz}=\int_0^m a^2\,dm=m\,a^2 \quad \rightarrow \quad I_{zz}=m\,a^2$$

Example: 11: Derive the Mass moment of inertia of Solid Cylinder?

Solution

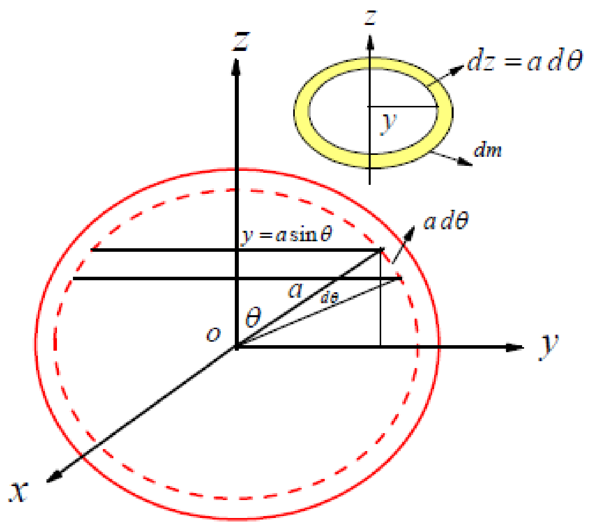
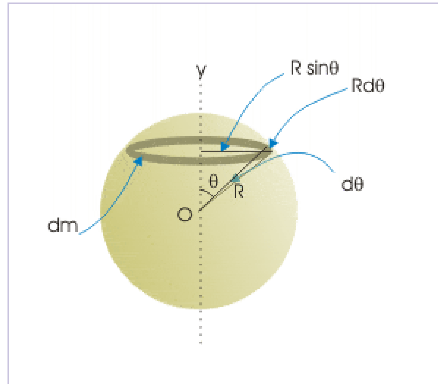
We divided the Solid Cylinder it into an infinite number of thin discs and take one of these discs with the mass (dm) and the radius (a).



Then the moment of inertia of this disc is given as. $dI_{zz}=\frac{1}{2}a^2\,dm$. Then the total moment of Hollow Cylinder $I_{zz}=\int_0^m \frac{1}{2}a^2\,dm=\frac{1}{2}m\,a^2 \quad \rightarrow \quad I_{zz}=\frac{1}{2}m\,a^2$

Example: 11: Derive the Mass moment of inertia of Hollow Sphere?

Solution

We divided the Hollow Sphere into a number of small circular rings and we consider one of them with the mass (dm), the radius (y) and thickness (dz).

$$dm = 2\pi\, y\, \rho\, dz = 2\pi\,(a\sin\theta)\,\rho\, a\, d\theta \rightarrow m = 2\pi\,\rho\, a^2\int_0^\pi \sin\theta\, d\theta \rightarrow m = -2\pi\,\rho\, a^2 \cos\theta\Big|_0^\pi =$$
$$= -2\pi\,\rho\, a^2\left(\cos(\pi)-\cos(0)\right) = -2\pi\,\rho\, a^2\left(-1-1\right) = 2\pi\,\rho\, a^2\left(1+1\right) == 4\pi\,\rho\, a^2$$

The moment of inertia of this circular ring is given as $dI_{zz} = y^2 dm$.

Then the total moment of Hollow Cylinder $I_{zz} = \int y^2\, dm$, then

$$I_{zz} = \int y^2\, dm = 2\pi\,\rho\, a^4\int_0^\pi (\sin)^2 \sin\theta\, d\theta = 2\pi\,\rho\, a^4\left[\int_0^\pi (1-\cos^2\theta)\sin\theta\, d\theta\right]$$
$$= 2\pi\,\rho\, a^4\left[\int_0^\pi \sin\theta\, d\theta - \int_0^\pi (\cos\theta)^2\, d(-\sin\theta)\right]$$
$$= 2\pi\,\rho\, a^4\left[-\cos\theta + \frac{1}{3}(\cos\theta)^3\right]_0^\pi = 2\pi\,\rho\, a^4\left[-\cos(\pi)+\frac{1}{3}(\cos(\pi))^3 - \left\{-\cos(0)+\frac{1}{3}(\cos(0))^3\right\}\right]$$
$$= 2\pi\,\rho\, a^4\left[1+\frac{1}{3}-\left\{-1+\frac{1}{3}\right\}\right] = 2\pi\,\rho\, a^4\left[1+\frac{1}{3}+1-\frac{1}{3}\right] = 2\pi\,\rho\, a^4\left[2-\frac{2}{3}\right] = \frac{8}{3}\pi\,\rho\, a^4$$
$$I_{zz} = \frac{8}{3}\pi\,\rho\, a^4 \frac{m}{4\pi a^2 \rho} = \frac{2}{3} m a^2 \qquad \textit{Then} \quad I_{zz} = \frac{2}{3} m a^2$$

For the symmetric of axes $I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} m a^2$

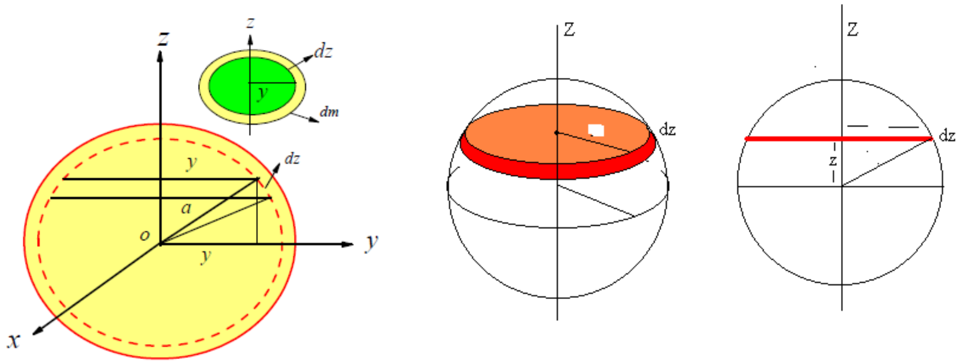
Also, we know $I_{xx} + I_{yy} + I_{zz} = 2 I_o$,

$$2 I_o = \frac{2}{3} m a^2 + \frac{2}{3} m a^2 + \frac{2}{3} m a^2 = \frac{6}{3} m a^2 = 2 m a^2$$

$$I_o = m a^2$$

Example: 12: Derive the Mass moment of inertia of Solid Sphere?

Solution



We divided the solid sphere into a number of hollow sphere and take one of these sphere with mass (dm), radius (r) and thickness (dr). Then the moment inertia of this sphere around oz axis is $dI_{zz} = \frac{2}{3}(dm)r^2$, for whole sphere the moment inertia is given as

$I_{zz} = \int \frac{2}{3}(dm)r^2$, where $dm = 4\pi r^2 \rho\, dr \rightarrow m = 4\pi \rho \int_0^a r^2\, dr = \frac{4}{3}\pi a^3 \rho$. Then

$$I_{zz} = \int \frac{2}{3}(dm)r^2 = \int_0^a \frac{2}{3}\left(4\pi r^2 \rho\, dr\right) r^2 = \frac{8}{3}\pi \rho \int_0^a r^4\, dr = \frac{8}{3}\pi \rho \left.\frac{r^4}{5}\right|_0^a = \frac{8}{15}\pi \rho a^5$$

$$I_{zz} = \frac{8}{15}\pi \rho a^5 \frac{m}{m} = \frac{8}{15}\pi \rho a^5 \frac{m}{\frac{4}{3}\rho a^3} = \frac{2}{5}m a^2$$ Then $$I_{zz} = \frac{2}{5}m a^2$$

Where the axes are Symmetrical $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}m a^2$

Also $I_{xx} + I_{yy} + I_{zz} = 2I_o$, Then $2I_o = \frac{2}{5}m a^2 + \frac{2}{5}m a^2 + \frac{2}{5}m a^2 = \frac{6}{5}m a^2$ $I_o = \frac{3}{5}m a^2$

Another solution

Dividing the solid sphere into a number of small disks and taking one of these disks with mass (dm), radius (y) and thickness (dz). The moment of inertial of this disc is around the oz-axis is given by $dI_{zz} = \frac{1}{2}(dm)\, y^2$

For the whole solid sphere is given $I_{zz} = \int \frac{1}{2}(dm)\, y^2$

$$I_{zz} = \int \frac{1}{2}(dm)\,y^2 = \int_{-a}^{a} \frac{1}{2}\left(\pi\, y^2 \rho\, dz\right) y^2 = \frac{1}{2}\pi\,\rho \int_{-a}^{a} y^4\, dz = \frac{1}{2}\pi\,\rho \int_{-a}^{a} (y^2)^2\, dz = \frac{1}{2}\pi\,\rho \int_{-a}^{a} (a^2 - z^2)^2\, dz$$

$$= \frac{1}{2}\pi\,\rho \int_{-a}^{a} (a^4 - 2a^2 z^2 + z^4)\, dz = \frac{1}{2}\pi\,\rho \left[a^4 z - \frac{2}{3}a^2 z^3 + \frac{1}{5} z^5 \right]_{-a}^{a}$$

$$= \frac{1}{2}\pi\,\rho \left[\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right) - \left(-a^5 + \frac{2}{3}a^5 - \frac{1}{5}a^5 \right) \right] = \frac{1}{2}\pi\,\rho \left[a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 + a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right]$$

$$= \frac{2}{2}\pi\,\rho \left[a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5 \right] = \frac{1}{15}\pi\,\rho[15 - 10 + 3]a^5 = \frac{8}{15}\pi\,\rho\, a^5 = \frac{8}{15}\pi\,\rho\, a^5 \frac{m}{\frac{4}{3}\pi\, a^3 \rho} = \frac{2}{5} m\, a^5$$

$$I_{zz} = \frac{2}{5} m\, a^2$$

Note that

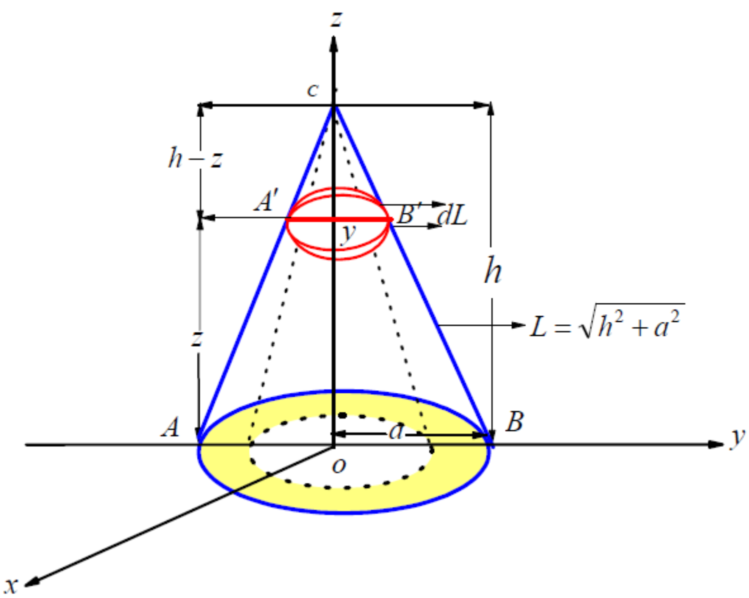
$$dm = \pi\, y^2\, \rho\, dz \rightarrow\ m = \pi\ \rho \int_{-a}^{a} y^2\ dz = \pi\ \rho \int_{-a}^{a} (a^2 - z^2)\ dz = \pi\ \rho \left[a^2 z - \frac{1}{3} z^3 \right]_{-a}^{a}$$

$$m = \pi\ \rho \left[a^3 - \frac{1}{3}a^3 - \left(-a^3 + \frac{1}{3}a^3 \right) \right] = \pi\ \rho \left[a^3 - \frac{1}{3}a^3 + a^3 - \frac{1}{3}a^3 \right] = 2\pi\ \rho \left[a^3 - \frac{1}{3}a^3 \right] = \frac{4}{3}\pi\ a^3 \rho$$

Example: 13: Find the Mass moment inertial for the Hollow Circular Cone ?

Solution

Divide the Hollow Circular Cone into a number of small circular rings and take one of these rings with mass (dm), radius (y) and thickness (dL), which is located higher (z) than the base of the cone with radius (a). Note that it is similar to triangles ABC and $A'B'C$, we have $\frac{h - z}{h} = \frac{y}{a} \rightarrow\ y = \frac{a}{h}(h - z) \rightarrow\ z = \frac{h}{a}(a - y)$



The moment of inertia of this circular ring is given as $dI_{zz} = y^2 dm$.

Then the total moment of Hollow Circular Cone $I_{zz} = \int y^2\, dm$

Note that , where $dm = 2\pi\, y\, \rho\, dL \rightarrow\ m = 2\pi\, \rho \int_0^h y\, dL$

$$dL = \sqrt{1 + \left(\frac{dz}{dy}\right)^2}\ dy = \sqrt{1 + \left(\frac{h}{a}\right)^2}\ dy = \frac{1}{a}\sqrt{a^2 + h^2}\ dy = \frac{L}{a}\, dy.$$ Then

$$dm = 2\pi\, \rho \int_0 y\, dL == 2\pi\, \rho \int_0^a y \frac{L}{a} dy = \ 2\pi\, \rho \frac{L}{a}\frac{y^2}{2}\bigg|_0^a = 2\pi\, \rho \frac{L}{a}\frac{a^2}{2} \rightarrow m = \pi\, a\, L\, \rho.$$ Then

$$I_{zz} = \int_0^a y^2\, dm = \int_0^a y^2 (2\pi\, y\, \rho\, dL) = 2\pi\ \ \rho \int_0^a y^3 \frac{L}{a} dy = 2\pi\ \ \rho \frac{L}{a}\frac{y^4}{4}\bigg|_0^a = 2\pi\ \ \rho \frac{L}{a}\frac{a^4}{4} = \pi\, L\rho\, \frac{a^3}{2}$$

$$= \pi\, L\rho\, \frac{a^3}{2}\frac{m}{m} = \pi\, L\rho\, \frac{a^3}{2}\ \frac{m}{\pi\, a\, L\, \rho} = \frac{1}{2} m\, a^2 \qquad\qquad I_{zz} = \frac{1}{2} m\, a^2$$

Again, $dL = \sqrt{1 + \left(\frac{dy}{dz}\right)^2}\ dz = \sqrt{1 + \left(\frac{a}{h}\right)^2}\ dz = \frac{1}{h}\sqrt{h^2 + a^2}\ dz = \frac{L}{h}\, dz$

$$dm = 2\pi\, \rho \int_0 y\, dL == 2\pi\, \rho \int_0^h \frac{a}{h} z \frac{L}{h} dz = \ 2\pi\, \rho\, a \frac{L}{h^2}\frac{z^2}{2}\bigg|_0^h = 2\pi\, \rho\, a \frac{L}{h^2}\frac{h^2}{2} \rightarrow m = \pi\, a\, L\, \rho$$

$$I_{zz} = \int_0^a y^2\, dm = \int_0^a y^2 (2\pi\, y\, \rho\, dL) = 2\pi\ \ \rho \int_0^h \left(\frac{a}{h} z\right)^3 \frac{L}{h} dz = 2\pi\ \ \rho \frac{a^3 L}{h^4} \int_0^h z^3\, dz == 2\pi\ \ \rho \frac{a^3 L}{h^4}\ \frac{z^4}{4}\bigg|_0^a$$

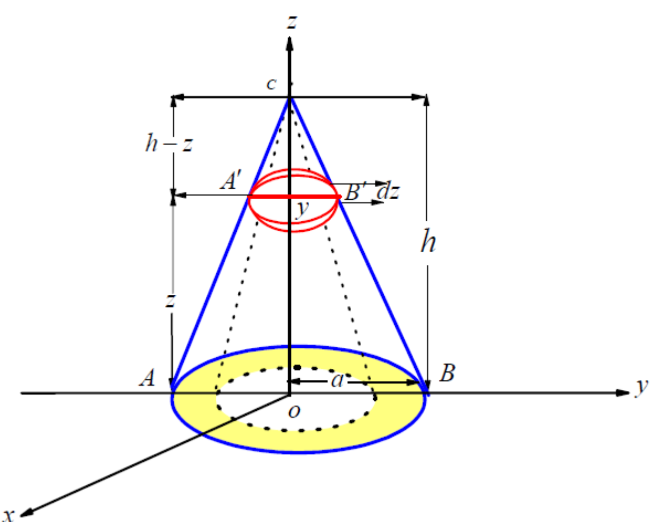
$$= 2\pi \ \rho \frac{a^3 L}{h^4} \frac{h^4}{4} = \pi L\rho \frac{a^3}{2} = \pi L\rho \frac{a^3}{2} \frac{m}{m} = \pi L\rho \frac{a^3}{2} \frac{m}{\pi a L \rho} = \frac{1}{2} m a^2 \qquad I_{zz} = \frac{1}{2} m a^2$$

Example 14: Find the Mass moment inertial for the Solid Circular Cone?

Solution

We divided the Solid Circular Cone into a number of small Disks and take one of them with mass (dm), radius (y) and thickness (dz), which is located higher (z) than the base of the cone with radius (a). Note that it is similar to triangles ABC and $A'B'C$, we have

$$\frac{h-z}{h} = \frac{y}{a} \rightarrow y = \frac{a}{h}(a-z) \rightarrow z = \frac{h}{a}(a-y)$$



Note that

$$dm = \pi y^2 \rho \, dz \rightarrow \ m = \pi \rho \int_0^h \left(\frac{a}{h}(h-z) \right)^2 dz = \pi \rho \frac{a^2}{h^2} \int_0^h \left(h^2 - 2hz + z^2 \right) dz$$

$$= \pi \rho \frac{a^2}{h^2} \left(h^2 z - 2h \frac{z^2}{2} + \frac{z^3}{3} \right) \Bigg|_0^h = \pi \rho \frac{a^2}{h^2} \left(h^3 - h^3 + \frac{h^3}{3} \right) \rightarrow m = \frac{1}{3} \pi a^2 h \rho$$

The moment of inertia of this Disk is given as $dI_{zz} = y^2 dm$.

Then the total moment of Solid Circular Cone $I_{zz} = \int y^2 \, dm$, that is given

$$I_{zz} = \frac{1}{2}\int_0^a y^2\, dm = \frac{1}{2}\int_0^h y^2\left(\pi\, y^2\, \rho\ dz\right) = \frac{1}{2}\pi\,\rho\int_0^h y^4\, dz = \frac{1}{2}\pi\,\rho\int_0^h\left(\frac{a}{h}(h-z)\right)^4 dz = \frac{1}{2}\pi\,\rho\left(\frac{a}{h}\right)^4\int_0^h (h-z)^4\, dz$$

$$= \frac{1}{2}\pi\,\rho\left(\frac{a}{h}\right)^4\ \frac{(h-z)^5}{-5}\Bigg|_0^h = \frac{1}{2}\pi\ \rho\frac{a^4}{h^4}\frac{h^5}{5} = \frac{1}{10}\pi\ \rho a^4\, h == \frac{1}{10}\pi\ \rho a^4\, h\ \frac{m}{m} == \frac{1}{10}\pi\ \rho a^4 h\ \frac{m}{\frac{1}{3}\pi\, a^2\, h\,\rho} = \frac{3}{10}m\, a^2$$

$I_{zz} = \frac{3}{10}m\, a^2$, Also

$$I_o = \int_0^h z^2\, dm = \int_0^h z^2\left(\pi\, y^2\ \rho\ dz\right) = \pi\,\rho\int_0^h z^2 y^2\, dz = \pi\,\rho\int_0^h z^2\left(\frac{a}{h}(h-z)\right)^2 dz$$

$$= \pi\,\rho\left(\frac{a}{h}\right)^2\int_0^h\left(h^2 z^2 - 2h\, z^3 + z^4\right)dz = \pi\,\rho\frac{a^2}{h^2}\left(h^2\frac{z^3}{3} - h\frac{z^4}{2} + \frac{z^5}{5}\right)\Bigg|_0^h = \pi\,\rho\frac{a^2}{h^2}h^5\left(\frac{10-15+6}{30}\right)$$

$$= \pi\ \rho\frac{a^2}{h^2}\frac{h^5}{30} = \frac{1}{30}\pi\ \rho a^2\, h^3 = \frac{1}{30}\pi\ \rho a^2\, h^3\ \frac{m}{m} = \frac{1}{30}\pi\ \rho a^2\, h^3\ \ \frac{m}{\frac{1}{3}\pi\, a^2\, h\,\rho} = \frac{1}{10}m\, h^2$$

Then $I_o = \frac{1}{10}m\, h^2$

Chapter 69

Double and Iterated Integrals

THE (SIMPLE) INTEGRAL $\int_a^b f(x)\,dx$ of a function $y = f(x)$ that is continuous over the finite interval $a \le x \le b$ of the x axis was defined in Chapter 38. Recall that

1. The interval $a \le x \le b$ was divided into n subintervals $h_1, h_2, \ldots, h_n$ of respective lengths $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ with λ_n the greatest of the $\Delta_k x$.
2. Points x_1 in h_1, x_2 in $h_2, \ldots, x_n$ in h_n were selected, and the sum $\sum_{k=1}^{n} f(x_k)\,\Delta_k x$ formed.
3. The interval was further subdivided in such a manner that $\lambda_n \to 0$ as $n \to +\infty$.
4. We defined $\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x$.

THE DOUBLE INTEGRAL. Consider a function $z = f(x, y)$ continuous over a finite region R of the xOy plane. Let this region be subdivided (see Fig. 69-1) into n subregions $R_1, R_2, \ldots, R_n$ of respective areas $\Delta_1 A, \Delta_2 A, \ldots, \Delta_n A$. In each subregion R_k, select a point $P_k(x_k, y_k)$ and form the sum

$$\sum_{k=1}^{n} f(x_k, y_k)\,\Delta_k A = f(x_1, y_1)\,\Delta_1 A + f(x_2, y_2)\,\Delta_2 A + \cdots + f(x_n, y_n)\,\Delta_n A \qquad (69.1)$$

Now, defining the diameter of a subregion to be the greatest distance between any two points within or on its boundary, and denoting by λ_n the maximum diameter of the subregions, suppose the number of subregions to be increased in such a manner that $\lambda_n \to 0$ as $n \to +\infty$. Then the *double integral* of the function $f(x, y)$ over the region R is defined as

$$\iint_R f(x, y)\,dA = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta_k A \qquad (69.2)$$

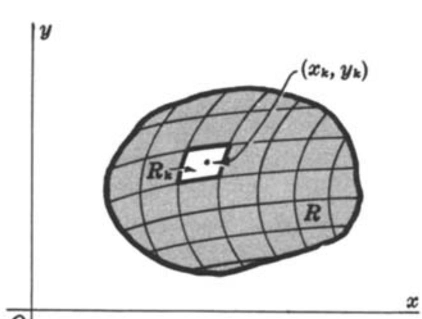


Fig. 69-1

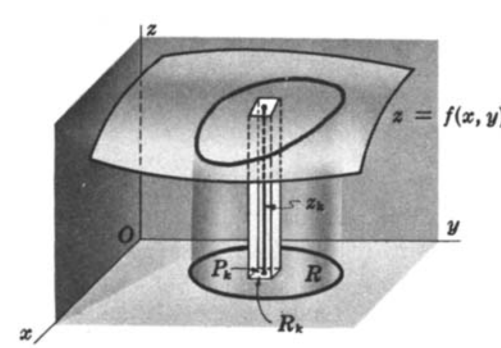


Fig. 69-2

When $z = f(x, y)$ is nonnegative over the region R, as in Fig. 69-2, the double integral (*69.2*) may be interpreted as a volume. Any term $f(x_k, y_k)\,\Delta_k A$ of (*69.1*) gives the volume of a vertical column whose parallel bases are of area $\Delta_k A$ and whose altitude is the distance z_k measured along the vertical from the selected point P_k to the surface $z = f(x, y)$. This, in turn, may be taken as an approximation of the volume of the vertical column whose lower base is the subregion R_k and whose upper base is the projection of R_k on the surface. Thus, (*69.1*) is an approximation of the volume "under the surface" (that is, the volume with lower base in the

xOy plane and upper base in the surface generated by moving a line parallel to the z axis along the boundary of R), and, intuitively, at least, (*69.2*) is the measure of this volume.

The evaluation of even the simplest double integral by direct summation is difficult and will not be attempted here.

THE ITERATED INTEGRAL. Consider a volume defined as above, and assume that the boundary of R is such that no line parallel to the x axis or to the y axis cuts it in more than two points. Draw (see Fig. 69-3) the tangents $x = a$ and $x = b$ to the boundary with points of tangency K and L, and the tangents $y = c$ and $y = d$ with points of tangency M and N. Let the equation of the plane arc LMK be $y = g_1(x)$, and that of the plane arc LNK be $y = g_2(x)$.

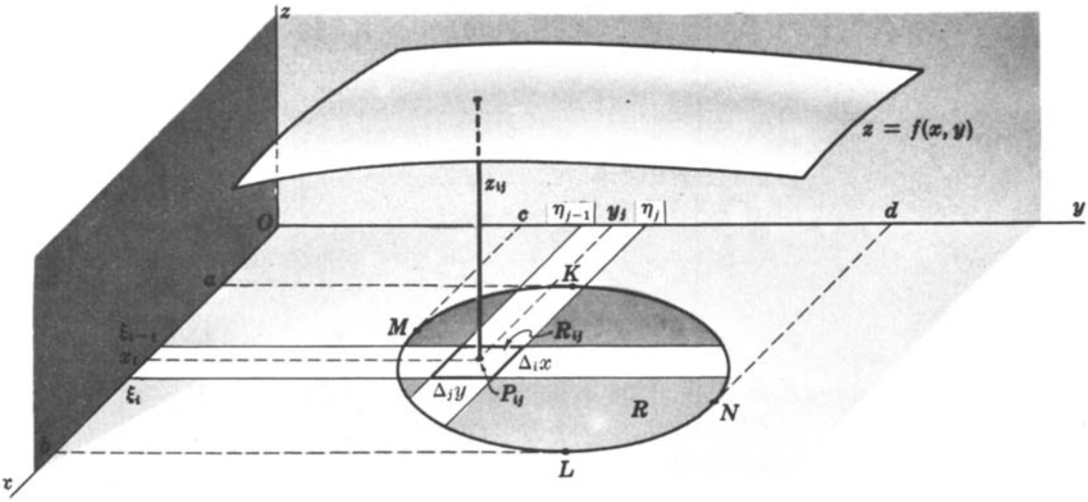


Fig. 69-3

Divide the interval $a \leq x \leq b$ into m subintervals $h_1, h_2, \ldots, h_m$ of respective lengths $\Delta_1 x$, $\Delta_2 x, \ldots, \Delta_m x$ by the insertion of points $x = \xi_1$, $x = \xi_2, \ldots, x = \xi_{m-1}$ (as in Chapter 38), and divide the interval $c \leq y \leq d$ into n subintervals $k_1, k_2, \ldots, k_n$ of respective lengths $\Delta_1 y$, $\Delta_2 y, \ldots, \Delta_n y$ by the insertion of points $y = \eta_1$, $y = \eta_2, \ldots, y = \eta_{n-1}$. Denote by λ_m the greatest $\Delta_i x$, and by μ_n the greatest $\Delta_j y$. Draw in the parallel lines $x = \xi_1$, $x = \xi_2, \ldots, x = \xi_{m-1}$ and the parallel lines $y = \eta_1$, $y = \eta_2, \ldots, y = \eta_{n-1}$, thus dividing the region R into a set of rectangles R_{ij} of areas $\Delta_i x \, \Delta_j y$ plus a set of nonrectangles that we shall ignore. On each subinterval h_i select a point $x = x_i$, and on each subinterval k_j select a point $y = y_j$, thereby determining in each subregion R_{ij} a point $P_{ij}(x_i, y_j)$. With each subregion R_{ij}, associate by means of the equation of the surface a number $z_{ij} = f(x_i, y_j)$, and form the sum

$$\sum_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}} f(x_i, y_j)\, \Delta_i x \, \Delta_j y \tag{69.3}$$

Now (*69.3*) is merely a special case of (*69.1*), so if the number of rectangles is indefinitely increased in such a manner that both $\lambda_m \to 0$ and $\mu_n \to 0$, the limit of (*69.3*) should be equal to the double integral (*69.2*).

In effecting this limit, let us first choose one of the subintervals, say h_i, and form the sum

$$\left[\sum_{j=1}^{n} f(x_i, y_j)\, \Delta_j y\right] \Delta_i x \quad (i \text{ fixed})$$

of the contributions of all rectangles having h_i as one dimension, that is, the contributions of all rectangles lying in the ith column. When $n \to +\infty$, $\mu_n \to 0$ and

$$\lim_{n\to+\infty}\left[\sum_{j=1}^{n} f(x_i, y_j)\,\Delta_j y\right]\Delta_i x = \left[\int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy\right]\Delta_i x = \phi(x_i)\,\Delta_i x$$

Now summing over the m columns and letting $m \to +\infty$, we have

$$\lim_{m\to+\infty}\sum_{i=1}^{m}\phi(x_i)\,\Delta_i x = \int_a^b \phi(x)\,dx = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx \qquad (69.4)$$

Although we shall not use the brackets hereafter, it must be clearly understood that (*69.4*) calls for the evaluation of two simple definite integrals in a prescribed order: first, the integral of $f(x, y)$ with respect to y (considering x as a constant) from $y = g_1(x)$, the lower boundary of R, to $y = g_2(x)$, the upper boundary of R, and then the integral of this result with respect to x from the abscissa $x = a$ of the leftmost point of R to the abscissa $x = b$ of the rightmost point of R. The integral (*69.4*) is called an *iterated* or *repeated integral.*

It will be left as an exercise to sum first for the contributions of the rectangles lying in each row and then over all the rows to obtain the equivalent iterated integral

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy \qquad (69.5)$$

where $x = h_1(y)$ and $x = h_2(y)$ are the equations of the plane arcs MKN and MLN, respectively.

In Problem 1 it is shown by a different procedure that the iterated integral (*69.4*) measures the volume under discussion. For the evaluation of iterated integrals see Problems 2 to 6.

The principal difficulty in setting up the iterated integrals of the next several chapters will be that of inserting the limits of integration to cover the region R. The discussion here assumed the simplest of regions; more complex regions are considered in Problems 7 to 9.

Solved Problems

1. Let $z = f(x, y)$ be nonnegative and continuous over the region R of the plane xOy whose boundary consists of the arcs of two curves $y = g_1(x)$ and $y = g_2(x)$ intersecting in the points K and L, as in Fig. 69-4. Find a formula for the volume V under the surface $z = f(x, y)$.

Let the section of this volume cut by a plane $x = x_i$, where $a < x_i < b$, meet the boundary of R in the points $S(x_i, g_1(x_i))$ and $T(x_i, g_2(x_i))$, and the surface $z = f(x, y)$ in the arc UV along which $z = f(x_i, y)$. The area of this section $STUV$ is given by

$$A(x_i) = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)\,dy$$

Thus, the areas of cross sections of the volume cut by planes parallel to the yOz plane are known functions $A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy$ of x, where x is the distance of the sectioning plane from the origin. By Chapter 42, the required volume is given by

$$V = \int_a^b A(x)\,dx = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\right]dx$$

This is the iterated integral of (*69.4*).

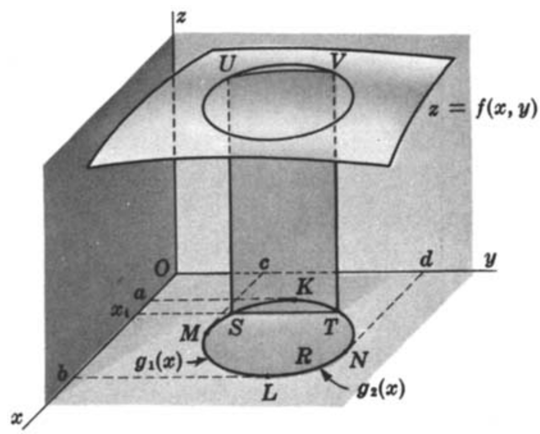


Fig. 69-4

In Problems 2 to 6, evaluate the integral at the left.

2. $$\int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 [y]_{x^2}^{x}\, dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

3. $$\int_1^2 \int_y^{3y} (x + y)\, dx\, dy = \int_1^2 [\tfrac{1}{2}x^2 + xy]_y^{3y}\, dy = \int_1^2 6y^2\, dy = [2y^3]_1^2 = 14$$

4. $$\int_{-1}^{2} \int_{2x^2-2}^{x^2+x} x\, dy\, dx = \int_{-1}^{2} [xy]_{2x^2-2}^{x^2+x}\, dx = \int_{-1}^{2} (x^3 + x^2 - 2x^3 + 2x)\, dx = \tfrac{9}{4}$$

5. $$\int_0^{\pi} \int_0^{\cos\theta} \rho \sin\theta\, d\rho\, d\theta = \int_0^{\pi} [\tfrac{1}{2}\rho^2 \sin\theta]_0^{\cos\theta}\, d\theta = \tfrac{1}{2}\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta = [-\tfrac{1}{6}\cos^3\theta]_0^{\pi} = \tfrac{1}{3}$$

6. $$\int_0^{\pi/2} \int_2^{4\cos\theta} \rho^3\, d\rho\, d\theta = \int_0^{\pi/2} \left[\frac{1}{4}\rho^4\right]_2^{4\cos\theta} d\theta = \int_0^{\pi/2} (64\cos^4\theta - 4)\, d\theta$$
$$= \left[64\left(\frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right) - 4\theta\right]_0^{\pi/2} = 10\pi$$

7. Evaluate $\iint_R dA$, where R is the region in the first quadrant bounded by the semicubical parabola $y^2 = x^3$ and the line $y = x$.

The line and parabola intersect in the points (0, 0) and (1, 1) which establish the extreme values of x and y on the region R.

Solution 1 (Fig. 69-5): Integrating first over a horizontal strip, that is, with respect to x from $x = y$ (the line) to $x = y^{2/3}$ (the parabola), and then with respect to y from $y = 0$ to $y = 1$, we get

$$\iint_R dA = \int_0^1 \int_y^{y^{2/3}} dx\, dy = \int_0^1 (y^{2/3} - y)\, dy = [\tfrac{3}{5}y^{5/3} - \tfrac{1}{2}y^2]_0^1 = \tfrac{1}{10}$$

Solution 2 (Fig. 69-6): Integrating first over a vertical strip, that is, with respect to y from $y = x^{3/2}$ (the parabola) to $y = x$ (the line), and then with respect to x from $x = 0$ to $x = 1$, we obtain

$$\iint_R dA = \int_0^1 \int_{x^{3/2}}^{x} dy\, dx = \int_0^1 (x - x^{3/2})\, dx = [\tfrac{1}{2}x^2 - \tfrac{2}{5}x^{5/2}]_0^1 = \tfrac{1}{10}$$

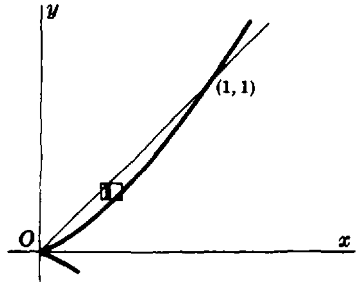


Fig. 69-5

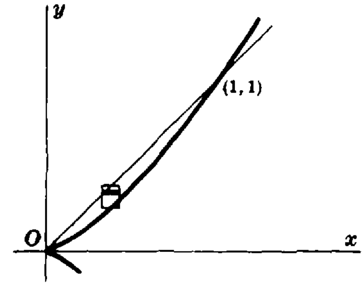


Fig. 69-6

8. Evaluate $\iint_R dA$ where R is the region between $y = 2x$ and $y = x^2$ lying to the left of $x = 1$.

Integrating first over the vertical strip (see Fig. 69-7), we have

$$\iint_R dA = \int_0^1 \int_{x^2}^{2x} dy\, dx = \int_0^1 (2x - x^2)\, dx = \tfrac{2}{3}$$

When horizontal strips are used (see Fig. 69-8), two iterated integrals are necessary. Let R_1 denote the part of R lying below AB, and R_2 the part above AB. Then

$$\iint_R dA = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{y/2}^{\sqrt{y}} dx\, dy + \int_1^2 \int_{y/2}^{1} dx\, dy = \tfrac{5}{12} + \tfrac{1}{4} = \tfrac{2}{3}$$

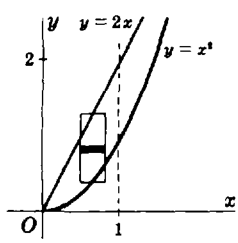


Fig. 69-7

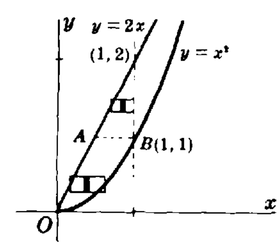


Fig. 69-8

9. Evaluate $\iint_R x^2\, dA$ where R is the region in the first quadrant bounded by the hyperbola $xy = 16$ and the lines $y = x$, $y = 0$, and $x = 8$. (See Fig. 69-9.)

It is evident from Fig. 69-9 that R must be separated into two regions, and an iterated integral evaluated for each. Let R_1 denote the part of R lying above the line $y = 2$, and R_2 the part below that line. Then

$$\iint_R x^2\, dA = \iint_{R_1} x^2\, dA + \iint_{R_2} x^2\, dA = \int_2^4 \int_y^{16/y} x^2\, dx\, dy + \int_0^2 \int_y^8 x^2\, dx\, dy$$

$$= \frac{1}{3} \int_2^4 \left(\frac{16^3}{y^3} - y^3\right) dy + \frac{1}{3} \int_0^2 (8^3 - y^3)\, dy = 448$$

As an exercise, you might separate R with the line $x = 4$ and obtain

$$\iint_R x^2\, dA = \int_0^4 \int_0^x x^2\, dy\, dx + \int_4^8 \int_0^{16/x} x^2\, dy\, dx$$

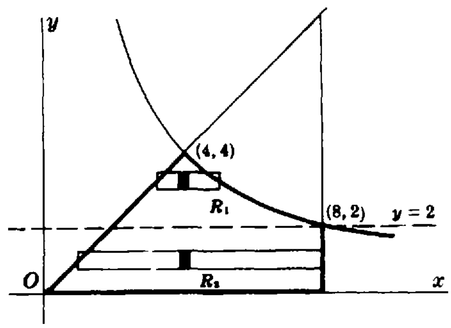


Fig. 69-9

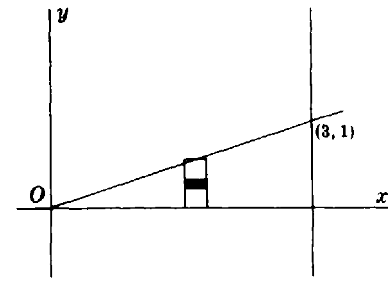


Fig. 69-10

10. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$ by first reversing the order of integration.

The given integral cannot be evaluated directly, since $\int e^{x^2}\, dx$ is not an elementary function. The region R of integration (see Fig. 69-10) is bounded by the lines $x = 3y$, $x = 3$, and $y = 0$. To reverse the order of integration, first integrate with respect to y from $y = 0$ to $y = x/3$, and then with respect to x from $x = 0$ to $x = 3$. Thus,

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 [e^{x^2} y]_0^{x/3}\, dx$$

$$= \tfrac{1}{3} \int_0^3 e^{x^2} x\, dx = [\tfrac{1}{6} e^{x^2}]_0^3 = \tfrac{1}{6}(e^9 - 1)$$

Supplementary Problems

11. Evaluate the iterated integral at the left:

(a) $\int_0^1 \int_1^2 dx\, dy = 1$

(b) $\int_1^2 \int_0^3 (x + y)\, dx\, dy = 9$

(c) $\int_2^4 \int_1^2 (x^2 + y^2)\, dy\, dx = \frac{70}{3}$

(d) $\int_0^1 \int_{x^2}^x xy^2\, dy\, dx = \frac{1}{40}$

(e) $\int_1^2 \int_0^{y^{3/2}} x/y^2\, dx\, dy = \frac{3}{4}$

(f) $\int_0^1 \int_x^{\sqrt{x}} (y + y^3)\, dy\, dx = \frac{7}{60}$

(g) $\int_0^1 \int_0^{x^2} xe^y\, dy\, dx = \frac{1}{2}e - 1$

(h) $\int_2^4 \int_y^{8-y} y\, dx\, dy = \frac{32}{3}$

(i) $\int_0^{\arctan 3/2} \int_0^{2 \sec \theta} \rho\, d\rho\, d\theta = 3$

(j) $\int_0^{\pi/2} \int_0^2 \rho^2 \cos \theta\, d\rho\, d\theta = \frac{8}{3}$

(k) $\int_0^{\pi/4} \int_0^{\tan \theta \sec \theta} \rho^3 \cos^2 \theta\, d\rho\, d\theta = \frac{1}{20}$

(l) $\int_0^{2\pi} \int_0^{1 - \cos \theta} \rho^3 \cos^2 \theta\, d\rho\, d\theta = \frac{49}{32}\pi$

12. Using an iterated integral, evaluate each of the following double integrals. When feasible, evaluate the iterated integral in both orders.

(*a*) x over the region bounded by $y = x^2$ and $y = x^3$ *Ans.* $\frac{1}{20}$

(*b*) y over the region of part (*a*) *Ans.* $\frac{1}{35}$

(*c*) x^2 over the region bounded by $y = x$, $y = 2x$, and $x = 2$ *Ans.* 4

(*d*) 1 over each first-quadrant region bounded by $2y = x^2$, $y = 3x$, and $x + y = 4$ *Ans.* $\frac{8}{3}$; $\frac{46}{3}$

(*e*) y over the region above $y = 0$ bounded by $y^2 = 4x$ and $y^2 = 5 - x$ *Ans.* 5

(*f*) $\dfrac{1}{\sqrt{2y - y^2}}$ over the region in the first quadrant bounded by $x^2 = 4 - 2y$ *Ans.* 4

13. In Problem 11(*a*) to (*h*), reverse the order of integration and evaluate the resulting iterated integral.

Chapter 70

Centroids and Moments of Inertia of Plane Areas

PLANE AREA BY DOUBLE INTEGRATION. If $f(x, y) = 1$, the double integral of Chapter 69 becomes $\iint_R dA$. In cubic units, this measures the volume of a cylinder of unit height; in square units, it measures the area of the region R. (See Problems 1 and 2.)

In polar coordinates, $A = \iint_R dA = \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} \rho \, d\rho \, d\theta$, where $\theta = \alpha$, $\theta = \beta$, $\rho_1(\theta)$, and $\rho_2(\theta)$ are chosen to cover the region R. (See Problems 3 to 5.)

CENTROIDS. The coordinates $(\bar{x}, \bar{y})$ of the centroid of a plane region R of area $A = \iint_R dA$ satisfy the relations

$$A\bar{x} = M_y \qquad \text{and} \qquad A\bar{y} = M_x$$

or

$$\bar{x} \iint_R dA = \iint_R x \, dA \qquad \text{and} \qquad \bar{y} \iint_R dA = \iint_R y \, dA$$

(See Problems 6 to 9.)

THE MOMENTS OF INERTIA of a plane region R with respect to the coordinate axes are given by

$$I_x = \iint_R y^2 \, dA \qquad \text{and} \qquad I_y = \iint_R x^2 \, dA$$

The polar moment of inertia (the moment of inertia with respect to a line through the origin and perpendicular to the plane of the area) of a plane region R is given by

$$I_0 = I_x + I_y = \iint_R (x^2 + y^2) \, dA$$

(See Problems 10 to 12.)

Solved Problems

1. Find the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.

Using vertical strips (see Fig. 70-1), we have

$$A = \int_{-1}^{3} \int_{x^2}^{2x+3} dy \, dx = \int_{-1}^{3} (2x + 3 - x^2) \, dx = 32/3 \text{ square units}$$

2. Find the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$.

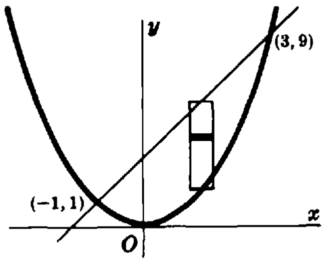


Fig. 70-1

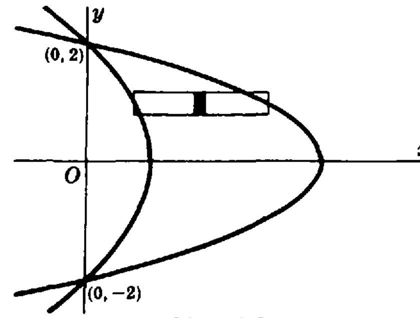


Fig. 70-2

Using horizontal strips (Fig. 70-2) and taking advantage of symmetry, we have

$$A = 2\int_0^2 \int_{1-y^2/4}^{4-y^2} dx\,dy = 2\int_0^2 [(4-y^2) - (1-\tfrac{1}{4}y^2)]\,dy$$

$$= 6\int_0^2 (1-\tfrac{1}{4}y^2)\,dy = 8 \text{ square units}$$

3. Find the area outside the circle $\rho = 2$ and inside the cardioid $\rho = 2(1+\cos\theta)$.

Owing to symmetry (see Fig. 70-3), the required area is twice that swept over as θ varies from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. Thus,

$$A = 2\int_0^{\pi/2} \int_2^{2(1+\cos\theta)} \rho\,d\rho\,d\theta = 2\int_0^{\pi/2} [\tfrac{1}{2}\rho^2]_2^{2(1+\cos\theta)}\,d\theta = 4\int_0^{\pi/2} (2\cos\theta + \cos^2\theta)\,d\theta$$

$$= 4[2\sin\theta + \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta]_0^{\pi/2} = (\pi + 8) \text{ square units}$$

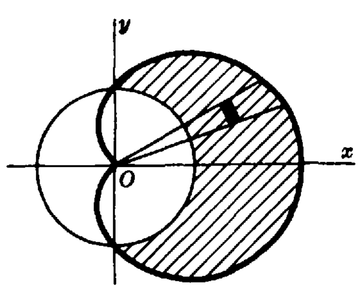


Fig. 70-3

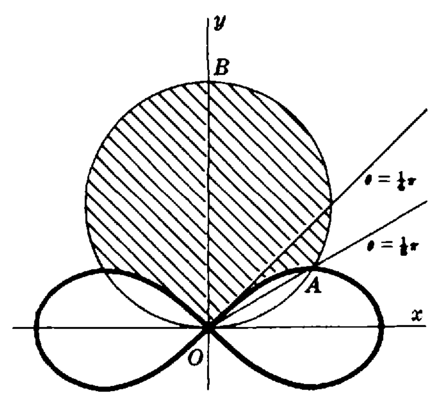


Fig. 70-4

4. Find the area inside the circle $\rho = 4\sin\theta$ and outside the lemniscate $\rho^2 = 8\cos 2\theta$.

The required area is twice that in the first quadrant bounded by the two curves and the line $\theta = \frac{1}{2}\pi$. Note in Fig. 70-4 that the arc AO of the lemniscate is described as θ varies from $\theta = \pi/6$ to $\theta = \pi/4$, while the arc AB of the circle is described as θ varies from $\theta = \pi/6$ to $\theta = \pi/2$. This area must then be considered as two regions, one below and one above the line $\theta = \pi/4$. Thus,

$$A = 2\int_{\pi/6}^{\pi/4} \int_{2\sqrt{2\cos 2\theta}}^{4\sin\theta} \rho\,d\rho\,d\theta + 2\int_{\pi/4}^{\pi/2} \int_0^{4\sin\theta} \rho\,d\rho\,d\theta$$

$$= \int_{\pi/6}^{\pi/4} (16\sin^2\theta - 8\cos 2\theta)\,d\theta + \int_{\pi/4}^{\pi/2} 16\sin^2\theta\,d\theta$$

$$= (\tfrac{8}{3}\pi + 4\sqrt{3} - 4) \text{ square units}$$

5. Evaluate $N = \int_0^{+\infty} e^{-x^2}\,dx$. (See Fig. 70-5.)

Since $\int_0^{+\infty} e^{-x^2}\,dx = \int_0^{+\infty} e^{-y^2}\,dy$, we have

$$N^2 = \int_0^{+\infty} e^{-x^2}\,dx \int_0^{+\infty} e^{-y^2}\,dy = \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy = \iint_R e^{-(x^2+y^2)}\,dA$$

Changing to polar coordinates ($x^2 + y^2 = \rho^2$, $dA = \rho\,d\rho\,d\theta$) yields

$$N^2 = \int_0^{\pi/2}\int_0^{+\infty} e^{-\rho^2}\rho\,d\rho\,d\theta = \int_0^{\pi/2} \lim_{a\to+\infty}\left[-\frac{1}{2}e^{-\rho^2}\right]_0^a d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}$$

and $N = \sqrt{\pi}/2$.

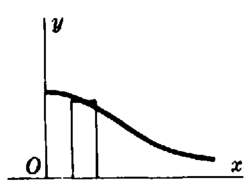


Fig. 70-5

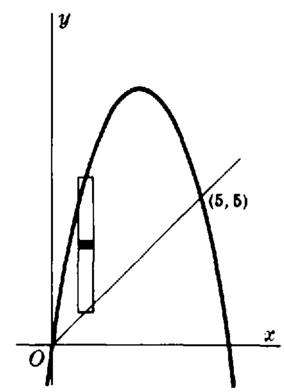


Fig. 70-6

6. Find the centroid of the plane area bounded by the parabola $y = 6x - x^2$ and the line $y = x$. (See Fig. 70-6.)

$$A = \iint_R dA = \int_0^5\int_x^{6x-x^2} dy\,dx = \int_0^5 (5x - x^2)\,dx = \tfrac{125}{6}$$

$$M_y = \iint_R x\,dA = \int_0^5\int_x^{6x-x^2} x\,dy\,dx = \int_0^5 (5x^2 - x^3)\,dx = \tfrac{625}{12}$$

$$M_x = \iint_R y\,dA = \int_0^5\int_x^{6x-x^2} y\,dy\,dx = \tfrac{1}{2}\int_0^5 [(6x - x^2)^2 - x^2]\,dx = \tfrac{625}{6}$$

Hence, $\bar{x} = M_y/A = \frac{5}{2}$, $\bar{y} = M_x/A = 5$, and the coordinates of the centroid are $(\frac{5}{2}, 5)$.

7. Find the centroid of the plane area bounded by the parabolas $y = 2x - x^2$ and $y = 3x^2 - 6x$. (See Fig. 70-7.)

$$A = \iint_R dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} dy\,dx = \int_0^2 (8x - 4x^2)\,dx = \tfrac{16}{3}$$

$$M_y = \iint_R x\,dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} x\,dy\,dx = \int_0^2 (8x^2 - 4x^3)\,dx = \tfrac{16}{3}$$

$$M_x = \iint_R y\,dA = \int_0^2\int_{3x^2-6x}^{2x-x^2} y\,dy\,dx = \tfrac{1}{2}\int_0^2 [(2x - x^2)^2 - (3x^2 - 6x)^2]\,dx = -\tfrac{64}{15}$$

Hence, $\bar{x} = M_y/A = 1$, $\bar{y} = M_x/A = -\frac{4}{5}$, and the centroid is $(1, -\frac{4}{5})$.

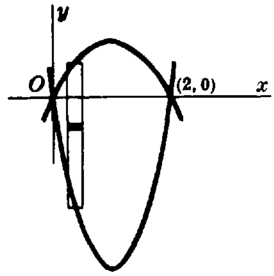


Fig. 70-7

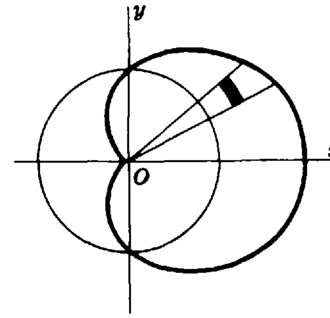


Fig. 70-8

8. Find the centroid of the plane area outside the circle $\rho = 1$ and inside the cardioid $\rho = 1 + \cos\theta$.

From Fig. 70-8 it is evident that $\bar{y} = 0$ and that $\bar{x}$ is the same whether computed for the given area or for the half lying above the polar axis. For the latter area,

$$A = \iint_R dA = \int_0^{\pi/2} \int_1^{1+\cos\theta} \rho\, d\rho\, d\theta = \frac{1}{2}\int_0^{\pi/2} [(1+\cos\theta)^2 - 1^2]\, d\theta = \frac{\pi+8}{8}$$

$$M_y = \iint_R x\, dA = \int_0^{\pi/2}\int_1^{1+\cos\theta} (\rho\cos\theta)\rho\, d\rho\, d\theta = \frac{1}{3}\int_0^{\pi/2} (3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)\, d\theta$$

$$= \frac{1}{3}\left[\frac{3}{2}\theta + \frac{3}{4}\sin 2\theta + 3\sin\theta - \sin^3\theta + \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_0^{\pi/2} = \frac{15\pi+32}{48}$$

The coordinates of the centroid are $\left(\dfrac{15\pi+32}{6(\pi+8)}, 0\right)$.

9. Find the centroid of the area inside $\rho = \sin\theta$ and outside $\rho = 1 - \cos\theta$. (See Fig. 70-9.)

$$A = \iint_R dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} \rho\, d\rho\, d\theta = \frac{1}{2}\int_0^{\pi/2} (2\cos\theta - 1 - \cos 2\theta)\, d\theta = \frac{4-\pi}{4}$$

$$M_y = \iint_R x\, dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} (\rho\cos\theta)\rho\, d\rho\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\sin^3\theta - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta)\cos\theta\, d\theta = \frac{15\pi - 44}{48}$$

$$M_x = \iint_R y\, dA = \int_0^{\pi/2}\int_{1-\cos\theta}^{\sin\theta} (\rho\sin\theta)\rho\, d\rho\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} (\sin^3\theta - 1 + 3\cos\theta - 3\cos^2\theta + \cos^3\theta)\sin\theta\, d\theta = \frac{3\pi - 4}{48}$$

The coordinates of the centroid are $\left(\dfrac{15\pi - 44}{12(4-\pi)}, \dfrac{3\pi-4}{12(4-\pi)}\right)$.

10. Find I_x, I_y, and I_0 for the area enclosed by the loop of $y^2 = x^2(2-x)$. (See Fig. 70-10.)

$$A = \iint_R dA = 2\int_0^2\int_0^{x\sqrt{2-x}} dy\, dx = 2\int_0^2 x\sqrt{2-x}\, dx$$

$$= -4\int_{\sqrt{2}}^0 (2z^2 - z^4)\, dz = -4\left[\frac{2}{3}z^3 - \frac{1}{5}z^5\right]_{\sqrt{2}}^0 = \frac{32\sqrt{2}}{15}$$

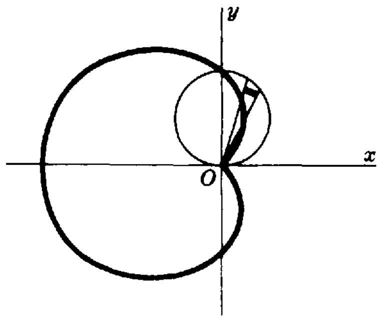


Fig. 70-9

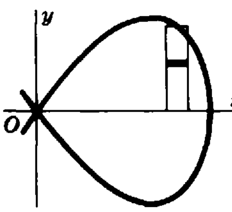


Fig. 70-10

where we have used the transformation $2 - x = z^2$. Then

$$I_x = \iint_R y^2\, dA = 2\int_0^2 \int_0^{x\sqrt{2-x}} y^2\, dy\, dx = \frac{2}{3}\int_0^2 x^3(2-x)^{3/2}\, dx$$

$$= -\frac{4}{3}\int_{\sqrt{2}}^0 (2-z^2)^3 z^4\, dz = -\frac{4}{3}\left[\frac{8}{5}z^5 - \frac{12}{7}z^7 + \frac{2}{3}z^9 - \frac{1}{11}z^{11}\right]_{\sqrt{2}}^0 = \frac{2048\sqrt{2}}{3465} = \frac{64}{231}A$$

$$I_y = \iint_R x^2\, dA = 2\int_0^2 \int_0^{x\sqrt{2-x}} x^2\, dy\, dx = 2\int_0^2 x^3\sqrt{2-x}\, dx$$

$$= -4\int_{\sqrt{2}}^0 (2-z^2)^3 z^2\, dz = -4\left[\frac{8}{3}z^3 - \frac{12}{5}z^5 + \frac{6}{7}z^7 - \frac{1}{9}z^9\right]_{\sqrt{2}}^0 = \frac{1024\sqrt{2}}{315} = \frac{32}{21}A$$

$$I_0 = I_x + I_y = \frac{13\,312\sqrt{2}}{3465} = \frac{416}{231}A$$

11. Find I_x, I_y, and I_0 for the first-quadrant area outside the circle $\rho = 2a$ and inside the circle $\rho = 4a\cos\theta$. (See Fig. 70-11.)

$$A = \iint_R dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} \rho\, d\rho\, d\theta = \frac{1}{2}\int_0^{\pi/3}[(4a\cos\theta)^2 - (2a)^2]\, d\theta = \frac{2\pi + 3\sqrt{3}}{3}a^2$$

$$I_x = \iint_R y^2\, dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} (\rho\sin\theta)^2\rho\, d\rho\, d\theta = \frac{1}{4}\int_0^{\pi/3}\{(4a\cos\theta)^4 - (2a)^4\}\sin^2\theta\, d\theta$$

$$= 4a^4\int_0^{\pi/3}(16\cos^4\theta - 1)\sin^2\theta\, d\theta = \frac{4\pi + 9\sqrt{3}}{6}a^4 = \frac{4\pi + 9\sqrt{3}}{2(2\pi + 3\sqrt{3})}a^2 A$$

$$I_y = \iint_R x^2\, dA = \int_0^{\pi/3}\int_{2a}^{4a\cos\theta} (\rho\cos\theta)^2\rho\, d\rho\, d\theta = \frac{12\pi + 11\sqrt{3}}{2}a^4 = \frac{3(12\pi + 11\sqrt{3})}{2(2\pi + 3\sqrt{3})}a^2 A$$

$$I_0 = I_x + I_y = \frac{20\pi + 21\sqrt{3}}{3}a^4 = \frac{20\pi + 21\sqrt{3}}{2\pi + 3\sqrt{3}}a^2 A$$

12. Find I_x, I_y, and I_0 for the area of the circle $\rho = 2(\sin\theta + \cos\theta)$. (See Fig. 70-12.)

Since $x^2 + y^2 = \rho^2$,

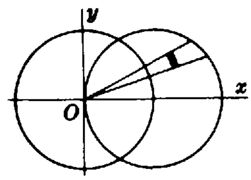


Fig. 70-11

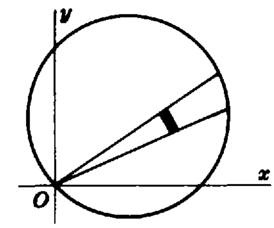


Fig. 70-12

$$I_0 = \iint_R (x^2+y^2)\,dA = \int_{-\pi/4}^{3\pi/4}\int_0^{2(\sin\theta+\cos\theta)} \rho^2\rho\,d\rho\,d\theta = 4\int_{-\pi/4}^{3\pi/4}(\sin\theta+\cos\theta)^4\,d\theta$$
$$= 4\left[\tfrac{3}{2}\theta - \cos 2\theta - \tfrac{1}{8}\sin 4\theta\right]_{-\pi/4}^{3\pi/4} = 6\pi = 3A$$

It is evident from Fig. 70-12 that $I_x = I_y$. Hence, $I_x = I_y = \frac{1}{2}I_0 = \frac{3}{2}A$.

Supplementary Problems

13. Use double integration to find the area:
(a) Bounded by $3x + 4y = 24$, $x = 0$, $y = 0$ *Ans.* 24 square units
(b) Bounded by $x + y = 2$, $2y = x + 4$, $y = 0$ *Ans.* 6 square units
(c) Bounded by $x^2 = 4y$, $8y = x^2 + 16$ *Ans.* $\frac{32}{3}$ square units
(d) Within $\rho = 2(1 - \cos\theta)$ *Ans.* 6π square units
(e) Bounded by $\rho = \tan\theta\sec\theta$ and $\theta = \pi/3$ *Ans.* $\frac{1}{2}\sqrt{3}$ square units
(f) Outside $\rho = 4$ and inside $\rho = 8\cos\theta$ *Ans.* $8(\frac{2}{3}\pi + \sqrt{3})$ square units

14. Locate the centroid of each of the following areas.
(a) The area of Problem 13(a) *Ans.* $(\frac{8}{3}, 2)$
(b) The first-quadrant area of Problem 13(c) *Ans.* $(\frac{3}{2}, \frac{8}{5})$
(c) The first-quadrant area bounded by $y^2 = 6x$, $y = 0$, $x = 6$ *Ans.* $(\frac{18}{5}, \frac{9}{4})$
(d) The area bounded by $y^2 = 4x$, $x^2 = 5 - 2y$, $x = 0$ *Ans.* $(\frac{13}{40}, \frac{26}{15})$
(e) The first-quadrant area bounded by $x^2 - 8y + 4 = 0$, $x^2 = 4y$, $x = 0$ *Ans.* $(\frac{3}{4}, \frac{2}{5})$
(f) The area of Problem 13(e) *Ans.* $(\frac{1}{2}\sqrt{3}, \frac{6}{5})$
(g) The first-quadrant area of Problem 13(f) *Ans.* $\left(\dfrac{16\pi + 6\sqrt{3}}{2\pi + 3\sqrt{3}}, \dfrac{22}{2\pi + 3\sqrt{3}}\right)$

15. Verify that $\frac{1}{2}\int_\alpha^\beta [g_2^2(\theta) - g_1^2(\theta)]\,d\theta = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \rho\,d\rho\,d\theta = \iint_R dA$; then infer that

$$\iint_R f(x, y)\,dA = \iint_R f(\rho\cos\theta, \rho\sin\theta)\rho\,d\rho\,d\theta$$

16. Find I_x and I_y for each of the following areas.
(a) The area of Problem 13(a) *Ans.* $I_x = 6A$; $I_y = \frac{32}{3}A$
(b) The area cut from $y^2 = 8x$ by its latus rectum *Ans.* $I_x = \frac{16}{5}A$; $I_y = \frac{12}{7}A$
(c) The area bounded by $y = x^2$ and $y = x$ *Ans.* $I_x = \frac{3}{14}A$; $I_y = \frac{3}{10}A$
(d) The area bounded by $y = 4x - x^2$ and $y = x$ *Ans.* $I_x = \frac{459}{70}A$; $I_y = \frac{27}{10}A$

17. Find I_x and I_y for one loop of $\rho^2 = \cos 2\theta$. *Ans.* $I_x = \left(\dfrac{\pi}{16} - \dfrac{1}{6}\right)A$; $I_y = \left(\dfrac{\pi}{16} + \dfrac{1}{6}\right)A$

18. Find I_0 for (a) the loop of $\rho = \sin 2\theta$ and (b) the area enclosed by $\rho = 1 + \cos\theta$. *Ans.* (a) $\frac{3}{8}A$; (b) $\frac{35}{24}A$

Chapter 71

Volume Under a Surface by Double Integration

THE VOLUME UNDER A SURFACE $z = f(x, y)$ or $z = f(\rho, \theta)$, that is, the volume of a vertical column whose upper base is in the surface and whose lower base is in the xOy plane, is defined by the double integral $V = \iint_R z\,dA$, the region R being the lower base of the column.

Solved Problems

1. Find the volume in the first octant between the planes $z = 0$ and $z = x + y + 2$, and inside the cylinder $x^2 + y^2 = 16$.

From Fig. 71-1, it is evident that $z = x + y + 2$ is to be integrated over a quadrant of the circle $x^2 + y^2 = 16$ in the xOy plane. Hence,

$$V = \iint_R z\,dA = \int_0^4 \int_0^{\sqrt{16-x^2}} (x + y + 2)\,dy\,dx = \int_0^4 \left(x\sqrt{16 - x^2} + 8 - \frac{1}{2}x^2 + 2\sqrt{16 - x^2}\right) dx$$

$$= \left[-\frac{1}{3}(16 - x^2)^{3/2} + 8x - \frac{x^3}{6} + x\sqrt{16 - x^2} + 16 \arcsin \frac{1}{4}x\right]_0^4 = \left(\frac{128}{3} + 8\pi\right) \text{ cubic units}$$

2. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

From Fig. 71-2, it is evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xOy plane. Hence,

$$V = \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (4 - y)\,dx\,dy = 2\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} (4 - y)\,dx\,dy = 16\pi \text{ cubic units}$$

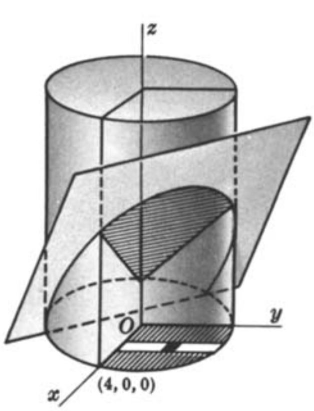


Fig. 71-1

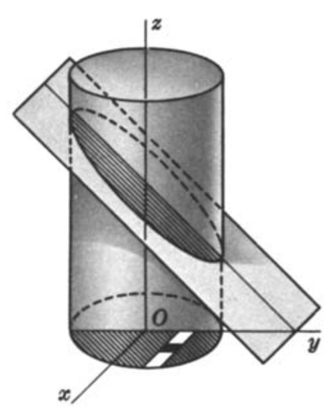


Fig. 71-2

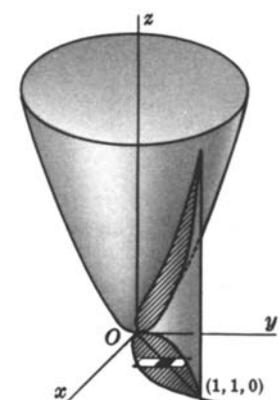


Fig. 71-3

3. Find the volume bounded above by the paraboloid $x^2 + 4y^2 = z$, below by the plane $z = 0$, and laterally by the cylinders $y^2 = x$ and $x^2 = y$. (See Fig. 71-3.)

The required volume is obtained by integrating $z = x^2 + 4y^2$ over the region R common to the parabolas $y^2 = x$ and $x^2 = y$ in the xOy plane. Hence,

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + 4y^2)\, dy\, dx = \int_0^1 [x^2y + \tfrac{4}{3}y^3]_{x^2}^{\sqrt{x}}\, dx = \tfrac{3}{7} \text{ cubic units}$$

4. Find the volume of one of the wedges cut from the cylinder $4x^2 + y^2 = a^2$ by the planes $z = 0$ and $z = my$. (See Fig. 71-4.)

The volume is obtained by integrating $z = my$ over half the ellipse $4x^2 + y^2 = a^2$. Hence,

$$V = 2\int_0^{a/2} \int_0^{\sqrt{a^2-4x^2}} my\, dy\, dx = m \int_0^{a/2} [y^2]_0^{\sqrt{a^2-4x^2}}\, dx = \frac{ma^3}{3} \text{ cubic units}$$

5. Find the volume bounded by the paraboloid $x^2 + y^2 = 4z$, the cylinder $x^2 + y^2 = 8y$, and the plane $z = 0$. (See Fig. 71-5.)

The required volume is obtained by integrating $z = \frac{1}{4}(x^2 + y^2)$ over the circle $x^2 + y^2 = 8y$. Using cylindrical coordinates, the volume is obtained by integrating $z = \frac{1}{4}\rho^2$ over the circle $\rho = 8 \sin \theta$. Then,

$$V = \iint_R z\, dA = \int_0^{\pi} \int_0^{8 \sin \theta} z\rho\, d\rho\, d\theta = \tfrac{1}{4} \int_0^{\pi} \int_0^{8 \sin \theta} \rho^3\, d\rho\, d\theta$$

$$= \tfrac{1}{16} \int_0^{\pi} [\rho^4]_0^{8 \sin \theta}\, d\theta = 256 \int_0^{\pi} \sin^4 \theta\, d\theta = 96\pi \text{ cubic units}$$

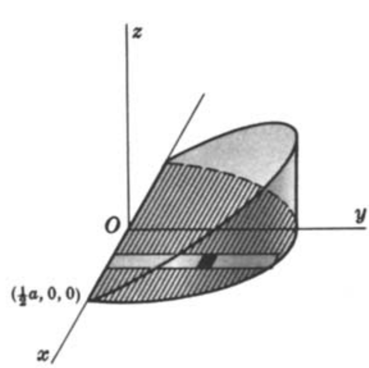


Fig. 71-4

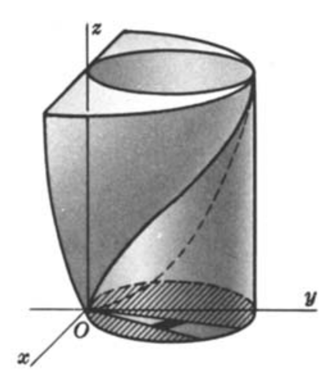


Fig. 71-5

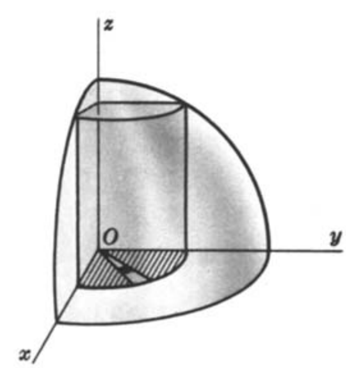


Fig. 71-6

6. Find the volume removed when a hole of radius a is bored through a sphere of radius $2a$, the axis of the hole being a diameter of the sphere. (See Fig. 71-6.)

From the figure, it is obvious that the required volume is eight times the volume in the first octant bounded by the cylinder $\rho^2 = a^2$, the sphere $\rho^2 + z^2 = 4a^2$, and the plane $z = 0$. The latter volume is obtained by integrating $z = \sqrt{4a^2 - \rho^2}$ over a quadrant of the circle $\rho = a$. Hence,

$$V = 8 \int_0^{\pi/2} \int_0^{a} \sqrt{4a^2 - \rho^2}\,\rho\, d\rho\, d\theta = \tfrac{8}{3} \int_0^{\pi/2} (8a^3 - 3\sqrt{3}a^3)\, d\theta = \tfrac{4}{3}(8 - 3\sqrt{3})a^3\pi \text{ cubic units}$$

Supplementary Problems

7. Find the volume cut from $9x^2 + 4y^2 + 36z = 36$ by the plane $z = 0$. *Ans.* 3π cubic units

8. Find the volume under $z = 3x$ and above the first-quadrant area bounded by $x = 0$, $y = 0$, $x = 4$, and $x^2 + y^2 = 25$. *Ans.* 98 cubic units

9. Find the volume in the first octant bounded by $x^2 + z = 9$, $3x + 4y = 24$, $x = 0$, $y = 0$, and $z = 0$.

Ans. 1485/16 cubic units

10. Find the volume in the first octant bounded by $xy = 4z$, $y = x$, and $x = 4$. *Ans.* 8 cubic units

11. Find the volume in the first octant bounded by $x^2 + y^2 = 25$ and $z = y$. *Ans.* $\frac{125}{3}$ cubic units

12. Find the volume common to the cylinders $x^2 + y^2 = 16$ and $x^2 + z^2 = 16$. *Ans.* $\frac{1024}{3}$ cubic units

13. Find the volume in the first octant inside $y^2 + z^2 = 9$ and outside $y^2 = 3x$. *Ans.* $27\pi/16$ cubic units

14. Find the volume in the first octant bounded by $x^2 + z^2 = 16$ and $x - y = 0$. *Ans.* $\frac{64}{3}$ cubic units

15. Find the volume in front of $x = 0$ and common to $y^2 + z^2 = 4$ and $y^2 + z^2 + 2x = 16$.

Ans. 28π cubic units

16. Find the volume inside $\rho = 2$ and outside the cone $z^2 = \rho^2$. *Ans.* $32\pi/3$ cubic units

17. Find the volume inside $y^2 + z^2 = 2$ and outside $x^2 - y^2 - z^2 = 2$. *Ans.* $8\pi(4 - \sqrt{2})/3$ cubic units

18. Find the volume common to $\rho^2 + z^2 = a^2$ and $\rho = a \sin\theta$. *Ans.* $2(3\pi - 4)a^2/9$ cubic units

19. Find the volume inside $x^2 + y^2 = 9$, bounded below by $x^2 + y^2 + 4z = 16$ and above by $z = 4$.

Ans. $81\pi/8$ cubic units

20. Find the volume cut from the paraboloid $4x^2 + y^2 = 4z$ by the plane $z - y = 2$. *Ans.* 9π cubic units

21. Find the volume generated by revolving the cardioid $\rho = 2(1 - \cos\theta)$ about the polar axis.

Ans. $V = 2\pi \int\int y\rho \, d\rho \, d\theta = 64\pi/3$ cubic units

22. Find the volume generated by revolving a petal of $\rho = \sin 2\theta$ about either axis.

Ans. $32\pi/105$ cubic units

23. A square hole 2 units on a side is cut symmetrically through a sphere of radius 2 units. Show that the volume removed is $\frac{4}{3}(2\sqrt{2} + 19\pi - 54 \arctan\sqrt{2})$ cubic units.

Chapter 72

Area of a Curved Surface by Double Integration

TO COMPUTE THE LENGTH OF A(PLANAR) ARC, (1) the arc is projected on a convenient coordinate axis, thus establishing an interval on the axis, and (2) an integrand function, $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ if the projection is on the x axis or $\sqrt{1+\left(\frac{dx}{dy}\right)^2}$ if the projection is on the y axis, is integrated over the interval.

A similar procedure is used to compute the area S of a portion R^* of a surface $z = f(x, y)$: (1) R^* is projected on a convenient coordinate plane, thus establishing a region R on the plane, and (2) an integrand function is integrated over R. Then,

$$\text{If } R^* \text{ is projected on } xOy,\ S = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

$$\text{If } R^* \text{ is projected on } yOz,\ S = \iint_R \sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\, dA.$$

$$\text{If } R^* \text{ is projected on } zOx,\ S = \iint_R \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\, dA.$$

Solved Problems

1. Derive the first of the formulas for the area S of a region R^* as given above.

Consider a region R^* of area S on the surface $z = f(x, y)$. Through the boundary of R^* pass a vertical cylinder (see Fig. 72-1) cutting the xOy plane in the region R. Now divide R into n subregions

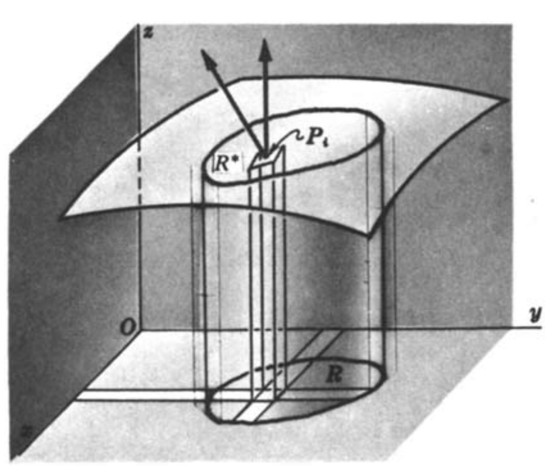


Fig. 72-1

ΔA_i (of areas ΔA_i), and denote by ΔS_i the area of the projection of ΔA_i on R^*. In each subregion ΔS_i, choose a point P_i and draw there the tangent plane to the surface. Let the area of the projection of ΔA_i on this tangent plane be denoted by ΔT_i. We shall use ΔT_i as an approximation of the corresponding surface area ΔS_i.

Now the angle between the xOy plane and the tangent plane at P_i is the angle γ_i between the z axis with direction numbers $[0, 0, 1]$, and the normal, $\left[-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right] = \left[-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right]$, to the surface at P_i; thus

$$\cos \gamma_i = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

Then (see Fig. 72-2),

$$\Delta T_i \cos \gamma_i = \Delta A_i \qquad \text{and} \qquad \Delta T_i = \sec \gamma_i \, \Delta A_i$$

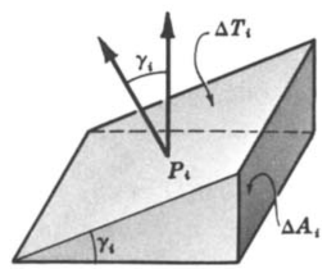


Fig. 72-2

Hence, an approximation of S is $\sum_{i=1}^{n} \Delta T_i = \sum_{i=1}^{n} \sec \gamma_i \, \Delta A_i$, and

$$S = \lim_{n \to +\infty} \sum_{i=1}^{n} \sec \gamma_i \, \Delta A_i = \iint_R \sec \gamma \, dA = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

2. Find the area of the portion of the cone $x^2 + y^2 = 3z^2$ lying above the xOy plane and inside the cylinder $x^2 + y^2 = 4y$.

Solution 1: Refer to Fig. 72-3. The projection of the required area on the xOy plane is the region R enclosed by the circle $x^2 + y^2 = 4y$. For the cone,

$$\frac{\partial z}{\partial x} = \frac{1}{3}\frac{x}{z} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{1}{3}\frac{y}{z}. \qquad \text{So} \qquad 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{9z^2 + x^2 + y^2}{9z^2} = \frac{12z^2}{9z^2} = \frac{4}{3}$$

Then $$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA = \int_0^4 \int_{-\sqrt{4y-y^2}}^{\sqrt{4y-y^2}} \frac{2}{\sqrt{3}} \, dx \, dy = 2 \frac{2}{\sqrt{3}} \int_0^4 \int_0^{\sqrt{4y-y^2}} dx \, dy$$

$$= \frac{4}{\sqrt{3}} \int_0^4 \sqrt{4y - y^2} \, dy = \frac{8\sqrt{3}}{3} \pi \text{ square units}$$

Solution 2: Refer to Fig. 72-4. The projection of one-half the required area on the yOz plane is the region R bounded by the line $y = \sqrt{3}z$ and the parabola $y = \frac{3}{4}z^2$, the latter obtained by eliminating x between the equations of the two surfaces. For the cone,

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \qquad \text{and} \qquad \frac{\partial x}{\partial z} = \frac{3z}{x}. \qquad \text{So} \qquad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + y^2 + 9z^2}{x^2} = \frac{12z^2}{x^2} = \frac{12z^2}{3z^2 - y^2}$$

Then $$S = 2 \int_0^4 \int_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}} \frac{2\sqrt{3}z}{\sqrt{3z^2 - y^2}} \, dz \, dy = \frac{4\sqrt{3}}{3} \int_0^4 \left[\sqrt{3z^2 - y^2}\right]_{y/\sqrt{3}}^{2\sqrt{y}/\sqrt{3}} dy = \frac{4\sqrt{3}}{3} \int_0^4 \sqrt{4y - y^2} \, dy$$

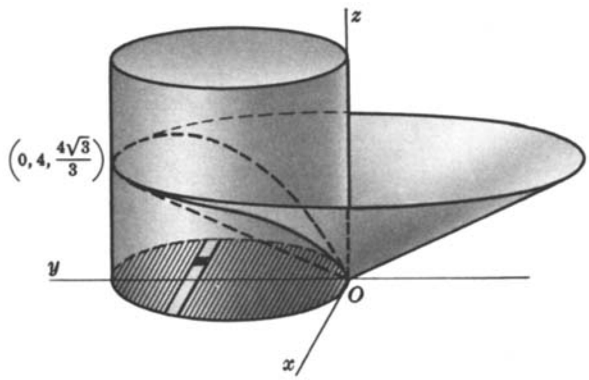


Fig. 72-3

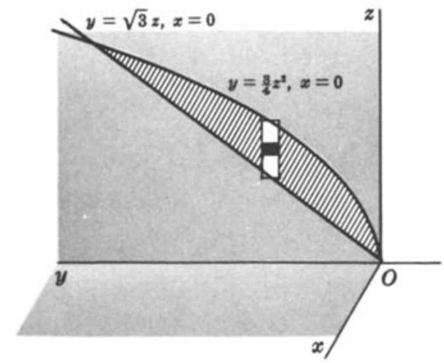


Fig. 72-4

Solution 3: Using polar coordinates in solution 1, we must integrate $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\frac{2}{\sqrt{3}}$ over the region R enclosed by the circle $\rho = 4\sin\theta$. Then,

$$S=\iint_R \frac{2}{\sqrt{3}}\,dA=\int_0^{\pi}\int_0^{4\sin\theta}\frac{2}{\sqrt{3}}\,\rho\,d\rho\,d\theta=\frac{1}{\sqrt{3}}\int_0^{\pi}[\rho^2]_0^{4\sin\theta}\,d\theta$$

$$=\frac{16}{\sqrt{3}}\int_0^{\pi}\sin^2\theta\,d\theta=\frac{8\sqrt{3}}{3}\pi \text{ square units}$$

3. **Find the area of the portion of the cylinder $x^2+z^2=16$ lying inside the cylinder $x^2+y^2=16$.**

Figure 72-5 shows one-eighth of the required area, its projection on the xOy plane being a quadrant of the circle $x^2+y^2=16$. For the cylinder $x^2+z^2=16$,

$$\frac{\partial z}{\partial x}=-\frac{x}{z} \quad \text{and} \quad \frac{\partial z}{\partial y}=0. \quad \text{So} \quad 1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{x^2+z^2}{z^2}=\frac{16}{16-x^2}$$

Then
$$S=8\int_0^4\int_0^{\sqrt{16-x^2}}\frac{4}{\sqrt{16-x^2}}\,dy\,dx=32\int_0^4 dx=128 \text{ square units}$$

4. **Find the area of the portion of the sphere $x^2+y^2+z^2=16$ outside the paraboloid $x^2+y^2+z=16$.**

Figure 72-6 shows one-fourth of the required area, its projection on the yOz plane being the region R bounded by the circle $y^2+z^2=16$, the y and z axes, and the line $z=1$. For the sphere,

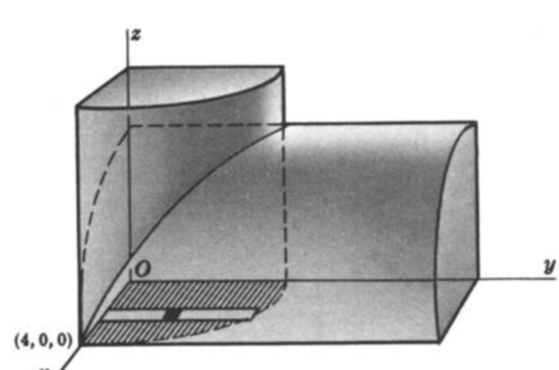


Fig. 72-5

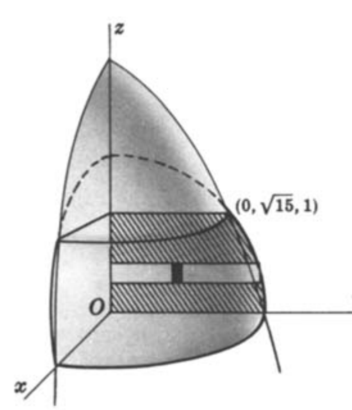


Fig. 72-6

$$\frac{\partial x}{\partial y} = -\frac{y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = -\frac{z}{x}. \quad \text{So} \quad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + y^2 + z^2}{x^2} = \frac{16}{16 - y^2 - z^2}$$

Then
$$S = 4 \iint_R \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dA = 4 \int_0^1 \int_0^{\sqrt{16-z^2}} \frac{4}{\sqrt{16 - y^2 - z^2}}\, dy\, dz$$
$$= 16 \int_0^1 \left[\arcsin \frac{y}{\sqrt{16 - z^2}}\right]_0^{\sqrt{16-z^2}} dz = 16 \int_0^1 \frac{1}{2}\pi\, dz = 8\pi \text{ square units}$$

5. Find the area of the portion of the cylinder $x^2 + y^2 = 6y$ lying inside the sphere $x^2 + y^2 + z^2 = 36$.

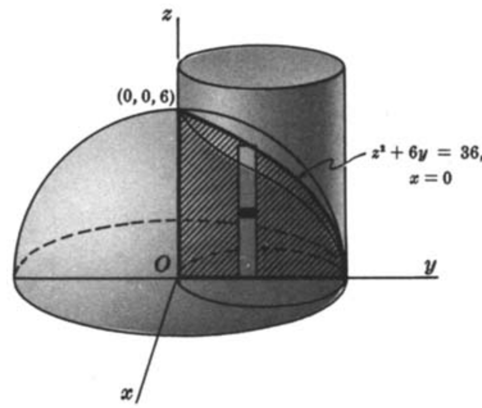


Fig. 72-7

Figure 72-7 shows one-fourth of the required area. Its projection on the yOz plane is the region R bounded by the z and y axes and the parabola $z^2 + 6y = 36$, the latter obtained by eliminating x from the equations of the two surfaces. For the cylinder,

$$\frac{\partial x}{\partial y} = \frac{3 - y}{x} \quad \text{and} \quad \frac{\partial x}{\partial z} = 0. \quad \text{So} \quad 1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \frac{x^2 + 9 - 6y + y^2}{x^2} = \frac{9}{6y - y^2}$$

Then
$$S = 4 \int_0^6 \int_0^{\sqrt{36-6y}} \frac{3}{\sqrt{6y - y^2}}\, dz\, dy = 12 \int_0^6 \frac{\sqrt{6}}{\sqrt{y}}\, dy = 144 \text{ square units}$$

Supplementary Problems

6. Find the area of the portion of the cone $x^2 + y^2 = z^2$ inside the vertical prism whose base is the triangle bounded by the lines $y = x$, $x = 0$, and $y = 1$ in the xOy plane. *Ans.* $\frac{1}{2}\sqrt{2}$ square units

7. Find the area of the portion of the plane $x + y + z = 6$ inside the cylinder $x^2 + y^2 = 4$.

Ans. $4\sqrt{3}\pi$ square units

8. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 36$ inside the cylinder $x^2 + y^2 = 6y$.

Ans. $72(\pi - 2)$ square units

9. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 4z$ inside the paraboloid $x^2 + y^2 = z$.

Ans. 4π square units

10. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 25$ between the planes $z = 2$ and $z = 4$.

Ans. 20π square units

11. Find the area of the portion of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$.

Ans. $2\pi(2\sqrt{2} - 1)/3$ square units

12. Find the area of the surface of the cone $x^2 + y^2 - 9z^2 = 0$ above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 6y$. *Ans.* $3\sqrt{10}\pi$ square units

13. Find the area of that part of the sphere $x^2 + y^2 + z^2 = 25$ that is within the elliptic cylinder $2x^2 + y^2 = 25$.

Ans. 50π square units

14. Find the area of the surface of $x^2 + y^2 - az = 0$ which lies directly above the lemniscate $4\rho^2 = a^2 \cos 2\theta$. *Ans.* $S = \frac{4}{a}\int\int \sqrt{4\rho^2 + a^2}\rho\, d\rho\, d\theta = \frac{a^2}{3}\left(\frac{5}{3} - \frac{\pi}{4}\right)$ square units

15. Find the area of the surface of $x^2 + y^2 + z^2 = 4$ which lies directly above the cardioid $\rho = 1 - \cos\theta$.

Ans. $8[\pi - \sqrt{2} - \ln(\sqrt{2} + 1)]$ square units

Chapter 73

Triple Integrals

CYLINDRICAL AND SPHERICAL COORDINATES. Assume that a point P has coordinates (x, y, z) in a right-handed rectangular coordinate system. The corresponding *cylindrical coordinates* of P are (r, θ, z), where (r, θ) are the polar coordinates for the point (x, y) in the xy plane. (Note the notational change here from (ρ, θ) to (r, θ) for the polar coordinates of (x, y); see Fig. 73-1.) Hence we have the relations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

In cylindrical coordinates, an equation $r = c$ represents a right circular cylinder of radius c with the z axis as its axis of symmetry. An equation $\theta = c$ represents a plane through the z axis.

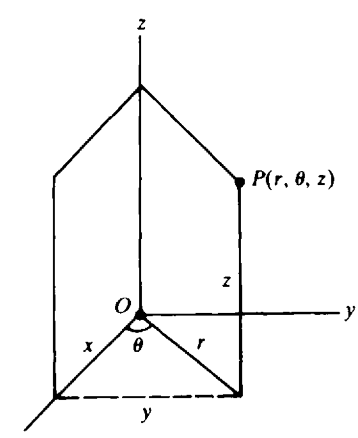


Fig. 73-1

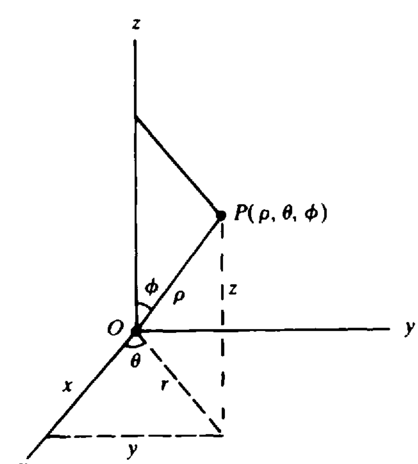


Fig. 73-2

A point P with rectangular coordinates (x, y, z) has the *spherical coordinates* (ρ, θ, ϕ), where $\rho = |OP|$, θ is the same as in cylindrical coordinates, and ϕ is the directed angle from the positive z axis to the vector **OP**. (See Fig. 73-2.) In spherical coordinates, an equation $\rho = c$ represents a sphere of radius c with center at the origin. An equation $\phi = c$ represents a cone with vertex at the origin and the z axis as its axis of symmetry.

The following additional relations hold among spherical, cylindrical, and rectangular coordinates:

$$r = \rho\sin\phi \qquad z = \rho\cos\phi \qquad \rho^2 = x^2 + y^2 + z^2$$
$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta$$

(See Problems 14 to 16.)

THE TRIPLE INTEGRAL $\iiint_R f(x, y, z)\, dV$ of a function of three independent variables over a closed region R of points (x, y, z), of volume V, on which the function is single-valued and continuous, is an extension of the notion of single and double integrals.

If $f(x, y, z) = 1$, then $\iiint_R f(x, y, z)\, dV$ may be interpreted as measuring the volume of the region R.

EVALUATION OF THE TRIPLE INTEGRAL. In rectangular coordinates,

$$\iiint_R f(x, y, z)\, dV = \int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

$$= \int_c^d \int_{x_1(y)}^{x_2(y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\, dz\, dx\, dy, \text{ etc.}$$

where the limits of integration are chosen to cover the region R.

In cylindrical coordinates,

$$\iiint_R f(r, \theta, z)\, dV = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta$$

where the limits of integration are chosen to cover the region R.

In spherical coordinates,

$$\iiint_R f(\rho, \phi, \theta)\, dV = \int_\alpha^\beta \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho, \phi, \theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

where the limits of integration are chosen to cover the region R.

Discussion of the definitions: Consider the function $f(x, y, z)$, continuous over a region R of ordinary space. After slicing R with planes $x = \xi_i$ and $y = \eta_j$ as in Chapter 69, let these subregions be further sliced by planes $z = \zeta_k$. The region R has now been separated into a number of rectangular parallelepipeds of volume $\Delta V_{ijk} = \Delta x_i\, \Delta y_j\, \Delta z_k$ and a number of partial parallelepipeds which we shall ignore. In each complete parallelepiped select a point $P_{ijk}(x_i, y_j, z_k)$; then compute $f(x_i, y_j, z_k)$ and form the sum

$$\sum_{\substack{i=1,\ldots,m\\ j=1,\ldots,n\\ k=1,\ldots,p}} f(x_i, y_j, z_k)\, \Delta V_{ijk} = \sum_{\substack{i=1,\ldots,m\\ j=1,\ldots,n\\ k=1,\ldots,p}} f(x_i, y_j, z_k)\, \Delta x_i\, \Delta y_j\, \Delta z_k \tag{73.1}$$

The triple integral of $f(x, y, z)$ over the region R is defined to be the limit of (*73.1*) as the number of parallelepipeds is indefinitely increased in such a manner that all dimensions of each go to zero.

In evaluating this limit, we may sum first each set of parallelepipeds having $\Delta_i x$ and $\Delta_j y$, for fixed i and j, as two dimensions and consider the limit as each $\Delta_k z \to 0$. We have

$$\lim_{p\to+\infty} \sum_{k=1}^{p} f(x_i, y_j, z_k)\, \Delta_k z\, \Delta_i x\, \Delta_j y = \int_{z_1}^{z_2} f(x_i, y_i, z)\, dz\, \Delta_i x\, \Delta_j y$$

Now these are the columns, the basic subregions, of Chapter 69; hence,

$$\lim_{\substack{m\to+\infty\\ n\to+\infty\\ p\to+\infty}} \sum_{\substack{i=1,\ldots,m\\ j=1,\ldots,n\\ k=1,\ldots,p}} f(x_i, y_j, z_k)\, \Delta V_{ijk} = \iiint_R f(x, y, z)\, dz\, dx\, dy = \iiint_R f(x, y, z)\, dz\, dy\, dx$$

CENTROIDS AND MOMENTS OF INERTIA. The coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the *centroid of a volume* satisfy the relations

$$\bar{x}\iiint_R dV=\iiint_R x\,dV \qquad \bar{y}\iiint_R dV=\iiint_R y\,dV$$

$$\bar{z}\iiint_R dV=\iiint_R z\,dV$$

The *moments of inertia of a volume* with respect to the coordinate axes are given by

$$I_x=\iiint_R (y^2+z^2)\,dV \qquad I_y=\iiint_R (z^2+x^2)\,dV \qquad I_z=\iiint_R (x^2+y^2)\,dV$$

Solved Problems

1. Evaluate the given triple integrals:

(*a*) $\int_0^1\int_0^{1-x}\int_0^{2-x} xyz\,dz\,dy\,dx$

$$=\int_0^1\left[\int_0^{1-x}\left(\int_0^{2-x} xyz\,dz\right)dy\right]dx$$

$$=\int_0^1\left[\int_0^{1-x}\left(\frac{xyz^2}{2}\bigg|_{z=0}^{z=2-x}\right)dy\right]dx=\int_0^1\left[\int_0^{1-x}\frac{xy(2-x)^2}{2}\,dy\right]dx$$

$$=\int_0^1\left[\frac{xy^2(2-x)^2}{4}\right]_{y=0}^{y=1-x}dx=\frac{1}{4}\int_0^1(4x-12x^2+13x^3-6x^4+x^5)\,dx=\frac{13}{240}$$

(*b*) $\int_0^{\pi/2}\int_0^1\int_0^2 zr^2\sin\theta\,dz\,dr\,d\theta$

$$=\int_0^{\pi/2}\int_0^1\left[\frac{z^2}{2}\right]_0^2 r^2\sin\theta\,dr\,d\theta=2\int_0^{\pi/2}\int_0^1 r^2\sin\theta\,dr\,d\theta$$

$$=\frac{2}{3}\int_0^{\pi/2}[r^3]_0^1\sin\theta\,d\theta=-\frac{2}{3}[\cos\theta]_0^{\pi/2}=\frac{2}{3}$$

(*c*) $\int_0^{\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\sin 2\phi\,d\rho\,d\phi\,d\theta$

$$=2\int_0^{\pi}\int_0^{\pi/4}\sin\phi\,d\phi\,d\theta=2\int_0^{\pi}(1-\tfrac{1}{2}\sqrt{2})\,d\theta=(2-\sqrt{2})\pi$$

2. Compute the triple integral of $F(x, y, z)=z$ over the region R in the first octant bounded by the planes $y=0$, $z=0$, $x+y=2$, $2y+x=6$, and the cylinder $y^2+z^2=4$. (See Fig. 73-3.)

Integrate first with respect to z from $z=0$ (the xOy plane) to $z=\sqrt{4-y^2}$ (the cylinder), then with respect to x from $x=2-y$ to $x=6-2y$, and finally with respect to y from $y=0$ to $y=2$. This yields

$$\iiint_R z\,dV=\int_0^2\int_{2-y}^{6-2y}\int_0^{\sqrt{4-y^2}} z\,dz\,dx\,dy=\int_0^2\int_{2-y}^{6-2y}[\tfrac{1}{2}z^2]_0^{\sqrt{4-y^2}}\,dx\,dy$$

$$=\tfrac{1}{2}\int_0^2\int_{2-y}^{6-2y}(4-y^2)\,dx\,dy=\tfrac{1}{2}\int_0^2[(4-y^2)x]_{2-y}^{6-2y}\,dy=\tfrac{26}{3}$$

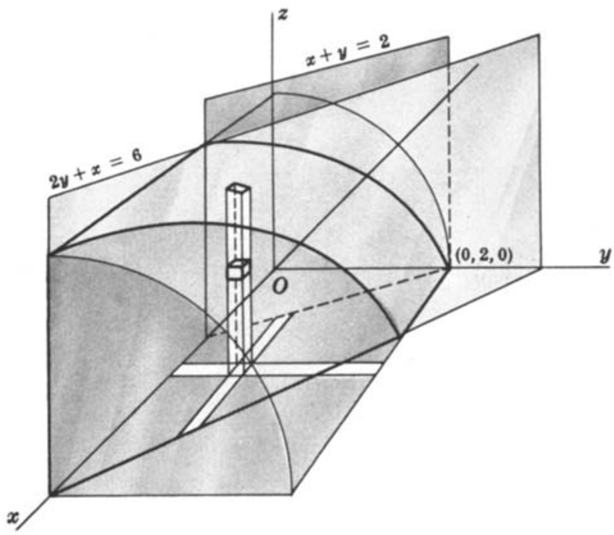


Fig. 73-3

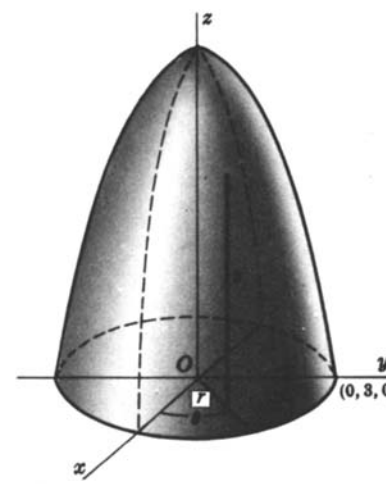


Fig. 73-4

3. Compute the triple integral of $f(r, \theta, z) = r^2$ over the region R bounded by the paraboloid $r^2 = 9 - z$ and the plane $z = 0$. (See Fig. 73-4.)

Integrate first with respect to z from $z = 0$ to $z = 9 - r^2$, then with respect to r from $r = 0$ to $r = 3$, and finally with respect to θ from $\theta = 0$ to $\theta = 2\pi$. This yields

$$\iiint_R r^2\, dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2(r\, dz\, dr\, d\theta) = \int_0^{2\pi}\int_0^3 r^3(9 - r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{9}{4}r^4 - \tfrac{1}{6}r^6\right]_0^3 d\theta = \int_0^{2\pi} \tfrac{243}{4}\, d\theta = \tfrac{243}{2}\pi$$

4. Show that the integrals (*a*) $4\int_0^4 \int_0^{\sqrt{16-x^2}} \int_{(x^2+y^2)/4}^4 dz\, dy\, dx$, (*b*) $4\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$, and (*c*) $4\int_0^4 \int_{y^2/4}^4 \int_0^{\sqrt{4z-y^2}} dx\, dz\, dy$ give the same volume.

(*a*) Here z ranges from $z = \frac{1}{4}(x^2 + y^2)$ to $z = 4$; that is, the volume is bounded below by the paraboloid $4z = x^2 + y^2$ and above the plane $z = 4$. The ranges of y and x cover a quadrant of the circle $x^2 + y^2 = 16$, $z = 0$, the projection of the curve of intersection of the paraboloid and the plane $z = 4$ on the xOy plane. Thus, the integral gives the volume cut from the paraboloid by the plane $z = 4$.

(*b*) Here y ranges from $y = 0$ to $y = \sqrt{4z - x^2}$; that is, the volume is bounded on the left by the zOx plane and on the right by the paraboloid $y^2 = 4z - x^2$. The ranges of x and z cover one-half the area cut from the parabola $x^2 = 4z$, $y = 0$, the curve of intersection of the paraboloid and the zOx plane, by the plane $z = 4$. The region R is that of (*a*).

(*c*) Here the volume is bounded behind by the yOz plane and in front by the paraboloid $4z = x^2 + y^2$. The ranges of z and y cover one-half the area cut from the parabola $y^2 = 4z$, $x = 0$, the curve of intersection of the paraboloid and the yOz plane, by the plane $z = 4$. The region R is that of (*a*).

5. Compute the triple integral of $F(\rho, \phi, \theta) = 1/\rho$ over the region R in the first octant bounded by the cones $\phi = \frac{1}{4}\pi$ and $\phi = \arctan 2$ and the sphere $\rho = \sqrt{6}$. (See Fig. 73-5.)

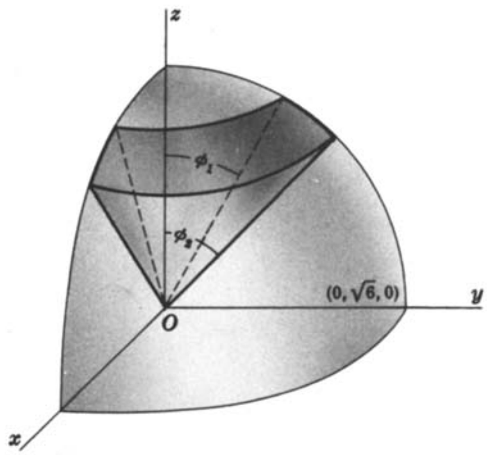


Fig. 73-5

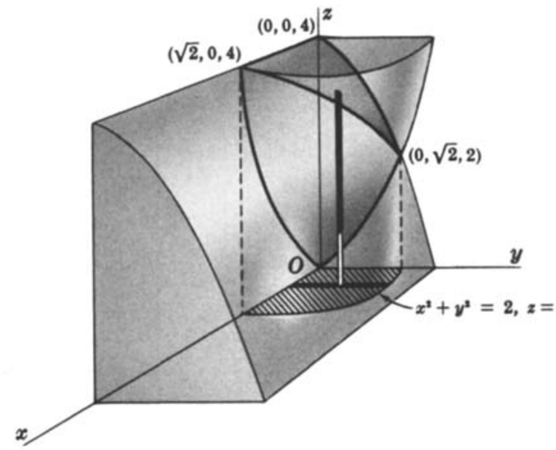


Fig. 73-6

Integrate first with respect to ρ from $\rho = 0$ to $\rho = \sqrt{6}$, then with respect to ϕ from $\phi = \frac{1}{4}\pi$ to $\phi = \arctan 2$, and finally with respect to θ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. This yields

$$\iiint_R \frac{1}{\rho}\, dV = \int_0^{\pi/2} \int_{\pi/4}^{\arctan 2} \int_0^{\sqrt{6}} \frac{1}{\rho} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 3\int_0^{\pi/2} \int_{\pi/4}^{\arctan 2} \sin\phi \, d\phi \, d\theta$$

$$= -3\int_0^{\pi/2} \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{2}}\right) d\theta = \frac{3\pi}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}\right)$$

6. Find the volume bounded by the paraboloid $z = 2x^2 + y^2$ and the cylinder $z = 4 - y^2$. (See Fig. 73-6.)

Integrate first with respect to z from $z = 2x^2 + y^2$ to $z = 4 - y^2$, then with respect to y from $y = 0$ to $y = \sqrt{2 - x^2}$ (obtain $x^2 + y^2 = 2$ by eliminating x between the equations of the two surfaces), and finally with respect to x from $x = 0$ to $x = \sqrt{2}$ (obtained by setting $y = 0$ in $x^2 + y^2 = 2$) to obtain one-fourth of the required volume. Thus,

$$V = 4\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{2x^2+y^2}^{4-y^2} dz \, dy \, dx = 4\int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} [(4 - y^2) + (2x^2 + y^2)] \, dy \, dx$$

$$= 4\int_0^{\sqrt{2}} \left[4y - 2x^2y - \frac{2y^3}{3}\right]_0^{\sqrt{2-x^2}} dx = \frac{16}{3}\int_0^{\sqrt{2}} (2 - x^2)^{3/2} \, dx = 4\pi \text{ cubic units}$$

7. Find the volume within the cylinder $r = 4\cos\theta$ bounded above by the sphere $r^2 + z^2 = 16$ and below by the plane $z = 0$. (See Fig. 73-7.)

Integrate first with respect to z from $z = 0$ to $z = \sqrt{16 - r^2}$, then with respect to r from $r = 0$ to $r = 4\cos\theta$, and finally with respect to θ from $\theta = 0$ to $\theta = \pi$ to obtain the required volume. Thus,

$$V = \int_0^{\pi} \int_0^{4\cos\theta} \int_0^{\sqrt{16-r^2}} r \, dz \, dr \, d\theta = \int_0^{\pi} \int_0^{4\cos\theta} r\sqrt{16 - r^2} \, dr \, d\theta$$

$$= -\tfrac{64}{3}\int_0^{\pi} (\sin^3\theta - 1) \, d\theta = \tfrac{64}{9}(3\pi - 4) \text{ cubic units}$$

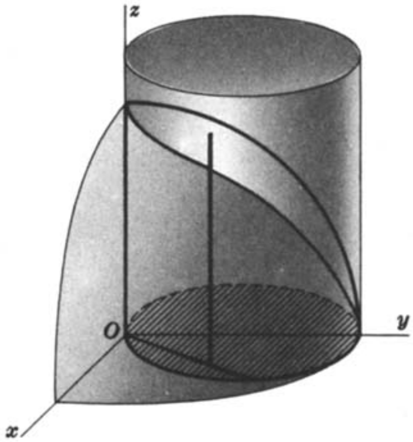


Fig. 73-7

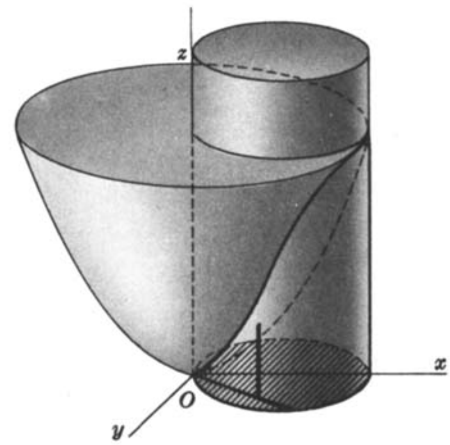


Fig. 73-8

8. Find the coordinates of the centroid of the volume within the cylinder $r = 2\cos\theta$, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$. (See Fig. 73-8.)

$$V = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_0^{\pi/2}[r^4]_0^{2\cos\theta}\,d\theta = 8\int_0^{\pi/2}\cos^4\theta\,d\theta = \tfrac{3}{2}\pi$$

$$M_{yz} = \iiint_R x\,dV = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2}(r\cos\theta)r\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^{2\cos\theta} r^4\cos\theta\,dr\,d\theta = \tfrac{64}{5}\int_0^{\pi/2}\cos^6\theta\,d\theta = 2\pi$$

Then $\bar{x} = M_{yz}/V = \frac{4}{3}$. By symmetry, $\bar{y} = 0$. Also,

$$M_{xy} = \iiint_R z\,dV = 2\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{r^2} zr\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r^5\,dr\,d\theta$$

$$= \tfrac{32}{3}\int_0^{\pi/2}\cos^6\theta\,d\theta = \tfrac{5}{3}\pi$$

and $\bar{z} = M_{xy}/V = \frac{10}{9}$. Thus, the centroid has coordinates $(\frac{4}{3}, 0, \frac{10}{9})$.

9. For the right circular cone of radius a and height h, find (a) the centroid, (b) the moment of inertia with respect to its axis (c), the moment of inertia with respect to any line through its vertex and perpendicular to its axis, (d) the moment of inertia with respect to any line through its centroid and perpendicular to its axis, an (e) the moment of inertia with respect to any diameter of its base.

Take the cone as in Fig. 73-9, so that its equation is $r = \dfrac{a}{h}z$. Then

$$V = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^a\left(hr - \frac{h}{a}r^2\right)dr\,d\theta$$

$$= \frac{2}{3}ha^2\int_0^{\pi/2}d\theta = \frac{1}{3}\pi ha^2$$

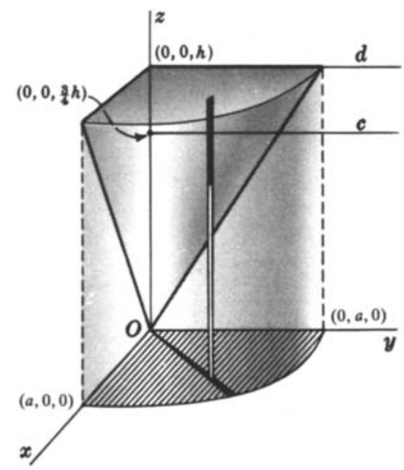


Fig. 73-9

(*a*) The centroid lies on the z axis, and we have

$$M_{xy} = \iiint_R z\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h zr\,dz\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_0^a \left(h^2r - \frac{h^2}{a^2}r^3\right) dr\,d\theta = \frac{1}{2}h^2a^2\int_0^{\pi/2} d\theta = \frac{1}{4}\pi h^2a^2$$

Then $\bar{z} = M_{xy}/V = \frac{3}{4}h$, and the centroid has coordinates $(0, 0, \frac{3}{4}h)$.

(*b*) $$I_z = \iiint_R (x^2 + y^2)\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h (r^2)r\,dz\,dr\,d\theta = \tfrac{1}{10}\pi ha^4 = \tfrac{3}{10}a^2V$$

(*c*) Take the line as the y axis. Then

$$I_y = \iiint_R (x^2 + z^2)\,dV = 4\int_0^{\pi/2}\int_0^a\int_{hr/a}^h (r^2\cos^2\theta + z^2)r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^a \left[\left(hr^3 - \frac{h}{a}r^4\right)\cos^2\theta + \frac{1}{3}\left(h^3r - \frac{h^3}{a^3}r^4\right)\right] dr\,d\theta$$

$$= \frac{1}{5}\pi ha^2\left(h^2 + \frac{1}{4}a^2\right) = \frac{3}{5}\left(h^2 + \frac{1}{4}a^2\right)V$$

(*d*) Let the line c through the centroid be parallel to the y axis. By the parallel-axis theorem,

$$I_y = I_c + V(\tfrac{3}{4}h)^2 \quad \text{and} \quad I_c = \tfrac{3}{5}(h^2 + \tfrac{1}{4}a^2)V - \tfrac{9}{16}h^2V = \tfrac{3}{80}(h^2 + 4a^2)V$$

(*e*) Let d denote the diameter of the base of the cone parallel to the y axis. Then

$$I_d = I_c + V(\tfrac{1}{4}h)^2 = \tfrac{3}{80}(h^2 + 4a^2)V + \tfrac{1}{16}h^2V = \tfrac{1}{20}(2h^2 + 3a^2)V$$

10. Find the volume cut from the cone $\phi = \frac{1}{4}\pi$ by the sphere $\rho = 2a\cos\phi$. (See Fig. 73-10.)

$$V = 4\iiint_R dV = 4\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{2a\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{32a^3}{3}\int_0^{\pi/2}\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi\,d\theta = 2a^3\int_0^{\pi/2} d\theta = \pi a^3 \text{ cubic units}$$

11. Locate the centroid of the volume cut from one nappe of a cone of vertex angle 60° by a sphere of radius 2 whose center is at the vertex of the cone.

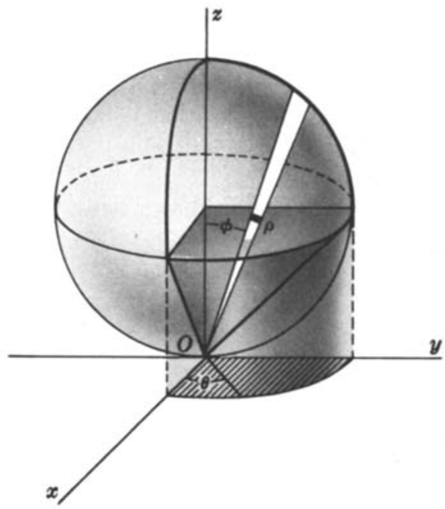


Fig. 73-10

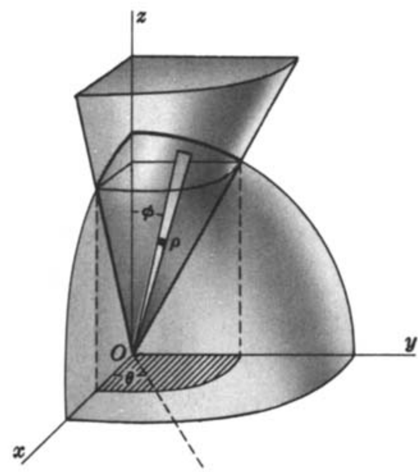


Fig. 73-11

Take the surfaces as in Fig. 73-11, so that $\bar{x} = \bar{y} = 0$. In spherical coordinates, the equation of the cone is $\phi = \pi/6$, and the equation of the sphere is $\rho = 2$. Then

$$V = \iiint_R dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{32}{3}\int_0^{\pi/2}\int_0^{\pi/6} \sin\phi \, d\phi \, d\theta$$

$$= -\frac{32}{3}\left(\frac{\sqrt{3}}{2} - 1\right)\int_0^{\pi/2} d\theta = \frac{8\pi}{3}(2 - \sqrt{3})$$

$$M_{xy} = \iiint_R z \, dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/6} \sin 2\phi \, d\phi \, d\theta = \pi$$

and $\bar{z} = M_{xy}/V = \frac{3}{8}(2 + \sqrt{3})$.

12. Find the moment of inertia with respect to the z axis of the volume of Problem 11.

$$I_z = \iiint_R (x^2 + y^2) \, dV = 4\int_0^{\pi/2}\int_0^{\pi/6}\int_0^2 (\rho^2 \sin^2\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{128}{5}\int_0^{\pi/2}\int_0^{\pi/6} \sin^3\phi \, d\phi \, d\theta = \frac{128}{5}\left(\frac{2}{3} - \frac{3}{8}\sqrt{3}\right)\int_0^{\pi/2} d\theta = \frac{8\pi}{15}(16 - 9\sqrt{3}) = \frac{5 - 2\sqrt{3}}{5} V$$

Supplementary Problems

13. Describe the curve determined by each of the following pairs of equations in cylindrical coordinates.
(*a*) $r = 1$, $z = 2$ (*b*) $r = 2$, $z = \theta$ (*c*) $\theta = \pi/4$, $r = \sqrt{2}$ (*d*) $\theta = \pi/4$, $z = r$

Ans. (*a*) circle of radius 1 in plane $z = 2$ with center having rectangular coordinates $(0, 0, 2)$; (*b*) helix on right circular cylinder $r = 2$; (*c*) vertical line through point having rectangular coordinates $(1, 1, 0)$; (*d*) line through origin in plane $\theta = \pi/4$, making an angle of 45° with xy plane

14. Describe the curve determined by each of the following pairs of equations in spherical coordinates.

(a) $\rho = 1,\ \theta = \pi$ (b) $\theta = \frac{\pi}{4},\ \phi = \frac{\pi}{6}$ (c) $\rho = 2,\ \phi = \frac{\pi}{4}$

Ans. (a) circle of radius 1 in xz plane with center at origin; (b) halfline of intersection of plane $\theta = \pi/4$ and cone $\phi = \pi/6$; (c) circle of radius $\sqrt{2}$ in plane $z = \sqrt{2}$ with center on z axis

15. Transform each of the following equations in either rectangular, cylindrical, or spherical coordinates into equivalent equations in the two other coordinate systems.

(a) $\rho = 5$ (b) $z^2 = r^2$ (c) $x^2 + y^2 + (z-1)^2 = 1$

Ans. (a) $x^2 + y^2 + z^2 = 25,\ r^2 + z^2 = 25$; (b) $z^2 = x^2 + y^2,\ \cos^2\phi = \frac{1}{2}$ (that is, $\phi = \pi/4$ or $\phi = 3\pi/4$); (c) $r^2 + z^2 = 2z,\ \rho = 2\cos\phi$

16. Evaluate the triple integral on the left in each of the following:

(a) $\int_0^1 \int_1^2 \int_2^3 dz\,dx\,dy = 1$

(b) $\int_0^1 \int_{x^2}^x \int_0^{xy} dz\,dy\,dx = \frac{1}{24}$

(c) $\int_0^6 \int_0^{12-2y} \int_0^{4-2y/3-x/3} x\,dz\,dx\,dy = 144 \left[= \int_0^{12} \int_0^{6-x/2} \int_0^{4-2y/3-x/3} x\,dz\,dy\,dx\right]$

(d) $\int_0^{\pi/2} \int_0^4 \int_0^{\sqrt{16-z^2}} (16 - r^2)^{1/2} rz\,dr\,dz\,d\theta = \frac{256}{5}\pi$

(e) $\int_0^{2\pi} \int_0^{\pi} \int_0^5 \rho^4 \sin\phi\,d\rho\,d\phi\,d\theta = 2500\pi$

17. Evaluate the integral of Problem 16(b) after changing the order to $dz\,dx\,dy$.

18. Evaluate the integral of Problem 16(c), changing the order to $dx\,dy\,dz$ and to $dy\,dz\,dx$.

19. Find the following volumes, using triple integrals in rectangular coordinates:

(a) Inside $x^2 + y^2 = 9$, above $z = 0$, and below $x + z = 4$ *Ans.* 36π cubic units

(b) Bounded by the coordinate planes and $6x + 4y + 3z = 12$ *Ans.* 4 cubic units

(c) Inside $x^2 + y^2 = 4x$, above $z = 0$, and below $x^2 + y^2 = 4z$ *Ans.* 6π cubic units

20. Find the following volumes, using triple integrals in cylindrical coordinates:

(a) The volume of Problem 4

(b) The volume of Problem 19(c)

(c) That inside $r^2 = 16$, above $z = 0$, and below $2z = y$ *Ans.* 64/3 cubic units

21. Find the centroid of each of the following volumes:

(a) Under $z^2 = xy$ and above the triangle $y = x,\ y = 0,\ x = 4$ in the plane $z = 0$ *Ans.* $(3, \frac{9}{5}, \frac{9}{8})$

(b) That of Problem 19(b) *Ans.* $(\frac{1}{2}, \frac{3}{4}, 1)$

(c) The first-octant volume of Problem 19(a) *Ans.* $\left(\frac{64 - 9\pi}{16(\pi - 1)}, \frac{23}{8(\pi - 1)}, \frac{73\pi - 128}{32(\pi - 1)}\right)$

(d) That of Problem 19(c) *Ans.* $(\frac{8}{3}, 0, \frac{10}{9})$

(e) That of Problem 20(c) *Ans.* $(0, 3\pi/4, 3\pi/16)$

22. Find the moments of inertia I_x, I_y, I_z of the following volumes:

(a) That of Problem 4 *Ans.* $I_x = I_y = \frac{32}{3}V;\ I_z = \frac{16}{3}V$

(b) That of Problem 19(b) *Ans.* $I_x = \frac{5}{2}V;\ I_y = 2V;\ I_z = \frac{13}{10}V$

(c) That of Problem 19(c) *Ans.* $I_x = \frac{55}{18}V;\ I_y = \frac{175}{18}V;\ I_z = \frac{80}{9}V$

(d) That cut from $z = r^2$ by the plane $z = 2$ *Ans.* $I_x = I_y = \frac{7}{3}V;\ I_z = \frac{2}{3}V$

23. Show that, in cylindrical coordinates, the triple integral of a function $f(r, \theta, z)$ over a region R may be represented by

$$\int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r, \theta, z) r \, dz \, dr \, d\theta$$

(*Hint*: Consider, in Fig. 73-12, a representative subregion of R bounded by two cylinders having Oz as axis and of radii r and $r + \Delta r$, respectively, cut by two horizontal planes through $(0, 0, z)$ and $(0, 0, z + \Delta z)$, respectively, and by two vertical planes through Oz making angles θ and $\theta + \Delta\theta$, respectively, with the xOz plane. Take $\Delta V = (r\,\Delta\theta)\,\Delta r\,\Delta z$ as an approximation of its volume.)

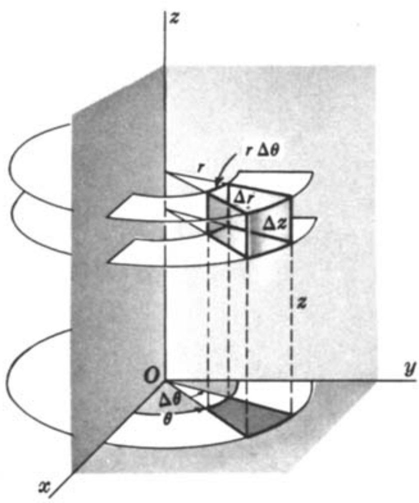


Fig. 73-12

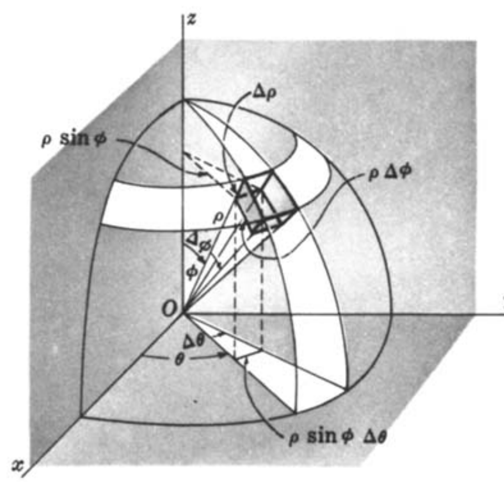


Fig. 73-13

24. Show that, in spherical coordinates, the triple integral of a function $f(\rho, \phi, \theta)$ over a region R may be represented by

$$\int_{\alpha}^{\beta} \int_{\phi_1(\theta)}^{\phi_2(\theta)} \int_{\rho_1(\phi,\theta)}^{\rho_2(\phi,\theta)} f(\rho, \phi, \theta) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(*Hint*: Consider, in Fig. 73-13, a representative subregion of R bounded by two spheres centered at O, of radii ρ and $\rho + \Delta\rho$, respectively, by two cones having O as vertex, Oz as axis, and semivertical angles ϕ and $\phi + \Delta\phi$, respectively, and by two vertical planes through Oz making angles θ and $\theta + \Delta\theta$, respectively, with the zOy plane. Take $\Delta V = (\rho\,\Delta\phi)(\rho \sin\phi\,\Delta\theta)(\Delta\rho) = \rho^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$ as an approximation of its volume.)

الباب الأول

مفاهيم أساسية

١- مقدمة :

تعريف

المعادلة التفاضلية هى علاقة تساوى بين متغير مستقل وليكن x ومتغير تابع ولـــيكن $y(x)$ وواحد أو أكثر من المشتقات التفاضلية ...... y', y'', أى أنها على الصورة العامة :

$$F(x, y, y', y'', \ldots\ldots) = 0$$

وهذه المعادلة تسمى معادلة تفاضلية عادية .

أما إذا كان عدد المتغيرات المستقلة أكثر من واحد وليكن x , y مستقلان ، وكان $z(x, y)$ ، متغير تابع قابل للاشتقاق بالنسبة لكل من $x, \; y$ جزئياً ، سميت المعادلة المشـــتملة علـــى المتغيرات المستقلة والمتغير التابع ومشتقاته الجزئية ، معادلة تفاضلية جزئية ، وهى على الصورة :

$$G\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \ldots\ldots\ldots\ldots\right) = 0$$

وعلى سبيل المثال المعادلات التفاضلية :

$$y'''^2 + 2y'^3 - 5y = \sin x \qquad (1)$$

$$y' + xy = x^2 \qquad (2)$$

$$\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial xy} + \frac{\partial z}{\partial y} = x \qquad (3)$$

نلاحظ أن المعادلتين (2) ,(1) كلاً منهما معادلة تفاضلية عادية بينما المعادلة (3) معادلـــة تفاضلية جزئية .

تعريف :

رتبة المعادلة *Order* : هى رتبة أعلى معامل تفاضلى فى المعادلة .

درجة المعادلة *Degree* : هى درجة (قوة) أعلى معامل تفاضلى فى المعادلة بشرط أن تكون جميع المعاملات التفاضلية خالية من القوى الكسرية .

مثال :

من مجموعة المعادلات التفاضلية السابقة نجد أن المعادلة *(1)* من الرتبة الثالثة والدرجة الثانية بينما المعادلة *(2)* من الرتبة الأولى والدرجة الأولى ، أما المعادلة *(3)* فهى تفاضلية جزئية (ليست محل دراستنا) وهى من الرتبة الثانية والدرجة الأولى .

مثال :

أوجد رتبة ودرجة المعادلة $y''=(5-2y')^{3/2}=0$.

الحـــل :

المعادلة من الرتبة الثانية والدرجة الثانية ...... لماذا ؟

تعريف :

حل المعادلة التفاضلية *Solution of D.E.* : تسمى الدالة $y = y(x)$ حلاً للمعادلة التفاضلية $F(x, y, y', y'', \ldots\ldots, y^n) = 0$ إذا كانت :

١) قابلة للاشتقاق n مرة .

٢) تحقق المعادلة التفاضلية أى : $F(x, y(x), y'(x), \ldots\ldots, y^{(n)}(x)) = 0$

مثال :

أثبت أن $y(x) = c\sin x$ حلاً للمعادلة التفاضلية $y'' + y = 0$ حيث c ثابت .

الحـــل :

$y(x) = c\sin x,\qquad y'(x) = c\cos x,\qquad y''(x) = -c\sin x$

وعلى ذلك نجد أن :

$y''(x) + y(x) = -x\sin x + x\sin x = 0$

مثال :

أثبت أن $\ln y + \dfrac{x}{y} = c,\quad y > 0$ ………… (1) حيـــث c ثابـــت هـــو حـــل للمعادلـــة

$(y-x)\dfrac{dy}{dx}+y=0$ ………………… (2) .

الحـــل :

بتفاضل طرفى $\ln y + \dfrac{x}{y} = c$ بالنسبة إلى x :

$$\frac{1}{y}\frac{dy}{dx}+\frac{y-x\dfrac{dy}{dx}}{y^2}=0$$

$$\left(\frac{1}{y}-\frac{x}{y^2}\right)\frac{dy}{dx}+\frac{1}{y}=0 \qquad y\neq 0$$

$$(y-x)\frac{dy}{dx}+y=0$$

أى أن المعادلة (١) حل للمعادلة (٢) .

الفصل الثانى

معادلات تفاضلية

من الرتبة الأولى والدرجة الأولى

مقدمة :

أى معادلة تفاضلية من الرتبة الأولى والدرجة الأولى تكون على الصورة .

$$\frac{dy}{dx} = F(x, y)$$

أو
$$M(x,y)\, dx + N(x,y)\, dy) = 0$$

ولحل مثل هذه المعادلة نستخدم إحدى الطرق التالية المتاحة :

١- طريقة فصل المتغيرات ***Separation of Variables*** :

إذا أمكن وضع المعادلة على الصورة

$$f(x)\, dx + g(y)\, dy = 0$$

حيث أن $f(x)$ دالة فى x فقط و $g(y)$ دالة فى y وبذلك فإن عملية فصل المتغيرات تكون تحققت ولحل المعادلة بعد عملية الفصل ، نستخدم التكامل المباشر فيكون الحل :

$$\int f(x)\, dx + \int g(y)\, dy = C$$

حيث C ثابت اختيارى ، ويسمى ذلك الحل بالحل العام ، ويمكن وضع الثابت الاختيارى على أى صورة حسب متطلبات تبسيط شكل الحل العام .

وإذا علم شرط ابتدائى ، نستطيع حذف الثابت الاختيارى والحل الناتج يكون حلاً خاصاً .

مثال :

أوجد الحل العام والمنحنى الخاص الذى يمر بالنقطة $(0,0)$ للمعادلة التفاضلية .

$$e^x \cos y \, dx + (1 + e^x) \sin y \, dy = 0$$

الحــــل :

بفصل المتغيرات ، وذلك بقسمة طرفى المعادلة المعطاة على $\cos y\,(1 + e^x)$ فنحصل على :

$$\therefore \frac{e^x}{1+e^x} dx + \frac{\sin y}{\cos y} dy = 0$$

بالتكامل المباشر

$$\therefore \ln(1 + e^x) - \ln|\cos y| = \ln c$$

$$\therefore \ln \frac{(1+e^x)}{|\cos y|} = \ln c$$

وبذلك يكون الحل العام للمعادلة هو :

$$1 + e^x = c\,|\cos y|$$

بالتعويض عن $x = 0, \;\; y = 0$

$$\therefore 1 + 1 = c \quad \text{و} \quad c = 2$$

ويكون الحل الخاص

$$1 + e^x = 2\,|\cos y|$$

٢- المعادلة التفاضلية المتجانسة *Homogeneous Equation*

يقال أن المعادلة التفاضلية $M(x,t)\,dx + N(x,y)\,dy = 0$

متجانسة إذا كان كل من M, N دالة متجانسة من نفس الدرجة ، علماً بأن :

$f(x,y)$ دالة متجانسة من درجة n إذا كان :

$$f(\lambda x, \lambda y) = \lambda^n f(x,y) \quad , \lambda \in R$$

ومثال ذلك :

$$1)\ f(x,y) = x^2 + 3xy - y^2 \quad \Rightarrow \quad f(\lambda x, \lambda y) = \lambda^2 x^2 + 3\lambda^2 xy - \lambda^2 y^2 = \lambda^2 f(x,y)$$

$\therefore$ $f(x,y)$ متجانسة من درجة 2 .

$$2)\ f(x,y) = \frac{x^2 - y^2}{\sqrt{x+y}} \quad \Rightarrow \quad f(\lambda x, \lambda y) = \frac{\lambda^2 x^2 - \lambda^2 y^2}{\sqrt{\lambda x + \lambda y}} = \lambda^{3/2} f(x,y)$$

$\therefore$ $f(x,y)$ متجانسة من درجة $3/2$.

على ذلك فإن المعادلة التفاضلية المتجانسة يمكن أن توضع على الصورة :

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x,y)$$

وحيث أن M, N متجانسة من نفس الدرجة نجد أن $f(x,y)$ متجانسة من درجة صفر .

أى أن من الممكن $f(x,y) = f(x/y)$.

الخلاصة :

المعادلة التفاضلية $M(x,y)\,dx + N(x,y)\,dy = 0$ تكون متجانسة إذا كانت كل من M, N متجانسة من نفس الدرجة .

أى أن المعادلة على الصورة $y' = f(x/y)$ تكون معادلة متجانسة .

فى هذه الحالة نستخدم التعويض $v = \frac{y}{x}$ أى $y = xv$ وبالتالى $dy = x\,dv + v\,dx$ ثم تتحول المعادلة إلى معادلة يمكن فصل متغيراتها ، ثم تحل كما سبق .

مثال :

أوجد الحل العام للمعادلة : $(x^2 + y^2)\,dx - 2\,xy\,dy = 0$

الحـــــل :

من الواضح أن المعادلة متجانسة .

$\therefore$ نستخدم التعويض $y = vx$ $\Leftarrow$ $dy = vdx + xdv$

$\therefore\ (x^2+v^2x^2)\,dx - 2x^2v\,(vdx + xdv) = 0$

$\therefore$ بالقسمة على x^2 نحصل على :

$(1+v^2)\,dx - 2v\,(v\,dx + x\,dv) = 0$

أى أن $\therefore\ [1+v^2-2v^2]\,dx - 2v\,x\,dv = 0$

أى $\therefore\ (1-v2)\,dx - 2v\,x\,dv = 0$

وبفصل المتغيرات نحصل على $\frac{1}{x}\,dx - \frac{2v}{1-v^2} = 0$

$\therefore$ بالتكامل المباشر $\ln x + \ln(1-v^2) = \ln c$

حيث أن : $v = y/x$

$\therefore x [1 - \frac{y^2}{x^2}] = c$ $\qquad x^2 - y^2 = cx$

هو الحل العام للمعادلة التفاضلية .

<u>مثال :</u>

أوجد الصورة العامة للمعادلة : $y' = f(\frac{y}{x})$

<u>الحـــــل :</u>

حيث أن المعادلة متجانسة ، نضع $v = (\frac{y}{x}) \Leftarrow y = vx$

$\therefore y` = v + xv`$

$\therefore \; v + x\,v` = f(v) \quad ) \; xv` = f(v) - v$

أى أن $x\frac{dv}{dx} = f(v) - v$

$\therefore \; \frac{dv}{f(v) - v} = \frac{dx}{x}$

أى أن الحل العام $\int \frac{dv}{f(v) - v} = \ln cx$

حيث $v = {}^{y}\!/_{x}$

<u>مثال :</u>

استخدم النتيجة السابقة فى حل المعادلة : $2x^2y` - y(2x+y) = 0$

Lecture III

Solution of first order equations

1 Separable equations

These are equations of the form

$$y' = f(x)g(y)$$

Assuing g is nonzero, we divide by g and integrate to find

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

What happens if $g(y)$ becomes zero at a point $y = y_0$?

Example 1. $xy' = y + y^2$

Solution: We write this as

$$\int \frac{dy}{y+y^2} = \int \frac{dx}{x} + C \Rightarrow \int \frac{dy}{y} - \int \frac{dy}{1+y} = \ln x + C \Rightarrow \ln y - \ln(1+y) = \ln x + C$$

Note: Strictly speaking, we should write the above solution as

$$\ln|y| - \ln|1+y| = \ln|x| + C$$

When we wrote the solution without the modulas sign, it was (implicitly) assumed that $x > 0, y > 0$. This is acceptable for problems in which the solution domain is not given explicitly. But for some problems, the modulas sign is necessary. For example, consider the following IVP:

$$xy' = y + y^2, \quad y(-1) = -2.$$

Try to solve this.

2 Reduction to separable form

2.1 Substitution method

Let the ODE be

$$y' = F(ax + by + c)$$

Suppose $b \neq 0$. Substituting $ax + by + c = v$ reduces the equation to a separable form. If $b = 0$, then it is already in separable form.

Example 2. $y' = (x + y)^2$

Solution: Let $v = x + y$. Then we find

$$v' = v^2 + 1 \Rightarrow \tan^{-1} v = x + C \Rightarrow x + y = \tan(x + C)$$

2.2 Homogeneous form

Let the ODE be of the form

$$y' = f(y/x)$$

In this case, substitution of $v = y/x$ reduces the above ODE to a seprable ODE.

Comment 1: Sometimes, substitution reduces an ODE to the homogeneous form. For example, if $ae \neq bd$, then h and k can be chosen so that $x = u + h$ and $y = v + k$ reduces the following ODE

$$y' = F\left(\frac{ax + by + c}{dx + ey + f}\right)$$

to a homeogeneous ODE. What happens if $ae = bd$?

Comment 2: Also, an ODE of the form

$$y' = y/x + g(x)h(y/x)$$

can be reduced to the separable form by substituting $v = y/x$.

Example 3. $xyy' = y^2 + 2x^2, \qquad y(1) = 2$

Solution: Substituting $v = y/x$ we find

$$v + xv' = v + 2/v \Rightarrow y^2 = 2x^2(C + \ln x^2)$$

Using $y(1) = 2$, we find $C = 2$. Hence, $y = 2x^2(1 + \ln x^2)$

3 Exact equation

A first order ODE of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is exact if there exits a function $u(x, y)$ such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}.$$

Then the above ODE can be written as $du = 0$ and hence the solution becomes $u = C$.

Theorem 1. *Let M and N be defined and continuously differentiable on a rectangle rectangle $R = \{(x, y) : |x - x_0| < a, |y - y_0| < b\}$. Then (1) is exact if and only if $\partial M/\partial y = \partial N/\partial x$ for all $(x, y) \in R$.*

Proof: We shall only prove the necessary part. Assume that (1) is exact. Then there exits a function $u(x, y)$ such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}.$$

Since M and N have continuous first partial derivatives, we have

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Now continuity of 2nd partial derivative implies $\partial M/\partial y = \partial N/\partial x$.

Example 4. *Solve* $(2x + \sin x \tan y)dx - \cos x \sec^2 y\, dy = 0$

Solution: Here $M = 2x + \sin x \tan y$ and $N = -\cos x \sec^2 y$. Hence, $M_y = N_x$. Hence, the solution is $u = C$, where $u = x^2 - \cos x \tan y$

4 Reduction to exact equation: integrating factor

An integrating factor $\mu(x, y)$ is a function such that

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2}$$

becomes exact on multiplying it by μ. Thus,

$$\mu M\, dx + \mu N\, dy = 0$$

is exact. Hence

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}.$$

Comment: If an equation has an integrating factor, then it has infinitely many integrating factors.
Proof: Let μ be an integrating factor. Then

$$\mu M\, dx + \mu N\, dy = du$$

Let $g(u)$ be any continuous function of u. Now multiplying by $\mu g(u)$, we find

$$\mu g(u) M\, dx + \mu g(u) N\, dy = g(u) du \Rightarrow \mu g(u) M\, dx + \mu g(u) N\, dy = d\left(\int^u g(u)\, du\right)$$

Thus,

$$\mu g(u) M\, dx + \mu g(u) N\, dy = dv, \quad \text{whare} \quad v = \int^u g(u)\, du$$

Hence, $\mu g(u)$ is an integrating factor. Since, g is arbitrary, there exists an infinite number of integrating factors.

Example 5. $xdy - ydx = 0$.

Solution: Clearly $1/x^2$ is an integrating factor since

$$\frac{xdy - ydx}{x^2} = 0 \Rightarrow d(y/x) = 0$$

Also, $1/xy$ is an integrating factor since

$$\frac{xdy - ydx}{xy} = 0 \Rightarrow d \ln(y/x) = 0$$

Similarly it can be shown that $1/y^2$, $1/(x^2 + y^2)$ etc. are integrating factors.

4.1 How to find intgrating factor

Theorem 2. *If (2) is such that*

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

is a function of x alone, say $F(x)$, then

$$\mu = e^{\int F\,dx}$$

is a function of x only and is an integrating factor for (2).

Example 6. $(xy-1)dx+(x^2-xy)dy=0$

Solution: Here $M=xy-1$ and $N=x^2-xy$. Also,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{1}{x}$$

Hence, $1/x$ is an integrting factor. Multiplying by $1/x$ we find

$$\frac{(xy-1)dx+(x^2-xy)dy}{x}=0 \Rightarrow xy-\ln x-y^2/2=C$$

Theorem 3. *If (2) is such that*

$$\frac{-1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

is a function of y alone, say $G(y)$, then

$$\mu = e^{\int G\,dy}$$

is a function of y only and is an integrating factor for (2).

Example 7. $y^3dx+(xy^2-1)dy=0$

Solution: Here $M=y^3$ and $N=xy^2-1$. Also,

$$-\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{2}{y}$$

Hence, $1/y^2$ is an integrting factor. Multiplying by $1/y^2$ we find

$$\frac{y^3dx+(xy^2-1)dy}{y^2}=0 \Rightarrow xy+\frac{1}{y}=C$$

Comment: Sometimes it may be possible to find integrating factor by inspection. For this, some known differential formulas are useful. Few of these are given below:

$$\begin{aligned}
d\left(\frac{x}{y}\right) &= \frac{ydx-xdy}{y^2}\\
d\left(\frac{y}{x}\right) &= \frac{xdy-ydx}{x^2}\\
d(xy) &= xdy+ydx\\
d\left(\ln\frac{x}{y}\right) &= \frac{ydx-xdy}{xy}
\end{aligned}$$

Example 8. $(2x^2y + y)dx + xdy = 0$

Obviously, we can write this as

$$2x^2ydx + (ydx + xdy) = 0 \Rightarrow 2x^2ydx + d(xy) = 0$$

Now if we divide this by xy, then the last term remains differential and the first term also becomes differential:

$$2xdx + \frac{d(xy)}{xy} = 0 \Rightarrow d\left(x^2 + \ln(xy)\right) = 0 \Rightarrow x^2 + \ln(xy) = C$$

2.3 Exact Differential Equations

A differential equation is called *exact* when it is written in the specific form

$$F_x\ dx + F_y\ dy = 0, \tag{2.4}$$

for some continuously differentiable function of two variables $F(x, y)$. (Note that in the above expressions $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$).

The solution to equation (2.3) is given implicitly by

$$F(x, y) + C = 0.$$

We see this by implicitly differentiating

$$F(x, y) + C = 0.$$

with respect to x (and using the chain rule from multivariable calculus) we see that an exact differential equation must be of the form:

$$F_x + F_y \frac{dy}{dx} = 0, \tag{2.5}$$

which can be written as

$$F_x\ dx + F_y\ dy = 0. \tag{2.6}$$

Example 2.6 *Find the exact differential equation that is solved by*

$$x^2y + y^3 \sin x + C = 0$$

Solution: Differentiating, we obtain

$$\left(2xy + y^3 \cos x\right) dx + \left(x^2 + 3y^2 \sin x\right) dy = 0 \ \ \square$$

Note that one needs to be extremely careful calling a differential equation exact, since performing algebra on an exact differential equation can make it no longer exact. In other words, the differential equation

$$\left(2xy^2 + y^4 \cos x\right) dx + \left(yx^2 + 3y^3 \sin x\right) dy = 0 \ \ \square$$

is algebraically equivalent to equation(2.3) but it is not exact, even though it is still solved by

$$x^2y + y^3 \sin x + C = 0.$$

One should recall that if F is continuously differentiable then the mixed partial derivatives of F must match namely, $F_{xy} = F_{yx}$. This gives us a method to detect if a differential equation is exact namely:

<u>Exactness Test and Method to Solve an Exact DE</u>

Consider the differential equation

$$M(x, y)\ dx + N(x, y)\ dy = 0$$

where M and N are both continuously differentiable functions with continuous partials M_y and N_x. If $M_y = N_x$, then the DE is exact. The implicit solutions are given by $F(x, y) + C = 0$ where $F = \int M\ dx$ and $F = \int N\ dy$, simultaneously, up to a constant C.

We first show that one can obtain a function so that $F = \int M\ dx = \int N\ dy$, simultaneously, up to a constant C. Given that $M_y = N_x$. Consider $\int M\ dx - \int N\ dy$. Rewrite this as:

$$\int (\int M_y\ dy)\ dx - \int (\int N_x\ dx)\ dy,$$

which equals

$$\int \int 0\ dx\ dy$$

which is a constant.

Suppose that such an F now exists so that $F = \int M\ dx$ and $F = \int N\ dy$, simultaneously. Then differentiating we obtain

$$F_x\ dx + F_y\ dy = 0, \tag{2.7}$$

or

$$M\ dx + N\ dy = 0. \tag{2.8}$$

Moreover, since $F_{xy} = F_{yx}$ we must have $M_y = N_x$. □

Example 2.7 *Use the test for exactness to show that the DE is exact, then solve it.*

$$\left(x^2 + xy - y^2\right)\ dx + \left(\frac{1}{2}x^2 - 2xy\right)\ dy = 0. \tag{2.9}$$

Solution:

In this problem, $M = x^2 + xy - y^2$ and $N = \frac{1}{2}x^2 - 2xy$. Thus,

$$M_y = x - 2y$$

and

$$N_x = x - 2y,$$

which implies that the differential equation is exact.

To obtain F we compute $F = \int M\ dx$ and $F = \int N\ dy$.

$$F = \int M\ dx = \int x^2 + xy - y^2\ dx = \frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2 + h_1(y)$$

where $h_1(y)$ is an unknown function of y. Similarly,

$$F = \int N\ dy = \int 12x^2 - 2xy\ dy = \frac{1}{2}x^2y - xy^2 + h_2(x)$$

where $h_2(x)$ is an unknown function of x.

For F to equal both simultaneously, we must have $h_2(x) = \frac{1}{3}x^3$ and $h_1(y) = 0$.

Thus $F(x, y) = \frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2$ and hence,

$$\frac{1}{3}x^3 + \frac{1}{2}x^2y - xy^2 + C = 0$$

is the solution to the DE. □

Example 2.8 *Use the test for exactness to show that the DE is exact, then solve the initial value problem.*

$$(ye^{xy})\ dx + (xe^{xy} + \sin y)\ dy = 0 \quad y(0) = \pi \tag{2.10}$$

Solution:

In this problem, $M = ye^{xy}$ and $N = xe^{xy} + \sin y$. Thus,

$$M_y = e^{xy} + xye^{xy}$$

and

$$N_x = e^{xy} + xye^{xy},$$

which implies that the differential equation is exact.

To obtain F we compute $F = \int M\ dx$ and $F = \int N\ dy$.

$$F = \int M\ dx = \int ye^{xy}\ dx = e^{xy} + h_1(y)$$

where $h_1(y)$ is an unknown function of y. Similarly,

$$F = \int N\ dy = \int xe^{xy} + \sin y\ dy = e^{xy} - \cos y + h_2(x)$$

where $h_2(x)$ is an unknown function of x.

For F to equal both simultaneously, we must have $h_2(x) = 0$ and $h_1(y) = -\cos y$.

Thus $F(x, y) = e^{xy} - \cos y$ and hence,

$$e^{xy} - \cos y + C = 0$$

is an implicit solution to the DE for any C.

To solve the initial value problem, when $x = 0$ we must have $y = \pi$ or $e^0 - \cos \pi + C = 0$ which implies that $C = -2$. Thus,

$$e^{xy} - \cos y - 2 = 0$$

solves the initial value problem. □

Exercises

Use the Exactness Test to Determine if the DE is exact.

1. $y^2\ dx + x\ dy = 0$
2. $\left(x^2 + y^2\right)\ dx + (2xy + \cos y)\ dy = 0$

3. $s\ dr + r\ ds = 0$

4. $\arctan(y)\ dx + \dfrac{x}{1+y^2}\ dy = 0$

Use the Exactness Test to show the DE is exact, then solve it.

5. $(\sqrt{y} + 2x \tan y)\, dx + \left(\dfrac{x}{2\sqrt{y}} + x^2 \sec^2 y\right) dy = 0$

6. $\left(2xy^4 - y^3 + \cos(2x)\right) dx + \left(4x^2y^3 - 3y^2x - 2y\right) dy = 0$

7. $\left(\dfrac{y}{x} - 3y^2 + x^3\right) dx + (\ln x - 6xy)\, dy = 0$

8. $\left(\sqrt{x^2+y^2} + \dfrac{x}{\sqrt{x^2+y^2}}\right) dx + \dfrac{xy}{\sqrt{x^2+y^2}} dy = 0$ (Hint: one integration is easier, use the easy one to backward engineer the harder one)

9. $(\cos(xy) - xy\sin(xy))\, dx + \left(-x^2\sin(xy) + y\right) dy = 0$ (Hint: one integration is easier, use the easy one to backward engineer the harder one))

Use the Exactness Test to show the DE is exact, then solve the initial value problem.

10. $2xy^3\ dx + 3x^2y^2\ dy = 0, \quad y(1) = 2$

11. $\left(y^2 - 2xe^y\right) dx + \left(2xy - x^2e^y\right) dy = 0, \quad y(2) = 0$

12. (a) Show that $xy^4 dx + 4x^2y^3\ dy = 0$ is not exact.

    (b) Multiply the DE by $\frac{1}{x}$ and show that the resulting DE is exact.

    (c) Solve the DE from (b). Does the solution in (b) solve the original DE (in (a))?

Lecture IV

Linear equations, Bernoulli equations, Orthogonal trajectories, Oblique trajectories

1 Linear equations

A first order linear equations is of the form

$$y' + p(x)y = r(x) \tag{1}$$

This can be written as

$$(p(x)y - r(x))dx + dy = 0.$$

Here $M = p(x)y - r(x)$ and $N = 1$. Now

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$$

Hence,

$$\mu(x) = e^{\int p(x)\,dx}$$

is an integrating factor. Multiplying (1) by $\mu(x)$ we get

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}y\right) = r(x)e^{\int p(x)\,dx}$$

Integrating we get

$$e^{\int p(x)\,dx}y = \int^x r(s)e^{\int p(s)\,ds}\,ds + C$$

which on simplification gives

$$y = e^{-\int p(x)\,dx}\left(C + \int^x r(s)e^{\int p(s)\,ds}\,ds\right)$$

Example 1. *Solve* $y' + 2xy = 2x$

Solution: An integrating factor is e^{x^2}. Hence,

$$ye^{x^2} = \int^x 2te^{t^2}\,dt + C \Rightarrow y = 1 + Ce^{-x^2}$$

Comment: The usual notation dy/dx implies that x is the independent variable and y is the dependent variable. In trying to solve first order ODE, it is sometimes helpful to reverse the role of x and y, and work on the resulting equations. Hence, the resulting equation

$$\frac{dx}{dy} + p(y)x = r(y)$$

is also a linear equation.

Example 2. *Solve* $(4y^3 - 2xy)y' = y^2, \quad y(2) = 1$

Solution: We write this as

$$\frac{dx}{dy} + \frac{2}{y}x = 4y$$

Clearly, y^2 is an integrating factor. Hence,

$$xy^2 = \int^y 4y^3\,dy + C \Rightarrow xy^2 = y^4 + C$$

Using initial condition, we find $xy^2 = y^4 + 1$.

2 Bernoulli's equation

This is of the form

$$y' + p(x)y = r(x)y^{\lambda}, \tag{2}$$

where λ is a real number. Equation (2) is linear for $\lambda = 0$ or 1. Otherwise, it is nonlinear and can be reduced to a linear form by substituting $z = y^{1-\lambda}$

Example 3. *Solve* $y' - y/x = y^3$

Solution: We write this as

$$y^{-3}y' - y^{-2}/x = 1$$

Substitute $y^{-2} = z \Rightarrow -2y^{-3}y' = z'$. This leads to

$$z' + 2z/x = -2$$

This is a linear equation whose solution is

$$zx^2 = -2x^3/3 + C$$

Replacing z we find

$$3\frac{x^2}{y^2} + 2x^3 = C$$

3 Reducible second order ODE

A general 2nd order ODE is of the form

$$F(x, y, y', y'') = 0$$

In some cases, by making substitution, we can reduce this 2nd order ODE to a 1st order ODE. Few cases are described below

Case I: If the *independent variable is missing*, then we have $F(y, y', y'') = 0$. If we substitute $w = y'$, then $y'' = w\frac{dw}{dy}$. Hence, the ODE becomes $F(y, w, w\frac{dw}{dy}) = 0$, which is a 1st order ODE.

Example 4. *Solve* $2y'' - y'^2 - 4 = 0$

Solution: With $w = y'$, the above equation becomes

$$2w\frac{dw}{dy} - w^2 - 4 = 0 \Rightarrow \ln[(w^2 + 4)/C] = y \Rightarrow w = \pm\sqrt{Ce^y - 4}$$

SInce $w = y'$, we find

$$\frac{dy}{\sqrt{Ce^y - 4}} = \pm x + D$$

The integral on the LHS can be evaluated by substitution.

Case II: If the *dependent variable is missing*, then we have $F(x, y', y'') = 0$. If we substitute $w = y'$, then $y'' = w'$. Hence, the ODE becomes $F(x, w, w') = 0$, which is a 1st order ODE.

Example 5. *Solve* $xy'' + 2y' = 0$

Solution: Substitute $w = y'$, then we find

$$\frac{dw}{dx} + \frac{2}{x}w = 0 \Rightarrow w = Cx^{-2}$$

Since $w = y'$, we further get

$$y' = C/x^2 \Rightarrow y = -C/x + D$$

4 Orthogonal trajectories

Definition 1. *Two families of curves are such that each curve in either family is orthogonal (whenever they intersect) to every curve in the other family. Each family of curves is orthogonal trajectories of the other. In case the two families are identical, they we say that the family is self-orthogonal.*

Comment: Orthogonal trajectories has important applications in the field of physics. For example, the equipotential lines and the streamlines in an irrotational 2D flow are orthogonal.

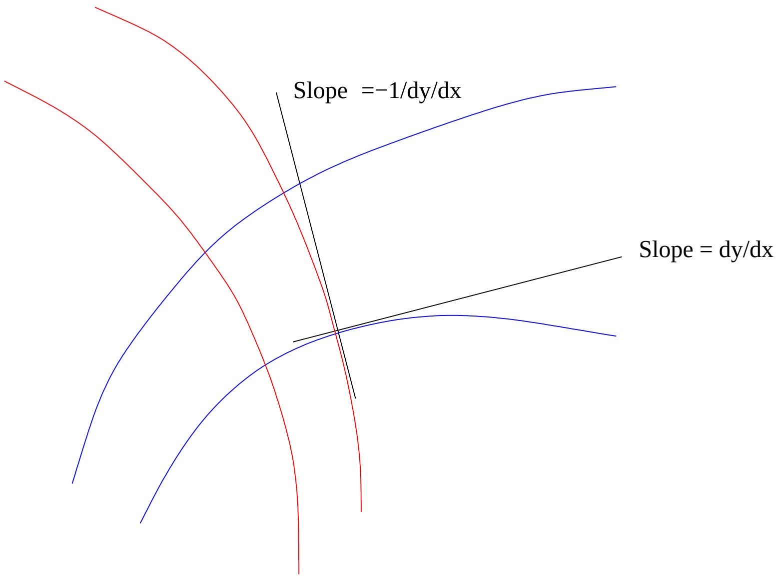


Figure 1: Orthogonal trajectories.

4.1 How to find orthonal trajectories

Suppose the first familiy

$$F(x, y, c) = 0. \tag{3}$$

To find the orthogonal trajectories of this family we proceed as follows. First, differentiate (3) w.r.t. x to find

$$G(x, y, y', c) = 0. \tag{4}$$

Now eliminate c between (3) and (4) to find the differential equation

$$H(x, y, y') = 0 \tag{5}$$

corresponding to the first family. As seen in Figure 1, the differential equation for the other family is obtained by replacing y' by $-1/y'$. Hence, the differetial equation of the orthogonal trajectories is

$$H(x, y, -1/y') = 0 \tag{6}$$

General solution of (6) gives the required orthogonal trajectories.

Example 6. *Find the orthogonal trajectories of familiy of straight lines through the origin.*

Solution: The familiy of straight lines through the origin is given by

$$y = mx$$

The ODE for this familiy is

$$xy' - y = 0$$

The ODE for the orthogonal family is

$$x + yy' = 0$$

Integrating we find

$$x^2 + y^2 = C,$$

which are family of circles with centre at the origin.

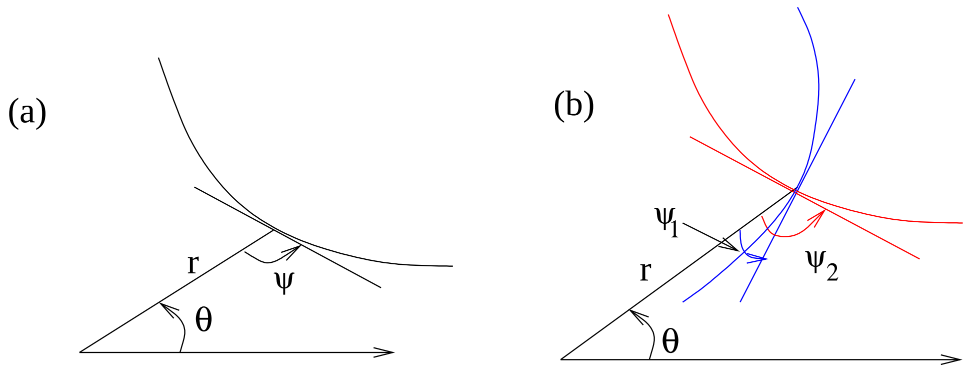


Figure 2: Orthogonal trajectories.

4.2 *Orthogonal trjactories in polar coordinates

Consider a curve in polar cordinate. The angle ψ between the radial and tangent directions is given by

$$\tan\psi = \frac{r\,d\theta}{dr}$$

Consider the curve with angle ψ_1. The curve that intersects it orthogonally has angle $\psi_2 = \psi_1 + \pi/2$. Now

$$\tan\psi_2 = -\frac{1}{\tan\psi_1}$$

Thus, at the point of orthogonal intersection, the value of

$$\frac{r\,d\theta}{dr} \tag{7}$$

for the second famility should be negative raciprocal of the value of (7) of the first family. To illustrate, consider the differential equation for the first family:

$$Pdr + Qd\theta = 0.$$

Thus we find $r\,d\theta/dr = -Pr/Q$. Hence, the differential equation of the orthogonal family is given by

$$\frac{r\,d\theta}{dr} = \frac{Q}{Pr}$$

or

$$Q\,dr - r^2\,P\,d\theta = 0$$

General solution of the last equation gives the orthogonal trajectories.

Example 7. *Find the orthogonal trajectories of familiy of straight lines through the origin.*

Solution: The familiy of straight lines through the origin is given by

$$\theta = A$$

The ODE for this familiy is

$$d\theta = 0$$

The ODE for the orthogonal family is

$$dr = 0$$

Integrating we find

$$r = C,$$

which are family of circles with centre at the origin.

4.3 Oblique trajectories

Here the two families of curves intersect at an arbitrary angle $\alpha \neq \pi/2$. Suppose the first familiy

$$F(x, y, c) = 0. \tag{8}$$

To find the oblique trajectories of this family we proceed as follows. First, differentiate (8) w.r.t. x to find

$$G(x, y, y', c) = 0. \tag{9}$$

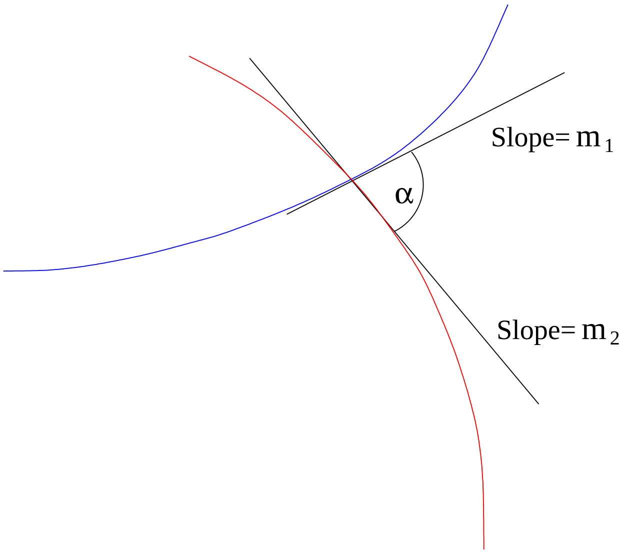


Figure 3: Oblique trajectories.

Now eliminate c between (8) and (9) to find the differential equation

$$H(x, y, y') = 0. \tag{10}$$

Now if m_1 is the slope of this family, then we write (10) as

$$H(x, y, m_1) = 0, \tag{11}$$

Let m_2 be the slope of the second famililly. Then

$$\pm \tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Thus, we find

$$m_1 = \frac{m_2 \pm \tan \alpha}{1 \mp m_2 \tan \alpha}$$

Hence, from (11), the ODE for the second family satisfies

$$H\left(x, y, \frac{m_2 \pm \tan \alpha}{1 \mp m_2 \tan \alpha}\right) = 0,$$

Replacing m_2 by y', the ODE for the second family is written as

$$H\left(x, y, \frac{y' \pm \tan \alpha}{1 \mp y' \tan \alpha}\right) = 0. \tag{12}$$

General solution of (12) gives the required oblique trajectories.
Note: If we let $\alpha \to \pi/2$, we obtained ODE for the orthogonal trajectories.

Example 8. *Find the oblique trajectories that intersects the familiy $y = x + A$ at an angle of* 60^o

Solution: The ODE for the given family is

$$y' = 1$$

For the oblique trajectories, we replace

$$y' \quad \text{by} \quad \frac{y' \pm \tan(\pi/3)}{1 \mp y' \tan(\pi/3)} = \frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'}$$

Thus, the ODE for the oblique trajectories is given by

$$\frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'} = 1$$

Simplifying we obtain

$$y' = \frac{1-\sqrt{3}}{1+\sqrt{3}} \quad \text{OR} \quad y' = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Hence, the oblique trajectories are either

$$y = \frac{1-\sqrt{3}}{1+\sqrt{3}}x + C_1$$

Or

$$y = \frac{1+\sqrt{3}}{1-\sqrt{3}}x + C_2$$

(3) → Integrating Factors Method

This method can be applied to ODEs of the form:

$$\boxed{y' + f(x)y = g(x)}$$

with f, g continuous on $\mathbb{R}$.

Solution method

Define $h(x) = \exp\left(\int f(x)\,dx\right)$ and note that $h'(x) = f(x)h(x)$.

Then we multiply both sides of the ODE with $h(x)$:

$$y' + f(x)y = g(x) \Leftrightarrow y'h(x) + h(x)f(x)\,y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow y'h(x) + h'(x)\,y = g(x)h(x) \Leftrightarrow$$

$$\Leftrightarrow \frac{d}{dx}\left[y\,h(x)\right] = h(x)g(x) \Leftrightarrow$$

$$\Leftrightarrow h(x)y = \int h(x)g(x)\,dx + c_1$$

$$\Leftrightarrow y = \frac{1}{h(x)}\int h(x)g(x)\,dx + \frac{c_1}{h(x)} \qquad (1)$$

↳ Note that for $g(x) = 0$, the above solution simplifies to

$$y = \frac{c_1}{h(x)} = c_1 \exp\left(-\int f(x)\,dx\right)$$

This is called the homogeneous term to Eq. (1).
The integral term is called the particular term.

EXAMPLE

a) Solve the ODE $y' + xy = x^2$ with $y(0) = y_0$.

Solution

Use the integrating factor

$$h(x) = \exp\left(\int x\,dx\right) = \exp(x^2/2) \Rightarrow h'(x) = x\,h(x)$$

and therefore:

$$y' + xy = x^2 \Leftrightarrow y'h(x) + xh(x)y = x^2 h(x) \Leftrightarrow y'h(x) + h'(x)y = x^2 h(x) \Leftrightarrow$$

$$\Leftrightarrow [y\,h(x)]' = x^2 h(x) \Leftrightarrow yh(x) = c + \int_0^x t^2 h(t)\,dt \qquad (1)$$

For $x = 0$: $y_0 h(0) = c + 0 \Leftrightarrow c = y_0 h(0) = y_0 \exp(0) = y_0$

and therefore,

$$(1) \Leftrightarrow yh(x) = y_0 + \int_0^x t^2 h(t)\,dt \Leftrightarrow$$

$$\Leftrightarrow y = \frac{y_0}{h(x)} + \frac{1}{h(x)} \int_0^x t^2 h(t)\,dt =$$

$$= \frac{y_0}{\exp(x^2/2)} + \frac{1}{\exp(x^2/2)} \int_0^x t^2 \exp(t^2/2)\,dt =$$

$$= y_0 \exp(-x^2/2) + \exp(-x^2/2) \int_0^x t^2 \exp(t^2/2)\,dt$$

→ The integrating factor method can be applied to the more general problem of the form

$$\boxed{f(x)y' + g(x)y = h(x)}$$

However, if $f(x_0) = 0$ for some $x_0 \in \mathbb{R}$, then x_0 is a <u>singular point</u> of the ODE and the ODE will only yield a unique solution if x is restricted to an interval between neighboring singular points.

EXAMPLE

Solve the ODE $(x^2-1)y' + xy = 0$ with $y(x_0) = y_0$.

<u>Solution</u>

We have

$$(x^2-1)y' + xy = 0 \Leftrightarrow y' + \frac{x}{x^2-1}y = 0 \qquad (1)$$

We will use the integrating factor

$$h(x) = \exp\left(\int \frac{x}{x^2-1}\,dx\right) = \exp\left(\frac{1}{2}\int \frac{(x^2-1)'}{x^2-1}\,dx\right) =$$

$$= \exp\left(\frac{1}{2}\ln|x^2-1|\right) = \exp\left(\ln\sqrt{|x^2-1|}\right) =$$

$$= \sqrt{|x^2-1|}$$

$\Rightarrow h'(x) = h(x)\dfrac{x}{x^2-1}$. It follows that

$$(1) \Leftrightarrow y' h(x) + \frac{x}{x^2-1} h(x) y = 0 \Leftrightarrow y' h(x) + y h'(x) = 0$$

$$\Leftrightarrow (d/dx)[y h(x)] = 0 \Leftrightarrow (d/dx)\left[y\sqrt{|x^2-1|}\right] = 0$$

$$\Leftrightarrow y\sqrt{|x^2-1|} = c_1 \Leftrightarrow y = \frac{c_1}{\sqrt{|x^2-1|}}$$

We note that the ODE has singular points on $x=1$ and $x=-1$. From the initial condition:

$$y(x_0) = y_0 \Leftrightarrow \frac{c_1}{\sqrt{|x_0^2-1|}} = y_0 \Leftrightarrow c = y_0\sqrt{|x_0^2-1|}$$

and therefore:

$$y = \frac{y_0\sqrt{|x_0^2-1|}}{\sqrt{|x^2-1|}}$$

We distinguish between the following cases:

Case 1 : If $x_0 \in (-\infty,-1)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (-\infty,-1)$$

Case 2 : If $x_0 \in (-1,1)$, then $|x_0^2-1| = 1-x_0^2$ and

$$y = \frac{y_0\sqrt{1-x_0^2}}{\sqrt{1-x^2}}, \quad \forall x \in (-1,1)$$

Case 3 : If $x_0 \in (1,+\infty)$, then $|x_0^2-1| = x_0^2-1$ and

$$y = \frac{y_0\sqrt{x_0^2-1}}{\sqrt{x^2-1}}, \quad \forall x \in (1,+\infty).$$