1. Introduction

Most natural phenomena, whether in the domain of fluid dynamics, electricity, magnetism, mechanics, optics, heat flow, economy, biology can be described in general by partial differential equations (PDEs). For example, the natural laws of physics, such as Maxwell's equations, Newton's law of cooling, the Navier-Stokes equations, Newton's equations of motion, and Schrodinger's equation of quantum mechanics, are stated (or can be) in terms of PDEs, that is, these laws describe physical phenomena by relating space and time derivatives. Derivatives occur in these equations because the derivatives represent natural things (like velocity, acceleration, force, friction, flux, current).

2. Basic Concepts and Definitions

Definition 1.1

A partial differential equation (usually denoted by PDE) is an equation that contains in addition to the dependent variable and independent variables , one or more partial derivatives of the dependent variable with respect to one or more independent variables. In general, it may be written in the form:

$$F\left(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots\right) = 0 \qquad (1.1)$$

involving several independent variables $x, y, \ldots$, an unknown function $u(x, y, \ldots)$ of these variables, and the partial derivatives $u_x, u_y, \ldots, u_{xx}, u_{xy}, \ldots$, of the unknown function.

A Few well-known PDEs

Heat equation: it is a partial differential equation gives the distribution of temperature in a specific region as a function of space and time when the temperature at the boundaries, the initial distribution of temperature, and the physical properties of the medium are given.

$$u_t = u_{xx} \qquad \text{(heat equation in one dimension)}$$

$$u_t = u_{xx} + u_{yy} \qquad \text{(heat equation in two dimensions)}$$

Laplace's equation: it is satisfied by the potential fields in source-free domains. For example, the Laplace equation is satisfied by the gravitational potential of the gravity force in domains free from attracting masses, the potential of an electrostatic field in a domain free from charges, etc.

$$u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{(Laplace's equation in Cartesian coordinates)}$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad \text{(Laplace's equation in polar coordinates)}$$

Wave equation: it is a partial differential equation describes various oscillatory processes and processes of wave propagation.

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} \qquad \text{(wave equation in three dimensions)}$$

Telegraph equation: is a partial differential equation describes the voltage on an electrical transmission line with distance and time

$$u_{tt} = u_{xx} + \alpha u_t + \beta u \qquad \text{(telegraph equation)},$$

Schrödinger equation: it is a partial differential equation that governs the wave function of a quantum-mechanical system

$$i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}\Delta u + Vu$$

Definition 1.2

The general solution of a partial differential equation constitute of arbitrary functions of independent variables involved in (PDE) rather than on arbitrary constants. These arbitrary functions are defined on some domain $D \subset \mathbb{R}^n$ which is continuously differentiable such that all its partial derivatives involved in equation (1.1) exist and satisfy (1.1) identically.

We recall that in the case of ordinary differential equations, the first task is to find the general solution, and then a particular solution is determined by finding the values of arbitrary constants from the prescribed conditions. But, for partial differential equations, selecting a particular solution satisfying the additional conditions from the general solution of a partial differential equation may be as difficult as, or even more difficult than, the problem of finding the general solution itself. This is so because the general solution of a partial differential equation involves arbitrary functions; the specialization of such a solution to the particular form which satisfies supplementary conditions requires the determination of these arbitrary functions, rather than merely the determination of constants.

As indicated above, the general solution of a linear partial differential equation contains arbitrary functions. This means that there are infinitely many solutions and only by specifying the initial and/or boundary conditions can we determine a specific solution of interest.

Usually, both initial and boundary conditions arise from the physical problems. In the case of partial differential equations in which one of the independent variables is the time t, an initial condition(s) specifies the physical state of the dependent variable $u(\mathrm{x},t)$ at a particular time $t = t_0$ or $t = 0$. Often $u(\mathrm{x},0)$ and/or $u_t(\mathrm{x},0)$ are specified to determine the function $u(\mathrm{x},t)$ at later times. Such conditions are called the Cauchy (or initial) conditions. It can be shown that these conditions are necessary and sufficient for the existence of a unique solution. The problem of finding the solution of the initial-value problem with prescribed Cauchy data on the line $t=0$ is called the Cauchy problem or the initial-value problem.

In each physical problem, the governing equation is to be solved within a given domain D of space with prescribed values of the dependent variable $u(\mathrm{x},t)$ given on the boundary ∂D of D. Often, the boundary need not enclose a finite volume in which case, part of the boundary is at infinity. For problems with a boundary at infinity, boundedness conditions on the behavior of the solution at infinity must be specified. This kind of problem is typically known as a boundary-value problem, and it is one of the most fundamental problems in applied mathematics and mathematical physics.
There are three important types of boundary conditions which arise frequently in formulating physical problems. These are

(i) Dirichlet conditions, where the solution u is prescribed at each point of a boundary ∂D of a domain D. The problem of finding the solution of a given equation parial differential equation inside D with prescribed values of u on ∂D is called the Dirichlet boundary-value problem;

(ii) Neumann conditions, where values of normal derivative $\dfrac{\partial u}{\partial n}$ of the solution on the boundary ∂D are specified. In this case, the problem is called the Neumann boundary-value problem;

(iii) Robin conditions, where $\left(\dfrac{\partial u}{\partial n}+au\right)$ is specified on ∂D. The corresponding problem is called the Robin boundary-value problem.

Definition 1.3

The order of a partial differential equation is the order of the highest ordered partial derivative appearing in the equation. For example

$$u_{xx}+2xu_{xy}+u_{yy}=\mathrm{e}^{y}$$

is a second-order partial differential equation, and

$$u_{xxy}+xu_{yy}+8u=7y$$

is a third-order partial differential equation, and

$$\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial x^4}=0,$$

is a fourth-order partial differential equation.

Definition 1.4

A partial differential equation is said to be linear if the function F is linear function in the dependent variable and all its derivatives with coefficients depending only on the independent variables, for example

$$x^2 y\frac{\partial u}{\partial x}+(x-y^2)\frac{\partial u}{\partial y}+yu=\sin(x+y)$$

is linear equation. While the equations

$$x^2 y\frac{\partial u}{\partial x}+(x-y^2)\frac{\partial u}{\partial y}+yu^2=0,$$

$$u\frac{\partial u}{\partial x}-\frac{\partial u}{\partial y}=x\cos u$$

are nonlinear equations.

In general, the linear partial differential equation of order n in two independent variables has the form

$$\sum_{i,j=0}^{i+j\le n} A_{ij}(x,y)\frac{\partial^{i+j}u}{\partial x^i\partial y^j}=G(x,y) \qquad (1.2)$$

Where $A_{ij}(x,y)$, $G(x,y)$ are functions of the independent variables.

Definition 1.5

A partial differential equation it is said to be quasi-linear if it is linear in the highest-ordered derivative of the dependent variable. That is the coefficients of terms involve functions of only lower order derivatives of the dependent variables. However, terms with lower order derivatives can occur in any manner. For example, the equation

$$uu_x+u_t=u$$

is first-order quasi-linear partial differential equation., while the equation

$$u_x u_{xx}+xuu_y=\sin y$$

is a second-order quasi-linear partial differential equation.

Definition 1.6

A quasi-linear partial differential equation it is said to be semi-linear if the coefficients of highest derivatives are functions of the independent variables alone, for example

$$u_{xx}+u_{yy}=u^2$$

Definition 1.7

Homogeneity: The equation (1.2) is called homogeneous if the right hand side $G(x,y)$ is identically zero for all x and y. If $G(x,y)$ is not identically zero, then the equation is called nonhomogeneous.

Definition 1.8

The linear partial differential equation is called of homogeneous terms if all the terms of the linear partial differential equation have the same order, for example

$$u_{xxx} + xy^2 u_{xyy} - \sin x\, u_{yyy} = e^{x+y}$$

3. Formation of partial differential equation

There are two methods to form a partial differential equation

(i) By elimination of arbitrary constants.

(ii) By elimination of arbitrary functions.

Elimination of arbitrary constants

Consider a system of geometrical surfaces described by the equation

$$\phi(x,y,z,a,b) = 0, \qquad (1.4)$$

where a and b are arbitrary parameters. We differentiate (1.4) with respect to x and y to obtain

$$\phi_x + p\phi_z = 0, \quad \phi_y + q\phi_z = 0, \qquad (1.5)$$

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$.

The set of three equations (1.4) and (1.5) involves two arbitrary parameters a and b. In general, these two parameters can be eliminated from this set to obtain a first-order equation of the form

$$\psi(x,y,z,p,q) = 0. \qquad (1.6)$$

Thus the system of surfaces (1.4) gives rise to a first-order partial differential equation (1.6).

In general, if the number of arbitrary constants to be eliminated is equal to the number of independent variables, then only one first-order partial differential equation arises. If the number of arbitrary constants to be eliminated is less than the number of independent variables, then more than one first-order partial differential equation is obtained. If the number of arbitrary constants to be eliminated is more than the number of independent variables, the partial differential equations obtained are of second or higher order.

Example (1)

Find the PDE corresponding to the family of spheres

$$x^2 + y^2 + (z-c)^2 = r^2 \qquad (1.7)$$

Solution

Differentiating the equation (1.7) with respect to x and y gives

$$x + p(z - c) = 0 \quad \text{and} \quad y + q(z - c) = 0.$$

Eliminating the arbitrary constant c from these equations, we obtain the first-order, partial differential equation

$$yp - xq = 0.$$

Example (2)

Find the PDE corresponding to the family of spheres

$$(x - a)^2 + (y - b)^2 + z^2 = r^2$$

Solution

We differentiate the equation of the family of spheres with respect to x and y to obtain

$$(x - a) + zp = 0, \quad (y - b) + zq = 0.$$

Eliminating the two arbitrary constants a and b, we find the non-linear partial differential equation

$$z^2\left(p^2 + q^2 + 1\right) = r^2.$$

Example (3)

Form the partial differential equation by eliminating the constants from

$$z = ax + by + ab. \qquad (1.8)$$

Solution

Differentiating Eq. (1.8) partially with respect to x and y, we obtain

$$\frac{\partial z}{\partial x} = a = p, \quad \frac{\partial z}{\partial y} = b = q$$

Substituting p and q for a and b in Eq. (1.8), we get the required PDE as

$$z = px + qy + pq$$

Example (4)

Find the partial differential equation of the family of planes, the sum of whose x, y, z intercepts is equal to unity.

Solution

Let

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

be the equation of the plane in intercept form, so that $a + b + c = 1$. Thus, we have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{1 - a - b} = 1, \qquad (1.9)$$

Differentiating Eq. (1.9) with respect to x and y, we have

$$\frac{p}{1-a-b} = -\frac{1}{a} \text{ and } \frac{q}{1-a-b} = -\frac{1}{b} \qquad (1.10)$$

From Eqs. (1.10), we get

$$\frac{p}{q} = \frac{b}{a} \qquad (1.11)$$

Also, from Eqs. (1.9) and (1.10), we get

$$pa = a + b - 1 = a + \frac{p}{q}a - 1 \text{ or } a\left(1 + \frac{p}{q} - p\right) = 1$$

Therefore,

$$a = \frac{q}{(p + q - pq)} \qquad (1.12)$$

Similarly, from Eqs. (1.9) and (1.10), we find

$$b = \frac{p}{(p + q - pq)} \qquad (1.13)$$

Substituting the values of a and b from Eqs. (1.12) and (1.13) respectively to Eq. (1.9), we have

$$\frac{p + q - pq}{q}x + \frac{p + q - pq}{p}y + \frac{p + q - pq}{-pq}z = 1$$

or

$$\frac{x}{q} + \frac{y}{p} - \frac{z}{pq} = \frac{1}{p + q - pq}$$

That is,

$$px + qy - z = \frac{pq}{p + q - pq}$$

which is the required PDE.

Example (5)

Find the differential equation of all spheres of radius $\lambda,$ having center in the xy -plane

Solution

Let

$$(x - a)^2 + (y - b)^2 + z^2 = \lambda^2 \qquad (1.14)$$

be the equation of the spheres having center at $(a,b,0)$ in the xy -plane.

Differentiating Eq. (1.14) with respect to x and y, we have

$$2(x - a) + 2pz = 0,\ 2(y - b) + 2qz = 0 \qquad (1.15)$$

Substituting of $(x - a)$ and $(y - b)$ from Eq. (1.15) to Eq. (1.14), we have

$$z^2\left(p^2 + q^2 + 1\right) = \lambda^2$$

which is the required PDE.

Elimination of arbitrary functions

Suppose u and v are any two given functions of x, y and z. Let F be an arbitrary function of u and v of the form

$$F(u,v)=0 \qquad (1.15)$$

We can form a differential equation by eliminating the arbitrary function F. For this, we differentiate Eq. (1.15) partially with respect to x and y to get

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right]=0 \qquad (1.16)$$

and

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right]+\frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right]=0 \qquad (1.17)$$

where $p=\dfrac{\partial z}{\partial x}$, $q=\dfrac{\partial z}{\partial y}$.

Now, eliminating $\partial F/\partial u$ and $\partial F/\partial v$ from Eqs. (1.16) and (1.17), we obtain

$$\begin{vmatrix} \dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial z}p & \dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial z}p \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}q & \dfrac{\partial v}{\partial y}+\dfrac{\partial v}{\partial z}q \end{vmatrix}=0$$

which simplifies to

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \qquad (1.18)$$

where,

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Eq. (1.18) is a linear PDE of the type

$$Pp+Qq=R$$

where

$$P=\frac{\partial(u,v)}{\partial(y,z)},\quad Q=\frac{\partial(u,v)}{\partial(z,x)},\quad R=\frac{\partial(u,v)}{\partial(x,y)}.$$

Eq. (1.18) is called Lagrange's PDE of first order.

If z is given in the form

$$z=\phi(u)+\psi(v) \qquad (*)$$

where ϕ, ψ are arbitrary functions of u, v respectively, and u, v are functions of x, y. Differentiating Eq. (*) with respect to x and y, we have

$z_x = \phi'(u)u_x + \psi'(v)v_x$

$z_y = \phi'(u)u_y + \psi'(v)v_y$

$z_{xx} = \phi''(u)u_x^2 + \psi''(v)v_x^2 + \phi'(u)u_{xx} + \psi'(v)v_{xx}$

$z_{xy} = \phi''(u)u_x u_y + \psi''(v)v_x v_y + \phi'(u)u_{xy} + \psi'(v)v_{xy}$

$z_{yy} = \phi''(u)u_y^2 + \psi''(v)v_y^2 + \phi'(u)u_{yy} + \psi'(v)v_{yy}$

Now, eliminating $\phi'(u), \phi''(u), \psi'(v), \psi''(v)$ from Eq. (**), we obtain

$$\begin{vmatrix} z_x & u_x & v_x & 0 & 0 \\ z_y & u_y & v_y & 0 & 0 \\ z_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ z_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ z_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0 \qquad (***)$$

Equation (***) is a second-order linear partial differential equation, can be written in the form

$Pz_{xx} + Qz_{xy} + Rz_{yy} + Sz_x + Tz_y = W$,

Where P, Q, R, S, T, W are certain functions of x, y. In general a relation of the form

$$z = \sum_{k=1}^{n} f_k(u_k)$$

where $f_1, f_2, \cdots, f_n$ are arbitrary function, and $u_1, u_2, \cdots, u_n$ are certain functions of x, y.

The following examples illustrate the idea of formation of PDE.

Example (6)

Form the PDE by eliminating the arbitrary function from

(i) $z = f(x + it) + g(x - it)$, where $i = \sqrt{-1}$

(ii) $f\left(x + y + z, x^2 + y^2 + z^2\right) = 0.$

Solution

(i) Given

$$z = f(x + it) + g(x - it)$$

Differentiating it twice partially with respect to x and t, we get

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it) \qquad (1.19)$$

Here, f' indicates derivative of f with respect to $(x + it)$ and g' indicates derivative of g with respect to $(x - it)$.Also, we have

$$\frac{\partial z}{\partial t} = if'(x + it) - ig'(x - it)$$

$$\frac{\partial^2 z}{\partial t^2} = -f''(x + it) - g''(x - it) \qquad (1.20)$$

From Eqs. (1.19) and (1.20), we at once, find that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$$

which is the required PDE.

(ii) The given relation can be written in the form

$$\phi(u, v) = 0,$$

where $u = x + y + z$, $v = x^2 + y^2 + z^2$

Hence, the required PDE is of the form

$$Pp + Qq = R, \text{ (Lagrange equation)}$$

where

$$P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2y \\ 1 & 2z \end{vmatrix} = 2(z - y)$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 2z \\ 1 & 2x \end{vmatrix} = 2(x - z)$$

and

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial v} & \frac{\partial v}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 2x \\ 1 & 2y \end{vmatrix} = 2(y - x)$$

Hence, the required PDE is

$$2(z - y)p + 2(x - z)q = 2(y - x)$$

or

$$(z - y)p + (x - z)q = y - x$$

Example (7)

Eliminate the arbitrary function from the following and hence, obtain the corresponding partial differential equation:

(i) $z = xy + f\left(x^2 + y^2\right)$

(ii) $z = f\left(xy/z\right)$

Solution

(i) Given $z = xy + f\left(x^2 + y^2\right)$

Differentiating it partially with respect to x and y, we obtain

$$\frac{\partial z}{\partial x} = y + 2xf'\left(x^2 + y^2\right) = p \qquad (1.21)$$

$$\frac{\partial z}{\partial y} = x + 2yf'\left(x^2 + y^2\right) = q \qquad (1.22)$$

Eliminating f' from Eqs. (1.21) and (1.22), we get

$$yp - xq = y^2 - x^2$$

which is the required PDE.

(ii) Given $z = f(xy/z)$

Differentiating it partially with respect to x and y, we get

$$\frac{\partial z}{\partial x} = \frac{y(z - xp)}{z^2} f'(xy/z) = p \qquad (1.23)$$

$$\frac{\partial z}{\partial y} = \frac{x(z - yq)}{z^2} f'(xy/z) = q \qquad (1.24)$$

Eliminating f' from Eqs. (1.23) and (1.24), we find

$$xp - yq = 0$$

or

$$px = qy$$

which is the required PDE.

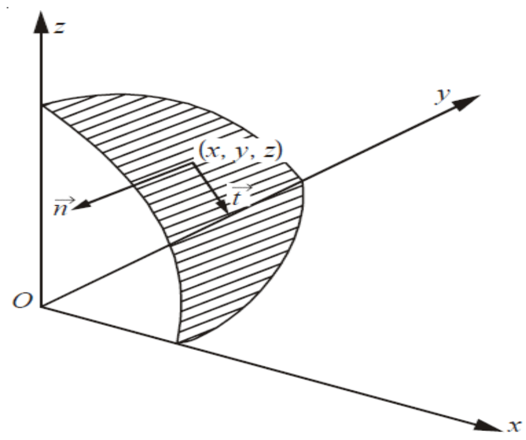

Fig. 1

Solution of partial differential equations of first order

In Section 1.4, we have observed that relations of the form

$$F(x,y,z,a,b)=0 \qquad (2.1)$$

give rise to PDE of first order of the form

$$f(x,y,z,p,q)=0 \qquad (2.2)$$

Thus, any relation of the form (2.1) containing two arbitrary constants a and b is a solution of the PDE of the form (2.2) and is called a complete solution or complete integral.

Consider a first order PDE of the form

$$P(x,y,z)\frac{\partial z}{\partial x}+Q(x,y,z)\frac{\partial z}{\partial y}=R(x,y,z) \qquad (2.3)$$

or simply

$$Pp+Qq=R \qquad (2.4)$$

where x and y are independent variables. The solution of Eq. (2.3) is a surface S lying in the $(x,\ y,\ z)-$space, called an integral surface. If we are given that $z=f(x,y)$ is an integral surface of the PDE (2.4). Then, the normal to this surface will have direction cosines proportional $(\partial z/\partial x,\partial z/\partial y,-1)$ or $(p,q,-1)$. Therefore, the direction of the normal is given by $\vec{n}=\{p,q,-1\}$. From the PDE (2.4), we observe that the normal $\vec{n}$ is perpendicular to the direction defined by the vector $\vec{t}=\{P,Q,R\}$. Therefore, any integral surface must be tangential to a vector with components $\{P,Q,R\}$, and hence, we will never leave the integral surface or solutions surface. Also, the total differential dz is given by

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \qquad (2.5)$$

From Eqs. (2.4) and (2.5), we find

$$\{P,Q,R\}=\{dx,dy,dz\}$$

Now, the solution to Eq. \$(0.37)\$ can be obtained using the following theorem:

Theorem 2.1 The general solution of the linear PDE

$$Pp+Qq=R$$

can be written in the form $F(u,v)=0$, where F is an arbitrary function, and $u(x,y,z)=C_1$ and $v(x,y,z)=C_2$ are a solution of the equation

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)} \qquad (2.6)$$

Proof

We observe that Eq. \$(2.6)\$ consists of a set of two independent ordinary differential equations, that is, a two parameter family of curves in space, one such set can be written as

$$\frac{dy}{dx}=\frac{Q(x,y,z)}{P(x,y,z)} \tag{2.7}$$

which is referred to as "characteristic curve". In quasi-linear case, Eq. (2.7) cannot be evaluated until $z(x,y)$ is known. Recalling Eqs. (2.4) and (2.5), we may recast them using matrix notation as

$$\begin{bmatrix} P & Q \\ dx & dy \end{bmatrix}\begin{pmatrix} \partial z/\partial x \\ \partial z/\partial y \end{pmatrix}=\begin{pmatrix} R \\ dz \end{pmatrix} \tag{2.8}$$

Both the equations must hold on the integral surface. For the existence of finite solutions of Eq. (2.8), we must have

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix}=\begin{vmatrix} P & R \\ dx & dz \end{vmatrix}=\begin{vmatrix} R & Q \\ dz & dy \end{vmatrix}=0$$

on expanding the determinants, we have

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)} \tag{2.9}$$

which are called auxiliary equations for a given PDE.

In order to complete the proof of the theorem, we have yet to show that any surface generated by the integral curves of Eq. (2.9) has an equation of the form $F(u,v)=0$.

Let

$$u(x,y,z)=C_1 \quad \text{and} \quad v(x,y,z)=C_2 \tag{*}$$

be two independent integrals of the ordinary differential equations (2.9). Then, we have

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=du=0$$

and

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=dv=0$$

Solving these equations, we find

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

which can be rewritten as

$$\frac{dx}{\frac{\partial(u,v)}{\partial(y,z)}}=\frac{dy}{\frac{\partial(u,v)}{\partial(z,x)}}=\frac{dz}{\frac{\partial(u,v)}{\partial(x,y)}} \tag{2.10}$$

Now, we may recall from Section 1.4 that the relation $F\left(u,\ v\right)=0$, where F is an arbitrary function, leads to the partial differential equation

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \tag{2.11}$$

By virtue of Eqs. (2.4) and (2.11), Eq. (2.10) can be written as

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{2.12}$$

The solution of these equations are known to be $u(x,y,z)=C_1$ and $v(x,y,z)=C_2$. Hence, $F(u,v)=0$ is the required solution of Eq. (2.4), if u and v are given by Eq. (*), We shall illustrate this method through following examples:

EXAMPLE (2.1)

Find the general integral of the following linear partial differential equations:

(i) $y^2p-xyq=x(z-2y)$

(ii) $(y+zx)p-(x+yz)q=x^2-y^2$.

Solution

(i) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$$

The first two members of the above equation give us

$$\frac{dx}{y}=\frac{dy}{-x} \quad \text{or} \quad xdx=-ydy$$

which on integration results in

$$\frac{x^2}{2}=-\frac{y^2}{2}+C \quad \text{or} \quad x^2+y^2=C_1$$

The last two members of Eq. (1) give

$$\frac{dy}{-y}=\frac{dz}{z-2y} \quad \text{or} \quad zdy-2ydy=-ydz$$

That is,

$$2ydy=ydz+zdy$$

which on integration yields

$$y^2=yz+C_2 \quad \text{or} \quad y^2-yz=C_2$$

Hence, the curves given by Eqs. (2) and (3) generate the required integral surface as

$$F\left(x^2+y^2,y^2-yz\right)=0$$

(ii) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{y+zx}=\frac{dy}{-(x+yz)}=\frac{dz}{x^2-y^2}$$

To get the first integral curve, let us consider the first combination as

$$\frac{xdx+ydy}{xy+zx^2-xy-y^2z}=\frac{dz}{x^2-y^2}$$

or

$$\frac{xdx+ydy}{z\left(x^2-y^2\right)}=\frac{dz}{x^2-y^2}$$

That is,

$$xdx+ydy=zdz$$

On integration, we get

$$\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=C \quad \text{or} \quad x^2+y^2-z^2=C_1$$

Similarly, for getting the second integral curve, let us consider the combination such as

Similarly, for getting the second integral curve, let us consider the combination such as

$$\frac{ydx+xdy}{y^2+xyz-x^2-xyz}=\frac{dz}{x^2-y^2}$$

or

$$ydx+xdy+dz=0$$

which on integration results in

$$xy+z=C_2$$

Thus, the curves given by Eqs. (2) and (3) generate the required integral surface as

$$F\left(x^2+y^2-z^2, xy+z\right)=0.$$

EXAMPLE (2.2) Use Lagrange's method to solve the equation

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix}=0$$

Where $z=z(x,y)$.

Solution

The given PDE can be written as

$$x\left[-\beta-\gamma\frac{\partial z}{\partial y}\right]-y\left[-\alpha-\gamma\frac{\partial z}{\partial x}\right]+z\left[\alpha\frac{\partial z}{\partial y}-\beta\frac{\partial z}{\partial x}\right]=0$$

or

The corresponding auxiliary equations are

$$(\gamma y-\beta z)\frac{\partial z}{\partial x}+(\alpha z-\gamma x)\frac{\partial z}{\partial y}=\beta x-\alpha y$$

$$\frac{dx}{(\gamma y-\beta z)}=\frac{dy}{(\alpha z-\gamma x)}=\frac{dz}{(\beta x-\alpha y)}$$

Using multipliers x, y, and z we find that each fraction is

$$=\frac{xdx+ydy+zdz}{0}.$$

Therefore,

$$xdx+ydy+zdz=0$$

which on integration yields

$$x^2+y^2+z^2=C_1$$

Similarly, using multipliers α, β, and γ, we find from Eq. (2) that each fraction is equal to

$$\alpha dx+\beta dy+\gamma dz=0$$

which on integration gives

$$\alpha x+\beta y+\gamma z=C_2$$

Thus, the general solution of the given equation is found

Thus, the general solution of the given equation is found to be

$$F\left(x^2+y^2+z^2, \alpha x+\beta y+\gamma z\right)=0$$

EXAMPLE (2.3)

Find the general integrals of the following linear PDEs:

(i) $pz-qz=z^2+(x+y)^2$

(ii) $\left(x^2-yz\right)p+\left(y^2-zx\right)q=z^2-xy$.

Solution

(i) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{z}=\frac{dy}{-z}=\frac{dz}{z^2+(x+y)^2}$$

The first two members of Eq. (1) give

$$dx+dy=0$$

which on integration yields

$$x+y=C_1$$

Now, considering Eq. (2) and the first and last members of Eq. (1), we obtain

$$\frac{2zdz}{z^2+C_1^2}=2dx$$

Or

$$dx = \frac{zdz}{z^2 + C_1^2}$$

which on integration yields

$$\ln\left(z^2 + C_1^2\right) = 2x + C_2$$

or

$$\ln\left[z^2 + (x+y)^2\right] - 2x = C_2$$

Thus, the curves given by Eqs. (2) and (3) generates the integral surface for the given PDE as

$$F\left(x+y, \log\left\{x^2 + y^2 + z^2 + 2xy\right\} - 2x\right) = 0$$

(ii) The integral surface of the given PDE is given by the integral curves of the auxiliary equation

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

Equation (1) can be rewritten as

$$\frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dz - dx}{(z-x)(x+y+z)}$$

Considering the first two terms of Eq. (2) and integrating, we get

$$\ln(x-y) = \ln(y-z) + \ln C_1$$

$$\frac{x-y}{y-z} = C_1$$

Similarly, considering the last two terms of Eq. (2) and integrating, we obtain

$$\frac{y-z}{z-x} = C_2$$

Thus, the integral curves given by Eqs. (3) and (4) generate the integral surface

$$F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0.$$

INTEGRAL SURFACES PASSING THROUGH A GIVEN CURVE

In the previous section, we have seen how a general solution for a given linear PDE can be obtained. Now, we shall make use of this general solution to find an integral surface containing a given curve as explained below:

Suppose, we have obtained two integral curves described by

$$\left.\begin{aligned} u(x,y,z) &= C_1 \\ v(x,y,z) &= C_2 \end{aligned}\right\}$$

from the auxiliary equations of a given PDE. Then, the solution of the given PDE can be written in the form

$$F(u,v)=0$$

Suppose, we wish to determine an integral surface, containing a given curve C described by the parametric equations of the form

$x=x(t), \quad y=y(t), \quad z=z(t),$

where t is a parameter. Then, the particular solution (0.48) must be like

$$\left.\begin{array}{l} u\{x(t),y(t),z(t)\}=C_1 \\ v\{x(t),y(t),z(t)\}=C_2 \end{array}\right\}$$

Thus, we have two relations, from which we can eliminate the parameter t to obtain a relation of the type

$$F\left(C_1,C_2\right)=0,$$

which leads to the solution given by Eq. (0.49). For illustration, let us consider the following couple of examples.

EXAMPLE (2.4)

Find the integral surface of the linear PDE

$$x\left(y^2+z\right)p-y\left(x^2+z\right)q=\left(x^2-y^2\right)z$$

containing the straight line $x+y=0,\ z=1.$

Solution

The auxiliary equations for the given PDE are

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{\left(x^2-y^2\right)z}$$

Using the multiplier $x\ y\ z,$ we have

$$yzdx+zxdy+xydz=0$$

On integration, we get

$$xyz=C_1$$

Suppose, we use the multipliers x, y and z. Then find that each fraction in Eq. (1) is equal to

$$xdx+ydy+zdz=0$$

which on integration yields

$$x^2+y^2+z^2=C_2$$

For the curve in question, we have the equations in parametric form as

$$x=t, \quad y=-t, \quad z=1$$

Substituting these values in Eqs. (2) and (3), we obtain

$$\left.\begin{array}{r} -t^2=C_1 \\ 2t^2+1=C_2 \end{array}\right\}$$

Eliminating the parameter t, we find

$$1-2C_1=C_2$$

or

$$2C_1 + C_2 - 1 = 0$$

Hence, the required integral surface is

$$x^2 + y^2 + z^2 + 2xyz - 1 = 0$$

EXAMPLE (2.5)

Find the integral surface of the linear PDE

$$xp + yq = z$$

which contains the circle defined by

$$x^2 + y^2 + z^2 = 4, \quad x + y + z = 2$$

Solution

The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Integration of the first two members of Eq. (1) gives

$$\ln x = \ln y + \ln C$$

or

$$\frac{x}{y} = C_1$$

Similarly, integration of the last two members of Eq. (1) yields

$$\frac{y}{z} = C_2$$

Hence, the integral surface of the given PDE is

$$F\left(\frac{x}{y}, \frac{y}{z}\right) = 0$$

If this integral surface also contains the given circle, then we have to find a relation between x/y and y/z .

The equation of the circle is

$$x^2 + y^2 + z^2 = 4$$

$$x + y + z = 2$$

From Eqs. (2) and (3), we have

$$y = x/C_1, \quad z = y/C_2 = x/C_1C_2$$

Substituting these values of y and z in Eqs. (5) and (6), we find

$$x^2 + \frac{x^2}{C_1^2} + \frac{x^2}{C_1^2C_2^2} = 4, \quad \text{or} \quad x^2\left(1 + \frac{1}{C_1^2} + \frac{1}{C_1^2C_2^2}\right) = 4$$

And

$$x + \frac{x}{C_1} + \frac{x}{C_1C_2} = 2, \quad \text{or} \quad x\left(1 + \frac{1}{C_1} + \frac{1}{C_1C_2}\right) = 2$$

From Eqs. (7) and (8) we observe

$$1+\frac{1}{C_1^2}+\frac{1}{C_1^2C_2^2}=\left(1+\frac{1}{C_1}+\frac{1}{C_1C_2}\right)^2$$

which on simplification gives us

$$\frac{2}{C_1}+\frac{2}{C_1C_2}+\frac{2}{C_1^2C_2}=0$$

That is,

$$C_1C_2+C_1+1=0$$

Now, replacing C_1 by x/y and C_2 by y/z, we get the required integral surface as

$$\frac{x}{y}\frac{y}{z}+\frac{x}{y}+1=0$$

Or

$$\frac{x}{z}+\frac{x}{y}+1=0$$

Or

$$xy+xz+yz=0$$

THE CAUCHY PROBLEM FOR FIRST ORDER EQUATIONS

Consider an interval I on the real line. If $x_0(s), y_0(s)$ and $z_0(s)$ are three arbitrary functions of a single variable $s\in I$ such that they are continuous in the interval I with their first derivatives.

Then, the Cauchy problem for a first order PDE of the form

$$F(x,y,z,p,q)=0$$

is to find a region $\mathbb{R}$ in (x,y), i.e. the space containing $\left(x_0(s), y_0(s)\right)$ for all $s\in I$, and a solution $z=\phi(x,y)$ of the PDE (0.53) such that

$$Z\left[x_0(s), y_0(s)\right]=Z_0(s)$$

and $\phi(x,y)$ together with its partial derivatives with respect to x and y are continuous functions of x and y in the region $\mathbb{R}$.

Geometrically, there exists a surface $z=\phi(x,y)$ which passes through the curve Γ, called datum curve, whose parametric equations are

$$x=x_0(s), \quad y=y_0(s), \quad z=z_0(s)$$

and at every point of which the direction $(p,q,-1)$ of the normal is such that

$$F(x,y,z,p,q)=0$$

This is only one form of the problem of Cauchy.

In order to prove the existence of a solution of Eq. (0.53) containing the curve Γ, we have to make further assumptions about the form of the function F and the nature of Γ. Based on these assumptions, we have a whole class of existence theorems which is beyond the scope of this book. However, we shall quote one form of the existence theorem without proof, which is due to Kowalewski (see Senddon, 1986).

Theorem 0.2 If

(i) $g(y)$, and all of its derivatives are continuous $|y-y_0|<\delta$

(ii) x_0 is a given number and $z_0=g(y_0), q_0=g'(y_0)$ and $f(x,y,z,q)$ and all of its partial derivatives are continuous in a region S defined by

$$|x-x_0|<\delta, |y-y_0|<\delta, |q-q_0|<\delta,$$

then, there exists a unique function $\phi(x,y)$ such that

(a) $\phi(x,y)$ and all of its partial derivatives are continuous in a region $\mathbb{R}$ defined by

$$|x-x_0|<\delta_1, \quad |y-y_0|<\delta_2,$$

(b) For all (x,y) in $\mathbb{R}$, $z=\phi(x,y)$ is a solution of the equation

$\dfrac{\partial z}{\partial x}=f\left(x,y,z,\dfrac{\partial z}{\partial y}\right)$ and

(c) For all values of y in the interval $|y-y_0|<\delta_1, \phi(x_0,y)=g(y)$.

Surfaces orthogonal to a given system of surfaces

One of the useful applications of the theory of linear first order PDE is to find the system of surfaces orthogonal to a given system of surfaces.

Let a one-parameter family of surfaces is described by the equation

$$F(x,y,z)=C \qquad (2.12)$$

Then, the task is to determine the system of surfaces which cut each of the given surfaces orthogonally. Let (x,y,z) be a point on the surface given by Eq. (2.12), where the normal to the surface will have direction ratios $(\partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$ which may be denoted by P , Q, R .

Let

$$z=\phi(x,y)$$

be the surface which cuts each of the given system orthogonally (see Fig. 0.2).

Then, its normal at the point (x,y,z) will have direction ratios $(\partial z/\partial x, \partial z/\partial y, -1)$ which, of course, will be perpendicular to the normal to the surfaces characterized by Eq. (2.12). As a consequence we have a relation

$$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}-R=0$$

or

$$Pp+Qq=R \qquad (2.13)$$

which is a linear PDE of Lagranges type, and can be recast into

$$\frac{\partial F}{\partial x}\frac{\partial z}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial z}{\partial y}=\frac{\partial F}{\partial z} \tag{2.14}$$

Thus, any solution of the linear first order PDE of the type given by either Eq. (2.13) or (2.14) is orthogonal to every surface of the system described by Eq. (2.12). In other words, the surfaces orthogonal to the system (2.12) are the surfaces generated by the integral curves of the auxiliary equations

$$\frac{dx}{\partial F/\partial x}=\frac{dy}{\partial F/\partial y}=\frac{dz}{\partial F/\partial z}$$

First order non-linear equations

In this section, we will discuss the problem of finding the solution of first order non-linear partial differential equations (PDEs) in three variables of the form

$F(x, y, z, p, q)=0,$

1.2 CLASSIFICATION OF SECOND ORDER PDE

The most general linear second order PDE, with one dependent function u on a domain Ω of points $X = (x_1, x_2, \ldots, x_n), n > 1,$ is

$$\sum_{i,\, j=1}^{n} a_{ij} u_{xi} x_j + \sum_{i=1}^{n} b_i u_{xi} + F(u) = G \tag{1.2}$$

The classification of a PDE depends only on the highest order derivatives present.

The classification of PDE is motivated by the classification of the quadratic equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1.3}$$

which is elliptic, parabolic, or hyperbolic according as the discriminant $B^2 - 4AC$ is negative, zero or positive. Thus, we have the following second order linear PDE in two variables x and y:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1.4}$$

where the coefficients A, B, C, … may be functions of x and y, however, for the sake of simplicity we assume them to be constants. Equation (1.4) is elliptic, parabolic or hyperbolic at a point (x_0, y_0) according as the discriminant

$$B^2(x_0, y_0) - 4A(x_0, y_0)\, C(x_0, y_0)$$

is negative, zero or positive. If this is true at all points in a domain Ω, then Eq. (1.4) is said to be elliptic, parabolic or hyperbolic in that domain. If the number of independent variables is two or three, a transformation can always be found to reduce the given PDE to a canonical form (also called normal form). In general, when the number of independent variables is greater than 3, it is not always possible to find such a transformation except in certain special cases. The idea of reducing the given PDE to a canonical form is that the transformed equation assumes a simple form so that the subsequent analysis of solving the equation is made easy.

1.3 CANONICAL FORMS

Consider the most general transformation of the independent variables x and y of Eq. (1.4) to new variables $\xi, \eta,$ where

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{1.5}$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = (\xi_x \eta_y - \xi_y \eta_x) \neq 0 \tag{1.6}$$

in the domain Ω where Eq. (1.4) holds. Using the chain rule of partial differentiation, the partial derivatives become

$$
\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}
\end{aligned} \tag{1.7}
$$

Substituting these expressions into the original differential equation (1.4), we get

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}u = \bar{G} \tag{1.8}$$

where

$$
\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\bar{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\bar{F} &= F, \qquad \bar{G} = G
\end{aligned} \tag{1.9}
$$

It may be noted that the transformed equation (1.8) has the same form as that of the original equation (1.4) under the general transformation (1.5).

Since the classification of Eq. (1.4) depends on the coefficients A, B and C, we can also rewrite the equation in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \tag{1.10}$$

It can be shown easily that under the transformation (1.5), Eq. (1.10) takes one of the following three canonical forms:

(i) $u_{\xi\xi} - u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ (1.11a)

or

$u_{\xi\eta} = \phi, (\xi, \eta, u, u_\xi, u_\eta)$ in the hyperbolic case

(ii) $u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the elliptic case (1.11b)

(iii) $u_{\xi\xi} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ (1.11c)

or

$$u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \text{ in the parabolic case}$$

We shall discuss in detail each of these cases separately.

Using Eq. (1.9) it can also be verified that

$$\bar{B}^2 - 4\bar{A}\bar{C} = (\xi_x\eta_y - \xi_y\eta_x)^2 \,(B^2 - 4AC)$$

and therefore we conclude that the transformation of the independent variables does not modify the type of PDE.

1.3.1 Canonical Form for Hyperbolic Equation

Since the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} > 0$ for hyperbolic case, we set $\bar{A} = 0$ and $\bar{C} = 0$ in Eq. (1.9), which will give us the coordinates ξ and η that reduce the given PDE to a canonical form in which the coefficients of $u_{\xi\xi}, u_{\eta\eta}$ are zero. Thus we have

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$\bar{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

which, on rewriting, become

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$$

Solving these equations for (ξ_x/ξ_y) and (η_x/η_y), we get

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{1.12}$$

The condition $B^2 > 4AC$ implies that the slopes of the curves $\xi(x, y) = C_1, \eta(x, y) = C_2$ are real. Thus, if $B^2 > 4AC$, then at any point (x, y), there exists two real directions given by the two roots (1.12) along which the PDE (1.4) reduces to the canonical form. These are called *characteristic equations*. Though there are two solutions for each quadratic, we have considered only one solution for each. Otherwise we will end up with the same two coordinates.

Along the curve $\xi(x, y) = c_1$, we have

$$d\xi = \xi_x \, dx + \xi_y \, dy = 0$$

Hence,

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) \tag{1.13}$$

Similarly, along the curve $\eta(x, y) = c_2$, we have

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) \tag{1.14}$$

Integrating Eqs. (1.13) and (1.14), we obtain the equations of family of characteristics $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$, which are called the characteristics of the PDE (1.4). Now to obtain the canonical form for the given PDE, we substitute the expressions of ξ and η into Eq. (1.8) which reduces to Eq. (1.11a).

To make the ideas clearer, let us consider the following example:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

Comparing with the standard PDE (1.4), we have $A = 3, B = 10, C = 3, B^2 - 4AC = 64 > 0.$ Hence the given equation is a hyperbolic PDE. The corresponding characteristics are:

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) = \frac{1}{3}$$

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right) = 3$$

To find ξ and η, we first solve for y by integrating the above equations. Thus, we get

$$y = 3x + c_1, \qquad y = \frac{1}{3}x + c_2$$

which give the constants as

$$c_1 = y - 3x, \qquad c_2 = y - x/3$$

Therefore,

$$\xi = y - 3x = c_1, \qquad \eta = y - \frac{1}{3}x = c_2$$

These are the characteristic lines for the given hyperbolic equation. In this example, the characteristics are found to be straight lines in the (x, y)-plane along which the initial data, impulses will propagate.

To find the canonical equation, we substitute the expressions for ξ and η into Eq. (1.9) to get

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 3(-3)^2 + 10(-3)(1) + 3 = 0$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(3)(-3)\left(-\frac{1}{3}\right) + 10\left[(-3)(1) + 1\left(-\frac{1}{3}\right)\right] + 2(3)(1)(1)$$

$$= 6 + 10\left(-\frac{10}{3}\right) + 6 = 12 - \frac{100}{3} = -\frac{64}{3}$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 0, \qquad \bar{F} = 0$$

Hence, the required canonical form is

$$\frac{64}{3}u_{\xi\eta} = 0 \quad \text{or} \quad u_{\xi\eta} = 0$$

On integration, we obtain

$$u(\xi, \eta) = f(\xi) + g(\eta)$$

where f and g are arbitrary. Going back to the original variables, the general solution is

$$u(x, y) = f(y - 3x) + g(y - x/3)$$

1.3.2 Canonical Form for Parabolic Equation

For the parabolic equation, the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} = 0$, which can be true if $\bar{B} = 0$ and $\bar{A}$ or $\bar{C}$ is equal to zero. Suppose we set first $\bar{A} = 0$ in Eq. (1.9). Then we obtain

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

or

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

which gives

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Using the condition for parabolic case, we get

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} \tag{1.15}$$

Hence, to find the function $\xi = \xi(x, y)$ which satisfies Eq. (1.15), we set

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A}$$

and get the implicit solution

$$\xi(x, y) = C_1$$

In fact, one can verify that $\bar{A} = 0$ implies $\bar{B} = 0$ as follows:

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

Since $B^2 - 4AC = 0$, the above relation reduces to

$$\bar{B} = 2A\xi_x\eta_x + 2\sqrt{AC}\,(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$
$$= 2(\sqrt{A}\xi_x + \sqrt{C}\xi_y)(\sqrt{A}\eta_x + \sqrt{C}\eta_y)$$

However,

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} = -\frac{2\sqrt{AC}}{2A} = -\sqrt{\frac{C}{A}}$$

Hence,

$$\bar{B} = 2(\sqrt{A}\xi_x - \sqrt{A}\xi_x)(\sqrt{A}\eta_x + \sqrt{C}\eta_y) = 0$$

We therefore choose ξ in such a way that both $\bar{A}$ and $\bar{B}$ are zero. Then η can be chosen in any way we like as long as it is not parallel to the ξ- coordinate. In other words, we choose η such that the Jacobian of the transformation is not zero. Thus we can write the canonical equation for parabolic case by simply substituting ξ and η into Eq. (1.8) which reduces to either of the forms (1.11c).

To illustrate the procedure, we consider the following example:

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$$

The discriminant $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$, and hence the given PDE is parabolic everywhere. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = -\frac{2xy}{2x^2} = -\frac{y}{x}$$

On integration, we have

$$xy = c$$

and hence $\xi = xy$ will satisfy the characteristic equation and we can choose $\eta = y$. To find the canonical equation, we substitute the expressions for ξ and η into Eq. (1.9) to get

$$\bar{A} = Ay^2 + Bxy + cx^2 = x^2y^2 - 2x^2y^2 + y^2x^2 = 0$$

$$\bar{B} = 0, \qquad \bar{C} = y^2, \qquad \bar{D} = -2xy$$

$$\bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = e^x$$

Hence, the transformed equation is

$$y^2 u_{\eta\eta} - 2xyu_\xi = e^x$$

or

$$\eta^2 u_{\eta\eta} = 2\xi u_\xi + e^{\xi/\eta}$$

The canonical form is, therefore,

$$u_{\eta\eta} = \frac{2\xi}{\eta^2} u_\xi + \frac{1}{\eta^2} e^{\xi/\eta}$$

1.3.3 Canonical Form for Elliptic Equation

Since the discriminant $B^2 - 4AC < 0$, for elliptic case, the characteristic equations

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

give us complex conjugate coordinates, say ξ and η. Now, we make another transformation from (ξ, η) to (α, β) so that

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

which give us the required canonical equation in the form (1.11b).

To illustrate the procedure, we consider the following example:

$$u_{xx} + x^2 u_{yy} = 0$$

The discriminant $B^2 - 4AC = -4x^2 < 0$. Hence, the given PDE is elliptic. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\sqrt{-4x^2}}{2} = -ix$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = ix$$

Integration of these equations yields

$$iy + \frac{x^2}{2} = c_1, \qquad -iy + \frac{x^2}{2} = c_2$$

Hence, we may assume that

$$\xi = \frac{1}{2}x^2 + iy, \qquad \eta = \frac{1}{2}x^2 - iy$$

Now, introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

we obtain

$$\alpha = \frac{x^2}{2}, \qquad \beta = y$$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + \beta\alpha_x\alpha_y + c\alpha_y^2 = x^2$$

$$\bar{B} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2c(\alpha_y\beta_y) = 0$$

$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + c\beta_y^2 = x^2$$

$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + c\alpha_{yy} + D\alpha_x + E\alpha_y = 1$$

$$\bar{E} = A\beta_{xx} + B\beta_{xy} + c\beta_{yy} + D\beta_x + E\beta_y = 0$$

$$\bar{F} = 0, \qquad \bar{G} = 0$$

Thus the required canonical equation is

$$x^2u_{\alpha\alpha} + x^2u_{\beta\beta} + u_\alpha = 0$$

or

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{2\alpha}$$

EXAMPLE 1.1 Classify and reduce the relation

$$y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$$

to a canonical form and solve it.

Solution The discriminant of the given PDE is

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$$

Hence the given equation is of a parabolic type. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = \frac{-2xy}{2y^2} = -\frac{x}{y}$$

Integration gives $x^2 + y^2 = c_1$. Therefore, $\xi = x^2 + y^2$ satisfies the characteristic equation. The η-coordinate can be chosen arbitrarily so that it is not parallel to ξ, i.e. the Jacobian of the transformation is not zero. Thus we choose

$$\xi = x^2 + y^2, \qquad \eta = y^2$$

To find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 4x^2y^2 - 8x^2y^2 + 4x^2y^2 = 0$$

$$\bar{B} = 0, \qquad \bar{C} = 4x^2y^2, \qquad \bar{D} = \bar{E} = \bar{F} = \bar{G} = 0$$

Hence, the required canonical equation is

$$4x^2y^2 u_{\eta\eta} = 0 \quad \text{or} \quad u_{\eta\eta} = 0$$

To solve this equation, we integrate it twice with respect to η to get

$$u_\eta = f(\xi), \qquad u = f(\xi)\eta + g(\xi)$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions of ξ. Now, going back to the original independent variables, the required solution is

$$u = y^2 f(x^2 + y^2) + g(x^2 + y^2)$$

EXAMPLE 1.2 Reduce the following equation to a canonical form:

$$(1 + x^2) u_{xx} + (1 + y^2) u_{yy} + x u_x + y u_y = 0$$

Solution The discriminant of the given PDE is

$$B^2 - 4AC = -4(1 + x^2)(1 + y^2) < 0$$

Hence the given PDE is an elliptic type. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\sqrt{-4(1+x^2)(1+y^2)}}{2(1+x^2)} = -i\sqrt{\frac{1+y^2}{1+x^2}}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = i\sqrt{\frac{1+y^2}{1+x^2}}$$

On integration, we get

$$\xi = \ln(x + \sqrt{x^2+1}) - i\ln(y + \sqrt{y^2+1}) = c_1$$

$$\eta = \ln(x + \sqrt{x^2+1}) + i\ln(y + \sqrt{y^2+1}) = c_2$$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\eta - \xi}{2i}$$

we obtain

$$\alpha = \ln(x + \sqrt{x^2+1})$$

$$\beta = \ln(y + \sqrt{y^2+1})$$

Then the canonical form can be obtained by computing

$$\bar{A} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 1, \qquad \bar{B} = 0, \qquad \bar{C} = 1, \qquad \bar{D} = \bar{E} = \bar{F} = \bar{G} = 0$$

Thus the canonical equation for the given PDE is

$$u_{\alpha\alpha} + u_{\beta\beta} = 0$$

EXAMPLE 1.3 Reduce the following equation to a canonical form and hence solve it:

$$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$$

Solution Comparing with the general second order PDE (1.4), we have

$$A = 1, \qquad B = -2\sin x, \qquad C = -\cos^2 x,$$

$$D = 0, \qquad E = -\cos x, \qquad F = 0, \qquad G = 0$$

The discriminate $B^2 - 4AC = 4(\sin^2 x + \cos^2 x) = 4 > 0$. Hence the given PDE is hyperbolic. The relevant characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\sin x - 1$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 1 - \sin x$$

On integration, we get

$$y = \cos x - x + c_1, \qquad y = \cos x + x + c_2$$

Thus, we choose the characteristic lines as

$$\xi = x + y - \cos x = c_1, \qquad \eta = -x + y - \cos x = c_2$$

In order to find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(\sin x + 1)(\sin x - 1) - 4\sin^2 x - 2\cos^2 x = -4$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = 0$$

Thus, the required canonical equation is

$$u_{\xi\eta} = 0$$

Integrating with respect to ξ, we obtain

$$u_\eta = f(\eta)$$

where f is arbitrary. Integrating once again with respect to η, we have

$$u = \int f(\eta)\, d\eta + g(\xi)$$

or

$$u = \psi(\eta) + g(\xi)$$

where $g(\xi)$ is another arbitrary function. Returning to the old variables x, y, the solution of the given PDE is

$$u(x, y) = \psi(y - x - \cos x) + g(y + x - \cos x)$$

EXAMPLE 1.4 Reduce the Tricomi equation

$$u_{xx} + xu_{yy} = 0, \qquad x \neq 0$$

for all x, y to canonical form.

Solution The discriminant $B^2 - 4AC = -4x$. Hence the given PDE is of mixed type: hyperbolic for $x < 0$ and elliptic for $x > 0$.

Case I In the half-plane $x < 0$, the characteristic equations are

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{-2\sqrt{-x}}{2} = -\sqrt{-x}$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \sqrt{-x}$$

Integration yields

$$y = \frac{2}{3}(-x)^{3/2} + c_1$$

$$y = -\frac{2}{3}(-x)^{3/2} + c_2$$

Therefore, the new coordinates are

$$\xi(x, y) = \frac{3}{2}y - (\sqrt{-x})^3 = c_1$$

$$\eta(x, y) = \frac{3}{2}y + (\sqrt{-x})^3 = c_2$$

which are cubic parabolas.

In order to find the canonical equation, we compute

$$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = -\frac{9}{4}x + 0 + \frac{9}{4}x = 0$$

$$\overline{B} = 9x, \qquad \overline{C} = 0, \qquad \overline{D} = -\frac{3}{4}(-x)^{-1/2} = -\overline{E}, \qquad \overline{F} = \overline{G} = 0$$

Thus, the required canonical equation is

$$9xu_{\xi\eta} - \frac{3}{4}(-x)^{-1/2}u_\xi + \frac{3}{4}(-x)^{-1/2}u_\eta = 0$$

or

$$u_{\xi\eta} = \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta)$$

Case II In the half-plane $x > 0$, the characteristic equations are given by

$$\frac{dy}{dx} = i\sqrt{x}, \qquad \frac{dy}{dx} = -i\sqrt{x}$$

On integration, we have

$$\xi(x, y) = \frac{3}{2}y - i(\sqrt{x})^3, \qquad \eta(x, y) = \frac{3}{2}y + i(\sqrt{x})^3$$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

we obtain

$$\alpha = \frac{3}{2}y, \qquad \beta = -(\sqrt{x})^3$$

The corresponding normal or canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\beta} u_\beta = 0$$

EXAMPLE 1.5 Find the characteristics of the equation

$$u_{xx} + 2u_{xy} + \sin^2(x) u_{yy} + u_y = 0$$

when it is of hyperbolic type.

Solution The discriminant $B^2 - 4AC = 4 - 4\sin^2 x = 4\cos^2 x$. Hence for all $x \neq (2n - 1)\pi/2$, the given PDE is of hyperbolic type. The characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = 1 \mp \cos x$$

On integration, we get

$$y = x - \sin x + c_1, \qquad y = x + \sin x + c_2$$

Thus, the characteristic equations are

$$\xi = y - x + \sin x, \qquad \eta = y - x - \sin x$$

EXAMPLE 1.6 Reduce the following equation to a canonical form and hence solve it:

$$yu_{xx} + (x + y) u_{xy} + xu_{yy} = 0$$

Solution The discriminant

$$B^2 - 4AC = (x + y)^2 - 4xy = (x - y)^2 > 0$$

Hence the given PDE is hyperbolic everywhere except along the line $y = x$; whereas on the line $y = x$, it is parabolic. When $y \neq x$, the characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = \frac{(x + y) \mp (x - y)}{2y}$$

Therefore,

$$\frac{dy}{dx} = 1, \qquad \frac{dy}{dx} = \frac{x}{y}$$

On integration, we obtain

$$y = x + c_1, \qquad y^2 = x^2 + c_2$$

Hence, the characteristic equations are

$$\xi = y - x, \qquad \eta = y^2 - x^2$$

These are straight lines and rectangular hyperbolas. The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = y - x - y + x = 0, \qquad \bar{B} = -2\,(x-y)^2,$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 2(x-y), \qquad \bar{F} = \bar{G} = 0$$

Thus, the canonical equation for the given PDE is

$$-2(x-y)^2 u_{\xi\eta} + 2(x-y)u_\eta = 0$$

or

$$-2\xi^2 u_{\xi\eta} + 2(-\xi)\,u_\eta = 0$$

or

$$\xi u_{\xi\eta} + u_\eta = \frac{\partial}{\partial \xi}\left(\xi \frac{\partial u}{\partial \eta}\right) = 0$$

Integration yields

$$\xi \frac{\partial u}{\partial \eta} = f(\eta)$$

Again integrating with respect to η, we obtain

$$u = \frac{1}{\xi}\int f(\eta)\, d\eta + g(\xi)$$

Hence,

$$u = \frac{1}{y-x}\int f(y^2 - x^2)\, d(y^2 - x^2) + g(y-x)$$

is the general solution.

EXAMPLE 1.7 Classify and transform the following equation to a canonical form:

$$\sin^2(x)u_{xx} + \sin(2x)u_{xy} + \cos^2(x)u_{yy} = x$$

Solution The discriminant of the given PDE is

$$B^2 - 4AC = \sin^2 2x - 4\sin^2 x\cos^2 x = 0$$

Hence, the given equation is of parabolic type. The characteristic equation is

$$\frac{dy}{dx} = \frac{B}{2A} = \cot x$$

Integration gives

$$y = \ln \sin x + c_1$$

Hence, the characteristic equations are:

$$\xi = y - \ln \sin x, \qquad \eta = y$$

η is chosen in such a way that the Jacobian of the transformation is nonzero. Now the canonical form can be obtained by computing

$$\bar{A} = 0, \qquad \bar{B} = 0, \qquad \bar{C} = \cos^2 x, \qquad \bar{D} = 1,$$

$$\bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = x$$

Hence, the canonical equation is

$$\cos^2(x)\, u_{\eta\eta} + u_\xi = x$$

or

$$[1 - e^{2(\eta-\xi)}]u_{\eta\eta} = \sin^{-1}(e^{\eta-\xi}) - u_\xi$$

EXAMPLE 1.8 Show that the equation

$$u_{xx} + \frac{2N}{x}u_x = \frac{1}{a^2}u_{tt}$$

where N and a are constants, is hyperbolic and obtain its canonical form.

Solution Comparing with the general PDE (1.4) and replacing y by t, we have $A = 1$, $B = 0$, $C = -1/a^2$, $D = 2N/x$, and $E = F = G = 0$. The discriminant $B^2 - 4AC = 4/a^2 > 0$. Hence, the given PDE is hyperbolic. The characteristic equations are

$$\frac{dt}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = \mp\frac{\sqrt{4/a^2}}{2} = \mp\frac{1}{a}$$

Therefore,

$$\frac{dt}{dx} = -\frac{1}{a}, \qquad \frac{dt}{dx} = \frac{1}{a}$$

On integration, we get

$$t = -\frac{x}{a} + c_1, \qquad t = \frac{x}{a} + c_2$$

Hence, the characteristic equations are

$$\xi = x + at, \qquad \eta = x - at$$

The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_t + C\xi_t^2 = 0,$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_t + \xi_t\eta_x) + 2C\xi_t\eta_t = 4,$$

$$\bar{C} = 0, \qquad \bar{D} = D\xi_x + E\xi_t = \frac{2N}{x}, \qquad \bar{E} = D\eta_x + E\eta_t = \frac{2N}{x}$$

Thus, the canonical equation for the given PDE is

$$4u_{\xi\eta} + \frac{2N}{x}(u_\xi + u_\eta) = 0$$

Expressing x in terms of ξ and η, the required canonical equation is

$$u_{\xi\eta} + \frac{N}{\xi + \eta}(u_\xi + u_\eta) = 0$$

EXAMPLE 1.9 Transform the following differential equation to a canonical form:

$$u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0$$

Solution The discriminant $B^2 - 4AC = -12 < 0$. Hence, the given PDE is elliptic. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = +1 - i\sqrt{3}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = +1 + i\sqrt{3}$$

Integration of these equations yields

$$y = +(1 - i\sqrt{3})x + c_1, \qquad y = +(1 + i\sqrt{3})x + c_2$$

Hence, we may take the characteristic equations in the form

$$\xi = y - (1 - i\sqrt{3})x, \qquad \eta = y - (1 + i\sqrt{3})x$$

In order to avoid calculations with complex variables, we introduce the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

Therefore,

$$\alpha = y - x, \qquad \beta = \sqrt{3x}$$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 3$$

$$\bar{B} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y = 0$$

$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 = 3$$

$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y = 1$$

$$\bar{E} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y = 2\sqrt{3}$$

$$\bar{F} = 0, \qquad \bar{G} = 0$$

Thus the required canonical form is

$$3u_{\alpha\alpha} + 3u_{\beta\beta} + u_\alpha + 2\sqrt{3}u_\beta = 0$$

or

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3}(u_\alpha + 2\sqrt{3}u_\beta)$$

1.4 ADJOINT OPERATORS

Let

$$Lu = \phi \tag{1.16}$$

where L is a differential operator given by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x)$$

One way of introducing the adjoint differential operator L^* associated with L is to form the product vLu and integrate it over the interval of interest. Let

$$\int_A^B vLu\,dx = [\;]_A^B + \int_A^B uL^*v\,dx \tag{1.17}$$

which is obtained after repeated integration by parts. Here, L^* is the operator adjoint to L, where the functions u and v are completely arbitrary except that Lu and L^*v should exist.

EXAMPLE 1.10 Let $Lu = a(x)(d^2u/dx^2) + b(x)(du/dx) + c(x)u$; construct its adjoint L^*.

Solution Consider the equation

$$\int_A^B vLu\,dx = \int_A^B v\left[a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u\right]dx$$

$$= \int_A^B (av)\frac{d^2u}{dx^2}dx + \int_A^B (bv)\frac{du}{dx}dx \int_A^B (cv)\,u\,dx$$

However,

$$\int_A^B (av)\frac{d^2u}{dx^2}dx = \int_A^B (av)\frac{d}{dx}(u')dx$$

$$= [u'va]_A^B - \int_A^B (av)'u'dx$$

$$= [u'av]_A^B - [u(av)']_A^B + \int_A^B u(av)''dx$$

$$\int_A^B (bv)\frac{du}{dx}dx = [u(bv)]_A^B - \int_A^B u(bv)'dx$$

$$\int_A^B (cv)u\,dx = \int_A^B u(cv)\,dx$$

Therefore,

$$\int_A^B vLu\,dx = [u'(av) - u(av)' + u(bv)]_A^B + \int_A^B u[(av)'' - (bv)' + (cv)]\,dx$$

Comparing this equation with Eq. (1.17), we get

$$L^*v = (av)'' - (bv)' + (cv) = av'' + (2a' - b)v' + (a'' - b' + c)v$$

Therefore,

$$L^* = a\frac{d^2}{dx^2} + (2a' - b)\frac{d}{dx} + (a'' - b' + c)$$

Consider the partial differential equation

$$L(u) = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = \phi \tag{1.18}$$

which is valid in a region S of the xy-plane, where A, B, C, …, ϕ are functions of x and y. In addition, linear boundary conditions of the general form

$$\alpha u + \beta u_x = f$$

are prescribed over the boundary curve ∂S of the region S. Let

$$\iint_S vLu\,d\sigma = [\quad] + \iint_S uL^*v\,d\sigma$$

where the integrated part [ ] is a line integral evaluated over ∂S, the boundary of S, then L^* is called the adjoint operator. In general, a second order linear partial differential operator L is denoted by

$$L(u) = \sum_{i,\,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i\frac{\partial u}{\partial x_i} + Cu \tag{1.19}$$

Its adjoint operator is defined by

$$L^*(v) = \sum_{i,\,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(A_{ij}v) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(B_i v) + Cv \tag{1.20}$$

Here it is assumed that $A_{ij} \in C^{(2)}$ and $B_i \in C^{(1)}$. For any pair of functions $u, v \in C^{(2)}$, it can be shown that

$$vL(u) - uL^*(v) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} A_{ij}\left(v\frac{\partial u}{\partial x_j} - u\frac{\partial v}{\partial x_j}\right) + uv\left(B_i - \sum_{j=1}^{n}\frac{\partial A_{ij}}{\partial x_j}\right)\right] \tag{1.21}$$

This is known as Lagrange's identity.

EXAMPLE 1.11 Construct an adjoint to the Laplace operator given by

$$L(u) = u_{xx} + u_{yy} \tag{1.22}$$

Solution Comparing Eq. (1.22) with the general linear PDE (1.19), we have $A_{11} = 1$, $A_{22} = 1$. From Eq. (1.20), the adjoint of (1.22) is given by

$$L^*(v) = \frac{\partial^2}{\partial x^2}(v) + \frac{\partial^2}{\partial y^2}(v) = v_{xx} + v_{yy}$$

Therefore,

$$L^*(u) = u_{xx} + u_{yy}$$

Hence, the Laplace operator is a self-adjoint operator.

EXAMPLE 1.12 Find the adjoint of the differential operator

$$L(u) = u_{xx} - u_t \tag{1.23}$$

Solution Comparing Eq. (1.23) with the general second order PDE (1.19), we have $A_{11} = 1$, $B_1 = -1$. From Eq. (1.20), the adjoint of (1.23) is given by

$$L^*(v) = \frac{\partial^2}{\partial x^2}(v) - \frac{\partial}{\partial t}(-v) = v_{xx} + v_t$$

Therefore,

$$L^*(u) = u_{xx} + u_t$$

It may be noted that the diffusion operator is not a self-adjoint operator.

1.5 RIEMANN'S METHOD

In Section 1.2, we have noted with interest that a linear second order PDE

$$L(u) = G(x, y)$$

is classified as hyperbolic if $B^2 > 4AC$, and it has two families of real characteristic curves in the *xy*-plane whose equations are

$$\xi = f_1(x, y) = c_1, \qquad \eta = f_2(x, y) = c_2$$

Here, (ξ, η) are the natural coordinates for the hyperbolic system. In the *xy*-plane, the curves $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$ are the characteristics of the given PDE as shown in Fig. 1.1(a), while in the $\xi\eta$-plane, the curves $\xi = c_1$ and $\eta = c_2$ are families of straight lines parallel to the axes as shown in Fig. 1.1(b).

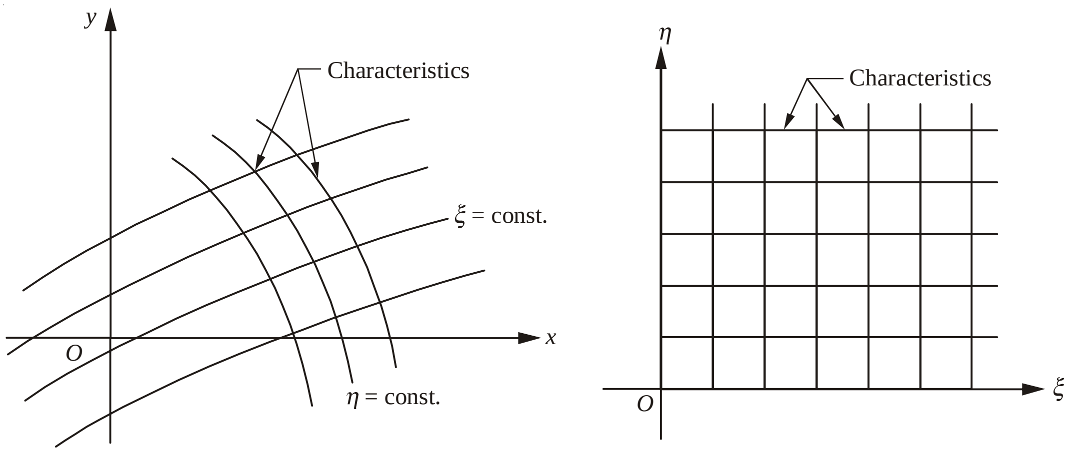

Fig. 1.1 Families of characteristic lines.

A linear second order partial differential equation in two variables, once classified as a hyperbolic equation, can always be reduced to the canonical form

$$\frac{\partial^2 u}{\partial x\,\partial y} = F(x, y, u, u_x, u_y)$$

In particular, consider an equation which is already reduced to its canonical form in the variables x, y:

$$L(u) = \frac{\partial^2 u}{\partial x\,\partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = F(x, y) \tag{1.24}$$

where L is a linear differential operator and a, b, c, F are functions of x and y only and are differentiable in some domain $\mathbb{R}$.

Let $v(x, y)$ be an arbitrary function having continuous second order partial derivatives. Let us consider the adjoint operator L^* of L defined by

$$L^*(v) = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv \tag{1.25}$$

Now we introduce

$$M = auv - u\frac{\partial v}{\partial y}, \qquad N = buv + v\frac{\partial u}{\partial x} \tag{1.26}$$

then

$$M_x + N_y = u_x(av) + u(av)_x - u_x v_y - uv_{xy} + u_y(bv) + u(bv)_y + v_y u_x + vu_{xy}$$

Adding and subtracting *cuv*, we get

$$M_x + N_y = -u\left[\frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv\right] + v\left[\frac{\partial^2 u}{\partial x\,\partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu\right]$$

i.e.

$$vLu - uL^*v = M_x + N_y \tag{1.27}$$

This is known as *Lagrange identity* which will be used in the subsequent discussion. The operator L is a self-adjoint if and only if $L = L^*$. Now we shall attempt to solve Cauchy's problem which is described as follows: Let

$$L(u) = F(x, y) \tag{1.28}$$

with the condition (Cauchy data)

(i) $u = f(x)$ on Γ, a curve in the xy-plane;

(ii) $\dfrac{\partial u}{\partial n} = g(x)$ on Γ.

That is u, and its normal derivatives are prescribed on a curve Γ which is not a characteristic line.

Let Γ be a smooth initial curve which is also continuous as shown in Fig. 1.2. Since Eq. (1.24) is in canonical form, x and y are the characteristic coordinates. We also assume that the tangent to Γ is nowhere parallel to the coordinate axes.

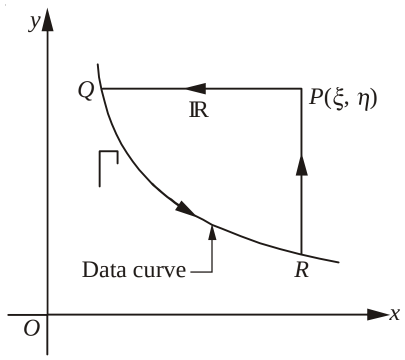

Fig. 1.2 Cauchy data.

Let $P(\xi, \eta)$ be a point at which the solution to the Cauchy problem is sought. Let us draw the characteristics PQ and PR through P to meet the curve Γ at Q and R. We assume that u, u_x, u_y are prescribed along Γ. Let $\partial\mathbb{R}$ be a closed contour $PQRP$ bounding $\mathbb{R}$. Since Eq. (1.28) is already in canonical form, the characteristics are lines parallel to x and y axes. Using Green's theorem, we have

$$\iint_R (M_x + N_y)\,dxdy = \oint_{\partial\mathbb{R}} (M\,dy - N\,dx) \tag{1.29}$$

where $\partial \mathbb{R}$ is the boundary of $\mathbb{R}$. Applying this theorem to the surface integral of Eq. (1.27), we obtain

$$\int_{\partial \mathbb{R}} (M\,dy - N\,dx) = \iint_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy \tag{1.30}$$

In other words,

$$\int_{\Gamma} (M\,dy - N\,dx) + \int_{RP} (M\,dy - N\,dx) + \int_{PQ} (M\,dy - N\,dx) = \iint_{R} [vL(u) - uL^*(v)]\,dx\,dy$$

Now using the fact that $dy = 0$ on PQ and $dx = 0$ on PR, we have

$$\int_{\Gamma} (M\,dy - N\,dx) + \int_{RP} M\,dy - \int_{PQ} N\,dx = \iint_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy \tag{1.31}$$

From Eq. (1.26), we find that

$$\int_{PQ} N\,dx = \int_P^Q buv\,dx + \int_P^Q vu_x\,dx$$

Integrating by parts the second term on the right-hand side and grouping, the above equation becomes

$$\int_{PQ} N\,dx = [uv]_p^Q + \int_P^Q u(bv - v_x)\,dx$$

Substituting this result into Eq. (1.31), we obtain

$$[uv]_p = [uv]_Q + \int_{PQ} u(bv - v_x)\,dx - \int_{RP} u(av - v_y)\,dy$$

$$- \int_{\Gamma} (M\,dy - N\,dx) + \iint_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy \tag{1.32}$$

Let us choose $v(x, y; \xi, \eta)$ to be a solution of the adjoint equation

$$L^*(v) = 0 \tag{1.33}$$

and at the same time satisfy the following conditions:

(i) $v_x = bv$ when $y = \eta$, i.e., on PQ (1.34a)

(ii) $v_y = av$ when $x = \xi$, i.e., on PR (1.34b)

(iii) $v = 1$ when $x = \xi$, $y = \eta$ (1.34c)

We call this function $v(x, y; \xi, \eta)$ as the *Riemann function* or the *Riemann-Green function*. Since $L(u) = F$, Eq. (1.32) reduces to

$$[u]_P = [uv]_Q - \int_\Gamma [u(av - v_y)\,dy - v(bu + u_x)\,dx] + \iint_{\mathbb{R}} (vF)\,dx\,dy \tag{1.35}$$

This is called the Riemann-Green solution for the Cauchy problem described by Eq. (1.28) when u and u_x are prescribed on Γ. Equation (1.35) can also be written as

$$[u]_P = [uv]_Q - \int_\Gamma uv(a\,dy - b\,dx) + \int_\Gamma (uv_y\,dy - vu_x\,dx) + \iint_{\mathbb{R}} (vF)\,dx\,dy \tag{1.36}$$

This relation gives us the value of u at a point P when u and u_x are prescribed on Γ. But when u and u_y are prescribed on Γ, we obtain

$$[u]_P = [uv]_R - \int_\Gamma uv(a\,dy - b\,dx) - \int_\Gamma (uv_x\,dx + vu_y\,dy) + \iint_{\mathbb{R}} (vF)\,dx\,dy \tag{1.37}$$

By adding Eqs. (1.36) and (1.37), the value of u at P is given by

$$[u]_P = \frac{1}{2}\{[uv]_Q + [uv]_R\} - \int_\Gamma uv(a\,dy - b\,dx) - \frac{1}{2}\int_\Gamma u(v_x\,dx - v_y\,dy)$$

$$+ \frac{1}{2}\int_\Gamma v(u_x\,dx - u_y\,dy) + \iint_{\mathbb{R}} (vF)\,dx\,dy \tag{1.38}$$

Thus, we can see that the solution to the Cauchy problem at a point (ξ, η) depends only on the Cauchy data on Γ. The knowledge of the Riemann-Green function therefore enables us to solve Eq. (1.28) with the Cauchy data prescribed on a noncharacteristic curve.

EXAMPLE 1.13 Obtain the Riemann solution for the equation

$$\frac{\partial^2 u}{\partial x\,\partial y} = F(x, y)$$

given

(i) $u = f(x)$ on Γ

(ii) $\dfrac{\partial u}{\partial n} = g(x)$ on Γ

where Γ is the curve $y = x$.

Solution Here, the given PDE is

$$L(u) = \frac{\partial^2 u}{\partial x\,\partial y} = F(x, y) \tag{1.39}$$

We construct the adjoint L^* of L as follows: setting

$$M = auv - uv_y, \qquad N = buv + vu_x$$

and comparing the given equation (1.39) with the standard canonical form of hyperbolic equation (1.24), we have

$$a = b = c = 0$$

Therefore,

$$M = -uv_y, \qquad N = vu_x \tag{1.40}$$

and

$$M_x + N_y = vu_{xy} - uv_{xy} = vL(u) - uL^*(v)$$

Thus,

$$L^*(v) = \frac{\partial^2 v}{\partial x\, \partial y} \tag{1.41}$$

Here, $L = L^*$ and is a self-adjoint operator. Using Green's theorem

$$\iint_{\mathbb{R}} (M_x + N_y)\, dx\, dy = \int_{\partial \mathbb{R}} (M\, dy - N\, dx)$$

we have

$$\iint_{\mathbb{R}} [vL(u) - uL^*(v)]\, dx\, dy = \int_{\partial \mathbb{R}} (M\, dy - N\, dx)$$

or

$$\iint_{\mathbb{R}} [vF - uL^*(v)]\, dx\, dy = \int_{\partial \mathbb{R}} (M\, dy - N\, dx) \tag{1.42}$$

But

$$\int_{\partial \mathbb{R}} (M\, dy - N\, dx) = \int_{\Gamma} (M\, dy - N\, dx) + \int_{QP} (M\, dy - N\, dx) + \int_{PR} (M\, dy - N\, dx) \tag{1.43}$$

where

$$\int_{\Gamma} (M\, dy - N\, dx) = \int_{\Gamma} \left(-u \frac{\partial v}{\partial y} dy - v \frac{\partial u}{\partial x} dx \right)$$

From Fig. 1.3, we have on Γ, $x = y$. Therefore, $dx = dy$. Hence

$$\int_{\Gamma} (M\, dy - N\, dx) = \int_{\Gamma} \left(-u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial x} \right) dx \tag{1.44}$$

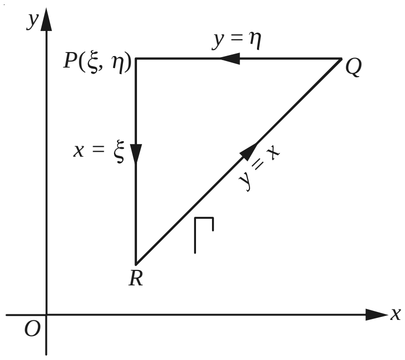

Fig. 1.3 An illustration of Example 1.13.

since on QP, y = constant. Therefore, $dy = 0$. Thus,

$$\int_{QP} (M\,dy - N\,dx) = \int_{QP} -N\,dx = \int_{QP} -v\frac{\partial u}{\partial x}\,dx \tag{1.45}$$

Similarly, on PR, x = constant. Hence, $dx = 0$. Thus,

$$\int_{PR} (M\,dy - N\,dx) = \int_{PR} M\,dy = \int_{PR} -u\frac{\partial v}{\partial y}\,dy \tag{1.46}$$

Substituting Eqs. (1.44)–(1.46) into Eq. (1.43), we obtain from Eq. (1.42), the relation

$$\iint_{\mathbb{R}} [vF - uL^*(v)]\,dx\,dy = \int_{\Gamma}\left(-u\frac{\partial v}{\partial y}\,dx - v\frac{\partial u}{\partial x}\,dx\right) + \int_{QP} -v\frac{\partial u}{\partial x}\,dx + \int_{PR} -u\frac{\partial v}{\partial y}\,dy$$

But

$$\int_{QP} -v\frac{\partial u}{\partial x}\,dx = [-vu]_Q^P + \int_{QP} u\frac{\partial v}{\partial x}\,dx$$

Therefore,

$$\iint_{\mathbb{R}} [vF - uL^*(v)]\,dx\,dy = \int_{\Gamma}\left(-u\frac{\partial v}{\partial y}\,dx - v\frac{\partial u}{\partial x}\,dx\right)$$
$$+ [-vu]_Q^P + \int_{QP} u\frac{\partial v}{\partial x}\,dx + \int_{PR} -u\frac{\partial v}{\partial y}\,dy \tag{1.47}$$

Now choosing $v(x, y; \xi, \eta)$ as the solution of the adjoint equation such that

(i) $L^*v = 0$ throughout the xy-plane

(ii) $\dfrac{\partial v}{\partial x} = 0$ when $y = \eta$, i.e., on QP

(iii) $\dfrac{\partial v}{\partial y} = 0$ when $x = \xi$, i.e., on PR

(iv) $v = 1$ at $P(\xi, \eta)$.

Equation (1.47) becomes

$$\iint_{\mathbb{R}} (vF)\, dx\, dy = \int_\Gamma \left(-u \frac{\partial v}{\partial y} dx - v \frac{\partial u}{\partial x} dx \right) + (uv)_Q - (u)_P$$

or

$$(u)_P = (uv)_Q + \int_\Gamma \left(-u \frac{\partial v}{\partial y} dx - v \frac{\partial u}{\partial x} dx \right) - \iint_{\mathbb{R}} (vF)\, dx\, dy \tag{1.48}$$

However,

$$(uv)_Q - (uv)_R = \int_\Gamma d\,(uv) = \int_\Gamma \left[\frac{\partial}{\partial x}(uv)\, dx + \frac{\partial}{\partial y}(uv)\, dy \right]$$

$$= \int_\Gamma (u_x v\, dx + uv_x\, dx + u_y v\, dy + uv_y\, dy)$$

Now Eq. (1.48) can be rewritten as

$$(u)_P = (uv)_R = \int_\Gamma (u_y v\, dy + uv_y\, dy) - \iint_{\mathbb{R}} (vF)\, dx\, dy \tag{1.49}$$

Finally, adding Eqs. (1.48) and (1.49), we get

$$(u)_P = \frac{1}{2}[(uv)_Q + (uv)_R] + \frac{1}{2}\int_\Gamma (-uv_x\, dx - vu_x\, dx)$$
$$+ \frac{1}{2}\int_\Gamma (u_y v\, dy + uv_y\, dy) - \iint_{\mathbb{R}} (vF)\, dx\, dy$$

EXAMPLE 1.14 Verify that the Green function for the equation

$$\frac{\partial^2 u}{\partial x\, \partial y} + \frac{2}{x+y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = 0$$

subject to $u = 0$, $\partial u/\partial x = 3x^2$ on $y = x$, is given by

$$v(x, y; \xi, \eta) = \frac{(x+y)\{2xy + (\xi - \eta)(x - y) + 2\xi\eta\}}{(\xi + \eta)^3}$$

and obtain the solution of the equation in the form

$$u = (x - y)(2x^2 - xy + 2y^2)$$

Solution In the given problem,

$$L(u) = \frac{\partial^2 u}{\partial x\,\partial y} + \frac{2}{x+y}\frac{\partial u}{\partial x} + \frac{2}{x+y}\frac{\partial u}{\partial y} = 0 \tag{1.50}$$

Comparing this equation with the standard canonical hyperbolic equation (1.24), we have

$$a = b = \frac{2}{x+y}, \qquad C = 0, \qquad F = 0$$

Its adjoint equation is $L^*(v) = 0$, where

$$L^*(v) = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial}{\partial x}\left(\frac{2v}{x+y}\right) - \frac{\partial}{\partial y}\left(\frac{2v}{x+y}\right). \tag{1.51}$$

such that

(i) $L^* v = 0$ throughout the xy-plane

(ii) $\dfrac{\partial v}{\partial x} = \dfrac{2}{x+y} v$ on PQ, i.e., on $y = \eta$

(iii) $\dfrac{\partial v}{\partial y} = \dfrac{2}{x+y} v$ on PR, i.e., on $x = \xi$

(1.52)

(iv) $v = 1$ at $P(\xi, \eta)$.

If v is defined by

$$v(x, y; \xi, \eta) = \frac{(x+y)}{(\xi+\eta)^3}[2xy + (\xi-\eta)(x-y) + 2\xi\eta] \tag{1.53}$$

Then

$$\frac{\partial v}{\partial x} = \frac{x+y}{(\xi+\eta)^3}[2y + (\xi-\eta)] + \left[\frac{2xy + (\xi-\eta)(x-y) + 2\xi\eta}{(\xi+\eta)^3}\right]$$

or

$$\frac{\partial v}{\partial x} = \frac{1}{(\xi+\eta)^3}[4xy + 2y^2 + 2x(\xi-\eta) + 2\xi\eta] \tag{1.54}$$

and

$$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{4(x+y)}{(\xi+\eta)^3} \tag{1.55}$$

$$\frac{\partial v}{\partial y} = \frac{1}{(\xi+\eta)^3}[4xy + 2x^2 - 2y(\xi-\eta) + 2\xi\eta] \tag{1.56}$$

Using the results described by Eqs. (1.53)–(1.56), Eq. (1.51) becomes

$$L^*(v) = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{2}{x+y}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{4v}{(x+y)^2}$$

$$= \frac{4(x+y)}{(\xi+\eta)^3} - \frac{2}{(x+y)(\xi+\eta)^3}[4xy + 2(x^2+y^2)]$$

or

$$L^*(v) = \frac{4(x+y)}{(\xi+\eta)^3} - \frac{4(x+y)}{(\xi+\eta)^3} = 0$$

Hence condition (i) of Eq. (1.52) is satisfied. Also, on $y = \eta$.

$$\frac{\partial v}{\partial x} = \frac{1}{(\xi+\eta)^3}[4x\eta + 2\eta^2 + 2x(\xi-\eta) + 2\xi\eta]$$

or

$$\left.\frac{\partial v}{\partial x}\right|_{y=\eta} = \frac{1}{(\xi+\eta)^3}[2\eta^2 + 2x(\xi+\eta) + 2\xi\eta] \tag{1.57}$$

Also, $2v/(x+y)$ at $y = \eta$ is given by

$$\frac{2v}{x+y} = \frac{2}{x+\eta}\,\frac{x+\eta}{(\xi+\eta)^3}[2x\eta + (\xi-\eta)\,(x-\eta) + 2\xi\eta]$$

$$= \frac{1}{(\xi+\eta)^3}[2\eta^2 + 2x(\xi+\eta) + 2\xi\eta] \tag{1.58}$$

From Eqs. (1.57) and (1.58), we get

$$\frac{\partial v}{\partial x} = \frac{2}{x+y}v \quad \text{at } y = \eta$$

Thus, property (ii) in Eq. (1.52) has been verified. Similarly, property (iii) can also be verified. Also, at $x = \xi$, $y = \eta$,

$$v = \frac{\xi+\eta}{(\xi+\eta)^3}[2\xi\eta + (\xi-\eta)^2 + 2\xi\eta] = \frac{(\xi+\eta)\,(\xi+\eta)^2}{(\xi+\eta)^3} = 1$$

Thus property (iv) in Eq. (1.52) has also been verified.

From Eqs. (1.50) and (1.51), we have

$$vL(u)-uL^*(v)=v\frac{\partial^2 u}{\partial x\,\partial y}-u\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left(\frac{2vu}{x+y}\right)+\frac{\partial}{\partial y}\left(\frac{2vu}{x+y}\right)$$

$$=\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial x}\left(\frac{2vu}{x+y}\right)+\frac{\partial}{\partial y}\left(\frac{2vu}{x+y}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{2vu}{x+y}-u\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{2vu}{x+y}+v\frac{\partial u}{\partial x}\right)$$

$$=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}$$

where

$$M=\frac{2uv}{x+y}-u\frac{\partial v}{\partial y},\qquad N=\frac{2vu}{x+y}+v\frac{\partial u}{\partial x}$$

Now using Green's theorem, we have

$$\iint_{\mathbb{R}}[vL(u)-uL^*(v)]\,dx\,dy=\int_{\partial\mathbb{R}}(M\;dy-N\;dx)=\int_R^Q(M\;dy-N\;dx)$$

$$+\int_Q^P(M\;dy-N\;dx)+\int_P^R(M\;dy-N\;dx) \tag{1.59}$$

(see Fig. 1.3) on QP, $y=C$. Hence, $dy=0$; on PR, $x=C$. Therefore, $dx=0$

$$=\int_R^Q\left[\left\{\frac{2uv}{x+y}-u\frac{\partial v}{\partial y}\right\}dy-\left\{\frac{2uv}{x+y}+v\frac{\partial u}{\partial x}\right\}dx\right]$$

$$-\int_Q^P\left[\left\{\frac{2uv}{x+y}+v\frac{\partial u}{\partial x}\right\}\right]dx+\int_P^R\left[\left\{\frac{2uv}{x+y}-u\frac{\partial v}{\partial y}\right\}\right]dy$$

However,

$$\int_Q^P\left(\frac{2uv}{x+y}+v\frac{\partial u}{\partial x}\right)dx=\int_Q^P\frac{2uv}{x+y}dx+(uv)_Q^P-\int_Q^P u\frac{\partial v}{\partial x}dx$$

Now, using the condition $u=0$ on $y=x$, Eq. (1.59) becomes

$$\iint_{\mathbb{R}}[vL(u)-uL^*(v)]\,dx\,dy=\int_R^Q\left(\frac{2uv}{x+y}-u\frac{\partial v}{\partial y}\right)dy-\int_R^Q\left(\frac{2uv}{x+y}+v\frac{\partial u}{\partial x}\right)dx$$

$$-\int_Q^P\frac{2uv}{x+y}dx-(uv)_P+(uv)_Q+\int_Q^P u\frac{\partial v}{\partial x}dx$$

$$+\int_Q^P\frac{2uv}{x+y}dy-\int_P^R\left(u\frac{\partial v}{\partial y}\right)dy$$

Also, using conditions (ii)–(iv) of Eq. (1.52), the above equation simplifies to

$$(u)_P = (uv)_Q - \int_R^Q v\frac{\partial u}{\partial x}dx$$

Now using the given condition, viz.

$$\frac{\partial u}{\partial x} = 3x^2 \text{ on } RQ$$

we obtain

$$(u)_P = (uv)_Q - 3\int_R^Q x^2\left[\frac{2x[2x^2 + 2\xi\eta]}{(\xi+\eta)^3}\right]dx$$

$$= -\frac{12}{(\xi+\eta)^3}\int_\xi^\eta (x^5 + x^3\xi\eta)\,dx$$

$$= -\frac{12}{(\xi+\eta)^3}\left[\frac{1}{6}(\eta^6 - \xi^6) + \frac{1}{4}\xi\eta(\eta^4 - \xi^4)\right]$$

$$= \frac{\xi^2 - \eta^2}{(\xi+\eta)^3}[2(\xi^4 + \xi^2\eta^2 + \eta^4) + 3\xi\eta(\xi^2 + \eta^2)]$$

$$= (\xi - \eta)(2\xi^2 - \xi\eta + 2\eta^2)$$

Therefore,

$$u(x, y) = (x - y)(2x^2 - xy + 2y^2)$$

Hence the result.

EXAMPLE 1.15 Show that the Green's function for the equation

$$\frac{\partial^2 u}{\partial x\,\partial y} + u = 0$$

is

$$v(x, y; \xi, \eta) = J_0\sqrt{2(x-\xi)(y-\eta)}$$

where J_0 denotes Bessel's function of the first kind of order zero.

Solution Comparing with the standard canonical hyperbolic equation (1.24), we have

$$a = b = 0, \qquad c = 1$$

It is a self-adjoint equation and, therefore, the Green's function v can be obtained from

$$\frac{\partial^2 v}{\partial x\,\partial y} + v = 0$$

subject to

$$\frac{\partial v}{\partial x} = 0 \qquad \text{on } y = \eta$$

$$\frac{\partial v}{\partial y} = 0 \qquad \text{on } x = \xi$$

$$v = 1 \qquad \text{at } x = \xi,\ y = \eta$$

Let

$$\phi^k = a(x-\xi)(y-\eta)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial x}$$

But

$$k\phi^{k-1}\frac{\partial \phi}{\partial x} = a(y-\eta)$$

Therefore,

$$\frac{\partial \phi}{\partial x} = \frac{a}{k}\phi^{1-k}(y-\eta)$$

Thus,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \phi}\frac{a}{k}\phi^{1-k}(y-\eta)$$

$$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial \phi}\frac{a}{k}\phi^{1-k}(y-\eta)\right]$$

$$= \frac{a}{k}\left[\phi^{1-k}\frac{\partial v}{\partial \phi} + (1-k)\phi^{-k}\frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial y}(y-\eta) + \phi^{1-k}(y-\eta)\frac{\partial^2 v}{\partial \phi^2}\frac{\partial \phi}{\partial y}\right]$$

However,

$$\frac{\partial \phi}{\partial y} = \frac{a}{k}\phi^{1-k}(x-\xi)$$

Therefore,

$$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{a}{k}\left[\phi^{1-k}\frac{\partial v}{\partial \phi} + (1-k)\phi^{-k}(x-\xi)(y-\eta)\frac{a}{k}\phi^{1-k}\frac{\partial v}{\partial \phi} + \phi^{1-k}(y-\eta)\frac{\partial^2 v}{\partial \phi^2}\frac{a}{k}\phi^{1-k}(x-\xi)\right]$$

Hence,

$$\frac{\partial^2 v}{\partial x\,\partial y} + v = 0$$

gives

$$\frac{a}{k}\left[\frac{\phi^k}{k}\phi^{2(1-k)}\frac{d^2v}{d\phi^2} + \frac{\phi^{1-k}}{k}(1-k)\frac{dv}{d\phi} + \phi^{1-k}\frac{dv}{d\phi}\right] + v = 0$$

or

$$\frac{a}{k^2}\left(\phi^{2-k}\frac{d^2v}{d\phi^2} + \phi^{1-k}\frac{dv}{d\phi}\right) + v = 0$$

or

$$\phi^2 v'' + \phi v' + \frac{k^2}{a}\phi^k v = 0$$

Let $k = 2,\ a = 4.$ Then the above equation reduces to

$$\phi^2 v'' + \phi v' + \phi^2 v = 0 = v'' + \frac{1}{\phi}v' + v \quad \text{(Bessel's equation)}$$

Its solution is known to be of the form

$$v = J_0(\phi) = J_0\sqrt{2(x-\xi)(y-\eta)}$$

which is the desired Green's function.

1.6 LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An nth order linear PDE with constant coefficients can be written in the form

$$a_0\frac{\partial^n u}{\partial x^n} + a_1\frac{\partial^n u}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n u}{\partial x^{n-2}\partial^2 y} + \cdots + a_n\frac{\partial^n u}{\partial y^n}$$
$$= f(x, y) \tag{1.60}$$

where $a_0,\ a_1,\ \ldots,\ a_n$ are constants; u is the dependent variable; x and y are independent variables. Introducing the standard differential operator notation, such as $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$, the above equation can be rewritten as

$$F(D, D')u = (a_0D^n + a_1D^{n-1}D' + a_2D^{n-2}D'^2 + \cdots + a_nD'^n)u = f(x, y) \tag{1.61}$$

It can also be written in more compact form as

$$F(D, D')u = \sum_i \sum_j C_{ij} D^i D'^j \tag{1.62}$$

where C_{ij} are constants, $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}, D^n = \frac{\partial^n}{\partial x^n}, D'^n = \frac{\partial^n}{\partial y^n}$, etc.

As in the case of linear ODE with constant coefficients, the complete solution of Eq. (1.61) consists of two parts:

(i) the complementary function (CF), which is the most general solution of the equation $F(D, D')u = 0$, the one containing, n arbitrary functions, where n is the order of the DE.

(ii) the particular integral (PI), is a particular solution, which is free from arbitrary constants or functions of the equation $F(D, D')u = f(x, y)$.

The complete solution of Eq. (1.61) is then

$$u = \text{CF} + \text{PI} \tag{1.63}$$

It may be noted that, if all the terms on the left-hand side of Eq. (1.61) are of the same order, it is said to be a homogeneous equation otherwise, it is a non-homogeneous equation. Now, we shall study few basic theorems as is the case in ODEs.

Theorem 1.1 If u_{CF} and u_{PI} are respectively the complementary function and particular integral of a linear PDE, then their sum $(u_{\text{CF}} + u_{\text{PI}})$ is a general solution of the given PDE.

Proof Since $$F(D, D')u_{\text{CF}} = 0,$$

and $$F(D, D')u_{\text{PI}} = f(x, y),$$

we arrive at

$$F(D, D')u_{\text{CF}} + F(D, D')u_{\text{PI}} = f(x, y).$$

showing that $(u_{\text{CF}} + u_{\text{PI}})$ is in fact a general solution of Eq. (1.61). Hence proved.

Theorem 1.2 If $u_1, u_2, \ldots, u_n$ are the solutions of the homogeneous PDE: $F(D, D')u = 0$, then $\sum_{i=1}^{n} C_i u_i$, where C_i are arbitrary constants, is also a solution.

Proof Since we observe that

$$F(D, D')(C_i u_i) = C_i F(D, D')u_i$$

and $$F(D, D')\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} F(D, D')v_i$$

For any set of functions v_i, we find at once,

$$F(D, D')\sum_{i=1}^{n} C_i u_i = \sum_{i=1}^{n} F(D, D')(C_i u_i)$$

$$= \sum_{i=1}^{n} C_i F(D, D') u_i = 0$$

Hence proved.

We shall now classify linear differential operator $F(D, D')$ into reducible and irreducible types, in the sense that $F(D, D')$ is reducible if it can be expressed as the product of linear factors of the form $(aD + bD' + c)$, where a, b and c are constants, otherwise $F(D, D')$ is irreducible. For example, the operator

$$F(D, D')u = (D^2 - D'^2 + 3D + 2D' + 2)u$$
$$= (D + D' + 1)(D - D' + 2)u$$

is reducible. While the operator $F(D, D')u = (D^2 - D')$, is irreducible, due to the fact that it cannot be factored into linear factors.

1.6.1 General Method for Finding CF of Reducible Non-homogeneous Linear PDE

The general strategy adopted for finding the CF of reducible equations is stated in the following theorems:

Theorem 1.3 If the operator $F(D, D')$ is reducible that is, if $(a_i D + b_i D' + c_i)$ is a factor of $F(D, D')$ and $\phi_i(\xi)$ is an arbitrary function of a single variable ξ, then, if $a_i \neq 0$,

$$u_i = \exp\left(-\frac{c_i}{a_i} x\right) \phi_i(b_i x - a_i y) \tag{1.64}$$

is a solution of the equation $F(D, D')u = 0$ (Sneddon, 1986).

Proof Using product rule of differentiation, Eq. (1.64) gives

$$Du_i = \left(-\frac{c_i}{a_i}\right)\exp\left(-\frac{c_i}{a_i} x\right)\phi_i(b_i x - a_i y) + b_i \exp\left(-\frac{c_i}{a_i} x\right)\phi_i'(b_i x - a_i y)$$

$$= -\frac{c_i}{a_i} u_i + b_i \exp\left(-\frac{c_i}{a_i} x\right)\phi'(b_i x - a_i y).$$

Similarly, we get

$$D'u_i = -a_i \exp\left(-\frac{c_i}{a_i} x\right)\phi'(b_i x - a_i y).$$

Thus, we observe that

$$(a_iD + b_iD' + c_i)u_i = 0 \tag{1.65}$$

That is, if the operator $F(D, D')$ is reducible, the order in which the linear factors appear is immaterial. Thus, if

$$F(D, D')u_i = \left\{ \prod_{j=1}^{n}{}' (a_jD + b_jD' + c_j) \right\} (a_iD + b_iD' + c_i)u_i \tag{1.66}$$

where, the prime on the product indicates that the factor corresponding to $i = j$ is omitted. Combining Eqs. (1.65) and (1.66), we arrive at the result $F(D, D')u_i = 0$. Hence proved.

It may be noted that if no two factors of Eq. (1.65) are linearly independent, then the general solution of Eq. (1.66) is the sum of the general solutions of the equations of the form (1.65). For illustration, we consider the following examples:

EXAMPLE 1.16 Solve the following equation $(D^2 + 2DD' + D'^2 - 2D - 2D')u = 0$.

Solution Observe that the given PDE is non-homogeneous and can be factored as

$$(D^2 + 2DD' + D'^2 - 2D - 2D')u = (D + D')(D + D' - 2)u.$$

Using the result of Theorem 1.3, we get the general solution or the CF as

$$u_{\text{CF}} = \phi_1(x - y) + e^{2x}\phi_2(x - y).$$

On similar lines, we can also establish the following result:

Theorem 1.4 (Sneddon, 1986) Let $(b_i D' + c_i)$ is a factor of $F(D, D')u$, and $\phi_i(\xi)$ is an arbitrary function of a single variable ξ, then, if $b_i \neq 0$, we have

$$u_i = \exp\left(-\frac{c_i}{b_i}y\right)\phi_i(b_ix) \tag{1.67}$$

as a solution of the equation $F(D, D')u = 0$.

Proof Suppose, the factorisation of $F(D, D') = 0$ gives rise to a multiple factors of the form $(a_iD + b_iD' + c_i)^n$, the solution of $F(D, D')u = 0$ can be obtained by the application of Theorems 1.3 and 1.4. For example, let us find the solution of

$$(a_iD + b_iD' + c_i)^2u = 0 \tag{1.68}$$

We set,

$$(a_iD + b_iD' + c_i)u = U,$$

then, Eq. (1.68) becomes

$$(a_iD + b_iD' + c_i)U = 0.$$

Using Theorem 1.3, its solution is found to be

$$U = \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(b_ix - a_iy).$$

Further, assume that $a_i \neq 0$, now, in order to find u, we have to solve

$$(a_iD + b_iD' + c_i)u = \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(b_ix - a_iy).$$

This is a first order PDE. Using the Lagrange's method (Section 0.8), its auxiliary equations are

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{du}{-c_iu + \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(b_ix - a_iy)} \tag{1.69}$$

One solution of which is given by

$$b_ix - a_iy = \lambda \text{ (a constant)} \tag{1.70}$$

Substituting this solution into the first and third of the auxiliary equations, we obtain

$$\frac{dx}{a_i} = \frac{du}{-c_iu + \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(\lambda)}$$

or

$$\frac{du}{dx} + \frac{c_i}{a_i}u = \frac{1}{a_i}\exp\left(-\frac{c_i}{a_i}x\right)\phi_i(\lambda).$$

This being an ODE, its solution can be readily written as

$$u\exp\left(\frac{c_i}{a_i}x\right) = \frac{1}{a_i}x\phi_i(\lambda) + \mu \text{ (constant)}$$

or

$$u = \frac{1}{a_i}[x\phi_i(\lambda) + \mu]\exp\left(-\frac{c_i}{a_i}x\right) \tag{1.71}$$

Thus, the solution of Eq. (1.68) is given by

$$u = [x\phi_i(b_ix - a_iy) + \psi_i(b_ix - a_iy)]e^{-c_i/a_ix} \tag{1.72}$$

where ϕ_i and ψ_i are arbitrary functions.

In general, if there are n, multiple factors of the form $(a_iD + b_iD' + c_i)$, then the solution of $(a_iD + b_iD' + c_i)^n u = 0$ can be written as

$$u = \exp\left(-\frac{c_i}{a_i}x\right)\left[\sum_{j=1}^{n} x^{j-1}\phi_{ij}(b_ix - a_iy)\right] \tag{1.73}$$

Here follows an example for illustration.

EXAMPLE 1.17 Find the solution of the equation $(2D - D' + 4)(D + 2D' + 1)^2 u = 0$.

Solution The complementary function (CF) corresponding to the factor $(2D - D' + 4)$ is $e^{-2x}\,\phi(-x - 2y)$. Similarly, CF corresponding to

$$(D + 2D' + 1)^2 \text{ is } e^{-x}[\psi_1(2x - y) + x\psi_2(2x - y)].$$

Thus, the CF for the given PDE is given by

$$u = e^{-2x}\phi(x + 2y) + e^{-x}[\psi_1(2x - y) + x\psi_2(2x - y)],$$

where, ϕ, ψ_1, ψ_2 are arbitrary functions.

1.6.2 General Method to Find CF of Irreducible Non-homogeneous Linear PDE

If the operator $F(D, D')$ is irreducible, we can find the complementary function, containing as many arbitrary functions as we wish by a method which is stated in the following theorem:

Theorem 1.5 The solution of irreducible PDE $F(D, D')u = 0$ is

$$u = \sum_{i=1}^{\infty} c_i \exp(a_i x + b_i y) \tag{1.74}$$

Proof Let us assume the solution of $F(D, D')u = 0$ in the form, $u = ce^{ax+by}$, where a, b and c are constants to be determined. Then, we have

$$D^i u = ca^i e^{ax+by},\quad D'^j u = cb^j\, e^{ax+by},$$

$$D^i D'^j u = ca^i b^j e^{ax+by}$$

Thus, $\quad F(D, D')u = 0$ yields

$$c[F(a, b)]e^{ax+by} = 0$$

where c is an arbitrary constant, not zero, holds true iff

$$F(a, b) = 0, \tag{1.75}$$

indicating that there exists infinite pair of values (a_i, b_i) satisfying Eq. (1.75). Hence,

$$u = \sum_{i=1}^{\infty} c_i\, e^{a_i x + b_i y} \tag{1.76}$$

is a solution of irreducible equation

$$F(D, D')u = 0, \tag{1.77}$$

provided $\quad F(a_i, b_i) = 0 \quad (1.78)$

It may be noted that this method is applicable even for reducible equations. Here follows an example for illustration:

EXAMPLE 1.18 Solve the following equation $(2D^2 - D'^2 + D)u = 0$.

Solution The given equation is an irreducible non-homogeneous PDE. Using the result of Theorem 1.5, it follows immediately that

$$u = u_{CF} = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$$

where a_i, b_i are related by $F(a_i, b_i) = 0$.

That is,

$$2a_i^2 - b_i^2 + a_i = 0$$

which gives $b_i^2 = 2a_i^2 + a_i$.

1.6.3 Methods for Finding the Particular Integral (PI)

To find the PI of Eq. (1.61), we rewrite the same in the form

$$u = \frac{1}{F(D, D')} f(x, y) \tag{1.79}$$

Very often, the operator $F^{-1}(D, D')$ can be expanded, using binomial theorem and interpret the operators D^{-1}, $(D')^{-1}$ as integrations. That is,

$$D^{-1} f(x, y) = \frac{1}{D} f(x, y) = \int_{y \text{ constant}} f(x, y) dx,$$

and

$$\frac{1}{D'} f(x, y) = \int_{x \text{ constant}} f(x, y) dy.$$

We present below different cases for finding the PI, depending on the nature of $f(x, y)$.

Case I Let $f(x, y) = \exp(ax + by)$, then

$$\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by} \tag{1.80}$$

By direct differentiation, we find $D^i D'^j e^{ax+by} = a^i b^j e^{ax+by}$.

In other words,

$$F(D, D') e^{ax+by} = F(a, b) e^{ax+by},$$

that is,

$$e^{ax+by} = F(a, b) \frac{1}{F(D, D')} e^{ax+by}.$$

Dividing both sides by $F(a, b)$, we get

$$\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by},$$

provided $F(a, b) \neq 0$.

Case II Let $f(x, y) = \sin(ax + by)$ or $\cos(ax + by)$, where a and b are constants, then, since

$$D^2 \sin(ax + by) = -a^2 \sin(ax + by)$$
$$DD' \sin(ax + by) = -ab \sin(ax + by)$$

$$D'^2 \sin(ax + by) = -b^2 \sin(ax + by)$$

We notice that

$$\frac{1}{F(D, D')} \sin(ax + by)$$

is obtained by setting, $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$ provided $F(D, D') \neq 0$. Thus,

$$F(D^2, DD', D'^2) \sin(ax + by) = F(-a^2, -ab, -b^2) \sin(ax + by)$$

or
$$\frac{1}{F(D^2, DD', D'^2)} \sin(ax + by) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by) \tag{1.81}$$

Similarly,

$$\frac{1}{F(D^2, DD', D'^2)} \cos(ax + by) = \frac{1}{F(-a^2, -ab, -b^2)} \cos(ax + by) \tag{1.82}$$

Case III Let $f(x, y)$ is of the form $x^p y^q$, where p and q are positive integers. Then, the PI can be obtained by expanding $F(D, D')$ in ascending powers of D or D'.

Case IV Let $f(x, y)$ is of the form $e^{ax+by}\, \phi(x, y)$.

Then,

$$F(D, D')[e^{ax+by}\, \phi(x, y)] = e^{ax+by}\, F(D + a, D' + b)\, \phi(x, y).$$

Let us recall Leibnitz's theorem for the nth derivative of a product of functions; thus, we have

$$D^n[e^{ax}\phi] = \sum_{r=0}^{n} nc_r (D^r e^{ax})(D^{n-r}\phi)$$

$$= e^{ax}\left(\sum_{r=0}^{n} nc_r a^r D^{n-r}\phi\right)$$

$$= e^{ax}(D + a)^n \phi$$

Applying this result, we arrive at

$$F(D, D')[e^{ax+by}\phi(x, y)] = e^{ax+by}\, F(D + a, D' + b)\, \phi(x, y) \tag{1.83}$$

Hence, it follows that

$$\frac{1}{F(D, D')}[e^{ax+by}\phi(x, y)] = e^{ax+by} \frac{1}{F(D + a, D' + b)} \phi(x, y)$$

$$= e^{ax} \cdot \frac{1}{F(D + a, D')} e^{by}\phi(x, y)$$

$$= e^{by} \cdot \frac{1}{F(D, D' + b)} e^{ax}\phi(x, y) \tag{1.84}$$

For illustration of various cases to find PI, here follows several examples:

EXAMPLE 1.19 Solve the equation $(D^2 + 3DD' + 2D'^2)u = x + y$.

Solution The given PDE is reducible, since it can be factored as

$$(D + D')(D + 2D')u = x + y \tag{1}$$

Therefore,

$$\text{CF} = \phi_1(x - y) + \phi_2(2x - y) \tag{2}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

The PI of the given PDE is obtained as follows:

$$\text{PI} = \frac{1}{(D^2 + 3DD' + 2D'^2)}(x + y)$$

$$= \frac{1}{D^2\left(1 + 3\dfrac{D'}{D} + 2\dfrac{D'^2}{D^2}\right)}(x + y)$$

$$= \frac{1}{D^2}\left[1 + \left(3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)\right]^{-1}(x + y)$$

$$= \frac{1}{D^2}\left[1 - 3\frac{D'}{D} - \cdots\right](x + y)$$

$$= \frac{1}{D^2}\left[x + y - \frac{3}{D}(1)\right]$$

$$= \frac{1}{D^2}[y - 2x] = y\frac{x^2}{2} - \frac{x^3}{3} \tag{3}$$

Adding Eqs. (2) and (3), we have the complete solution of the given PDE as

$$u = \phi_1(y - x) + \phi_2(y - 2x) + y\frac{x^2}{2} - \frac{x^3}{3}\,.$$

EXAMPLE 1.20 Solve the following equation $(D - D' - 1)(D - D' - 2)u = e^{2x-y} + x$.

Solution The CF of the given PDE is

$$\text{CF} = e^x\phi_1(x + y) + e^{2x}\phi_2(x + y) \tag{1}$$

The PI corresponding to the term e^{2x-y} is

$$= \frac{1}{(2 + 1 - 1)(2 + 1 - 2)}e^{2x-y} = \frac{1}{2}e^{2x-y} \tag{2}$$

while the PI corresponding to the term x is

$$= \frac{1}{2}(1 - D + D')^{-1}\left(1 - \frac{D}{2} + \frac{D'}{2}\right)^{-1} x$$

$$= \frac{1}{2}\left[(1 + D - D' + \cdots)\left(1 + \frac{D}{2} - \frac{D'}{2} + \cdots\right)\right] x$$

$$= \frac{1}{2}(1 + D - D')\left(x + \frac{1}{2}\right) = \frac{1}{2}\left(x + \frac{3}{2}\right) \quad (3)$$

Combining Eqs. (1), (2) and (3), the complete solution of the given PDE is found to be

$$u = e^x \phi_1(x + y) + e^{2x}\phi_2(x + y) + \frac{1}{2}e^{2x-y} + \frac{x}{2} + \frac{3}{4}.$$

EXAMPLE 1.21 Solve the following equation

$$(D^2 + 2DD' + D'^2 - 2D - 2D')u = \sin(x + 2y).$$

Solution The given PDE can be factored and rewritten as

$$(D + D')(D + D' - 2)u = \sin(x + 2y) \quad (1)$$

for which the CF is given by

$$\text{CF} = \phi_1(x - y) + e^{2x}\phi_2(x - y) \quad (2)$$

while

$$\text{PI} = \frac{1}{(D^2 + 2DD' + D'^2 - 2D - 2D')} \sin(x + 2y).$$

Setting $D^2 = -1$, $DD' = -2$, $D'^2 = -4$, we get

$$\text{PI} = -\frac{1}{(2D + 2D' + 9)} \sin(x + 2y)$$

$$= -\frac{[2(D + D') - 9]}{[4(D^2 + 2DD' + D'^2) - 81]} \sin(x + 2y)$$

$$= \frac{2(D + D') - 9}{117} \sin(x + 2y)$$

$$= \frac{1}{117}\ [2\cos(x + 2y) + 4\cos(x + 2y) - 9\sin(x + 2y)]$$

$$= \frac{1}{39}\ [2\cos(x + 2y) - 3\sin(x + 2y)] \quad (3)$$

Adding Eqs. (1), (2) and (3), we find that the complete solution of the given PDE as

$$u = \phi_1(x - y) + e^{2x}\phi_2(x - y) + \frac{1}{39}[2\cos(x + 2y) - 3\sin(x + 2y)].$$

EXAMPLE 1.22 Solve the following equation

$$(D^2 - DD')u = \cos x \cos 2y.$$

Solution The given PDE can be rewritten as

$$D(D - D')u = \cos x \cos 2y \tag{1}$$

Its CF is given by

$$\text{CF} = \phi_1(y) + \phi_2(y + x), \tag{2}$$

while its PI is given by

$$\text{PI} = \frac{1}{(D^2 - DD')}\cdot\frac{1}{2}[\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{(-1+2)}\cos(x+2y) + \frac{1}{(-1-2)}\cos(x-2y)\right]$$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y) \tag{3}$$

Hence, the complete solution of the given PDE is given by

$$u = \phi_1(y) + \phi_2(y + x) + \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y).$$

EXAMPLE 1.23 Find the solution of

$$(D^2 + DD' - 6D'^2)u = y \cos x.$$

Solution The given PDE can be rewritten as

$$(D + 3D')(D - 2D')u = y \cos x \tag{1}$$

Its CF is given by

$$\text{CF} = \phi_1(3x - y) + \phi_2(2x + y) \tag{2}$$

The PI of the given PDE is

$$\text{PI} = \frac{1}{(D+3D')}\cdot\frac{1}{(D-2D')}y\cos x \tag{3}$$

Applying the operator $\frac{1}{(D-2D')}$ first on $y \cos x$

$$(D - 2D')u = y \cos x$$

Its auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{du}{y \cos x}.$$

The first two members give

$$y + 2x = \lambda \text{ (constant).}$$

From the first and the third members, we have

$$du = y \cos x \, dx = (\lambda - 2x) \cos x \, dx,$$

on integration, we get

$$u = \int (\lambda - 2x) \cos x \, dx$$

where λ is to be replaced by $(y + 2x)$ after integration. Now, Eq. (3) gives

$$\text{PI} = \frac{1}{(D + 3D')}\left[(\lambda - 2x) \sin x - \int \sin x(-2) dx\right]$$

$$= \frac{1}{(D + 3D')}[(\lambda - 2x) \sin x - 2 \cos x]$$

$$= \frac{1}{(D + 3D')}[y \sin x - 2 \cos x]$$

$$= \int [(\lambda + 3x) \sin x - 2 \cos x] dx$$

$$= (\lambda + 3x)(-\cos x) + 3\int \cos x \, dx - 2 \sin x.$$

Now, replacing λ by $(y - 3x)$, we get

$$\text{PI} = -y \cos x + \sin x \tag{4}$$

Hence, the solution of the given PDE is found to be

$$u = \text{CI} + \text{PI}$$

$$= \phi_1(3x - y) + \phi_2(2x + y) - y \cos x + \sin x.$$

EXAMPLE 1.24 Show that a linear PDE of the type

$$\sum_i \sum_j a_{ij} \, x^i y^j \frac{\partial^{i+j} u}{\partial x^i \partial y^j} = f(x, y)$$

can be reduced to a one with constant coefficients by the substitution

$$\xi = \log x, \ \eta = \log y.$$

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{x}\frac{\partial u}{\partial \xi}$$

or
$$x\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}$$

That is,

$$x\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} = D \text{ (say)} \tag{1}$$

Therefore,

$$x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1}u}{\partial x^{n-1}}\right) = x^n\frac{\partial^n u}{\partial x^n} + (n-1)x^{n-1}\frac{\partial^{n-1}u}{\partial x^{n-1}}$$

or
$$x^n\frac{\partial^n u}{\partial x^n} = \left(x\frac{\partial}{\partial x} - n + 1\right)x^{n-1}\frac{\partial^{n-1}u}{\partial x^{n-1}} \tag{2}$$

By setting n = 2, 3, 4, … in Eq. (2), we obtain

$$x^2\frac{\partial^2 u}{\partial x^2} = (D-1)x\frac{\partial u}{\partial x} = D(D-1)u,$$

$$x^3\frac{\partial^3 u}{\partial x^3} = D(D-1)(D-2)u,$$

and so on. Similarly, we can show that

$$y\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} = D'u,$$

$$y^2\frac{\partial^2 u}{\partial y^2} = D'(D'-1)u,$$

and
$$xy\frac{\partial^2 u}{\partial x \partial y} = DD'u$$

and so on. Substituting these results into the given PDE, it becomes

$$F(D,\ D')u = f(e^{\xi},\ e^{\eta}) = f(\xi,\ \eta) \tag{3}$$

where,

$$D = \frac{\partial}{\partial \xi},\ D' = \frac{\partial}{\partial \eta}$$

Thus, Eq. (3) can be seen as a PDE with constant coefficients.

For illustration, here follows an example.

EXAMPLE 1.25 Solve the following PDE

$$(x^2D^2 + 2xy\ DD' + y^2D'^2)u = x^2y^2 \tag{1}$$

Solution Using the substitution

$\xi = \log x,\ \eta = \log y$ and using the notation

$$\frac{\partial}{\partial \xi} = D, \frac{\partial}{\partial \eta} = D'$$

respectively, the given PDE reduces to a PDE with constant coefficients, in the form

$$[D(D - 1) + 2DD' + D'(D' - 1)]u = e^{2\xi+2\eta}.$$

On rewriting, we have

$$(D + D')(D + D' - 1)u = e^{2\xi+2\eta} \tag{2}$$

The CF of Eq. (2) is given by

$$\text{CF} = \phi_1(\xi - \eta) + e^{\xi}\ \phi_2\ (\xi - \eta), \tag{3}$$

while, its PI is obtained as

$$\text{PI} = \frac{1}{(2+2)(2+2-1)} e^{2\xi+2\eta}$$

$$= \frac{1}{12} e^{2\xi+2\eta}. \tag{4}$$

Transforming back from (ξ, η) to (x, y), we find the complete solution of the given PDE as

$$u = \phi_1(\log x - \log y) + x\ \phi_2(\log x - \log y) + \frac{x^2y^2}{12}$$

or

$$u = \psi_1\left(\frac{x}{y}\right) + x\ \psi_2\left(\frac{x}{y}\right) + \frac{1}{12}x^2y^2.$$

1.7 HOMOGENEOUS LINEAR PDE WITH CONSTANT COEFFICIENTS

Equation (1.61) is said to be linear PDE of nth order with constant coefficients. It is also called *homogeneous*, because all the terms containing derivatives are of the same order. Now, Eq. (1.61) can be rewritten in operator notation as

$$[a_0D^n + a_1D^{n-1}\ D' + a_2D^{n-2}D'^2 + \cdots + a_nD'^n]u = F(D, D')u = f(x, y) \tag{1.85}$$

As in the case of ODE, the complete solution of Eq. (1.85) consists of the sum of CF and PI.

1.7.1 Finding the Complementary Function

Let us assume that the solution of the equation $F(D, D')u = 0$ in the form

$$u = \phi(y + mx) \tag{1.86}$$

Then,

$$D^i u = m^i \phi^i(y + mx),\ D'^j u = \phi^j(y + mx),$$

and $$D^i D'^j u = m^i \phi^{i+j}\ (y + mx).$$

Substituting these results into $F(D, D')u = 0$, we obtain

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n)\phi^n(y + mx) = 0,$$

which will be true, iff

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0 \tag{1.87}$$

This equation is called an *auxiliary equation* for $F(D, D')u = 0$.

Let $m_1, m_2, \ldots, m_n$ are the roots of Eq. (1.87). Depending on the nature of these roots, several cases arise:

Case I When the roots $m_1, m_2, \ldots, m_n$ are distinct. Corresponding to $m = m_1$, the CF is $u = \phi_1(y + m_1 x)$. Similarly, $u = \phi_2(y + m_2 x)$, $u = \phi_3(y + m_3 x)$, etc. are all complementary functions. Since, the given PDE is linear, using the principle of superposition, the CF of Eq. (1.85) can be written as

$$\text{CF} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

where, $\phi_1, \phi_2, \ldots, \phi_n$ are arbitrary.

Case II When some of the roots are repeated. Let two roots say m_1 and m_2 are repeated, and each equal to m. Consider the equation

$$(D - mD')(D - mD')u = 0.$$

Setting $(D - mD')u = z$, the above equation becomes

$$(D - mD')z = 0,$$

or $$\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = 0$$

which is of Lagrange's form. Writing down its auxiliary equations, we have

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}.$$

The first two members gives $y + mx$ = constant = a(say).

The third member yields z = constant.

Therefore, $z = \phi(y + mx)$ is a solution.

Substituting for z, we get

$$(D - mD')u = \phi(y + mx)$$

which is again in the Lagranges form, whose auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{du}{\phi(y+mx)}$$

which gives,

$$y + mx = \text{constant}$$

and $$u = x\phi(y + mx) + \text{constant}$$

Hence, the CF is

$$u = x\phi(y + mx) + \psi(y + mx).$$

In general, if the root m is repeated r times, then the CF is given by

$$u = x^{r-1}\phi_1(y + mx) + x^{r-2}\phi_2(y + mx) + \cdots + \phi_r(y + mx)$$

where, $\phi_1, \phi_2, \ldots, \phi_r$ are arbitrary.

For illustration, we consider the following couple of examples.

EXAMPLE 1.26 Solve the following PDE $(D^3 - 3D^2D' + 4D'^3)u = 0$.

Solution Observe that the given PDE is a linear homogeneous PDE. Dividing throughout by D' and denoting (D/D') by m, its auxiliary equation can be written as

$$m^3 - 3m^2 + 4 = (m + 1)(m - 2)^2 = 0.$$

Therefore, the roots of the auxiliary equation are –1, 2, 2. Thus,

$$\text{CF} = \phi_1(y - x) + \phi_2(y + 2x) + x\,\phi_3(y + 2x).$$

EXAMPLE 1.27 Solve the following PDE

$$(D^3 + DD'^2 - 10D'^3)u = 0.$$

Solution Observe that the given equation is a linear homogeneous PDE. Denoting (D/D') by m. The auxiliary equation for the given PDE is given by

$$m^3 + m - 10 = (m - 2)(m^2 + 2m + 5) = 0.$$

Its roots are: 2, $(-1 + 2i)$, $(-1 - 2i)$.

Hence, the required CF is found to be

$$u = \phi_1(y + 2x) + \phi_2(y - x + 2ix) + \phi_3(y - x - 2ix).$$

1.7.2 Methods for Finding the PI

Methods for finding the PI, in the case of linear homogeneous PDE's are, similar to the one's developed in the Section 1.6 for linear non-homogeneous PDEs. That is, the PI for the equation

$$F(D, D')u = f(x, y)$$

is obtained from

$$u_{\text{PI}} = \text{PI} = \frac{1}{F(D, D')} f(x, y).$$

However, when the above stated methods fail we have a general method, which is applicable whatever may be the form of $f(x, y)$, and is presented below:

We have already assumed that $F(D, D')$ can be factorised, in general, say into n linear factors. Thus,

$$\text{PI} = \frac{1}{F(D, D')} f(x, y)$$

$$\text{PI} = \frac{1}{(D - m_1 D')(D - m_2 D') \ldots (D - m_n D')} f(x, y)$$

$$= \frac{1}{(D - m_1 D')} \cdot \frac{1}{(D - m_2 D')} \cdots \frac{1}{(D - m_n D')} f(x, y)\,.$$

In general, to evaluate

$$\frac{1}{(D - mD')} f(x, y),$$

we consider the equation

$$(D - mD')u = f(x, y)$$

or

$$p - mq = f(x, y) \qquad \text{(Lagrange's form)}$$

for which the auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{du}{f(x, y)}\,.$$

Its first two members, yield

$$y + mx = c \text{ (constant)}$$

The first and last members gives us

$$du = f(x, y)dx = f(x, c - mx).$$

On integration, we get

$$u = \int f(x, c - mx)dx$$

or

$$\frac{1}{(D - mD')} f(x, y) = \int f(x, c - mx)dx$$

After integration, we shall immediately replace c by $(y + mx)$. Applying this procedure repeatedly, we can find the PI for the given PDE. For illustration, we consider the following examples:

EXAMPLE 1.28 Solve the following PDE

$$(D^2 - 4DD' + 4D'^2)u = e^{2x+y}.$$

Solution The given equation is a linear homogeneous PDE. Its auxiliary equation can be written as

$$m^2 - 4m + 4 = (m - 2)^2 = 0.$$

In this example, the roots are repeated and they are 2, 2. The complementary function and particular integral are obtained as

$$\text{CF} = \phi_1(y + 2x) + x\phi_2(y + 2x) \tag{1}$$

and

$$\text{PI} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

If we set $D = 2$, $D' = 1$, we observe that $F(D, D') = 0$, which is a failure case. Therefore, we shall adopt the general method to find PI. Now, noting that

$$\frac{1}{(D - mD')} f(x, y) = \int f(x, c - mx)dx.$$

$$\text{PI} = \frac{1}{(D - 2D')} \cdot \frac{1}{(D - 2D')} e^{2x+y}$$

$$= \frac{1}{(D - 2D')} \int e^{(2x+c-2x)} dx$$

$$= \frac{1}{(D - 2D')} xe^c = \frac{1}{(D - 2D')} xe^{y+2x}$$

$$= \int xe^{(c-2x+2x)} dx = e^c \int xdx$$

$$= e^c \frac{x^2}{2} = \frac{x^2}{2} e^{y+2x} \tag{2}$$

From Eqs. (1) and (2), the complete solution of the given PDE is found to be

$$u = \phi_1(y + 2x) + x\phi_2(y + 2x) + \frac{x^2}{2} e^{y+2x}.$$

EXAMPLE 1.29 Find a real function $u(x, y)$, which reduces to zero when $y = 0$ and satisfy the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\pi(x^2 + y^2).$$

Solution In symbolic form, the given PDE can be written as

$$(D^2 + D'^2)u = -\pi(x^2 + y^2)$$

Its auxiliary equation is given by

$$(m^2 + 1) = 0, \text{ which gives } m = \pm i.$$

Hence,

$$\text{CF} = \phi_1(y + ix) + \phi_2(y - ix) \tag{1}$$

and

$$\text{PI} = \frac{-1}{(D^2 + D'^2)}\pi(x^2 + y^2) = -\frac{\pi}{D'^2}\frac{(x^2 + y^2)}{(1 + D^2/D'^2)}$$

$$= -\frac{\pi}{D'^2}\left(1 + \frac{D^2}{D'^2}\right)^{-1}(x^2 + y^2)$$

$$= -\frac{\pi}{D'^2}\left[1 - \frac{D^2}{D'^2} + \cdots\right](x^2 + y^2)$$

$$= -\frac{\pi}{D'^2}(x^2 + y^2) + \frac{2\pi}{D'^4}$$

$$= -\frac{\pi}{D'}\left(x^2 y + \frac{y^3}{3}\right) + \frac{2\pi}{D'^3}y$$

or

$$\text{PI} = -\pi\left(\frac{x^2 y^2}{2} + \frac{y^4}{12}\right) + \frac{2\pi}{D'^2}\frac{y^2}{2}$$

$$= -\pi\left(\frac{x^2 y^2}{2} + \frac{y^4}{12}\right) + 2\pi\frac{y^4}{24}$$

$$= -\frac{\pi}{2}x^2 y^2 \tag{2}$$

Hence, the complete solution of the given PDE is found to be

$$u = \phi_1(y + ix) + \phi_2(y - ix) - \frac{\pi}{2}x^2 y^2 \tag{3}$$

Finally, the real function satisfying the given PDE is given by

$$u = -\frac{\pi}{2}x^2 y^2 \tag{4}$$

which of course $\to 0$ as $y \to 0$.

EXERCISES

1. Find the region in the xy-plane in which the following equation is hyperbolic:

$$[(x - y)^2 - 1]u_{xx} + 2u_{xy} + [(x - y)^2 - 1]u_{yy} = 0$$

2. Find the families of characteristics of the PDE

$$(1-x^2)u_{xx} - u_{yy} = 0$$

in the elliptic and hyperbolic cases.

3. Reduce the following PDE to a canonical form

$$u_{xx} + xyu_{yy} = 0$$

4. Classify and reduce the following equations to a canonical form:

(a) $y^2u_{xx} - x^2u_{yy} = 0, \quad x > 0, \quad y > 0.$

(b) $u_{xx} + 2u_{xy} + u_{yy} = 0.$

(c) $e^x u_{xx} + e^y u_{yy} = u.$

(d) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0.$

(e) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2.$

5. Reduce the following equation to a canonical form and hence solve it:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

6. If $L(u) = c^2u_{xx} - u_{tt}$, then show that its adjoint operator is given by

$$L^* = c^2v_{xx} - v_{tt}$$

7. Determine the adjoint operator L^* corresponding to

$$L(u) = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu$$

where A, B, C, D, E and F are functions of x and y only.

8. Find the solution of the following Cauchy problem

$$u_{xy} = F(x, y)$$

given

$$u = f(x), \qquad \frac{\partial u}{\partial n} = g(x) \quad \text{on the line } y = x$$

using Riemann's method which is of the form

$$u(x_0, y_0) = \frac{1}{2}[f(x_0) + f(y_0)] + \frac{1}{\sqrt{2}}\int_{x_0}^{y_0} g(x)\,dx - \iint_{\mathbb{R}} F(x, y)\,dx\,dy$$

where $\mathbb{R}$ is the triangular region in the xy-plane bounded by the line $y = x$ and the lines $x = x_0$, $y = y_0$ through (x_0, y_0).

9. The characteristics of the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} + \cos^2 x\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 0$$

when it is of hyperbolic type are… and… (GATE-Maths, 1997)

10. Using $\eta = x + y$ as one of the transformation variable, obtain the canonical form of

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(GATE-Maths, 1998)

Choose the correct answer in the following questions (11–14):

11. The PDE

$$y^3 u_{xx} - (x^2 - 1)u_{yy} = 0 \quad \text{is}$$

(A) parabolic in $\{(x, y) : x < 0\}$

(B) hyperbolic in $\{(x, y) : y > 0\}$

(C) elliptic in $\mathbb{R}^2$

(D) parabolic in $\{(x, y) : x > 0\}$. (GATE-Maths, 1998)

12. The equation

$$x^2(y-1)z_{xx} - x(y^2 - 1)z_{xy} + y(y^2 - 1)z_{yy} + z_x = 0$$

is hyperbolic in the entire *xy*-plane except along

(A) *x*-axis
(B) *y*-axis
(C) A line parallel to *y*-axis
(D) A line parallel to *x*-axis.

(GATE-Maths, 2000)

13. The characteristic curves of the equation

$$x^2 u_{xx} - y^2 u_{yy} = x^2 y^2 + x, \quad x > 0 \text{ are}$$

(A) rectangular hyperbola
(B) parabola
(C) circle
(D) straight line.

(GATE-Maths, 2000)

14. Pick the region in which the following PDE is hyperbolic:

$$yu_{xx} + 2xyu_{xy} + xu_{yy} = u_x + u_y$$

(A) $xy \neq 1$
(B) $xy \neq 0$
(C) $xy > 1$
(D) $xy > 0$.

(GATE-Maths, 2003)

15. Solve the following PDEs:

(i) $(D - D' - 1)(D - D' - 2)u = 0$

(ii) $(D + D' - 1)(D + 2D' - 3)u = 0$

16. Solve the following PDE:

$$(D^2 + DD' + D + D' + 1)u = 0$$

17. Solve the following PDEs:

(i) $(D^2 - DD' + D' - 1)u = \cos(x + 2y)$

(ii) $D(D - 2D' - 3)u = e^{x+2y}$

(iii) $(2D + D' - 1)^2(D - 2D' + 2)^3 = 0$

18. Find the complete solution of the following PDEs:

(i) $(x^2D^2 - 2xy\, DD' + y^2D'^2 - xD + 3yD')u = 8(y/x)$

(ii) $D(D - 2D' - 3)u = e^{x+2y}$

19. Find the complete solution of the following PDEs:

(i) $(D^2 + 3DD' + 2D'^2)u = x + y$

(ii) $(D^2 + D'^2)u = \cos px \cos qy$

(iii) $(D^2 - DD' - 2D'^2)u = (y - 1)e^x$

(iv) $(4D^2 - 4DD' + D'^2)u = 16 \log (x + 2y)$

(v) $(D^2 - 3DD' + 2D'^2)u = e^{2x+3y} + \sin (x - 2y)$

CHAPTER 2

Elliptic Differential Equations

2.1 OCCURRENCE OF THE LAPLACE AND POISSON EQUATIONS

In Chapter 1, we have seen the classification of second order partial differential equation into elliptic, parabolic and hyperbolic types. In this chapter we shall consider various properties and techniques for solving Laplace and Poisson equations which are elliptic in nature.

Various physical phenomena are governed by the well known Laplace and Poisson equations. A few of them, frequently encountered in applications are: steady heat conduction, seepage through porous media, irrotational flow of an ideal fluid, distribution of electrical and magnetic potential, torsion of prismatic shaft, bending of prismatic beams, distribution of gravitational potential, etc. In the following two sub-sections, we shall give the derivation of Laplace and Poisson equations in relation to the most frequently occurring physical situation, namely, the gravitational potential.

2.1.1 Derivation of Laplace Equation

Consider two particles of masses m and m_1 situated at Q and P separated by a distance r as shown in Fig. 2.1. According to Newton's universal law of gravitation, the magnitude of the force, proportional to the product of their masses and inversely proportional to the square of the distance, between them is given by

$$F = G\frac{mm_1}{r^2} \tag{2.1}$$

where G is the gravitational constant. It $\mathbf{r}$ represents the vector $\overrightarrow{PQ}$, assuming unit mass at Q and $G = 1$, the force at Q due to the mass at P is given by

$$\mathbf{F} = -\frac{m_1\mathbf{r}}{r^3} = \nabla\left(\frac{m_1}{r}\right) \tag{2.2}$$

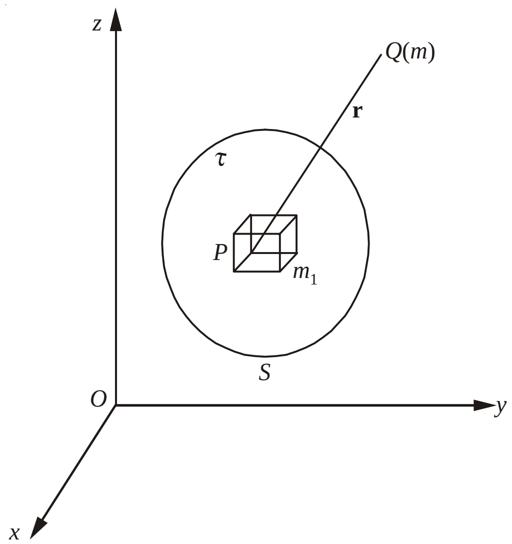

Fig. 2.1 Illustration of Newton's universal law of gravitation.

which is called the intensity of the gravitational force. Suppose a particle of unit mass moves under the attraction of a particle of mass m_1 at P from infinity up to Q; then the work done by the force $\mathbf{F}$ is

$$\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = \int_{\infty}^{r} \nabla\left(\frac{m_1}{r}\right) \cdot d\mathbf{r} = \frac{m_1}{r} \tag{2.3}$$

This is defined as the potential V at Q due to a particle at P and is denoted by

$$V = -\frac{m_1}{r} \tag{2.4}$$

From Eq. (2.2), the intensity of the force at P is

$$\mathbf{F} = -\nabla V \tag{2.5}$$

Now, if we consider a system of particles of masses $m_1, m_2, \ldots, m_n$ which are at distances $r_1, r_2, \ldots, r_n$ respectively, then the force of attraction per unit mass at Q due to the system is

$$\mathbf{F} = \sum_{i=1}^{n} \nabla \frac{m_i}{r_i} = \nabla \sum_{i=1}^{n} \frac{m_i}{r_i} \tag{2.6}$$

The work done by the force acting on the particle is

$$\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \frac{m_i}{r_i} = -V \tag{2.7}$$

Therefore,

$$\nabla^2 V = -\nabla^2 \sum_{i=1}^{n} \frac{m_i}{r_i} = \sum_{i=1}^{n} \nabla^2 \frac{m_i}{r_i} = 0, \qquad r_i \neq 0 \tag{2.8}$$

where

$$\nabla^2 = \text{div}\,\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplace operator.

In the case of continuous distribution of matter of density ρ in a volume τ, we have

$$V(x, y, z) = \iiint_{\tau} \frac{\rho(\xi, \eta, \zeta)}{r} d\tau \tag{2.9}$$

where $r = \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{1/2}$ and Q is outside the body. It can be verified that

$$\nabla^2 V = 0 \tag{2.10}$$

which is called the Laplace equation.

2.1.2 Derivation of Poisson Equation

Consider a closed surface S consisting of particles of masses $m_1, m_2, \ldots, m_n$. Let Q be any point on S. Let $\sum_{i=1}^{n} m_i = M$ be the total mass inside S, and let $g_1, g_2, \ldots, g_n$ be the gravity field at Q due to the presence of $m_1, m_2, \ldots, m_n$ respectively within S. Also, let $\sum_{i=1}^{n} g_i = g$, the entire gravity field at Q. Then, according to Gauss law, we have

$$\iint_S \mathbf{g} \cdot d\mathbf{S} = -4\pi GM \tag{2.11}$$

where $M = \iiint_{\tau} \rho\, d\tau, \rho$ is the mass density function and τ is the volume in which the masses are distributed throughout. Since the gravity field is conservative, we have

$$\mathbf{g} = \nabla V \tag{2.12}$$

where V is a scalar potential. But the Gauss divergence theorem states that

$$\iint_S \mathbf{g} \cdot d\mathbf{S} = \iiint_{\tau} \nabla \cdot \mathbf{g}\, d\tau \tag{2.13}$$

Also, Eq. (2.11) gives

$$\iint_S \mathbf{g} \cdot d\mathbf{S} = -4\pi G \iiint_{\tau} \rho\, d\tau \tag{2.14}$$

Combining Eqs. (2.13) and (2.14), we have

$$\iiint_\tau (\nabla \cdot \mathbf{g} + 4\pi G\rho)\, d\tau = 0$$

implying

$$\nabla \cdot \mathbf{g} = -4\pi G\rho = \nabla \cdot \nabla V$$

Therefore,

$$\nabla^2 V = -4\pi G\rho \tag{2.15}$$

This equation is known as *Poisson's equation*.

2.2 BOUNDARY VALUE PROBLEMS (BVPs)

The function V, whose analytical form we seek for the problems stated in Section 2.1, in addition to satisfying the Laplace and Poisson equations in a bounded region $\mathbb{R}$ in R^3, should also satisfy certain boundary conditions on the boundary $\partial\mathbb{R}$ of this region. Such problems are referred to as boundary value problems (BVPs) for Laplace and Poisson equations. We shall denote the set of all boundary points of $\mathbb{R}$ by $\partial\mathbb{R}$. By the closure of $\mathbb{R}$, we mean the set of all interior points of $\mathbb{R}$ together with its boundary points and is denoted by $\overline{\mathbb{R}}$. Symbolically, $\overline{\mathbb{R}}. = \mathbb{R} U \partial\mathbb{R}$.

If a function $f \in c^{(n)}$ (f "belongs to" $c^{(n)}$), then all its derivatives of order n are continuous. If it belongs to $c^{(0)}$, then we mean f is continuous.

There are mainly three types of boundary value problems for Laplace equation. If $f \in c^{(0)}$ and is specified on the boundary $\partial\mathbb{R}$ of some finite region $\mathbb{R}$, the problem of determining a function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within $\mathbb{R}$ and satisfying $\psi = f$ on $\partial\mathbb{R}$ is called the boundary value problem of first kind, or the Dirichlet problem. For example, finding the steady state temperature within the region $\mathbb{R}$ when no heat sources or sinks are present and when the temperature is prescribed on the boundary $\partial\mathbb{R}$, is a Dirichlet problem. Another example would be to find the potential inside the region $\mathbb{R}$ when the potential is specified on the boundary $\partial\mathbb{R}$. These two examples correspond to the interior Dirichlet problem.

Similarly, if $f \in c^{(0)}$ and is prescribed on the boundary $\partial\mathbb{R}$ of a finite simply connected region $\mathbb{R}$, determining a function $\psi(x, y, z)$ which satisfies $\nabla^2\psi = 0$ outside $\mathbb{R}$ and is such that $\psi = f$ on $\partial\mathbb{R}$, is called an exterior Dirichlet problem. For example, determination of the distribution of the potential outside a body whose surface potential is prescribed, is an exterior Dirichlet problem. The second type of BVP is associated with von Neumann. The problem is to determine the function $\psi(x, y, z)$ so that $\nabla^2\psi = 0$ within $\mathbb{R}$ while $\partial\psi/\partial n$ is specified at every point of $\partial\mathbb{R}$, where $\partial\psi/\partial n$ denotes the normal derivative of the field variable ψ. This problem is called the Neumann problem. If ψ is the temperature, $\partial\psi/\partial n$ is the heat flux representing the amount of heat crossing per unit volume per unit time along the normal direction, which is zero when insulated. The third type of BVP is concerned with the determination of the function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within $\mathbb{R}$, while a boundary condition of the form $\partial\psi/\partial n + h\psi = f$, where $h \geq 0$ is specified at every point of $\partial\mathbb{R}$. This is called a mixed BVP or Churchill's problem. If we assume Newton's law of cooling, the heat lost is $h\psi$, where ψ is the temperature difference from the surrounding medium and $h > 0$ is a constant depending on the medium. The heat f supplied at a point of the boundary is partly conducted into the medium and partly lost by radiation to the surroundings. Equating these amounts, we get the third boundary condition.

2.3 SOME IMPORTANT MATHEMATICAL TOOLS

Among the mathematical tools we employ in deriving many important results, the Gauss divergence theorem plays a vital role, which can be stated as follows: Let $\partial\mathbb{R}$ be a closed surface in the *xyz*-space and $\mathbb{R}$ denote the bounded region enclosed by $\partial\mathbb{R}$ in which $\mathbf{F}$ is a vector belonging to $c^{(1)}$ in $\mathbb{R}$ and continuous on $\mathbb{R}$. Then

$$\iint_{\partial\mathbb{R}} \mathbf{F}\cdot\hat{n}\, dS = \iiint_{\mathbb{R}} \nabla\cdot\mathbf{F}\, dV \tag{2.16}$$

where dV is an element of volume, dS is an element of surface area, and $\hat{n}$ the outward drawn normal.

Green's identities which follow from divergence theorem are also useful and they can be derived as follows: Let $\mathbf{F} = \mathbf{f}\phi$, where $\mathbf{f}$ is a vector function of position and ϕ is a scalar function of position. Then,

$$\iiint_{\mathbb{R}} \nabla\cdot(\mathbf{f}\phi)\, dV = \iint_{\partial\mathbb{R}} \hat{n}\cdot\mathbf{f}\phi\, dS$$

Using the vector identity

$$\nabla\cdot(\mathbf{f}\phi) = \mathbf{f}\cdot\nabla\phi + \phi\nabla\cdot\mathbf{f}$$

we have

$$\iiint_{\mathbb{R}} \mathbf{f}\cdot\nabla\phi\, dV = \iint_{\partial\mathbb{R}} \hat{n}\cdot\mathbf{f}\phi\, dS - \iiint_{\mathbb{R}} \phi\nabla\cdot\mathbf{f}\, dV$$

If we choose $\mathbf{f} = \nabla\psi$, the above equation yields

$$\iint_{\mathbb{R}} \nabla\phi \cdot \nabla\psi \, dV = \iint_{\partial\mathbb{R}} \phi\hat{n} \cdot \nabla\psi \, dS - \iiint_{\mathbb{R}} \phi\nabla^2\psi \, dV \tag{2.17}$$

Noting that $\hat{n} \cdot \nabla\psi$ is the derivative of ψ in the direction of $\hat{n}$, we introduce the notation

$$\hat{n} \cdot \nabla\psi = \partial\psi/\partial n$$

into Eq. (2.17) to get

$$\iiint_{\mathbb{R}} \nabla\phi \cdot \nabla\psi \, dV = \iint_{\partial\mathbb{R}} \phi \frac{\partial\psi}{\partial n} dS - \iiint_{\mathbb{R}} \phi\nabla^2\psi \, dV \tag{2.18a}$$

This equation is known as *Green's first identity*. Of course, it is assumed that both ϕ and ψ possess continuous second derivatives.

Interchanging the role of ϕ and ψ, we obtain from relation (2.18a) the equation

$$\iiint_{\mathbb{R}} \nabla\psi \cdot \nabla\phi \, dV = \iint_{\partial\mathbb{R}} \psi \frac{\partial\phi}{\partial n} dS - \iiint_{\mathbb{R}} \psi\nabla^2\phi \, dV \tag{2.18b}$$

Now, subtracting Eq. (2.18b) from Eq. (2.18a), we get

$$\iiint_{\mathbb{R}} (\phi\nabla^2\psi - \psi\nabla^2\phi) \, dV = \iint_{\partial\mathbb{R}} \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS \tag{2.19}$$

This is known as *Green's second identity*. If we set $\phi = \psi$ in Eq. (2.18a) we get

$$\iiint_{\mathbb{R}} (\nabla\phi)^2 \, dV = \iint_{\partial\mathbb{R}} \phi \frac{\partial\phi}{\partial n} dS - \iiint_{\mathbb{R}} \phi\nabla^2\phi \, dV \tag{2.20}$$

which is a special case of Green's first identity.

2.4 PROPERTIES OF HARMONIC FUNCTIONS

Solutions of Laplace equation are called harmonic functions which possess a number of interesting properties, and they are presented in the following theorems.

Theorem 2.1 If a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere.

Proof If ϕ is a harmonic function, then $\nabla^2\phi = 0$ in $\mathbb{R}$. Also, if $\phi = 0$ on $\partial\mathbb{R}$, we shall show that $\phi = 0$ in $\overline{\mathbb{R}} = \mathbb{R} U \partial\mathbb{R}$. Recalling Green's first identity, i.e., Eq. (2.20), we get

$$\iiint_{\mathbb{R}} (\nabla\phi)^2 \, dV = \iint_{\partial\mathbb{R}} \phi \frac{\partial\phi}{\partial n} dS - \iiint_{\mathbb{R}} \phi\nabla^2\phi \, dV$$

and using the above facts we have, at once, the relation

$$\iiint_{\mathbb{R}} (\nabla\phi)^2\, dV = 0$$

Since $(\nabla\phi)^2$ is positive, it follows that the integral will be satisfied only if $\nabla\phi = 0$. This implies that ϕ is a constant in $\mathbb{R}$. Since ϕ is continuous in $\overline{\mathbb{R}}$ and ϕ is zero on $\partial\mathbb{R}$, it follows that $\phi = 0$ in $\mathbb{R}$.

Theorem 2.2 If ϕ is a harmonic function in $\mathbb{R}$ and $\partial\phi/\partial n = 0$ on $\partial\mathbb{R}$, then ϕ is a constant in $\overline{\mathbb{R}}$.

Proof Using Green's first identity and the data of the theorem, we arrive at

$$\iiint_{\mathbb{R}} (\nabla\phi)^2\, dV = 0$$

implying $\nabla\phi = 0$, i.e., ϕ is a constant in $\mathbb{R}$. Since the value of ϕ is not known on the boundary $\partial\mathbb{R}$ while $\partial\phi/\partial n = 0$, it is implied that ϕ is a constant on $\partial\mathbb{R}$ and hence on $\overline{\mathbb{R}}$.

Theorem 2.3 If the Dirichlet problem for a bounded region has a solution, then it is unique.

Proof If ϕ_1 and ϕ_2 are two solutions of the interior Dirichlet problem, then

$$\nabla^2\phi_1 = 0 \quad \text{in } \mathbb{R}; \qquad \phi_1 = f \quad \text{on } \partial\mathbb{R}$$

$$\nabla^2\phi_2 = 0 \quad \text{in } \mathbb{R}; \qquad \phi_2 = f \quad \text{on } \partial\mathbb{R}$$

Let $\psi = \phi_1 - \phi_2$. Then

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 \qquad \text{in } \mathbb{R};$$

$$\psi = \phi_1 - \phi_2 = f - f = 0 \qquad \text{on } \partial\mathbb{R}$$

Therefore,

$$\nabla^2\psi = 0 \text{ in } \mathbb{R}, \qquad \psi = 0 \quad \text{on } \partial\mathbb{R}$$

Now using Theorem 2.1, we obtain $\psi = 0$ on $\overline{\mathbb{R}}$, which implies that $\phi_1 = \phi_2$. Hence, the solution of the Dirichlet problem is unique.

Theorem 2.4 If the Neumann problem for a bounded region has a solution, then it is either unique or it differs from one another by a constant only.

Proof Let ϕ_1 and ϕ_2 be two distinct solutions of the Neumann problem. Then we have

$$\nabla^2\phi_1 = 0 \quad \text{in} \quad \mathbb{R}; \qquad \frac{\partial \phi_1}{\partial n} = f \quad \text{on } \partial\mathbb{R},$$

$$\nabla^2\phi_2 = 0 \quad \text{in} \quad \mathbb{R}; \qquad \frac{\partial \phi_2}{\partial n} = f \quad \text{on } \partial\mathbb{R}$$

Let $\psi = \phi_1 - \phi_2$. Then

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 \quad \text{in} \quad \mathbb{R}$$

$$\frac{\partial \psi}{\partial n} = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n} = 0 \qquad \text{on } \partial\mathbb{R}$$

Hence from Theorem 2.2, ψ is a constant on $\overline{\mathbb{R}}$, i.e., $\phi_1 - \phi_2$ = constant. Therefore, the solution of the Neumann problem is not unique. Thus, the solutions of a certain Neumann problem can differ from one another by a constant only.

2.4.1 The Spherical Mean

Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$ and let $P(x, y, z)$ be any point in $\mathbb{R}$. Also, let $S(P, r)$ represent a sphere with centre at P and radius r such that it lies entirely within the domain $\mathbb{R}$ as depicted in Fig. 2.2. Let u be a continuous function in $\mathbb{R}$. Then the spherical mean of u denoted by $\bar{u}$ is defined as

$$\bar{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q)\, dS \tag{2.21}$$

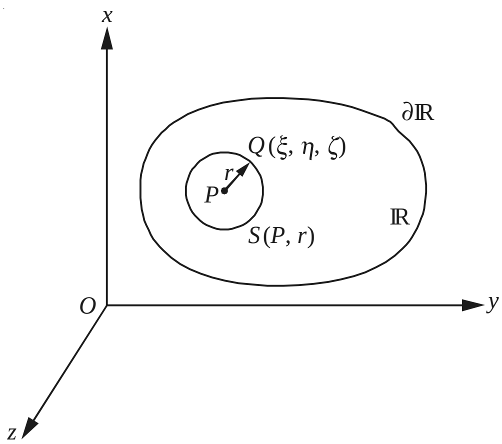

Fig. 2.2 Spherical mean.

where $Q(\xi, \eta, \zeta)$ is any variable point on the surface of the sphere $S(P, r)$ and dS is the surface element of integration. For a fixed radius r, the value $\bar{u}(r)$ is the average of the values of u taken over the sphere $S(P, r)$, and hence it is called the spherical mean. Taking the origin at P, in terms of spherical polar coordinates, we have

$$\xi = x + r \sin\theta \cos\phi$$
$$\eta = y + r \sin\theta \sin\phi$$
$$\zeta = z + r \cos\theta$$

Then, the spherical mean can be written as

$$\bar{u}(r) = \frac{1}{4\pi r^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} u(x + r\sin\theta\cos\phi,\, y + r\sin\theta\sin\phi,\, z + r\cos\theta) r^2 \sin\theta \, d\theta \, d\phi$$

Also, since u is continuous on $S(P, r), \bar{u}$ too is a continuous function of r on some interval $0 < r \le R$, which can be verified as follows:

$$\bar{u}(r) = \frac{1}{4\pi} \iint u(Q) \sin\theta \, d\theta \, d\phi = \frac{u(Q)}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi = u(Q)$$

Now, taking the limit as $r \to 0$, $Q \to P$, we have

$$\operatorname*{Lt}_{r \to 0} \bar{u}(r) = u(P) \tag{2.22}$$

Hence, $\bar{u}$ is continuous in $0 \le r \le R$.

2.4.2 Mean Value Theorem for Harmonic Functions

Theorem 2.5 Let u be harmonic in a region $\mathbb{R}$. Also, let $P(x, y, z)$ be a given point in $\mathbb{R}$ and $S(P, r)$ be a sphere with centre at P such that $S(P, r)$ is completely contained in the domain of harmonicity of u. Then

$$u(P) = \bar{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q) \, dS$$

Proof Since u is harmonic in $\mathbb{R}$, its spherical mean $\bar{u}(r)$ is continuous in $\mathbb{R}$ and is given by

$$\bar{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q) \, dS = \frac{1}{4\pi r^2} \int_0^{2\pi} \int_0^{\pi} u(\xi, \eta, \zeta) r^2 \sin\theta \, d\theta \, d\phi$$

Therefore,

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} (u_\xi \xi_r + u_\eta \eta_r + u_\zeta \zeta_r)\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} (u_\xi \sin\theta\cos\phi + u_\eta \sin\theta\sin\phi + u_\zeta \cos\theta)\sin\theta\, d\theta\, d\phi \quad (2.23)$$

Noting that $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ and $\cos\theta$ are the direction cosines of the normal $\hat{n}$ on $S(P, r)$,

$$\nabla u = iu_\xi + ju_\eta + ku_\zeta, \qquad \hat{n} = (in_1, jn_2, kn_3),$$

the expression within the parentheses of the integrand of Eq. (2.23) can be written as $\nabla u \cdot \hat{n}$. Thus

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi r^2}\iint_{S(P,r)} \nabla u \cdot \hat{n} r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi r^2}\iint_{S(P,r)} \nabla u \cdot \hat{n}\, dS$$

$$= \frac{1}{4\pi r^2}\iiint_{V(P,r)} \nabla \cdot \nabla u\, dV \text{ (by divergence theorem)}$$

$$= \frac{1}{4\pi r^2}\iiint_{V(P,r)} \nabla^2 u\, dV = 0 \text{ (since } u \text{ is harmonic)}$$

Therefore, $\dfrac{d\bar{u}}{dr} = 0$, implying $\bar{u}$ is constant.

Now the continuity of $\bar{u}$ at $r = 0$ gives, from Eq.(2.22), the relation

$$\bar{u}(r) = u(P) = \frac{1}{4\pi r^2}\iint_{S(P,r)} u(Q)\, dS \quad (2.24)$$

2.4.3 Maximum-Minimum Principle and Consequences

Theorem 2.6 Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$. Also, let u be a function which is continuous in a closed region $\overline{\mathbb{R}}$ and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$. Further, if u is not constant everywhere on $\overline{\mathbb{R}}$, then the maximum and minimum values of u must occur only on the boundary $\partial\mathbb{R}$.

Proof Suppose u is a harmonic function but not constant everywhere on $\overline{\mathrm{IR}}$. If possible, let u attain its maximum value M at some interior point P in IR. Since M is the maximum of u which is not a constant, there should exist a sphere $S(P, r)$ about P such that some of the values of u on $S(P, r)$ must be less than M. But by the mean value property, the value of u at P is the average of the values of u on $S(P, r)$, and hence it is less than M. This contradicts the assumption that $u = M$ at P. Thus u must be constant over the entire sphere $S(P, r)$.

Let Q be any other point inside IR which can be connected to P by an arc lying entirely within the domain IR. By covering this arc with spheres and using the Heine-Borel theorem to choose a finite number of covering spheres and repeating the argument given above, we can arrive at the conclusion that u will have the same constant value at Q as at P. Thus u cannot attain a maximum value at any point inside the region IR. Therefore, u can attain its maximum value only on the boundary $\partial\mathrm{IR}$. A similar argument will lead to the conclusion that u can attain its minimum value only on the boundary $\partial\mathrm{IR}$.

Some important consequences of the maximum-minimum principle are given in the following theorems.

Theorem 2.7 (Stability theorem). The solutions of the Dirichlet problem depend continuously on the boundary data.

Proof Let u_1 and u_2 be two solutions of the Dirichlet problem and let f_1 and f_2 be given continuous functions on the boundary $\partial\mathrm{IR}$ such that

$$\nabla^2 u_1 = 0 \quad \text{in } \mathrm{IR}; \quad u_1 = f_1 \quad \text{on } \partial\mathrm{IR},$$

$$\nabla^2 u_2 = 0 \quad \text{in } \mathrm{IR}; \quad u_2 = f_2 \quad \text{on } \partial\mathrm{IR}$$

Let $u = u_1 - u_2$. Then,

$$\nabla^2 u = \nabla^2 u_1 - \nabla^2 u_2 = 0 \quad \text{in } \mathrm{IR}; \qquad u = f_1 - f_2 \quad \text{on } \partial\mathrm{IR}$$

Hence, u is a solution of the Dirichlet problem with boundary condition $u = f_1 - f_2$ on $\partial\mathrm{IR}$. By the maximum-minimum principle, u attains the maximum and minimum values on $\partial\mathrm{IR}$. Thus at any interior point in IR, we shall have, for a given $\varepsilon > 0$,

$$-\varepsilon < u_{\min} \le u \le u_{\max} < \varepsilon$$

Therefore,

$$|u| < \varepsilon \quad \text{in } \mathrm{IR}, \text{ implying } |u_1 - u_2| < \varepsilon$$

Hence, if

$$|f_1 - f_2| < \varepsilon \quad \text{on } \partial\mathbb{R}, \quad \text{then } |u_1 - u_2| < \varepsilon \quad \text{on } \mathbb{R}$$

Thus, small changes in the initial data bring about an arbitrarily small change in the solution. This completes the proof of the theorem.

Theorem 2.8 Let $\{f_n\}$ be a sequence of functions, each of which is continuous on $\overline{\mathbb{R}}$ and harmonic on $\mathbb{R}$. If the sequence $\{f_n\}$ converges uniformly on $\partial\mathbb{R}$, then it converges uniformly on $\overline{\mathbb{R}}$.

Proof Since the sequence $\{f_n\}$ converges uniformly on $\partial\mathbb{R}$, for a given $\varepsilon > 0$, we can always find an integer N such that

$$|f_n - f_m| < \varepsilon \quad \text{for } n, m > N$$

Hence, from stability theorem, for all $n, m > N$, it follows immediately that

$$|f_n - f_m| < \varepsilon \quad \text{in } \mathbb{R}$$

Therefore, $\{f_n\}$ converges uniformly on $\overline{\mathbb{R}}$.

EXAMPLE 2.1 Show that if the two-dimensional Laplace equation $\nabla^2 u = 0$ is transformed by introducing plane polar coordinates r, θ defined by the relations $x = r\cos\theta, y = r\sin\theta$, it takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solution In many practical problems, it is necessary to write the Laplace equation in various coordinate systems. For instance, if the boundary of the region $\partial\mathbb{R}$ is a circle, then it is natural to use polar coordinates defined by $x = r\cos\theta,\ y = r\sin\theta$. Therefore,

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x)$$

$$r_x = \cos\theta, \qquad r_y = \sin\theta, \qquad \theta_x = -\frac{\sin\theta}{r}, \qquad \theta_y = \frac{\cos\theta}{r}$$

since

$$u = u(r, \theta) \qquad u_x = u_r r_x + u_\theta \theta_x = \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)$$

Similarly,

$$u_y = u_r r_y + u_\theta \theta_y = \left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right)$$

Now for the second order derivatives,

$$u_{xx} = (u_x)_x = (u_x)_r r_x + (u_x)_\theta \theta_x = \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_r \cos\theta + \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_\theta \left(-\frac{\sin\theta}{r}\right)$$

Therefore,

$$u_{xx} = \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right)\cos\theta$$
$$+\left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right) \tag{2.25}$$

Similarly, we can show that

$$u_{yy} = \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right)\sin\theta$$
$$+\left(u_{r\theta}\sin\theta + u_r \cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right) \tag{2.26}$$

By adding Eqs. (2.25) and (2.26) and equating to zero, we get

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{2.27}$$

which is the Laplace equation in polar coordinates. One can observe that the Laplace equation in Cartesian coordinates has constant coefficients only, whereas in polar coordinates, it has variable coefficients.

EXAMPLE 2.2 Show that in cylindrical coordinates r, θ, z defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the Laplace equation $\nabla^2 u = 0$ takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Solution The Laplace equation in Cartesian coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The relations between Cartesian and cylindrical coordinates give

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x), \qquad z = z$$

Since

$$u = u(r, \theta, z)$$

$$u_x = u_r r_x + u_\theta \theta_x + u_z z_x = u_r \cos\theta - u_\theta \left(\frac{\sin\theta}{r}\right)$$

$$u_y = u_r r_y + u_\theta \theta_y + u_z z_y = u_r \sin\theta + u_\theta \left(\frac{\cos\theta}{r}\right)$$

$$u_z = u_r r_z + u_\theta \theta_z + u_z = u_z$$

for the second order derivatives, we find

$$u_{xx} = (u_x)_x = (u_x)_r r_x + (u_x)_\theta \theta_x + (u_x)_z z_x$$

$$= \left[u_r \cos\theta - u_\theta \left(\frac{\sin\theta}{r}\right)\right]_r \cos\theta + \left[u_r \cos\theta - u_\theta \left(\frac{\sin\theta}{r}\right)\right]_\theta \left(-\frac{\sin\theta}{r}\right)$$

$$= \left(u_{rr} \cos\theta - u_{r\theta} \frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right) \cos\theta$$

$$+ \left(u_{r\theta} \cos\theta - u_r \sin\theta - u_{\theta\theta} \frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right) \tag{2.28}$$

Similarly

$$u_{yy} = (u_y)_y = (u_y)_r r_y + (u_y)_\theta \theta_y + (u_y)_z z_y$$

$$= \left[u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right]_r \sin\theta + \left[u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right]_\theta \left(\frac{\cos\theta}{r}\right)$$

$$= \left(u_{rr} \sin\theta + u_{\theta r} \frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right) \sin\theta$$

$$+ \left(u_{r\theta} \sin\theta + u_r \cos\theta + u_{\theta\theta} \frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right) \tag{2.29}$$

$$u_{zz} = u_{zz} \tag{2.30}$$

Adding Eqs. (2.28)–(2.30), we obtain

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} \tag{2.31}$$

EXAMPLE 2.3 Show that in spherical polar coordinates r, θ, ϕ defined by the relations $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, the Laplace equations $\nabla^2 u = 0$ takes the form

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

Solution In Cartesian coordinates, the Laplace equation is

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

In spherical coordinates, $u = u(r, \theta, \phi)$, $r^2 = x^2 + y^2 + z^2$, $\cos\theta = z/r$, $\tan\phi = y/x$.

It can be easily verified that

$$\theta_x = \frac{\cos\theta\cos\phi}{r}, \qquad \theta_y = \frac{\cos\theta\sin\phi}{r}, \qquad \theta_z = -\frac{\sin\theta}{r}$$

$$\phi_x = -\frac{\sin\phi}{r\sin\theta}, \qquad \phi_y = \frac{\cos\phi}{r\sin\theta}, \qquad \phi_z = 0$$

Now,

$$u_x = u_r r_x + u_\theta \theta_x + u_\phi \phi_x = u_r \sin\theta\cos\phi + u_\theta \frac{\cos\theta\cos\phi}{r} - u_\phi \frac{\sin\phi}{r\sin\theta}$$

$$u_y = u_r r_y + u_\theta \theta_y + u_\phi \phi_y = u_r \sin\theta\sin\phi + u_\theta \frac{\cos\theta\sin\phi}{r} + \frac{u_\phi \cos\phi}{r\sin\theta}$$

$$u_z = u_r r_z + u_\theta \theta_z + u_\phi \phi_z = u_r \cos\theta + u_\theta \left(-\frac{\sin\theta}{r}\right)$$

For the second order derivatives,

$$\begin{aligned}
u_{xx} &= (u_x)_r r_x + (u_x)_\theta \theta_x + (u_x)_\phi \phi_x \\
&= \left(u_r \sin\theta\cos\phi + u_\theta \frac{\cos\theta\cos\phi}{r} - u_\phi \frac{\sin\phi}{r\sin\theta}\right)_r \cdot (\sin\theta\cos\phi) \\
&\quad + \left(u_r \sin\theta\cos\phi + u_\theta \frac{\cos\theta\cos\phi}{r} - u_\phi \frac{\sin\phi}{r\sin\theta}\right)_\theta \left(\frac{\cos\theta\cos\phi}{r}\right) \\
&\quad + \left(u_r \sin\theta\cos\phi + u_\theta \frac{\cos\theta\cos\phi}{r} - u_\phi \frac{\sin\phi}{r\sin\theta}\right)_\phi \left(-\frac{\sin\phi}{r\sin\theta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (\sin^2\theta\cos^2\phi)u_{rr} + \frac{\cos^2\theta\cos^2\phi}{r^2}u_{\theta\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}u_{\phi\phi} \\
&+ u_{r\theta}\left(\frac{2\sin\theta\cos\theta\cos^2\phi}{r}\right) + u_{r\phi}\left(-\frac{2\sin\phi\cos\phi}{r}\right) \\
&+ u_{\theta\phi}\left(-\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right) + u_r\left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\sin^2\phi}{r}\right) \\
&+ u_\phi\left(\frac{\sin\phi\cos\phi}{r^2} + \frac{\cos^2\theta\cos\phi\sin\phi}{r^2\sin^2\theta} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\right) \\
&+ u_\theta\left(\frac{\cos\theta\sin^2\phi}{r^2\sin\theta} - \frac{2\cos\theta\sin\theta\cos^2\phi}{r^2}\right)
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
u_{yy} &= (u_y)_r r_y + (u_y)_\theta\theta_y + (u_y)_\phi\phi_y = \left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_r(\sin\theta\sin\phi) \\
&+ \left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_\theta\frac{\cos\theta\sin\phi}{r} \\
&+ \left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_\phi\frac{\cos\phi}{r\sin\theta} \\
&= (\sin^2\theta\sin^2\phi)u_{rr} + \frac{\cos^2\theta\sin^2\phi}{r^2}u_{\theta\theta} + \frac{\cos^2\phi}{r^2\sin^2\theta}u_{\phi\phi} \\
&+ u_{r\theta}\left(\frac{2\sin\theta\cos\theta\sin^2\phi}{r}\right) + u_{r\phi}\left(\frac{2\cos\phi\sin\phi}{r}\right) \\
&+ u_{\theta\phi}\left(\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right) + u_r\left(\frac{\cos^2\theta\sin^2\phi}{r} + \frac{\cos^2\phi}{r}\right) \\
&+ u_\theta\left(-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2} + \frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\right) \\
&+ u_\phi\left(-\frac{\sin\phi\cos\phi}{r^2} - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta} - \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\right)
\end{aligned}
\tag{2.33}
$$

Similarly,

$$u_{zz} = (u_z)_r r_z + (u_z)_\theta \theta_z + (u_z)_\phi \phi_z$$

$$= \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_r (\cos\theta) + \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_\theta - \left(-\frac{\sin\theta}{r}\right)$$

$$= u_{rr}\cos^2\theta - u_{r\theta}\frac{2\sin\theta\cos\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2}$$

$$+ u_r \frac{\sin^2\theta}{r} + u_\theta \frac{\cos\theta\sin\theta}{r^2} \tag{2.34}$$

Adding Eqs. (2.32)–(2.34), we obtain

$$\nabla^2 u = u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + \frac{2}{r}u_r + \frac{\cos\theta}{r^2\sin\theta}u_\theta = 0$$

which can be rewritten as

$$\nabla^2 u = \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \tag{2.35}$$

2.5 SEPARATION OF VARIABLES

The method of separation of variables is applicable to a large number of classical linear homogeneous equations. The choice of the coordinate system in general depends on the shape of the boundary. For example, consider a two-dimensional Laplace equation in Cartesian coordinates.

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{2.36}$$

We assume the solution in the form

$$u(x, y) = X(x)\,Y(y) \tag{2.37}$$

Substituting in Eq. (2.36), we get

$$X''Y + Y''X = 0$$

i.e.

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

where k is a separation constant. Three cases arise.

Case I Let $k = p^2$, p is real. Then

$$\frac{d^2X}{dx^2} - p^2X = 0 \qquad \text{and} \qquad \frac{d^2Y}{dy^2} + p^2Y = 0$$

whose solution is given by

$$X = c_1e^{px} + c_2e^{-px}$$

and

$$Y = c_3 \cos py + c_4 \sin py$$

Thus, the solution is

$$u(x, y) = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \tag{2.38}$$

Case II Let $k = 0$. Then

$$\frac{d^2X}{dx^2} = 0 \qquad \text{and} \qquad \frac{d^2Y}{dy^2} = 0$$

Integrating twice, we get

$$X = c_5x + c_6$$

and

$$Y = c_7y + c_8$$

The solution is therefore,

$$u(x, y) = (c_5x + c_6)(c_7y + c_8) \tag{2.39}$$

Case III Let $k = -p^2$. Proceeding as in Case I, we obtain

$$X = c_9 \cos px + c_{10} \sin px$$

$$Y = c_{11}e^{py} + c_{12}e^{-py}$$

Hence, the solution in this case is

$$u(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11}e^{py} + c_{12}e^{-py}) \tag{2.40}$$

In all these cases, $c_i \, (i = 1, 2, \ldots, 12)$ refer to integration constants, which are calculated by using the boundary conditions.

2.6 DIRICHLET PROBLEM FOR A RECTANGLE

The Dirichlet problem for a rectangle is defined as follows:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le x \le a, \; 0 \le y \le b$$

$$\text{BCs: } u(x, b) = u(a, y) = 0, \qquad u(0, y) = 0, \qquad u(x, 0) = f(x) \tag{2.41}$$

This is an interior Dirichlet problem. The general solution of the governing PDE, using the method of variables separable, is discussed in Section 2.5. The various possible solutions of the Laplace equation are given by Eqs. (2.38–2.40). Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions as depicted in Fig. 2.3.

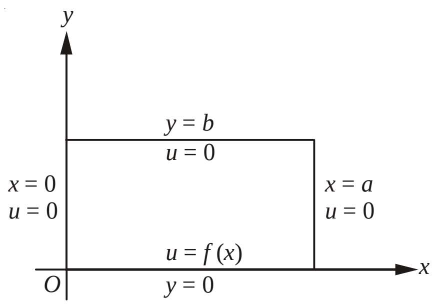

Fig. 2.3 Dirichlet boundary conditions.

Consider the solution given by Eq. (2.38):

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Using the boundary condition: $u(0, y) = 0$, we get

$$(c_1 + c_2)(c_3 \cos py + c_4 \sin py) = 0$$

which means that either $c_1 + c_2 = 0$ or $c_3 \cos py + c_4 \sin py = 0$. But $c_3 \cos py + c_4 \sin py \ne 0$; therefore,

$$c_1 + c_2 = 0 \tag{2.42}$$

Again, using the BC; $u(a, y) = 0$, Eq. (2.38) gives

$$(c_1 e^{ap} + c_2 e^{-ap})(c_3 \cos py + c_4 \sin py) = 0$$

implying thereby

$$c_1 e^{ap} + c_2 e^{-ap} = 0 \tag{2.43}$$

To determine the constants c_1, c_2, we have to solve Eqs. (2.42) and (2.43); being homogeneous, the determinant

$$\begin{vmatrix} 1 & 1 \\ e^{ap} & e^{-ap} \end{vmatrix} = 0$$

for the existence of non-trivial solution, which is not the case. Hence, only the trivial solution $u(x, y) = 0$ is possible.

If we consider the solution given by Eq. (2.39) $u(x, y) = (c_5 x + c_6)(c_7 y + c_8)$, the boundary conditions: $u(0, y) = u(a, y) = 0$ again yield a trivial solution. Hence, the possible solutions given by Eqs. (2.38) and (2.39) are ruled out. Therefore, the only possible solution obtained from Eq. (2.40) is

$$u(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11} e^{py} + c_{12} e^{-py})$$

Using the BC: $u(0, y) = 0$, we get $c_9 = 0$. Also, the other BC: $u(a, y) = 0$ yields

$$c_{10} \sin pa \, (c_{11} e^{py} + c_{12} e^{-py}) = 0$$

For non-trivial solution, c_{10} cannot be zero, implying $\sin pa = 0$, which is possible if $pa = n\pi$ or $p = n\pi/a$, $n = 1, 2, 3, \ldots$ Therefore, the possible non-trivial solution after using the superposition principle is

$$u(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} [a_n \exp(n\pi y/a) + b_n \exp(-n\pi y/a)] \tag{2.44}$$

Now, using the BC: $u(x, b) = 0$, we get

$$\sin \frac{n\pi x}{a} [a_n \exp(n\pi b/a) + b_n \exp(-n\pi b/a)] = 0$$

implying thereby

$$a_n \exp(n\pi b/a) + b_n \exp(-n\pi b/a) = 0$$

which gives

$$b_n = -a_n \frac{\exp(n\pi b/a)}{\exp(-n\pi b/a)}, \qquad n = 1, 2, \ldots, \infty$$

The solution (2.44) now becomes

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2a_n \sin(n\pi x/a)}{\exp(-n\pi b/a)} \left[\frac{\exp\{n\pi (y-b)/a\} - \exp\{-n\pi(y-b)/a\}}{2} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2a_n}{\exp(-n\pi b/a)} \sin(n\pi x/a) \sin h\{n\pi(y-b)/a\}$$

Let $2a_n/[\exp(-n\pi b/a)] = A_n$. Then the solution can be written in the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\} \tag{2.45}$$

Finally, using the non-homogeneous boundary condition: $u(x, 0) = f(x)$, we get

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh(-n\pi b/a) = f(x)$$

which is a half-range Fourier series. Therefore,

$$A_n \sinh(-n\pi b/a) = \frac{2}{a}\int_0^a f(x) \sin(n\pi x/a)\, dx \tag{2.46}$$

Thus, the required solution for the given Dirichlet problem is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\} \tag{2.47}$$

where

$$A_n = \frac{2}{a}\frac{1}{\sinh(-n\pi b/a)}\int_0^a f(x) \sin(n\pi x/a)\, dx$$

2.7 THE NEUMANN PROBLEM FOR A RECTANGLE

The Neumann problem for a rectangle is defined as follows:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le x \le a,\ 0 \le y \le b$$

$$\text{BCs: } u_x(0, y) = u_x(a, y) = 0, \qquad u_y(x, 0) = 0, \qquad u_y(x, b) = f(x) \tag{2.48}$$

The general solution of the Laplace equation using the method of variables separable is given in Section 2.5, and is found to be

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

The BC: $u_x(0, y) = 0$ gives

$$0 = c_2 p(c_3 e^{py} + c_4 e^{-py})$$

implying $c_2 = 0$. Therefore,

$$u(x, y) = c_1 \cos px(c_3 e^{py} + c_4 e^{-py}) \tag{2.49}$$

The BC: $u_x(a, y) = 0$ gives

$$0 = -c_1 p \sin pa\,(c_3 e^{py} + c_4 e^{-py})$$

For non-trivial solution, $c_1 \neq 0$, implying

$$\sin pa = 0, \qquad pa = n\pi, \qquad p = \frac{n\pi}{a} \qquad (n = 0, 1, 2, \ldots)$$

Thus the possible solution is

$$u(x, y) = \cos\frac{n\pi x}{a}(Ae^{n\pi y/a} + Be^{-n\pi y/a}) \tag{2.50}$$

Now, using the BC: $u_y(x, 0) = 0$, we get

$$0 = \cos\frac{n\pi x}{a}\left(A\frac{n\pi}{a} - B\frac{n\pi}{a}\right)$$

implying $B = A$. Thus, the solution is

$$u(x, y) = A\cos\frac{n\pi x}{a}[\exp(n\pi y/a) + \exp(-n\pi y/a)]$$

$$= 2A\cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}$$

Using the superposition principle and defining $2A = A_n$, we get

$$u = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a} \tag{2.51}$$

Finally, using the BC: $u_y(x, b) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\frac{n\pi}{a}\sinh\frac{n\pi b}{a}$$

which is the half-range Fourier cosine series. Therefore,

$$A_n \frac{n\pi}{a}\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx$$

Hence, the required solution is

$$u = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a} \tag{2.52}$$

where

$$A_n = \frac{2}{n\pi}\frac{1}{\sinh(n\pi b/a)}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

2.8 INTERIOR DIRICHLET PROBLEM FOR A CIRCLE

The Dirichlet problem for the circle is defined as follows:

$$\text{PDE}: \nabla^2 u = 0, \qquad 0 \le r \le a,\ 0 \le \theta \le 2\pi$$
$$\text{BC}:\ u(a, \theta) = f(\theta), \qquad 0 \le \theta \le 2\pi \tag{2.53}$$

where $f(\theta)$ is a continuous function on $\partial\mathbb{R}$. The task is to find the value of u at any point in the interior of the circle $\mathbb{R}$ in terms of its values on $\partial\mathbb{R}$ such that u is single valued and continuous on $\overline{\mathbb{R}}$.

In view of circular geometry, it is natural to choose polar coordinates to solve this problem and then use the variables separable method. The requirement of single-valuedness of u in $\overline{\mathbb{R}}$ implies the periodicity condition, i.e.,

$$u(r, \theta + 2\pi) = u(r, \theta), \qquad 0 \le r \le a, \tag{2.54}$$

From Eq. (2.27), $\nabla^2 u = 0$ which in polar coordinates can be written as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

If $u(r, \theta) = R(r)H(\theta)$, the above equation reduces to

$$R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0$$

This equation can be rewritten as

$$\frac{r^2R'' + rR'}{R} = -\frac{H''}{H} = k \tag{2.55}$$

which means that a function of r is equal to a function of θ and, therefore, each must be equal to a constant k (a separation constant).

Case I Let $k = \lambda^2$. Then

$$r^2R'' + rR' - \lambda^2 R = 0 \tag{2.56}$$

which is a Euler type of equation and can be solved by setting $r = e^z$. Its solution is

$$R = c_1e^{\lambda z} + c_2e^{-\lambda z} = c_1r^{\lambda} + c_2r^{-\lambda}$$

Also,

$$H'' + \lambda^2 H = 0$$

whose solution is

$$H = c_3 \cos \lambda\theta + c_4 \sin \lambda\theta$$

Therefore,

$$u(r, \theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(c_3 \cos \lambda\theta + c_4 \sin \lambda\theta) \tag{2.57}$$

Case II Let $k = -\lambda^2$. Then

$$r^2 R'' + rR' + \lambda^2 R = 0, \qquad H'' - \lambda^2 H = 0$$

Their respective solutions are

$$R = c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r)$$
$$H = c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}$$

Thus

$$u(r, \theta) = [c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r)](c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}) \tag{2.58}$$

Case III Let $k = 0$. Then we have

$$rR'' + R' = 0$$

Setting $R'(r) = V(r)$, we obtain

$$r\frac{dV}{dr} + V = 0, \quad \text{i.e.,} \frac{dV}{V} + \frac{dr}{r} = 0$$

Integrating, we get $\ln Vr = \ln c_1$. Therefore,

$$V = \frac{c_1}{r} = \frac{dR}{dr}$$

On integration,

$$R = c_1 \ln r + c_2$$

Also,

$$H'' = 0$$

After integrating twice, we get

$$H = c_3\theta + c_4$$

Thus,

$$u(r, \theta) = (c_1 \ln r + c_2)(c_3\theta + c_4) \tag{2.59}$$

Now, for the interior problem, $r = 0$ is a point in the domain $\mathbb{R}$ and since ln r is not defined at $r = 0$, the solutions (2.58) and (2.59) are not acceptable. Thus the required solution is obtained from Eq. (2.57). The periodicity condition in θ implies

$$c_3 \cos \lambda\theta + c_4 \sin \lambda\theta = c_3 \cos (\lambda(\theta + 2\pi)) + c_4 \sin (\lambda(\theta + 2\pi))$$

i.e.

$$c_3[\cos \lambda\theta - \cos (\lambda\theta + 2\lambda\pi)] + c_4[\sin \lambda\theta - \sin (\lambda\theta + 2\lambda\pi)] = 0$$

or

$$2 \sin \lambda\pi [c_3 \sin (\lambda\theta + \lambda\pi) - c_4 \cos (\lambda\theta + \lambda\pi)] = 0$$

implying $\sin \lambda\pi = 0$, $\lambda\pi = n\pi$, $\lambda = n (n = 0, 1, 2, \ldots)$. Using the principle of superposition and renaming the constants, the acceptable general solution can be written as

$$u(r, \theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n}) (a_n \cos n\theta + b_n \sin n\theta) \tag{2.60}$$

At $r = 0$, the solution should be finite, which requires $d_n = 0$. Thus the appropriate solution assumes the form

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

For $n = 0$, let the constant A_0 be $A_0/2$. Then the solution is

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{2.61}$$

which is a full-range Fourier series. Now we have to determine A_n and B_n so that the BC: $u(a, \theta) = f(\theta)$ is satisfied, i.e.,

$$f(\theta) = \sum_{n=0}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

Hence,

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta \\ a^n A_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \\ a^n B_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \; n = 1, 2, 3, \ldots \end{aligned} \tag{2.62}$$

In Eqs. (2.62) we replace the dummy variable θ by ϕ to distinguish this variable from the current variable θ in Eq. (2.61). Substituting Eq. (2.62) into Eq. (2.61), we obtain the relation

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi+\sum_{n=1}^{\infty}\left[\frac{r^n}{a^n}\frac{\cos n\theta}{\pi}\int_0^{2\pi}\cos(n\phi)\,f(\phi)\,d\phi\right.$$

$$\left.+\frac{r^n}{a^n}\frac{\sin n\theta}{\pi}\int_0^{2\pi}\sin(n\phi)\,f(\phi)\,d\phi\right]$$

Interchanging the order of summation and integration, we get

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi+\frac{1}{\pi}\int_0^{2\pi} f(\phi)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\{\cos n\phi\cos n\theta+\sin n\phi\sin n\theta\}\,d\phi$$

$$=\frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\phi-\theta)\right]d\phi \tag{2.63}$$

To obtain an alternative expression for $u(r,\theta)$ in closed integral form, we can proceed as follows:

Let

$$c=\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\phi-\theta)$$

$$s=\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\sin n(\phi-\theta)$$

so that

$$c+is=\sum_{n=1}^{\infty}\left[\frac{r}{a}e^{i(\phi-\theta)}\right]^n$$

Since $r<a, (r/a)<1$ and $|e^{i(\phi-\theta)}|\le 1$,

$$c+is=\sum_{n=1}^{\infty}\left[\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right]^n=\frac{(r/a)\,e^{i(\phi-\theta)}}{[1-(r/a)\,e^{i(\phi-\theta)}]}$$

$$=\frac{(r/a)\{e^{i(\phi-\theta)}-(r/a)\}}{[1-(r/a)e^{i(\phi-\theta)}][1-(r/a)\,e^{-i(\phi-\theta)}]}$$

Equating the real part on both sides, we get

$$c = \frac{[r/a) \cos(\phi - \theta) - (r^2/a^2)]}{[1 - (2r/a) \cos(\phi - \theta) + (r^2/a^2)]}$$

Thus, the expression in the square brackets of Eq. (2.63) becomes

$$\frac{1}{2} + \frac{[(r/a) \cos(\phi - \theta) - (r^2/a^2)]}{[1 - (2r/a) \cos(\phi - \theta) + (r^2/a^2)]} = \frac{a^2 - r^2}{2[a^2 - 2ar \cos(\phi - \theta) + r^2]}$$

Thus, the required solution takes the form

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2) f(\phi)}{[a^2 - 2ar \cos(\phi - \theta) + r^2]} d\phi \tag{2.64}$$

This is known as Poisson's integral formula for a circle, which gives a unique solution for the Dirichlet problem. The solution (2.64) can be interpreted physically in many ways: It can be thought of as finding the potential $u(r, \theta)$ as a weighted average of the boundary potentials $f(\phi)$ weighted by the Poisson kernel P, given by

$$P = \frac{a^2 - r^2}{[a^2 - 2ar \cos(\phi - \theta) + r^2]}$$

It can also be thought of as a steady temperature distribution $u(r, \theta)$ in a circular disc, when the temperature u on its boundary $\partial\mathbb{R}$ is given by $u = f(\phi)$ which is independent of time.

2.9 EXTERIOR DIRICHLET PROBLEM FOR A CIRCLE

The exterior Dirichlet problem is described by

$$\text{PDE: } \nabla^2 u = 0$$

$$\text{BC: } u(a, \theta) = f(\theta) \tag{2.65}$$

u must be bounded as $r \to \infty$.

By the method of separation of variables, the general solution (2.60) of $\nabla^2 u = 0$ in polar coordinates can be written as

$$u(r, \theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

Now as, $r \to \infty$, we require u to be bounded, and, therefore, $c_n = 0$.

After adjusting the constants, the general solution now reads

$$u(r, \theta) = \sum_{n=0}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta)$$

With no loss of generality, it can also be written as

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta) \tag{2.66}$$

Using the BC: $u(a, \theta) = f(\theta)$, we obtain

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^{-n} (A_n \cos n\theta + B_n \sin n\theta)$$

This is a full-range Fourier series in $f(\theta)$, where

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta \\ a^{-n} A_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \\ a^{-n} B_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta\, d\theta \end{aligned} \tag{2.67}$$

In Eq. (2.67) we replace the dummy variable θ by ϕ so as to distinguish it from the current variable θ. We then introduce the changed variable into solution (2.66) which becomes

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi + \sum_{n=1}^{\infty} \left[\frac{r^{-n} a^n}{\pi} \cos n\theta \int_0^{2\pi} \cos (n\phi)\, f(\phi)\, d\phi + \frac{r^{-n} a^n}{\pi} \sin n\theta \int_0^{2\pi} \sin (n\phi)\, f(\phi)\, d\phi \right]$$

or

$$u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{a}{r} \right)^n \cos n(\phi - \theta) \right] d\phi \tag{2.68}$$

Let

$$C = \sum_{n=1}^{\infty} \left(\frac{a}{r} \right)^n \cos n(\phi - \theta)$$

$$S = \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \sin n(\phi - \theta)$$

Then,

$$C + iS = \sum_{n=1}^{\infty} \left[\left(\frac{a}{r}\right) e^{i(\phi-\theta)}\right]^n$$

Since $\frac{a}{r} < 1$, $|e^{i(\phi-\theta)}| \le 1$. We have

$$C + iS = \frac{a}{r} \frac{e^{i(\phi-\theta)}}{[1-(a/r)e^{i(\phi-\theta)}]} = \frac{(a/r)[e^{i(\phi-\theta)} - (a/r)]}{[1-(a/r)e^{i(\phi-\theta)}][1-(a/r)e^{-i(\phi-\theta)}]}$$

Hence,

$$C = \frac{[(a/r)\cos(\phi-\theta) - (a^2/r^2)]}{[1-(2a/r)\cos(\phi-\theta) + (a^2/r^2)]}$$

Thus the quantity in the square brackets on the right-hand side of Eq. (2.68) becomes

$$\frac{1}{2} + \frac{[(a/r)\cos(\phi-\theta) - (a^2/r^2)]}{[1-(2a/r)\cos(\phi-\theta) + (a^2/r^2)]} = \frac{r^2 - a^2}{2[r^2 - 2ar\cos(\phi-\theta) + a^2]}$$

Therefore, the solution of the exterior Dirichlet problem reduces to that of an integral equation of the form

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - a^2) f(\phi)}{[r^2 - 2ar\cos(\phi-\theta) + a^2]} d\phi \tag{2.69}$$

EXAMPLE 2.4 Find the steady state temperature distribution in a semi-circular plate of radius a, insulated on both the faces with its curved boundary kept at a constant temperature U_0 and its bounding diameter kept at zero temperature as described in Fig. 2.4.

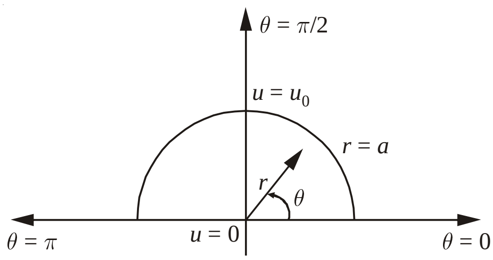

Fig. 2.4 Semi-circular plate.

Solution The governing heat flow equation is

$$u_t = \nabla^2 u$$

In the steady state, the temperature is independent of time; hence $u_t = 0$, and the temperature satisfies the Laplace equation. The problem can now be stated as follows: To solve

$$\text{PDE: } \nabla^2 u(r, \theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$\text{BCs: } u(a, \theta) = U_0, \qquad u(r, 0) = 0, \qquad u(r, \pi) = 0$$

the acceptable general solution is

$$u(r, \theta) = (cr^{\lambda} + Dr^{-\lambda})(A\cos\lambda\theta + B\sin\lambda\theta) \tag{2.70}$$

From the BC: $u(r, 0) = 0$, we get $A = 0$; however, the BC: $u(r, \pi) = 0$ also gives

$$B\sin\lambda\pi(cr^{\lambda} + Dr^{-\lambda}) = 0$$

implying either $B = 0$ or $\sin\lambda\pi = 0$. $B = 0$ gives a trivial solution. For a non-trivial solution, we must have $\sin\lambda\pi = 0$, implying

$$\lambda\pi = n\pi, \qquad n = 1, 2, \ldots$$

meaning thereby $\lambda = n$. Hence, the possible solution is

$$u(r, \theta) = B\sin n\theta(Cr^{\lambda} + Dr^{-\lambda}) \tag{2.71}$$

In Eq. (2.71), we observe that as $r \to 0$, the term $r^{-\lambda} \to \infty$. But the solution should be finite at $r = 0$, and so $D = 0$. Then after adjusting the constants, it follows from the superposition principle that,

$$u(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta$$

Finally, using the first BC: $u(a, \theta) = U_0$, we get

$$u(a, \theta) = U_0 = \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

which is a half-range Fourier sine series. Therefore,

$$B_n a^n = \frac{2}{\pi}\int_0^{\pi} U_0 \sin n\theta \, d\theta = \begin{cases} \dfrac{4U_0}{n\pi}, & \text{for } n = 1, 3, \ldots \\ 0, & \text{for } n = 2, 4, \ldots \end{cases}$$

Hence,

$$B_n = \frac{4U_0}{n\pi a^n}, \qquad n = 1, 3, \ldots$$

With these values of B_n, the required solution is

$$u(r, \theta) = \frac{4U_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \sin n\theta$$

2.10 INTERIOR NEUMANN PROBLEM FOR A CIRCLE

The interior Neumann problem for a circle is described by

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le r < a;\ 0 \le \theta \le 2\pi \tag{2.72}$$

$$\text{BC: } \frac{\partial u}{\partial n} = \frac{\partial u(a, \theta)}{\partial r} = g(\theta), \qquad r = a$$

Following the method of separation of variables, the general solution (2.60) of equation $\nabla^2 u = 0$ in polar coordinates is given by

$$u(r, \theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

At $r = 0$, the solution should be finite and, therefore, $d_n = 0$. Hence, after adjusting the constants, the general solution becomes

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

With no loss of generality, this equation can be written as

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{2.73}$$

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} (A_n \cos n\theta + B_n \sin n\theta)$$

Using the BC:

$$\frac{\partial u}{\partial r}(a, \theta) = g(\theta)$$

we get

$$g(\theta) = \sum_{n=1}^{\infty} na^{n-1}(A_n \cos n\theta + B_n \sin n\theta) \tag{2.74}$$

which is a full-range Fourier series in $g(\theta)$, where

$$na^{n-1}A_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta) \cos n\theta \, d\theta$$

$$na^{n-1}B_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta) \sin n\theta \, d\theta \tag{2.75}$$

Here, we replace the dummy variable θ by ϕ to distinguish from the current variable θ in Eq. (2.74). Now introducing Eq. (2.75) into Eq. (2.73), we obtain

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{n\pi a^{n-1}} \int_0^{2\pi} g(\phi)\,(\cos n\phi \cos n\theta + \sin n\phi \sin n\theta)\, d\phi$$

or

$$u(r, \theta) = \frac{A_0}{2} + \int_0^{2\pi} g(\phi) \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \cos n(\phi - \theta)\, d\phi \tag{2.76}$$

This solution can also be expressed in an alternative integral form as follows: Let

$$C = \sum \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \cos n(\phi - \theta)$$

$$S = \sum \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \sin n(\phi - \theta)$$

Therefore,

$$C + iS = \sum \left(\frac{r}{a}\right)^n \frac{a}{n\pi} e^{in(\phi-\theta)} = \frac{a}{\pi} \sum_{n=1}^{\infty} \left[\frac{r}{a} e^{i(\phi-\theta)}\right]^n \frac{1}{n}$$

$$= \frac{a}{\pi}\left[\frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}}{1} + \frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}^2}{2} + \frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}^3}{3} + \cdots\right]$$

or

$$C + iS = -\frac{a}{\pi} \ln \left[1 - \frac{r}{a} e^{i(\phi-\theta)}\right] = -\frac{a}{\pi} \ln \left[1 - \frac{r}{a}\cos(\phi-\theta) - i\frac{r}{a}\sin(\phi-\theta)\right] \quad (2.77)$$

To get the real part of ln z, we may note that

$$W = \ln z \ \text{ or } \ z = e^W$$

i.e., $x + iy = e^{u+iv} = e^u \cos v + ie^u \sin v.$ Therefore,

$$x = e^u \cos v, \qquad y = e^u \sin v$$

$$e^{2u} = x^2 + y^2 = |z|^2$$

i.e., $u = \ln|z|$. Therefore,

$$C = -\frac{a}{\pi} \ln \sqrt{\left(1 - \frac{r}{a}\cos(\phi-\theta)\right)^2 + \left(\frac{r}{a}\sin(\phi-\theta)\right)^2}$$

$$= -\frac{a}{\pi} \ln \sqrt{\frac{a^2 - 2ar\cos(\phi-\theta) + r^2}{a^2}}$$

Thus the required solution is

$$u(r,\theta) = \frac{A_0}{2} - \frac{a}{\pi}\int_0^{2\pi} \ln \sqrt{\frac{a^2 - 2ar\cos(\phi-\theta) + r^2}{a^2}}\, g(\phi)\, d\phi \quad (2.78)$$

which is again an integral equation.

2.11 SOLUTION OF LAPLACE EQUATION IN CYLINDRICAL COORDINATES

The Laplace equation in cylindrical coordinates assumes the following form:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0 \quad (2.79)$$

We now seek a separable solution of the form

$$u(r,\theta,z) = F(r,\theta)\, Z(z) \quad (2.80)$$

Substituting Eq. (2.80) into Eq. (2.79), we get

$$\frac{\partial^2 F}{\partial r^2} Z + \frac{1}{r}\frac{\partial F}{\partial r} Z + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} Z + F\frac{d^2 Z}{dz^2} = 0$$

or

$$\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\frac{1}{F} = -\frac{d^2 Z}{dz^2}\frac{1}{Z} = k \ \text{(say)}$$

where k is a separation constant. Therefore, either

$$\frac{d^2Z}{dz^2} + kZ = 0 \tag{2.81}$$

or

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - KF = 0 \tag{2.82}$$

If k is real and positive, the solution of Eq. (2.81) is

$$Z = c_1 \cos \sqrt{k}z + c_2 \sin \sqrt{k}z$$

If k is negative, the solution of Eq. (2.81) is

$$Z = c_1 e^{\sqrt{k}z} + c_2 e^{-\sqrt{k}z}$$

If k is equal to zero, the solution of Eq. (2.81) is

$$Z = c_1 z + c_2$$

From physical considerations, one would expect a solution which decays with increasing z and, therefore, the solution corresponding to negative k is acceptable. Let $k = -\lambda^2$. Then

$$Z = c_1 e^{\lambda z} + c_2 e^{-\lambda z} \tag{2.83}$$

Equation (2.82) now becomes

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \lambda^2 F = 0$$

Let $F(r, \theta) = f(r)H(\theta)$. Substituting into the above equations, we get

$$f''H + \frac{1}{r}f'H + \frac{1}{r^2}fH'' + \lambda^2 fH = 0$$

or

$$(r^2 f'' + rf' + \lambda^2 r^2 f)\frac{1}{f} = -\frac{H''}{H} = k' \quad \text{(say)}$$

From physical consideration, we expect the solution to be periodic in θ, which can be obtained when k' is positive and $k' = n^2$. Therefore, the acceptable solution will be

$$H = c_3 \cos n\theta + c_4 \sin n\theta \tag{2.84}$$

When $k' = n^2$, we will also have

$$r^2 \frac{d^2 f}{dr^2} + r\frac{df}{dr} + (\lambda^2 r^2 - n^2) f = 0 \tag{2.85}$$

which is a Bessel's equation whose general solution is given by

$$f = AJ_n(\lambda r) + BY_n(\lambda r) \tag{2.86}$$

Here, $J_n(\lambda r)$ and $Y_n(\lambda r)$ are the nth order Bessel functions of first and second kind, respectively. Since $Y_n(\lambda r) \to -\infty$ as $r \to 0$, $Y_n(\lambda r)$ becomes unbounded at $r = 0$. Continuity of the solution demands $B = 0$. Hence the most general and acceptable solution of $\nabla^2 u = 0$ is

$$u(r, \theta, z) = J_n(\lambda r)\,(c_1 e^{\lambda z} + c_2 e^{-\lambda z})\,(c_3 \cos n\theta + c_4 \sin n\theta) \tag{2.87}$$

EXAMPLE 2.5 A homogeneous thermally conducting cylinder occupies the region $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le h$, where r, θ, z are cylindrical coordinates. The top $z = h$ and the lateral surface $r = a$ are held at 0°, while the base $z = 0$ is held at 100°. Assuming that there are no sources of heat generation within the cylinder, find the steady-temperature distribution within the cylinder.

Solution The temperature u must be a single valued continuous function. The steady state temperature satisfies the Laplace equation inside the cylinder. To compute the temperature distribution inside the cylinder, we have to solve the following BVP:

PDE: $\nabla^2 u = 0$

BCs: $u = 0°$ on $z = h$,

$u = 0°$ on $r = a$,

$u = 100°$ on $z = 0$

The general solution of the Laplace equation in cylindrical coordinates as given in Section 2.11 is

$$r(r, \theta, z) = J_n(\lambda r)\,(c_1 \cos n\theta + c_2 \sin n\theta)\,(c_3 e^{\lambda z} + c_4 e^{-\lambda z})$$

Since the face $z = 0$ is maintained at 100° and since the other face and lateral surface of the cylinder are maintained at 0°, the temperature at any point inside the cylinder is obviously independent of θ. This is possible only when $n = 0$ in the general solution. Thus,

$$u(r, z) = J_0(\lambda r)\,(Ae^{\lambda z} + Be^{-\lambda z})$$

Using the BC: $u = 0$ on $z = h$, we get

$$0 = J_0(\lambda r)\,(Ae^{\lambda h} + Be^{-\lambda h})$$

implying thereby $Ae^{\lambda h} + Be^{-\lambda h} = 0$, from which

$$B = -\frac{Ae^{\lambda h}}{e^{-\lambda h}}$$

Therefore, the solution is

$$u(r, z) = \frac{J_0(\lambda r)\, A}{e^{-\lambda h}}[e^{\lambda(z-h)} - e^{-\lambda(z-h)}]$$

or

$$u(r, z) = J_0(\lambda r) A_1 \sinh \lambda(z-h)$$

where $A_1 = 2A/e^{-\lambda h}$. Now using the BC: $u = 0$ on $r = a$, we have

$$0 = A_1 J_0(\lambda a) \sinh \lambda(z-h)$$

implying $J_0(\lambda a) = 0$, which has infinitely many positive roots. Denoting them by ξ_n, we have $\xi_n = \lambda a$, and therefore,

$$\lambda = \frac{\xi_n}{a}$$

Thus the solution is

$$u(r, z) = A_1 J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z-h)\right], \qquad n = 1, 2, \ldots$$

Using the principle of superposition, we have

$$u(r, z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z-h)\right]$$

The BC: $u = 100°$ on $z = 0$ gives

$$100 = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) J_0\left(\frac{\xi_n r}{a}\right)$$

which is a Fourier-Bessel series. Multiplying both sides with $rJ_0(\xi_m r/a)$ and integrating, we get

$$100 \int_0^a rJ_0\left(\frac{\xi_m r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \int_0^a rJ_0\left(\frac{\xi_n r}{a}\right) J_0\left(\frac{\xi_m r}{a}\right) dr$$

Using the orthogonality property of Bessel's function, namely,

$$\int_0^a xJ_n(\alpha_i x) J_n(\alpha_j x)\, dx = \begin{cases} 0, & \text{if } i \neq j \\ \dfrac{a^2}{2} J_{n+1}^2(\alpha_i), & \text{if } i = j \end{cases}$$

where α_i, α_j are the zeros at $J_n(x) = 0$, we have

$$100 \int_0^a r J_0\left(\frac{\xi_n r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \frac{a^2}{2} J_1^2(\xi_n)$$

Therefore,

$$A_n = \frac{200}{a^2 \sinh\left(-\dfrac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^a r J_0\left(\frac{\xi_n r}{a}\right) dr$$

Setting

$$\frac{\xi_n r}{a} = x, \qquad dr = \frac{a}{\xi_n} dx$$

the relation for A_n can also be written as

$$A_n = \frac{200}{\xi_n^2 \sinh\left(-\dfrac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} x J_0(x)\, dx$$

Using the recurrence relation

$$x^n J_{n-1}(x) = \frac{d}{dx}[x^n J_n(x)],$$

For $n = 1$, we get

$$\int x J_0(x)\, dx = x J_1(x)$$

Now, A_n can be written as

$$A_n = \left[\frac{200\, x J_1(x)}{\xi_n^2 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)}\right]_0^{\xi_n} = \frac{200}{\xi_n \sinh(-\xi_n h/a)\, J_1(\xi_n)}$$

Hence, the required temperature distribution inside the cylinder is

$$u(r, z) = 200 \sum_{n=1}^{\infty} \frac{J_0(\xi_n r/a) \sinh[(\xi_n/a)(z-h)]}{\xi_n \sinh(-\xi_n h/a)\, J_1(\xi_n)}$$

where ξ_n are the positive zeros of $J_0(\xi)$.

EXAMPLE 2.6 Find the potential u inside the cylinder $0 \le r \le a,\ 0 \le \theta \le 2\pi,\ 0 \le z \le h,$ if the potential on the top $z = h,$ and on the lateral surface $r = a$ is held at zero, while on the base $z = 0$, the potential is given by $u(r, \theta, 0) = V_0(1 - r^2/a^2),$ where V_0 is a constant; r, θ, z are cylindrical polar coordinates.

Solution The potential u must be a single-valued continuous function and satisfy the Laplace equation inside the cylinder. To compute the potential inside the cylinder, we have to solve the following BVP:

$$\text{PDE:} \quad \nabla^2 u = 0$$

$$\text{BCs:} \quad u = 0 \quad \text{on } z = h,$$

$$u = 0 \quad \text{on } r = a,$$

$$u = V_0\left(1 - \frac{r^2}{a^2}\right) \quad \text{on } z = 0$$

In cylindrical coordinates, the general solution of the Laplace equation as given in Section 2.11 is

$$u(r, \theta, z) = J_n(\lambda r)\,(c_1 \cos n\theta + c_2 \sin n\theta)\,(c_3 e^{\lambda z} + c_4 e^{-\lambda z})$$

Since the face $z = 0$ has potential $V_0(1 - r^2/a^2),$ which is purely a function of r and is independent of θ and since the other faces of the cylinder are at zero potential, the potential at any point inside the cylinder will obviously be independent of θ. This is possible only when $n = 0$ in the general solution. Thus,

$$u(r, z) = J_0(\lambda r)\,(Ae^{\lambda z} + Be^{-\lambda z})$$

Using the BC: $u = 0$ on $z = h,$ we obtain

$$0 = J_0(\lambda r)\,(Ae^{\lambda h} + Be^{-\lambda h})$$

implying $Ae^{\lambda h} + Be^{-\lambda h} = 0,$ which yields

$$B = -\frac{Ae^{\lambda h}}{e^{-\lambda h}}$$

Hence, the solution is

$$u(r, z) = \frac{A}{e^{-\lambda h}} J_0(\lambda r)[e^{\lambda(z-h)} - e^{-\lambda(z-h)}]$$

or

$$u(r, z) = A_1 J_0(\lambda r) \sinh \lambda (z-h)$$

where $A_1 = A/e^{-\lambda h}$. Now, using the BC: $u = 0$ on the lateral surface, i.e., on $r = a$, we get

$$0 = A_1 J_0(\lambda a) \sinh \lambda (z-h)$$

implying $J_0(\lambda a) = 0$. This has infinitely many positive roots; denoting them by ξ_n we shall have

$$\xi_n = \lambda a \quad \text{or} \quad \lambda = \xi_n/a$$

The solution now takes the form

$$u(r, z) = A_1 J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z-h)\right], \qquad n = 1, 2, \ldots$$

The principle of superposition gives

$$u(r, z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z-h)\right]$$

The last BC: $u = V_0\left(1 - \dfrac{r^2}{a^2}\right)$ on $z = 0$ yields

$$V_0\left(1 - \frac{r^2}{a^2}\right) = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) J_0\left(\frac{\xi_n r}{a}\right)$$

This is a Fourier-Bessel's series. Multiplying both sides by $rJ_0(\xi_m r/a)$ and integrating, we get

$$V_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\frac{\xi_m r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \int_0^a r J_0\left(\frac{\xi_n r}{a}\right) J_0\left(\frac{\xi_m r}{a}\right) dr$$

Using the orthogonality property of the Bessel functions

$$\int_0^a x J_n(\alpha_i x)\, J_n(\alpha_j x)\, dx = \begin{cases} 0, & \text{if } i \neq j \\ \dfrac{a^2}{2} J_{n+1}^2(\alpha_i), & \text{if } i = j \end{cases}$$

where α_i, α_j are the zeros of $J_n(x) = 0$, we get

$$V_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\frac{\xi_n r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \frac{a^2}{2} J_1^2(\xi_n)$$

which gives

$$A_n = \frac{2V_0}{a^2 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\frac{\xi_n r}{a}\right) dr$$

By letting $\xi_n r/a = x$, this equation can be modified to

$$A_n = \frac{2V_0}{\xi_n^4 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} (\xi_n^2 - x^2)\, x J_0(x)\, dx$$

Using the well-known recurrence relation

$$x^\alpha J_{\alpha-1}(x) = \frac{d}{dx}[x^a J_\alpha(x)] \quad \text{for } \alpha = 1, 2, \ldots$$

we get

$$\int x J_0(x) = x J_1(x), \qquad \int x^2 J_1(x) = x^2 J_2(x)$$

From these relations, we obtain

$$A_n = \frac{2V_0}{\xi_n^4 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} (\xi_n^2 - x^2)\, d[x J_1(x)]$$

Integrating by parts, we get

$$A_n = \frac{4V_0}{\xi_n^4 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)} \int_0^{\xi_n} x^2 J_1(x)\, dx$$

$$= \frac{4V_0}{\xi_n^4 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)} \int_0^{\xi_n} d[x^2 J_2(x)]$$

$$= \frac{4V_0}{\xi_n^4 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)} [x^2 J_2(x)]_0^{\xi_n}$$

Thus,

$$A_n = \frac{4V_0 J_2(\xi_n)}{\xi_n^2 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)}$$

The recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

for $n = 1$ gives

$$J_0(\xi_n) + J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n)$$

Hence,

$$J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n)$$

since $J_0(\xi_n) = 0$. Therefore,

$$A_n = \frac{8V_0 J_1(\xi_n)}{\xi_n^3 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)}$$

Thus, the required potential inside the cylinder is

$$u(r, z) = \sum_{n=1}^{\infty} \frac{8V_0 J_0\left(\dfrac{\xi_n r}{a}\right) \sinh\left[\dfrac{\xi_n}{a}(z - h)\right]}{\xi_n^3 J_1(\xi_n) \sinh\left(-\dfrac{\xi_n h}{a}\right)}$$

2.12 SOLUTION OF LAPLACE EQUATION IN SPHERICAL COORDINATES

In Example 2.3, the Laplace equation is expressed in spherical coordinates and has the following form:

$$\nabla^2 u = \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \tag{2.88}$$

Let us assume the separable solution in the form

$$u(r, \theta, \phi) = R(r)\, F(\theta, \phi) \tag{2.89}$$

Substituting Eq. (2.89) into Eq. (2.88), we get

$$F\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial F}{\partial\theta}\right) + \frac{R}{\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2} = 0$$

Separation of variables gives

$$\frac{\dfrac{d}{dr}\left(r^2 \dfrac{dR}{dr}\right)}{R} = \frac{-\dfrac{1}{\sin\theta}\left\{\dfrac{\partial}{\partial\theta}\left(\sin\theta \dfrac{\partial F}{\partial\theta}\right) + \dfrac{1}{\sin\theta}\dfrac{\partial^2 F}{\partial\phi^2}\right\}}{F} = -\mu$$

where μ is a separation constant. Therefore,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\mu \tag{2.90}$$

$$\frac{1}{F\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}\right] = \mu \tag{2.91}$$

Equation (2.90) gives

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}+\mu R = 0$$

which is a Euler's equation. Hence, using the transformation $r = e^z$, the auxiliary equation can be written as

$$D(D-1)+2D+\mu = D^2+D+\mu = 0$$

where $D = d/dz$. Its roots are given by

$$D = \frac{-1\pm\sqrt{1-4\mu}}{2}$$

Let $\mu = -\alpha(\alpha+1)$; then we get

$$D = \frac{\left\{-1\pm 2\sqrt{\left(\alpha+\frac{1}{2}\right)^2}\right\}}{2} = -\frac{1}{2}\pm\left(\alpha+\frac{1}{2}\right)$$

Hence, $D = \alpha$ and $-(\alpha+1)$. Therefore, the solution of Euler's equation is

$$R = c_1 r^{\alpha} + c_2 r^{-(\alpha+1)} \tag{2.92}$$

Taking $-\mu = \alpha(\alpha+1)$, Eq. (2.91) becomes

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}+\alpha(\alpha+1)\,F\sin\theta = 0$$

Inserting

$$F = H(\theta)\Phi(\phi)$$

into the above equation and separating the variables, we obtain

$$\frac{\sin\theta}{H}\left[\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)+\alpha(\alpha+1)\sin\theta H\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \nu^2$$

where ν^2 is another separation constant. Then

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0 \tag{2.93}$$

$$\frac{\sin\theta}{H}\left[\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \alpha(\alpha+1)\sin(\theta)H\right] = \nu^2 \tag{2.94}$$

The general solution of Eq. (2.93) is

$$\Phi = c_3\cos\nu\phi + c_4\sin\nu\phi \tag{2.95}$$

provided $\nu \neq 0.$ If $\nu = 0,$ the solution is independent of ϕ which corresponds to the axisymmetric case. Equation (2.94) becomes, for the axisymmetric, case,

$$\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \alpha(\alpha+1)\sin(\theta)H = 0$$

Transforming the independent variable θ to x and by letting $x = \cos\theta,$ the abvoe equation becomes

$$\frac{d}{dx}\left(\sin\theta\frac{dH}{dx}\frac{dx}{d\theta}\right)\frac{dx}{d\theta} + \alpha(\alpha+1)\sin(\theta)H = 0$$

i.e.,

$$\frac{d}{dx}\left[(1-\cos^2\theta)\frac{dH}{dx}\right] + \alpha(\alpha+1)H = 0$$

or

$$\frac{d}{dx}\left[(1-x^2)\frac{dH}{dx}\right] + \alpha(\alpha+1)H = 0 \tag{2.96}$$

This is the well-known Legendre equation. Its general solution is given by

$$H = c_5P_\alpha(x) + c_6Q_\alpha(x), \qquad -1 \le x \le 1 \tag{2.97}$$

where P_α, Q_α are Legendre functions of the first and second kind respectively. For convenience let α be a positive integer, say $\alpha = n.$ Then

$$H = c_5P_n(\cos\theta) + c_6Q_n(\cos\theta) \tag{2.98}$$

Continuity of $H(\theta)$ at $\theta = 0, \pi$ implies the continuity of $H(x)$ at $x = \pm 1.$ Since $Q_n(x)$ has a singularity at $x = 1$, we choose $c_6 = 0.$ Therefore, in axisymmetric case the solution of Laplace equation in spherical coordinates is given by

$$u(r,\theta,\phi) = \{c_1r^\alpha + c_2r^{-(\alpha+1)}\}(c_3)[c_5P_n(\cos\theta)]$$

After renaming the constants and using the principle of superposition, we find the solution to be

$$u(r,\theta) = \sum_{n=0}^{\infty} [A_n r^n + B_n r^{-(n+1)}]\, P_n(\cos\theta) \tag{2.99}$$

EXAMPLE 2.7 In a solid sphere of radius '*a*', the surface is maintained at the temperature given by

$$f(\theta) = \begin{cases} k\cos\theta, & 0 \le \theta < \pi/2 \\ 0, & \pi/2 < \theta < \pi \end{cases}$$

Prove that the steady state temperature within the solid is

$$u(r,\theta) = k\left[\frac{1}{4}P_0(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta) + \frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{3}{32}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \cdots\right]$$

Solution It is known that the steady state temperature distribution is governed by the Laplace equation. In spherical polar coordinates, the axisymmetric solution of the Laplace equation in general with the assumption that the temperature should be finite at the origin is given by Eq. (2.99) in the form

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \tag{2.100}$$

Using the given BC: $u(a,\theta) = f(\theta)$, we have

$$u(a,\theta) = f(\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) = \sum_{n=0}^{\infty} b_n P_n(\cos\theta)$$

where $A_n a^n = b_n$. This is a Fourier-Legendre series, where

$$b_n = \frac{2n+1}{2}\int_{-1}^{1} f(\theta)\, P_n(\cos\theta)\, d\theta$$

In the present problem,

$$b_n = \frac{2n+1}{2}\int_{0}^{1} f(\theta) P_n(\cos\theta)\, d\theta$$

Let $\cos\theta = x$,

$$b_0 = \frac{1}{2}\int_0^1 kx \cdot P_0(x)\, dx = \frac{1}{2}\int_0^1 kx \cdot 1 \cdot dx = \frac{k}{4}$$

Hence, we get

$$A_0 = \frac{k}{4}$$

Also,

$$b_1 = \frac{3}{2}\int_0^1 kx \cdot x \cdot dx = \frac{k}{2} = A_1 a$$

Therefore,

$$A_1 = \frac{k}{2}\left(\frac{1}{a}\right)$$

$$b_2 = \frac{5}{2}\int_0^1 kxP_2(x)\,dx = \frac{5}{2}\int_0^1 kx\frac{3x^2-1}{2}dx = \frac{5}{16}k$$

Thus,

$$A_2 = \frac{5}{16}k \cdot \frac{1}{a^2}$$

Further,

$$b_3 = \frac{7}{2}\int_0^1 kxP_3(x)\,dx = \frac{7}{2}\int_0^1 kx\frac{5x^3-3x}{2}dx = 0$$

Similarly, noting that $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, we get

$$b_4 = -\frac{3}{32}k = A_4 a^4$$

Hence,

$$A_4 = -\frac{3}{32}k \cdot \frac{1}{a^4}, \cdots$$

Substituting these values of A_0, A_1, A_2,… into Eq. (2.100), we obtain, finally, the required temperature as

$$u(r,\theta) = k\left[\frac{1}{4}P_0(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta)\right.$$

$$\left. + \frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) + \left(-\frac{3}{32}\right)\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \cdots\right].$$

EXAMPLE 2.8 Find the potential at all points of space inside and outside of a sphere of radius $R = 1$ which is maintained at a constant distribution of electric potential $u(R, \theta) = f(\theta) = \cos 2\theta$.

Solution It is known that the potential on the surface of a sphere is governed by the Laplace equation. The Laplace equation in spherical polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

The possible general solution by variables separable method, after using superposition principle, is given by Eq. (2.99). Thus we have two possible solutions:

$$u_1(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \tag{2.101}$$

$$u_2(r, \theta) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta) \tag{2.102}$$

For points inside the sphere, we take the series (2.101). Why is this so? Applying the BC: $u(R, \theta)$ $= f(\theta) = \cos 2\theta$, we obtain

$$f(\theta) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta)$$

which is a generalized Fourier series of $f(\theta)$ in terms of the Legendre polynomials. Using the orthogonality property, we get

$$A_n R^n = \frac{2n+1}{2}\int_{-1}^{1} f(\theta)\, P_n(x)\, dx$$

Let $x = \cos\theta$. Then we have

$$A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta$$

For points outside the sphere, we take the series (2.102). Why is this so? Using the BC: $u(R, \theta) = f(\theta)$, we get

$$f(\theta) = \sum_{n=0}^{\infty} \frac{B_n}{R^{n+1}} P_n(\cos\theta)$$

Again, using the orthogonality property of Legendre polynomials, we have

$$B_n = \frac{2n+1}{2} R^{n+1} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta \, d\theta$$

In the present problem, it is assumed that at $R = 1,\; f(\theta) = \cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1.$ Hence,

$$A_n = \frac{2n+1}{2}\int_{-1}^{1}(2x^2 - 1)\,P_n(x)\,dx$$

However,

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Therefore,

$$2x^2 - 1 = \frac{4}{3}P_2(x) - \frac{1}{3}$$

Thus,

$$A_n = \frac{2n+1}{2}\left[\frac{4}{3}\int_{-1}^{1}P_2(x)\,P_n(x)\,dx - \frac{1}{3}\int_{-1}^{1}P_0(x)P_n(x)\,dx\right]$$

Using the orthogonality property of Legendre polynomials, all integrals vanish except those corresponding to $n = 0$ and $n = 2$. We obtain, therefore,

$$A_0 = -\frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1}P_0^2(x)\,dx = -\frac{1}{3}$$

$$A_2 = \frac{5}{2}\cdot\frac{4}{3}\int_{-1}^{1}P_2^2(x)\,dx = \frac{4}{3}$$

Also,

$$B_n = \frac{2n+1}{2}\int_{-1}^{1}(2x^2 - 1)\,P_n(x)\,dx$$

$$= \frac{2n+1}{2}\left[\frac{4}{3}\int_{-1}^{1}P_2(x)\,P_n(x)\,dx - \frac{1}{3}\int_{-1}^{1}P_0(x)\,P_n(x)\,dx\right]$$

which, on using the orthogonality property, gives the non-vanishing coefficients as

$$B_0 = -\frac{1}{3}, \qquad B_2 = \frac{4}{3}$$

Substituting these values of A_0 and A_2 into Eq. (2.101), we obtain

$$u_1(r, \theta) = -\frac{1}{3} + \frac{4}{3}r^2 P_2(\cos\theta)$$

which gives the potential everywhere inside the sphere. Similarly, substituting the values of B_0 and B_2 into Eq. (2.102), we get

$$u_2(r,\theta) = -\frac{1}{3r} + \frac{4}{3r^3} P_2(\cos\theta)$$

which gives the potential outside the sphere.

EXAMPLE 2.9 Find a general spherically symmetric solution of the following Helmholtz equation:

$$(\nabla^2 - k^2)u = 0$$

Solution In spherical polar coordinates, the Helmholtz equation can be written as

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} - k^2 u = 0 \tag{2.103}$$

In view of spherical symmetry, we look for u to be a function of r alone. Hence, Eq. (2.103) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - k^2 u = 0$$

Therefore, we have to solve

$$r^2\frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} - k^2 r^2 u = 0 \tag{2.104}$$

Let

$$u = \frac{1}{\sqrt{r}} F(r)$$

Differentiating twice with respect to r and rearranging, we obtain

$$2r\frac{\partial u}{\partial r} = -\frac{F(r)}{\sqrt{r}} + 2\sqrt{r}F'(r)$$

$$r^2\frac{\partial^2 u}{\partial r^2} = -\frac{3}{4}r^{-1/2}F(r) - r^{1/2}F'(r) + r^{3/2}F''(r)$$

Substituting the above relations, Eq. (2.104) becomes

$$r^2 F''(r) + rF'(r) - \left(k^2 r^2 + \frac{1}{4}\right)F(r) = 0$$

or

$$r^2 F''(r) + rF'(r) + \left[(ik)^2 r^2 - \left(\frac{1}{2}\right)^2\right]F(r) = 0$$

This is the Bessel equation whose solution is

$$F(r) = AJ_{1/2}(ikr) + BY_{1/2}(ikr)$$

where $J_{1/2}, Y_{1/2}$ are Bessel functions with imaginary arguments, and is rewritten as

$$F(r) = AI_{1/2}(kr) + BK_{1/2}(kr)$$

Therefore,

$$u(r) = r^{-1/2}[AI_{1/2}(kr) + BK_{1/2}(kr)]$$

But as $r \to \infty$, the solution should be finite, which is possible only if $A = 0$. It is also known that for large z,

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$$

Thus the acceptable spherically symmetric solution of the Helmholtz equation is given by

$$u(r) = Br^{-1/2}\sqrt{\frac{\pi}{2kr}}e^{-kr} = \frac{c}{r}e^{-kr}$$

where

$$c = B\sqrt{\frac{\pi}{2k}}$$

2.13 MISCELLANEOUS EXAMPLES

EXAMPLE 2.10 Show that the velocity potential for an irrotational flow of an incompressible fluid satisfies the Laplace solution.

Solution Let us consider a closed surface S enclosing a fixed volume V in the region occupied by a moving fluid as shown in Fig. 2.5.

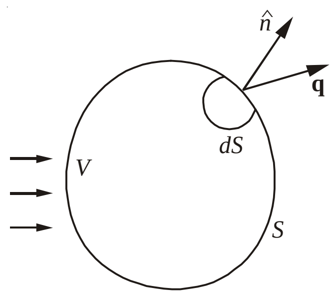

Fig. 2.5 Conservation of mass.

Let ρ be the density of the fluid. If $\hat{n}$ is a unit vector in the direction of the normal to the surface element dS and $\mathbf{q}$ the velocity of the fluid at that point, then the inward normal velocity is $(-\mathbf{q}\cdot\hat{n})$. Hence the mass of the fluid entering per unit time through the element dS is $(-\mathbf{q}\cdot\hat{n})\,dS$. It follows therefore that the mass of the fluid entering the surface S in unit time is

$$-\iint_S \rho(\mathbf{q}\cdot\hat{n})\,dS$$

Also, the mass of the fluid within S is

$$\iiint_V \rho\,dV$$

So the rate at which the mass goes on increasing is given by

$$\frac{\partial}{\partial t}\iiint_V \rho\,dV = \iiint_V \frac{\partial\rho}{\partial t}\,dV$$

By conservation of mass, the rate of generation of mass within a given volume under the assumption that no internal sources are present is equal to the net inflow of mass through the surface enclosing the given volume. Thus,

$$\iiint_V \frac{\partial\rho}{\partial t}\,dV = -\iint_S \rho(\mathbf{q}\cdot\hat{n})\,dS$$

$$= -\iiint_V \operatorname{div}(\rho\mathbf{q})\,dV \quad \text{[using the divergence theorem]}$$

Therefore,

$$\iiint_V \left[\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{q})\right] dV = 0$$

Since the integrand is a continuous function and since this result is true for any arbitrary volume element dV, it follows that the integrand is zero. Therefore,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{q} = 0$$

which is called the equation of continuity. For an incompressible fluid, $\rho = \text{constant}$ and, therefore,

$$\nabla\cdot\mathbf{q} = 0$$

Further, if the flow is irrotational, i.e., there exists a velocity potential ϕ such that

$$\mathbf{q} = -\nabla\phi$$

Hence,

$$\nabla\cdot\mathbf{q} = \nabla\cdot\nabla\phi = \nabla^2\phi = 0$$

Thus, an incompressible irrotational fluid satisfies the Laplace equation.

EXAMPLE 2.11 A thin rectangular homogeneous thermally conducting plate lies in the *xy*-plane defined by $0 \le x \le a$, $0 \le y \le b$. The edge $y = 0$ is held at the temperature $Tx(x-a)$, where T is a constant, while the remaining edges are held at 0°. The other faces are insulated and no internal sources and sinks are present. Find the steady state temperature inside the plate.

Solution Since no heat sources and sinks are present in the plate, the steady state temperature u must satisfy $\nabla^2 u = 0$. Hence the problem is to solve

$$\text{PDE: } \nabla^2 u = 0$$

$$\text{BCs: } u(0, y) = 0, \qquad u(a, y) = 0, \qquad u(x, b) = 0, \qquad u(x, 0) = Tx(x-a)$$

This is a typical Dirichlet's problem. The general solution satisfying the first three BCs is given by Eq. (2.47). Therefore,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sinh\left[\frac{n\pi}{a}(y-b)\right]$$

where

$$A_n \sinh \frac{-n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a} x\right) dx$$

Using the last BC: $u(x, 0) = Tx(x-a) = f(x)$, we get

$$\begin{aligned}
A_n \sinh \frac{-n\pi b}{a} &= \frac{2}{a}\int_0^a Tx(x-a) \sin\left(\frac{n\pi}{a} x\right) dx \\
&= \frac{2T}{a}\int_0^a x(x-a) \sin\left(\frac{n\pi}{a} x\right) dx \\
&= -\frac{a}{n\pi} \cdot \frac{2T}{a}\left[\int_0^a x(x-a)\, d\left\{\cos\left(\frac{n\pi}{a} x\right)\right\}\right] \\
&= -\frac{2T}{n\pi}\left[(x-a)\cos\left(\frac{n\pi}{a} x\right)\right]_0^a - \frac{a}{n\pi}\int_0^a (2x-a)\, d\left[\sin\left(\frac{n\pi}{a} x\right)\right] \\
&= \frac{2aT}{n^2\pi^2}\left[(2x-a)\sin\left(\frac{n\pi}{a} x\right)\right]_0^a - \int_0^a 2\sin\left(\frac{n\pi}{a} x\right) dx \\
&= \frac{2aT}{n^2\pi^2}\left\{a \sin n\pi + \frac{2a}{n\pi}\left[\cos\left(\frac{n\pi}{a} x\right)\right]_0^a\right\} \\
&= \frac{2aT}{n^2\pi^2}\,\frac{2a}{n\pi}(\cos n\pi - 1) = \frac{4a^2 T}{n^3\pi^3}[(-1)^n - 1]
\end{aligned}$$

Thus the required temperature distribution is given by

$$u(x, y) = \sum_{n=1}^{\infty} \operatorname{cosech}\left(-\frac{n\pi}{a}b\right)\frac{4Ta^2}{n^3\pi^3}[(-1)^n - 1]\sin\left(\frac{n\pi}{a}x\right)\sinh\left[\frac{n\pi}{a}(y-b)\right]$$

EXAMPLE 2.12 Solve

$$\nabla^2 u = 0, \qquad 0 \le x \le a, \qquad 0 \le y \le b$$

satisfying the BCs:

$$u(0, y) = 0, \qquad u(x, 0) = 0, \qquad u(x, b) = 0$$

$$\frac{\partial u}{\partial x}(a, y) = T\sin^3\frac{\pi y}{a}$$

Solution Using the variables separable method, one of the acceptable general solutions is given by Eq. (2.38). Hence

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Using the BC: $u(x, 0) = 0$, we get

$$0 = c_3(c_1 e^{px} + c_2 e^{-px})$$

implying $c_3 = 0$. Therefore,

$$u(x, y) = c_4 \sin py(c_1 e^{px} + c_2 e^{-px})$$

Now, using the BC: $u(x, b) = 0$, we obtain

$$0 = c_4 \sin pb(c_1 e^{px} + c_2 e^{-px})$$

$c_4 \ne 0$ (why?) implying $\sin pb = 0$ which gives

$$pb = n\pi \quad \text{or} \quad p = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots$$

Thus,

$$u(x, y) = c_4 \sin\left(\frac{n\pi}{b}y\right)(c_1 e^{px} + c_2 e^{-px})$$

Renaming the constants, we have

$$u(x, y) = \sin\left(\frac{n\pi}{b}y\right)\left[A\exp\left(\frac{n\pi}{b}x\right) + B\exp\left(-\frac{n\pi}{b}x\right)\right], \qquad n = 1, 2, \ldots$$

If we use the BC: $u(0, y) = 0$, we get

$$0 = \sin\left(\frac{n\pi}{b}y\right)(A + B)$$

giving $A + B = 0$; therefore, $A = -B$. Thus,

$$u(x, y) = A \sin\left(\frac{n\pi}{b} y\right)\left[\exp\left(\frac{n\pi}{b} x\right) - \exp\left(-\frac{n\pi}{b} x\right)\right]$$

$$= 2A \sin\left(\frac{n\pi}{b} y\right) \sinh\left(\frac{n\pi}{b} x\right), \qquad n = 1, 2, \ldots$$

Differentiating with respect to x, we obtain

$$\frac{\partial u}{\partial x} = 2A \frac{n\pi}{b} \sin\left(\frac{n\pi}{b} y\right) \cosh\left(\frac{n\pi}{b} x\right)$$

The last BC yields

$$T \sin^3 \frac{\pi y}{a} = 2A \frac{n\pi}{b} \sin\left(\frac{n\pi}{b} y\right) \cosh\left(\frac{n\pi}{b} a\right)$$

from which we can determine $2A$. Hence, the required solution is

$$u(x, y) = \frac{bT}{n\pi} \sin^3 \frac{\pi y}{a} \operatorname{sech} \frac{n\pi}{b} a \sinh\left(\frac{n\pi}{b} x\right)$$

The principle of superposition gives the required solution as

$$u(x, y) = \sum_{n=1}^{\infty} \frac{bT}{n\pi} \sin^3 \frac{\pi y}{a} \operatorname{sech}\left(\frac{n\pi}{b} a\right) \sinh\left(\frac{n\pi}{b} x\right)$$

EXAMPLE 2.13 Find the potential function $u(x, y, z)$ in a rectangular box defined by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ (see Fig. 2.6), if the potential is zero on all sides and the bottom, while $u = f(x, y)$ on the top of the box.

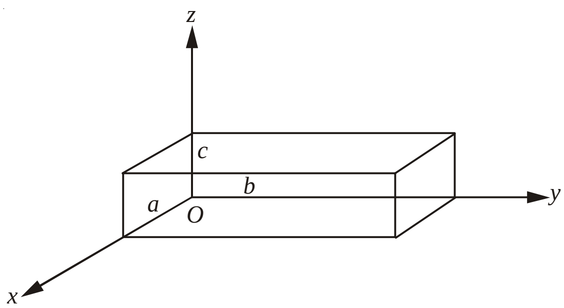

Fig. 2.6 Rectangular box.

Solution The potential distribution in the rectangular box satisfies the Laplace equation. Thus the problem is to solve

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

subject to the BCs:

$$u(0, y, z) = u(a, y, z) = 0$$

$$u(x, 0, z) = u(x, b, z) = 0$$

$$u(x, y, 0) = 0$$

$$u(x, y, c) = f(x, y)$$

Following the variables separable method, let us assume the solution in the form

$$u(x, y, z) = X(x)Y(y)Z(z)$$

Substituting into the Laplace equation, we get

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

which can also be written as

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \lambda_1^2$$

where λ_1 is a separation constant. Thus we have

$$X''(x) + \lambda_1^2 X(x) = 0 \tag{2.105}$$

After the second separation, we also have

$$\frac{Z''(z)}{Z(z)} - \lambda_1^2 = -\frac{Y''(y)}{Y(y)} = \lambda_2^2$$

$$Y''(y) + \lambda_2^2 Y(y) = 0 \tag{2.106}$$

$$Z''(z) - \lambda_3^2 Z(z) = 0 \tag{2.107}$$

where $\lambda_3^2 = \lambda_1^2 + \lambda_2^2$. The general solutions of Eqs. (2.105)–(2.107) are

$$X(x) = c_1 \cos \lambda_1 x + c_2 \sin \lambda_1 x$$

$$Y(y) = c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y$$

$$Z(z) = c_5 \cosh \lambda_3 z + c_6 \sinh \lambda_3 z$$

From the BCs,

$$X(0) = X(a) = 0$$

$$Y(0) = Y(b) = 0$$

$$Z(0) = 0$$

$$X(0) = 0 \text{ gives } c_1 = 0$$

$$X(a) = 0 \text{ gives } \lambda_1 a = m\pi.$$

Therefore,

$$\lambda_1 = \frac{m\pi}{a}, \qquad m = 1, 2, \ldots$$

Similarly,

$$Y(0) = 0 \text{ gives } c_3 = 0$$

$$Y(b) = 0 \text{ gives } \lambda_2 b = n\pi$$

Therefore,

$$\lambda_2 = \frac{n\pi}{b}, \qquad n = 1, 2, \ldots$$

Also, $Z(0) = 0$ gives $c_5 = 0$. Further, we note that

$$\lambda_3^2 = \lambda_1^2 + \lambda_2^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) = \lambda_{mn}^2 \quad \text{(say)}$$

Then

$$\lambda_3 = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \lambda_{mn}$$

The solutions now take the form

$$X(x) = c_{2m} \sin\frac{m\pi x}{a}, \qquad m = 1, 2, \ldots$$

$$Y(y) = c_{4n} \sin\frac{n\pi y}{b}, \qquad n = 1, 2, \ldots$$

$$Z(z) = c_{6mn} \sinh \lambda_{mn} z$$

Let $c_{mn} = c_{2m}c_{4n}c_{6mn}$; then, after using the principle of superposition, the required solution is

$$u(x, y, z) = X(x)Y(y)Z(z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh \lambda_{mn} z \qquad (2.108)$$

Using the final BC: $f(x, y) = u(x, y, c)$, we get

$$f(x, y) = \sum \sum c_{mn} \sinh \lambda_{mn} c \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

which is a double Fourier sine series. Thus, we have

$$c_{mn} \sinh \lambda_{mn} c = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx\, dy \qquad (2.109)$$

Therefore, Eqs. (2.108) and (2.109) constitute the required potential.

EXAMPLE 2.14 Find the electrostatic potential u in the annular region bounded by the concentric spheres $r = a$, $r = b$, $0 < a < b$ (see Fig. 2.7), if the inner and outer surfaces are kept at constant potentials u_1 and u_2, $u_1 \neq u_2$.

Solution The electrostatic potential satisfies the Laplace equation

$$\nabla^2 u = 0$$

It is natural that we choose spherical polar coordinates. From the problem, it is evident that we are looking for a solution with spherical symmetry which is independent of θ and ϕ. Hence, $u = u(r)$.

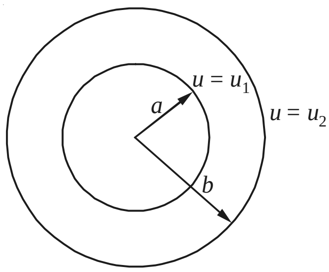

Fig. 2.7 Annular region.

Thus, we have to solve

$$\text{PDE: } \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0 \qquad (2.110)$$

subject to

$$\text{BCs: } u = u_1 \quad \text{at } r = a$$

$$u = u_2 \quad \text{at } r = b$$

Integrating Eq. (2.110) with respect to r, we obtain

$$r^2 \frac{\partial u}{\partial r} = A \quad \text{(a constant of integration)}$$

Again, integrating, we get

$$u = -\frac{A}{r} + B$$

Now, using the BCs, we have

$$u_1 = -\frac{A}{a} + B, \qquad u_2 = -\frac{A}{b} + B$$

Solving these equations, we get

$$A = \frac{u_2 - u_1}{(1/a) - (1/b)}, \qquad B = \frac{(u_1/b) - (u_2/a)}{(1/b) - (1/a)}$$

Hence, using these values, the required potential is

$$u = \frac{1}{(1/a) - (1/b)}\left[u_1\left(\frac{1}{r} - \frac{1}{b}\right) + u_2\left(\frac{1}{a} - \frac{1}{r}\right)\right]$$

EXAMPLE 2.15 A thermally conducting solid bounded by two concentric spheres of radii a and b as shown in Fig. 2.8, $a < b$, is such that the internal boundary is kept at $f_1(\theta)$ and the outer boundary at $f_2(\theta)$. Find the steady state temperature in the solid.

Solution It is known that the steady temperature T satisfies the Laplace equation. In the present problem,

$$T = T(r, \theta)$$

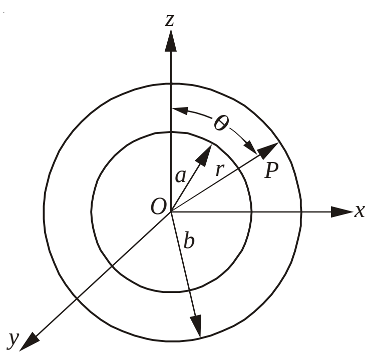

Fig. 2.8 Region bounded by two concentric spheres.

Thus, we have to solve

$$\text{PDE}: \nabla^2 T = 0$$

subject to the boundary conditions

$$T = f_1(\theta) \quad \text{at } r = a$$

$$T = f_2(\theta) \quad \text{at } r = b$$

In spherical polar coordinates, for axially symmetric case, the solution of the Laplace equation is given by Eq. (2.99) as follows:

$$T(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

Using the BCs:

$$f_1(\theta) = \sum_{n=0}^{\infty} \left(A_n a^n + \frac{B_n}{a^{n+1}} \right) P_n(\cos\theta) \tag{2.111}$$

$$f_2(\theta) = \sum_{n=0}^{\infty} \left(A_n b^n + \frac{B_n}{b^{n+1}} \right) P_n(\cos\theta) \tag{2.112}$$

In order to find the coefficients A_n and B_n, we have to express $f_1(\theta)$ and $f_2(\theta)$ in terms of Legendre polynomials and compare the coefficients. In this process, the following orthogonality relation is useful:

$$\int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n+1}, & \text{if } m = n \end{cases}$$

Thus, multiplying both sides of Eq. (2.111) by $P_m(\cos\theta)\sin\theta$ and integrating, we obtain

$$\int_0^{\pi} f_1(\theta) P_m(\cos\theta) \sin\theta \, d\theta = \sum_{n=0}^{\infty} \left(A_n a^n + \frac{B_n}{a^{n+1}} \right) \int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta$$

$$= \left(A_m a^m + \frac{B_m}{a^{m+1}} \right) \frac{2}{2m+1} \tag{2.113}$$

Similarly, Eq. (2.112) gives

$$\int_0^{\pi} f_2(\theta) P_m(\cos\theta) \sin\theta \, d\theta = \sum_{n=0}^{\infty} \left(A_n b^n + \frac{B_n}{b^{n+1}} \right) \int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta$$

$$= \left(A_m b^m + \frac{B_m}{b^{m+1}} \right) \frac{2}{2m+1} \tag{2.114}$$

Let

$$\frac{2m+1}{2}\int_0^{\pi} f_1(\theta)P_m(\cos\theta)\sin\theta\, d\theta = C_m$$

$$\frac{2m+1}{2}\int_0^{\pi} f_2(\theta)P_m(\cos\theta)\sin\theta\, d\theta = D_m$$

Then Eqs. (2.113) and (2.114) reduce to

$$A_m a^m + \frac{B_m}{a^{m+1}} = C_m$$

$$A_m b^n + \frac{B_m}{b^{m+1}} = D_m$$

Solving this pair of equations, we obtain

$$A_m = \frac{C_m a^{m+1} - D_m b^{m+1}}{a^{2m+1} - b^{2m+1}} \tag{2.115}$$

$$B_m = \frac{a^{m+1}b^{m+1}(C_m b^m - D_m a^m)}{b^{2m+1} - a^{2m+1}} \tag{2.116}$$

Hence, the required steady temperature is

$$T(r,\theta) = \sum_{m=0}^{\infty}\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)P_m(\cos\theta)$$

where A_m and B_m are given by Eqs. (2.115) and (2.116).

EXAMPLE 2.16 A thin annulus occupies the region $0 < a \le r \le b,\ 0 \le \theta \le 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at 0°, while along the outer edge the temperature is held at $T = K\cos(\theta/2)$, where K is a constant. Determine the temperature distribution in the annulus.

Solution Mathematically, the problem is to solve

$$\text{PDE:} \quad \nabla^2 T = 0, \qquad a \le r \le b,\ 0 \le \theta \le 2\pi$$

$$\text{BCs:} \ T(a,\theta) = 0$$

$$T(b,\theta) = k\cos\theta/2$$

The required general solution is given by Eq. (2.57) in the form

$$T(r,\theta) = (c_1 r^n + c_2 r^{-n})(c_3\cos n\theta + c_4\sin n\theta)$$

Using the first BC, we get

$$0 = (c_1 a^n + c_2 a^{-n})\,(c_3 \cos n\theta + c_4 \sin n\theta)$$

implying thereby $c_1 a^n + c_2 a^{-n} = 0$, or $c_2 = -c_1 a^{2n}$. After adjusting the constants suitably, we have

$$T(r, \theta) = \left(r^n - \frac{a^{2n}}{r^n} \right) (A \cos n\theta + B \sin n\theta)$$

The principle of superposition gives

$$T(r, \theta) = \sum_{n=1}^{\infty} \left(r^n - \frac{a^{2n}}{r^n} \right) (A_n \cos n\theta + B_n \sin n\theta)$$

Now, using the second boundary condition, we obtain

$$T(b, \theta) = K \cos \frac{\theta}{2} = \sum_{n=1}^{\infty} (b^n - b^{-n} a^{2n})\,(A_n \cos n\theta + B_n \sin n\theta)$$

which is a full-range Fourier series. Hence,

$$\begin{aligned} A_n (b^n - b^{-n} a^{2n}) &= \frac{1}{\pi} \int_0^{2\pi} K \cos \frac{\theta}{2} \cos n\theta \, d\theta \\ &= \frac{k}{2\pi} \int_0^{2\pi} \left[\cos \left(n + \frac{1}{2} \right)\theta + \cos \left(n - \frac{1}{2} \right)\theta \right] d\theta \\ &= \frac{k}{2\pi} \left[\frac{\sin\left(n + \frac{1}{2} \right)\theta}{n + \frac{1}{2}} + \frac{\sin\left(n - \frac{1}{2} \right)\theta}{n - \frac{1}{2}} \right]_0^{2\pi} \\ &= 0 \end{aligned}$$

implying $A_n = 0$. Also,

$$\begin{aligned} B_n (b^n - b^{-n} a^{2n}) &= \frac{k}{\pi} \int_0^{2\pi} \cos \frac{\theta}{2} \sin n\theta \, d\theta \\ &= \frac{k}{2\pi} \int_0^{2\pi} \left[\sin \left(n + \frac{1}{2} \right)\theta + \sin \left(n - \frac{1}{2} \right)\theta \right] d\theta \\ &= -\frac{k}{2\pi} \left[\frac{\cos\left(n + \frac{1}{2} \right)\theta}{n + \frac{1}{2}} + \frac{\cos\left(n - \frac{1}{2} \right)\theta}{n - \frac{1}{2}} \right]_0^{2\pi} \end{aligned}$$

$$= -\frac{k}{2\pi}\left(-\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} \right)$$

$$= \frac{k}{\pi}\left(\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}} \right) = \frac{k}{\pi}\,\frac{2n}{n^2-\frac{1}{4}}$$

or

$$B_n(b^n - b^{-n}a^{2n}) = \frac{8kn}{\pi(4n^2-1)}$$

Thus the temperature distribution in the annulus is given by

$$T(r,\theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\left[\frac{(r/a)^n-(a/r)^n}{(b/a)^n-(a/b)^{-n}}\right]\sin n\theta$$

EXAMPLE 2.17 V is a function of r and θ satisfying the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

within the region of the plane bounded by $r = a$, $r = b$, $\theta = 0$, $\theta = \pi/2$. Its value along the boundary $r = a$ is $\theta(\pi/2 - \theta)$, along the other boundaries is zero. Prove that

$$V = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(r/b)^{4n-2}-(b/r)^{4n-2}}{(a/b)^{4n-2}-(b/a)^{4n-2}}\left[\frac{\sin(4n-2)\theta}{(2n-1)^3}\right]$$

Solution The task is to solve the PDE

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

subject to the following boundary conditions:

(i) $V(b,\theta) = 0, \qquad 0 < \theta < \pi/2$

(ii) $V(r,\pi/2) = 0, \qquad a < r \le b$

(iii) $V(r,0) = 0, \qquad a < r \le b$

(iv) $V(a,\theta) = \theta(\pi/2 - \theta), \qquad 0 < \theta < \pi/2.$

The three possible solutions (see Section 2.8) are given as follows:

$$V = (c_1 r^p + c_2 r^{-p})\,(c_3 \cos p\theta + c_4 \sin p\theta)$$

$$V = [c_1 \cos(p \ln r) + c_2 \sin(p \ln r)]\,(c_3 e^{p\theta} + c_4 e^{-p\theta})$$

$$V = (c_1 \ln r + c_2)\,(c_3\theta + c_4)$$

Since the problem is not defined for $r = 0,\ \infty,$ the second and third solutions are not acceptable. Hence, the generally acceptable solution is the first one. The boundary condition (iii) gives

$$0 = c_3\,(c_1 r^p + c_2 r^{-p})$$

implying $c_3 = 0$. The boundary condition (ii) implies

$$0 = c_4 \sin p\frac{\pi}{2}\,(c_1 r^p + c_2 r^{-p})$$

Therefore,

$$\sin p\frac{\pi}{2} = 0 \quad \text{or} \quad p = 2n, \qquad n = 1, 2, \ldots$$

Thus, the possible solution of the given equation has the form

$$V(r, \theta) = c_4 \sin(2n\theta)(c_1 r^{2n} + c_2 r^{-2n})$$

Now, applying the boundary condition (i), we get

$$0 = c_4 \sin(2n\theta)\,(c_1 b^{2n} + c_2 b^{-2n})$$

which gives $c_2 = -c_1 b^{4n}$. Therefore,

$$V(r, \theta) = c_1 c_4 \sin(2n\theta)[r^{2n} - r^{-2n} b^{4n}]$$

Superposing all the solutions, we obtain

$$V(r, \theta) = \sum_{n=1}^{\infty} c_n \sin(2n\theta)(r^{2n} - r^{-2n} b^{4n})$$

Satisfying boundary conditions (iv), we get

$$\theta\left(\frac{\pi}{2} - \theta\right) = \sum c_n \sin(2n\theta)\left(\frac{a^{4n} - b^{4n}}{a^{2n}}\right)$$

which is a Fourier sine series. Thus, we have

$$\frac{2}{\pi/2}\int_0^{\pi/2} \theta\left(\frac{\pi}{2} - \theta\right)\sin(2n\theta) = c_n\left(\frac{a^{4n} - b^{4n}}{a^{2n}}\right)$$

Integrating by parts, we obtain

$$\left[\left(\frac{\pi}{2}\theta-\theta^2\right)\left\{-\frac{\cos(2n\theta)}{2n}\right\}-\left(\frac{\pi}{2}-2\theta\right)\left\{-\frac{\sin(2n\theta)}{4n^2}\right\}+(-2)\left\{\frac{\cos(2n\theta)}{8n^3}\right\}\right]_0^{\pi/2}=c_n\,\frac{\pi}{4}\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)$$

On simplification, we get

$$-\frac{1}{4n^3}\{(-1)^n-1\}=\frac{\pi}{4}c_n\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)$$

Thus,

$$\frac{\pi}{4}c_n\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)=\begin{cases}\dfrac{1}{2n^3}, & \text{for } n \text{ odd}\\[2mm] 0, & \text{for } n \text{ even}\end{cases}$$

Hence, the required solution is

$$V(r,\theta)=\sum_{1}^{\infty}\frac{2}{\pi}\frac{1}{(2n-1)^3}\left(\frac{a}{r}\right)^{4n-2}\sin(4n-2)\theta\left(\frac{r^{8n-4}-b^{8n-4}}{a^{8n-4}-b^{8n-4}}\right)$$

which can be recast in the form given in the problem.

EXAMPLE 2.18 Determine the potential of a grounded conducting sphere in a uniform field defined by

$$\text{PDE: } \nabla^2 u=0, \qquad 0\le r<a,\ 0<\theta<\pi,\ 0\le\phi<2\pi$$

$$\text{BCs: (i) } u(a,\theta)=0.$$

$$\text{(ii) } u\to -E_0 r\cos\theta \text{ as } r\to\infty.$$

Solution In spherical polar coordinates, with axial symmetry, the solution of the Laplace equation is given by Eq. (2.99) in the form

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Using the boundary condition (ii), we have

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)=-E_0 r\cos\theta$$

which is true only for $n=1$, when $P_1(\cos\theta)=\cos\theta$. Also, $A_n=0$ for $n\ge 2$. Therefore,

$$u(r,\theta)=A_1 r\cos\theta=-E_0 r\cos\theta$$

implying $A_1 = -E_0$. Hence,

$$u(r,\theta) = -E_0 r\cos\theta + \sum_{n=1}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

Now, applying the boundary condition (i), we get

$$0 = -E_0 a\cos\theta + \sum_{n=1}^{\infty} \frac{B_n}{a^{n+1}} P_n(\cos\theta)$$

Multiplying both sides by $P_m(\cos\theta)\sin\theta$ and integrating between the limits 0 to π, we have

$$E_0 a\int_0^{\pi} \cos(\theta) P_m(\cos\theta)\sin\theta\, d\theta = \sum_{n=1}^{\infty} \frac{B_n}{a^{n+1}} \int_0^{\pi} P_n(\cos\theta)\, P_m(\cos\theta)\sin\theta\, d\theta \quad (2.117)$$

Using the orthogonality property

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0, & \text{for } m \neq n \\ \dfrac{2}{2m+1}, & \text{for } m = n \end{cases}$$

we obtain

$$\frac{B_m}{a^{m+1}}\frac{2}{2m+1} = E_0 a\int_0^{\pi} \cos(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta$$

or

$$B_m = \frac{2m+1}{2} E_0 a^{m+2} \int_0^{\pi} \cos(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta$$

It can be verified that the integral on the right-hand side of the Eq. (2.117) vanishes for all m except when $m = 1$, in which case

$$B_1 = E_0 a^3$$

Therefore, the required potential is given by

$$u(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

EXAMPLE 2.19 The steady, two-dimensional, incompressible viscous fluid flow past a circular cylinder, when the inertial terms are neglected (Stokes flow), is governed by the biharmonic PDE: $\nabla^4\psi = 0$, where ψ is the stream function. Find its solution subject to the BCs:

(i) $\psi(r,\theta) = \partial\psi/\partial r = 0$ on $r = 1$

(ii) $\psi(r,\theta) \to r\sin\theta$ as $r \to \infty$.

Solution In view of the cylindrical geometry, we can write

$$\nabla^4\psi = \nabla^2(\nabla^2\psi) = 0$$

where

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi$$

Using the variables separable method, let us look for a solution of the form

$$\psi(r, \theta) = f(r)\sin\theta$$

Therefore,

$$\frac{\partial\psi}{\partial r} = f'(r)\sin\theta, \qquad \frac{\partial\psi}{\partial\theta} = f(r)\cos\theta$$

$$\frac{\partial^2\psi}{\partial r^2} = f''(r)\sin\theta, \qquad \frac{\partial^2\psi}{\partial\theta^2} = -f(r)\sin\theta$$

Hence,

$$\nabla^2\psi = \left[f''(r) + \frac{1}{r}f'(r) - \frac{1}{r^2}f(r)\right]\sin\theta$$

which can also be written in the form

$$\nabla^2\psi = F(r)\sin\theta$$

where

$$F(r) = f''(r) + \frac{1}{r}f'(r) - \frac{1}{r^2}f(r)$$

Therefore,

$$\nabla^4\psi = \nabla^2(\nabla^2\psi) = \nabla^2[F(r)\sin\theta] = 0$$

i.e.,

$$\left[F''(r) + \frac{1}{r}F'(r) - \frac{1}{r^2}F(r)\right]\sin\theta = 0$$

implying

$$F''(r) + \frac{1}{r}F'(r) - \frac{1}{r^2}F(r) = 0$$

Introducing the transformation $r = e^z$, $D = d/dz$, the above equation becomes

$$[D(D-1) + D - 1]F(r) = 0$$

or

$$(D^2 - 1)\, F(r) = 0$$

Its complementary function is

$$F(r) = Ae^z + Be^{-z} = Ar + \frac{B}{r}$$

or

$$f''(r) + \frac{1}{r} f'(r) - \frac{1}{r^2} f(r) = Ar + \frac{B}{r}$$

i.e.

$$r^2 f''(r) + rf'(r) - f(r) = Ar^3 + Br$$

which is a homogeneous ordinary differential equation. Again using the transformation $r = e^z$, $D = d/dz$, we get

$$[D(D-1) + D - 1]\, f = Ae^{3z} + Be^z$$

or

$$(D^2 - 1) f = Ae^{3z} + Be^z$$

Its complementary function is

$$f(r) = Ce^z + De^{-z}$$

while its particular integral is

$$\frac{1}{D^2 - 1}(Ae^{3z} + Be^z) = \frac{Ae^{3z}}{8} + \frac{Bze^z}{2}$$

Therefore,

$$f(r) = Cr + \frac{D}{r} + \frac{A}{8} r^3 + \frac{B}{2} r \ln r$$

Thus, we have

$$\psi = \left(\frac{A}{8} r^3 + \frac{B}{2} r \ln r + Cr + \frac{D}{r} \right) \sin\theta$$

Now to satisfy the BC: $\psi \to r \sin\theta$, as $r \to \infty$ and from physical considerations, we choose $A = 0$, Therefore,

$$\psi = \left(\frac{B}{2} r \ln r + Cr + \frac{D}{r} \right) \sin\theta$$

The boundary condition $\psi = 0$ on $r = 1$ gives $(C + D)\sin\theta = 0$, implying $C = -D$. Also, the boundary condition $\partial\psi/\partial r = 0$ on $r = 1$ gives

$$\frac{B}{2} + C - D = 0 = \frac{B}{2} - 2D$$

implying $B = 4D$. Hence, the general solution is

$$\psi = 2D\left(r \ln r - \frac{r}{2} + \frac{1}{2r}\right)\sin\theta$$

EXAMPLE 2.20 The problem of axisymmetric fluid flow in a semi-infinite or in a finite circular pipe of radius a is described as follows in cylindrical coordinates:

$$\text{PDE:} \quad \nabla^2 u = 0, \qquad 0 < r < a$$

$$\text{BCs:} \quad \text{(i)} \quad \frac{\partial u}{\partial r} = 0 \quad \text{at } r = 0$$

$$\text{(ii)} \quad \frac{\partial u}{\partial z} = 0, \qquad \frac{\partial u}{\partial r} = V(z) \quad \text{at } r = a$$

Show that the speed of suction is given by

$$V(z) = -\sum_{n=1}^{\infty} \alpha_n (A_n \cosh \alpha_n z + B_n \sinh \alpha_n z)\, J_1(\alpha_n a)$$

Solution In cylindrical coordinates (r, θ, z),

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

In axisymmetric case, the above equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2.118}$$

Let $u(r, z) = f(r)\,\phi(z)$ which, when substituted into Eq. (2.118), gives

$$\frac{f'' + (1/r) f'}{f} = -\frac{\phi''}{\phi} = -\alpha^2 \text{ (say)}$$

Then

$$\phi'' - \alpha^2 \phi = 0 \tag{2.119}$$

$$f'' + \frac{1}{r} f' + \alpha^2 f = 0 \tag{2.120}$$

The solution of Eq. (2.119) is

$$\phi = A\cosh\alpha z + B\sinh\alpha z$$

Equation (2.120) can be rewritten as

$$r^2 f'' + rf' + \alpha^2 r^2 f = 0$$

which is a Bessel's equation of zeroth order whose general solution is

$$f = J_0(\alpha r) + DY_0(\alpha r)$$

Here, $J_0(\alpha r)$ and $Y_0(\alpha r)$ are zeroth order Bessel functions of first and second kind respectively. Therefore, the typical solution is

$$u = (A\cosh\alpha z + B\sinh\alpha z)\,[J_0(\alpha r) + DY_0(\alpha r)]$$

Now, $Y_0(\alpha r)$ is infinite at $r = 0$, and hence $D = 0$. Therefore, the possible solution is

$$u = (A\cosh\alpha z + B\sinh\alpha z)\,J_0(\alpha r)$$

The condition $\partial u/\partial r = 0$ at $r = 0$ is automatically satisfied, since $J_0'(\alpha r) = 0$ at $r = 0$. Now the boundary condition $\partial u/\partial z = 0$ at $r = a$ gives $J_0(\alpha a) = 0$, implying that αa are the zeros of the Bessel function J_0. Let these zeros be $\alpha_n a\,(n = 0, 1, 2, \ldots)$. Thus the appropriate solution is

$$u(r, z) = \sum_{n=1}^{\infty} \alpha_n (A_n \cosh\alpha_n z + B_n \sinh\alpha_n z)\,J_0(\alpha_n r)$$

Using the fact that, $J_0' = -J_1$, the speed of suction is given by

$$V(z) = \left(\frac{\partial u}{\partial r}\right)_{r=a} = -\sum_{n=1}^{\infty} \alpha_n (A_n \cosh\alpha_n z + B_n \sinh\alpha_n z)\,J_1(\alpha_n a)$$

EXAMPLE 2.21 Solve the following Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$$

subject to the boundary conditions

$$u(0, y) = u(5, y) = u(x, 0) = u(x, 4) = 0$$

Solution We assume the solution of the form

$$u = v + \omega \tag{2.121}$$

where v is a particular solution of the Poisson equation and ω is the solution of the corresponding homogeneous Laplace equation. That is,

$$\nabla^2 v = 2 \tag{2.122}$$

$$\nabla^2 \omega = 0 \tag{2.123}$$

It is customary to assume that v has the form

$$v(x, y) = a + bx + cy + dx^2 + exy + fy^2$$

Substituting this into Eq. (2.122), we get

$$2d + 2f = 2$$

Let $f = 0$. Then $d = 1$. The remaining coefficients can be chosen arbitrarily. Thus we take

$$v(x, y) = -5x + x^2 \tag{2.124}$$

so that v reduces to zero (satisfies the boundary conditions) on the sides $x = 0$ and $x = 5$.

Now, we shall find ω from

$$\nabla^2 \omega = 0, \qquad 0 < x < 5, \qquad 0 < y < 4 \tag{2.125}$$

satisfying

$$\omega(0, y) = -v(0, y) = 0$$

$$\omega(5, y) = -v(5, y) = 0$$

$$\omega(x, 0) = -v(x, 0) = -(-5x + x^2)$$

$$\omega(x, 4) = -v(x, 4) = -(-5x + x^2)$$

The above conditions are obtained by using Eqs. (2.121), (2.124) and the given boundary conditions. By using the superposition principle (see Section 2.5), the general solution of Eq. (2.125) is found to be

$$\omega(x, y) = \sum_{n=1}^{\infty} \sin(n\pi x/5)\,[a_n \exp(n\pi y/5) + b_n \exp(-n\pi y/5)] \tag{2.126}$$

Now, applying the non-homogeneous BC: $\omega(x, 0) = -(-5x + x^2)$, we get, after renaming the constants, the equation

$$\omega(x, 0) = -(-5x + x^2) = \sum A_n \sin(n\pi x/5)$$

Also, applying the BC: $\omega(x, 4) = -(-5x + x^2)$, Eq. (2.126) can be rewritten in the form

$$-(-5x + x^2) = \sum_{n=1}^{\infty} \left(a_n \cosh \frac{4n\pi}{5} + b_n \sinh \frac{4n\pi}{5} \right) \sin \frac{n\pi x}{5} \tag{2.127}$$

which gives

$$a_n = \frac{2}{5}\int_0^5 (5x - x^2)\sin\frac{n\pi x}{5}\,dx$$

Now, integrating by parts, the right-hand side yields

$$a_n = \frac{2}{5}\left[(5x-x^2)\left(-\frac{5}{n\pi}\cos\frac{n\pi}{5}x\right) - (5-2x)\left(-\frac{5^2}{n^2\pi^2}\sin\frac{n\pi x}{5}\right) + (-2)\left(\frac{5^3}{n^3\pi^3}\cos\frac{n\pi}{5}x\right)\right]_0^5$$

$$= \frac{2}{5}\left[-\frac{2(5^3)}{n^3\pi^3}\cos n\pi + \frac{2}{n^3}\left(\frac{5^3}{\pi^3}\right)\right]$$

$$= \frac{4(5^2)}{\pi^3}\left[\frac{1}{n^3} - \frac{\cos n\pi}{n^3}\right] = \frac{4(5)^2}{\pi^3}\left[\frac{1}{n^3} - \frac{(-1)^n}{n^3}\right]$$

Hence,

$$a_n = \begin{cases} \dfrac{8(5^2)}{\pi^3 n^3}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even} \end{cases} \tag{2.128}$$

Also, from Eq. (2.127), we have

$$a_n \cosh\frac{4n\pi}{5} + b_n \sinh\frac{4n\pi}{5} = \frac{2}{5}\int_0^5 (5x-x^2)\sin\left(\frac{n\pi}{5}x\right)dx = a_n$$

Therefore,

$$b_n = \frac{a_n\left[1-\cosh\left(\frac{4}{5}n\pi\right)\right]}{\sinh\left(\frac{4}{5}n\pi\right)} \tag{2.129}$$

Substituting a_n, b_n from Eqs. (2.128) and (2.129) into Eq. (2.126), we get

$$\omega(x, y) = \sum_{n=1}^{\infty} a_n\left[\cosh\left(\frac{n\pi}{5}y\right)\sinh\left(\frac{4}{5}n\pi\right) - \cosh\left(\frac{4}{5}n\pi\right)\sinh\left(\frac{n\pi}{5}y\right)\right.$$
$$\left. + \sinh\left(\frac{n\pi}{5}y\right)\sin\left(\frac{n\pi}{5}x\right)\Big/\sinh\left(\frac{4}{5}n\pi\right)\right]$$

or

$$\omega(x, y) = \sum_{n=1}^{\infty} \left[\sinh\left\{\frac{n\pi}{5}(4-y)\right\} + \sinh\left(\frac{n\pi}{5}y\right)\right]\sin\left(\frac{n\pi}{5}x\right) \Big/ \sinh\left(\frac{4}{5}n\pi\right) \qquad (2.130)$$

Combining Eqs. (2.124), (2.128) and (2.130), the solution of the given Poisson equation is $u(x, y)$ =

$$x(x-5) + \frac{8\times 5^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{\sinh\,(2n-1)\,\pi(4-y)/5 + \sinh\,[(2n-1)\pi y/5]}{\sinh\left[(2n-1)\frac{4}{5}\pi\right]}\right] \times \left\{\frac{\sin\,[(2n-1)\pi x/5]}{(2n-1)^3}\right\}$$

EXAMPLE 2.22 Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$. Let $P(x, y, z)$ be any point in the interior of $\mathbb{R}$, as shown in Fig. 2.9. Let ϕ be a harmonic function in $\overline{\mathbb{R}}$; also, let $\psi = 1/r$, where r is the distance from P. Applying Green's second identity, show that

$$\phi(P) = \frac{1}{4\pi}\iint_{\partial\mathbb{R}}\left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]ds$$

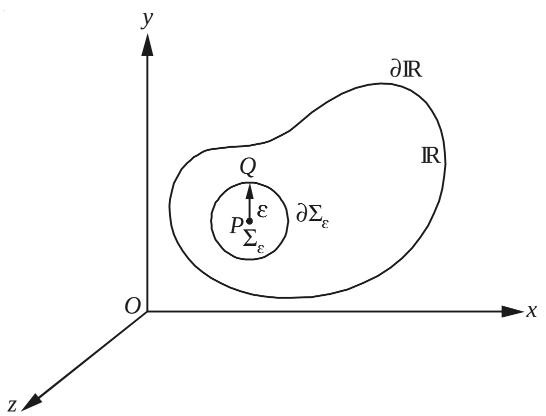

Fig. 2.9 An illustration of Example 2.22.

Solution Since ψ possesses a point of discontinuity in $\mathbb{R}$ at $P(x, y, z)$, Green's second identity cannot be directly applied to ϕ and ψ. However, $\psi = 1/r$ is bounded in $\mathbb{R}-\Sigma_\varepsilon$ with the boundary $\partial\mathbb{R}\cup\partial\Sigma_\varepsilon$, where Σ_ε is a sphere of radius ε with centre at P. Now applying Green's second identity (2.19) to functions ϕ and ψ in $\mathbb{R}-\Sigma_\varepsilon$, we get

$$\iiint_{\mathbb{R}-\Sigma_\varepsilon}\left[\phi\nabla^2(1/r) - \frac{1}{r}\nabla^2\phi\right]dV = \iint_{\partial\mathbb{R}}\left[\phi\frac{\partial}{\partial n}(1/r) - (1/r)\frac{\partial\phi}{\partial n}\right]dS$$

$$+ \iint_{\partial\Sigma_\varepsilon}\phi\frac{\partial}{\partial n}(1/r)dS - \iint_{\partial\Sigma_\varepsilon}(1/r)\left(\frac{\partial\phi}{\partial n}\right)dS \qquad (2.131)$$

From the right-hand side of Eq. (2.131), we observe that the last two integrals depend only on ε. But in the direction of the exterior normal to $\partial\Sigma_\varepsilon$, we find that

$$\frac{\partial}{\partial n}(1/r)\bigg|_{\partial\Sigma_\varepsilon} = -\frac{\partial}{\partial r}(1/r)\bigg|_{r=\varepsilon} = \frac{1}{\varepsilon^2}$$

Therefore,

$$\iint_{\partial\Sigma_\varepsilon} \phi\frac{\partial}{\partial n}(1/r)\,dS = \frac{1}{\varepsilon^2}\iint_{\partial\Sigma_\varepsilon} \phi\,dS = \frac{4\pi\varepsilon^2}{\varepsilon^2}\phi(Q) = 4\pi\phi^*(Q)$$

where $\phi^*(Q)$ is the average value of $\phi(Q)$ on $\partial\Sigma_\varepsilon$. Further, the third integral

$$-\iint_{\partial\Sigma_\varepsilon} (1/r)\frac{\partial\phi}{\partial n}\,dS = -\frac{1}{\varepsilon}\iint_{\partial\Sigma_\varepsilon}\left(\frac{\partial\phi}{\partial n}\right)dS = -4\pi\varepsilon\left(\frac{\partial\phi^*}{\partial n}\right)$$

where $(\partial\phi/\partial n)^*$ is an average value of the normal derivative on $\partial\Sigma_\varepsilon$. Substituting these results and using the fact that $\nabla^2(1/r) = 0$ in $\mathbb{R} - \Sigma_\varepsilon$, we obtain

$$\iiint_{\mathbb{R}-\Sigma_\varepsilon} (-1/r)\nabla^2\phi\,dV = \iint_{\partial\mathbb{R}}\left[\phi\frac{\partial}{\partial n}(1/r) - (1/r)\frac{\partial\phi}{\partial n}\right]dS + 4\pi\phi^*(P) + 4\pi\varepsilon\left(\frac{\partial\phi}{\partial n}\right)^* \quad (2.132)$$

Now, taking the limit as $\varepsilon \to 0$, and using the fact that ϕ is harmonic in $\mathbb{R} - \Sigma_\varepsilon$, we arrive at the fundamental result

$$\phi(P) = \frac{1}{4\pi}\iint_{\partial\mathbb{R}}\left[(1/r)\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}(1/r)\right]dS \quad (2.133)$$

Thus, the value of a harmonic function at any point of $\mathbb{R}$ can be obtained in terms of the values of ϕ and $\partial\phi/\partial n$ on the boundary $\partial\mathbb{R}$ of the region $\mathbb{R}$.

EXAMPLE 2.23 Find the solution of the following Helmholtz equation, using separation of variables method:

$$\nabla^2 u + K^2 u = u_{xx} + u_{yy} + u_{zz} + K^2 u = 0 \quad (2.134)$$

Solution It may be noted that the Laplacian in cartesian coordinates is a PDE with constant coefficients, while in cylindrical or spherical coordinates, it is a PDE with variable coefficients. Thus, let us assume the solution of the given Helmholtz equation in the form

$$u(x, y, z) = X(x)\,Y(y)\,Z(z)$$

where $X(x)$ is a function of x alone, $Y(y)$ is a function of y alone, $Z(z)$ is a function of z only. Substituting into the given Helmholtz equation, we get

$$X''(x)\,Y(y)\,Z(z) + X(x)\,Y''(y)\,Z(z) + X(x)\,Y(y)\,Z''(z) + K^2X(x)\,Y(y)\,Z(z) = 0,$$

which can be rewritten as

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + K^2 = 0\,.$$

This equation is satisfied iff

$$\frac{X''(x)}{X(x)} = -K_1^2, \frac{Y''(y)}{Y(y)} = -K_2^2, \frac{Z''(z)}{Z(z)} = -K_3^2 \tag{2.135}$$

and $$K^2 = K_1^2 + K_2^2 + K_3^2$$

The sign of these separation constants K_1, K_2 and K_3 need not be same, of course depends on the physical considerations. The solution of the three ODEs in Eq. (2.135) can be written in the form

$$\left.\begin{aligned} X(x) &= C_1 e^{iK_1 x} + C_2 e^{-iK_1 x} \\ Y(y) &= C_3 e^{iK_2 y} + C_4 e^{-iK_2 y} \\ Z(z) &= C_5 e^{iK_3 z} + C_6 e^{-iK_3 z} \end{aligned}\right\} \tag{2.136}$$

Hence, in general, the solution of Eq. (2.134) can be written as

$$u(x, y, z) = Ae^{iK\mathbf{r}} + Be^{-iK\mathbf{r}}.$$

However, if K^2 is positive, the solution is of the form

$$u(x, y, z) = A\cos(K_1x + K_2y + K_3z) + B\sin(K_1x + K_2y + K_3z),$$

while, if K^2 is negative, the solution is found to be

$$u(x, y, z) = A\cosh(K_1x + K_2y + K_3z) + B\sinh(K_1x + K_2y + K_3z).$$

EXERCISES

1. Solve the following boundary value problem:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le r \le 10,\ 0 \le \theta \le \pi$$

$$\text{BCs: } u(10, \theta) = \frac{400}{\pi}(\pi\theta - \theta^2)$$

$$u(r, 0) = 0 = u(r, \pi)$$

$$u(0, \theta) \text{ is finite}$$

2. A homogeneous thermally conducting solid is bounded by the concentric spheres $r = a$, $r = b$, $0 < a < b$. There are no heat sources within the solid. The inner surface $r = a$ is held at constant temperature T_1, and at the outer surface there is radiation into the medium $r > b$ which is at a constant temperature T_2. Find the steady temperature T in the solid.

3. A thermally conducting solid bounded by two concentric spheres of radii a and b, $a < b$, is such that the internal boundary is kept at T_1 and the outer boundary at $T_2(1-\cos\theta)$. Find the steady state temperature in the solid.

4. A thin annulus occupies the region $0 < a \le r \le b,\ 0 \le \theta \le 2\pi$, where $b > a$. The faces are insulated, and along the inner edge, the temperature is maintained at 0°, while along the outer edge, the temperature is held at 100°. Find the temperature distribution in the annulus.

5. A thermally conducting homogeneous disc with insulated faces occupies the region $0 \le r \le a$ in the xy-plane. The temperature u on the rim $r = a$, is

$$u = \begin{cases} C, & 0 < \theta < \alpha \\ 0, & \alpha < \theta < 2\pi \end{cases}$$

where α is a given angle $0 < \alpha < 2\pi$. Find the series expression for temperature at interior points of the disc. In particular, consider the case when $C = 100,\ \alpha = \pi/2$.

6. If ψ is a harmonic function which is zero on the cone $\theta = \alpha$ and takes the value $\Sigma \alpha_n r^n$ on the cone $\theta = \beta$, show that, when $\alpha < \theta < \beta$,

$$\psi = \sum_{n=0}^{\infty} \alpha_n \left\{ \frac{Q_n(\cos\alpha)\, P_n(\cos\theta) - P_n(\cos\alpha)\, Q_n(\cos\theta)}{Q_n(\cos\alpha) P_n(\cos\beta) - P_n(\cos\alpha)\, Q_n(\cos\beta)} \right\} r^n$$

7. Show that

$$\psi = \frac{q}{|\mathbf{r} - \mathbf{r}'|}, \qquad (q \text{ is constant})$$

is a solution of the Laplace equation.

8. Solve the following

$$\text{PDE: } \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\text{BCs: } v_r = \frac{\partial \phi}{\partial r} = 0 \qquad \text{at } r = a$$

$$v_r = U_\infty \cos\theta \qquad \text{at } r = \infty$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U_\infty \sin\theta$$

9. In the theory of elasticity, the stress function ψ, in the problem of torsion of a beam satisfies the Poisson equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2, \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

with the boundary conditions $\psi = 0$ on sides $x = 0$, $x = 1$, $y = 0$ and on $y = 1$. Find the stress function ψ.

10. For an infinitely long conducting cylinder of radius a, with its axis coincident with its z-axis, the voltage $u(r, \theta)$ obeys the Laplace equation

$$\nabla^2 u = 0, \qquad 0 \le r \le \infty,\ 0 \le \theta \le 2\pi$$

Find the voltage $u(r, \theta)$ for $r \ge a$ if $\underset{r \to \infty}{\text{Lt}}\ u(r, \theta) = 0$, subject to the condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = \frac{u_0}{a} \sin 3\theta$$

11. Hadamard's example:

(a) Consider the Cauchy problem for the Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{E11.1}$$

subject to $u(x, 0) = 0, u_y(x, 0) = \frac{1}{n} \sin nx$, where n is a positive integer. Show that its solution is

$$u_n(x, y) \ \frac{1}{n^2} \sinh ny \sin nx \tag{E11.2}$$

(b) Show that for large n, the absolute value of the initial data in (a) can be made arbitrarily small, while the solution (E11.2) takes arbitrarily large values even at the points (x, y) with $|y|$ as small as we want.

(c) Let f and g be analytic, and let u_1 be the solution to the Cauchy problem described by

$$u_{xx} + u_{yy} = 0$$

subject to

$$u(x, 0) = f(x), u_y(x, 0) = g(x) \tag{E11.3}$$

and let u_2 be the solution of the Laplace equation (E11.1) subject to $u(x, 0) = f(x), u_y(x, 0) = g(x) + (1/n) \sin nx$. Show that

$$u_2(x, y) - u_1(x, y) = \frac{1}{n^2} \sinh ny \sin nx \tag{E11.4}$$

(d) Conclude that the solution to the Cauchy problem for Laplace equation does not depend continuously on the initial data. In other words, the initial value problem (Cauchy problem) for the Laplace equation is not well-posed. It may be noted that a problem involving a PDE is well-posed if the following three properties are satisfied:

(i) The solution to the problem exists.

(ii) The solution is unique.

(iii) The solution depends continuously on the data of the problem.

Fortunately, many a physical phenomena give rise to initial or boundary or IBVPs which are well-posed.

12. Find the solution of the following PDE using separation of variables method

$$u_{xx} - u_y + u = 0.$$

CHAPTER 3

Parabolic Differential Equations

3.1 OCCURRENCE OF THE DIFFUSION EQUATION

The diffusion phenomena such as conduction of heat in solids and diffusion of vorticity in the case of viscous fluid flow past a body are governed by a partial differential equation of parabolic type. For example, the flow of heat in a conducting medium is governed by the parabolic equation

$$\rho C \frac{\partial T}{\partial t} = \text{div}(K \nabla T) + H(\mathbf{r}, T, t) \tag{3.1}$$

where ρ is the density, C is the specific heat of the solid, T is the temperature at a point with position vector $\mathbf{r}$, K is the thermal conductivity, t is the time, and $H(\mathbf{r}, T, t)$ is the amount of heat generated per unit time in the element dV situated at a point (x, y, z) whose position vector is $\mathbf{r}$. This equation is known as diffusion equation or heat equation. We shall now derive the heat equation from the basic concepts.

Let V be an arbitrary domain bounded by a closed surface S and let $\overline{V} = V \cup S$. Let $T(x, y, z, t)$ be the temperature at a point (x, y, z) at time t. If the temperature is not constant, heat flows from a region of high temperature to a region of low temperature and follows the Fourier law which states that heat flux $\mathbf{q}(\mathbf{r}, t)$ across the surface element dS with normal $\hat{n}$ is proportional to the gradient of the temperature. Therefore,

$$\mathbf{q}(\mathbf{r}, t) = -K \nabla T(\mathbf{r}, t) \tag{3.2}$$

where K is the thermal conductivity of the body. The negative sign indicates that the heat flux vector points in the direction of decreasing temperature. Let $\hat{n}$ be the outward unit normal vector and $\mathbf{q}$ be the heat flux at the surface element dS. Then the rate of heat flowing out through the elemental surface dS in unit time as shown in Fig. 3.1 is

$$dQ = (\mathbf{q} \cdot \hat{n})\, dS \tag{3.3}$$

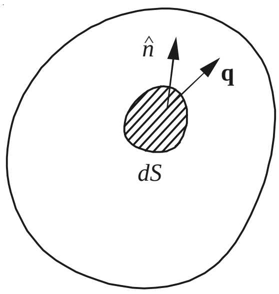

Fig. 3.1 The heat flow across a surface.

Heat can be generated due to nuclear reactions or movement of mechanical parts as in inertial measurement unit (IMU), or due to chemical sources which may be a function of position, temperature and time and may be denoted by $H(\mathbf{r}, T, t)$. We also define the specific heat of a substance as the amount of heat needed to raise the temperature of a unit mass by a unit temperature. Then the amount of heat dQ needed to raise the temperature of the elemental mass $dm = \rho\, dV$ to the value T is given by $dQ = C\rho T\, dV$. Therefore,

$$Q = \iiint_V C\rho T\, dV$$

$$\frac{dQ}{dt} = \iiint_V C\rho \frac{\partial T}{\partial t} dV$$

The energy balance equation for a small control volume V is: The rate of energy storage in V is equal to the sum of rate of heat entering V through its bounding surfaces and the rate of heat generation in V. Thus,

$$\iiint_V C\rho \frac{\partial T(\mathbf{r}, t)}{\partial t} dV = -\iint_S \mathbf{q} \cdot \hat{n}\, dS + \iiint_V H(\mathbf{r}, T, t)\, dV \tag{3.4}$$

Using the divergence theorem, we get

$$\iiint_V \left[C\rho \frac{\partial T}{\partial t}(\mathbf{r}, t) + \operatorname{div} \mathbf{q}(\mathbf{r}, t) - H(\mathbf{r}, T, t) \right] dV = 0 \tag{3.5}$$

Since the volume is arbitrary, we have

$$\rho C \frac{\partial T(\mathbf{r}, t)}{\partial t} = -\operatorname{div} \mathbf{q}(\mathbf{r}, t) + H(\mathbf{r}, T, t) \tag{3.6}$$

Substituting Eq. (3.2) into Eq. (3.6), we obtain

$$\rho C \frac{\partial T(\mathbf{r}, t)}{\partial t} = \nabla \cdot [K \nabla T(\mathbf{r}, t)] + H(\mathbf{r}, T, t) \tag{3.7}$$

If we define thermal diffusivity of the medium as

$$\alpha = \frac{K}{\rho C}$$

then the differential equation of heat conduction with heat source is

$$\frac{1}{\alpha}\frac{\partial T(\mathbf{r}, t)}{\partial t} = \nabla^2 T(\mathbf{r}, t) + \frac{H(\mathbf{r}, T, t)}{K} \tag{3.8}$$

In the absence of heat sources, Eq. (3.8) reduces to

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} = \alpha \nabla^2 T(\mathbf{r}, t) \tag{3.9}$$

This is called Fourier heat conduction equation or diffusion equation. The fundamental problem of heat conduction is to obtain the solution of Eq. (3.8) subject to the initial and boundary conditions which are called initial boundary value problems, hereafter referred to as IBVPs.

3.2 BOUNDARY CONDITIONS

The heat conduction equation may have numerous solutions unless a set of initial and boundary conditions are specified. The boundary conditions are mainly of three types, which we now briefly explain.

Boundary Condition I: *The temperature is prescribed all over the boundary surface*. That is, the temperature $T(\mathbf{r}, t)$ is a function of both position and time. In other words, $T = G(\mathbf{r}, t)$ which is some prescribed function on the boundary. This type of boundary condition is called the *Dirichlet condition*. Specification of boundary conditions depends on the problem under investigation. Sometimes the temperature on the boundary surface is a function of position only or is a function of time only or a constant. A special case includes $T(\mathbf{r}, t) = 0$ on the surface of the boundary, which is called a *homogeneous boundary condition*.

Boundary Condition II: *The flux of heat, i.e., the normal derivative of the temperature $\partial T/\partial n$, is prescribed on the surface of the boundary*. It may be a function of both position and time, i.e.,

$$\frac{\partial T}{\partial n} = f(\mathbf{r}, t)$$

This is called the Neumann condition. Sometimes, the normal derivatives of temperature may be a function of position only or a function of time only. A special case includes

$$\frac{\partial T}{\partial n} = 0 \text{ on the boundary}$$

This homogeneous boundary condition is also called insulated boundary condition which states that the heat flow is zero.

Boundary Condition III: *A linear combination of the temperature and its normal derivative is prescribed on the boundary*, i.e.,

$$K\frac{\partial T}{\partial n} + hT = G(\mathbf{r}, t)$$

where K and h are constants. This type of boundary condition is called *Robin's condition*. It means that the boundary surface dissipates heat by convection. Following Newton's law of cooling, which states that the rate at which heat is transferred from the body to the surroundings is proportional to the difference in temperature between the body and the surroundings, we have

$$-K\frac{\partial T}{\partial n} = h(T - T_a)$$

As a special case, we may also have

$$K\frac{\partial T}{\partial n} + hT = 0$$

which is a homogeneous boundary condition. This means that heat is convected by dissipation from the boundary surface into a surrounding maintained at zero temperature.

The other boundary conditions such as the heat transfer due to radiation obeying the fourth power temperature law and those associated with change of phase, like melting, ablation, etc. give rise to non-linear boundary conditions.

3.3 ELEMENTARY SOLUTIONS OF THE DIFFUSION EQUATION

Consider the one-dimensional diffusion equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad -\infty < x < \infty,\ t > 0 \tag{3.10}$$

The function

$$T(x, t) = \frac{1}{\sqrt{4\pi\alpha t}}\exp\left[-(x-\xi)^2/(4\alpha t)\right] \tag{3.11}$$

where ξ is an arbitrary real constant, is a solution of Eq. (3.10). It can be verified easily as follows:

$$\frac{\partial T}{\partial t} = \frac{1}{\sqrt{4\pi\alpha t}}\left[\frac{(x-\xi)^2}{4\alpha t^2} - \frac{1}{2t}\right]\exp\left[-(x-\xi)^2/(4\alpha t)\right]$$

$$\frac{\partial T}{\partial x} = \frac{1}{\sqrt{4\pi\alpha t}}\,\frac{-2(x-\xi)}{4\alpha t}\,\exp\left[-(x-\xi)^2/(4\alpha t)\right]$$

Therefore,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sqrt{4\pi\alpha t}}\left[-\frac{1}{2\alpha t} + \frac{(x-\xi)^2}{4\alpha^2 t^2}\right]\exp\left[-(x-\xi)^2/(4\alpha t)\right] = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which shows that the function (3.11) is a solution of Eq. (3.10). The function (3.11), known as Kernel, is the elementary solution or the fundamental solution of the heat equation for the infinite interval. For $t > 0$, the Kernel $T(x, t)$ is an analytic function of x and t and it can also be noted that $T(x, t)$ is positive for every x. Therefore, the region of influence for the diffusion equation includes the entire x-axis. It can be observed that as $|x| \to \infty$, the amount of heat transported decreases exponentially.

In order to have an idea about the nature of the solution to the heat equation, consider a one-dimensional infinite region which is initially at temperature $f(x)$. Thus the problem is described by

$$\text{PDE: } \frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0 \tag{3.12}$$

$$\text{IC: } T(x, 0) = f(x), \qquad -\infty < x < \infty,\ t = 0 \tag{3.13}$$

Following the method of variables separable, we write

$$T(x, t) = X(x)\beta(t) \tag{3.14}$$

Substituting into Eq. (3.12), we arrive at

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{\beta'}{\beta} = \lambda \tag{3.15}$$

where λ is a separation constant. The separated solution for β gives

$$\beta = Ce^{\alpha\lambda t} \tag{3.16}$$

If $\lambda > 0$, we have β and, therefore, T growing exponentially with time. From realistic physical considerations, it is reasonable to assume that $f(x) \to 0$ as $|x| \to \infty$, while $|T(x, t)| < M$ as $|x| \to \infty$. But, for $T(x, t)$ to remain bounded, λ should be negative and thus we take $\lambda = -\mu^2$. Now from Eq. (3.15) we have

$$X'' + \mu^2 X = 0$$

Its solution is found to be

$$X = c_1 \cos \mu x + c_2 \sin \mu x$$

Hence

$$T(x, t, \mu) = (A\cos \mu x + B\sin \mu x)e^{-\alpha\mu^2 t} \tag{3.17}$$

is a solution of Eq. (3.12), where A and B are arbitrary constants. Since $f(x)$ is in general not periodic, it is natural to use Fourier integral instead of Fourier series in the present case. Also, since A and B are arbitrary, we may consider them as functions of μ and take $A = A(\mu), B = B(\mu)$. In this particular problem, since we do not have any boundary conditions which limit our choice of μ, we should consider all possible values. From the principles of superposition, this summation of all the product solutions will give us the relation

$$T(x,t) = \int_0^\infty T(x,t,\mu)\,d\mu = \int_0^\infty [A(\mu)\cos \mu x + B(\mu)\sin \mu x]e^{-\alpha\mu^2 t}\,d\mu \tag{3.18}$$

which is the solution of Eq. (3.12). From the initial condition (3.13), we have

$$T(x,0) = f(x) = \int_0^\infty [A(\mu)\cos \mu x + B(\mu)\sin \mu x]\,d\mu \tag{3.19}$$

In addition, if we recall the Fourier integral theorem, we have

$$f(t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(x)\cos \omega(t-x)\,dx\right]d\omega \tag{3.20}$$

Thus, we may write

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\cos \mu(x-y)\,dy\right]d\mu$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\,(\cos \mu x \cos \mu y + \sin \mu x \sin \mu y)\,dy\right]d\mu$$

$$= \frac{1}{\pi}\int_0^\infty \left[\cos \mu x \int_{-\infty}^\infty f(y)\cos \mu y\,dy + \sin \mu x \int_{-\infty}^\infty f(y)\sin \mu y\,dy\right]d\mu \tag{3.21}$$

Let

$$A(\mu) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\cos \mu y\,dy$$

$$B(\mu) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\sin \mu y\,dy$$

Then Eq. (3.21) can be written in the form

$$f(x) = \int_0^\infty [A(\mu)\cos \mu x + B(\mu)\sin \mu x]\,d\mu \tag{3.22}$$

Comparing Eqs. (3.19) and (3.22), we shall write relation (3.19) as

$$T(x,0) = f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\cos \mu(x-y)\,dy\right]d\mu \tag{3.23}$$

Thus, from Eq. (3.18), we obtain

$$T(x, t) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cos\mu(x-y)\exp(-\alpha\mu^2 t)\,dy\right]d\mu \tag{3.24}$$

Assuming that the conditions for the formal interchange of orders of integration are satisfied, we get

$$T(x, t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\left[\int_0^{\infty}\exp(-\alpha\mu^2 t)\cos\mu(x-y)\,d\mu\right]dy \tag{3.25}$$

Using the standard known integral

$$\int_0^{\infty}\exp(-s^2)\cos(2bs)\,ds = \frac{\sqrt{\pi}}{2}\exp(-b^2) \tag{3.26}$$

Setting $s = \mu\sqrt{\alpha t}$, and choosing

$$b = \frac{x-y}{2\sqrt{\alpha t}}$$

Equation (3.26) becomes

$$\int_0^{\infty} e^{-\alpha\mu^2 t}\cos\mu(x-y)\,d\mu = \frac{\sqrt{\pi}}{\sqrt{4\alpha t}}\exp[-(x-y)^2/(4\alpha t)] \tag{3.27}$$

Substituting Eq. (3.27) into Eq. (3.25), we obtain

$$T(x, t) = \frac{1}{\sqrt{4\alpha\pi t}}\int_{-\infty}^{\infty} f(y)\exp[-(x-y)^2/(4\alpha t)]\,dy \tag{3.28}$$

Hence, if $f(y)$ is bounded for all real values of y, Eq. (3.28) is the solution of the problem described by Eqs. (3.12) and (3.13).

EXAMPLE 3.1 In a one-dimensional infinite solid, $-\infty < x < \infty$, the surface $a < x < b$ is initially maintained at temperature T_0 and at zero temperature everywhere outside the surface. Show that

$$T(x, t) = \frac{T_0}{2}\left[\operatorname{erf}\left(\frac{b-x}{\sqrt{4\alpha t}}\right) - \operatorname{erf}\left(\frac{a-x}{\sqrt{4\alpha t}}\right)\right]$$

where erf is an error function.

Solution The problem is described as follows:

$$\text{PDE: } T_t = \alpha T_{xx}, \qquad -\infty < x < \infty$$

$$\text{IC: } T = T_0, \qquad a < x < b$$

$$= 0 \text{ outside the above region}$$

The general solution of PDE is found to be

$$T(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} f(\xi) \exp[-(x-\xi)^2/(4\alpha t)]\, d\xi$$

Substituting the IC, we obtain

$$T(x, t) = \frac{T_0}{\sqrt{4\pi\alpha t}} \int_a^b \exp[-(x-\xi)^2/(4\alpha t)]\, d\xi$$

Introducing the new independent variable η defined by

$$\eta = -\frac{x-\xi}{\sqrt{4\alpha t}}$$

and hence

$$d\xi = \sqrt{4\alpha t}\, d\eta$$

the above equation becomes

$$T(x, t) = \frac{T_0}{\sqrt{\pi}} \int_{(a-x)/\sqrt{(4\alpha t)}}^{(b-x)/\sqrt{(4\alpha t)}} e^{-\eta^2} d\eta = \frac{T_0}{2}\left[\frac{2}{\sqrt{\pi}} \int_0^{(b-x)/\sqrt{(4\alpha t)}} e^{-\eta^2} d\eta - \frac{2}{\sqrt{\pi}} \int_0^{(a-x)/\sqrt{(4\alpha t)}} e^{-\eta^2} d\eta\right]$$

Now we introduce the error function defined by

$$\text{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\eta^2)\, d\eta$$

Therefore, the required solution is

$$T(x, t) = \frac{T_0}{2}\left[\text{erf}\left(\frac{b-x}{\sqrt{4\alpha t}}\right) - \text{erf}\left(\frac{a-x}{\sqrt{4\alpha t}}\right)\right]$$

3.4 DIRAC DELTA FUNCTION

According to the notion in mechanics, we come across a very large force (ideally infinite) acting for a short duration (ideally zero time) known as impulsive force. Thus we have a function which is non-zero in a very short interval. The Dirac delta function may be thought of as a generalization of this concept. This Dirac delta function and its derivative play a useful role in the solution of initial boundary value problem (IBVP).

Consider the function having the following property:

$$\delta_\varepsilon(t) = \begin{cases} 1/2\varepsilon, & |t| < \varepsilon \\ 0, & |t| > \varepsilon \end{cases} \tag{3.29}$$

Thus,

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)\,dt = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,dt = 1 \tag{3.30}$$

Let $f(t)$ be any function which is integrable in the interval $(-\varepsilon, \varepsilon)$. Then using the Mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t)\,dt = f(\xi), \qquad -\varepsilon < \xi < \varepsilon \tag{3.31}$$

Thus, we may regard $\delta(t)$ as a limiting function approached by $\delta_\varepsilon(t)$ as $\varepsilon \to 0$, i.e.

$$\delta(t) = \underset{\varepsilon\to 0}{\text{Lt}}\ \delta_\varepsilon(t) \tag{3.32}$$

As $\varepsilon \to 0$, we have, from Eqs. (3.29) and (3.30), the relations

$$\delta(t) = \underset{\varepsilon\to 0}{\text{Lt}}\ \delta_\varepsilon(t) = \begin{cases} & \text{(in the sense of being very large)} \\ \infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \tag{3.33}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{3.34}$$

This limiting function $\delta(t)$ defined by Eqs. (3.33) and (3.34) is known as Dirac delta function or the unit impulse function. Its profile is depicted in Fig. 3.2. Dirac originally called it an improper function as there is no proper function with these properties. In fact, we can observe that

$$1 = \int_{-\infty}^{\infty} \delta(t)\,dt = \underset{\varepsilon\to 0}{\text{Lt}} \int_{|t|>\varepsilon} \delta_\varepsilon(t)\,dt = \underset{\varepsilon\to 0}{\text{Lt}}\ 0 = 0$$

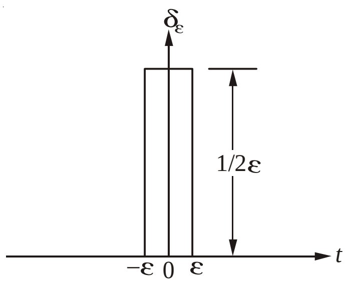

Fig. 3.2 Profile of Dirac delta function.

Obviously, this contradiction implies that $\delta(t)$ cannot be a function in the ordinary sense. Some important properties of Dirac delta function are presented now:

PROPERTY I: $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

PROPERTY II: For any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

Proof Consider the equation

$$\underset{\varepsilon\to 0}{\text{Lt}} \int_{-\infty}^{\infty} f(t)\,\delta_{\varepsilon}(t)\,dt = \underset{\xi\to 0}{\text{Lt}}\, f(\xi), \qquad -\varepsilon < \xi < \varepsilon$$

As $\varepsilon \to 0$, we have $\xi \to 0$. Therefore,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)$$

PROPERTY III: Let $f(t)$ be any continuous function. Then

$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\,dt = f(a)$$

Proof Consider the function

$$\delta_{e}(t-a) = \begin{cases} 1/\varepsilon, & a < t < a+\varepsilon \\ 0, & \text{elsewhere} \end{cases}$$

Using the mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} \delta_{\varepsilon}(t-a)\, f(t)\,dt = \frac{1}{\varepsilon}\int_{a}^{a+\varepsilon} f(t)\,dt = f(a+\theta\varepsilon), \qquad 0<\theta<1$$

Now, taking the limit as $\varepsilon \to 0$, we obtain

$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\,dt = f(a)$$

Thus, the operation of multiplying $f(t)$ by $\delta(t-a)$ and integrating over all t is equivalent to substituting a for t in the original function.

PROPERTY IV: $\delta(-t) = \delta(t)$

PROPERTY V: $\delta(at) = \dfrac{1}{a}\delta(t), \qquad a > 0$

PROPERTY VI: If $\delta(t)$ is a continuously differentiable. Dirac delta function vanishing for large t, then

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -f'(0)$$

Proof Using the rule of integration by parts, we get

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = [f(t)\,\delta(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt$$

Using Eq. (3.33) and property (III), the above equations becomes

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -f'(0)$$

PROPERTY VII: $$\int_{-\infty}^{\infty} \delta'(t-a)\,f(t)\,dt = -f'(a)$$

Having discussed the one-dimensional Dirac delta function, we can extend the definition to two dimensions. Thus, for every f which is continuous over the region S containing the point (ξ, η), we define $\delta(x-\xi, y-\eta)$ in such a way that

$$\iint_S \delta(x-\xi, y-\eta)\, f(x, y)\, d\sigma = f(\xi, \eta) \tag{3.35}$$

Note that $\delta(x-\xi, y-\eta)$ is a formal limit of a sequence of ordinary functions, i.e.,

$$\delta(x-\xi, y-\eta) = \operatorname*{Lt}_{\varepsilon\to 0} \delta_\varepsilon(r) \tag{3.36}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$. Also observe that

$$\iint \delta(x-\xi)\,\delta(y-\eta)\, f(x, y)\, dx\, dy = f(\xi, \eta) \tag{3.37}$$

Now, comparing Eqs. (3.35) and (3.37), we see that

$$\delta(x-\xi, y-\eta) = \delta(x-\xi)\,\delta(y-\eta) \tag{3.38}$$

Thus, a two-dimensional Dirac delta function can be expressed as the product of two one-dimensional delta functions. Similarly, the definition can be extended to higher dimensions.

EXAMPLE 3.2 A one-dimensional infinite region $-\infty < x < \infty$ is initially kept at zero temperature. A heat source of strength g_s units, situated at $x = \xi$ releases its heat instantaneously at time $t = \tau$. Determine the temperature in the region for $t > \tau$.

Solution Initially, the region $-\infty < x < \infty$ is at zero temperature. Since the heat source is situated at $x = \xi$ and releases heat instantaneously at $t = \tau$, the released temperature at $x = \xi$ and $t = \tau$ is a δ- function type. Thus, the given problem is a boundary value problem described by

$$\text{PDE: } \frac{\partial^2 T}{\partial x^2} + \frac{g(x, t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad -\infty < x < \infty, t > 0$$

$$\text{IC: } \quad T(x, t) = F(x) = 0, \qquad -\infty < x < \infty, t = 0$$

$$g(x, t) = g_s \delta(x - \xi)\, \delta(t - \tau)$$

The general solution to this problem as given in Example 7.25, after using the initial condition $F(x) = 0$, is

$$T(x, t) = \frac{\alpha}{k} \int_{t'=0}^{t} \frac{dt'}{\sqrt{4\pi\alpha(t - t')}} \int_{x'=-\infty}^{\infty} g(x', t') \exp[-(x - x')^2/\{4\alpha(t - t')\}]\, dx' \tag{3.39}$$

Since the heat source term is of the Dirac delta function type, substituting

$$g(x, t) = g_s \delta(x - \xi)\, \delta(t - \tau)$$

into Eq. (3.39), and integrating we get, with the help of properties of delta function, the relation

$$T(x, t) = \frac{\alpha}{k} \frac{g_s}{\sqrt{4\pi\alpha}} \int_0^t \frac{\exp[-(x - \xi)^2/\{4\alpha(t - t')\}]}{\sqrt{t - t'}} \delta(t - \tau)\, dt'$$

Therefore, the required temperature is

$$T(x, t) = \frac{\alpha g_s}{k} \frac{\exp[-(x - \xi)^2/\{4\alpha(t - \tau)\}]}{\sqrt{4\pi\alpha(t - \tau)}} \qquad \text{for } t > \tau$$

EXAMPLE 3.3 An infinite one-dimensional solid defined by $-\infty < x < \infty$ is maintained at zero temperature initially. There is a heat source of strength $g_s(t)$ units, situated at $x = \xi$, which releases constant heat continuously for $t > 0$. Find an expression for the temperature distribution in the solid for $t > 0$.

Solution This problem is similar to Example 3.2, except that $g(x, t) = g_s(t)\, \delta(x - \xi)$ is a Dirac delta function type. The solution to this IBVP is

$$T(x, t) = \frac{\alpha}{K} \int_{t'=0}^{t} \frac{g_s(t')}{\sqrt{4\pi\alpha(t - t')}} \exp[-(x - \xi)^2/\{4\alpha(t - t')\}]\, dt' \tag{3.40}$$

It is given as $g_s(t) = \text{constant} = g_s$ (say). Let us introduce a new variable η defined by

$$\eta = \frac{x-\xi}{\sqrt{4\alpha(t-t')}} \quad \text{or} \quad t - t' = \frac{1(x-\xi)^2}{\eta^2 4\alpha}$$

Therefore,

$$dt' = \frac{1}{\eta^3}\frac{(x-\xi)^2}{2\alpha}d\eta$$

Thus, Eq. (3.40) becomes

$$T(x,t) = g_s \frac{x-\xi}{2K\sqrt{\pi}} \int_{(x-\xi)/\sqrt{4\alpha t}}^{\infty} \frac{\exp(-\eta^2)}{\eta^2} d\eta$$

However,

$$\frac{d}{d\eta}\left(-\frac{e^{-\eta^2}}{\eta}\right) = \frac{e^{-\eta^2}}{\eta^2} + 2e^{-\eta^2}$$

Hence,

$$T(x,t) = g_s \frac{x-\xi}{2K\sqrt{\pi}} \left[\left(-\frac{e^{-\eta^2}}{\eta}\right)_{(x-\xi)/\sqrt{4\alpha t}}^{\infty} - 2\int_{(x-\xi)/\sqrt{(4\alpha t)}}^{\infty} e^{-\eta^2} d\eta \right]$$

Recalling the definitions of error function and its complement

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta, \qquad \text{erf}(\infty) = 1$$

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}} \left(\int_0^\infty \exp(-\eta^2)\, d\eta - \int_0^x \exp(-\eta^2)\, d\eta \right)$$

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\eta^2)\, d\eta$$

the temperature distribution can be expressed as

$$T(x,t) = \frac{\alpha g_s}{K} \left[\sqrt{\frac{t}{2\pi}} \exp[-(x-\xi)^2/(4\alpha t)] - \frac{|x-\xi|}{2\alpha} \left(1 - \text{erf}\,\frac{x-\xi}{\sqrt{4\alpha t}}\right) \right]$$

Alternatively, the required temperature is

$$T(x,t) = \frac{\alpha g_s}{K} \left[\sqrt{\frac{t}{2\pi}} \exp[-(x-\xi)^2/(4\alpha t)] - \frac{|x-\xi|}{2\alpha} \,\text{erfc}\frac{x-\xi}{\sqrt{4\alpha t}} \right]$$

3.5 SEPARATION OF VARIABLES METHOD

Consider the equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.41}$$

Among the many methods that are available for the solution of the above parabolic partial differential equation, the method of separation of variables is very effective and straightforward. We separate the space and time variables of $T(x, t)$ as follows: Let

$$T(x, t) = X(x)\beta(t) \tag{3.42}$$

be a solution of the differential Eq. (3.41). Substituting Eq. (3.42) into (3.41), we obtain

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{\beta'}{\beta} = K, \text{ a separation constant}$$

Then we have

$$\frac{d^2X}{dx^2} - KX = 0 \tag{3.43}$$

$$\frac{d\beta}{dt} - \alpha K \beta = 0 \tag{3.44}$$

In solving Eqs. (3.43) and (3.44), three distinct cases arise:

Case I When K is positive, say λ^2, the solution of Eqs. (3.43) and (3.44) will have the form

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}, \qquad \beta = c_3 e^{\alpha\lambda^2 t} \tag{3.45}$$

Case II When K is negative, say $-\lambda^2$, then the solution of Eqs. (3.43) and (3.44) will have the form

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x, \qquad \beta = c_3 e^{-\alpha\lambda^2 t} \tag{3.46}$$

Case III When K is zero, the solution of Eqs. (3.43) and (3.44) can have the form

$$X = c_1 x + c_2, \qquad \beta = c_3 \tag{3.47}$$

Thus, various possible solutions of the heat conduction equation (3.41) could be the following:

$$\begin{aligned} T(x, t) &= (c_1' e^{\lambda x} + c_2' e^{-\lambda x}) e^{\alpha\lambda^2 t} \\ T(x, t) &= (c_1' \cos \lambda x + c_2' \sin \lambda x) e^{-\alpha\lambda^2 t} \\ T(x, t) &= c_1' x + c_2' \end{aligned} \tag{3.48}$$

where

$$c_1' = c_1 c_3, \qquad c_2' = c_2 c_3$$

EXAMPLE 3.4 Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi, t \ge 0$, subject to the conditions

(i) T remains finite as $t \to \infty$

(ii) $T = 0$, if $x = 0$ and π for all t

(iii) At $t = 0$, $T = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \dfrac{\pi}{2} \le x \le \pi. \end{cases}$

Solution Since T should satisfy the diffusion equation, the three possible solutions are:

$$T(x, t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x}) e^{\alpha \lambda^2 t}$$

$$T(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\alpha \lambda^2 t}$$

$$T(x, t) = (c_1 x + c_2)$$

The first condition demands that T should remain finite as $t \to \infty$. We therefore reject the first solution. In view of BC (ii), the third solution gives

$$0 = c_1 \cdot 0 + c_2, \qquad 0 = c_1 \cdot \pi + c_2$$

implying thereby that both c_1 and c_2 are zero and hence $T = 0$ for all t. This is a trivial solution. Since we are looking for a non-trivial solution, we reject the third solution also. Thus, the only possible solution satisfying the first condition is

$$T(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\alpha \lambda^2 t}$$

Using the BC (ii), we have

$$0 = (c_1 \cos \lambda x + c_2 \sin \lambda x)\big|_{x=0}$$

implying $c_1 = 0$. Therefore, the possible solution is

$$T(x, t) = c_2 e^{-\alpha \lambda^2 t} \sin \lambda x$$

Applying the BC: $T = 0$ when $x = \pi$, we get

$$\sin \lambda \pi = 0 \Rightarrow \lambda \pi = n\pi$$

where n is an integer. Therefore,

$$\lambda = n$$

Hence the solution is found to be of the form

$$T(x, t) = c e^{-\alpha n^2 t} \sin nx$$

Noting that the heat conduction equation is linear, its most general solution is obtained by applying the principle of superposition. Thus,

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha n^2 t} \sin nx$$

Using the third condition, we get

$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$$

which is a half-range Fourier-sine series and, therefore,

$$c_n = \frac{2}{\pi}\int_0^{\pi} T(x,0)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$$

Integrating by parts, we obtain

$$c_n = \frac{2}{\pi}\left[\left(-x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_0^{\pi/2} + \left\{-(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

or

$$c_n = \frac{4\sin(n\pi/2)}{n^2\pi}$$

Thus, the required solution is

$$T(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha n^2 t}\sin(n\pi/2)}{n^2}\sin nx$$

EXAMPLE 3.5 A uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.

Solution The initial boundary value problem IBVP of heat conduction is given by

$$\text{PDE: } \frac{\partial\theta}{\partial t} = \alpha\frac{\partial^2\theta}{\partial x^2}, \qquad 0 \le x \le L,\ t > 0$$

$$\text{BCs: } \theta(0,t) = 0, \qquad t \ge 0$$

$$\frac{\partial\theta}{\partial x}(L,t) = 0, \qquad t > 0$$

$$\text{IC: } \theta(x,0) = \theta_0, \qquad 0 \le x \le L$$

From Section 3.5, it can be noted that the physically meaningful and non-trivial solution is

$$\theta(x, t) = e^{-\alpha\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

Using the first boundary condition, we obtain $A = 0$. Thus the acceptable solution is

$$\theta = Be^{-\alpha\lambda^2 t}\sin\lambda x$$

$$\frac{\partial\theta}{\partial x} = \lambda Be^{-\alpha\lambda^2 t}\cos\lambda x$$

Using the second boundary condition, we have

$$0 = \lambda Be^{-\alpha\lambda^2 t}\cos\lambda L$$

implying $\cos\lambda L = 0$. Therefore,

The eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \frac{(2n+1)\pi}{2L}, \qquad n = 0, 1, 2, \ldots$$

Thus, the acceptable solution is of the form

$$\theta = B\exp[-\alpha\{(2n+1)/2L\}^2\pi^2 t]\sin\left(\frac{2n+1}{2L}\right)\pi x$$

Using the principle of superposition, we obtain

$$\theta(x, t) = \sum_{n=0}^{\infty} B_n\exp[-\alpha\{(2n+1)/2L\}^2\pi^2 t]\sin\left(\frac{2n+1}{2L}\pi x\right)$$

Finally, using the initial condition, we have

$$\theta_0 = \sum_{n=0}^{\infty} B_n\sin\left(\frac{2n+1}{2L}\pi x\right)$$

which is a half-range Fourier-sine series and, thus,

$$B_n = \frac{2}{L}\int_0^L \theta_0\sin\left(\frac{2n+1}{2L}\pi x\right)dx$$

$$= \frac{2}{L}\left[-\theta_0\frac{2L}{(2n+1)\pi}\left\{\cos\left(\frac{2n+1}{2L}\pi x\right)\right\}_0^L\right]$$

$$= -\frac{4\theta_0}{(2n+1)\pi}[\cos\{(2n+1)\pi/2\} - \cos 0] = \frac{4\theta_0}{(2n+1)\pi}$$

Thus, the required temperature distribution is

$$\theta(x,t)=\sum_{n=0}^{\infty}\frac{4\theta_0}{(2n+1)\pi}\exp\left[-\alpha\{(2n+1)/2L\}^2\pi^2t\right]\sin\left(\frac{2n+1}{2L}\pi x\right)$$

EXAMPLE 3.6 A conducting bar of uniform cross-section lies along the x-axis with ends at $x=0$ and $x=L.$ It is kept initially at temperature 0° and its lateral surface is insulated. There are no heat sources in the bar. The end $x = 0$ is kept at 0°, and heat is suddenly applied at the end $x = L$, so that there is a constant flux q_0 at $x = L$. Find the temperature distribution in the bar for $t > 0$.

Solution The given initial boundary value problem can be described as follows:

$$\text{PDE:}\ \frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}$$

$$\text{BCs:}\ T(0,t)=0,\qquad t>0$$

$$\frac{\partial T}{\partial x}(L,t)=q_0,\qquad t>0$$

$$\text{IC:}\quad T(x,0)=0,\qquad 0\le x\le L$$

Prior to applying heat suddenly to the end $x = L$, when $t = 0$, the heat flow in the bar is independent of time (steady state condition). Let

$$T(x,t)=T_{(s)}(x)+T_1(x,t)$$

where $T_{(s)}$ is a steady part and T_1 is the transient part of the solution. Therefore,

$$\frac{\partial^2 T_{(s)}}{\partial x^2}=0$$

whose general solution is

$$T_{(s)}=Ax+B$$

when $x=0, T_{(s)}=0,$ implying $B=0.$ Therefore,

$$T_{(s)}=Ax$$

Using the other BC: $\dfrac{\partial T_{(s)}}{\partial x}=q_0,$ we get $A=q_0.$ Hence, the steady state solution is

$$T_{(s)}=q_0x$$

For the transient part, the BCs and IC are redefined as

(i) $T_1(0, t) = T(0, t) - T_{(s)}(0) = 0 - 0 = 0$

(ii) $\partial T_1(L, t)/\partial x = \partial T(L, t)/\partial x - \partial T_s(L, t)/\partial x = q_0 - q_0 = 0$

(iii) $T_1(x, 0) = T(x, 0) - T_{(s)}(x) = -q_0 x, 0 < x < L.$

Thus, for the transient part, we have to solve the given PDE subject to these conditions. The acceptable solution is given by Eq. (3.48), i.e.

$$T_1(x, t) = e^{-\alpha\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

Applying the BC (i), we get $A = 0$. Therefore,

$$T_1(x, t) = Be^{-\alpha\lambda^2 t}\sin\lambda x$$

and using the BC (ii), we obtain

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=L} = B\lambda e^{-\alpha\lambda^2 t}\cos\lambda L = 0$$

implying $\lambda L = (2n-1)\dfrac{\pi}{2}, n = 1, 2, \ldots$ Using the superposition principle, we have

$$T_1(x, t) = \sum_{n=1}^{\infty} B_n \exp[-\alpha\{(2n-1)/2L\}^2\pi^2 t]\sin\left(\frac{2n-1}{2L}\pi x\right)$$

Now, applying the IC (iii), we obtain

$$T_1(x, 0) = -q_0 x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n-1}{2L}\pi x\right)$$

Multiplying both sides by $\sin\left(\dfrac{2m-1}{2L}\pi x\right)$ and integrating between 0 to L and noting that

$$\int_0^L B_n \sin\left(\frac{2n-1}{2L}\right)\pi x \sin\left(\frac{2m-1}{2L}\pi x\right)dx = \begin{cases} 0, & n \neq m \\ \dfrac{B_m L}{2}, & n = m \end{cases}$$

we get at once, after integrating by parts, the equation

$$-q_0 \frac{4L^2}{(2m-1)^2\pi^2}\left[\sin\left(\frac{2m-1}{2}\pi\right)\right] = B_m \frac{L}{2}$$

or

$$-q_0 \frac{4L^2}{(2m-1)^2 \pi^2} (-1)^{m-1} = B_m \frac{L}{2}$$

which gives

$$B_m = \frac{(-1)^m 8Lq_0}{(2m-1)^2 \pi^2}$$

Hence, the required temperature distribution is

$$T(x, t) = q_0 x + \frac{8Lq_0}{\pi^2} \sum_{m=1}^{\infty} \left[\frac{(-1)^m}{(2m-1)^2} \exp[-\alpha\{(2m-1)/L\}^2 \pi^2 t] \sin\left(\frac{2m-1}{2L} \pi x \right) \right]$$

EXAMPLE 3.7 The ends *A* and *B* of a rod, 10 cm in length, are kept at temperatures 0°C and 100°C until the steady state condition prevails. Suddenly the temperature at the end *A* is increased to 20°C, and the end *B* is decreased to 60°C. Find the temperature distribution in the rod at time *t*.

Solution The problem is described by

$$\text{PDE: } \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < 10$$

$$\text{BCs: } T(0, t) = 0, \qquad T(10, t) = 100$$

Prior to change in temperature at the ends of the rod, the heat flow in the rod is independent of time as steady state condition prevails. For steady state,

$$\frac{d^2 T}{dx^2} = 0$$

whose solution is

$$T_{(s)} = Ax + B$$

When $x = 0$, $T = 0$, implying $B = 0$. Therefore,

$$T_{(s)} = Ax$$

When $x = 10$, $T = 100$, implying $A = 10$. Thus, the initial steady temperature distribution in the rod is

$$T_{(s)}(x) = 10x$$

Similarly, when the temperature at the ends *A* and *B* are changed to 20 and 60, the final steady temperature in the rod is

$$T_{(s)}(x) = 4x + 20$$

which will be attained after a long time. To get the temperature distribution $T(x, t)$ in the intermediate period, counting time from the moment the end temperatures were changed, we assume that

$$T(x, t) = T_1(x, t) + T_{(s)}(x)$$

where $T_1(x, t)$ is the transient temperature distribution which tends to zero as $t \to \infty$. Now, $T_1(x, t)$ satisfies the given PDE. Hence, its general solution is of the form

$$T(x, t) = (4x + 20) + e^{-\alpha\lambda^2 t}(B \cos \lambda x + c \sin \lambda x)$$

Using the BC: $T = 20$ when $x = 0$, we obtain

$$20 = 20 + Be^{-\alpha\lambda^2 t}$$

implying $B = 0$. Using the BC: $T = 60$ when $x = 10$, we get

$$\sin 10\lambda = 0, \text{ implying } \lambda = \frac{n\pi}{10}, \qquad n = 1, 2, \ldots$$

The principle of superposition yields

$$T(x, t) = (4x + 20) + \sum_{n=1}^{\infty} c_n \exp[-\alpha(n\pi/10)^2 t] \sin\left(\frac{n\pi}{10}\right)x$$

Now using the IC: $T = 10x$, when $t = 0$, we obtain

$$10x = 4x + 20 + \sum c_n \sin\left(\frac{n\pi}{10}x\right)$$

or

$$6x - 20 = \sum c_n \sin\left(\frac{n\pi}{10}x\right)$$

where

$$c_n = \frac{2}{10}\int_0^{10} (6x - 20) \sin\left(\frac{n\pi}{10}x\right)dx = -\frac{1}{5}\left[(-1)^n \frac{800}{n\pi} - \frac{200}{n\pi}\right]$$

Thus, the required solution is

$$T(x, t) = 4x + 20 - \frac{1}{5}\sum_{n=1}^{\infty}\left[(-1)^n \frac{800}{n\pi} - \frac{200}{n\pi}\right] \exp\left[-\alpha\left(\frac{n\pi}{10}\right)^2 t\right] \sin\left(\frac{n\pi}{10}x\right).$$

EXAMPLE 3.8 Assuming the surface of the earth to be flat, which is initially at zero temperature and for times $t > 0$, the boundary surface is being subjected to a periodic heat flux $g_0 \cos \omega t$. Investigate the penetration of these temperature variations into the earth's surface and show that at a depth x, the temperature fluctuates and the amplitude of the steady temperature is given by

$$\frac{g_0}{\sqrt{2}} \sqrt{\frac{2\alpha}{\omega}} \exp[-\sqrt{(\omega/2\alpha)}\, x]$$

Solution The given IBVP is described by

$$\text{PDE:} \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.49}$$

$$\text{BC:} -\frac{\partial T}{\partial x} = g_0 \cos \omega t \quad \text{at} \quad x = 0,\ t > 0 \tag{3.50}$$

$$\text{IC:} T(x, 0) = 0 \tag{3.51}$$

We shall introduce an auxiliary function $\tilde{T}$ satisfying Eqs. (3.49)–(3.51) and then define the complex function Z such that

$$Z = T + i\tilde{T}$$

We can easily verify that Z satisfies

$$\text{PDE:} \frac{\partial Z}{\partial t} = \alpha \frac{\partial^2 Z}{\partial x^2} \tag{3.52}$$

$$\text{BC:} -\frac{\partial Z}{\partial x} = g_0\, e^{i\omega t} \quad \text{at} \quad x = 0,\ t > 0$$

$$\text{IC:} Z = 0 \text{ in the region, } t = 0$$

Let us assume the solution of Eq. (3.52) in the form

$$Z = f(x) e^{i\omega t}$$

where $f(x)$ satisfies

$$\frac{d^2 f(x)}{dx^2} - i\frac{\omega}{\alpha} f(x) = 0 \tag{3.53}$$

$$-\frac{df(x)}{dx} = g_0 \quad \text{at} \quad x = 0 \tag{3.54}$$

Also,

$$f(x) \text{ is finite for large } x. \tag{3.55}$$

The solution of Eq. (3.53), satisfying the BC (3.55), is

$$f(x) = A \exp[-\sqrt{(i\omega/\alpha)\,x}]$$

The constant A can be determined by using the BC (3.54). Therefore,

$$f(x) = \frac{1}{\sqrt{i}} g_0 \sqrt{\frac{\alpha}{\omega}} \exp[-\sqrt{(i\omega/\alpha)}\; x]$$

Thus,

$$Z = g_0 \sqrt{\frac{\alpha}{\omega}} \frac{1}{\sqrt{i}} \exp[i\omega t - \sqrt{(i\omega/\alpha)}\; x] \tag{3.56}$$

It can be shown for convenience that

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}, \qquad \frac{1}{\sqrt{i}} = \frac{1-i}{\sqrt{2}}$$

Thus, Eq. (3.56) can be written in the form

$$Z = \frac{g_0}{2}\sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}}x\right)(1-i)\exp\left[i\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}\,x\right)\right]$$

$$= \frac{g_0}{2}\sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}}x\right)(1-i)\left[\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right) + i\sin\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x\right)\right]$$

Its real part gives the fluctuation in temperature is

$$T(x,t) = \frac{g_0}{2}\sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}}x\right)\left[\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}\,x\right) + \sin\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}\,x\right)\right]$$

$$= \frac{g_0}{2}\sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}}x\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}}x - \frac{\pi}{4}\right)$$

Hence, the amplitude of the steady temperature is given by the factor

$$\frac{g_0}{\sqrt{2}}\sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}}x\right)$$

EXAMPLE 3.9 Find the solution of the one-dimensional diffusion equation satisfying the following BCs:

(i) T is bounded as $t \to \infty$

(ii) $\left.\dfrac{\partial T}{\partial x}\right|_{x=0} = 0, \quad \text{for all } t$

(iii) $\left.\dfrac{\partial T}{\partial x}\right|_{x=a} = 0, \quad \text{for all } t$

(iv) $T(x, 0) = x(a - x), \qquad 0 < x < a.$

Solution This is an example with insulated boundary conditions. From Section 3.5, it can be seen that a physically acceptable general solution of the diffusion equation is

$$T(x, t) = \exp(-\alpha\lambda^2 t)(A \cos \lambda x + B \sin \lambda x)$$

Thus,

$$\frac{\partial T}{\partial x} = \exp(-\alpha\lambda^2 t)(-A\lambda \sin \lambda x + B\lambda \cos \lambda x) \tag{3.57}$$

Using BC (ii), Eq. (3.57), gives $B = 0$. Since we are looking for a non-trivial solution, the use of BC (iii) into Eq. (3.57) at once gives

$$\sin \lambda a = 0 \text{ implying } \lambda a = n\pi, \qquad n = 0, 1, 2, \ldots$$

Using the principle of superposition, we get

$$T(x, t) = \sum A_n \exp(-\alpha\lambda^2 t) \cos \lambda x = \sum_{n=0}^{\infty} A_n \exp\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right] \cos\left(\frac{n\pi}{a}\right)x.$$

The boundary condition (iv) gives

$$T(x, 0) = x(a - x) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right] \cos\left(\frac{n\pi}{a}\right)x$$

where

$$A_0 = \frac{2}{a}\int_0^a (ax - x^2)\, dx = \frac{a^2}{6}$$

$$A_n = \frac{2}{a}\int_0^a (ax - x^2) \cos\left(\frac{n\pi}{a}x\right) dx$$

$$= \frac{2a^2}{n^2\pi^2}(1 + \cos n\pi) = \frac{2a^2}{n^2\pi^2}[1 + (-1)^n]$$

Therefore,

$$A_n = \begin{cases} -\dfrac{4a^2}{n^2\pi^2}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

Hence, the required solution is

$$T(x,t) = \frac{a^2}{6} - \frac{4a^2}{\pi^2} \sum_{n=2,\,4,\ldots\text{even}}^{\infty} \frac{1}{n^2} \cos\left(\frac{n\pi}{a}\right) x \exp\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right]$$

EXAMPLE 3.10 The boundaries of the rectangle $0 \le x \le a,\ 0 \le y \le b$ are maintained at zero temperature. If at $t = 0$ the temperature T has the prescribed value $f(x, y)$, show that for $t > 0$, the temperature at a point within the rectangle is given by

$$T(x, y, t) = \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) \exp(-\alpha\lambda_{mn}^2 t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

where

$$f(m, n) = \int_0^a \int_0^b f(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

and

$$\lambda_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

Solution The problem is to solve the diffusion equation described by

$$\text{PDE: } \frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \qquad 0 < x < a,\ 0 < y < b,\ t > 0$$

$$\text{BCs: } T(0, y, t) = T(a, y, t) = 0, \qquad 0 < y < b,\ t > 0$$

$$T(x, 0, t) = T(x, b, t) = 0, \qquad 0 < x < a,\ t > 0$$

$$\text{IC: } T(x, y, 0) = f(x, y), \qquad 0 < x < a,\ 0 < y < b$$

Let the separable solution be

$$T = X(x)Y(y)\beta(t)$$

Substituting into PDE, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2$$

Then $\beta' + \alpha\lambda^2\beta = 0$

$$\frac{X''}{X} = -\left(\lambda^2 + \frac{Y''}{Y}\right) = -p^2 \text{ (say)}$$

Hence,

$$X'' + p^2 X = 0$$

$$\frac{Y''}{Y} = -\lambda^2 + p^2 = -q^2 \text{ (say)}$$

Therefore,

$$Y'' + q^2 Y = 0$$

Thus, the general solution of the given PDE is

$$T(x, y, t) = (A \cos px + B \sin px)\,(c \cos qy + D \sin qy)\, e^{-\alpha\lambda^2 t}$$

where

$$\lambda^2 = p^2 + q^2$$

Using the BC: $T(0, y, t) = 0$, we get $A = 0$. Then, the solution is of the form

$$T(x, y, t) = B \sin px (c \cos qy + D \sin qy)\, e^{-\alpha\lambda^2 t}$$

Applying the BC: $T(x, 0, t) = 0$, we get $c = 0$. Thus, the solution is given by

$$T(x, y, t) = BD \sin px \sin qy e^{-\alpha\lambda^2 t}$$

Application of the BC: $T(a, y, t) = 0$ gives

$$\sin pa = 0, \text{ implying } pa = n\pi$$

or

$$p = \frac{n\pi}{a}, \qquad n = 1, 2, \ldots$$

Using the principle of superposition, the solution can be written in the form

$$T(x, y, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a} x\right) \sin qy e^{-\alpha\lambda^2 t}$$

Using the last BC: $T(x, b, t) = 0$, we obtain

$$q = \frac{m\pi}{b}, \qquad m = 1, 2, \ldots$$

Thus, the solution is found to be

$$T(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) e^{-\alpha\lambda^2 t}$$

where

$$\lambda^2 = p^2 + q^2 = \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{a^2} \right)$$

Finally, using the IC, we get

$$T(x, y, 0) = f(x, y) = \sum A_{mn} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)$$

which is a double Fourier series, where

$$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b f(x, y) \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) dx\, dy$$

Hence, the required general solution is

$$T(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) e^{-\alpha\lambda^2 t} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right)$$

where

$$f(m, n) = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) dx\, dy$$

and

$$\lambda^2 = \pi^2 \left(\frac{m^2}{b^2} + \frac{n^2}{a^2} \right)$$

3.6 SOLUTION OF DIFFUSION EQUATION IN CYLINDRICAL COORDINATES

Consider a three-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

In cylindrical coordinates (r, θ, z), it becomes

$$\frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \tag{3.58}$$

where $T = T(r, \theta, z, t)$.

Let us assume separation of variables in the form

$$T(r, \theta, z, t) = R(r) H(\theta) Z(z) \beta(t)$$

Substituting into Eq. (3.58), it becomes

$$R''HZ\beta + \frac{1}{r}R'HZ\beta + \frac{1}{r^2}H''RZ\beta + Z''RH\beta = \frac{\beta'}{\alpha}RHZ$$

or

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \frac{Z''}{Z} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2$$

where $-\lambda^2$ is a separation constant. Then

$$\beta' + \alpha\lambda^2\beta = 0 \tag{3.59}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 = -\frac{Z''}{Z} = -\mu^2 \text{ (say)}$$

Thus, the equations determining Z, R and H become

$$Z'' - \mu^2 Z = 0 \tag{3.60}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H} + \lambda^2 + \mu^2 = 0$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} + (\lambda^2 + \mu^2)r^2 = -\frac{H''}{H} = v^2 \text{ (say)}$$

Therefore,

$$H'' + v^2 H = 0 \tag{3.61}$$

$$R'' + \frac{1}{r}R' + \left[(\lambda^2 + \mu^2) - \frac{v^2}{r^2}\right]R = 0 \tag{3.62}$$

Equations (3.59)–(3.61) have particular solutions of the form

$$\beta = e^{-\alpha\lambda^2 t}$$

$$H = c\cos v\theta + D\sin v\theta$$

$$Z = Ae^{\mu z} + Be^{-\mu z}$$

The differential equation (3.62) is called Bessel's equation of order v and its general solution is known as

$$R(r) = c_1 J_v(\sqrt{\lambda^2 + \mu^2 r}) + c_2 Y_v(\sqrt{\lambda^2 + \mu^2 r})$$

where $J_\nu(r)$ and $Y_\nu(r)$ are Bessel functions of order ν of the first and second kind, respectively. Of course, Eq. (3.62) is singular when $r = 0$. The physically meaningful solutions must be twice continuously differentiable in $0 \le r \le a$. Hence, Eq. (3.62) has only one bounded solution, i.e.

$$R(r) = J_\nu(\sqrt{\lambda^2 + \mu^2 r})$$

Finally, the general solution of Eq. (3.58) is given by

$$T(r, \theta, z, t) = e^{-\alpha\lambda^2 t}\,[Ae^{\mu z} + Be^{-\mu z}][C\cos\nu\theta + D\sin\nu\theta]\,J_\nu(\sqrt{\lambda^2 + \mu^2} r)$$

EXAMPLE 3.11 Determine the temperature $T(r, t)$ in the infinite cylinder $0 \le r \le a$ when the initial temperature is $T(r, 0) = f(r)$, and the surface $r = a$ is maintained at 0° temperature.

Solution The governing PDE from the data of the problem is

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T$$

where T is a function of r and t only. Therefore,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.63}$$

The corresponding boundary and initial conditions are given by

$$\begin{aligned} &\text{BC: } T(a, t) = 0 \\ &\text{IC: } T(r, 0) = f(r) \end{aligned} \tag{3.64}$$

The general solution of Eq. (3.63) is

$$T(r, t) = A\exp(-\alpha\lambda^2 t)\,J_0(\lambda r)$$

Using the BC (3.64), we obtain

$$J_0(\lambda a) = 0$$

which has an infinite number of roots, $\xi_n a\,(n = 1, 2, \ldots, \infty)$. Thus, we get from the superposition principle the equation

$$T(r, t) = \sum_{n=1}^{\infty} A_n \exp(-\alpha\xi_n^2 t)\,J_0(\xi_n r)$$

Now using the IC: $T(r, 0) = f(r)$, we get

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\xi_n r)$$

To compute A_n, we multiply both sides of the above equation by $rJ_0(\xi_m r)$ and integrate with respect to r to get

$$\int_0^a rf(r)J_0(\xi_m r)\,dr = \sum_{n=1}^{\infty} A_n \int_0^a rJ_0(\xi_m r)\,J_0(\xi_n r)\,dr$$

$$= \begin{cases} 0 & \text{for } n \neq m \\ A_m\left(\dfrac{a^2}{2}\right)J_1^2(\xi_m a) & \text{for } n = m \end{cases}$$

which gives

$$A_m = \frac{2}{a^2 J_1^2(\xi_m a)}\int_0^a uf(u)J_0(\xi_m u)\,du$$

Hence, the final solution of the problem is given by

$$T(r,t) = \frac{2}{a^2}\sum_{m=1}^{\infty}\frac{J_0(\xi_m r)}{J_1^2(\xi_m a)}\exp\left(-\alpha\xi_m^2 t\right)\left[\int_0^a uf(u)J_0(\xi_m u)\,du\right]$$

3.7 SOLUTION OF DIFFUSION EQUATION IN SPHERICAL COORDINATES

In this section, we shall examine the solution of diffusion or heat conduction equation in the spherical coordinate system. Let us consider the three-dimensional diffusion Eq. (3.9), and let $T = T(r, \theta, \phi, t)$. In the spherical coordinate system, Eq. (3.9) can be written as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.65}$$

This equation is separated by assuming the temperature function T in the form

$$T = R(r)\,H(\theta)\,\Phi(\phi)\,\beta(t) \tag{3.66}$$

Substituting Eq. (3.66) into Eq. (3.65), we get

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2\sin\theta}\frac{1}{H}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2 \quad \text{(say)}$$

where λ^2 is a separation constant. Thus,

$$\frac{d\beta}{dt} + \lambda^2\alpha\beta = 0$$

whose solution is

$$\beta = c_1 e^{-\alpha\lambda^2 t} \tag{3.67}$$

Also,

$$r^2 \sin^2\theta \left[\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{Hr^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dH}{d\theta}\right) + \lambda^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2 \text{ (say)}$$

which gives

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

whose solution is

$$\Phi(\phi) = c_1 e^{im\phi} + c_2 e^{-im\phi} \tag{3.68}$$

Now, the other separated equation is

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{Hr^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dH}{d\theta}\right) + \lambda^2 = \frac{m^2}{r^2 \sin^2\theta}$$

or

$$\frac{r^2}{R}\left(R'' + \frac{2}{r}R'\right) + \lambda^2 r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{H \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dH}{d\theta}\right)$$

$$= n(n+1) \text{ (say)}$$

On re-arrangement, this equation can be written as

$$R'' + \frac{2}{r}R' + \left\{\lambda^2 - \frac{n(n+1)}{r^2}\right\}R = 0 \tag{3.69}$$

and

$$-\frac{1}{H \sin\theta}\left(\sin\theta \frac{d^2H}{d\theta^2} + \cos\theta \frac{dH}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = n(n+1)$$

or

$$\frac{d^2H}{d\theta^2} + \cot\theta \frac{dH}{d\theta} + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}H = 0 \tag{3.70}$$

Let $R = (\lambda r)^{-1/2}\psi(r)$; then Eq. (3.69) becomes

$$(\lambda r)^{-1/2}\left[\psi''(r) + \frac{1}{r}\psi'(r) + \left\{\lambda^2 - \frac{(n+1/2)^2}{r^2}\right\}\psi\right] = 0$$

Since $(\lambda r) \neq 0$, we have

$$\psi''(r) + \frac{1}{r}\psi'(r) + \left\{\lambda^2 - \frac{(n+1/2)^2}{r^2}\right\}\psi(r) = 0$$

which is Bessel's differential equation of order $(n+1/2)$, whose solution is

$$\psi(r) = AJ_{n+1/2}(\lambda r) + BY_{n+1/2}(\lambda r)$$

Therefore,

$$R(r) = (\lambda r)^{-1/2}[AJ_{n+1/2}(\lambda r) + BY_{n+1/2}(\lambda r)] \tag{3.71}$$

where J_n and Y_n are Bessel functions of first and second kind, respectively. Now, Eq. (3.70) can be put in a more convenient form by introducing a new independent variable

$$\mu = \cos\theta$$

so that

$$\cot\theta = \mu/\sqrt{1-\mu^2}$$

$$\frac{dH}{d\theta} = -\sqrt{1-\mu^2}\,\frac{dH}{d\mu}$$

$$\frac{d^2H}{d\theta^2} = (1-\mu^2)\frac{d^2H}{d\mu^2} - \mu\frac{dH}{d\mu}$$

Thus, Eq. (3.70) becomes

$$(1-\mu^2)\frac{d^2H}{d\mu^2} - 2\mu\frac{dH}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]H = 0 \tag{3.72}$$

which is an associated Legendre differential equation whose solution is

$$H(\theta) = A'P_n^m(\mu) + B'Q_n^m(\mu) \tag{3.73}$$

where $P_n^m(\mu)$ and $Q_n^m(\mu)$ are associated Legendre functions of degree n and of order m, of first and second kind, respectively. Hence the physically meaningful general solution of the diffusion equation in spherical geometry is of the form

$$T(r,\theta,\phi,t) = \sum_{\lambda,m,n} A_{\lambda mn}(\lambda r)^{-1/2} J_{n+1/2}(\lambda r)P_n^m(\cos\theta)e^{\pm im\phi-\alpha\lambda^2 t} \tag{3.74}$$

In this general solution, the functions $Q_n^m(\mu)$ and $(\lambda r)^{-1/2}Y_{n+1/2}(\lambda r)$ are excluded because these functions have poles at $\mu = \pm 1$ and $r = 0$ respectively.

EXAMPLE 3.12 Find the temperature in a sphere of radius a, when its surface is kept at zero temperature and its initial temperature is $f(r, \theta)$.

Solution Here, the temperature is governed by the three-dimensional heat equation in spherical polar coordinates independent of ϕ. Therefore, the task is to find the solution of PDE

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) \tag{3.75}$$

subject to

$$\text{BC:} \, T(a, \theta, t) = 0 \tag{3.76}$$

$$\text{IC:} \, T(r, \theta, 0) = f(r, \theta) \tag{3.77}$$

The general solution of Eq. (3.75), with the help of Eq. (3.74), can be written as

$$T(r, \theta, t) = \sum_{\lambda, n} A_{\lambda n} (\lambda r)^{-1/2} J_{n+1/2}(\lambda r) P_n(\cos\theta) e^{-\alpha \lambda^2 t} \tag{3.78}$$

Applying the BC (3.76), we get

$$J_{n+1/2}(\lambda a) = 0$$

This equation has infinitely many positive roots. Denoting them by ξ_i, we have

$$T(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r)\, P_n(\cos\theta) \exp(-\alpha \xi_i^2 t) \tag{3.79}$$

Now, applying the IC and denoting $\cos\theta$ by μ, we get

$$f(r, \cos^{-1}(\mu)) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r)\, P_n(\mu).$$

Multiplying both sides by $P_m(\mu)\, d\mu$ and integrating between the limits, –1 to 1, we obtain

$$\begin{aligned}\int_{-1}^{1} f(r, \cos^{-1}(\mu))\, P_m(\mu)\, d\mu &= \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \left(\frac{2}{2n+1}\right)\end{aligned}$$

or

$$\left(\frac{2n+1}{2}\right)\int_{-1}^{1} P_n(\mu)\, f(r, \cos^{-1}(\mu))\, d\mu = \sum_{i=1}^{\infty} A_{ni}\, (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Now, to evaluate the constants A_{ni}, we multiply both sides of the above equation by $r^{3/2} J_{n+1/2}(\xi_j r)$ and integrate with respect to r between the limits 0 to a and use the orthogonality property of Bessel functions to get

$$\xi_i^{1/2}\left(\frac{2n+1}{2}\right)\int_0^a r^{3/2} J_{n+1/2}(\xi_j r)\, dr \left[\int_{-1}^{1} P_n(\mu)\, f(r, \cos^{-1}(\mu))\, d\mu\right]$$

$$= \sum_{i=1}^{\infty} A_{ni} \int_0^a r J_{n+1/2}(\xi_i r)\, J_{n+1/2}(\xi_j r)\, dr$$

$$= \frac{1}{2}\sum_{i=1}^{\infty} A_{ni}\, [J'_{n+1/2}(\xi_i r)]^2, \quad n = 0, 1, 2, 3, \ldots \tag{3.80}$$

Thus, Eqs. (3.79) and (3.80) together constitutes the solution for the given problem.

3.8 MAXIMUM-MINIMUM PRINCIPLE AND CONSEQUENCES

Theorem 3.1 (Maximum-minimum principle).

Let $u(x, t)$ be a continuous function and a solution of

$$u_t = \alpha u_{xx} \tag{3.81}$$

for $0 \le x \le l,\ 0 \le t \le T,$ where $T > 0$ is a fixed time. Then the maximum and minimum values of u are attained either at time $t = 0$ or at the end points $x = 0$ and $x = l$ at some time in the interval $0 \le t \le T$.

Proof To start with, let us assume that the assertion is false. Let the maximum value of $u(x, t)$ for $t = 0 (0 \le x \le l)$ or for $x = 0$ or $x = l\,(0 \le t \le T)$ be denoted by M. We shall assume that the function $u(x, t)$ attains its maximum at some interior point (x_0, t_0), in the rectangle defined by $0 \le x_0 \le l,\ 0 \le t_0 \le T,$ and then arrive at a contradiction. This means that

$$u(x_0, t_0) = M + \varepsilon \tag{3.82}$$

Now, we shall compare the signs in Eq. (3.81) at the point (x_0, t_0). It is well known from calculus that the necessary condition for the function $u(x, t)$ to possess maximum at (x_0, t_0) is

$$\frac{\partial u}{\partial x}(x_0, t_0) = 0, \qquad \frac{\partial^2 u}{\partial x^2}(x_0, t_0) \le 0 \tag{3.83}$$

In addition, $u(x_0, t_0)$ attains maximum for $t = t_0$, implying

$$\frac{\partial u}{\partial t}(x_0, t_0) \geq 0 \tag{3.84}$$

Thus, with the help of Eqs. (3.83) and (3.84) we observe that the signs on the left- and right-hand sides of Eq. (3.81) are different. However, we cannot claim that we have reached a contradiction, since the left- and right-hand sides can simultaneously be zero.

To complete the proof, let us consider another point (x_1, t_1) at which $\partial^2 u/\partial x^2 \leq 0$ and $\partial u/\partial t > 0$.

Now, we construct an auxiliary function

$$v(x, t) = u(x, t) + \lambda(t_0 - t) \tag{3.85}$$

where λ is a constant. Obviously, $v(x_0, t_0) = u(x_0, t_0) = M + \varepsilon$ and $\lambda(t_0 - t) \leq \lambda T$. Suppose we choose $\lambda > 0$, such that $\lambda < \varepsilon/2T$; then the maximum of $v(x, t)$ for $t = 0$ or for $x = 0$, $x = l$ cannot exceed the value $M + \varepsilon/2$. But $v(x, t)$ is a continuous function and, therefore, a point (x_1, t_1) exists at which it assumes its maximum. It implies

$$M + \varepsilon/2 \leq v(x_1, t_1) \geq v(x_0, t_0) = M + \varepsilon$$

This pair of inequalities is inconsistent and therefore contradicts the assumption that v takes on its maximum at (x_0, t_0). Therefore, the assertion that u attains its maximum either at $t = 0$ or at the end points is true.

We can establish a similar result for minimum values of $u(x, t)$. If u satisfies Eq. (3.81), $-u$ also satisfies Eq. (3.81). Hence, both maximum and minimum values are attained either initially or at the end points. Thus the proof is complete. We shall give some of the consequences of the maximum-minimum principle in the following theorems.

Theorem 3.2 (Uniqueness theorem). Given a rectangular region defined by $0 \leq x \leq l$, $0 \leq t \leq T$, and a continuous function $u(x, t)$ defined on the boundary of the rectangle satisfying the heat equation

$$u_t = \alpha u_{xx}$$

This equation possesses one and only one solution satisfying the initial and boundary conditions

$$u(x, 0) = f(x)$$

$$u(0, t) = g_1(t),\ u(l, t) = g_2(t)$$

where $f(x)$, $g_1(t)$, $g_2(t)$ are continuous on their domains of definition.

Proof Suppose there are two solutions $u_1(x, t)$, $u_2(x, t)$ satisfying the heat equation as well as the same initial and boundary conditions. Now let us consider the difference

$$v(x, t) = u_2(x, t) - u_1(x, t)$$

It is also a solution of the heat conduction equation for $0 \le x \le l,\ 0 \le t \le T$ and is continuous in x and t. Also, $v(x, t) = 0,\ 0 \le x \le l$ and $v(0, t) = v(l, t) = 0,\ 0 \le t \le T$. Hence, $v(x, t)$ satisfies the conditions required for the application of maximum-minimum principle. Thus, $v(x, t) = 0$ in the rectangular region defined by $0 \le x \le l,\ 0 \le t \le T$. It follows therefore that $u_1(x, t) = u_2(x, t)$.

Another important consequence of the maximum-minimum principle is the stability property which is stated in the following theorem without proof.

Theorem 3.3 (Stability theorem). The solution $u(x, t)$ of the Dirichlet problem

$$\begin{aligned} u_t &= \alpha u_{xx}, && 0 \le x \le l,\ 0 \le t \le T \\ u(x, 0) &= f(x), && 0 \le x \le l \\ u(0, t) &= g(t), && u(l, t) = h(t),\ 0 \le t \le T \end{aligned}$$

depends continuously on the initial and boundary conditions.

3.9 NON-LINEAR EQUATIONS (MODELS)

Today various studies of fluid behaviour are available which encompass virtually any type of phenomena of practical importance. However, there are many unresolved important problems in fluid dynamics due to the non-linear nature of the governing PDEs and due to difficulties encountered in many of the conventional, analytical and numerical techniques in solving them. In the following, we shall present few non-linear model equations to have a feel for this vast field of study.

3.9.1 Semilinear Equations

Reaction–diffusion equations that appear in the literature are frequently semilinear and are of the form $u_t = \nabla^2 u + f(u, x, t)$.

Typically, they appear as models in population dynamics, with inhomogeneous term depending on the density of local population. In chemical engineering, f varies with temperature and/or chemical concentration in a reaction like $f(u) = \lambda u^N$.

3.9.2 Quasi-linear Equations

Many problems in fluid mechanics, when formulated mathematically, give rise to quasi-linear parabolic PDEs. A simple example concerns the flow of compressible fluid through a porous medium. Let ρ denotes fluid density. Following Darcy's law, which relates the velocity **V** to the pressure p as

$$\mathbf{V} = -\left(\frac{K}{\mu}\right)\nabla p.$$

Then, the equation of conservation of mass is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

and the equation of state $p = p(\rho)$ can be combined to get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [K(\rho) \nabla \rho]$$

where $K(\rho)$ is proportional to $\rho \dfrac{dp}{d\rho}$. The model can then be written as the porous medium equation in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho^n \nabla \rho)$$

where n is a positive constant.

3.9.3 Burger's Equation

The well-known Burger's equation is non-linear and finding its solution has been the subject of active research for many years. For simplicity, let us consider one-dimensional Burger's equation in the form

$$u_t + uu_x = vu_{xx}$$

or

$$u_t - \left(vu_x - \frac{1}{2}u^2 \right)_x = 0 \tag{3.85a}$$

which is actually the non-linear momentum equation in *fluid mechanics* without the pressure term. v is the physical viscosity. Here vu_{xx} measures dissipative term and uu_x measures convective term, while u_t is the unsteady term. Hopf (1950) and Cole (1951) gave independently the analytical solution for a model problem using a two-step Hopf–Cole transformation described by

$$u(x, t) = \psi_x$$
$$\psi = -2v \log \phi(x, t)$$

That is,

$$u = -2v \frac{\phi_x}{\phi} \tag{3.85b}$$

Thus,

$$u_t = -2v \left(\frac{\phi_{xt}}{\phi} \right) + 2v \frac{\phi_x \phi_t}{\phi^2},$$

$$u_x = -2v\left(\frac{\phi_{xx}}{\phi}\right) + 2v\left(\frac{\phi_x^2}{\phi^2}\right)$$

$$u_{xx} = -2v\left(\frac{\phi_{xxx}}{\phi}\right) + 6v\left(\frac{\phi_x \phi_{xx}}{\phi^2}\right) - 4v\left(\frac{\phi_x^3}{\phi^3}\right).$$

Inserting, these derivative expressions into Eq. (3.85a) and on simplification, we arrive at

$$\frac{\phi_x}{\phi}(v\phi_{xx} - \phi_t) - (v\phi_{xx} - \phi_t)_x = 0. \tag{3.85c}$$

Therefore, we have to solve Eq. (3.85c) to find $\phi(x, t)$, and using this result in Eq. (3.85b), we obtain an expression for $u(x, t)$ which of course satisfies Eq. (3.85a). Thus, if $\phi(x, t)$ satisfies heat conduction equation

$$\phi_t = v\phi_{xx} \tag{3.85d}$$

which also means solving trivially Eq. (3.85c). This is also called *linearised Burger's equation*. Equivalently, we may introduce the transformation:

$$\left.\begin{aligned} \psi_x &= u, \\ \psi_t &= vu_x - \frac{u^2}{2} \end{aligned}\right\} \tag{3.85e}$$

in such a way, satisfying that $\psi_{xt} = \psi_{tx}$. Then, the above transformation can be rewritten as

$$\psi_t = v\psi_{xx} - \frac{\psi_x^2}{2} \tag{3.85f}$$

Also, Eq. (3.85b) can be recast in the form

$$\phi(x, t) = e^{[-\psi(x,\, t)/2v]} \tag{3.85g}$$

Thus, knowing $\phi(x, t)$, we can find $u(x, t)$ from Eq. (3.85b). It may also be observed that Eqs. (3.85d) and (3.85f) are equivalent.

Hence, the transformation of non-linear Burger's equation into heat conduction equation, made life easy to get analytical solution to the Burger's equation.

3.9.4 Initial Value Problem for Burger's Equation

The IVP for Burger's equation can be stated as follows. Solve

$$\text{PDE: } u_t + uu_x = vu_{xx},\ -\infty < x < \infty,\ t > 0$$

$$\text{IC: } u(x, 0) = f(x) \tag{3.85h}$$

Under the transformation defined by Eq. (3.85b) and using (3.85g), the given IVP can be restated as a Cauchy problem, described by

$$\text{PDE: } \phi_t = v\phi_{xx},$$

$$\text{IC: } \phi(x, 0) = \bar{\phi}(x) = e^{\left[-\frac{1}{2\nu}\int_0^x f(\eta)d\eta\right]} \tag{3.85i}$$

Using, the standard, separation of variables, method of solution, as given in Eq. (3.28), the solution to Eq. (3.85i) is found to be

$$\phi(x, t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} \bar{\phi}(x)\exp\left[-\frac{(x-\xi)^2}{4\nu t}\right]d\xi \tag{3.85j}$$

Substituting the expression for $\bar{\phi}(x)$, the above equation can be rewritten as

$$\phi(x, t) = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} \exp\left[-\frac{f(\xi, x, t)}{2\nu}\right]d\xi$$

where,

$$f(\xi, x, t) = \int_0^{\xi} f(\alpha)d\alpha + \frac{(x-\xi)^2}{2t}.$$

Finally, using Eq. (3.85b), the exact solution of the IVP for Burger's equation as stated in Eq. (3.85h) is found to be

$$u(x, t) = \frac{\left[\int_{-\infty}^{\infty} \frac{(x-\xi)}{t}\exp\left\{-\frac{f(\xi, x, t)}{2\nu}\right\}d\xi\right]}{\exp\left[-\frac{f(\xi, x, t)}{2\nu}\right]d\xi} \tag{3.85k}$$

Here, the function $f(\xi, x, t)$ is known as Hopf–Cole function.

3.10 MISCELLANEOUS EXAMPLES

EXAMPLE 3.13 A homogeneous solid sphere of radius R has the initial temperature distribution $f(r)$, $0 \le r \le R$, where r is the distance measured from the centre. The surface temperature is maintained at 0°. Show that the temperature $T(r, t)$ in the sphere is the solution of

$$T_t = c^2\left(T_{rr} + \frac{2}{r}T_r\right)$$

where c^2 is a constant. Show also that the temperature in the sphere for $t > 0$ is given by

$$T(r, t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right)\exp(-\lambda_n^2 t), \qquad \lambda_n = \frac{cn\pi}{R}$$

Solution The temperature distribution in a solid sphere is governed by the parabolic heat equation

$$T_t = c^2 \nabla^2 T$$

From the data given, T is a function of r and t alone. In view of the symmetry of the sphere, the above equation with the help of Eq. (3.65) reduces to

$$T_t = c^2 \left(T_{rr} + \frac{2}{r} T_r \right) \tag{3.86}$$

Setting $v = rT$, the given BC gives

$$v(R, t) = rT(R, t) = 0$$

while the IC gives

$$v(r, 0) = rT(r, 0) = rf(r)$$

Since T must be bounded at $r = 0$, we require

$$v(0, t) = 0$$

Now,

$$v_t = rT_t, \qquad T_r = \left(\frac{v}{r} \right)_r = \frac{v_r r - v}{r^2}$$

Similarly, finding T_{rr} and substituting into Eq. (3.86), we obtain

$$v_t = c^2 v_{rr}$$

Using the variables separable method, we may write $v(r, t) = R(r)\tau(t)$ and get

$$R(r) = A \cos kr + B \sin kr$$

$$\tau(t) = \exp(-c^2 k^2 t)$$

Thus, using the principle of superposition, we get

$$v(r, t) = \sum_{n=1}^{\infty} (A_n \cos kr + B_n \sin kr) \exp(-c^2 k^2 t)$$

Also, using $v(0, t) = 0$, we have

$$(A_n \cos kr + B_n \sin kr)|_{r=0} = 0$$

implying $A_n = 0$. Also, $v(R, t) = 0$ gives $B_n \sin kR = 0$, implying $\sin kR = 0$, as $B_n \neq 0$. Therefore,

$$kR = n\pi, \qquad k = \frac{n\pi}{R}, \qquad n = 1, 2, \ldots$$

Thus, the possible solution is

$$v(r,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right) \exp\left(-\frac{c^2 n^2 \pi^2 t}{R^2}\right)$$

Finally, applying the IC: $v(r,0) = rf(r)$, we get

$$rf(r) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right)$$

which is a half-range Fourier series. Therefore,

$$B_n = \frac{2}{R}\int_0^R rf(r)\sin\left(\frac{n\pi}{R} r\right) dr$$

But $v(r,t) = rT(r,t)$. Hence, the temperature in the sphere is given by

$$T(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right) \exp\left(-\frac{c^2 n^2 \pi^2 t}{R^2}\right)$$

EXAMPLE 3.14 A circular cylinder of radius a has its surface kept at a constant temperature T_0. If the initial temperature is zero throughout the cylinder, prove that for $t > 0$.

$$T(r,t) = T_0\left\{1 - \frac{2}{a}\sum_{n=1}^{\infty} \frac{J_0(\xi_n a)}{\xi_n J_1(\xi_n a)} \exp(-\xi_n^2 kt)\right\}$$

where $\pm\xi_1, \pm\xi_2, \ldots, \pm\xi_n$ are the roots of $J_0(\xi a) = 0$, and k is the thermal conductivity which is a constant.

Solution It is evident that T is a function of r and t alone and, therefore, the PDE to be solved is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{3.87}$$

subject to

$$\text{IC}: T(r,0) = 0, \qquad 0 \le r < a$$

$$\text{BC}: T(a,t) = T_0, \qquad t \ge 0$$

Let

$$T(r,t) = T_0 + T_1(r,t)$$

so that

$$T_1(r, 0) = -T_0 \tag{3.88}$$

$$T_1(a, t) = 0 \tag{3.89}$$

where T_1 is the solution of Eq. (3.87). By the variables separable method we have (see Example 3.11),

$$T_1(r, t) = AJ_0(\lambda r) \exp(-\lambda^2 kt)$$

Using the BC: $T_1(a, t) = 0$, we get

$$AJ_0(\lambda a) \exp(-\lambda^2 kt) = 0$$

which gives $J_0(\lambda a) = 0$ *as* $A \neq 0$. Let $\xi_1, \xi_2, \ldots, \xi_n$, be the roots of $J_0(\lambda a) = 0$. Then the possible solution using the superposition principle is

$$T_1(r, t) = \sum_{n=1}^{\infty} A_n J_0(\xi_n r) \exp(-\xi_n^2 kt) \tag{3.90}$$

Using the IC: $T_1(r, 0) = -T_0$ into Eq. (3.90), we obtain

$$\sum_{n=1}^{\infty} A_n J_0(\xi_n r) = -T_0$$

Multiplying both sides by $rJ_0(\xi_m r)$ and integrating, we get

$$-T_0 \int_0^a rJ_0(\xi_m r)\, dr = \sum_{n=1}^{\infty} A_n \int_0^a rJ_0(\xi_m r)\, J_0(\xi_n r)\, dr$$

$$= A_m \int_0^a rJ_0^2(\xi_m r) \quad \text{if } m = n; \text{ otherwise } 0$$

$$= A_m \frac{a^2}{2} J_1^2(\xi_m a)$$

But,

$$-T_0 \int_0^a rJ_0(\xi_m r)\, dr = -T_0 \int_0^{\xi_m a} \frac{x}{\xi_m} J_0(x) \frac{dx}{\xi_m} \qquad (x = \xi_m r)$$

$$= -\frac{T_0}{\xi_m^2} \int_0^{\xi_m a} \frac{d}{dx}[xJ_1(x)]\, dx$$

$$= -\frac{T_0}{\xi_m^2} [xJ_1(x)]_0^{\xi_m a} = -\frac{aT_0}{\xi_m} J_1(\xi_m a)$$

Therefore,

$$A_m \frac{a^2}{2} J_1^2(\xi_m a) = -\frac{aT_0}{\xi_m} J_1(\xi_m a)$$

or

$$A_n = -\frac{2T_0}{a\xi_n} \frac{1}{J_1(\xi_n a)}$$

Hence, Eq. (3.90) becomes

$$T_1(r, t) = -\frac{2}{a} T_0 \sum_{n=1}^{\infty} \frac{J_0(\xi_n r)}{J_1(\xi_n a)} \frac{\exp(-\xi_n^2 kt)}{\xi_n}$$

Finally, the complete solution is found to be

$$T(r, t) = T_0 \left[1 - \frac{2}{a} \sum_{n=1}^{\infty} \frac{J_0(\xi_n r)}{J_1(\xi_n a)} \frac{\exp(-\xi_n^2 kt)}{\xi_n} \right]$$

EXAMPLE 3.15 Determine the temperature in a sphere of radius a, when its surface is maintained at zero temperature while its initial temperature is $f(r, \theta)$.

Solution Here the temperature is governed by the three-dimensional heat equation in polar coordinates independent of ϕ, which is given by

$$\frac{1}{k} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) \tag{3.91}$$

Let

$$u(r, \theta, t) = R(r) H(\theta) T(t)$$

By the variables separable method (see Section 3.7), the general solution of Eq. (3.91) is found to be

$$u(r, \theta, t) = \sum_{\lambda} \sum_{n} A_{\lambda n} (\lambda r)^{-1/2} J_{n+1/2}(\lambda r) \, P_n(\cos\theta) \exp(-k\lambda^2 t) \tag{3.92}$$

In the present problem, the boundary and initial conditions are

$$\text{BC: } u(a, \theta, t) = 0 \tag{3.93}$$

$$\text{IC: } u(r, \theta, 0) = f(r, \theta) \tag{3.94}$$

Substituting the BC (3.93) into Eq. (3.92), we get

$$J_{n+1/2}(\lambda a) = 0 \tag{3.95}$$

Let $\xi_1 a, \xi_2 a, \ldots, \xi_i a, \ldots$ be the roots of Eq. (3.95). Then the general solution can be put in the form

$$u(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) P_n(\cos\theta) \exp(-k\xi_i^2 t) \tag{3.96}$$

Now using the IC, we obtain

$$f(r, \theta) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) P_n(\cos\theta)$$

Multiplying both sides by $P_n(\cos\theta)\, d(\cos\theta)$ and integrating, we have

$$\int_{-1}^{1} P_n(\cos\theta) f(r, \theta)\, d(\cos\theta) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \int_{-1}^{1} P_n^2(\cos\theta)\, d(\cos\theta)$$

Using the orthogonality property of Legendre polynomials, we get

$$\int_{-1}^{1} P_n(\cos\theta) f(r, \theta)\, d(\cos\theta) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \left(\frac{2}{2n+1}\right)$$

Rearranging and multiplying both sides of the above equation by $r^{3/2} J_{n+1/2}(\xi_i r)$ and integrating between the limits 0 to a with respect to r, we get

$$\begin{aligned} \frac{2n+1}{2} \int_0^a r^{3/2} J_{n+1/2}(\xi_i r)\, dr \int_{-1}^{1} P_n(\cos\theta)\; f(r, \theta)\, d(\cos\theta) &= A_{ni} \int_0^a r J_{n+1/2}^{\,2}(\xi_i r) \xi_i^{-1/2} dr \\ &= A_{ni} \frac{a^2}{2} \{J'_{n+1/2}(\xi_i a)\}^2 \end{aligned}$$

Therefore,

$$A_{ni} = \frac{(2n+1)\xi_i^{1/2}}{a^2 \{J'_{n+1/2}(\xi_i a)\}^2} \int_0^a r^{3/2} J_{n+1/2}(\xi_i r)\, dr \int_{-1}^{1} P_n(\cos\theta)\; f(r, \theta)\, d(\cos\theta) \tag{3.97}$$

Hence, we obtain the solution to the given problem from Eq. (3.96), where A_{ni} is given by Eq. (3.97).

EXAMPLE 3.16 The heat conduction in a thin round insulated rod with heat sources present is described by the PDE

$$u_t - \alpha u_{xx} = F(x, t)/\rho c, \qquad 0 < x < l,\; t > 0 \tag{3.98}$$

subject to

$$\begin{aligned} &\text{BCs: } u(0, t) = u(l, t) = 0 \\ &\text{IC: } \; u(x, 0) = f(x), \qquad 0 \le x \le l \end{aligned} \tag{3.99}$$

where ρ and c are constants and F is a continuous function of x and t. Find $u(x, t)$.

Solution It can be noted that the boundary conditions are of homogeneous type. Let us consider the homogeneous equation

$$u_t - \alpha u_{xx} = 0 \tag{3.100}$$

Setting $u(x, t) = X(x)T(t)$, we get

$$\frac{T'}{\alpha T} = \frac{X''}{X} = -\lambda^2 \text{ (say)} \tag{3.101}$$

which gives $X'' + \lambda^2 X = 0$. The corresponding BCs are

$$X(0) = X(l) = 0$$

The solution of Eq. (3.101) gives the desired eigenfunctions and eigenvalues, which are

$$X_n(x) = \sin \lambda_n x, \qquad \lambda_n^2 = \left(\frac{n\pi}{l}\right)^2, \qquad n \geq 1 \tag{3.102}$$

For the non-homogeneous problem (3.98), let us propose a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t)\, X_n(x) \tag{3.103}$$

It is clear that Eq. (3.103) satisfies the BCs (3.99). From the orthogonality of eigenfunctions, it follows that

$$T_m(t) = \frac{2}{l}\int_0^l u(x, t)\, X_m(x)\, dx$$

However,

$$T_m(0) = \frac{2}{l}\int_0^1 f(x) \sin\left(\frac{m\pi}{l} x\right) dx \tag{3.104}$$

which is an IC for $T_m(t)$. Introducing Eq. (3.103) into the governing equation (3.98), we get

$$\sum_{n=1}^{\infty} T_n' X_n - \alpha \sum_{n=1}^{\infty} T_n X_n'' = \frac{F(x, t)}{\rho c} \tag{3.105}$$

Now, we shall expand $F(x, t)/\rho c$, so that it is represented by a convergent series on $0 < x < l,\ t > 0$ in the form

$$\frac{F}{\rho c} = \sum_{n=1}^{\infty} q_n(t)\, X_n(x) \tag{3.106}$$

where

$$q_n(t) = \frac{2}{l}\int_0^l \frac{F(x,t)}{\rho c}\sin\left(\frac{n\pi}{l}x\right)dx \tag{3.107}$$

Thus, $q_n(t)$ is known. Now, Eq. (3.105), with the help of Eq. (3.101), becomes

$$\sum_{n=1}^{\infty} X_n(T_n' + \lambda_n^2 \alpha T_n - q_n) = 0$$

Therefore, it follows that

$$T_n'(t) + \lambda_n^2 \alpha T_n(t) = q_n(t) \tag{3.108}$$

Its solution with the help of IC (3.104) is

$$T_n(t) = T_n(0)\exp(-\lambda_n^2 \alpha t) + \int_0^t \exp[\lambda_n^2 \alpha(\tau - t)]q_n(\tau)\,d\tau \tag{3.109}$$

From Eqs. (3.103) and (3.109), the complete solution is found to be

$$u(x,t) = \sum_{n=1}^{\infty}\left[T_n(0)\exp(-\lambda_n^2 \alpha t) + \int_0^1 \exp[\lambda_n^2 \alpha(\tau - t)\, q_n(\tau)\,d\tau\right]X_n(x)$$

In the expanded form, it becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left\{\frac{2}{l}\int_0^l f(\xi)X_n(\xi)\,d\xi\right\}\exp(-\lambda_n^2 \alpha t)\right.$$

$$\left. + \frac{2}{l}\int_0^l \exp\left\{-\lambda_n^2 \alpha(\tau - t)\right\}\int_0^l \frac{F(\xi,\tau)}{\rho c}X_n(\xi)\,d\xi\,d\tau\right]X_n(x) \tag{3.110}$$

It can be verified that the series in Eq. (3.110) converges uniformly for $t > 0$. By changing the order of integration and summation in Eq. (3.110), we get

$$u(x,t) = \int_0^l\left[\sum_{n=1}^{\infty}\frac{\exp(-\lambda_n^2 \alpha t)\, X_n(x)X_n(\xi)}{1/2}\right]f(\xi)\,d\xi$$

$$+ \int_0^l \int_0^l\left[\sum_{n=1}^{\infty}\frac{\exp\{-\lambda_n^2 \alpha(t-\tau)\}\, X_n(x)\, X_n(\xi)}{l/2}\right]\frac{F(\xi,t)}{\rho c}\,d\xi\,d\tau$$

which can also be written in the form

$$u(x,t) = \int_0^l G(x,\xi;t)\, f(\xi)\,d\xi + \int_0^l \int_0^l G(x,\xi;t-\tau)\frac{F(\xi,\tau)}{\rho c}\,d\xi\,d\tau \tag{3.111}$$

where

$$G(x, \xi; t) = \sum_{n=1}^{\infty} \frac{\exp(-\lambda_n^2 \alpha t)\, X_n(x)\, X_n(\xi)}{l/2}$$

is called Green's function. More details on Green's function are given in Chapter 5.

EXAMPLE 3.17 The temperature distribution of a homogeneous thin rod, whose surface is insulated is described by the following IBVP:

$$\text{PDE: } v_t - v_{xx} = 0,\ 0 < x < L,\ 0 < t < \infty \tag{3.112}$$

$$\text{BCS: } v(0, t) = v(L, t) = 0 \tag{3.113}$$

$$\text{IC: } v(x, 0) = f(x),\ 0 \le x \le L \tag{3.114}$$

Find its formal solution.

Solution Let us assume the solution in the form

$$v(x, t) = X(x)T(t)$$

Eq. (3.112) gives

$$XT' = X''T$$

or

$$\frac{X''}{X} = \frac{T'}{T} = -\alpha^2 \text{ (say)}$$

where α is a positive constant. Then, we have

$$X'' + \alpha^2 X = 0$$

and

$$T' + \alpha^2 T = 0$$

From the BCS

$$v(0, t) = X(0)T(t) = 0,$$

and

$$v(L, t) = X(L)T(t) = 0,$$

we obtain, $X(0) = X(L) = 0$ for arbitrary t. Thus, we have to solve the eigenvalue problem

$$X'' + \alpha^2 X = 0$$

subject to $X(0) = X(L) = 0$.

The solution of the differential equation is

$$X(x) = A \cos \alpha x + B \sin \alpha x.$$

Since $X(0) = 0$, $A = 0$. The second condition yields

$$X(L) = B \sin \alpha L = 0$$

For non-trivial solution, $B \neq 0$ and therefore we have

$$\sin \alpha L = 0, \text{ implying } \alpha = n\pi/L, \text{ for } n = 1, 2, 3, \ldots$$

Thus, the solution is obtained as

$$X_n(x) = B_n \sin \frac{n\pi x}{L}.$$

Next, we consider the equation

$$T' + \alpha^2 T = 0$$

whose solution can be written as

$$T(t) = Ce^{-\alpha^2 t}$$

or

$$T_n(t) = C_n e^{-(n\pi/L)^2 t}.$$

Hence, the non-trivial solution of the given heat equation satisfying both the boundary conditions is found to be

$$v_n(x, t) = a_n e^{-(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{3.115}$$

where $a_n = b_n c_n$ (arbitrary constant).

To satisfy the IC, we should have

$$v(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

which holds good, if $f(x)$ is representable as Fourier Sine series with Fourier coefficients

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Hence, the required formal solution is

$$v(x, t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(\tau)\sin\frac{n\pi\tau}{L}\,d\tau\right] e^{-(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

EXERCISES

1. A conducting bar of uniform cross-section lies along the x-axis, with its ends at $x = 0$ and $x = l$. The lateral surface is insulated. There are no heat sources within the body. The ends are also insulated. The initial temperature is $lx - x^2, 0 \le x \le l$. Find the temperature distribution in the bar for $t > 0$.

2. The faces $x = 0$, $x = a$ of a finite slab are maintained at zero temperature. The initial distribution of temperature in the slab is given by $T(x, 0) = f(x), 0 \le x \le a$. Determine the temperature at subsequent times.

3. Show that the solution of the equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

satisfying the conditions:

(i) $T \to 0$ as $t \to \infty$

(ii) $T = 0$ for $x = 0$ and $x = a$ for all $t > 0$

(iii) $T = x$ when $t = 0$ and $0 < x < a$

is

$$T(x, t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{n\pi}{a}\right) x \exp[-(n\pi/a)^2 t]$$

4. Solve the equation $\dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}$ satisfying the conditions:

(i) $T = 0$ when $x = 0$ and 1 for all t

(ii) $T = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & \dfrac{1}{2} \le x \le 1 \text{ when } t = 0. \end{cases}$

5. Solve the diffusion equation

$$\frac{\partial \theta}{\partial t} = \nu\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right)$$

subject to

$$r = 0, \quad \theta \text{ is finite}, \quad t > 0$$

$$r = a, \ \theta = 0, \quad t > 0$$

$$\theta = \frac{P}{4\mu}(a^2 - r^2), \quad t = 0$$

Here, P, μ and ν are constants.

6. A homogeneous solid sphere of radius R has the initial temperature distribution $f(r), 0 \le r \le R,$ where r is the distance measured from the centre. The surface temperature is maintained at 0°. Show that the temperature $T(r, t)$ in the sphere is the solution of $T_t = c^2\left(T_{rr} + \dfrac{2}{r}T_r\right)$. Show that the temperature in the sphere for $t > 0$ is given by

$$T(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right)\exp(-\lambda_n^2 t)$$

where $\lambda_n = cn\pi/R$ and c^2 is a constant.

7. If $\phi(x)$ is bounded for all real values of x, show that

$$T(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \phi(\xi)\exp[-(x-\xi)^2/(4kt)]\,d\xi$$

is a solution of $T_t = kT_{xx}$ such that $T(x,0) = \phi(x)$.

8. An infinite homogeneous solid circular cylinder of radius a is thermally insulated to prevent heat escape. At any time t, the temperature $T(r, t)$ at a distance r from the axis of symmetry is given by the heat conduction equation with axial symmetry. At time $t = 0$, the initial temperature distribution at a distance r from the axis is known to be a function of r. Find the temperature distribution at any subsequent time.

9. Let $\bar{r} = (x, y, z)$ represent a point in three-dimensional Euclidean space R_3. Find a formal solution $u(\bar{r}, t)$ which satisfies the diffusion equation

$$u_t = \alpha\nabla^2 u, \qquad t > 0$$

and the BC: $u(\bar{r}, 0) = f(\bar{r})$, where $\bar{r} \in R_3$.

10. Solve $\dfrac{\partial\theta}{\partial t} = \dfrac{\partial^2\theta}{\partial x^2}$, $0 \le x \le a,\ t > 0$ subject to the conditions

$$\theta(0,t) = \theta(a,t) = 0 \text{ and } \theta(x,0) = \theta_0 \text{ (constant).}$$

(GATE-Maths, 1996)

CHAPTER 4

Hyperbolic Differential Equations

4.1 OCCURRENCE OF THE WAVE EQUATION

One of the most important and typical homogeneous hyperbolic differential equations is the wave equation. It is of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{4.1}$$

where c is the wave speed. This differential equation is used in many branches of Physics and Engineering and is seen in many situations such as transverse vibrations of a string or membrane, longitudinal vibrations in a bar, propagation of sound waves, electromagnetic waves, sea waves, elastic waves in solids, and surface waves as in earthquakes. The solution of a wave equation is called a *wave function*.

An example for inhomogeneous wave equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = F \tag{4.2}$$

where F is a given function of spatial variables and time. In physical problems F represents an external driving force such as gravity force. Another related equation is

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} - c^2 \nabla^2 u = F \tag{4.3}$$

where γ is a real positive constant. This equation is called a wave equation with damping term, the amplitude of which decreases exponentially as t increases. In Section 4.2, we shall derive the partial differential equation describing the transverse vibration of a string.

4.2 DERIVATION OF ONE-DIMENSIONAL WAVE EQUATION

Suppose a flexible string is stretched under tension τ between two points at a distance L apart as shown in Fig. 4.1. We assume the following:

1. The motion takes place in one plane only and in this plane each particle moves in a direction perpendicular to the equilibrium position of the string.
2. The tension τ in the string is constant.

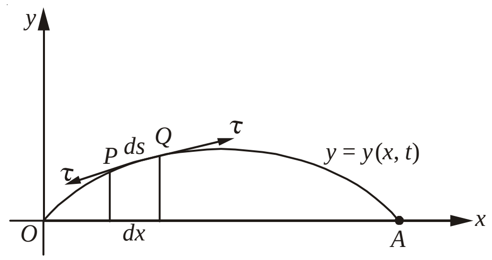

Fig. 4.1 Flexible string.

3. The gravitational force is neglected as compared with tension τ of the string.
4. The slope of the deflection curve is small.

Let the two fixed ends of the string be at the origin O and $A(L, 0)$ which lies along the x-axis in its equilibrium position. Consider an infinitesimal segment PQ of the string. Let ρ be the mass per unit length of the string. If the string is set vibrating in the xy-plane, the subsequent displacement, y from the equilibrium position of a point P of the string will be a function of x and time t, while an element of length dx is stretched into an element of length ds given by

$$ds = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,dx \approx \left[1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]dx$$

The elementary elongation is given by

$$dL = ds - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

while the work done by this element against the tension τ is

$$\frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Therefore, the total work done, W, for the whole string is

$$W = \frac{1}{2}\int_0^L \tau\left(\frac{\partial y}{\partial x}\right)^2 dx$$

If U is the potential energy of the string, then

$$U = W = \frac{1}{2}\int_0^L \tau\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Also, the total kinetic energy T of the string is given by

$$T = \frac{1}{2}\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx$$

Using Hamilton's principle (See—Sankara Rao, 2005), we have

$$\delta\int_{t_0}^{t_1} (T - U)\, dt = 0$$

i.e.

$$\int_{t_0}^{t_1} (T - U)\, dt$$

is stationary. In other words,

$$\frac{1}{2}\int_{t_0}^{t_1}\int_0^L \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 - \tau\left(\frac{\partial y}{\partial x}\right)^2\right] dx\, dt$$

is stationary, and is of the form

$$\iint F(x, t, y, y_x, y_t)\, dx\, dt$$

Noting that x and t are independent variables, from the Euler-Ostrogradsky equation, we have

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) = 0$$

which gives

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right) - \frac{\partial}{\partial x}\left(\tau\frac{\partial y}{\partial x}\right) = 0$$

If the string is homogeneous, then ρ and τ are constants, in which case the governing equation representing the transverse vibration of a string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad (4.4)$$

where

$$c^2 = \tau/\rho \qquad (4.5)$$

EXAMPLE 4.1 Consider Maxwell's equations of electromagnetic theory given by

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{4\pi i}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where **E** is an electric field, ρ is electric charge density, **H** is the magnetic field, i is the current density, and c is the velocity of light. Show that in the absence of charges, i.e., when $\rho = i = 0$, **E** and **H** satisfy the wave equations.

Solution Given

$$\text{curl}\,\mathbf{E} = \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

Taking its curl again, we get

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{c}\nabla \times \mathbf{H}\right)$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Moreover, using the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

it follows directly that

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly, we can observe that the magnetic field **H** also satisfies

$$\nabla^2 \mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

which is also a wave equation.

4.3 SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION BY CANONICAL REDUCTION

The one-dimensional wave equation is

$$u_{tt} - c^2 u_{xx} = 0 \tag{4.6}$$

Choosing the characteristic lines

$$\xi = x - ct, \quad \eta = x + ct \tag{4.7}$$

the chain rule of partial differentiation gives

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

$$u_t = u_\xi \xi_t + u_\eta \eta_t = c(u_\eta - u_\xi)$$

In the operator notation we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$$

Thus, we get

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 u = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \tag{4.8}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \tag{4.9}$$

Substituting Eqs. (4.8) and (4.9) into Eq. (4.6), we obtain

$$4u_{\xi\eta} = 0 \tag{4.10}$$

Integrating, we get

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta),$$

where ϕ and ψ are arbitrary functions. Replacing ξ and η as defined in Eq. (4.7), we have the general solution of the wave equation (4.6) in the form

$$u(x, t) = \phi(x - ct) + \psi(x + ct) \tag{4.11}$$

The two terms in Eq. (4.11) can be interpreted as waves travelling to the right and left, respectively. Consider

$$u_1(x, t) = \phi(x - ct)$$

This represents a wave travelling to the right with speed c whose shape does not change as it travels, the initial shape being given by a known function $\phi(x)$. In fact, by setting $t = 0$ in the argument of ϕ, it can be observed that the initial wave profile is given by

$$u_1(x, 0) = \phi(x)$$

At time $t = 1/c$,

$$u_1(x, 1/c) = \phi(x - 1)$$

Let $x' = x - 1$. Then $\phi(x-1) = \phi(x')$. That is, the same shape is retained even if the origin is shifted by one unit along the x-axis. In other words, the graph of $u_1(x, 1/c)$ is the same as the graph of the original wave profile translated one unit to the right. At $t = 2/c$, the graph of $u_1(x, 2/c)$ is the graph of the wave profile translated two units to the right. Thus, in particular, at $t = 1$, we have $u_1(x, 1) = \phi(x - c)$. Hence in one unit of time, the profile has moved c units to the right. Therefore, c is the wave speed or speed of propagation. Using similar argument, we can conclude that the equation $u_2(x, t) = \psi(x + ct)$ is also a wave profile travelling to the left with speed c along the x-axis. Hence the general solution (4.11) of one-dimensional wave equation represents the superposition of two arbitrary wave profiles, both of which are travelling with a common speed but in the opposite directions along the x-axis, while their forms remain unaltered as they travel. This situation is described in Fig. 4.2.

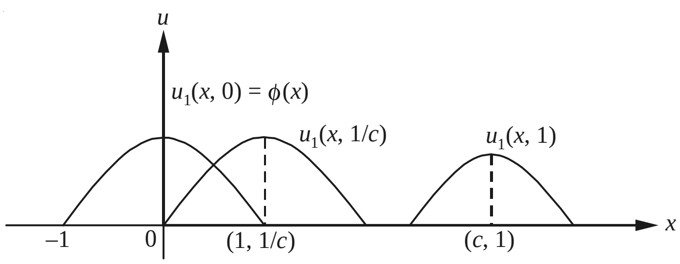

Fig. 4.2 Travelling wave profile.

Let k be an arbitrary real parameter. Consider then

$$u(x, t) = \phi[k(x - ct)] + \psi[k(x + ct)] \tag{4.12}$$

This is also a solution of the one-dimensional wave equation. Further, let $\omega = kc$. Then

$$u(x, t) = \phi[kx - \omega t] + \psi[kx + \omega t] \tag{4.13}$$

A function of the type given in Eq. (4.13) is a solution of one-dimensional wave equation iff $\omega = kc$. Therefore, waves travelling with speeds which are not the same as c cannot be described by the solution of the wave equation (4.6). Here, $(kx + \omega t)$ is called the phase for the left travelling wave. We have already noted that $x \pm ct$ are the characteristics of the one-dimensional wave equation.

EXAMPLE 4.2 Obtain the periodic solution of the wave equation in the form

$$u(x, t) = Ae^{i(kx \pm \omega t)}$$

where $i = \sqrt{-1}, k = \pm\,\omega/c$, A is constant; and hence define various terms involved in wave propagation.

Solution Let $u(x, t) = f(x)e^{\pm i\omega t}$ be a solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

Then

$$u_{xx} = f''(x)e^{\pm i\omega t}, \quad u_{tt} = -f(x)\omega^2 e^{\pm i\omega t}$$

Substituting into the wave equation, we get

$$f''(x) + \frac{\omega^2}{c^2} f(x) = 0$$

Its general solution is found to be

$$f(x) = c_1 \exp[i(\omega/c)x] + c_2 \exp[-i(\omega/c)x]$$

Therefore, the required solution of the wave equation is

$$u(x, t) = [c_1 \exp\{i(\omega/c)x\} + c_2 \exp\{-i(\omega/c)x\}]e^{\pm i\omega t}$$

Since $k = \pm\,\omega/c$, the time-dependent wave functions are of the form

$$u(x, t) = Ae^{i(kx \pm \omega t)}$$

Hence, $u(x, t) = Ae^{i(kx \pm \omega t)}$ is a solution of the wave equation, and is called a wave function. It is also called a plane harmonic wave or monochromatic wave. Here, A is called the amplitude, ω the angular or circular frequency, and k is the wave number, defined as the number of waves per unit distance. By taking the real and imaginary parts of the solution, we find the linear combination of terms of the form

$$A\cos(kx \pm \omega t), \quad A\sin(kx \pm \omega t)$$

representing periodic plane waves. For instance, consider the function $u(x, t) = A\sin(kx - \omega t)$. This is a sinusoidal wave profile moving towards the right along the x-axis with speed c. Defining the wave length λ as the length over which one full cycle is completed, we have $\lambda = 2\pi/k$, thereby implying that $k = 2\pi/\lambda$.

Suppose an observer is stationed at a fixed point x_0; then,

$$u\left(x_0, t + \frac{\lambda}{c}\right) = A\sin\left(kx_0 - \omega t - \omega\frac{\lambda}{c}\right)$$

$$= A\sin(kx_0 - \omega t - 2\pi) = A\sin(kx_0 - \omega t)$$

Thus, we have

$$u(x_0, t + \lambda/c) = u(x_0, t)$$

Hence, exactly one complete wave passes the observer in time $T = \lambda/c$, which is called the period of the wave. The reciprocal of the period is called frequency and is denoted by

$$f = 1/T$$

The function, $u = A\cos(kx - \omega t) = A\sin(\pi/2 + kx - \omega t)$, also represents a wave train except that it differs in phase by $\pi/2$ from the sinusoidal wave. Now consider the superposition of the sinusoidal waves having the same amplitude, speed, frequency, but moving in opposite directions. Thus, we have

$$u(x, t) = A\sin[k(x - ct)] + A\sin[k(x + ct)]$$

$$= 2A\sin kx\cos(kct) = 2A\cos(kct)\sin kx$$

Its amplitude factor $[2A\cos(kct)]$ varies sinusoidally with frequency ω. This situation is described as a standing wave. The points $x_n = n\pi/k$, $n = 0, \pm 1, \pm 2, \ldots$ are called nodes. No displacement takes place at a node. Therefore,

$$u(x_n, t) = 0 \quad \text{for all } t$$

The nth standing wave profile will have $(n-1)$ equally spaced nodes in a given interval as shown in Fig. 4.3.

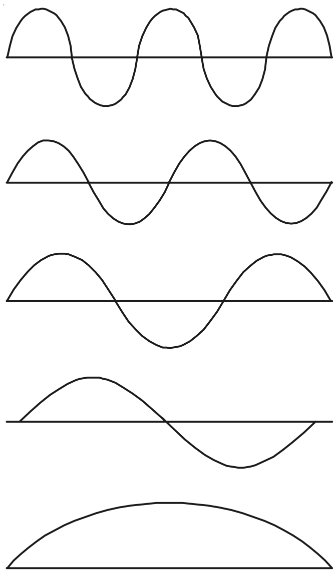

Fig. 4.3 Standing wave profiles.

4.4 THE INITIAL VALUE PROBLEM; D'ALEMBERT'S SOLUTION

Consider the initial value problem of Cauchy type described as

$$\text{PDE: } u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < \infty, t \geq 0 \tag{4.14}$$

$$\text{ICs: } u(x, 0) = \eta(x), \qquad u_t(x, 0) = v(x) \tag{4.15}$$

where the curve on which the initial data $\eta(x)$ and $v(x)$ are prescribed is the x-axis. $\eta(x)$ and $v(x)$ are assumed to be twice continuously differentiable. Here, the string considered is of an infinite extent. Let $u(x, t)$ denote the displacement for any x and t. At $t = 0$, let the displacement and velocity of the string be prescribed. We have already noted in Section 4.3 that the general solution of the wave equation is given by

$$u(x,t) = \phi(x + ct) + \psi(x - ct) \tag{4.16}$$

where ϕ and ψ are arbitrary functions. Substituting the ICs (4.15) into Eq. (4.16), we obtain

$$\begin{aligned} \phi(x) + \psi(x) &= \eta(x) \\ c[\phi'(x) - \psi'(x)] &= v(x) \end{aligned} \tag{4.17}$$

Integrating the second equation of (4.17), we have

$$\phi(x) - \psi(x) = \frac{1}{c}\int_0^x v(\xi)\, d\xi$$

Addition and substraction of this equation with the first relation of Eqs. (4.17) yield

$$\phi(x) = \frac{\eta(x)}{2} + \frac{1}{2c}\int_0^x v(\xi)\, d\xi$$

$$\psi(x) = \frac{\eta(x)}{2} - \frac{1}{2c}\int_0^x v(\xi)\, d\xi$$

respectively. Substituting these relations for $\phi(x)$ and $\psi(x)$ into Eq. (4.16), we at once have

$$u(x, t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v\,(\xi)\, d\xi \tag{4.18}$$

This is known as the D'Alembert's solution of the one-dimensional wave equation. If $v = 0$, i.e., if the string is released from rest, the required solution is

$$u(x, t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] \tag{4.19}$$

The D'Alembert's solution has an interesting interpretation as given in Fig. 4.4.

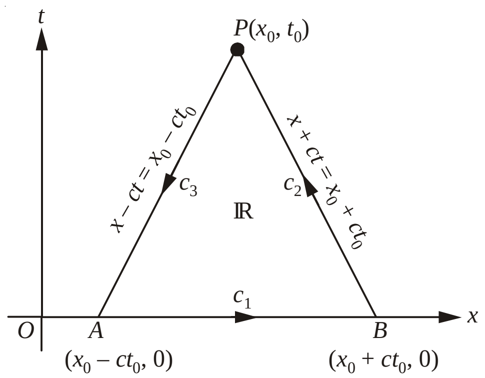

Fig. 4.4 Characteristic triangle.

Consider the xt-plane and a point $P(x_0, t_0)$. Draw two characteristics through P backwards, until they intersect the initial line, i.e., the x-axis at A and B. The equation of these two characteristics are

$$x \pm ct = x_0 \pm ct_0$$

Equation (4.18) reveals that the solution $u(x, t)$ at $P(x_0, t_0)$ can be obtained by averaging the value of η at A and B and integrating v along the x-axis between A and B. Thus, to find the solution of the wave equation at a given point P in the xt-plane, we should know the initial data on the segment AB of the initial line which is obtained by drawing the characteristics backward from P to the initial line. Here the segment AB of the initial line, on which the value of $u(x, t)$ at P depends, is called the domain of dependence of P, and the triangle PAB is called the characteristic triangle (see Fig. 4.4), which is also called the domain of determinancy of the interval.

EXAMPLE 4.3 A stretched string of finite length L is held fixed at its ends and is subjected to an initial displacement $u(x, 0) = u_0 \sin(\pi x/L)$. The string is released from this position with zero initial velocity. Find the resultant time dependent motion of the string.

Solution One of the practical applications of the theory of wave motion is the vibration of a stretched string, say, that of a musical instrument. In the present problem, let us consider a stretched string of finite length L, which is subjected to an initial disturbance. The governing equation of motion is

$$\text{PDE: } u_{tt} - c^2 u_{xx} = 0, 0 \le x \le L, t > 0 \tag{1}$$

$$\text{BCs: } u(0, t) = u(L, t) = 0 \tag{2}$$

$$\text{ICs: } u(x, 0) = u_0 \sin(\pi x/L), \tag{3}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \tag{4}$$

In Section 4.3, we have shown the solution of the one-dimensional wave equation by canonical reduction as

$$u(x, t) = \phi(x - ct) + \psi(x + ct) \tag{5}$$

One of the known methods for solving this problem is based on trial function approach. Let us choose a trial function of the form

$$u(x, t) = A\left[\sin\frac{\pi}{L}(x + ct) + \sin\frac{\pi}{L}(x - ct)\right] \tag{6}$$

where A is an arbitrary constant. Now, we rewrite Eq. (6) as

$$u(x, t) = 2A \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right) \tag{7}$$

obviously, Eq. (7) satisfies the initial condition (3) with $A = u_0/2$, while the second initial condition (4) is satisfied identically. In fact Eq. (7) also satisfies the boundary condition (2). Therefore, the final solution is found to be

$$u(x, t) = u_0 \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right) \tag{8}$$

It may be noted that the trial function approach is easily adoptable if the initial condition is specified as a sin function. However, it is difficult if the initial conditions are specified as a general function such as $f(x)$. In such case, it is better to follow variables separable method as explained in Section 4.5.

EXAMPLE 4.4 Solve the following initial value problem of the wave equation (Cauchy problem), described by the inhomogeneous wave equation

$$\text{PDE: } u_{tt} - c^2 u_{xx} = f(x, t)$$

subject to the initial conditions

$$u(x, 0) = \eta(x), \qquad u_t(x, 0) = v(x)$$

Solution To make the task easy, we shall set $u = u_1 + u_2$, so that u_1 is a solution of the homogeneous wave equation subject to the general initial conditions given above. Then u_2 will be a solution of

$$\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} = f(x, t) \tag{4.20}$$

subject to the homogeneous ICs

$$u_2(x, 0) = 0, \quad \frac{\partial u_2}{\partial t}(x, 0) = 0 \tag{4.21}$$

To obtain the value of u at $P(x_0, t_0)$, we integrate the partial differential equation (4.20) over the region $\mathbb{R}$ as shown in Fig. 4.4, to obtain

$$\iint_{\mathbb{R}} \left(\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} \right) dx\, dt = \iint_{\mathbb{R}} f(x, t)\, dx\, dt$$

Using Green's theorem in a plane to the left-hand side of the above equation to replace the surface integral over $\mathbb{R}$ by a line integral around the boundary $\partial\mathbb{R}$ of $\mathbb{R}$, the above equation reduces to

$$-\iint_{\mathbb{R}} \left[\frac{\partial}{\partial x}\left(c^2 \frac{\partial u_2}{\partial x} \right) - \frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial t} \right) \right] dx\, dt = \iint_{\mathbb{R}} f(x, t)\, dx\, dt$$

and finally to

$$\int_{\partial\mathbb{R}} \left(\frac{\partial u_2}{\partial t} dx + c^2 \frac{\partial u_2}{\partial x} dt \right) = \iint_{\mathbb{R}} f(x, t)\, dx\, dt \tag{4.22}$$

Now, the boundary $\partial \mathbb{R}$ comprises three segments BP, PA and AB. Along BP, $dx/dt = -c$; along PB, $dx/dt = c$. Using these results, Eq. (4.22) becomes

$$\int_{BP} c\left(\frac{\partial u_2}{\partial t}dt + \frac{\partial u_2}{\partial x}dx\right) - \int_{PA} c\left(\frac{\partial u_2}{\partial t}dt + \frac{\partial u_2}{\partial x}dx\right)$$

$$-\int_{AB}\left(\frac{\partial u_2}{\partial t}dx + c^2\frac{\partial u_2}{\partial x}dt\right) = \iint_{\mathbb{R}} f(x,t)\,dx\,dt$$

The integrands of the first two integrals are simply the total differentials, while in the third integral, the first term vanishes on AB in view of the second IC in Eq. (4.21), and the second term also vanishes because AB is directed along the x-axis on which $dt/dx = 0$. Then we arrive at the result

$$\int_{BP} c\,du_2 - \int_{PA} c\,du_2 = \iint_{\mathbb{R}} f(x,t)\,dx\,dt$$

which can be rewritten as

$$cu_2(P) - cu_2(B) + cu_2(P) - cu_2(A) = \iint_{\mathbb{R}} f(x,t)dx\,dt \tag{4.23}$$

Using the first IC of Eq. (4.21), we get $u_2(A) = u_2(B) = 0$, and hence Eq. (4.23) becomes

$$u_2(P) = \frac{1}{2c}\iint_{\mathbb{R}} f(x,t)\,dx\,dt$$

with the help of Fig. 4.4, we deduce

$$u_2(P) = \frac{1}{2c}\int_0^{t_0}\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} f(x,t)\,dx\,dt \tag{4.24}$$

Now, using the fact that $u = u_1 + u_2$, as also using Eq. (4.24) and D'Alembert's solution (4.18), the required solution of the inhomogeneous wave equation subject to the given ICs is given by

$$u(x,t) = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi$$

$$+\frac{1}{2}\int_0^{t_0}\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} f(x,t)\,dx\,dt \tag{4.25}$$

This solution is known as the Riemann-Volterra solution.

4.5 VIBRATING STRING—VARIABLES SEPARABLE SOLUTION

Following Tychonov and Samarski, it is known that transverse vibration of a string is normally generated in musical instruments. We distinguish the string instruments depending on whether the string is plucked as in the case of guitar or struck as in the case of harmonium or piano. In the case of strings which are struck we give a fixed initial velocity but does not undergo any initial displacement. In the case of plucked instruments, the strings vibrate from a fixed initial displacement without any initial velocity. The vibrations of stretched strings of musical instruments, vocal cards, power transmission cables, guy wires for antennae structures, etc. can be examined by considering the basic form of wave equations as discussed in Sections 4.3 and 4.4.

Let a thin homogeneous string which is perfectly flexible under uniform tension lie in its equilibrium position along the x-axis. The ends of the string are fixed at $x = 0$ and $x = L$. The string is pulled aside a short distance and released. If no external forces are present which correspond to the case of free vibrations, the subsequent motion of the string is described by the solution $u(x, t)$ of the following problem:

$$\text{PDE: } u_{tt} - c^2 u_{xx} = 0, \qquad 0 \le x \le L,\ t > 0 \tag{4.26}$$

$$\text{BCs: } u(0, t) = 0, \qquad t > 0$$

$$u(L, t) = 0, \qquad t > 0 \tag{4.27}$$

$$\text{ICs: } u(x, 0) = f(x), \qquad u_t(x, 0) = g(x) \tag{4.28}$$

To obtain the variables separable solution, we assume

$$u(x, t) = X(x)T(t) \tag{4.29}$$

and substituting into Eq. (4.26), we obtain

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2}$$

i.e.,

$$\frac{d^2X/dx^2}{X} = \frac{d^2T/dt^2}{c^2T} = k \text{ (a separation constant)}$$

Case I When $k > 0$, we have $k = \lambda^2$. Then

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

$$\frac{d^2T}{dt^2} - c^2\lambda^2 T = 0$$

Their solution can be put in the form

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \tag{4.30}$$

$$T = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t} \tag{4.31}$$

Therefore,

$$u(x, t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \tag{4.32}$$

Now, use the BCs:

$$u(0,t) = 0 = (c_1 + c_2)(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \tag{4.33}$$

which imply that $c_1 + c_2 = 0$. Also, $u(L,t) = 0$ gives

$$c_1 e^{-\lambda L} + c_2 e^{-\lambda L} = 0 \tag{4.34}$$

Equations (4.33) and (4.34) possess a non-trivial solution iff

$$\begin{vmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda L} \end{vmatrix} = e^{-\lambda L} - e^{\lambda L} = 0$$

or

$$1 - e^{2\lambda L} = 0 \text{ implying } e^{2\lambda L} = 1 \quad \text{or} \quad \lambda L = 0$$

This implies that $\lambda = 0$, since L cannot be zero, which is against the assumption as in Case I. Hence, this solution is not acceptable.

Case II Let $k = 0$. Then we have

$$\frac{d^2X}{dx^2} = 0, \quad \frac{d^2T}{dt^2} = 0$$

Their solutions are found to be

$$X = Ax + B, \qquad T = ct + D$$

Therefore, the rquired solution of the PDE (4.26) is

$$u(x, t) = (Ax + B)(ct + D)$$

Using the BCs, we have

$$u(0, t) = 0 = B(ct + D), \quad \text{implying } B = 0$$

$$u(L, t) = 0 = AL(ct + D), \text{ implying } A = 0$$

Hence, only a trivial solution is possible. Since we are looking for a non-trivial solution, consider the following case.

Case III When $k < 0$, say $k = -\lambda^2$, the differential equations are

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad \frac{d^2T}{dt^2} + c^2\lambda^2 T = 0$$

Their general solutions give

$$u(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)\,(c_3 \cos c\lambda t + c_4 \sin c\lambda t) \tag{4.35}$$

Using the BC: $u(0, t) = 0$ we obtain $c_1 = 0$. Also, using the BC: $u(L, t) = 0$, we get $\sin \lambda L = 0$, implying that $\lambda_n = n\pi/L$, $n = 1, 2, \ldots$, which are the eigenvalues. Hence the possible solution is

$$u_n(x, t) = \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right), \quad n = 1, 2, \ldots \tag{4.36}$$

Using the superposition principle, we have

$$u(x, t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right) \tag{4.37}$$

The initial conditions give

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

which is a half-range Fourier sine series, where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \tag{4.38}$$

Also,

$$u_t(x, 0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\left(\frac{n\pi}{L}c\right)$$

which is also a half-range sine series, where

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\frac{n\pi x}{L}\, dx \tag{4.39}$$

Hence the required physically meaningful solution is obtained from Eq. (4.37), where A_n and B_n are given by Eqs. (4.38) and (4.39). $u_n(x, t)$ given by Eq. (4.36) are called normal modes of vibration and $n\pi c/L = \omega_n, n = 1, 2, \ldots$ are called normal frequencies.

The following comments may be noted:

(i) The displaced form of the stretched string defined by Eq. (4.36) is referred to as the *n*th eigenfunction or the *n*th normal mode of vibration.

(ii) The period of the *n*th normal mode is $2L/nC$, which means $nC/2L$ cycles per second, called its frequency.

(iii) The frequency can be expressed as

$$f = \frac{n}{2L}\left(\frac{\tau}{\rho}\right)^{1/2}$$

Thus, the frequency can be increased either by reducing L or by increasing the tension τ.

(iv) For a given L, τ and ρ, the first normal mode $n = 1$, vibrates with the lowest frequency

$$f = \sqrt{\frac{\tau}{4L^2\rho}} = \frac{C}{2L},$$

is called the *fundamental frequency*.

(v) If the stretched string can be made to vibrate in a higher normal mode, the frequency is increased by an integer multiple. The deflected configuration of the stretched string corresponding to the given normal mode at a specified time $t = t^*$, can be obtained from Eq. (4.36). The deflected shapes corresponding to the first three normal modes and the associated frequencies are depicted in Fig. 4.4.

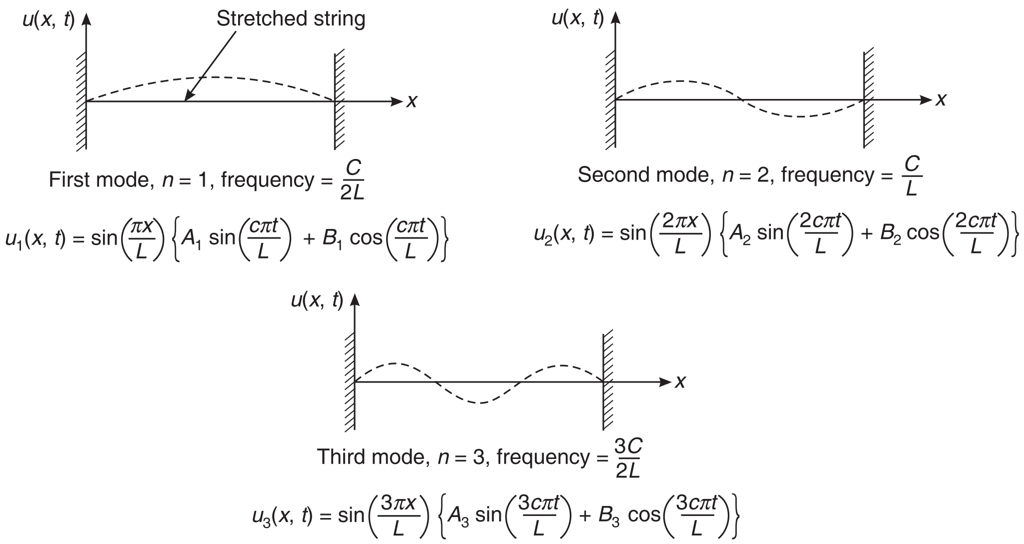

Fig. 4.4(a) Normal modes of a vibrating stretched string.

EXAMPLE 4.5 Obtain the solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

under the following conditions:

(i) $u(0, t) = u(2, t) = 0$

(ii) $u(x, 0) = \sin^3 \pi x/2$

(iii) $u_t(x, 0) = 0.$

Solution We have noted in Example 4.4 that the physically acceptable solution of the wave equation is given by Eq. (4.35), and is of the form

$$u(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)\,[c_3 \cos (c\lambda t) + c_4 \sin (c\lambda t)]$$

Using the condition $u(0, t) = 0$, we obtain $c_1 = 0$. Also, condition (iii) implies $c_4 = 0$. The condition $u(2, t) = 0$ gives

$$\sin 2\lambda = 0,$$

implying that

$$\lambda = n\pi/2, \qquad n = 1, 2, \ldots$$

Thus, the possible solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} \cos \frac{n\pi ct}{2} \tag{4.40}$$

Finally, using condition (ii), we obtain

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} = \sin^3 \frac{\pi x}{2} = \frac{3}{4} \sin \frac{\pi x}{2} - \frac{1}{4} \sin \frac{3\pi x}{2}$$

which gives $A_1 = 3/4$, $A_3 = -1/4$, while all other A_n's are zero. Hence, the required solution is

$$u(x, t) = \frac{3}{4} \sin \frac{\pi x}{2} \cos \frac{\pi ct}{2} - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \frac{3\pi ct}{2}$$

EXAMPLE 4.6 Prove that the total energy of a string, which is fixed at the points $x = 0, x = L$ and executing small transverse vibrations, is given by

$$\frac{1}{2} T \int_0^L \left[\left(\frac{\partial y}{\partial x} \right)^2 + \frac{1}{c^2} \left(\frac{\partial y}{\partial t} \right)^2 \right] dx$$

where $c^2 = T/\rho$, ρ is the uniform linear density and T is the tension. Show also that if $y = f(x - ct), 0 \le x \le L,$ then the energy of the wave is equally divided between potential energy and kinetic energy.

Solution The kinetic energy (KE) of an element dx of the string executing small transverse vibrations is given by (see Fig. 4.5)

$$\frac{1}{2}(\rho\, dx)\left(\frac{\partial y}{\partial t}\right)^2$$

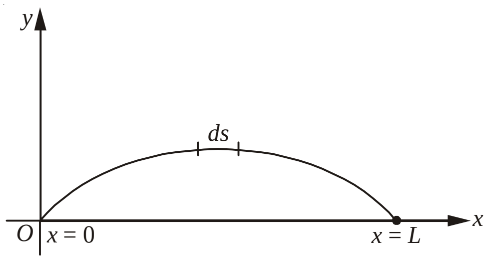

Fig. 4.5 Vibrating string.

Therefore,

$$\text{Total KE} = \frac{T}{2}\int_0^L \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 dx \tag{4.41}$$

However, $ds^2 = dx^2 + dy^2$, which gives

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \approx \left[1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] dx$$

Hence, the stretch in the string is given by

$$ds - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Now, the potential energy (PE) of this element is given by

$$\text{PE} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Therefore,

$$\text{Total PE} = \frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \tag{4.42}$$

When added, Eqs. (4.41) and (4.42) will yield the required total energy of the string. If $y = f(x - ct)$, then

$$\frac{\partial y}{\partial t} = -cf'(x - ct), \quad \frac{\partial y}{\partial x} = f'(x - ct)$$

$$\left(\frac{\partial y}{\partial t}\right)^2 = c^2 (f')^2$$

From Eq. (4.41),

$$\text{Total KE} = \frac{1}{2} T \int_0^L (f')^2 \, dx$$

From Eq. (4.42),

$$\text{Total PE} = \frac{1}{2} T \int_0^L (f')^2 \, dx$$

which clearly demonstrates that the total KE = total PE.

EXAMPLE 4.7 A string of length L is released from rest in the position $y = f(x)$. Show that the total energy of the string is

$$\frac{\pi^2 T}{4L} \sum_{n=1}^{\infty} s^2 k_s^2$$

where

$$k_s = \frac{2}{L} \int_0^L f(x) \sin(s\pi x/L) \, dx$$

T-tension in the string

If the mid-point of a string is pulled aside through a small distance and then released, show that in the subsequent motion the fundamental mode contributes $8/\pi^2$ of the total energy.

Solution If $f(x)$ can be expressed in Fourier series, then

$$f(x) = (a_0/2) + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Here, $(a_1 \cos x + b_1 \sin x)$ is called the fundamental mode. Following the variables separable method and using the superposition principle, the general solution of the wave equation is

$$y(x, t) = \sum_{n=1}^{\infty} k_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L} \tag{4.43}$$

where

$$k_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

and the total energy is obtained from

$$E = \frac{T}{2}\int_0^L \left[\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2\right]dx \tag{4.44}$$

From Eq. (4.43),

$$\frac{\partial y}{\partial x} = \sum_{n=1}^{\infty} k_n \frac{n\pi}{L}\cos\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}$$

$$\frac{\partial y}{\partial t} = -\sum_{n=1}^{\infty} k_n \frac{cn\pi}{L}\sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}$$

Using the standard integrals

$$\int_0^{2\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_0^{2\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

We have

$$\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{\pi^2}{L^2}\int_0^L \sum k_n^2 n^2 \cos^2\frac{n\pi x}{L}\cos^2\frac{cn\pi t}{L}\,dx$$

$$= \frac{\pi^2}{2L}\sum k_n^2 n^2 \cos^2\frac{cn\pi t}{L} \tag{4.45}$$

Also,

$$\int_0^L \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{\pi^2}{L^2}\sum k_n^2 n^2 \sin^2\frac{cn\pi t}{L}\left(\frac{L}{2}\right)$$

$$= \frac{\pi^2}{2L}\sum k_n^2 n^2 \sin^2\frac{cn\pi t}{L} \tag{4.46}$$

Substituting Eqs. (4.45) and (4.46) into Eq. (4.44), we obtain

$$E = \frac{T}{2}\frac{\pi^2}{2L}\sum_{n=1}^{\infty} n^2 k_n^2 \tag{4.47}$$

Now, the transverse motion of the string is described by the equation

$$y(x, t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

where

$$k_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

But the equation of the line *OP* is (see Fig. 4.6)

$$y = \frac{2\varepsilon}{L}x, \quad 0 \le x \le L/2$$

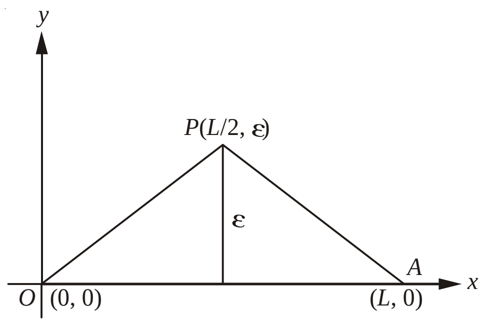

Fig. 4.6 An illustration of Example 4.7.

while the equation for the line of *PA* is

$$y = -\frac{2\varepsilon}{L}(x - L), \quad L/2 \le x \le L$$

Therefore,

$$k_n = \frac{2}{L}\left[\int_0^{L/2}\frac{2\varepsilon}{L}x\sin\frac{n\pi x}{L}dx + \int_{L/2}^{L} -\frac{2\varepsilon}{L}(x-L)\sin\frac{n\pi x}{L}dx\right]$$

Integration by parts yields

$$k_n = \frac{2}{L}\left[\frac{2\varepsilon}{L}\frac{L^2}{n^2\pi^2} + \frac{2\varepsilon}{L}\frac{L^2}{n^2\pi^2}\right]\sin\frac{n\pi}{2}$$

$$= \frac{8\varepsilon}{n^2\pi^2}\sin\frac{n\pi}{2} \quad \text{for } n \text{ odd} \tag{4.48}$$

Substituting this value of k_n into Eq. (4.47), we obtain

$$E = \frac{\pi^2 T}{4L} \sum_{n \text{ odd}} n^2 \frac{64\varepsilon^2}{\pi^4} \frac{1}{n^4} = \frac{16T\varepsilon^2}{L\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2}$$

but we know that

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Hence,

$$\text{Total energy, } E = \frac{16T\varepsilon^2}{L\pi^2}\left(\frac{\pi^2}{8}\right)$$

while the total energy due to the fundamental mode is $16T\varepsilon^2/L\pi^2$. Hence the result.

4.6 FORCED VIBRATIONS—SOLUTION OF NON-HOMOGENEOUS EQUATION

Consider the problems of forced vibrations of a finite string due to an external driving force. If we assume that the string is released from rest, from its equilibrium position, the resulting motion of the string is governed by

$$\text{PDE: } u_{tt} - c^2 u_{xx} = F(x, t) \qquad 0 \le x \le L, \ t \ge 0 \tag{4.49}$$

$$\text{BCs: } u(0, t) = u(L, t) = 0, \qquad t \ge 0 \tag{4.50}$$

$$\text{ICs: } u(x, 0) = u_t(x, 0) = 0, \qquad 0 \le x \le L \tag{4.51}$$

Here, $F(x, t)$ is the external driving force. To obtain the solution of the above problem, we proceed as follows: Taking the solution of vibrating string in the absence of applied external forces as a guideline, we assume the solution to this case to be

$$u(x, t) = \sum_{n=1}^{\infty} \phi_n(t) \sin\frac{n\pi x}{L} \tag{4.52}$$

It can be seen easily that the BCs are satisfied. The function $u(x, t)$ defined by Eq. (4.52) also satisfies the ICs (4.51), provided.

$$\phi_n(0) = \phi_n'(0) = 0, \quad n = 1, 2, \ldots \tag{4.53}$$

Substituting the assumed solution (4.52) into the governing PDE (4.49), we obtain

$$\sum_{n=1}^{\infty} \left[\ddot{\phi}_n(t) + \frac{n^2\pi^2}{L^2} c^2 \phi_n(t)\right] \sin\frac{n\pi x}{L} = F(x, t)$$

or

$$\sum_{n=1}^{\infty}\left[\ddot{\phi}_n(t)+\omega_n^2\,\phi_n(t)\right]\sin\frac{n\pi x}{L}=F(x,\ t) \tag{4.54}$$

where

$$\omega_n=\frac{n\pi c}{L} \tag{4.55}$$

and the dots over ϕ denote differentiation with respect to t. Multiplying Eq. (4.54) by $\sin k\pi x/L$ and integrating with respect to x from $x=0$ to $x=L$ and interchanging the order of summation and integration, we get

$$\sum_{n=1}^{\infty}[\ddot{\phi}_n(t)+\omega_n^2\phi_n(t)]\int_0^L\sin\frac{n\pi x}{L}\sin\frac{k\pi x}{L}dx=\int_0^L F(x,\ t)\sin\frac{k\pi x}{L}dx=\bar{F}_k(t)$$

From the orthogonality property of the function $\sin(n\pi x/L)$, we have

$$[\ddot{\phi}_k(t)+\omega_k^2\,\phi_k(t)]\int_0^L\sin^2\frac{k\pi x}{L}dx=\bar{F}_k(t)$$

or

$$[\ddot{\phi}_k(t)+\omega_k^2\phi_k(t)]=\frac{2}{L}\bar{F}_k(t),\quad k=1,2,\ldots \tag{4.56}$$

This is a linear second order ODE which, for instance, can be solved by using the method of variation of parameters. Thus, we solve

$$\ddot{\phi}_k(t)+\omega_k^2\,\phi_k(t)=\bar{F}_k(t) \tag{4.57}$$

subject to

$$\phi_k(0)=\dot{\phi}_k(0)=0$$

where

$$\bar{F}_k(t)=\frac{2}{L}\int_0^L F(x,\ t)\sin\frac{k\pi x}{L}dx$$

The complementary function for the homogeneous part is $A\cos\omega_k t+B\sin\omega_k t.$ Taking A and B as functinos of t, let

$$\phi_k(t)=A(t)\cos\omega_k t+B(t)\sin\omega_k t$$

$$\dot{\phi}_k(t)=\dot{A}\cos\omega_k t+\dot{B}\sin\omega_k t-A\omega_k\sin\omega_k t+B\omega_k\cos\omega_k t$$

We choose A and B such that

$$\dot{A}\cos\omega_k t + \dot{B}\sin\omega_k t = 0 \tag{4.58}$$

Therefore,

$$\ddot{\phi}_k(t) = A\omega_k^2\cos\omega_k t - B\omega_k^2\sin\omega_k t - \dot{A}\omega_k\sin\omega_k t + \dot{B}\omega_k\cos\omega_k t$$

Substituting these expressions into Eq. (4.57), we get

$$\omega_k(\dot{B}\cos\omega_k t - \dot{A}\sin\omega_k t) = \bar{F}_k(t) \tag{4.59}$$

Solving Eqs. (4.58) and (4.59) for $\dot{A}$ and $\dot{B}$, we obtain

$$\dot{A}(t) = -\frac{\bar{F}_k(t)\sin\omega_k t}{\omega_k}$$

$$\dot{B}(t) = \frac{\bar{F}_k(t)\cos\omega_k t}{\omega_k}$$

Integrating, we get

$$A = -\frac{1}{\omega_k}\int_0^t \bar{F}_k(\xi)\sin\omega_k\xi\,d\xi$$

$$B = \frac{1}{\omega_k}\int_0^t \bar{F}_k(\xi)\cos\omega_k\xi\,d\xi$$

Thus,

$$\phi = \frac{1}{\omega_k}\int_0^t \bar{F}_k(\xi)\sin[\omega_k(t-\xi)]\,d\xi \tag{4.60}$$

It can be verified easily that zero ICs are also satisfied. Hence the formal solution to the problem described by Eqs. (4.49) to (4.51), using the superposition principle, is

$$u(x,\ t) = \sum_{n=1}^{\infty}\left\{\frac{1}{\omega_n}\int_0^t \bar{F}_n(\xi)\sin[\omega_n(t-\xi)]\,d\xi\right\}\sin\frac{n\pi x}{L} \tag{4.61}$$

Thus, if u_1 is a solution of the problem defined by Eqs. (4.26) to (4.28) and if u_2 is a solution of the problem described by Eqs. (4.49)–(4.51), then (u_1+u_2) is a solution of the IBVP described by

$$\text{PDE}: u_{tt} - c^2u_{xx} = F(x,t), \qquad 0 \le x \le L, t \ge 0 \tag{4.62}$$

$$\text{BCs}: u(0,t) = u(L,t) = 0, \qquad t \ge 0 \tag{4.63}$$

$$\text{ICs}: u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{4.64}$$

Hence, the solution of this nonhomogeneous problem is found to be

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi x}{L}$$

$$+ \sum_{n=1}^{\infty} \left[\frac{1}{\omega_n} \int_0^t \bar{F}_n(\xi) \sin[\omega_n (t - \xi)]\, d\xi \right] \sin \frac{n\pi x}{L} \tag{4.65}$$

This solution may be termed as formal solution, because it has not been proved that the series actually converages and represents a function which satisfies all the conditions of the given physical problem.

4.7 BOUNDARY AND INITIAL VALUE PROBLEMS FOR TWO-DIMENSIONAL WAVE EQUATIONS—METHOD OF EIGENFUNCTION

Let $\mathbb{R}$ be a region in the *xy*-plane bounded by a simple closed curve $\partial \mathbb{R}$. Let $\bar{\mathbb{R}} = \mathbb{R}\, U \partial \mathbb{R}$. Consider the problem described by

$$\text{PDE:} \quad u_{tt} - c^2 \nabla^2 u = F(x, y, t), \qquad x, y \in \mathbb{R},\ t \geq 0 \tag{4.66}$$

$$\text{BCs:} \quad B(u) = 0 \text{ on } \partial \mathbb{R}, \qquad t \geq 0 \tag{4.67}$$

$$\text{ICs:} \quad u(x, y, 0) = f(x, y) \quad \text{in } \bar{\mathbb{R}}$$

$$u_t(x, y, 0) = g(x, y) \quad \text{in } \bar{\mathbb{R}} \tag{4.68}$$

where $B(u) = 0$ stands for any one of the following boundary conditions:

(i) $u = 0$ on $\partial \mathbb{R}$ (Dirichlet condition)

(ii) $\dfrac{\partial u}{\partial n} = 0$ on $\partial \mathbb{R}$ (Neumann condition)

(iii) $u = \dfrac{\partial u}{\partial n} = 0$ on $\partial \mathbb{R}$ (Robin/Mixed condition)

Before we discuss the method of eigenfunctions, it is appropriate to introduce Helmholtz equation or the space form of the wave equation. The wave equation in three dimensions may be written in vectorial form as

$$u_{tt} = c^2 \nabla^2 u$$

By the variables separable method, we assume the solution in the form

$$u(x, y, z\ t) = \phi(x, y, z)\, T(t)$$

Substituting into the above wave equation, we obtain

$$\phi T'' = c^2 T \nabla^2 \phi$$

which gives

$$\frac{T''}{c^2 T} = \frac{\nabla^2 \phi}{\phi} = -\lambda \text{ (a separation constant)}$$

thereby implying

$$T'' + \lambda c^2 T = 0 \tag{4.69}$$

$$\nabla^2 \phi + \lambda \phi = 0 \tag{4.70}$$

Equation (4.70) is the space form of the wave equation or Helmholtz equation. Of course, $\phi = 0$ is the trivial solution Eq. (4.70). But a nontrivial solution ϕ exists only for certain values of $\{\lambda_n\}$, called eigenvalues and the corresponding solution $\{\phi_n\}$, are the eigenfunctions. Corresponding to each eigenvalue λ_n, there exists at least one real-valued twice continuously differentiable function ϕ_n such that

$$\nabla^2 \phi_n + \lambda_n \phi_n = 0 \text{ in } \mathbb{R}$$

$$\phi_n = 0 \text{ on } \partial\mathbb{R}$$

It may be noted that the sequence of eigenfunctions (ϕ_n) satisfies the orthogonality property

$$\iint_{\mathbb{R}} \phi_n \phi_m \, dA = 0 \text{ for all } n \neq m$$

As in the case of one-dimensional wave equation, each continuously differentiable function in $\mathbb{R}$, which vanishes on $\partial\mathbb{R}$, can have a Fourier series expansion relative to the orthogonal set $\{\phi_n\}$. Thus the solution to the proposed problem can be written as

$$u(x, y, t) = \sum_{n=1}^{\infty} C_n(t)\, \phi_n(x, y) \tag{4.71}$$

where $C_n(t)$ has to be found out. Substitution of the Fourier series into the PDE (4.66) yields

$$\sum_{n=1}^{\infty} [\ddot{C}_n(t) \phi_n(x, y) - C^2 C_n(t) \nabla^2 \phi_n(x, y)] = F(x, y, t)$$

But

$$\nabla^2 \phi_n = -\lambda_n \phi_n$$

Therefore,

$$\sum_{n=1}^{\infty} [\ddot{C}_n(t) + \omega_n^2 C_n(t)]\, \phi_n = F(x, y, t) \tag{4.72}$$

where

$$\omega_n^2 = C^2 \lambda_n, \quad n = 1, 2, \ldots \tag{4.73}$$

Multiplying both sides of Eq. (4.72) by ϕ_m and integrating over the ragion $\mathbb{R}$ and interchaning the order of integration and summation, we obtain

$$\sum_{n=1}^{\infty} [\ddot{C}_n(t) + \omega_n^2 C_n(t)] \iint_{\mathbb{R}} \phi_n(x, y)\phi_m(x, y)\, dA = \iint_{\mathbb{R}} F\phi_m\, dA$$

Using the orthogonality property, this equation can be reduced to

$$\ddot{C}_m(t) + \omega^2 C_m(t) = F_m(t) \tag{4.74}$$

where

$$F_m(t) = \frac{1}{\|\phi_m\|^2} \iint_{\mathbb{R}} F(x, y, t)\phi_m(x, y)\, dA \tag{4.75}$$

and

$$\|\phi_m\|^2 = \iint_{\mathbb{R}} |\phi_m|^2 dA \tag{4.76}$$

The series (4.71) satisfies the ICs (4.68) is

$$\sum C_n(0)\phi_n(x, y) = f(x, y)$$

$$\sum \dot{C}_n(0)\phi_n(x, y) = g(x, y)$$

In order to determine $C_n(0)$ and $\dot{C}_n(0)$, we multiply both sides of the above two equations by ϕ_m and integrate over $\mathbb{R}$ and use the orthogonality property to get

$$C_m(0) = \frac{1}{\|\phi_m\|^2} \iint_{\mathbb{R}} f(x, y)\phi_m(x, y)\, dA \tag{4.77}$$

$$\dot{C}_m(0) = \frac{1}{\|\phi_m\|^2} \iint_{\mathbb{R}} g(x, y)\phi_m(x, y)\, dA$$

Using the method of variation of parameters, the general solution of Eq. (4.74) is given by

$$C_m(t) = A_m \cos\omega_m t + B_m \sin\omega_m t + \frac{1}{\omega_n}\int_0^t F_m(\xi)\sin\omega_m(t-\xi)\,d\xi$$

Using Eq. (4.77), we obtain

$$A_m = \frac{1}{\|\phi_m\|^2}\iint_{\mathbb{R}} f(x, y)\phi_m(x, y)\,dA \tag{4.78}$$

$$B_m = \frac{1}{\omega_m \|\phi_m\|^2}\iint_{\mathbb{R}} g(x, y)\phi_m(x, y)\,dA, \quad m = 1, 2, \ldots$$

Hence, the formal series solution of the general problem represented by Eqs. (4.66)–(4.68) is

$$u(x, y, t) = \sum_{n=1}^{\infty} (A_n \cos\omega_n t + B_n \sin\omega_n t)\phi_n(x, y) + \sum_{n=1}^{\infty} \frac{1}{\omega_n}\left[\int_0^t F_n(\xi)\sin\{\omega_n(t-\xi)\}\,d\xi\right]\phi_n(x, y) \tag{4.79}$$

4.8 PERIODIC SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION IN CYLINDRICAL COORDINATES

In cylindrical coordinates with u depending only on r, the one-dimensional wave equation assumes the following form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{4.80}$$

If we are looking for a periodic solution in time, we set

$$u = F(r)\,e^{i\omega t} \tag{4.81}$$

Then

$$\frac{\partial u}{\partial r} = F'(r)e^{i\omega t}, \quad \frac{\partial^2 u}{\partial t^2} = -\omega^2 F(r)\,e^{i\omega t}$$

Inserting these expressions, Eq. (4.80) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}[rF'(r)e^{i\omega t}] = -\frac{\omega^2}{c^2}F(r)\,e^{i\omega t}$$

or

$$F''(r) + \frac{F'(r)}{r} + \frac{\omega^2}{c^2} F(r) = 0 \tag{4.82}$$

which has the form of Bessel's equation and hence its solution can at once be written as

$$F = AJ_0\left(\frac{\omega r}{c}\right) + BY_0\left(\frac{\omega r}{c}\right) \tag{4.83}$$

In complex form, we can write this equation as

$$F = C_1\left[J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right)\right] + C_2\left[J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right)\right] \tag{4.84}$$

It can be rewritten as

$$F = C_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + C_2 H_0^{(2)}\left(\frac{\omega r}{c}\right) \tag{4.85}$$

where $H_0^{(1)}, H_0^{(2)}$ are Hankel functions defined by

$$H_0^{(1)} = J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right) \tag{4.86}$$

$$H_0^{(2)} = J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right) \tag{4.87}$$

which behave like damped trigonometric functions for large r. Thus the solution of one-dimensinoal wave equation becomes

$$u = C_1 e^{i\omega t} H_0^{(1)}\left(\frac{\omega r}{c}\right) + C_2 e^{i\omega t} H_0^{(2)}\left(\frac{\omega r}{c}\right) \tag{4.88}$$

Using asymptotic expressions, for $H_0^{(1)}$ and $H_0^{(2)}$ defined by

$$\left.\begin{aligned} H_0^{(1)}(x) &= \sqrt{\frac{2}{\pi x}}\, e^{i(x-\pi/4)} \\ H_0^{(2)}(x) &= \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\pi/4)} \quad \text{for large } x \end{aligned}\right\} \tag{4.89}$$

the general periodic solution to the given wave equation in cylindrical coordinates is

$$u(r, t) = \sqrt{\frac{2c}{\pi\omega}}\left[C_1 e^{-i\pi/4}\frac{\exp[i(\omega/c)(r+ct)]}{\sqrt{r}} + C_2 e^{i\pi/4}\frac{\exp[i(\omega/c)(r-ct)]}{\sqrt{r}}\right] \tag{4.90}$$

4.9 PERIODIC SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION IN SPHERICAL POLAR COORDINATES

In spherical polar coordinates, with u depending only on r, the source distance, the wave equation assumes the following form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad r > 0 \tag{4.91}$$

We look for a periodic solution in time in the form

$$u = F(r)e^{i\omega t} \tag{4.92}$$

Then

$$\frac{\partial u}{\partial r} = F'(r)e^{i\omega t}, \qquad \frac{\partial^2 u}{\partial t^2} = -\omega^2 F(r)e^{i\omega t}$$

Substituting these derivatives into Eq. (4.91), we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F' e^{i\omega t}) = -\frac{\omega^2}{c^2}F(r)e^{i\omega t}$$

i.e.

$$\frac{1}{r^2}e^{i\omega t}[r^2 F'' + 2rF'] = -\frac{\omega^2}{c^2}e^{i\omega t}F$$

Therefore,

$$F'' + \frac{2}{r}F' + \frac{\omega^2}{c^2}F = 0 \tag{4.93}$$

Let

$$F = \left(\frac{\omega}{c}r\right)^{-1/2}\psi(r)$$

Then

$$F' = -\frac{\omega}{2c}\left(\frac{\omega}{c}r\right)^{-3/2}\psi(r) + \left(\frac{\omega}{c}r\right)^{-1/2}\psi'(r)$$

$$F'' = \frac{3}{4}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega}{c}r\right)^{-5/2}\psi(r) - \frac{\omega}{c}\left(\frac{\omega}{c}r\right)^{-3/2}\psi'(r) + \left(\frac{\omega}{c}r\right)^{-1/2}\psi''(r)$$

Substituting into Eq. (4.93), we obtain

$$\left(\frac{\omega}{c}r\right)^{-1/2}\left[\psi''(r)+\frac{1}{r}\psi'(r)+\left\{\left(\frac{\omega}{c}\right)^2-\left(\frac{1}{2r}\right)^2\right\}\psi(r)\right]=0$$

Since $\left(\frac{\omega}{c}r\right)\neq 0,$ we have

$$\left[\psi''(r)+\frac{1}{r}\psi'(r)+\left\{\left(\frac{\omega}{c}\right)^2-\left(\frac{1}{2r}\right)^2\right\}\psi(r)\right]=0$$

which is a form of Bessel's equation, whose solution is given by

$$\psi(r)=A'J_{1/2}\left(\frac{\omega}{c}r\right)+B'J_{-1/2}\left(\frac{\omega}{c}r\right)$$

where A' and B' are constants. Therefore,

$$F(r)=\left(\frac{\omega}{c}r\right)^{-1/2}\left[A'J_{1/2}\left(\frac{\omega}{c}r\right)+B'J_{-1/2}\left(\frac{\omega}{c}r\right)\right] \tag{4.94}$$

or

$$F(r)=\frac{A}{\sqrt{r}}J_{1/2}\left(\frac{\omega}{c}r\right)+\frac{B}{\sqrt{r}}J_{-1/2}\left(\frac{\omega}{c}r\right) \tag{4.95}$$

But, we know that

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

Therefore,

$$F(r)=\sqrt{\frac{2c}{\pi\omega}}\left[A\frac{\sin(\omega r/c)}{r}+B\frac{\cos(\omega r/c)}{r}\right] \tag{4.96}$$

In complex form,

$$F(r)=C_1\frac{\exp(i\omega r/c)}{r}+C_2\frac{\exp(-i\omega r/c)}{r} \tag{4.97}$$

Thus, the required solution of the wave equation is

$$u(r,\ t) = C_1 \frac{\exp(i\omega/c)(r+ct)}{r} + C_2 \frac{\exp(-i\omega/c)(r-ct)}{r} \tag{4.98}$$

4.10 VIBRATION OF A CIRCULAR MEMBRANE

To find the solution of the wave equation representing the vibration of a circular membrane, it is natural that we introduce polar coordinates $(r,\ \theta), 0 \le r \le a, 0 \le \theta \le 2\pi.$ Thus, the governing two-dimensional wave equation is given by

$$\text{PDE: } \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{4.99}$$

and the boundary and initial conditions are given by

$$\text{BCs: } u(a,\ \theta,\ t) = 0, \qquad t \ge 0 \tag{4.100}$$

i.e. the boundary is held fixed, and

$$\text{ICs: } u(r, \theta, 0) = f_1(r,\ \theta), \qquad \frac{\partial u}{\partial t}(r,\ \theta,\ 0) = f_2(r,\ \theta) \tag{4.101}$$

Let us look for a solution of Eq. (4.99) in the following variables separable form:

$$u = R(r)\ H(\theta)\ T(t) \tag{4.102}$$

Substituting into Eq. (4.99), we obtain

$$\frac{RHT''}{c^2} = R''HT + \frac{1}{r}R'HT + \frac{1}{r^2}RH''T$$

Dividing throughout by RHT/c^2, we get

$$\frac{T''}{T} = c^2\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H}\right] = -\mu^2 \text{(say)}$$

Then

$$T'' + \mu^2 T = 0 \tag{4.103}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\mu^2}{c^2}r^2 = -\frac{H''}{H} = k^2 \text{(say)}$$

i.e.

$$r^2 R'' + rR' + \left(\frac{u^2}{c^2}r^2 - k^2\right)R = 0 \tag{4.104}$$

$$H'' + k^2 H = 0 \tag{4.105}$$

Here, μ^2 and k^2 are arbitrary separation constants. The general solutions of Eqs. (4.103)–(4.105) respectively are

$$T = A\cos\mu t + B\sin\mu t$$

$$R = PJ_k\left(\frac{\mu r}{c}\right) + QY_k\left(\frac{\mu r}{c}\right) \tag{4.106}$$

$$H = E\cos k\theta + F\sin k\theta$$

where J_k, Y_k are Bessel functions of first and second kind respectively of order k. Thus, the general solution of the wave equation (4.99) is

$$u(r,\theta,t) = (A\cos\mu t + B\sin\mu t)\left\{PJ_k\left(\frac{\mu r}{c}\right) + QY_k\left(\frac{\mu r}{c}\right)\right\}(E\cos k\theta + F\sin k\theta) \tag{4.107}$$

Since the deflection is a single-valued periodic function in θ of period 2π, k must be integral, say $k = n$. Also, since $Y_k(\mu r/c) \to -\infty$ as $r \to 0$, we can avoid infinite deflections at the centre $(r = 0)$ by taking $Q = 0$. Again noting that the BC: (4.100) implies that the deflection u is zero on the boundary of the circular membrane, we obtain

$$J_n\left(\frac{\mu a}{c}\right) = 0 \tag{4.108}$$

which has an inifinte number of positive zeros. These zeros (roots) are tabulated for several values of n in many handbooks. Their representation requires two indices. The first one indicates the order of the Bessel function, and the second, the solution. Thus denoting the roots by $\mu_{nm}(n = 0, 1, 2, \ldots; m = 1, 2, 3, \ldots)$, we have, after using the principle of superposition, the solution of the circular membrane in the form

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} PJ_n\left(\frac{\mu_{nm} r}{c}\right)(A\cos\mu t + B\sin\mu t)(E\cos n\theta + F\sin n\theta)$$

Alternatively,

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n\left(\frac{\mu_{nm} r}{c}\right)\{[a_{nm}\cos n\theta + b_{nm}\sin n\theta]\cos\mu t + [c_{nm}\cos n\theta + d_{nm}\sin n\theta]\sin\mu t\} \tag{4.109}$$

Now, to determine the constants, we shall use the prescribed ICs which yield

$$f_1(r,\theta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{nm} \cos n\theta + b_{nm} \sin n\theta) J_n\left(\frac{\mu_{nm} r}{c}\right) \tag{4.110}$$

$$f_2(r,\theta) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu_{nm} (c_{nm} \cos n\theta + d_{nm} \sin n\theta) J_n\left(\frac{\mu_{nm} r}{c}\right)$$

Hence, the solution of the circular membrance is given by Eq. (4.109), where

$$a_{nm} = \frac{2}{\pi a^2 [J_n'(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_1(r,\theta)\, J_n\left(\mu_{nm} \frac{r}{c}\right) \cos n\theta r\, dr\, d\theta$$

$$b_{nm} = \frac{2}{\pi a^2 [J_n'(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_1(r,\theta)\, J_n\left(\mu_{nm} \frac{r}{c}\right) \sin n\theta r\, dr\, d\theta$$

$$c_{nm} = \frac{2}{\pi a^2 [J_n'(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_2(r,\theta)\, J_n\left(\mu_{nm} \frac{r}{c}\right) \cos n\theta r\, dr\, d\theta,$$

$$d_{nm} = \frac{2}{\pi a^2 [J_n'(\mu_{nm})]^2} \int_0^{2\pi} \int_0^a f_2(r,\theta)\, J_n\left(\mu_{nm} \frac{r}{c}\right) \sin n\theta r\, dr\, d\theta$$

4.11 UNIQUENESS OF THE SOLUTION FOR THE WAVE EQUATION

In Section 4.5, we have developed the variables separable method to find solutions to the wave equation with certain initial and boundary conditions. A formal solution for the non-homogeneous equation is also given in Section 4.6. In this section, we shall show that the solution to the wave equation is unique.

Uniqueness Theorem The solution to the wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{4.111}$$

satisfying the ICs:

$$u(x,0) = f(x), \quad 0 \le x \le L$$

$$u_t(x,0) = g(x), \quad 0 \le x \le L$$

and the BCs:

$$u(0,t) = u(L,t) = 0$$

where $u(x,t)$ is twice continuously differentiable function with respect to x and t, is unique.

Proof Suppose u_1 and u_2 are two solutions of the given wave equation (4.111) and let $v = u_1 - u_2$. Obviously $v(x, t)$ is the solution of the following problem:

$$v_{tt} = c^2 v_{xx}, \quad 0 < x < L, \quad t > 0 \tag{4.112}$$

$$v(x, 0) = 0, \quad v_t(x, 0) = 0, \quad 0 \le x \le L$$

and

$$v(0, t) = v(L, t) = 0$$

It is required to prove that $v(x, t)$ is identically zero, implying $u_1 = u_2$. For, let us consider the function

$$E(t) = \frac{1}{2}\int_0^L (c^2 v_x^2 + v_t^2)\, dx \tag{4.113}$$

which, in fact, represents the total energy of the vibrating string at time t. It may be noted that $E(t)$ is differentiable with respect to t, as $v(x, t)$ is twice continuously differentiable. Thus,

$$\frac{dE}{dt} = \int_0^L \left[c^2 v_x v_{xt} + v_t\, v_{tt}\right] dx \tag{4.114}$$

Integrating by parts, the right-hand side of the above equation gives us

$$\int_0^L c^2 v_x v_{xt}\, dx = \left[c^2 v_x v_t\right]_0^L - \int_0^L c^2 v_t v_{xx}\, dx$$

But, $v(0, t) = 0$ implies $v_t(0, t) = 0$ for $t \ge 0$ and $v(L, t) = 0$ implying $v_t(L, t) = 0$ for $t \ge 0$. Hence, Eq. (4.114) reduces to

$$\frac{dE}{dt} = \int_0^L v_t (v_{tt} - c^2 v_{xx})\, dx = 0$$

In other words, $E(t) = \text{constant} = c$ (say). Since $v(x, 0) = 0$ implies $v_x(x, 0) = 0$, and $v_t(x, 0) = 0$, we can evaluate c and find that

$$E(0) = c = \int_0^L \left[c^2 v_x^2 + v_t^2\right]\Big|_{t=0} dx = 0$$

which gives $E(t) = 0$, which is possible if and only if $v_x \equiv 0$ and $v_t \equiv 0$ for all $t > 0, 0 \le x \le L$ which is possible only if $v(x, t) = \text{constant}$. However, since $v(x, 0) = 0$, we find $v(x, t) \equiv 0$. Hence, $u_1(x, t) = u_2(x, t)$. This means that the solution $u(x, t)$ of the given wave equation is unique.

4.12 DUHAMEL'S PRINCIPLE

With the help of Duhamel's principle, one can find the solution of an inhomogeneous equation, in terms of the general solution of the homogeneous equation. We shall illustrate this principle for wave equation. Let the Euclidean three-dimensional space be denoted by R_3, and a point in R_3 be represented by $X = (x_1, x_2, x_3)$. If $v(X, t, \tau)$ satisfies for each fixed τ the PDE

$$v_{tt}(X, t) - c^2\nabla^2 v(X, t) = 0, \ X \text{ in } R_3,$$

with the conditions

$$v(X, 0, \tau) = 0, \qquad v_t(X, 0, \tau) = F(X, \tau)$$

where $F(X, \tau)$ denotes a continuous function defined for X in R_3, and if u satisfies

$$u(X, t) = \int_0^t v(X, t-\tau, \tau)\, d\tau$$

then $u(X, t)$ satisfies

$$u_{tt} - c^2\nabla^2 u = F(x, t), \quad X \text{ in } R_3, \ t > 0$$

$$u(X, 0) = u_t(X, 0) = 0$$

Proof Consider the equation

$$u_{tt} - c^2\nabla^2 u = F(X, t) \tag{4.115}$$

with

$$u(X, 0) = u_t(X, 0) = 0$$

Let us assume the solution of the problem (4.115) in the form

$$u(x, t) = \int_0^t v(X, t-\tau, \tau)\, d\tau \tag{4.116}$$

where $v(X, t-\tau, \tau)$ is a one-parameter family solution of

$$v_{tt} - c^2\nabla^2 v = 0 \quad \text{for all } \tau \tag{4.117}$$

Further, we assume that at $t = \tau$,

$$v(X, 0, \tau) = 0 \quad \text{for all values of } \tau \tag{4.118}$$

Now, differentiating with respect to t under integral sign and using the Liebnitz rule, from Eq. (4.116), we have

$$u_t = v(X, 0, t) + \int_0^t v_t(X, t-\tau, \tau)\, d\tau$$

Using Eq. (4.118), we get

$$u_t = \int_0^t v_t(X,\ t-\tau,\ \tau)\, d\tau$$

Differentiating this result once again with respect to t, we obtain

$$u_{tt} = v_t(X,\ 0,\ t) + \int_0^t v_{tt}(X,\ t-\tau,\ \tau)\, d\tau \tag{4.119}$$

Noting that u satisfies Eq. (4.115), v satisfies Eq. (4.117), and after using Eq. (4.117), the above equation reduces to

$$u_{tt} = v_t\,(X,\ 0,\ t) + \int_0^t c^2\nabla^2 v\, d\tau$$

Finally, using Eq. (4.116), the above equation reduces to

$$u_{tt} - c^2\nabla^2 u = v_t(X,\ 0,\ t) \tag{4.120}$$

Comparing Eqs. (4.115) and (4.120), we obtain

$$v_t(X,\ 0,\ t) = F(X,\ t) \tag{4.121}$$

Therefore, if v satisfies the equation

$$v_{tt} - c^2\nabla^2 v = 0$$

with the conditions

$$v(X, 0, \tau) = 0, v_t(X, 0, \tau) = F(X, \tau) \quad \text{at } t = \tau$$

then, u defined by Eq. (4.116) satisfies the given inhomogeneous equation (4.115) and the specified conditions. Here, the function $v(X,\ t)$ is called the pulse function or the force function.

EXAMPLE 4.8 Use Duhamel's principle to solve the heat equation problem described by

$$u_t(x,\ t) = ku_{xx}(x,\ t) + f(x,\ t), \quad -\infty < x < \infty,\ t > 0 \tag{4.122}$$

$$u(x,\ 0) = 0, \qquad -\infty < x < \infty$$

Solution We have obtained, in Section 3.3, the unique solution of the problem

$$v_t(x,\ t) = kv_{xx}(x,\ t), \qquad -\infty < x < \infty,\ t > 0$$

$$v(x,\ 0) = f(x,\ \tau)$$

in the form

$$v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\,[-(x-\xi)^2/(4kt)]\, f(\xi)\, d\xi$$

Hence, using Duhamel's principle, the solution of the corresponding inhomogeneous problem described by Eq. (4.122) is given by

$$u(x, t, \tau) = \int_0^t v(x, t-\tau, \tau)\, d\tau$$

or

$$u(x, t, \tau) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-\tau)}} \exp\left[\frac{-(x-\xi)^2}{4k(t-\tau)}\right] f(\xi)\, d\xi\, d\tau \tag{4.123}$$

4.13 MISCELLANEOUS EXAMPLES

EXAMPLE 4.9 A uniform string of line density ρ is stretched to tension ρc^2 and executes a small transverse vibtration in a plane through the undisturbed line of string. The ends $x = 0, L$ of the string are fixed. The string is at rest, with the point $x = b$ drawn aside through a small distance ε and released at time $t = 0.$ Find an expression for the displacement $y(x, t)$.

Solution The transverse vibration of the string is described by

$$\text{PDE: } y_{xx} = \frac{1}{c^2} y_{tt} \tag{4.124}$$

The boundary and initial conditions are

$$\text{BCs: } y(0, t) = y(L, t) = 0$$

$$\text{IC: } y_t(x, 0) = 0$$

Using the variables separable method, let

$$y(x, t) = X(x)\, T(t)$$

then, we have from Eq. (4.124),

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = \pm\lambda^2$$

The equation of the string is given by (see Fig. 4.7)

$$y(x,0) = \begin{cases} \dfrac{\varepsilon x}{b}, & 0 \le x \le b \\[2mm] \dfrac{\varepsilon(x-L)}{(b-L)}, & b \le x \le L \end{cases}$$

The solution to the given problem is discussed now for various values of λ.

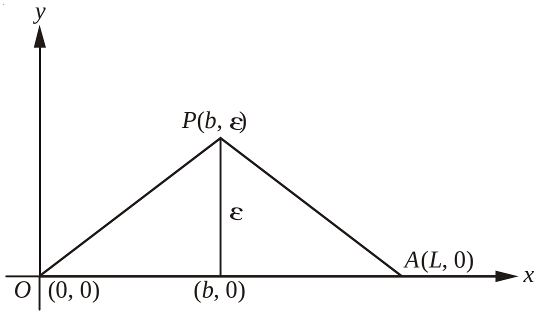

Fig. 4.7 Illustration of Example 4.9.

Case I Taking the constant $\lambda = 0$, we have

$$X'' = T'' = 0$$

whose general solution is

$$X = Ax + B, \qquad T = Ct + D$$

Therefore,

$$y(x, t) = (Ax + B)(Ct + D)$$

Using the BCs at $x = 0, L$, we can observe that $A = B = 0$, implies a trivial solution.

Case II Taking the constant as $+\lambda^2$, we have

$$X'' - \lambda^2 X = 0 = T'' - c^2\lambda^2 T$$

Thus, the general solution is

$$y(x, t) = (A \cosh \lambda x + B \sinh \lambda x)(C \cosh c\lambda t + D \sinh c\lambda t)$$

Now the BCs:

$$y(0, t) = 0 \text{ gives } A = 0$$

and

$$y(L, t) = 0 \text{ gives } B \sinh \lambda L = 0$$

which is possible only if $B = 0$. Thus we are again getting only a trivial solution.

Case III If the constant is $-\lambda^2$, then we have

$$X'' + \lambda^2 X = 0 = T'' + c^2\lambda^2 T = 0$$

In this case, the general solution is

$$y(x, t) = (A \cos \lambda x + B \sin \lambda x)(P \cos c\lambda t + Q \sin c\lambda t)$$

using the BCs:

$$y(0, t) = 0 \text{ gives } A = 0$$

$$y(L, t) = 0 \text{ gives } B \sin \lambda L = 0$$

For a non-trivial solution, $B \neq 0 \Rightarrow \lambda L = n\pi$. Therefore, $\lambda = n\pi/L, n = 1, 2, \ldots$ Also, using the IC: $y_t(x, 0) = 0$, we can notice that $Q = 0$. Hence, the acceptable non-trivial solution is

$$y(x, t) = BP \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}, \qquad n = 1, 2, \ldots$$

Using the principle of superposition, we have

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}$$

which gives

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

This is half-range sine series, where

$$b_n = \frac{2}{L}\int_0^L y(x, 0) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L}\int_0^b \frac{\varepsilon}{b} x \sin \frac{n\pi x}{L} dx + \frac{2}{L}\int_0^L \frac{\varepsilon}{b-L}(x-L) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2\varepsilon}{Lb}\left[-\frac{\cos(n\pi x/L)}{n\pi/L} x\right]_0^b - \frac{2\varepsilon}{Lb}\left[-\frac{\sin(n\pi/L)\, x}{n^2\pi^2/L^2}\right]_0^b$$

$$+ \frac{2\varepsilon}{L(b-L)}\left[-\frac{\cos(n\pi x/L)}{n\pi/L}(x-L)\right]_b^L - \frac{2\varepsilon}{L(b-L)}\left[-\frac{\sin(n\pi x/L)}{n^2\pi^2/L^2}\right]_b^L$$

or

$$b_n = \frac{2\varepsilon L^2}{n^2\pi^2 b(L-b)} \sin \frac{n\pi b}{L}$$

Hence the subsequent motion of the string is given by

$$y(x, t) = \sum_{n=1}^{\infty} \frac{2\varepsilon L^2}{n^2\pi^2 b(L-b)} \sin \frac{n\pi b}{L} \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}.$$

EXAMPLE 4.10 Find a particular solution of the problem described by

$$\text{PDE: } y_{tt} - c^2 y_{xx} = g(x)\cos\omega t, \qquad 0 < x < L, \quad t > 0$$

$$\text{BCs: } y(0, t) = y(L, t) = 0, \qquad t > 0$$

where $g(x)$ is a piecewise smooth function and ω is a positive constant.

Solution Taking the clue from Example 4.9, we assume the solution in the form

$$y(x, t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{n\pi x}{L}$$

To determine $A_n(t)$, we consider the Fourier sine expansion of $g(x)$ in the form

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

and substitute into the given PDE which yields

$$\sum_{n=1}^{\infty} \left[A_n''(t) + \left(\frac{n\pi c}{L}\right)^2 A_n(t)\right] \sin\frac{n\pi x}{L} = \cos\omega t \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

Choosing $A_n(t)$ as the solution of the ODE

$$A_n''(t) + \left(\frac{n\pi c}{L}\right)^2 A_n(t) = B_n \cos\omega t$$

we have for any n, the particular solution

$$A_n(t) = A_n \cos\omega t \quad \text{if} \quad \omega \neq \frac{n\pi c}{L}$$

Therefore,

$$A_n\left[-\omega^2 + \left(\frac{n\pi c}{L}\right)^2\right] = B_n$$

Hence, the required particular solution is given by

$$y(x, t) = \cos\omega t \sum_{n=1}^{\infty} \frac{B_n \sin(n\pi x/L)}{(n\pi c/L)^2 - \omega^2}$$

EXAMPLE 4.11 A rectangular membrane with fastened edges makes free transverse vibrations. Explain how a formal series solution can be found.

Solution Mathematically, the problem can be posed as follows:
Solve

$$\text{PDE: } u_{tt} - c^2(u_{xx} + u_{yy}) = 0, \quad 0 \le x \le a, 0 \le y \le b$$

subject to the BCs:

(i) $u(0, y, t) = 0$

(ii) $u(a, y, t) = 0$

(iii) $u(x, 0, t) = 0$

(iv) $u(x, b, t) = 0$

and ICs:

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y)$$

We look for a separable solution of the form

$$u(x, y, t) = X(x)\, Y(y)\, T(t)$$

Substituting into the given PDE, we obtain

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \quad \text{(say)}$$

Then

$$T'' + c^2\lambda^2 T = 0$$

$$\frac{X''}{X} + \lambda^2 = -\frac{Y''}{Y} = \mu^2 \quad \text{(say)}$$

thus yielding

$$Y'' + \mu^2 Y = 0, \quad X'' + (\lambda^2 - \mu^2)\, X = 0$$

Let $\lambda^2 - \mu^2 = p^2$, $\mu^2 = q^2$. Then $\lambda^2 = p^2 + q^2 = r^2$. Therefore, we have

$$X'' + p^2 X = 0, \quad Y'' + q^2 Y = 0, \quad T'' + r^2 c^2 T = 0$$

The possible separable solution is

$$u(x, y, t) = (A \cos px + B \sin px)\, (C \cos qy + D \sin qy)\, (E \cos (rct) + F \sin (rct))$$

Using the BCs: $u(0, y, t) = 0$ gives $A = 0$

$u(x, 0, t) = 0$ gives $C = 0$

$u(a, y, t) = 0$ gives $p = m\pi/a, \quad m = 1, 2, \ldots$

$u(x, b, t) = 0$ gives $q = n\pi/b, \quad n = 1, 2, \ldots$

Using the principle of superposition, we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos(rct) + B_{mn} \sin(rct)] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{4.125}$$

where

$$r^2 = p^2 + q^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

Applying the initial condition: $u(x, y, 0) = f(x, y)$, Eq. (4.125) gives

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \tag{4.126}$$

Finally, applying the initial condition: $u_t(x, y, 0) = g(x, y)$, Eq. (4.125) gives

$$g(x, y) = cr \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$B_{mn} = \frac{4}{abcr} \int_0^a \int_0^b g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \tag{4.127}$$

Hence, the required series solution is given by Eq. (4.125), where A_{mn} and B_{mn} are given by Eqs. (4.126) and (4.127).

EXAMPLE 4.12 Solve the IVP described by

$$\text{PDE: } u_{tt} - c^2 u_{xx} = F(x, t), \qquad -\infty < x < \infty, t \geq 0$$

with the data

(i) $F(x, t) = 4x + t$, (ii) $u(x, 0) = 0$, (iii) $u_t(x, 0) = \cosh bx$.

Solution In Example 4.4, we have obtained the Riemann-Volterra solution for the inhomogeneous wave equation in the following form:

$$u(x, t) = \frac{1}{2}[\eta(x - ct) + \eta(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} v(\xi) \, d\xi + \frac{1}{2c} \iint_{\mathbb{R}} F(x, t) \, dx \, dt \tag{4.128}$$

Since $u(x, 0) = \eta(x) = 0$, the first term on the right-hand side of Eq. (4.128) vanishes. Also,

$$\frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\, d\xi = \frac{1}{2c}\int_{x-ct}^{x+ct} \cosh b\xi\, d\xi$$

$$= \frac{1}{2c}\left(\frac{\sin b\xi}{b}\right)_{x-ct}^{x+ct} = \frac{1}{2bc}[\sinh b(x+ct) - \sinh b(x-ct)]$$

$$= (\cosh bx \sinh (bct))/bc$$

and

$$\frac{1}{2c}\iint_{\mathbb{R}} F(x, t)\, dx\, dt = \frac{1}{2c}\iint_{\mathbb{R}} (4x + t)\, dx\, dt$$

From Fig. 4.4, we can write the equation of the line PA in the form

$$t = \frac{x - x_0 + ct_0}{c}$$

or

$$x = x_0 + ct - ct_0$$

Similarly, the equation of the line PB is

$$x = x_0 + ct_0 - ct$$

Thus

$$\frac{1}{2c}\iint_{\mathbb{R}} F(x, t)\, dt = \int_0^{t_0}\int_{x_0+ct-ct_0}^{x_0+ct_0-ct} (4x + t)\, dx\, dt$$

$$= \int_0^{t_0} (4x_0t_0 - 4tx_0 + tt_0 - t^2)\, dt = 2x_0t_0^2 + t_0^3/6$$

The required solution at any point (x, t) is, therefore, given by

$$u(x, t) = \frac{\cosh bx \sinh (bct)}{bc} + 2xt^2 + \frac{t^3}{6}$$

EXAMPLE 4.13 Derive the wave equation representing the transverse vibration of a string in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left\{1 + \left(\frac{\partial u}{\partial x}\right)^2\right\}^{-2} \frac{\partial^2 u}{\partial x^2}$$

Solution Consider the motion of an element $PQ = \delta s$ of the string as shown in Fig. 4.8. In equilibrium position, let the string lie along the *x*-axis, such that *PQ* is originally at $P'Q'$. Let the displacement of *PQ* from the *x*-axis, be denoted by *u*. Let *T* be the tension in the string and ρ be the density of the string. Writing down the equation of motion of the element *PQ* of the string in the *u*-direction, we have

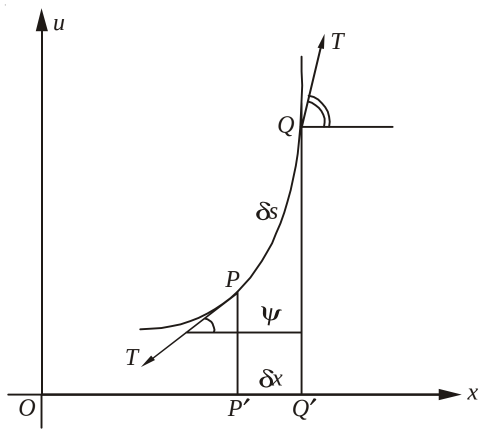

Fig. 4.8 An Illustration of Example 4.13.

$$T \sin (\psi + \delta\psi) - T \sin\psi = \rho\delta_s \frac{\partial^2 u}{\partial t^2}$$

Neglecting squares of small quantities, we get

$$T \cos \psi\delta\psi = \rho\delta_s \frac{\partial^2 u}{\partial t^2} \tag{4.129}$$

by noting that

$$\tan\psi = \frac{\partial u}{\partial x}, \quad \sec^2\psi\delta\psi = \frac{\partial^2 u}{\partial x^2}\delta x$$

Equation (4.129) becomes

$$\rho\frac{\partial^2 u}{\partial t^2} = T\cos^3\psi\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s} = T\cos^4\psi\frac{\partial^2 u}{\partial x^2} \tag{4.130}$$

but

$$\cos^2\psi = \frac{1}{1+\tan^2\psi} = \left\{1+\left(\frac{\partial u}{\partial x}\right)^2\right\}^{-1} \tag{4.131}$$

Using Eq. (4.131) into Eq. (4.130), we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left\{1+\left(\frac{\partial u}{\partial x}\right)^2\right\}^{-2} \frac{\partial^2 u}{\partial x^2}$$

If we define $c^2 = T/\rho$, the required wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left\{1+\left(\frac{\partial u}{\partial x}\right)^2\right\}^{-2} \frac{\partial^2 u}{\partial x^2} \qquad (4.132)$$

This is a non-linear second order partial differential equation.

EXAMPLE 4.14 Using Duhamel's principle solve the following IBVP:

PDE: $u_t - u_{xx} = f(x, t),\ 0 < x < L,\ 0 < t < \infty$

BCS: $u(0, t) = u(L, t) = 0$

IC: $u(x, 0) = 0,\ 0 \le x \le L.$

Solution Using Duhamel's principle, the required solution is given by

$$u(x, t) = \int_0^t v(x, t-\tau, \tau)\, d\tau$$

where $v(x, t, \tau)$ is the solution of the homogeneous problem described as

$$v_t - v_{xx} = 0,\ 0 < x < L,\ 0 < t < \infty$$

$$v(0, t, \tau) = v(L, t, \tau) = 0$$

and $$u(x, 0, \tau) = f(x, \tau).$$

Now, recalling Example 3.17, the solution to this homogeneous problem is obtained as

$$v(x, t, \tau) = \sum_{n=1}^{\infty} a_n e^{-(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

Observe that, the Fourier coefficients a_n depends on the parameter τ, so that

$$a_n = a_n(\tau) = \frac{2}{L}\int_0^L f(x, \tau) \sin\left(\frac{n\pi x}{L}\right) dx$$

Hence, the solution to the gives IBVP is found to be

$$u(x, t) = \int_0^t \sum_{n=1}^{\infty} a_n(\tau) e^{-(n\pi/L)^2 (t-\tau)} \sin\left(\frac{n\pi x}{L}\right) d\tau.$$

EXERCISES

1. A homogeneous string is stretched and its ends are at $x = 0$ and $x = l$. Motion is started by displacing the string into the form $f(x) = u_0 \sin(\pi x/l)$, from which it is released at time $t = 0$. Find the displacement at any point x and time t.

2. Solve the boundary value problem described by

$$\begin{aligned} &\text{PDE: } u_{tt} - c^2 u_{xx} = 0, && 0 \le x \le l, t \ge 0 \\ &\text{BCs: } u(0, t) = u(l, t) = 0, && t \ge 0 \\ &\text{ICs: } u(x,0) = 10 \sin(\pi x/l), && 0 \le x \le l \\ &\quad u_t(x, 0) = 0 && \end{aligned}$$

3. Solve the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le \pi, t \ge 0$$

subject to

$$\begin{aligned} u &= 0 && \text{when } x = 0 \text{ and } x = \pi \\ u_t &= 0 && \text{when } t = 0 \text{ and } u(x, 0) = x, \quad 0 < x < \pi \end{aligned}$$

4. Solve

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le l, \ t \ge 0$$

subject to

$$u(0, t) = 0, \quad u(l, t) = 0 \text{ for all } t$$

$$u(x, 0) = 0, \quad u_t(x, 0) = b \sin^3(\pi x/l)$$

5. Solve the vibrating string problem described by

$$\begin{aligned} &\text{PDE: } u_{tt} - c^2 u_{xx} = 0, && 0 < x < l, \ t > 0 \\ &\text{BCs: } u(0, t) = u(l, t) = 0, && t > 0 \\ &\text{ICs: } u(x, 0) = f(x), && 0 \le x \le l \\ &\quad u_t(x, 0) = 0, && 0 \le x \le l \end{aligned}$$

6. In spherical coordinates, if u is a spherical wave, i.e. $u = u(r,t)$, then the wave equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

which is called the Euler-Poisson-Darboux equation. Find its general solution.

7. Solve the initial value problem described by

$$\text{PDE}: u_{tt} - c^2 u_{xx} = e^x$$

with the given data

$$u(x,0) = 5, \qquad u_t(x,0) = x^2$$

8. Sovle the initial value problem described by

$$\text{PDE}: u_{tt} - c^2 u_{xx} = xe^t$$

with the data

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 0$$

9. Solve the initial boundary value problem described by

$$\text{PDE}: u_{tt} = c^2 u_{xx}, \qquad x > 0, \ t > 0$$

with the data:

$$u(x, 0) = 0, \qquad u_t(x, 0) = 0, \ x > 0$$

$$u(0, t) = \sin t, \qquad t > 0$$

10. Determine the solution of the one-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \quad 0 < x < a, t > 0$$

with c as a constant, under the following initial and boundary conditions:

(i) $\phi(x, 0) = f(x) = \begin{cases} x/b, & 0 \le x \le b \\ (a-x)/(a-b), & b < x \le a \end{cases}$

(ii) $\dfrac{\partial \phi}{\partial t}(x, 0) = 0, \ 0 < x < a$

(iii) $\phi(0, t) = \phi(a, t) = 0, \quad t \ge 0.$

11. A piano string of length L is fixed at both ends. The string has a linear density ρ and is under tension τ. At time $t = 0$, the string is pulled a distance s from equilibrium position at its mid-point so that it forms an isosceles triangle and is then released $(s \le L)$. Find the subsequent motion of the string.

12. Obtain the normal frequencies and normal modes for the vibrating string of Problem 11.

13. A flexible stretched string is constrained to move with zero slope at one end $x = 0,$ while the other end $x = L$ is held fixed against any movement. Find an expression for the time-dependent motion of the string if it is subjected to the initial displacement given by

$$y(x, 0) = y_0 \cos\left(\frac{\pi x}{2L}\right)$$

and is released from this position with zero velocity.

14. Show that if f and g are arbitrary functions, then

$$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$$

is a solution of the equation

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$$

provided $\alpha^2 = 1 - v^2/c^2.$

Choose the correct answer in the following questions (15 and 16):

15. The solution of the initial value problem

$$u_{tt} = 4u_{xx}, \qquad t > 0, \qquad -\infty < x < \infty$$

satisfying the conditions $u(x, 0) = x, \quad u_t(x, 0) = 0$ is

(A) x (B) $x^2/2$ (C) $2x$ (D) $2t$ (GATE-Maths, 2001)

16. Let $u = \psi(x, t)$ be the solution to the initial value problem

$$u_{tt} = u_{xx} \qquad \text{for } -\infty < x < \infty, t > 0$$

with $u(x, 0) = \sin x, u_t(x, 0) = \cos x,$ then the value of $\psi(\pi/2, \pi/6)$ is

(A) $\sqrt{3}/2$ (B) $1/2$ (C) $1/\sqrt{2}$ (D) 1 (GATE-Maths, 2003)

17. Solve the following IBVP

$$\text{PDE: } u_t = u_{xx} + f(x, t),\ 0 < x < \pi,\ 0 < t < \infty$$

$$\text{BCS: } u(0, t) = u(\pi, t) = 0$$

$$\text{IC: } u(x, 0) = 0,\ 0 \le x \le \pi$$

Using Duhamel’s principle.