

Definition: A complex number is a number consisting of a real and imaginary part.

Its standard form is $\text{Re}(z) = x$, $\text{Im}(z) = y$. , $i = \sqrt{-1}$; $z = x + iy$

✓ The complex conjugate of a complex number $z = x + iy$, denoted by \bar{z} is given by $\bar{z} = x - iy$.

✓ The complex number $-z = -x - iy$ is the addition inverse of a complex number $z = x + iy$, and the multiplication inverse of a complex number

$$0 \neq z = x + iy \text{ is } z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} .$$

Examples: Find $\text{Re}(z)$, $\text{Im}(z)$, \bar{z} , $-z$, z^{-1} for each complex number z of the following:

$$1 - 2i , \quad 2 + i , \quad i , \quad 2i , \quad \frac{1}{1 + i} , \quad -1$$

Solution: $z = 1 - 2i$

$$\text{Re}(z) = 1 , \quad \text{Im}(z) = -2 , \quad \bar{z} = 1 + 2i , \quad -z = -1 + 2i ,$$

$$z^{-1} = \frac{1}{1 - 2i} = \frac{1 + 2i}{(1 - 2i)(1 + 2i)} = \frac{1 + 2i}{1^2 - (2i)^2} = \frac{1}{5}(1 + 2i)$$

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✓ Two complex numbers are equal if their real parts are equal and their imaginary parts are equal

(i.e. If $x_1 + iy_1 = x_2 + iy_2$ Then $x_1 = x_2$ and $y_1 = y_2$).

The polar form of a complex number:

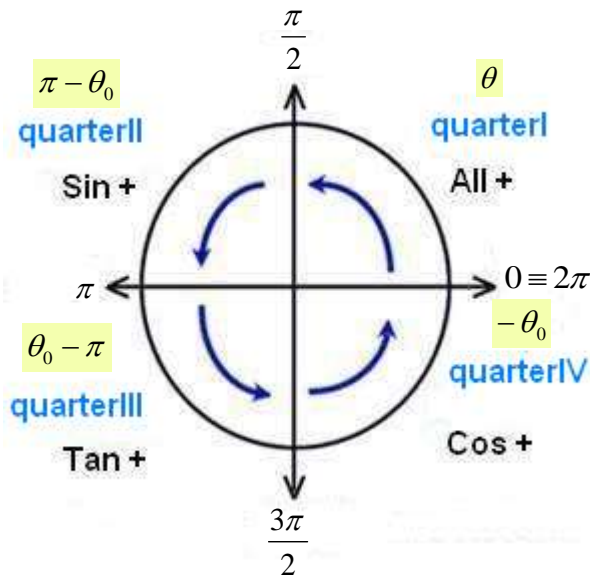
$z = r(\cos\theta + i\sin\theta)$ is called the *polar form* of a complex number $z = x + iy$ such that:

$$x = r \cos\theta, \quad y = r \sin\theta, \quad r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}$$

θ is called the *argument* of z , denoted by $\arg(z)$.

The *principal argument* of z is $-\pi \leq \theta \leq \pi$
(determined according to in which quarter lies?)

As shown in the following diagram:



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(θ_0 will be one of the famous angles $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}, \dots$ rad)

In other words:

- ✓ The complex number $z = x + iy$ lies in quarter I.
- ✓ The complex number $z = -x + iy$ lies in quarter II.
- ✓ The complex number $z = -x - iy$ lies in quarter III.
- ✓ The complex number $z = x - iy$ lies in quarter IV.

Examples: Write each of the following complex number z in polar form:

$$1+i, -\sqrt{3}+i, -1-i\sqrt{3}, 1-i$$

(1) $z = 1+i$

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2},$$

$$\sin \theta = \frac{y}{r} = \frac{1}{\sqrt{2}},$$

$$\cos \theta = \frac{x}{r} = \frac{1}{\sqrt{2}},$$

$$\tan \theta = \frac{y}{x} = \frac{1}{1} = 1.$$

$$\therefore \theta = \frac{\pi}{4},$$

$$\therefore 1+i = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

(2) $z = -\sqrt{3}+i$

$$r = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2,$$

$$\sin \theta = \frac{y}{r} = \frac{1}{2},$$

$$\cos \theta = \frac{x}{r} = \frac{-\sqrt{3}}{2},$$

$$\tan \theta = \frac{y}{x} = \frac{1}{-\sqrt{3}}.$$

$$\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6},$$

$$\therefore -\sqrt{3}+i = 2 \left[\cos \left(\frac{5\pi}{6} \right) + i \sin \left(\frac{5\pi}{6} \right) \right].$$

(3) $z = -1 - i\sqrt{3}$

$$r = \sqrt{x^2 + y^2} = \sqrt{1+3} = 2,$$

$$\sin \theta = \frac{y}{r} = \frac{-\sqrt{3}}{2},$$

$$\cos \theta = \frac{x}{r} = \frac{-1}{2},$$

$$\tan \theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1} = \sqrt{3}.$$

$$\therefore \theta = \frac{\pi}{3} - \pi = -\frac{2\pi}{3},$$

$$\therefore -1 - i\sqrt{3} = 2\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right].$$

(4) $z = 1 - i$

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2},$$

$$\sin \theta = \frac{y}{r} = \frac{-1}{\sqrt{2}},$$

$$\cos \theta = \frac{x}{r} = \frac{1}{\sqrt{2}},$$

$$\tan \theta = \frac{y}{x} = \frac{-1}{1} = -1.$$

$$\therefore \theta = -\frac{\pi}{4},$$

$$\therefore 1 - i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right].$$

✓ **H.W.:**

1- Write the complex number $z = \frac{2}{1+i}$ in the form $z = x + iy$

$\operatorname{Re}(z)$, $\operatorname{Im}(z)$, \bar{z} , $|z|$, $\arg(z)$.

2- Write the complex number $z = \frac{4}{-\sqrt{3} + i}$ in the form $z = x + iy$

$\operatorname{Re}(z)$, $\operatorname{Im}(z)$, \bar{z} , $|z|$, $\arg(z)$.

De Moivre`s Theorem: Let $z = r(\cos\theta + i \sin\theta)$ be a complex number and n be any real number. Then $z^n = r^n(\cos n\theta + i \sin n\theta)$.

Examples:

(1) Using De Moivre`s Theorem, find the value of $(1+i)^8$

Solution:

we put the complex number $z = 1+i$ in the polar form as follows:

$$r = \sqrt{1+1} = \sqrt{2}, \theta = \tan^{-1} \frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore 1+i = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}),$$

$$\therefore (1+i)^8 = (\sqrt{2})^8 (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})^8 = 16(\cos 2\pi + i \sin 2\pi) = 16.$$

(2) Using De Moivre`s Theorem, reduce the complex number:

$$z = \frac{(\cos 2\theta - i \sin 2\theta)^5 (\cos 3\theta + i \sin 3\theta)^7}{(\cos 4\theta + i \sin 4\theta)^{11} (\cos 5\theta - i \sin 5\theta)^9}, \text{ and find its value at } \theta = \frac{\pi}{6}.$$

Solution:

$$z = \frac{[\cos(-2\theta) + i \sin(-2\theta)]^5 [\cos 3\theta + i \sin 3\theta]^7}{[\cos 4\theta + i \sin 4\theta]^{11} [\cos(-5\theta) + i \sin(-5\theta)]^9}$$

$$= \frac{[\cos \theta + i \sin \theta]^{-10} [\cos \theta + i \sin \theta]^{21}}{[\cos \theta + i \sin \theta]^{44} [\cos \theta + i \sin \theta]^{-45}}$$

$$= (\cos \theta + i \sin \theta)^{12} = \cos 12\theta + i \sin 12\theta.$$

and at $\theta = \frac{\pi}{6}$: $z = \cos(12)(\frac{\pi}{6}) + i \sin(12)(\frac{\pi}{6}) = \cos 2\pi + i \sin 2\pi = 1$

(3) Using De Moivre`s Theorem, reduce the complex number:

$$\frac{(1 + i \tan \theta)^5}{(1 - i \tan \theta)^7}, \text{ and find its value at } \theta = \frac{\pi}{6}.$$

Solution:

$$\begin{aligned} z &= \frac{(1 + i \tan \theta)^5}{(1 - i \tan \theta)^7} = \frac{(1 + i \frac{\sin \theta}{\cos \theta})^5}{(1 - i \frac{\sin \theta}{\cos \theta})^7} \\ &= \frac{(\cos \theta)^2 (\cos \theta + i \sin \theta)^5}{(\cos \theta - i \sin \theta)^7} \\ &= \frac{(\cos \theta)^2 (\cos \theta + i \sin \theta)^5}{(\cos \theta + i \sin \theta)^{-7}} \\ &= (\cos \theta)^2 (\cos \theta + i \sin \theta)^{12} \\ &= (\cos \theta)^2 [\cos(12\theta) + i \sin(12\theta)]. \end{aligned}$$

and at $\theta = \frac{\pi}{6}$:

$$z = (\cos(\frac{\pi}{6}))^2 [\cos(12)(\frac{\pi}{6}) + i \sin(12)(\frac{\pi}{6})] = (\frac{\sqrt{3}}{2})^2 [\cos 2\pi + i \sin 2\pi] = \frac{3}{4}.$$

✓ **H.W.:**

Using De Moivre`s Theorem, find the value of $(1 + i\sqrt{3})^6$, $(\sqrt{3} + i)^{12}$