Mathematical Induction:
is a special method of proof used to prove a Statement, a Theorem, or a Formula, that is asserted about every natural number.
The natural numbers are the counting numbers: $1,2,3,4,...\ etc.$, also called positive integers.

Principle of Mathematical Induction:
Let $P(n)$ be a statement involving the positive integer n .
IF the statement is true when $n=1$, and whenever the statement is true for $n=k$, then it is also true for $n=k+1$, Then the statement is true for all integers $n\geq 1$.
There is nothing special about the integer 1 in the statement above.
It can be replaced (in both places it occurs) by any other positive integer, and the Principle still works.

Steps of Mathematical Induction:
(STEP 1): We show that $P(1)$ is true.
(STEP 2): We assume that $P(k)$ is true.
(STEP 3): We show that $P(k+1)$ is true.

As shown in the following examples:

1- Use mathematical induction to prove that:

$$1+2+3+....+n=\frac{n(n+1)}{2} .$$

Solution: Let the statement $P(n)$ be $1+2+3+....+n=\dfrac{n(n+1)}{2}$

(STEP 1): We show that $P(1)$ is true:

$$L.H.S.=1 \quad , \quad R.H.S.=\frac{1(1+1)}{2}=1$$

Both sides of the statement are equal hence $P(1)$ is true.
(STEP 2): We assume that $P(k)$ is true:

$$1+2+3+....+k=\frac{k(k+1)}{2} .$$

(STEP 3): We show that $P(k+1)$ is true:

$$\begin{aligned} L.H.S. &= 1+2+3+...+k+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{(k+1)}{2}[k+2] \\ &= R.H.S. \end{aligned}$$

Which is the statement $P(k+1)$.
Then the statement $P(n)$ is true for all positive integers n .

<u>We can rewrite the solution as follow</u>:

<u>Solution</u>: Let $P(n)$ be $1+2+3+....+n=\frac{n(n+1)}{2}$

(1) at $n=1$:

$L.H.S.=1$, $R.H.S.=\frac{1(1+1)}{2}=1$

$\therefore P(1)$ is true.

(2) let $n=k$:

$1+2+3+....+k=\frac{k(k+1)}{2}$.

(3) at $n=k+1$:

$L.H.S.=1+2+3+...+k+(k+1)$

$=\frac{k(k+1)}{2}+(k+1)$

$=\frac{(k+1)}{2}[k+2]$

$=R.H.S.$

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n .

2- Use mathematical induction to prove that:

$$1^2+2^2+3^2+....+n^2=\frac{n(n+1)(2n+1)}{6}.$$

Solution: Let $P(n)$ be $1^2+2^2+3^2+....+n^2=\frac{n(n+1)(2n+1)}{6}$

(1) at $n=1$: $L.H.S.=1^2=1$ **,** $R.H.S.=\frac{1(1+1)(2+1)}{6}=1$

$\therefore P(1)$ is true.

(2) let $n=k$: $1^2+2^2+3^2+....+k^2=\frac{k(k+1)(2k+1)}{6}$.

(3) at $n=k+1$:

$$L.H.S.=1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=\frac{(k+1)}{6}[k(2k+1)+6(k+1)]$$

$$=\frac{(k+1)}{6}[2k^2+k+6k+6]$$

$$=\frac{(k+1)}{6}[2k^2+7k+6]$$

$$=\frac{(k+1)}{6}[(2k+3)(k+2)]$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=R.H.S.$$

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n .

3- Prove that (n^3+2n) is divisible by 3 for all positive integers n .

Solution: Let $P(n)$ be "(n^3+2n) is divisible by 3"

(1) at $n=1$:

$1^3+2(1)=3$ is divisible by 3.

$\therefore P(1)$ is true.

(2) let $n=k$:

"(k^3+2k) is divisible by 3".

(3) at $n=k+1$:

$$\begin{aligned}(k+1)^3+2(k+1) &= (k^3+3k^2+3k+1)+(2k+2)\\ &= k^3+3k^2+5k+3\\ &= (k^3+2k)+(3k^2+3k+3)\\ &= (k^3+2k)+3(k^2+k+1)\end{aligned}$$

(k^3+2k) is divisible by 3 from (2), and $3(k^2+k+1)$ is also divisible by 3

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n .

4- Prove that $2^{n-1} \le n!$ for all positive integers n .

Solution: Let $P(n)$ be $2^{n-1} \le n!$

(1) at $n=1$:

$2^{1-1}=2^0=1 \le 1!=1$

$\therefore P(1)$ is true.

(2) let $n=k$:

$2^{k-1} \le k!$

(3) at $n=k+1$:

$2^{k-1} \le k! \Rightarrow (2)(2^{k-1}) \le (2)(k!) \Rightarrow (2)(2^{k-1}) \le (k+1)(k!) \Rightarrow 2^k \le (k+1)!$;

$2 \le k+1 \;\; \forall \; k \in Z^+$

$\therefore P(k+1)$ is true.

Then $P(n)$ is true for all positive integers n .

H.W:

1- Use mathematical induction to prove that:

$(i)\ 2+4+6+...+2n = n(n+1).$

$(ii)\ 1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^{n-1}} = 2-\frac{1}{2^{n-1}}.$

2- Prove that $(x^n - 1)$ is divisible by $(x-1)$ for all positive integers n .

<u>Exercises</u>:

1- Use mathematical induction to prove that:

$(i)\quad 1+3+5+....+(2n-1)=n^2$

$(ii)\quad 1+4+7+....+(3n-2)=\dfrac{n(3n-1)}{2}$

$(iii)\quad 2+6+12+....+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$

2- Prove that $(3n^2-n)$ is divisible by 2 for all positive integers n.

3- Prove that (7^n-2^n) is divisible by 5 for all positive integers n.

4- Prove that (x^n-y^n) is divisible by $(x-y)$ for all positive integers n.
