Definition: A *complex number* is a number consisting of a real and imaginary part.

Its standard form is $\operatorname{Re}(z) = x$, $\operatorname{Im}(z) = y.$, $i = \sqrt{-1}$; $z = x + iy$

✓ The complex conjugate of a complex number $z = x + iy$, denoted by $\bar{z}$ is given by $\bar{z} = x - iy$.

✓ The complex number $-z = -x - iy$ is the addition inverse of a complex number $z = x + iy$, and the multiplication inverse of a complex number $0 \neq z = x + iy$ is $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{(x+iy)(x-iy)} = \frac{x - iy}{x^2 + y^2}$.

Examples: Find $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, $-z$, z^{-1} for each comlex number z of the following:

$1-2i$, $2+i$, i , $2i$, $\frac{1}{1+i}$, -1

<u>Solution</u>: $z = 1 - 2i$

$\operatorname{Re}(z) = 1$, $\operatorname{Im}(z) = -2$, $\bar{z} = 1 + 2i$, $-z = -1 + 2i$,

$$z^{-1} = \frac{1}{1-2i} = \frac{1+2i}{(1-2i)(1+2i)} = \frac{1+2i}{1^2 - (2i)^2} = \frac{1}{5}(1+2i)$$

✓ Two complex numbers are equal if their real parts are equal and their imaginary parts are equal

(i.e. If $x_1 + iy_1 = x_2 + iy_2$ Then $x_1 = x_2$ and $y_1 = y_2$).

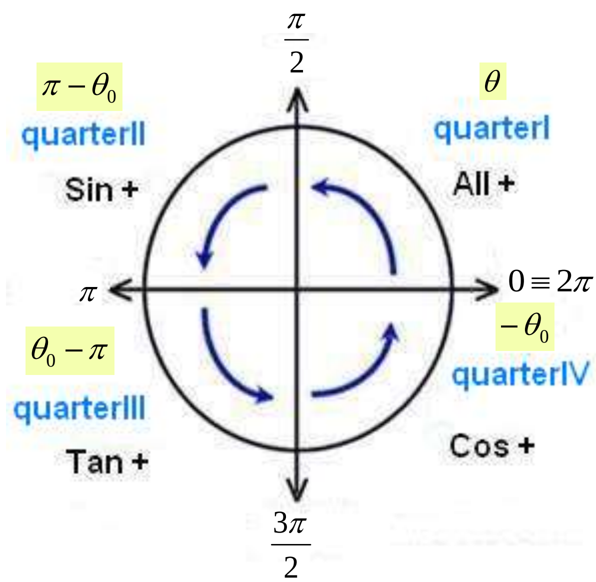
<u>The polar form of a complex number</u>:

$z = r(\cos\theta + i\sin\theta)$ is called the *polar form* of a comlex number $z = x + iy$ such that:

$$x = r\cos\theta\ ,\ \ y = r\sin\theta\ \ ,\ \ r = |z| = \sqrt{x^2 + y^2}\ \ ,\ \ \theta = \tan^{-1}\frac{y}{x}$$

θ is called the *argument* of z, denoted by $\arg(z)$.

The *principal argument* of z is $-\pi \le \theta \le \pi$
(determined according to in which quarter lies?)
As shown in the following diagram:



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(θ_0 will be one of the famous angles $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{4}, \ldots$ rad)

<u>In other words</u>:

- ✓ The comlex number $z = x + iy$ lies in quarterI.
- ✓ The comlex number $z = -x + iy$ lies in quarterII.
- ✓ The comlex number $z = -x - iy$ lies in quarterIII.
- ✓ The comlex number $z = x - iy$ lies in quarterIV.

Examples: Write each of the following comlex number z in polar form:

$1+i$, $-\sqrt{3}+i$, $-1-i\sqrt{3}$, $1-i$

(1) $z=1+i$

$$r=\sqrt{x^2+y^2}=\sqrt{1+1}=\sqrt{2},$$

$$\sin\theta=\frac{y}{r}=\frac{1}{\sqrt{2}},$$

$$\cos\theta=\frac{x}{r}=\frac{1}{\sqrt{2}},$$

$$\tan\theta=\frac{y}{x}=\frac{1}{1}=1.$$

$$\therefore\theta=\frac{\pi}{4},$$

$$\therefore 1+i=\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}).$$

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(2) $z=-\sqrt{3}+i$

$$r=\sqrt{x^2+y^2}=\sqrt{3+1}=2,$$

$$\sin\theta=\frac{y}{r}=\frac{1}{2},$$

$$\cos\theta=\frac{x}{r}=\frac{-\sqrt{3}}{2},$$

$$\tan\theta=\frac{y}{x}=\frac{1}{-\sqrt{3}}.$$

$$\therefore\theta=\pi-\frac{\pi}{6}=\frac{5\pi}{6},$$

$$\therefore -\sqrt{3}+i=2[\cos(\frac{5\pi}{6})+i\sin(\frac{5\pi}{6})].$$

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(3) $z=-1-i\sqrt{3}$

$r=\sqrt{x^2+y^2}=\sqrt{1+3}=2,$

$\sin\theta=\frac{y}{r}=\frac{-\sqrt{3}}{2}$,

$\cos\theta=\frac{x}{r}=\frac{-1}{2}$,

$\tan\theta=\frac{y}{x}=\frac{-\sqrt{3}}{-1}=\sqrt{3}.$

$\therefore\theta=\frac{\pi}{3}-\pi=-\frac{2\pi}{3}$,

$\therefore -1-i\sqrt{3}=2[\cos(-\frac{2\pi}{3})+i\sin(-\frac{2\pi}{3})].$

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(4) $z=1-i$

$r=\sqrt{x^2+y^2}=\sqrt{1+1}=\sqrt{2},$

$\sin\theta=\frac{y}{r}=\frac{-1}{\sqrt{2}}$,

$\cos\theta=\frac{x}{r}=\frac{1}{\sqrt{2}}$,

$\tan\theta=\frac{y}{x}=\frac{-1}{1}=-1.$

$\therefore\theta=-\frac{\pi}{4}$,

$\therefore 1-i=\sqrt{2}[\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})].$

✓ **H.W:**

1- Write the complex number $z=\frac{2}{1+i}$ in the form , and find $z=x+iy$

$\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$, $|z|$, $\arg(z)$.

2- Write the complex number $z=\frac{4}{-\sqrt{3}+i}$ in the form , and find $z=x+iy$

$\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$, $|z|$, $\arg(z)$.

<u>De Moivre`s Theorem</u>: Let $z = r(\cos\theta + i\sin\theta)$ be a complex number and n be any real number. Then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

<u>Examples</u>:

(1) Using De Moivre`s Theorem, find the value of $(1+i)^8$

<u>Solution</u>:

we put the complex number $z = 1+i$ in the polar form as follows:

$$r = \sqrt{1+1} = \sqrt{2}\,, \theta = \tan^{-1}\frac{1}{1} = \tan^{-1}1 = \frac{\pi}{4}$$

$$\therefore 1+i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})\ ,$$

$$\therefore (1+i)^8 = (\sqrt{2})^8(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^8 = 16(\cos 2\pi + i\sin 2\pi) = 16.$$

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(2) Using De Moivres` Theorem, reduce the complex number:

$$z = \frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^7}{(\cos 4\theta + i\sin 4\theta)^{11}(\cos 5\theta - i\sin 5\theta)^9}$$, and find its value at $\theta = \frac{\pi}{6}$.

<u>Solution</u>:

$$z = \frac{[\cos(-2\theta) + i\sin(-2\theta)]^5[\cos 3\theta + i\sin 3\theta]^7}{[\cos 4\theta + i\sin 4\theta]^{11}[\cos(-5\theta) + i\sin(-5\theta)]^9}$$

$$= \frac{[\cos\theta + i\sin\theta]^{-10}[\cos\theta + i\sin\theta]^{21}}{[\cos\theta + i\sin\theta]^{44}[\cos\theta + i\sin\theta)]^{-45}}$$

$$= (\cos\theta + i\sin\theta)^{12} = \cos 12\theta + i\sin 12\theta.$$

and at $\theta = \frac{\pi}{6}$: $z = \cos(12)(\frac{\pi}{6}) + i\sin(12)(\frac{\pi}{6}) = \cos 2\pi + i\sin 2\pi = 1$

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(3) Using De Moivre`s Theorem, reduce the complex number:

$\dfrac{(1+i\tan\theta)^5}{(1-i\tan\theta)^7}$, and find its value at $\theta=\dfrac{\pi}{6}$.

<u>Solution</u>:

$$z=\frac{(1+i\tan\theta)^5}{(1-i\tan\theta)^7}=\frac{(1+i\dfrac{\sin\theta}{\cos\theta})^5}{(1-i\dfrac{\sin\theta}{\cos\theta})^7}$$

$$=\frac{(\cos\theta)^2(\cos\theta+i\sin\theta)^5}{(\cos\theta-i\sin\theta)^7}$$

$$=\frac{(\cos\theta)^2(\cos\theta+i\sin\theta)^5}{(\cos\theta+i\sin\theta)^{-7}}$$

$$=(\cos\theta)^2(\cos\theta+i\sin\theta)^{12}$$

$$=(\cos\theta)^2[\cos(12\theta)+i\sin(12\theta)].$$

and at $\theta=\dfrac{\pi}{6}$:

$$z=(\cos(\frac{\pi}{6}))^2[\cos(12)(\frac{\pi}{6})+i\sin(12)(\frac{\pi}{6})]=(\frac{\sqrt{3}}{2})^2[\cos 2\pi+i\sin 2\pi]=\frac{3}{4}.$$

✓ **<u>H.W</u>:**

Using De Moivre`s Theorem, find the value of $(1+i\sqrt{3})^6$, $(\sqrt{3}+i)^{12}$
