Basic Concepts of Alternating Current

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CHAPTER 1

CONCEPTS OF ALTERNATING CURRENT

LEARNING OBJECTIVES

Upon completion of this chapter you will be able to:

1. State the differences between ac and dc voltage and current.
2. State the advantages of ac power transmission over dc power transmission.
3. State the "left-hand rule" for a conductor.
4. State the relationship between current and magnetism.
5. State the methods by which ac power can be generated.
6. State the relationship between frequency, period, time, and wavelength.
7. Compute peak-to-peak, instantaneous, effective, and average values of voltage and current.
8. Compute the phase difference between sine waves.

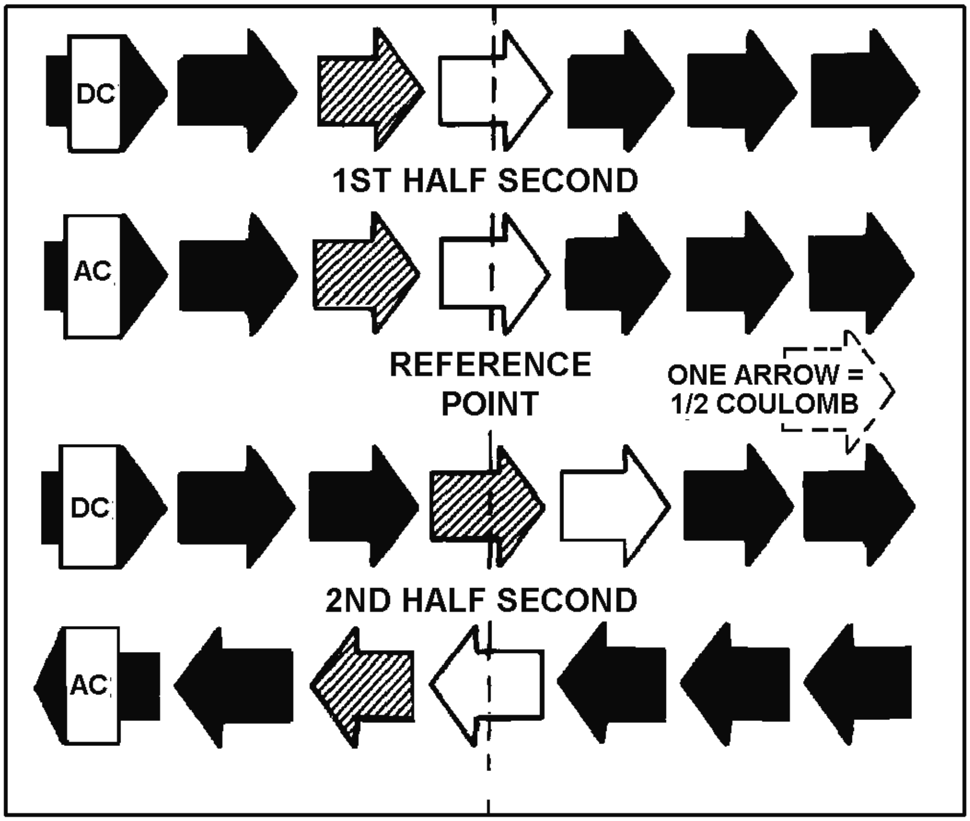
CONCEPTS OF ALTERNATING CURRENT

All of your study thus far has been with direct current (dc), that is, current which does not change direction. However, as you saw in module 1 and will see later in this module, a coil rotating in a magnetic field actually generates a current which regularly changes direction. This current is called ALTERNATING CURRENT or ac.

AC AND DC

Alternating current is current which constantly changes in amplitude, and which reverses direction at regular intervals. You learned previously that direct current flows only in one direction, and that the amplitude of current is determined by the number of electrons flowing past a point in a circuit in one second. If, for example, a coulomb of electrons moves past a point in a wire in one second and all of the electrons are moving in the same direction, the amplitude of direct current in the wire is one ampere. Similarly, if half a coulomb of electrons moves in one direction past a point in the wire in half a second, then reverses direction and moves past the same point in the opposite direction during the next half-second, a total of one coulomb of electrons passes the point in one second. The amplitude of the alternating current is one ampere. The preceding comparison of dc and ac as illustrated. Notice that one white arrow plus one striped arrow comprise one coulomb.

COMPARING DC & AC CURRENT FLOW IN A WIRE



Q1. Define direct current.

Q2. Define alternating current.

DISADVANTAGES OF DC COMPARED TO AC

When commercial use of electricity became wide-spread in the United States, certain disadvantages in using direct current in the home became apparent. If a commercial direct-current system is used, the voltage must be generated at the level (amplitude or value) required by the load. To properly light a 240-volt lamp, for example, the dc generator must deliver 240 volts. If a 120-volt lamp is to be supplied power from the 240-volt generator, a resistor or another 120-volt lamp must be placed in series with the 120-volt lamp to drop the extra 120 volts. When the resistor is used to reduce the voltage, an amount of power equal to that consumed by the lamp is wasted.

Another disadvantage of the direct-current system becomes evident when the direct current (I) from the generating station must be transmitted a long distance over wires to the consumer. When this happens, a large amount of power is lost due to the resistance (R) of the wire. The power loss is equal to I^2R. However, this loss can be greatly reduced if the power is transmitted over the lines at a very high voltage level and a low current level. This is not a practical solution to the power loss in the dc system since the load would then have to be operated at a dangerously high voltage. Because of the disadvantages related to transmitting and using direct current, practically all modern commercial electric power companies generate and distribute alternating current (ac).

Unlike direct voltages, alternating voltages can be stepped up or down in amplitude by a device called a TRANSFORMER. (The transformer will be explained later in this module.) Use of the transformer permits efficient transmission of electrical power over long-distance lines. At the electrical power station, the transformer output power is at high voltage and low current levels. At the consumer end of the transmission lines, the voltage is stepped down by a transformer to the value required by the load. Due to its inherent advantages and versatility, alternating current has replaced direct current in all but a few commercial power distribution systems.

Q3. What is a disadvantage of a direct-current system with respect to supply voltage?

Q4. What disadvantage of a direct current is due to the resistance of the transmission wires?

Q5. What kind of electrical current is used in most modern power distribution systems?

VOLTAGE WAVEFORMS

You now know that there are two types of current and voltage, that is, direct current and voltage and alternating current and voltage. If a graph is constructed showing the amplitude of a dc voltage across the terminals of a battery with respect to time, it will appear in figure 1-1 view A. The dc voltage is shown to have a constant amplitude. Some voltages go through periodic changes in amplitude like those shown in figure 1-1 view B. The pattern which results when these changes in amplitude with respect to time are plotted on graph paper is known as a WAVEFORM. Figure 1-1 view B shows some of the common electrical waveforms. Of those illustrated, the sine wave will be dealt with most often.

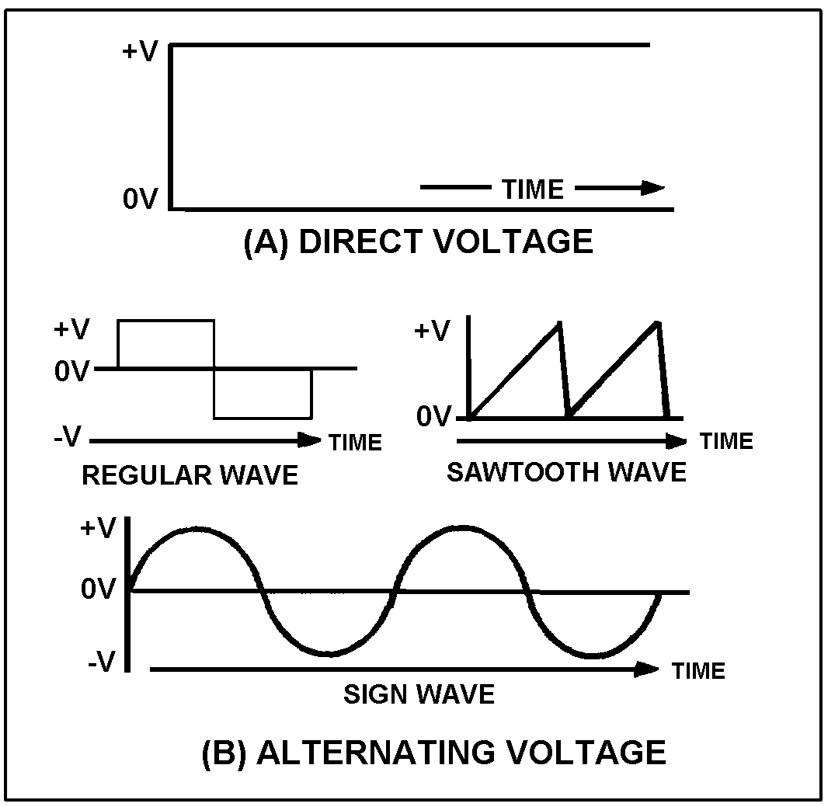


Figure 1-1.—Voltage waveforms: (A) Direct voltage; (B) Alternating voltage.

ELECTROMAGNETISM

The sine wave illustrated in figure 1-1 view B is a plot of a current which changes amplitude and direction. Although there are several ways of producing this current, the method based on the principles of electromagnetic induction is by far the easiest and most common method in use.

The fundamental theories concerning simple magnets and magnetism were discussed in Module 1, but how magnetism can be used to produce electricity was only briefly mentioned. This module will give you a more in-depth study of magnetism. The main points that will be explained are how magnetism is affected by an electric current and, conversely, how electricity is affected by magnetism. This general subject area is most often referred to as ELECTROMAGNETISM. To properly understand electricity you must first become familiar with the relationships between magnetism and electricity. For example, you must know that:

- An electric current always produces some form of magnetism.
- The most commonly used means for producing or using electricity involves magnetism.
- The peculiar behavior of electricity under certain conditions is caused by magnetic influences.

MAGNETIC FIELDS

In 1819 Hans Christian Oersted, a Danish physicist, found that a definite relationship exists between magnetism and electricity. He discovered that an electric current is always accompanied by certain magnetic effects and that these effects obey definite laws.

MAGNETIC FIELD AROUND A CURRENT-CARRYING CONDUCTOR

If a compass is placed in the vicinity of a current-carrying conductor, the compass needle will align itself at right angles to the conductor, thus indicating the presence of a magnetic force. You can demonstrate the presence of this force by using the arrangement illustrated in figure 1-2. In both (A) and (B) of the figure, current flows in a vertical conductor through a horizontal piece of cardboard. You can determine the direction of the magnetic force produced by the current by placing a compass at various points on the cardboard and noting the compass needle deflection. The direction of the magnetic force is assumed to be the direction in which the north pole of the compass points.

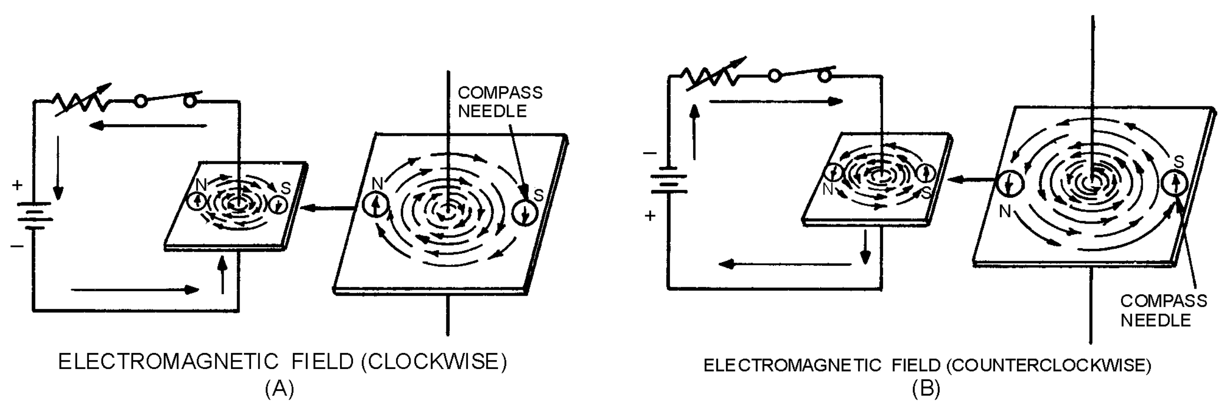


Figure 1-2.—Magnetic field around a current-carrying conductor.

In figure 1-2 (A), the needle deflections show that a magnetic field exists in circular form around the conductor. When the current flows upward (see figure 1-2(A)), the direction of the field is clockwise, as viewed from the top. However, if you reverse the polarity of the battery so that the current flows downward (see figure 1-2(B)), the direction of the field is counterclockwise.

The relation between the direction of the magnetic lines of force around a conductor and the direction of electron current flow in the conductor may be determined by means of the LEFT-HAND RULE FOR A CONDUCTOR: if you grasp the conductor in your left hand with the thumb extended in the direction of the electron flow (current) (– to +), your fingers will point in the direction of the magnetic lines of force. Now apply this rule to figure 1-2. Note that your fingers point in the direction that the north pole of the compass points when it is placed in the magnetic field surrounding the wire.

An arrow is generally used in electrical diagrams to denote the direction of current in a length of wire (see figure 1-3(A)). Where a cross section of a wire is shown, an end view of the arrow is used. A cross-sectional view of a conductor that is carrying current toward the observer is illustrated in figure 1-3(B). Notice that the direction of current is indicated by a dot, representing the head of the arrow. A conductor that is carrying current away from the observer is illustrated in figure 1-3(C). Note that the direction of current is indicated by a cross, representing the tail of the arrow. Also note that the magnetic field around a current-carrying conductor is perpendicular to the conductor, and that the magnetic lines of force are equal along all parts of the conductor.

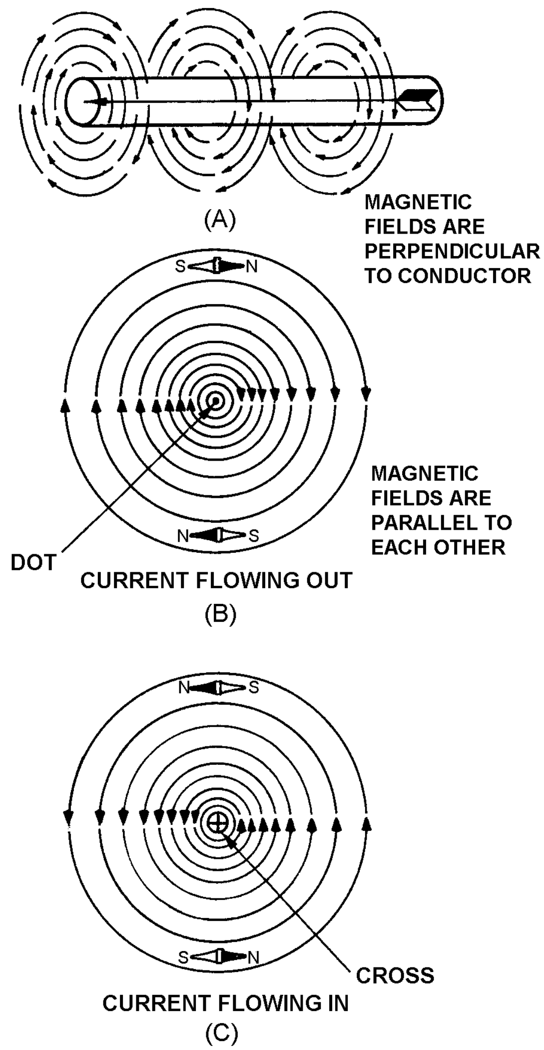


Figure 1-3.—Magnetic field around a current-carrying conductor, detailed view.

When two adjacent parallel conductors are carrying current in the same direction, the magnetic lines of force combine and increase the strength of the field around the conductors, as shown in figure 1-4(A). Two parallel conductors carrying currents in opposite directions are shown in figure 1-4(B). Note that the field around one conductor is opposite in direction to the field around the other conductor. The resulting lines of force oppose each other in the space between the wires, thus deforming the field around each conductor. This means that if two parallel and adjacent conductors are carrying currents in the same direction, the fields about the two conductors aid each other. Conversely, if the two conductors are carrying currents in opposite directions, the fields about the conductors repel each other.

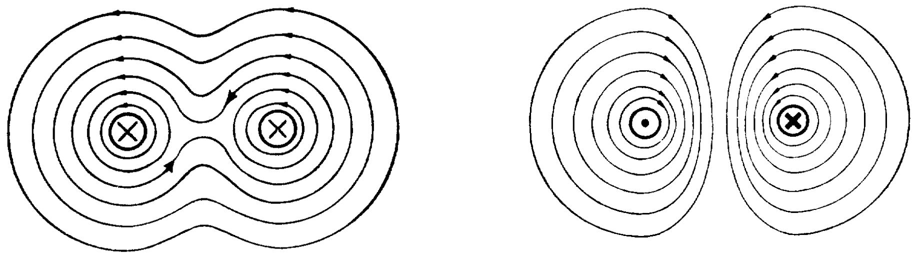


Figure 1-4.—Magnetic field around two parallel conductors.

Q6. When placed in the vicinity of a current-carrying conductor, the needle of a compass becomes aligned at what angle to the conductor?

Q7. What is the direction of the magnetic field around a vertical conductor when (a) the current flows upward and (b) the current flows downward.

Q8. The "left-hand rule" for a conductor is used for what purpose

Q9. In what direction will the compass needle point when the compass is placed in the magnetic field surrounding a wire?

Q10. When two adjacent parallel wires carry current in the same direction, the magnetic field about one wire has what effect on the magnetic field about the other conductor?

Q11. When two adjacent parallel conductors carry current in opposite directions, the magnetic field about one conductor has what effect on the magnetic field about the other conductor?

MAGNETIC FIELD OF A COIL

Figure 1-3(A) illustrates that the magnetic field around a current-carrying wire exists at all points along the wire. Figure 1-5 illustrates that when a straight wire is wound around a core, it forms a coil and that the magnetic field about the core assumes a different shape. Figure 1-5(A) is actually a partial cutaway view showing the construction of a simple coil. Figure 1-5(B) shows a cross-sectional view of the same coil. Notice that the two ends of the coil are identified as X and Y.

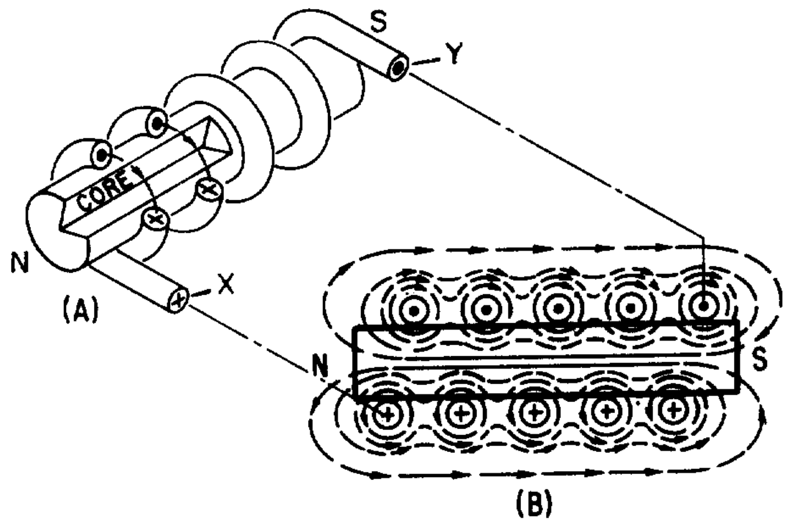


Figure 1-5.—Magnetic field produced by a current-carrying coil.

When current is passed through the coil, the magnetic field about each turn of wire links with the fields of the adjacent turns. (See figure 1-4(A)). The combined influence of all the turns produces a two-pole field similar to that of a simple bar magnet. One end of the coil is a north pole and the other end is a south pole.

Polarity of an Electromagnetic Coil

Figure 1-2 shows that the direction of the magnetic field around a straight wire depends on the direction of current in that wire. Thus, a reversal of current in a wire causes a reversal in the direction of the magnetic field that is produced. It follows that a reversal of the current in a coil also causes a reversal of the two-pole magnetic field about the coil.

When the direction of the current in a coil is known, you can determine the magnetic polarity of the coil by using the LEFT-HAND RULE FOR COILS. This rule, illustrated in figure 1-6, is stated as follows:

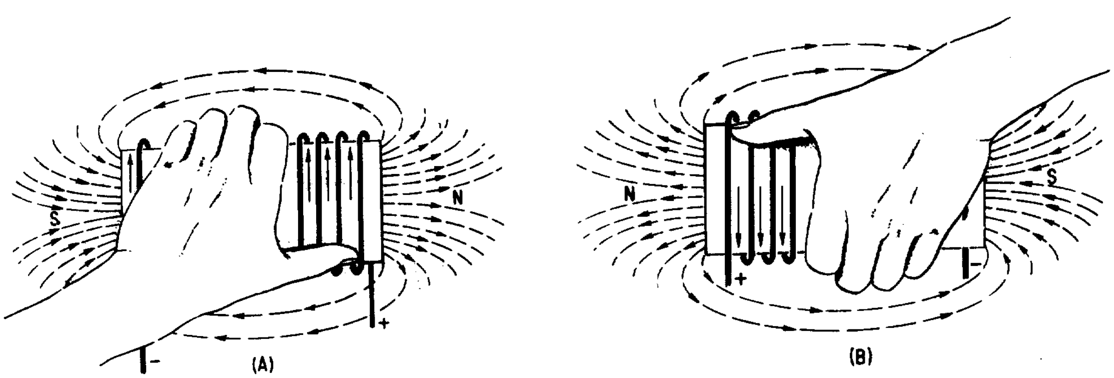


Figure 1-6.—Left-hand rule for coils.

Grasp the coil in your left hand, with your fingers "wrapped around" in the direction of the electron current flow. Your thumb will then point toward the north pole of the coil.

Strength of an Electromagnetic Field

The strength or intensity of a coil's magnetic field depends on a number of factors. The main ones are listed below and will be discussed again later.

- The number of turns of wire in the coil.
- The amount of current flowing in the coil.
- The ratio of the coil length to the coil width.
- The type of material in the core.

Losses in an Electromagnetic Field

When current flows in a conductor, the atoms in the conductor all line up in a definite direction, producing a magnetic field. When the direction of the current changes, the direction of the atoms' alignment also changes, causing the magnetic field to change direction. To reverse all the atoms requires that power be expended, and this power is lost. This loss of power (in the form of heat) is called HYSTERESIS LOSS. Hysteresis loss is common to all ac equipment; however, it causes few problems except in motors, generators, and transformers. When these devices are discussed later in this module, hysteresis loss will be covered in more detail.

Q12. What is the shape of the magnetic field that exists around (a) a straight conductor and (b) a coil?

Q13. What happens to the two-pole field of a coil when the current through the coil is reversed?

Q14. What rule is used to determine the polarity of a coil when the direction of the electron current flow in the coil is known?

Q15. State the rule whose purpose is described in Q14.

BASIC AC GENERATION

From the previous discussion you learned that a current-carrying conductor produces a magnetic field around itself. In module 1, under producing a voltage (emf) using magnetism, you learned how a changing magnetic field produces an emf in a conductor. That is, if a conductor is placed in a magnetic field, and either the field or the conductor moves, an emf is induced in the conductor. This effect is called electromagnetic induction.

CYCLE

Figures 1-7 and 1-8 show a suspended loop of wire (conductor) being rotated (moved) in a clockwise direction through the magnetic field between the poles of a permanent magnet. For ease of explanation, the loop has been divided into a dark half and light half. Notice in (A) of the figure that the dark half is moving along (parallel to) the lines of force. Consequently, it is cutting NO lines of force. The same is true of the light half, which is moving in the opposite direction. Since the conductors are cutting no lines of force, no emf is induced. As the loop rotates toward the position shown in (B), it cuts more and more lines of force per second (inducing an ever-increasing voltage) because it is cutting more directly across the field (lines of force). At (B), the conductor is shown completing one-quarter of a complete revolution, or 90º , of a complete circle. Because the conductor is now cutting directly across the field, the voltage induced in the conductor is maximum. When the value of induced voltage at various points during the rotation from (A) to (B) is plotted on a graph (and the points connected), a curve appears as shown below.

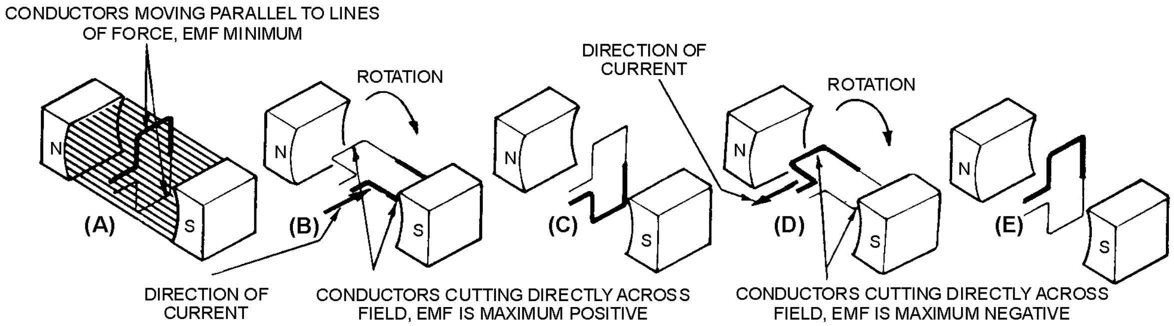


Figure 1-7.—Simple alternating-current generator.

As the loop continues to be rotated toward the position shown below in (C), it cuts fewer and fewer lines of force. The induced voltage decreases from its peak value. Eventually, the loop is once again moving in a plane parallel to the magnetic field, and no emf is induced in the conductor.

The loop has now been rotated through half a circle (one alternation or 180º). If the preceding quarter-cycle is plotted, it appears as shown below.

When the same procedure is applied to the second half of rotation (180º through 360º), the curve appears as shown below. Notice the only difference is in the polarity of the induced voltage. Where previously the polarity was positive, it is now negative.

The sine curve shows the value of induced voltage at each instant of time during rotation of the loop. Notice that this curve contains 360º , or two alternations. TWO ALTERNATIONS represent ONE complete CYCLE of rotation.

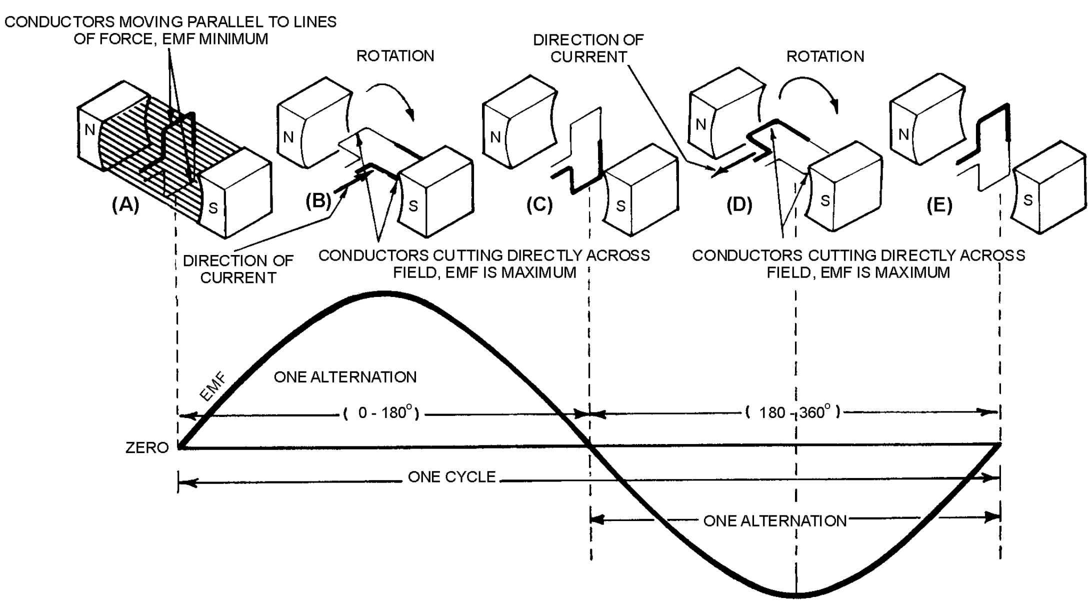


Figure 1-8.—Basic alternating-current generator.

Assuming a closed path is provided across the ends of the conductor loop, you can determine the direction of current in the loop by using the LEFT-HAND RULE FOR GENERATORS. Refer to figure 1-9. The left-hand rule is applied as follows: First, place your left hand on the illustration with the fingers as shown. Your THUMB will now point in the direction of rotation (relative movement of the wire to the magnetic field); your FOREFINGER will point in the direction of magnetic flux (north to south); and your MIDDLE FINGER (pointing out of the paper) will point in the direction of electron current flow.

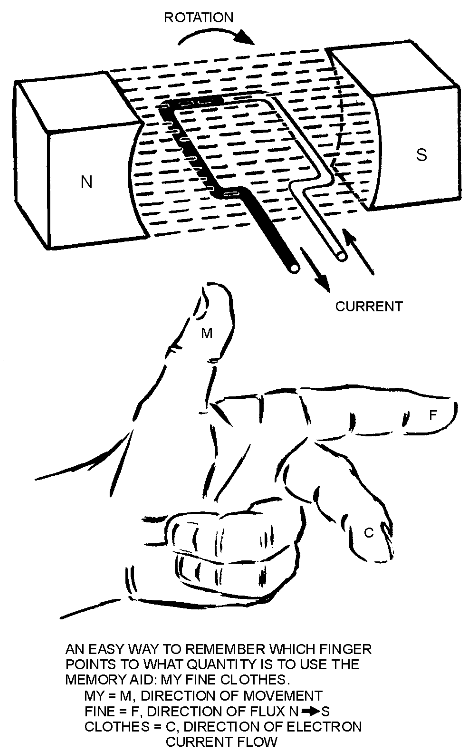


Figure 1-9.—Left-hand rule for generators.

By applying the left-hand rule to the dark half of the loop in (B) in figure 1-8, you will find that the current flows in the direction indicated by the heavy arrow. Similarly, by using the left-hand rule on the light half of the loop, you will find that current therein flows in the opposite direction. The two induced voltages in the loop add together to form one total emf. It is this emf which causes the current in the loop.

When the loop rotates to the position shown in (D) of figure 1-8, the action reverses. The dark half is moving up instead of down, and the light half is moving down instead of up. By applying the left-hand rule once again, you will see that the total induced emf and its resulting current have reversed direction. The voltage builds up to maximum in this new direction, as shown by the sine curve in figure 1-8. The loop finally returns to its original position (E), at which point voltage is again zero. The sine curve represents one complete cycle of voltage generated by the rotating loop. All the illustrations used in this chapter show the wire loop moving in a clockwise direction. In actual practice, the loop can be moved clockwise or counterclockwise. Regardless of the direction of movement, the left-hand rule applies.

If the loop is rotated through 360º at a steady rate, and if the strength of the magnetic field is uniform, the voltage produced is a sine wave of voltage, as indicated in figure 1-9. Continuous rotation of the loop will produce a series of sine-wave voltage cycles or, in other words, an ac voltage.

As mentioned previously, the cycle consists of two complete alternations in a period of time. Recently the HERTZ (Hz) has been designated to indicate one cycle per second. If ONE CYCLE PER SECOND is ONE HERTZ, then 100 cycles per second are equal to 100 hertz, and so on. Throughout the NEETS, the term cycle is used when no specific time element is involved, and the term hertz (Hz) is used when the time element is measured in seconds.

Q16. When a conductor is rotated in a magnetic field, at what points in the cycle is emf (a) at maximum amplitude and (b) at minimum amplitude?

Q17. One cycle is equal to how many degrees of rotation of a conductor in a magnetic field?

Q18. State the left-hand rule used to determine the direction of current in a generator.

Q19. How is an ac voltage produced by an ac generator?

FREQUENCY

If the loop in the figure 1-8 (A) makes one complete revolution each second, the generator produces one complete cycle of ac during each second (1 Hz). Increasing the number of revolutions to two per second will produce two complete cycles of ac per second (2 Hz). The number of complete cycles of alternating current or voltage completed each second is referred to as the FREQUENCY. Frequency is always measured and expressed in hertz.

Alternating-current frequency is an important term to understand since most ac electrical equipments require a specific frequency for proper operation.

Q20. Define Frequency.

PERIOD

An individual cycle of any sine wave represents a definite amount of TIME. Notice that figure 1-10 shows 2 cycles of a sine wave which has a frequency of 2 hertz (Hz). Since 2 cycles occur each second, 1 cycle must require one-half second of time. The time required to complete one cycle of a waveform is called the PERIOD of the wave. In figure 1-10, the period is one-half second. The relationship between time (t) and frequency (f) is indicated by the formulas

$$t = \frac{1}{f} \quad \text{and} \quad f = \frac{1}{t}$$

where t = period in seconds and
f = frequency in hertz

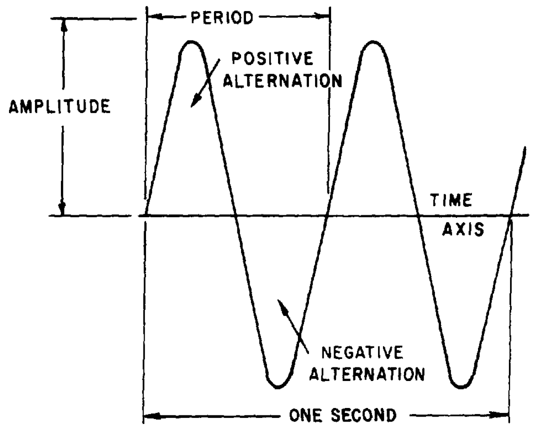


Figure 1-10.—Period of a sine wave.

Each cycle of the sine wave shown in figure 1-10 consists of two identically shaped variations in voltage. The variation which occurs during the time the voltage is positive is called the POSITIVE ALTERNATION. The variation which occurs during the time the voltage is negative is called the NEGATIVE ALTERNATION. In a sine wave, these two alternations are identical in size and shape, but opposite in polarity.

The distance from zero to the maximum value of each alternation is called the AMPLITUDE. The amplitude of the positive alternation and the amplitude of the negative alternation are the same.

WAVELENGTH

The time it takes for a sine wave to complete one cycle is defined as the period of the waveform. The distance traveled by the sine wave during this period is referred to as WAVELENGTH. Wavelength, indicated by the symbol λ (Greek lambda), is the distance along the waveform from one point to the same point on the next cycle. You can observe this relationship by examining figure 1-11. The point on the waveform that measurement of wavelength begins is not important as long as the distance is measured to the same point on the next cycle (see figure 1-12).

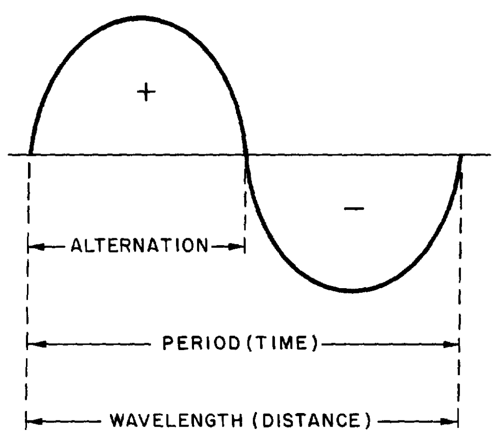


Figure 1-11.—Wavelength.

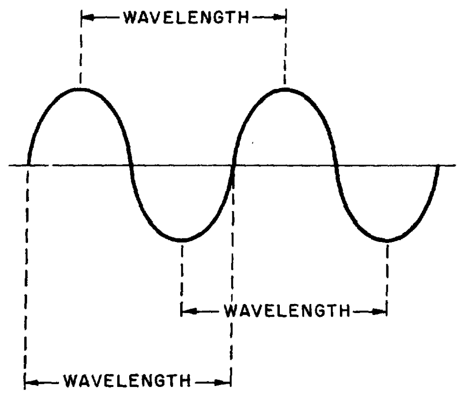


Figure 1-12.—Wavelength measurement.

Q21. What term is used to indicate the time of one complete cycle of a waveform?

Q22. What is a positive alternation?

Q23. What do the period and the wavelength of a sine wave measure, respectively?

ALTERNATING CURRENT VALUES

In discussing alternating current and voltage, you will often find it necessary to express the current and voltage in terms of MAXIMUM or PEAK values, PEAK-to-PEAK values, EFFECTIVE values, AVERAGE values, or INSTANTANEOUS values. Each of these values has a different meaning and is used to describe a different amount of current or voltage.

PEAK AND PEAK-TO-PEAK VALUES

Refer to figure 1-13. Notice it shows the positive alternation of a sine wave (a half-cycle of ac) and a dc waveform that occur simultaneously. Note that the dc starts and stops at the same moment as does the positive alternation, and that both waveforms rise to the same maximum value. However, the dc values are greater than the corresponding ac values at all points except the point at which the positive alternation passes through its maximum value. At this point the dc and ac values are equal. This point on the sine wave is referred to as the maximum or peak value.

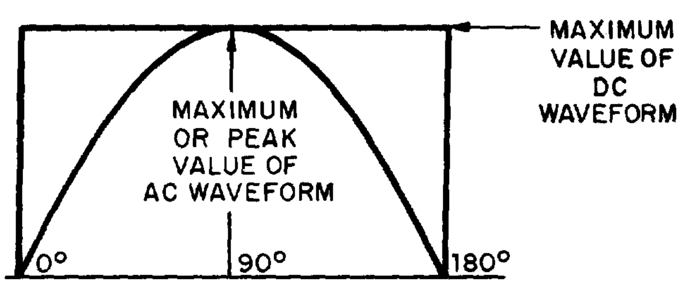


Figure 1-13.—Maximum or peak value.

During each complete cycle of ac there are always two maximum or peak values, one for the positive half-cycle and the other for the negative half-cycle. The difference between the peak positive value and the peak negative value is called the peak-to-peak value of the sine wave. This value is twice the maximum or peak value of the sine wave and is sometimes used for measurement of ac voltages. Note the difference between peak and peak-to-peak values in figure 1-14. Usually alternating voltage and current are expressed in EFFECTIVE VALUES (a term you will study later) rather than in peak-to-peak values.

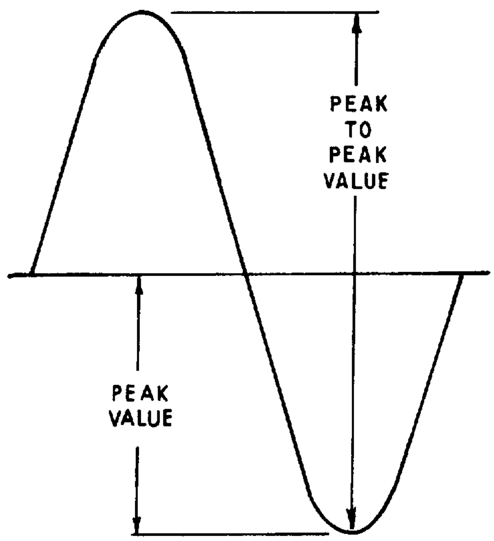


Figure 1-14.—Peak and peak-to-peak values.

Q24. What is meant by peak and peak-to-peak values of ac?

Q25. How many times is the maximum or peak value of emf or current reached during one cycle of ac?

INSTANTANEOUS VALUE

The INSTANTANEOUS value of an alternating voltage or current is the value of voltage or current at one particular instant. The value may be zero if the particular instant is the time in the cycle at which the polarity of the voltage is changing. It may also be the same as the peak value, if the selected instant is the time in the cycle at which the voltage or current stops increasing and starts decreasing. There are actually an infinite number of instantaneous values between zero and the peak value.

AVERAGE VALUE

The AVERAGE value of an alternating current or voltage is the average of ALL the INSTANTANEOUS values during ONE alternation. Since the voltage increases from zero to peak value and decreases back to zero during one alternation, the average value must be some value between those two limits. You could determine the average value by adding together a series of instantaneous values of the alternation (between 0º and 180º), and then dividing the sum by the number of instantaneous values used. The computation would show that one alternation of a sine wave has an average value equal to 0.636 times the peak value. The formula for average voltage is

$$E_{avg} = 0.636 \times E_{max}$$

where E_{avg} is the average voltage of one alternation, and E_{max} is the maximum or peak voltage. Similarly, the formula for average current is

$$I_{avg} = 0.636 \times I_{max}$$

where I_{avg} is the average current in one alternation, and I_{max} is the maximum or peak current.

Do not confuse the above definition of an average value with that of the average value of a complete cycle. Because the voltage is positive during one alternation and negative during the other alternation, the average value of the voltage values occurring during the complete cycle is zero.

Q26. If any point on a sine wave is selected at random and the value of the current or voltage is measured at that one particular moment, what value is being measured?

Q27. What value of current or voltage is computed by averaging all of the instantaneous values during the negative alternation of a sine wave?

Q28. What is the average value of all of the instantaneous currents or voltages occurring during one complete cycle of a sine wave?

Q29. What mathematical formulas are used to find the average value of current and average value of voltage of a sine wave?

Q30. If E_{max} is 115 volts, what is E_{avg}?

Q31. If I_{avg} is 1.272 ampere, what is I_{max}?

EFFECTIVE VALUE OF A SINE WAVE

E_{max}, E_{avg}, I_{max}, and I_{avg} are values used in ac measurements. Another value used is the EFFECTIVE value of ac This is the value of alternating voltage or current that will have the same effect on a resistance as a comparable value of direct voltage or current will have on the same resistance.

In an earlier discussion you were told that when current flows in a resistance, heat is produced. When direct current flows in a resistance, the amount of electrical power converted into heat equals I^2R watts. However, since an alternating current having a maximum value of 1 ampere does not maintain a constant value, the alternating current will not produce as much heat in the resistance as will a direct current of 1 ampere.

Figure 1-15 compares the heating effect of 1 ampere of dc to the heating effect of 1 ampere of ac.

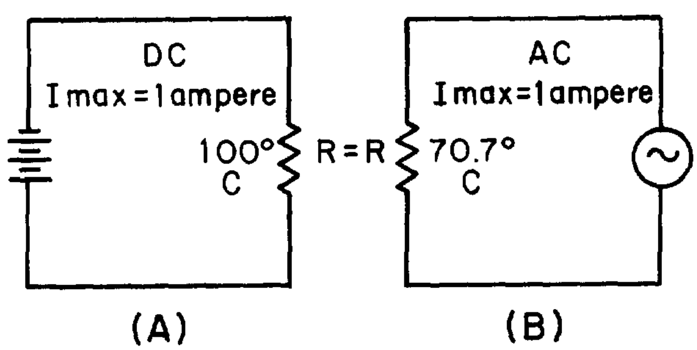


Figure 1-15.—Heating effect of ac and dc.

Examine views A and B of figure 1-15 and notice that the heat (70.7º C) produced by 1 ampere of alternating current (that is, an ac with a maximum value of 1 ampere) is only 70.7 percent of the heat (100º C) produced by 1 ampere of direct current. Mathematically,

$$\frac{\begin{array}{l}\text{The heating effect}\\ \text{of 1 maximum}\\ \text{a.c. ampere}\end{array}}{\begin{array}{l}\text{The heating effect}\\ \text{of 1 maximum}\\ \text{d.c. ampere}\end{array}} = \frac{70.7^\circ\,C}{100^\circ\,C} = 0.707$$

Therefore, for effective value of ac (I_{eff}) = $0.707 \times I_{max}$.

The rate at which heat is produced in a resistance forms a convenient basis for establishing an effective value of alternating current, and is known as the "heating effect" method. An alternating current is said to have an effective value of one ampere when it produces heat in a given resistance at the same rate as does one ampere of direct current.

You can compute the effective value of a sine wave of current to a fair degree of accuracy by taking equally-spaced instantaneous values of current along the curve and extracting the square root of the average of the sum of the squared values.

For this reason, the effective value is often called the "root-mean-square" (rms) value. Thus,

$$I_{eff} = \sqrt{\text{Average of the sum of the squares of } I_{inst}}.$$

Stated another way, the effective or rms value (I_{eff}) of a sine wave of current is 0.707 times the maximum value of current (I_{max}). Thus, $I_{eff} = 0.707 \times I_{max}$. When I_{eff} is known, you can find I_{max} by using the formula $I_{max} = 1.414 \times I_{eff}$. You might wonder where the constant 1.414 comes from. To find out, examine figure 1-15 again and read the following explanation. Assume that the dc in figure 1-15(A) is maintained at 1 ampere and the resistor temperature at 100º C. Also assume that the ac in figure 1-15(B) is increased until the temperature of the resistor is 100º C. At this point it is found that a maximum ac value of 1.414 amperes is required in order to have the same heating effect as direct current. Therefore, in the ac circuit the maximum current required is 1.414 times the effective current. It is important for you to remember the above relationship and that the effective value (I_{eff}) of any sine wave of current is always 0.707 times the maximum value (I_{max}).

Since alternating current is caused by an alternating voltage, the ratio of the effective value of voltage to the maximum value of voltage is the same as the ratio of the effective value of current to the maximum value of current. Stated another way, the effective or rms value (E_{eff}) of a sine-wave of voltage is 0.707 times the maximum value of voltage (E_{max}), Thus,

$$E_{eff} = \sqrt{\text{Average of the sum of the squares of } E_{inst}}$$

or,

$$E_{eff} = 0.707 \times E_{max}$$

and,

$$E_{max} = 1.414 \times E_{eff}$$

When an alternating current or voltage value is specified in a book or on a diagram, the value is an effective value unless there is a definite statement to the contrary. Remember that all meters, unless marked to the contrary, are calibrated to indicate effective values of current and voltage.

Problem: A circuit is known to have an alternating voltage of 120 volts and a peak or maximum current of 30 amperes. What are the peak voltage and effective current values?

Given:

$$E_s = 120 \text{ V}$$
$$E_{max} = 30 \text{ A}$$

Solution:

$$E_{max} = 1.414 \times E_{eff}$$
$$E_{max} = 1.414 \times 120 \text{ volts}$$
$$E_{max} = 169.68 \text{ volts}$$
$$I_{eff} = 0.707 \times I_{max}$$
$$I_{eff} = 0.707 \times 30 \text{ amperes}$$
$$I_{eff} = 21.21 \text{ amperes}$$

Figure 1-16 shows the relationship between the various values used to indicate sine-wave amplitude. Review the values in the figure to ensure you understand what each value indicates.

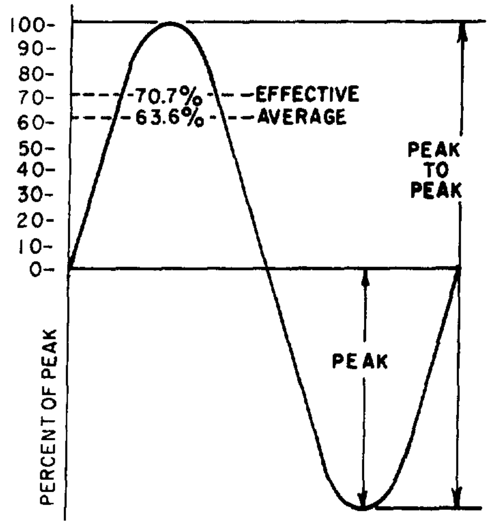


Figure 1-16.—Various values used to indicate sine-wave amplitude.

Q32. What is the most convenient basis for comparing alternating and direct voltages and currents?

Q33. What value of ac is used as a comparison to dc?

Q34. What is the formula for finding the effective value of an alternating current?

Q35. If the peak value of a sine wave is 1,000 volts, what is the effective (E_{eff}) value?

Q36. If I_{eff} = 4.25 ampere, what is I_{max}?

SINE WAVES IN PHASE

When a sine wave of voltage is applied to a resistance, the resulting current is also a sine wave. This follows Ohm's law which states that current is directly proportional to the applied voltage. Now examine figure 1-17. Notice that the sine wave of voltage and the resulting sine wave of current are superimposed on the same time axis. Notice also that as the voltage increases in a positive direction, the current increases along with it, and that when the voltage reverses direction, the current also reverses direction. When two sine waves, such as those represented by figure 1-17, are precisely in step with one another, they are said to be IN PHASE. To be in phase, the two sine waves must go through their maximum and minimum points at the same time and in the same direction.

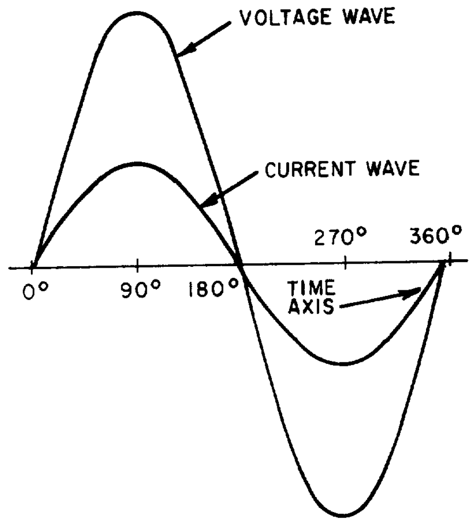


Figure 1-17.—Voltage and current waves in phase.

In some circuits, several sine waves can be in phase with each other. Thus, it is possible to have two or more voltage drops in phase with each other and also be in phase with the circuit current.

SINE WAVES OUT OF PHASE

Figure 1-18 shows voltage wave E_1 which is considered to start at 0º (time one). As voltage wave E_1 reaches its positive peak, voltage wave E_2 starts its rise (time two). Since these voltage waves do not go through their maximum and minimum points at the same instant of time, a PHASE DIFFERENCE exists between the two waves. The two waves are said to be OUT OF PHASE. For the two waves in figure 1-18 the phase difference is 90º .

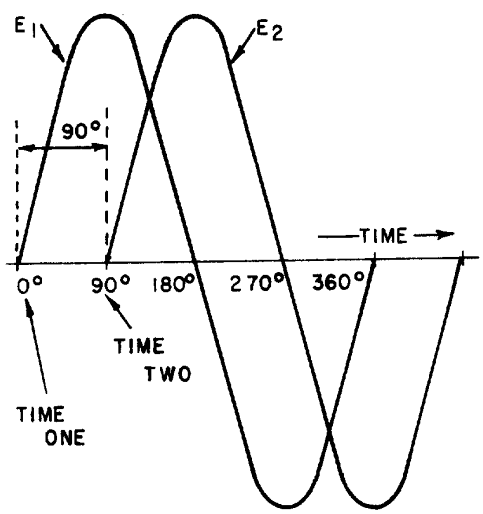


Figure 1-18.—Voltage waves 90º out of phase.

To further describe the phase relationship between two sine waves, the terms LEAD and LAG are used. The amount by which one sine wave leads or lags another sine wave is measured in degrees. Refer again to figure 1-18. Observe that wave E_2 starts 90º later in time than does wave E_1. You can also describe this relationship by saying that wave E_1 leads wave E_2 by 90º , or that wave E_2 lags wave E_1 by 90º . (Either statement is correct; it is the phase relationship between the two sine waves that is important.)

It is possible for one sine wave to lead or lag another sine wave by any number of degrees, except 0º or 360º . When the latter condition exists, the two waves are said to be in phase. Thus, two sine waves that differ in phase by 45º are actually out of phase with each other, whereas two sine waves that differ in phase by 360º are considered to be in phase with each other.

A phase relationship that is quite common is shown in figure 1-19. Notice that the two waves illustrated differ in phase by 180º . Notice also that although the waves pass through their maximum and minimum values at the same time, their instantaneous voltages are always of opposite polarity. If two such waves exist across the same component, and the waves are of equal amplitude, they cancel each other. When they have different amplitudes, the resultant wave has the same polarity as the larger wave and has an amplitude equal to the difference between the amplitudes of the two waves.

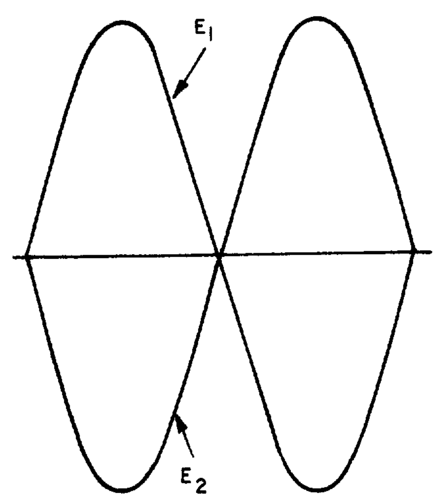


Figure 1-19.—Voltage waves 180º out of phase.

To determine the phase difference between two sine waves, locate the points on the time axis where the two waves cross the time axis traveling in the same direction. The number of degrees between the crossing points is the phase difference. The wave that crosses the axis at the later time (to the right on the time axis) is said to lag the other wave.

Q37. When are the voltage wave and the current wave in a circuit considered to be in phase?

Q38. When are two voltage waves considered to be out of phase?

Q39. What is the phase relationship between two voltage waves that differ in phase by 360° ?

Q40. How do you determine the phase difference between two sine waves that are plotted on the same graph?

OHM'S LAW IN AC CIRCUITS

Many ac circuits contain resistance only. The rules for these circuits are the same rules that apply to dc circuits. Resistors, lamps, and heating elements are examples of resistive elements. When an ac circuit contains only resistance, Ohm's Law, Kirchhoff's Law, and the various rules that apply to voltage, current, and power in a dc circuit also apply to the ac circuit. The Ohm's Law formula for an ac circuit can be stated as

$$I_{eff} = \frac{E_{eff}}{R} \text{ or } I = \frac{E}{R}$$

Remember, unless otherwise stated, all ac voltage and current values are given as effective values. The formula for Ohm's Law can also be stated as

$$I_{avg} = \frac{E_{avg}}{R} \text{ or } I_{max} = \frac{E_{max}}{R}$$

$$I_{peak-to-peak} = \frac{E_{peak-to-peak}}{R}$$

The important thing to keep in mind is: Do Not mix ac values. When you solve for effective values, all values you use in the formula must be effective values. Similarly, when you solve for average values, all values you use must be average values. This point should be clearer after you work the following problem: A series circuit consists of two resistors (R1 = 5 ohms and R2 = 15 ohms) and an alternating voltage source of 120 volts. What is I_{avg}?

Given: R1 = 5 ohms

R2 = 15 ohms

E_s = 120 ohms

Solution: First solve for total resistance R_T.

$$R_T = R1 + R2$$

$$R_T = 5 \text{ ohms} + 15 \text{ ohms}$$

$$R_T = 20 \text{ ohms}$$

The alternating voltage is assumed to be an effective value (since it is not specified to be otherwise). Apply the Ohm's Law formula.

$$I_{eff} = \frac{E_{eff}}{R}$$
$$I_{eff} = \frac{120\text{ volts}}{20\text{ ohms}}$$
$$I_{eff} = 6\text{ amperes}$$

The problem, however, asked for the average value of current (I_{avg}). To convert the effective value of current to the average value of current, you must first determine the peak or maximum value of current, I_{max}.

$$I_{max} = 1.414 \times I_{eff}$$
$$I_{max} = 1.414 \times 6\text{ amperes}$$
$$I_{max} = 8.484\text{ amperes}$$

You can now find I_{avg}. Just substitute 8.484 amperes in the I_{avg} formula and solve for I_{avg}.

$$I_{avg} = 0.636 \times I_{max}$$
$$I_{avg} = 0.636 \times 8.484\text{ amperes}$$
$$I_{avg} = 5.4\text{ amperes (rounded off to one decimal place)}$$

Remember, you can use the Ohm's Law formulas to solve any purely resistive ac circuit problem. Use the formulas in the same manner as you would to solve a dc circuit problem.

Q41. A series circuit consists of three resistors (R1 = 10Ω, R2 = 20Ω, R3 = 15Ω) and an alternating voltage source of 100 volts. What is the effective value of current in the circuit?

Q42. If the alternating source in Q41 is changed to 200 volts peak-to-peak, what is I_{avg}?

Q43. If E_{eff} is 130 volts and I_{eff} is 3 amperes, what is the total resistance (R_T) in the circuit?

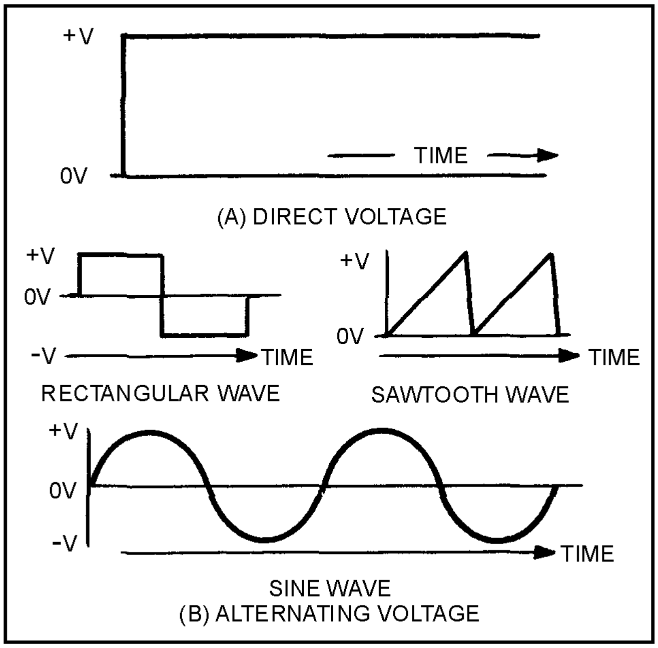
SUMMARY

Before going on to chapter 2, read the following summary of the material in chapter 1. This summary will reinforce what you have already learned.

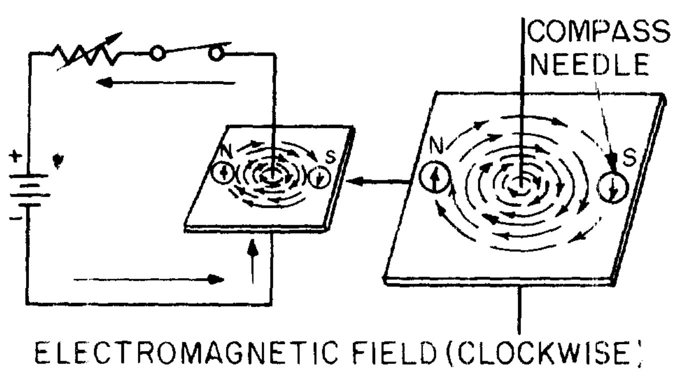
DC AND AC—Direct current flows in one direction only, while alternating current is constantly changing in amplitude and direction.

ADVANTAGES AND DISADVANTAGES OF AC AND DC—Direct current has several disadvantages compared to alternating current. Direct current, for example, must be generated at the voltage level required by the load. Alternating current, however, can be generated at a high level and stepped down at the consumer end (through the use of a transformer) to whatever voltage level is required by the load. Since power in a dc system must be transmitted at low voltage and high current levels, the I^2R power loss becomes a problem in the dc system. Since power in an ac system can be transmitted at a high voltage level and a low current level, the I^2R power loss in the ac system is much less than that in the dc system.

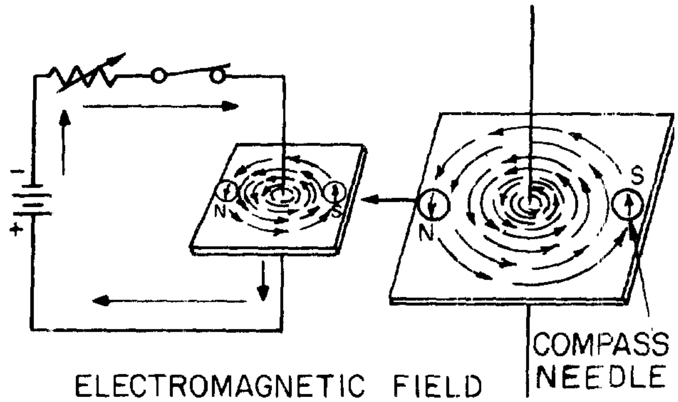
VOLTAGE WAVEFORMS—The waveform of voltage or current is a graphical picture of changes in voltage or current values over a period of time.



ELECTROMAGNETISM—When a compass is placed in the vicinity of a current-carrying conductor, the needle aligns itself at right angles to the conductor. The north pole of the compass indicates the direction of the magnetic field produced by the current. By knowing the direction of current, you can use the left-hand rule for conductors to determine the direction of the magnetic lines of force.



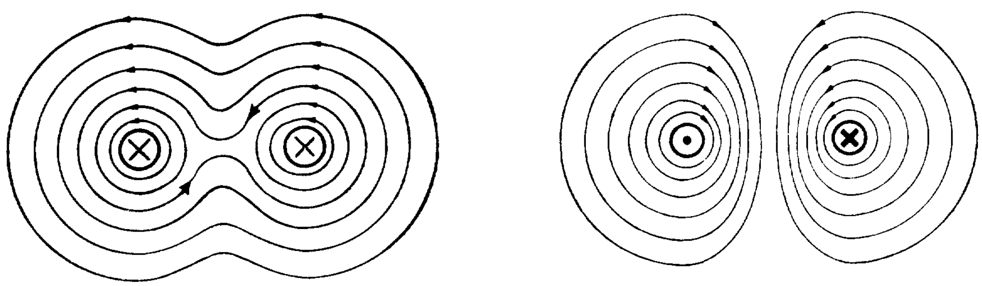
ELECTROMAGNETIC FIELD (CLOCKWISE)
(A)



ELECTROMAGNETIC FIELD
(COUNTERCLOCKWISE)
(B)

Arrows are generally used in electrical diagrams to indicate the direction of current in a wire. A cross (+) on the end of a cross-sectional view of a wire indicates that current is flowing away from you, while a dot (·) indicates that current is flowing toward you.

When two adjacent parallel conductors carry current in the same direction, the magnetic fields around the conductors aid each other. When the currents in the two conductors flow in opposite directions, the fields around the conductors oppose each other.



CURRENT FLOWING IN THE SAME DIRECTION (A)

CURRENT FLOWING IN OPPOSITE DIRECTION (B)

<u>MAGNETIC FIELD OF A COIL</u>—When wire is wound around a core, it forms a COIL. The magnetic fields produced when current flows in the coil combine. The combined influence of all of the fields around the turns produce a two-pole field similar to that of a simple bar magnet.

When the direction of current in the coil is reversed, the polarity of the two-pole field of the coil is reversed.

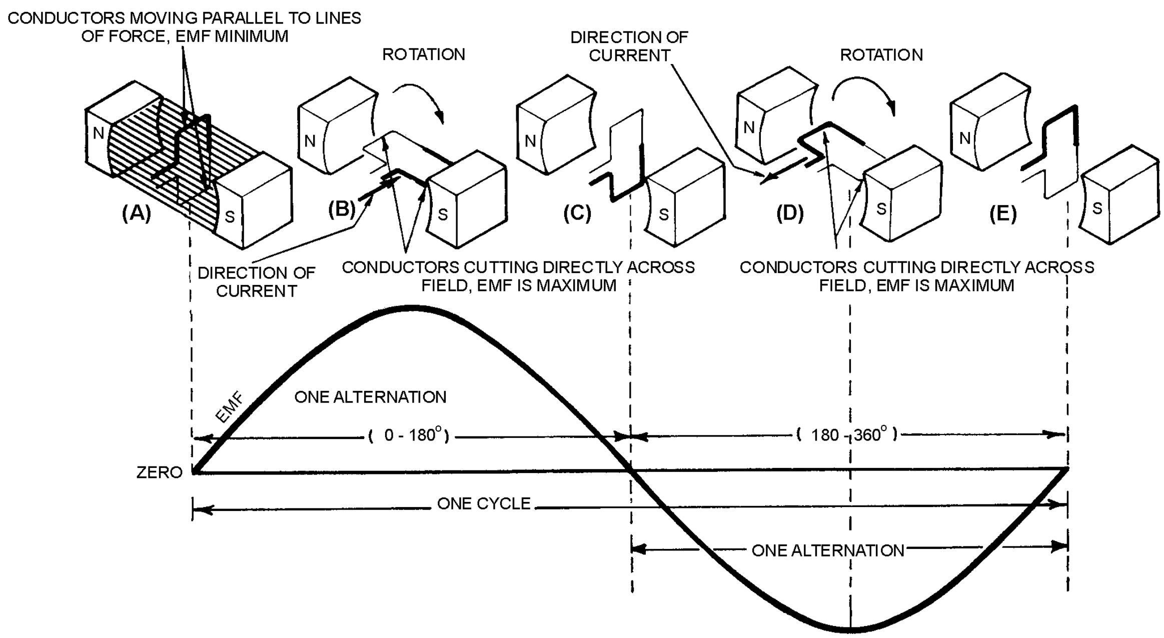
The strength of the magnetic field of the coil is dependent upon:

- The number of turns of the wire in the coil.
- The amount of current in the coil.
- The ratio of the coil length to the coil width.

- The type of material in the core.

BASIC AC GENERATION—When a conductor is in a magnetic field and either the field or the conductor moves, an emf (voltage) is induced in the conductor. This effect is called electromagnetic induction.

A loop of wire rotating in a magnetic field produces a voltage which constantly changes in amplitude and direction. The waveform produced is called a sine wave and is a graphical picture of alternating current (ac). One complete revolution (360º) of the conductor produces one cycle of ac. The cycle is composed of two alternations: a positive alternation and a negative alternation. One cycle of ac in one second is equal to 1 hertz (1 Hz).



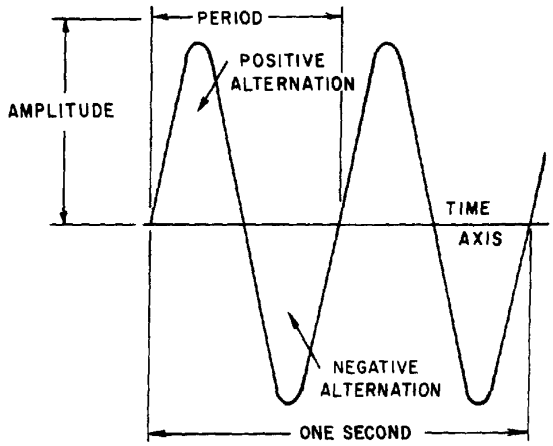
FREQUENCY—The number of cycles of ac per second is referred to as the FREQUENCY. AC frequency is measured in hertz. Most ac equipment is rated by frequency as well as by voltage and current.

PERIOD—The time required to complete one cycle of a waveform is called the PERIOD OF THE WAVE.

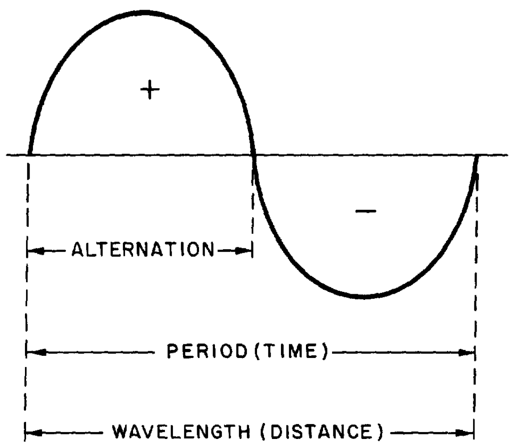
Each ac sine wave is composed of two alternations. The alternation which occurs during the time the sine wave is positive is called the positive alternation. The alternation which occurs during the time the sine wave is negative is called the negative alternation. In each cycle of sine wave, the two alternations are identical in size and shape, but opposite in polarity.

The period of a sine wave is inversely proportional to the frequency; e.g., the higher the frequency, the shorter the period. The mathematical relationships between time and frequency are

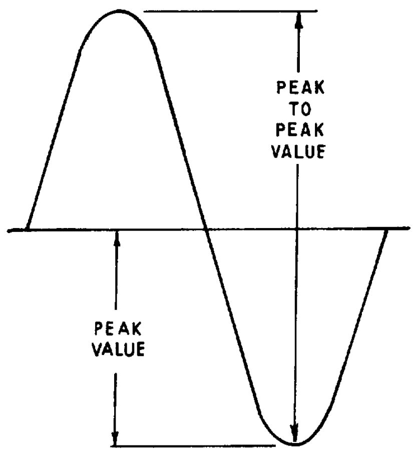
$$t = \frac{1}{f} \text{ and } f = \frac{1}{t}$$



WAVELENGTH—The period of a sine wave is defined as the time it takes to complete one cycle. The distance the waveform covers during this period is referred to as the wavelength. Wavelength is indicated by lambda (λ) and is measured from a point on a given waveform (sine wave) to the corresponding point on the next waveform.



PEAK AND PEAK-TO-PEAK VALUES—The maximum value reached during one alternation of a sine wave is the peak value. The maximum reached during the positive alternation to the maximum value reached during the negative alternation is the peak-to-peak value. The peak-to-peak value is twice the peak value.



INSTANTANEOUS VALUE—The instantaneous value of a sine wave of alternating voltage or current is the value of voltage or current at one particular instant of time. There are an infinite number of instantaneous values between zero and the peak value.

AVERAGE VALUE—The average value of a sine wave of voltage or current is the average of all the instantaneous values during one alternation. The average value is equal to 0.636 of the peak value. The formulas for average voltage and average current are:

$$E_{avg} = 0.636 \times E_{max}$$
$$I_{avg} = 0.636 \times I_{max}$$

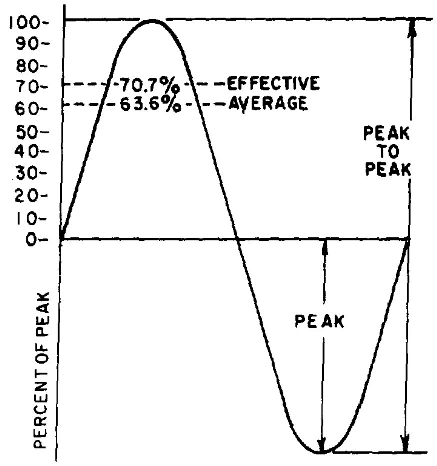
Remember: The average value (E_{avg} or I_{avg}) is for one alternation only. The average value of a complete sine wave is zero.

EFFECTIVE VALUE—The effective value of an alternating current or voltage is the value of alternating current or voltage that produces the same amount of heat in a resistive component that would be produced in the same component by a direct current or voltage of the same value. The effective value of a sine wave is equal to 0.707 times the peak value. The effective value is also called the root mean square or rms value.

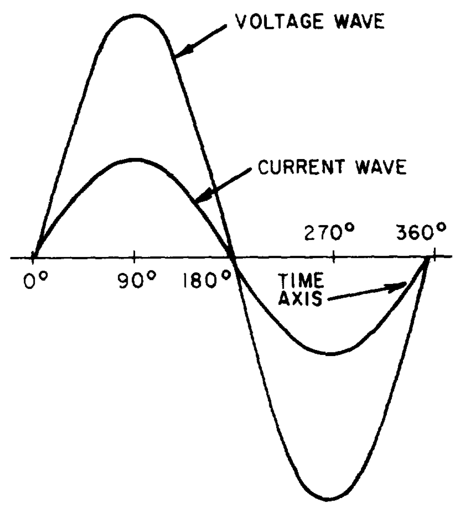
The term rms value is used to describe the process of determining the effective value of a sine wave by using the instantaneous value of voltage or current. You can find the rms value of a current or voltage by taking equally spaced instantaneous values on the sine wave and extracting the square root of the average of the sum of the instantaneous values. This is where the term "Root-Mean-Square" (rms) value comes from.

The formulas for effective and maximum values of voltage and current are:

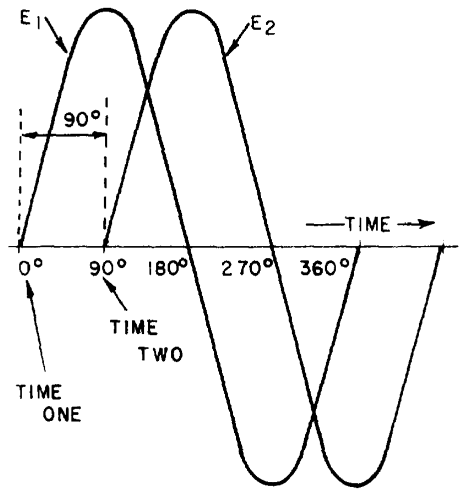
$$E_{eff} = 0.707 \times E_{max}$$
$$E_{max} = 1.414 \times E_{eff}$$
$$I_{eff} = 0.707 \times I_{max}$$
$$I_{max} = 1.414 \times I_{eff}$$



SINE WAVES IN PHASE—When two sine waves are exactly in step with each other, they are said to be in phase. To be in phase, both sine waves must go through their minimum and maximum points at the same time and in the same direction.



SINE WAVES OUT OF PHASE—When two sine waves go through their minimum and maximum points at different times, a phase difference exists between them. The two waves are said to be out of phase with each other. To describe this phase difference, the terms lead and lag are used. The wave that reaches its minimum (or maximum) value first is said to lead the other wave. The term lag is used to describe the wave that reaches its minimum (or maximum) value some time after the first wave does. When a sine wave is described as leading or lagging, the difference in degrees is usually stated. For example, wave E_1 leads wave E_2 by 90° , or wave E_2 lags wave E_1 by 90° . Remember: Two sine waves can differ by any number of degrees except 0° and 360° . Two sine waves that differ by 0° or by 360° are considered to be in phase. Two sine waves that are opposite in polarity and that differ by 180° are said to be out of phase, even though they go through their minimum and maximum points at the same time.



<u>OHM'S LAW IN AC CIRCUIT</u>—All dc rules and laws apply to an ac circuit that contains <u>only resistance</u>. The important point to remember is: Do not <u>mix ac values</u>. Ohm's Law formulas for ac circuits are given below:

$$I = \frac{E}{R}$$

$$I_{eff} = \frac{E_{eff}}{R}$$

$$I_{avg} = \frac{E_{avg}}{R}$$

$$I_{max} = \frac{E_{max}}{R}$$

$$I_{Peak\text{-}to\text{-}Peak} = \frac{E_{Peak\text{-}to\text{-}Peak}}{R}$$

ANSWERS TO QUESTIONS Q1. THROUGH Q43.

A1. *An electrical current which flows in one direction only.*

A2. *An electrical current which is constantly varying in amplitude, and which changes direction at regular intervals.*

A3. *The dc voltage must be generated at the level required by the load.*

A4. *The I^2R power loss is excessive.*

A5. *Alternating current (ac).*

A6. *The needle aligns itself at right angles to the conductor.*

A7. *(a) clockwise (b) counterclockwise.*

A8. *It is used to determine the relation between the direction of the magnetic lines of force around a conductor and the direction of current through the conductor.*

A9. *The north pole of the compass will point in the direction of the magnetic lines of force.*

A10. *It combines with the other field.*

A11. *It deforms the other field.*

A12. *(a) The field consists of concentric circles in a plane perpendicular to the wire (b) the field of each turn of wire links with the fields of adjacent turns producing a two-pole field similar in shape to that of a simple bar magnet.*

A13. *The polarity of the two-pole field reverses.*

A14. *Use the left-hand rule for coils.*

A15. *Grasp the coil in your left hand, with your fingers "wrapped around" in the direction of electron flow. The thumb will point toward the north pole.*

A16. *(a) When the conductors are cutting directly across the magnetic lines of force (at the 90º and 270º points). (b) When the conductors are moving parallel to the magnetic lines of force (at the 0°, 180°, and 360° points).*

A17. *360°.*

A18. *Extend your left hand so that your thumb points in the direction of conductor movement, and your forefinger points in the direction of the magnetic flux (north to south). Now point your middle finger 90° from the forefinger and it will point in the direction of electron current flow in the conductor.*

A19. *Continuous rotation of the conductor through magnetic fines of force produces a series of cycles of alternating voltage or, in other words, an alternating voltage or a sine wave of voltage.*

A20. *Frequency is the number of complete cycles of alternating voltage or current completed each second.*

A21. Period.

A22. A positive alternation is the positive variation in the voltage or current of a sine curve.

A23. The period measures time and the wavelength measures distance.

A24. The peak value is the maximum value of one alternation; the peak-to-peak value is twice the maximum or peak value.

A25. Twice.

A26. The instantaneous value (E_{inst} or I_{inst})

A27. Average value (E_{avg} or I_{avg})

A28. Zero

A29.

$$I_{avg} = 0.636 \times I_{max}$$

$$E_{avg} = 0.636 \times E_{max}$$

A30.

$$E_{avg} = 0.636 \times 115 \text{ volts}$$

$$E_{avg} = 73.14 \text{ volts}$$

A31.

$$\text{If } I_{avg} = I_{max} \times 0.636, \text{ then } I_{max} = \frac{I_{avg}}{0.636}$$

Thus,

$$I_{max} = \frac{1.272}{0.636} \text{ ampere} = 2 \text{ amperes}$$

A32. The power (heat) produced in a resistance by a dc voltage is compared to that produced in the same resistance by an ac voltage of the same peak amplitude.

A33. The effective value.

A34.

$$I_{eff} = 0.707 \times I_{max}$$

A35.

$$
\begin{aligned}
E_{eff} &= 0.707 \times E_{max} \\
&= 0.707 \times E_{max} \\
&= 0.707 \times 1{,}000 \text{ volts} \\
E_{eff} &= 707 \text{ volts}
\end{aligned}
$$

A36.

$$
\begin{aligned}
I_{max} &= 1.414 \times I_{eff} \\
&= 1.414 \times 4.25 \text{ amperes} \\
&= 6 \text{ amperes.}
\end{aligned}
$$

(Remember: Unless specified otherwise, the voltage or current value is always considered to be the effective value.)

A37. When the two waves go through their maximum and minimum points at the same time and in the same direction.

A38. When the waves do not go through their maximum and minimum points at the same time, a PHASE DIFFERENCE exists, and the two waves are said to be out of phase. (Two waves are also considered to be out of phase if they differ in phase by 180° and their instantaneous voltages are always of opposite polarity, even though both waves go through their maximum and minimum points at the same time).

A39. They are in phase with each other.

A40. Locate the points on the time axis where the two waves cross traveling in the same direction. The number of degrees between these two points is the phase difference.

A41.

$$
I_{eff} = \frac{100}{45} = 2.22 \text{ ampers}
$$

A42. $I_{avg} = 0.636 \times I_{max} = 1.41$ *amperes.*

A43. 43.3 ohms.

Chapter 31 – Alternating Current

- Phasors and Alternating Currents

- Resistance and Reactance

- Magnetic-Field Energy

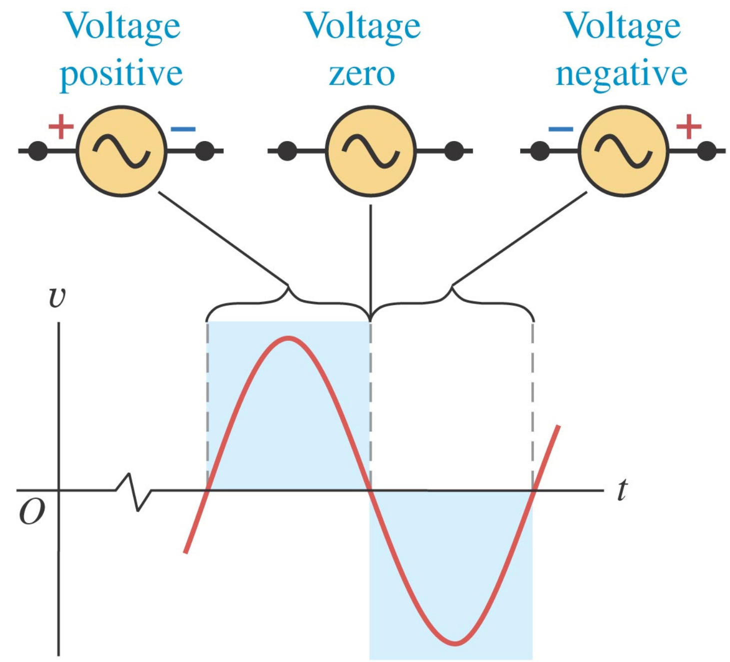
- The L-R-C Series Circuit

- Power in Alternating-Current Circuits

- Resonance in Alternating-Current Circuits

- Transformers

1. Phasors and Alternating Currents



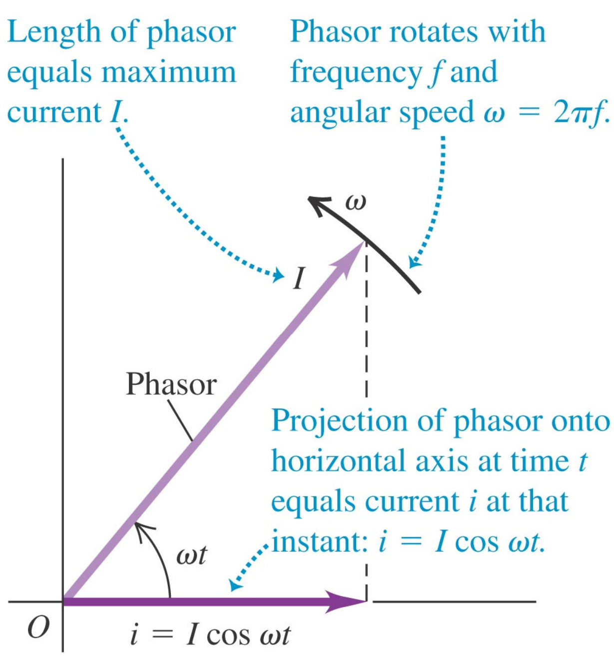
Ex. source of ac: coil of wire rotating with constant ω in a magnetic field → sinusoidal alternating emf.

$$v = V \cos \omega t$$

$$i = I \cos \omega t$$

v, i = instantaneous potential difference / current.
V, I = maximum potential difference / current → voltage/current amplitude.

$\omega = 2\pi f$



Phasor Diagrams

- Represent sinusoidally varying voltages / currents through the projection of a vector, with length equal to the amplitude, onto a horizontal axis.

- Phasor: vector that rotates counterclockwise with constant ω.

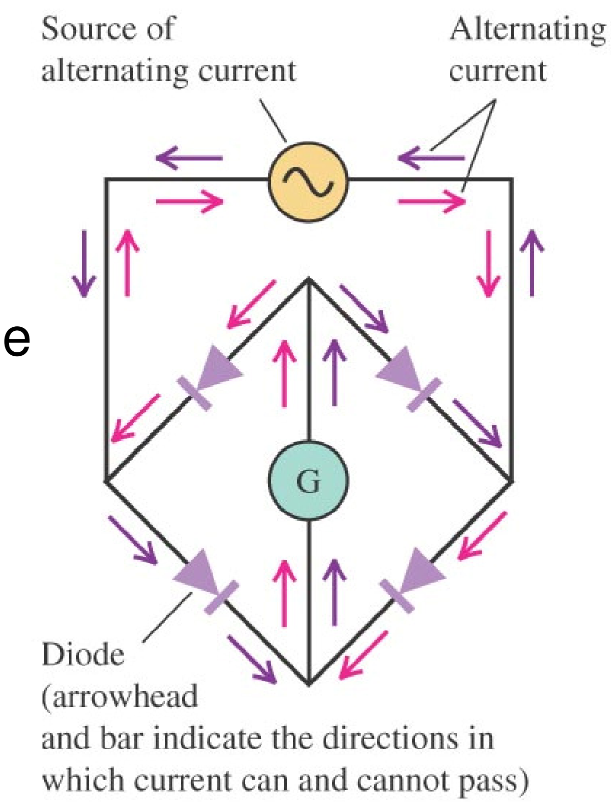
- Diode (rectifier): device that conducts better in one direction than in the other. If ideal, R = 0 in one direction and R = ∞ in other.

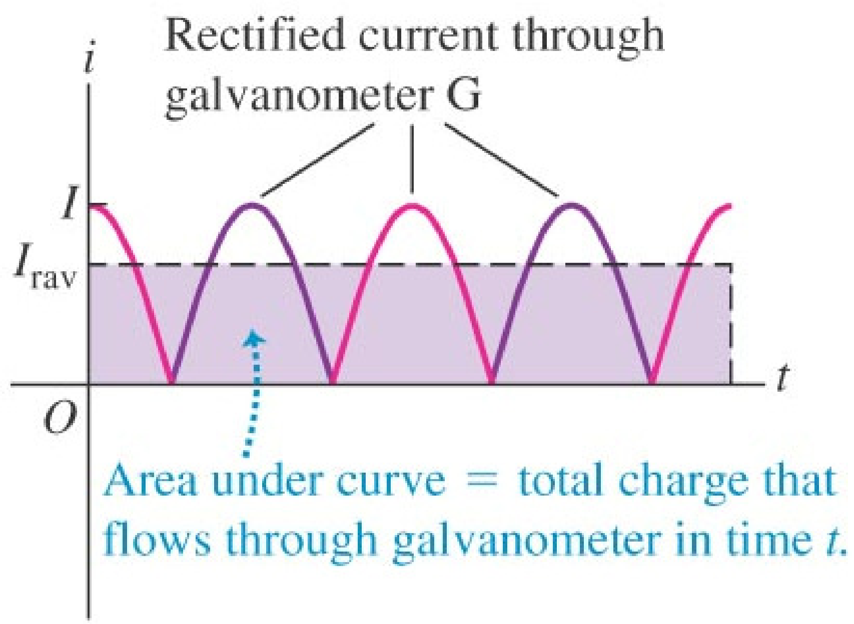
Rectified average current (I_{rav}): during any whole number of cycles, the total charge that flows is same as if current were constant (I_{rav}).

$$i_{rav} = \frac{2}{\pi} I$$

average value of Icos ωtI or Isin ωtI

full wave rectifier circuit





Root-Mean Square (rms) values:

$$i_{rms} = \sqrt{(i^2)_{av}} = \frac{I}{\sqrt{2}}$$

$$V_{rms} = \frac{V}{\sqrt{2}}$$

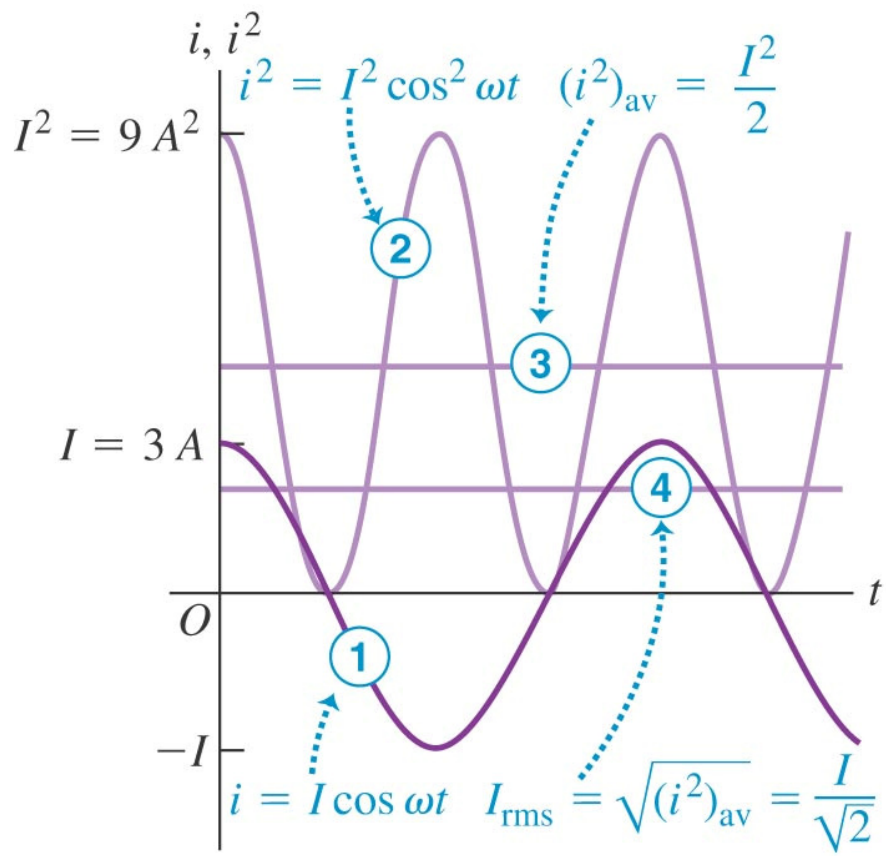
Meaning of the rms value of a sinusoidal quantity (here, ac current with $I = 3$ A):

1. Graph current i versus time.
2. *Square* the instantaneous current i.
3. Take the *average* (mean) value of i^2.
4. Take the *square root* of that average.

$$i^2 = I^2 \cos^2 \omega t$$

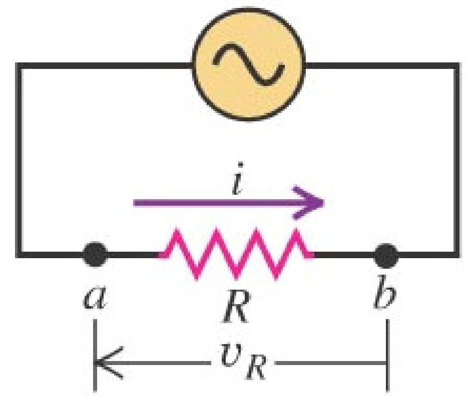
$$\cos^2 \omega t = 0.5 \cdot (1 + \cos 2\omega t)$$

$$i^2 = 0.5I^2 + 0.5I^2 \cos(2\omega t)$$



2. Resistance and Reactance

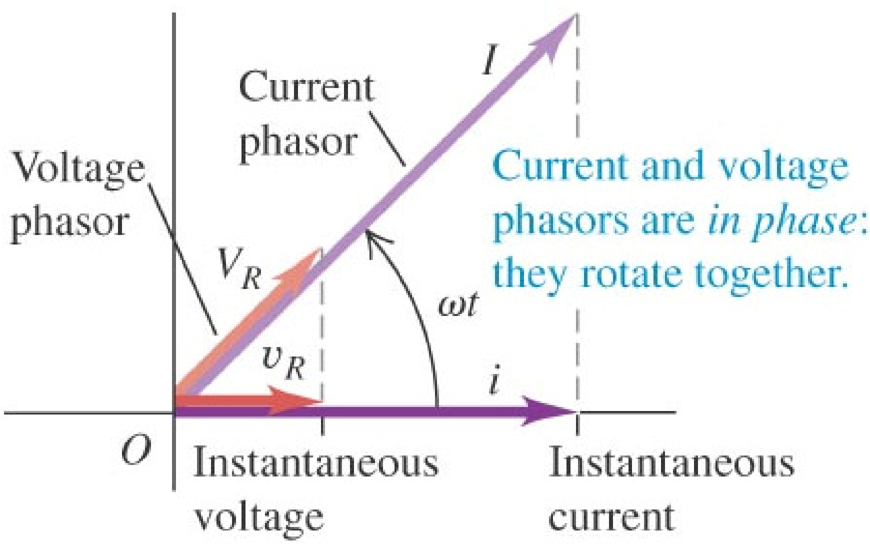
Resistor in an ac circuit

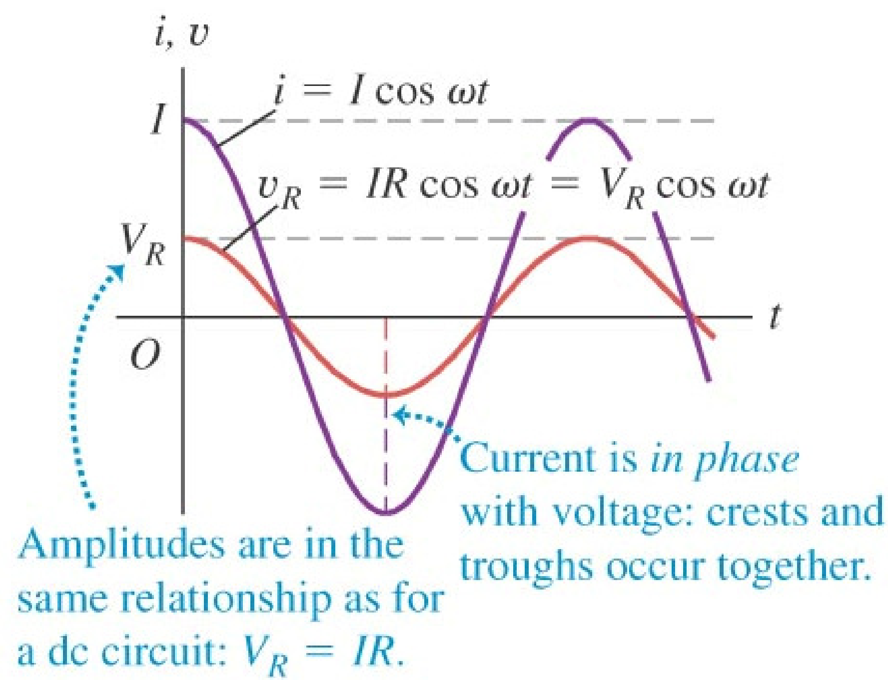


$$v_R = iR = (IR)\cos\omega t = V_R \cos\omega t$$ (instantaneous potential)

$$V_R = IR$$ (amplitude –max- of voltage across R)

- Current in phase with voltage → phasors rotate together



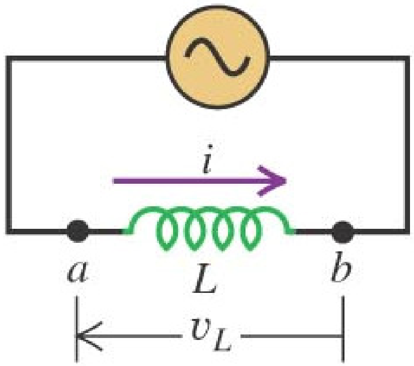


Inductor in an ac Circuit

- Current varies with time → self-induced emf → di/dt > 0 → ε < 0

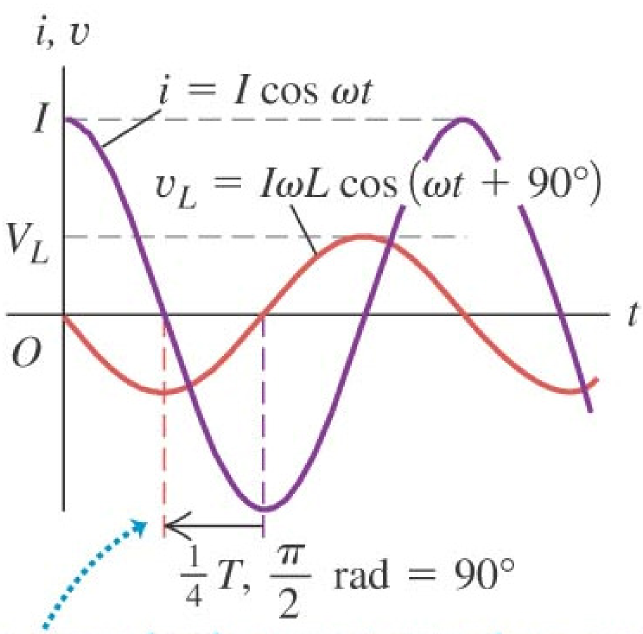
$$\varepsilon = -L\frac{di}{dt}$$

$V_a > V_b$ → $V_{ab} = V_a - V_b = V_L = L\,di/dt > 0$



$$v_L = L\frac{di}{dt} = L\frac{d}{dt}(I\cos\omega t)$$

$$\boxed{v_L = -I\omega L\sin\omega t = I\omega L\cos\left(\omega t + 90^\circ\right)}$$



Voltage curve *leads* current curve by a quarter-cycle (corresponding to $\phi = \pi/2$ rad = 90°).

v_L has 90° "head start" with respect to i.

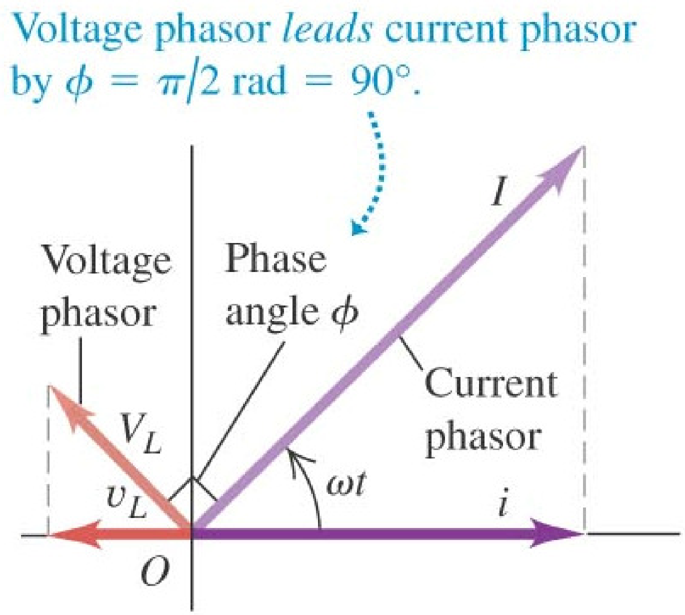
Inductor in an ac circuit

$$i = I\cos\omega t$$

$$v_L = \boxed{I\omega L}\cos(\omega t + 90^\circ)$$

V_L

$$v = V\cos(\omega t + \varphi)$$



φ = phase angle = phase of voltage relative to current

Pure resistor: φ = 0

Pure inductor: φ = 90º

Inductive reactance: $\boxed{X_L = \omega L}$

$$I = \frac{V_L}{\omega L}$$

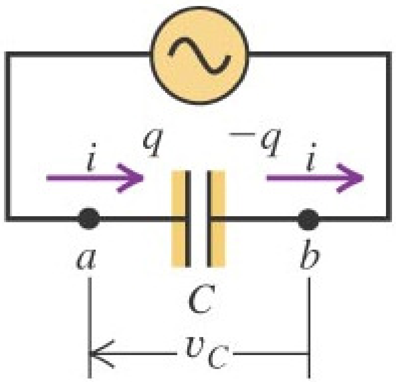
High ω → low I
Low ω → high I

Voltage amplitude: $\boxed{V_L = IX_L = I\omega L}$

Inductors used to block high ω

Capacitor in an ac circuit

As the capacitor charges and discharges → at each t, there is "i" in each plate, and equal displacement current between the plates, as though charge was conducted through C.



$$i = \frac{dq}{dt} = I\cos\omega t \quad \rightarrow \int dq = \int I\cos\omega t\, dt$$

$C = q / v_C$

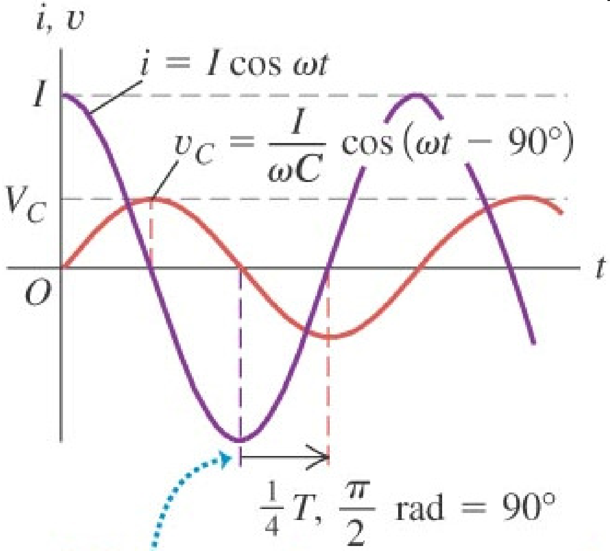
$$q = \frac{I}{\omega}\sin\omega t$$

$$v_c = \frac{q}{C} = \frac{I}{\omega C}\sin\omega t = \frac{I}{\omega C}\cos(\omega t - 90^\circ)$$

$$V_C = \frac{I}{\omega C}$$

Pure capacitor: φ = 90º

v_c lags current by 90º.



Voltage curve *lags* current curve by a quarter-cycle (corresponding to $\phi = \pi/2$ rad = 90°).

Capacitive reactance: $X_C = \dfrac{1}{\omega C}$

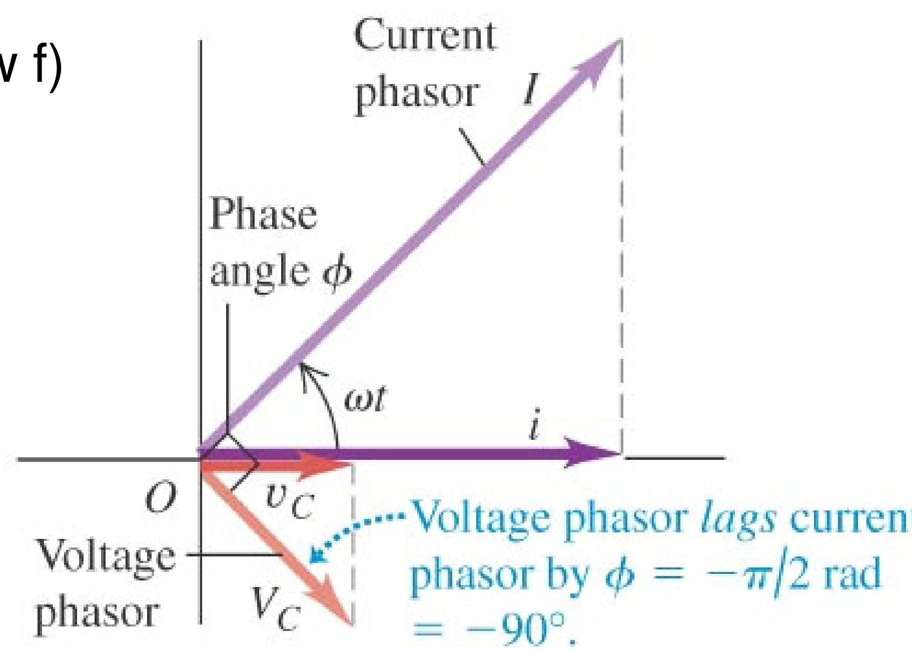
$V_C = IX_C$ (amplitude of voltage across C)

$I = V_C \omega C$

High ω → high I
Low ω → low I

Capacitor in an ac circuit

Capacitors used to block low ω (or low f)
→ high-pass filter



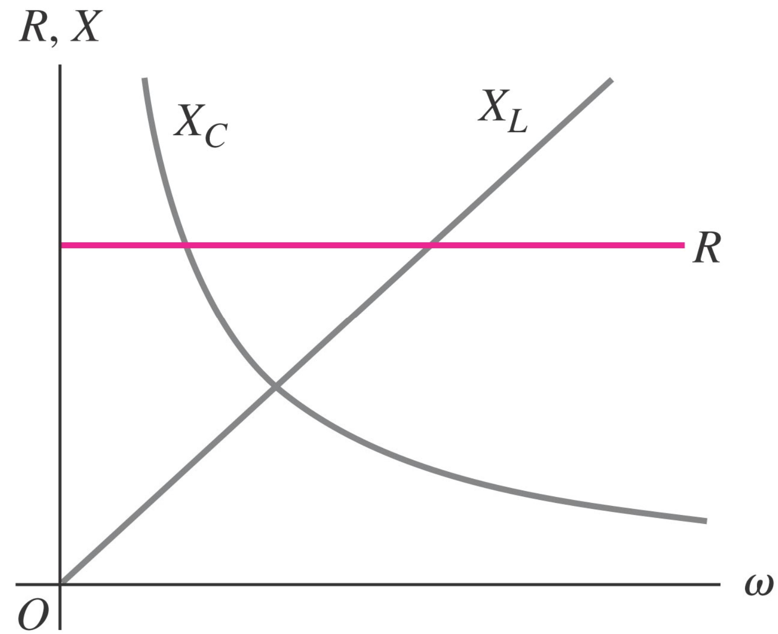
Comparing ac circuit elements:

- R is independent of ω.
- X_L and X_C depend on ω.
- If ω = 0 (dc circuit) → $X_c = 1/\omega C \rightarrow \infty$
  → $i_c = 0$

  $X_L = \omega L = 0$

- If ω → ∞, $X_L \rightarrow \infty \rightarrow i_L = 0$

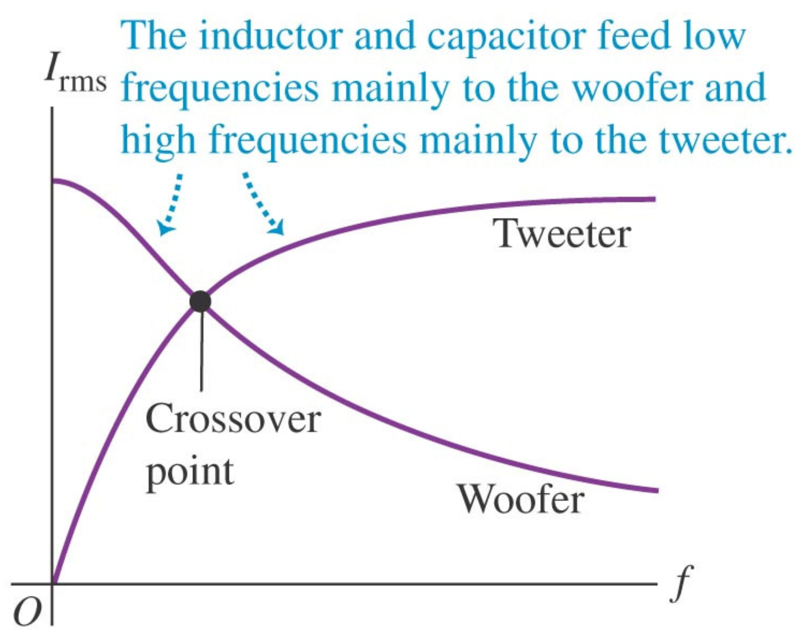
  $X_C = 0 \rightarrow V_C = 0$ → current changes direction so rapidly that no charge can build up on each plate.



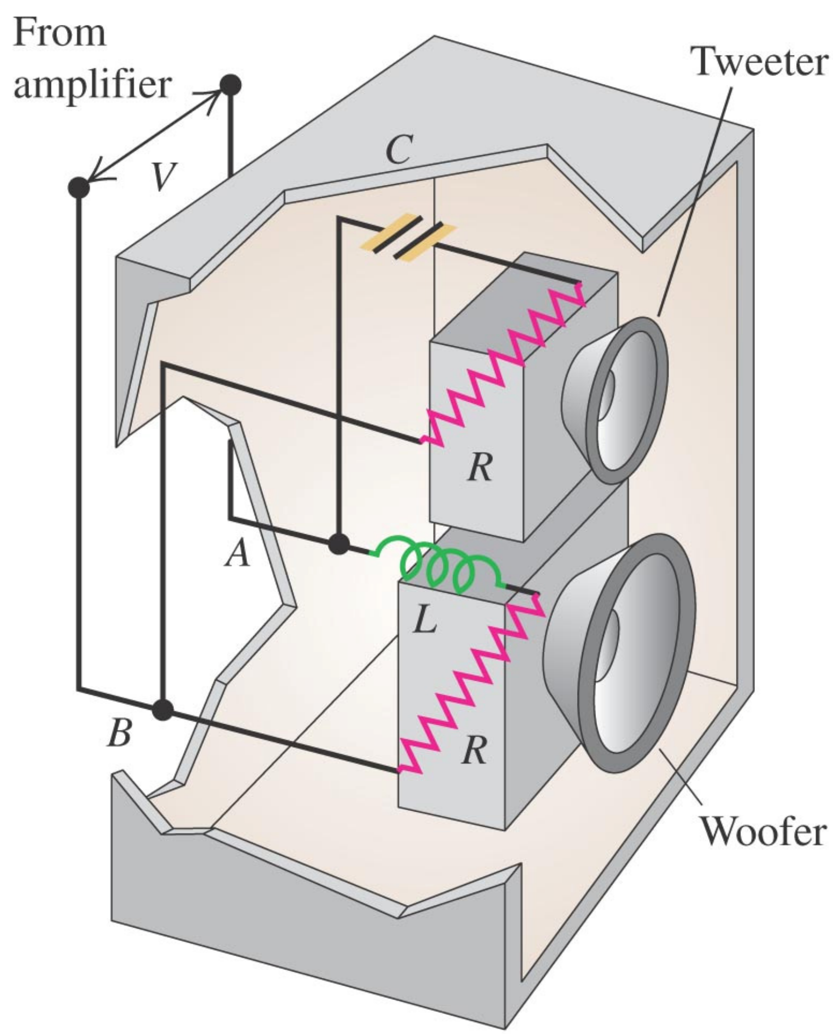
Circuit Element	Amplitude Relationship	Circuit Quantity	Phase of *v*
Resistor	$V_R = IR$	R	In phase with *i*
Inductor	$V_L = IX_L$	$X_L = \omega L$	Leads *i* by 90°
Capacitor	$V_C = IX_C$	$X_C = 1/\omega C$	Lags *i* by 90°

Example: amplifier → C in tweeter branch blocks low-f components of sound but passes high-f; L in woofer branch does the opposite.

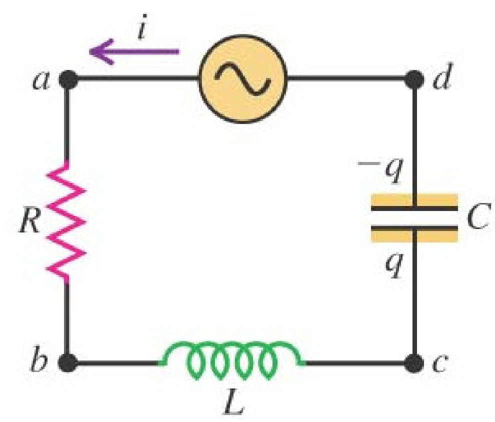
Graphs of rms current as functions of frequency for a given amplifier voltage



A crossover network in a loudspeaker system



3. The L-R-C Series Circuit

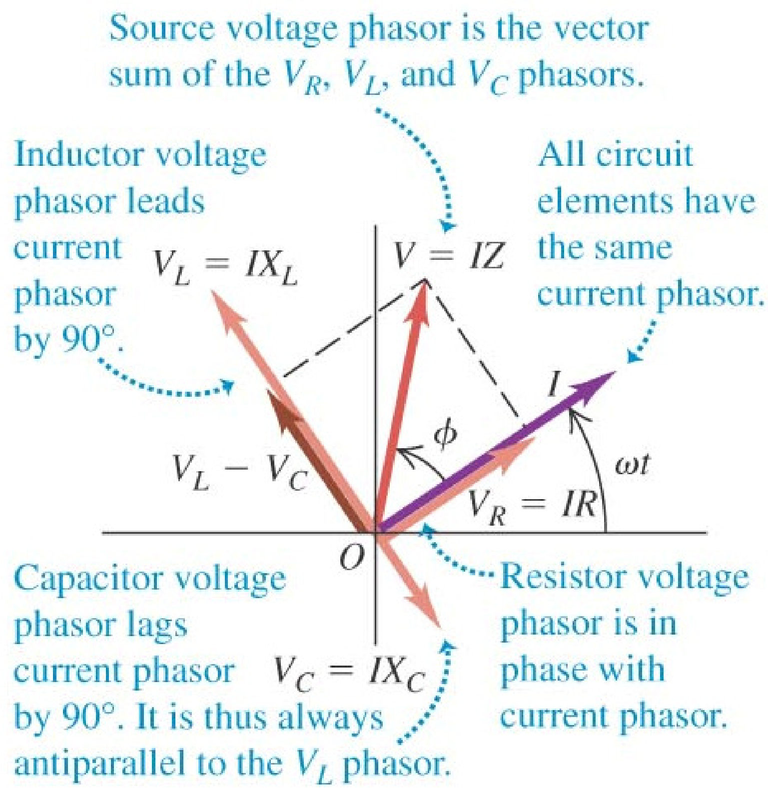


- Instantaneous v across L, C, R = v_{ad} = v source

- Total voltage phasor = vector sum of phasors of individual voltages.

- C, R, L in series → same current, i = I cosωt → only one phasor (I) for three circuit elements, amplitude I.

- The projections of I and V phasors onto horizontal axis at t give rise to instantaneous i and v.



$$V_C = IR$$

$$V_L = IX_L$$

$$V_C = IX_C$$

(amplitudes = maximum values)

-The instantaneous potential difference between terminals a,d =

= algebraic sum of v_R, v_C, v_L (instantaneous voltages) =

= sum of projections of phasors V_R, V_C, V_L

= projection of their vector sum (V) that represents the source voltage v and instantaneous voltage v_{ad} across series of elements.

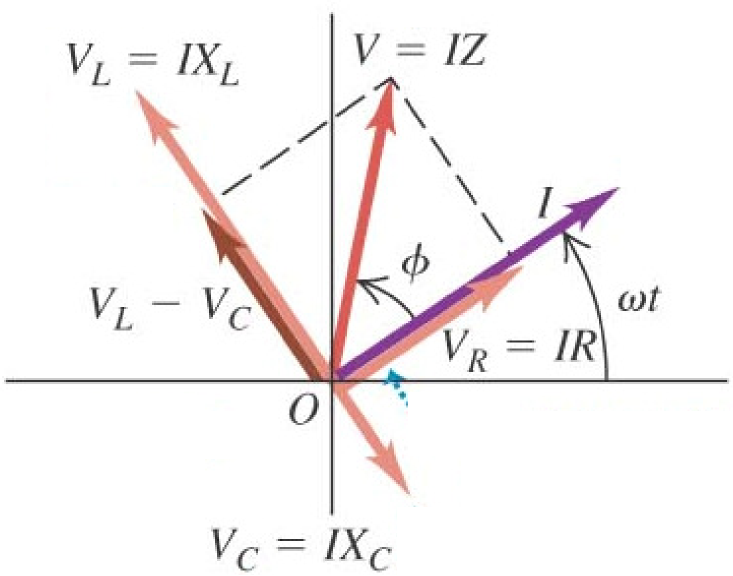
$$V = \sqrt{V_R^2 + (V_L - V_c)^2} = \sqrt{(IR)^2 + (IX_L - IX_c)^2} = I\sqrt{R^2 + (X_L - X_c)^2}$$

Impedance: $$Z = \sqrt{R^2 + (X_L - X_c)^2}$$

$$V = IZ$$

$$Z = \sqrt{R^2 + [\omega L - (1/\omega C)]^2}$$

Impedance of R-L-C series circuit



$$\tan\varphi = \frac{V_L - V_C}{V_R} = \frac{I(X_L - X_C)}{IR} = \frac{X_L - X_C}{R}$$

Phase angle of the source voltage with respect to current

$$\tan\varphi = \frac{\omega L - 1/\omega C}{R}$$

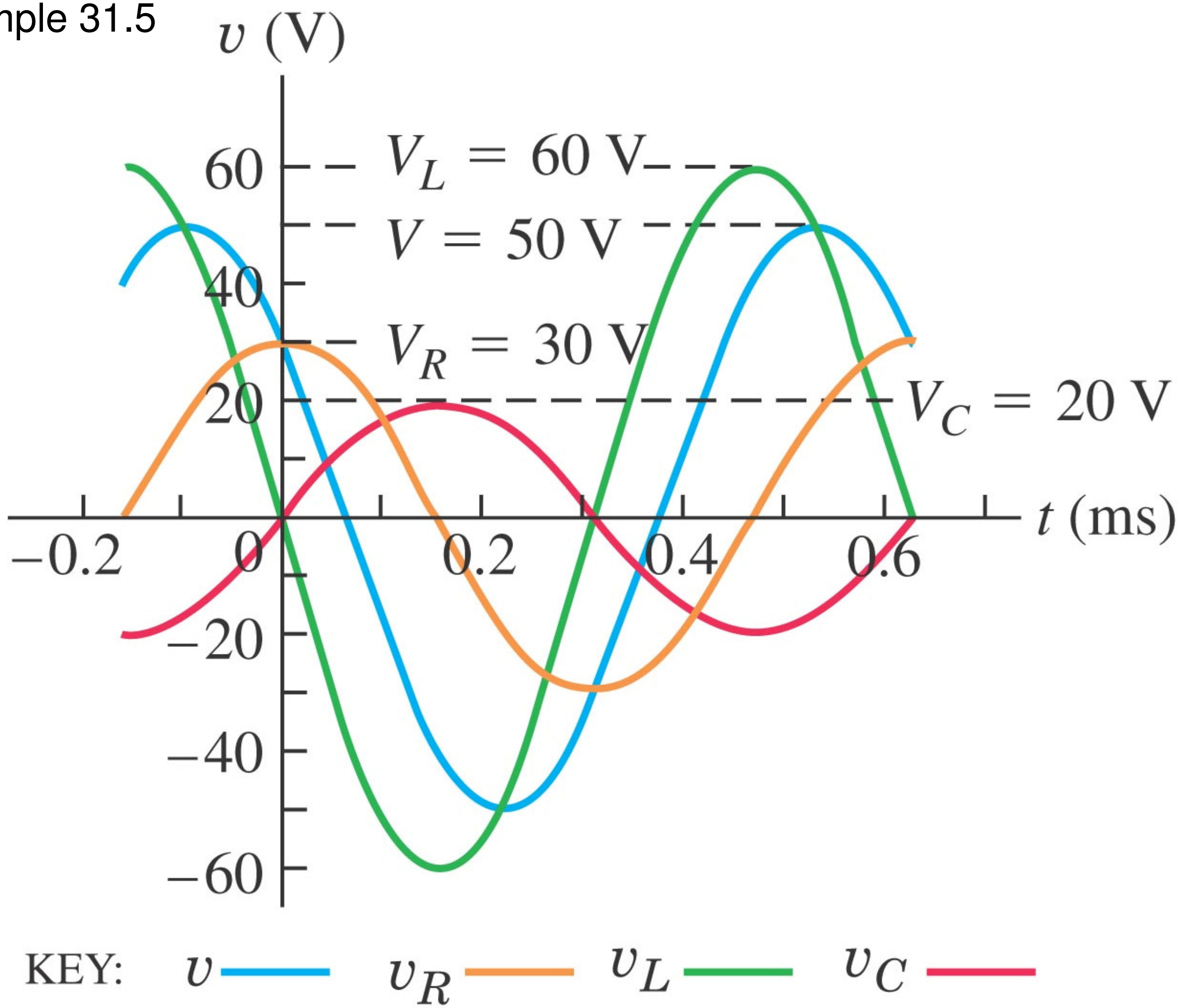
$$i = I\cos\omega t$$

$$v = V\cos(\omega t + \varphi)$$

$$V_{rms} = I_{rms} Z$$

$$\frac{V}{\sqrt{2}} = \frac{I}{\sqrt{2}} Z$$

Example 31.5

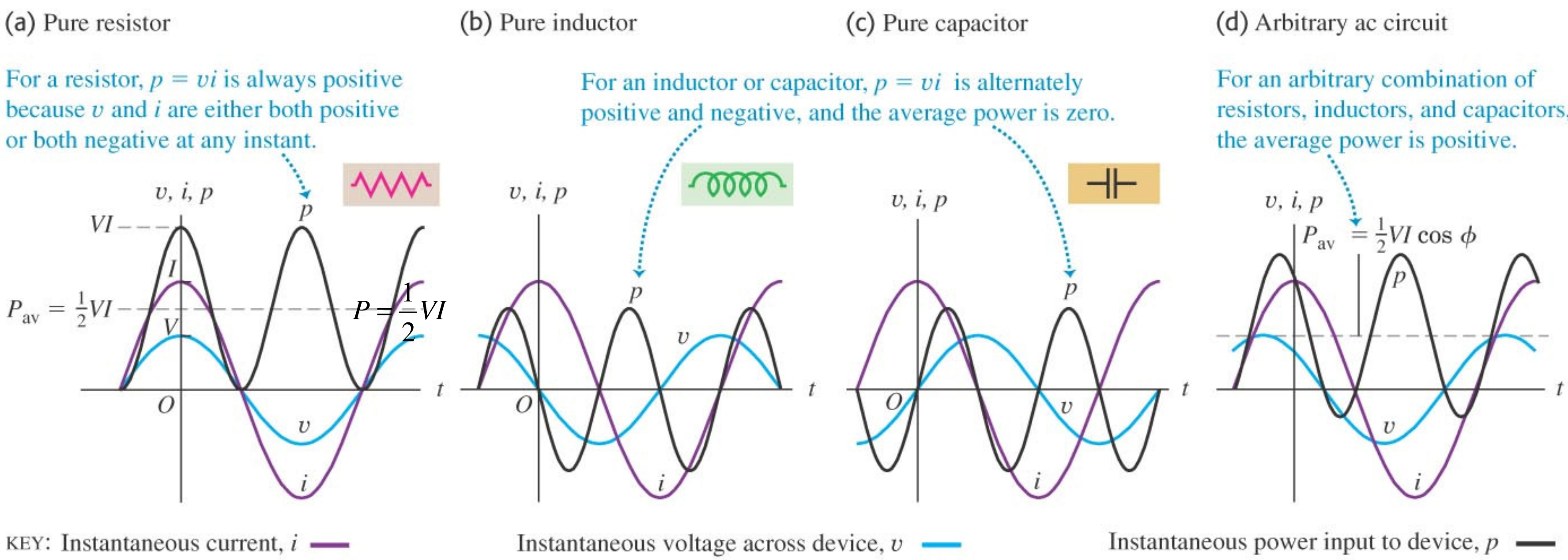


v (V)
$V_L = 60$ V
$V = 50$ V
$V_R = 30$ V
$V_C = 20$ V
60
40
20
−20
−40
−60
−0.2
0
0.2
0.4
0.6
t (ms)
KEY: v v_R v_L v_C

4. Power in Alternating-Current Circuits

$$P = \frac{1}{2}VI$$

$$P_{av} = \frac{V}{\sqrt{2}}\frac{I}{\sqrt{2}} = V_{rms}I_{rms} = I_{rms}{}^2 R = \frac{V_{rms}{}^2}{R}$$



Power in a General Circuit

$$P = vi = [V\cos(\omega t + \varphi)][I\cos\omega t] = [V(\cos\omega t\cos\varphi - \sin\omega t\sin\varphi)][I\cos\omega t]$$

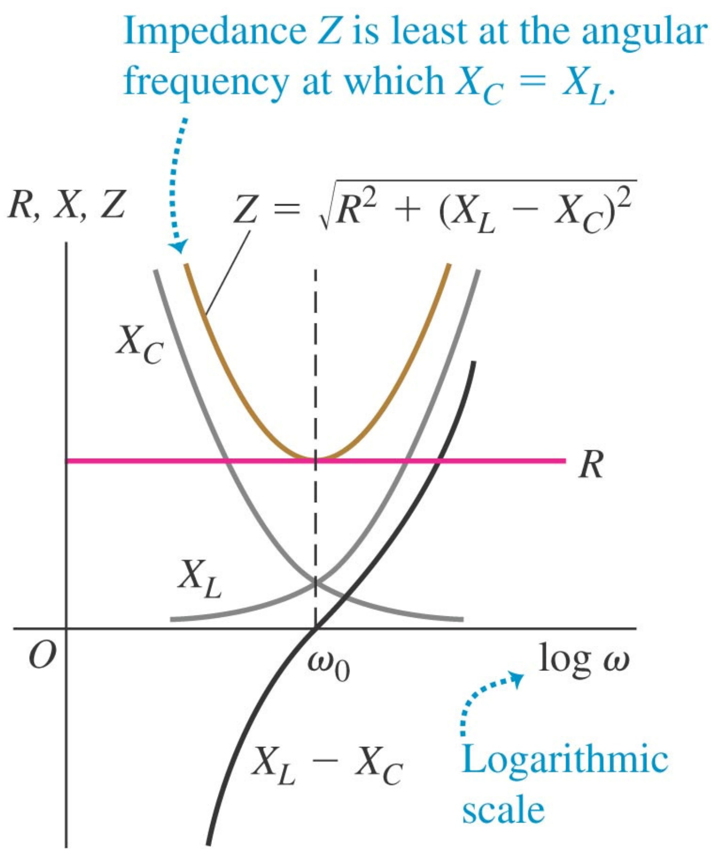
$$= VI\cos\varphi\cos^2\omega t - VI\sin\varphi\cos\omega t\sin\omega t$$

$$P_{av} = \frac{1}{2}VI\cos\varphi = V_{rms}I_{rms}\cos\varphi$$

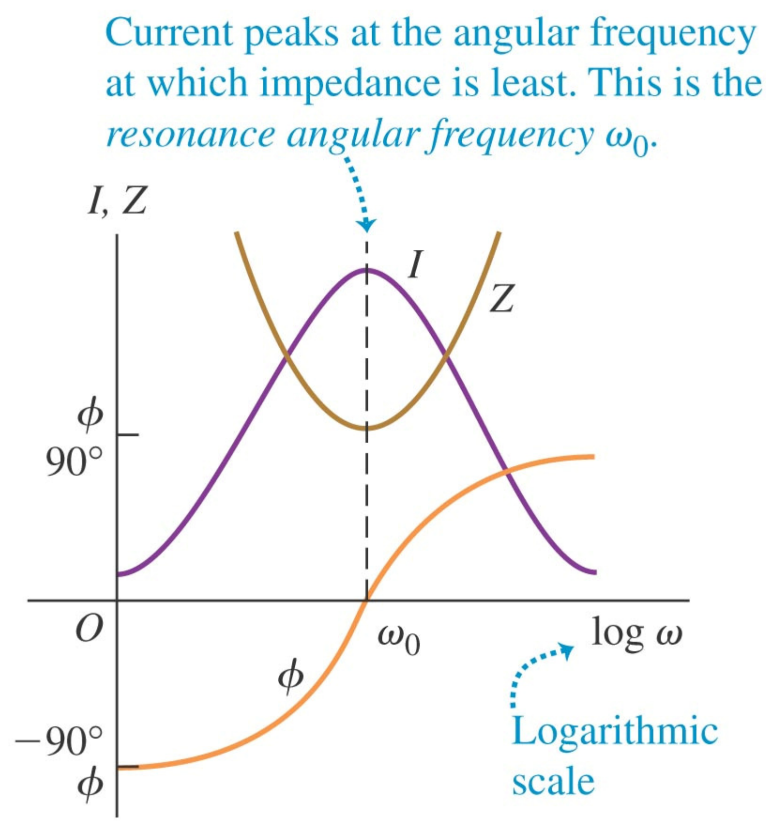
5. Resonance in Alternating-Current Circuits

$$X_L = X_C \qquad \omega_0 L = \frac{1}{\omega_0 C} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Reactance, resistance, and impedance as functions of angular frequency



Impedance, current, and phase angle as functions of angular frequency



6. Transformers

$$\mathcal{E}_1 = -N_1 \frac{d\Phi_B}{dt} \qquad \mathcal{E}_2 = -N_2 \frac{d\Phi_B}{dt}$$

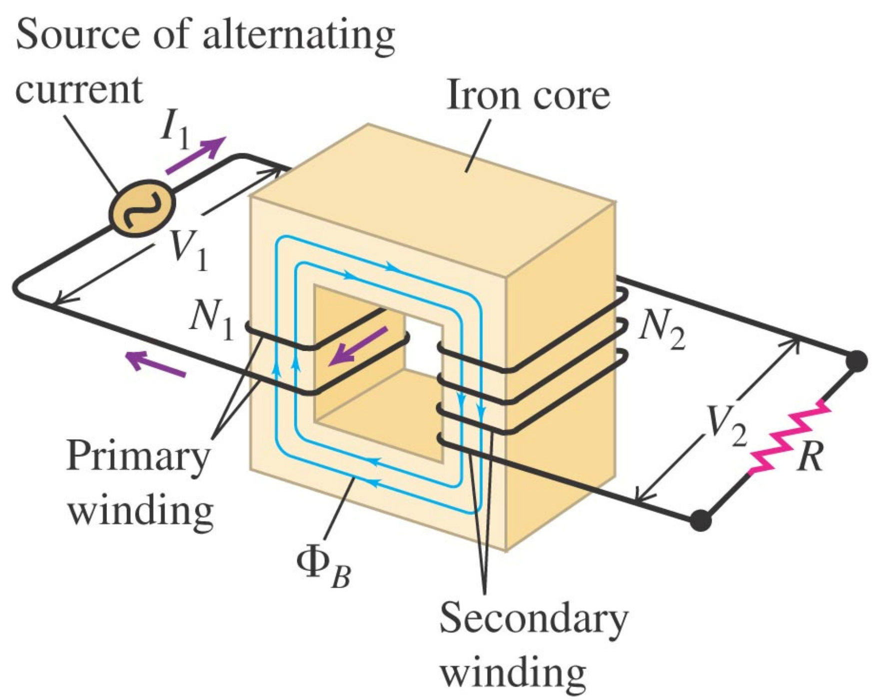
$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}$$

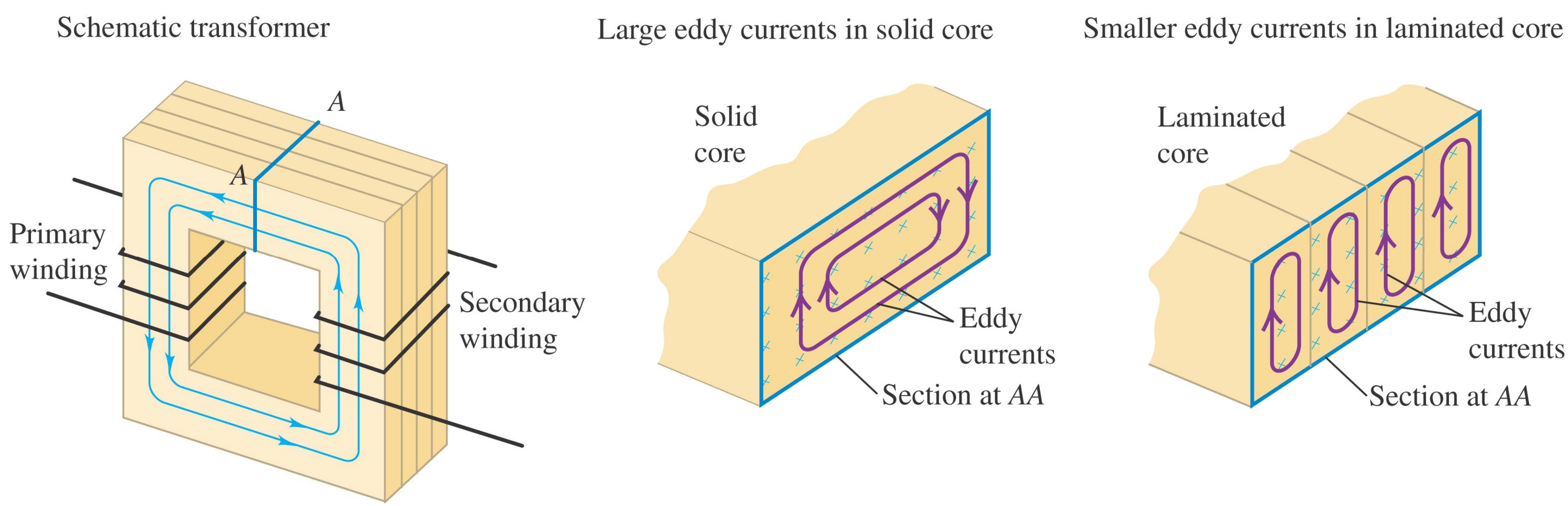
$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$

$$\frac{V_2}{I_1} = \frac{R}{(N_2 / N_1)}$$

The induced emf *per turn* is the same in both coils, so we adjust the ratio of terminal voltages by adjusting the ratio of turns:

$$\frac{V_2}{V_1} = \frac{N_2}{N_1}$$




Schematic transformer
A
A
Primary winding
Secondary winding
Large eddy currents in solid core
Solid core
Eddy currents
Section at AA
Smaller eddy currents in laminated core
Laminated core
Eddy currents
Section at AA

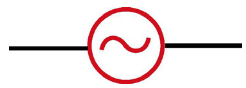
Chapter 12

Alternating-Current Circuits

Alternating-Current Circuits

12.1 AC Sources

In Chapter 10 we learned that changing magnetic flux can induce an emf according to Faraday's law of induction. In particular, if a coil rotates in the presence of a magnetic field, the induced emf varies sinusoidally with time and leads to an alternating current (AC), and provides a source of AC power. The symbol for an AC voltage source is



An example of an AC source is

$$V(t) = V_0 \sin \omega t \tag{12.1.1}$$

where the maximum value V_0 is called the *amplitude*. The voltage varies between V_0 and $-V_0$ since a sine function varies between +1 and −1. A graph of voltage as a function of time is shown in Figure 12.1.1.

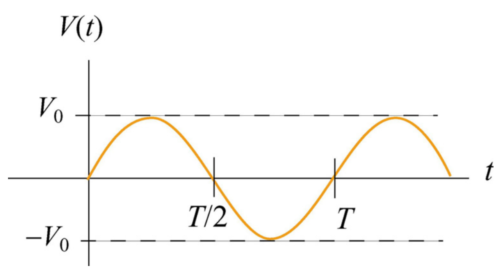


Figure 12.1.1 Sinusoidal voltage source

The sine function is periodic in time. This means that the value of the voltage at time t will be exactly the same at a later time $t' = t + T$ where T is the *period*. The *frequency,* f, defined as $f = 1/T$, has the unit of inverse seconds (s^{-1}), or hertz (Hz). The angular frequency is defined to be $\omega = 2\pi f$.

When a voltage source is connected to an *RLC* circuit, energy is provided to compensate the energy dissipation in the resistor, and the oscillation will no longer damp out. The oscillations of charge, current and potential difference are called driven or forced oscillations.

After an initial "transient time," an AC current will flow in the circuit as a response to the driving voltage source. The current, written as

$$I(t) = I_0 \sin(\omega t - \phi) \tag{12.1.2}$$

will oscillate with the same frequency as the voltage source, with an amplitude I_0 and phase ϕ that depends on the driving frequency.

12.2 Simple AC circuits

Before examining the driven *RLC* circuit, let's first consider the simple cases where only one circuit element (a resistor, an inductor or a capacitor) is connected to a sinusoidal voltage source.

12.2.1 Purely Resistive load

Consider a purely resistive circuit with a resistor connected to an AC generator, as shown in Figure 12.2.1. (As we shall see, a purely resistive circuit corresponds to infinite capacitance $C = \infty$ and zero inductance $L = 0$.)

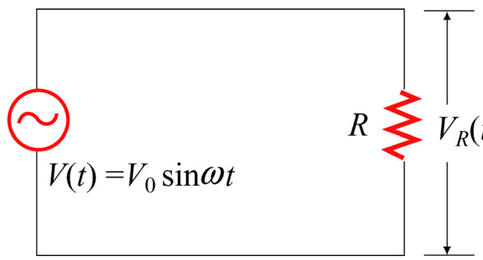


Figure 12.2.1 A purely resistive circuit

Applying Kirchhoff's loop rule yields

$$V(t) - V_R(t) = V(t) - I_R(t)R = 0 \tag{12.2.1}$$

where $V_R(t) = I_R(t)R$ is the instantaneous voltage drop across the resistor. The instantaneous current in the resistor is given by

$$I_R(t) = \frac{V_R(t)}{R} = \frac{V_{R0}\sin\omega t}{R} = I_{R0}\sin\omega t \tag{12.2.2}$$

where $V_{R0} = V_0$, and $I_{R0} = V_{R0}/R$ is the maximum current. Comparing Eq. (12.2.2) with Eq. (12.1.2), we find $\phi = 0$, which means that $I_R(t)$ and $V_R(t)$ are in phase with each other, meaning that they reach their maximum or minimum values at the same time. The time dependence of the current and the voltage across the resistor is depicted in Figure 12.2.2(a).

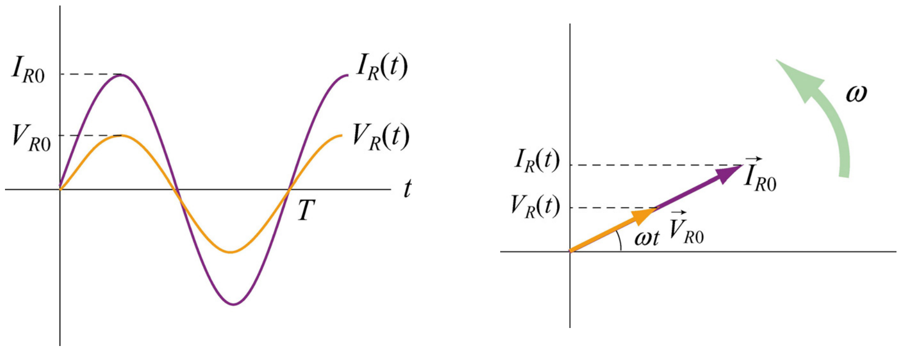


Figure 12.2.2 (a) Time dependence of $I_R(t)$ and $V_R(t)$ across the resistor. (b) Phasor diagram for the resistive circuit.

The behavior of $I_R(t)$ and $V_R(t)$ can also be represented with a phasor diagram, as shown in Figure 12.2.2(b). A phasor is a rotating vector having the following properties:

(i) length: the length corresponds to the amplitude.

(ii) angular speed: the vector rotates counterclockwise with an angular speed ω.

(iii) projection: the projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time t.

We shall denote a phasor with an arrow above it. The phasor $\vec{V}_{R0}$ has a constant magnitude of V_{R0}. Its projection along the vertical direction is $V_{R0}\sin\omega t$, which is equal to $V_R(t)$, the voltage drop across the resistor at time t. A similar interpretation applies to $\vec{I}_{R0}$ for the current passing through the resistor. From the phasor diagram, we readily see that both the current and the voltage are in phase with each other.

The average value of current over one period can be obtained as:

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)dt = \frac{1}{T}\int_0^T I_{R0}\sin\omega t\ dt = \frac{I_{R0}}{T}\int_0^T \sin\frac{2\pi t}{T}\ dt = 0 \qquad (12.2.3)$$

This average vanishes because

$$\langle \sin\omega t\rangle = \frac{1}{T}\int_0^T \sin\omega t\ dt = 0 \qquad (12.2.4)$$

Similarly, one may find the following relations useful when averaging over one period:

$$\begin{aligned}
\langle \cos\omega t \rangle &= \frac{1}{T}\int_0^T \cos\omega t \; dt = 0 \\
\langle \sin\omega t \cos\omega t \rangle &= \frac{1}{T}\int_0^T \sin\omega t \cos\omega t \; dt = 0 \\
\langle \sin^2 \omega t \rangle &= \frac{1}{T}\int_0^T \sin^2 \omega t \; dt = \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2} \\
\langle \cos^2 \omega t \rangle &= \frac{1}{T}\int_0^T \cos^2 \omega t \; dt = \frac{1}{T}\int_0^T \cos^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2}
\end{aligned} \qquad (12.2.5)$$

From the above, we see that the average of the square of the current is non-vanishing:

$$\langle I_R^2(t) \rangle = \frac{1}{T}\int_0^T I_R^2(t) dt = \frac{1}{T}\int_0^T I_{R0}^2 \sin^2 \omega t \; dt = I_{R0}^2 \frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right) dt = \frac{1}{2} I_{R0}^2 \qquad (12.2.6)$$

It is convenient to define the root-mean-square (rms) current as

$$I_{\text{rms}} = \sqrt{\langle I_R^2(t) \rangle} = \frac{I_{R0}}{\sqrt{2}} \qquad (12.2.7)$$

In a similar manner, the rms voltage can be defined as

$$V_{\text{rms}} = \sqrt{\langle V_R^2(t) \rangle} = \frac{V_{R0}}{\sqrt{2}} \qquad (12.2.8)$$

The rms voltage supplied to the domestic wall outlets in the United States is $V_{\text{rms}} = 120\ \text{V}$ at a frequency $f = 60\ \text{Hz}$.

The power dissipated in the resistor is

$$P_R(t) = I_R(t) V_R(t) = I_R^2(t) R \qquad (12.2.9)$$

from which the average over one period is obtained as:

$$\langle P_R(t) \rangle = \langle I_R^2(t) R \rangle = \frac{1}{2} I_{R0}^2 R = I_{\text{rms}}^2 R = I_{\text{rms}} V_{\text{rms}} = \frac{V_{\text{rms}}^2}{R} \qquad (12.2.10)$$

12.2.2 Purely Inductive Load

Consider now a purely inductive circuit with an inductor connected to an AC generator, as shown in Figure 12.2.3.

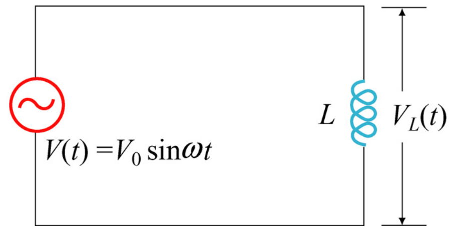


Figure 12.2.3 A purely inductive circuit

As we shall see below, a purely inductive circuit corresponds to infinite capacitance $C = \infty$ and zero resistance $R = 0$. Applying the modified Kirchhoff's rule for inductors, the circuit equation reads

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0 \tag{12.2.11}$$

which implies

$$\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{L0}}{L}\sin\omega t \tag{12.2.12}$$

where $V_{L0} = V_0$. Integrating over the above equation, we find

$$I_L(t) = \int dI_L = \frac{V_{L0}}{L}\int \sin\omega t\ dt = -\left(\frac{V_{L0}}{\omega L}\right)\cos\omega t = \left(\frac{V_{L0}}{\omega L}\right)\sin\left(\omega t - \frac{\pi}{2}\right) \tag{12.2.13}$$

where we have used the trigonometric identity

$$-\cos\omega t = \sin\left(\omega t - \frac{\pi}{2}\right) \tag{12.2.14}$$

for rewriting the last expression. Comparing Eq. (12.2.14) with Eq. (12.1.2), we see that the amplitude of the current through the inductor is

$$I_{L0} = \frac{V_{L0}}{\omega L} = \frac{V_{L0}}{X_L} \tag{12.2.15}$$

where

$$X_L = \omega L \tag{12.2.16}$$

is called the *inductive reactance*. It has SI units of ohms (Ω), just like resistance. However, unlike resistance, X_L depends linearly on the angular frequency ω. Thus, the resistance to current flow increases with frequency. This is due to the fact that at higher frequencies the current changes more rapidly than it does at lower frequencies. On the other hand, the inductive reactance vanishes as ω approaches zero.

By comparing Eq. (12.2.14) to Eq. (12.1.2), we also find the phase constant to be

$$\phi = +\frac{\pi}{2} \tag{12.2.17}$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 12.2.4 below.

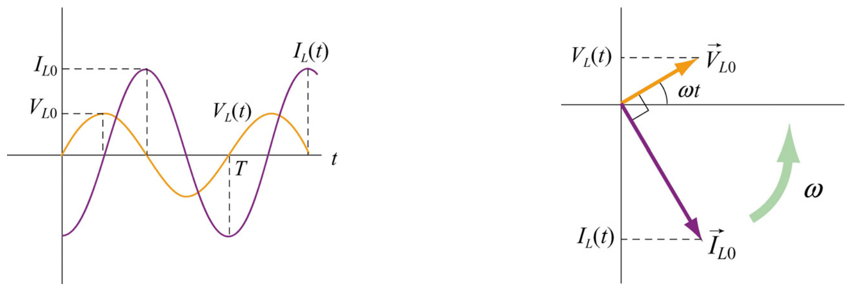


Figure 12.2.4 (a) Time dependence of $I_L(t)$ and $V_L(t)$ across the inductor. (b) Phasor diagram for the inductive circuit.

As can be seen from the figures, the current $I_L(t)$ is out of phase with $V_L(t)$ by $\phi = \pi/2$; it reaches its maximum value after $V_L(t)$ does by one quarter of a cycle. Thus, we say that

The current lags voltage by $\pi/2$ in a purely inductive circuit

12.2.3 Purely Capacitive Load

In the purely capacitive case, both resistance R and inductance L are zero. The circuit diagram is shown in Figure 12.2.5.

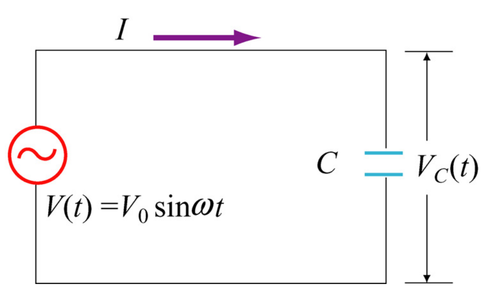


Figure 12.2.5 A purely capacitive circuit

Again, Kirchhoff's voltage rule implies

$$V(t) - V_C(t) = V(t) - \frac{Q(t)}{C} = 0 \qquad (12.2.18)$$

which yields

$$Q(t) = CV(t) = CV_C(t) = CV_{C0} \sin \omega t \qquad (12.2.19)$$

where $V_{C0} = V_0$. On the other hand, the current is

$$I_C(t) = +\frac{dQ}{dt} = \omega C V_{C0} \cos \omega t = \omega C V_{C0} \sin\left(\omega t + \frac{\pi}{2}\right) \qquad (12.2.20)$$

where we have used the trigonometric identity

$$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right) \qquad (12.2.21)$$

The above equation indicates that the maximum value of the current is

$$I_{C0} = \omega C V_{C0} = \frac{V_{C0}}{X_C} \qquad (12.2.22)$$

where

$$X_C = \frac{1}{\omega C} \qquad (12.2.23)$$

is called the *capacitance reactance*. It also has SI units of ohms and represents the effective resistance for a purely capacitive circuit. Note that X_C is inversely proportional to both C and ω, and diverges as ω approaches zero.

By comparing Eq. (12.2.21) to Eq. (12.1.2), the phase constant is given by

$$\phi = -\frac{\pi}{2} \qquad (12.2.24)$$

The current and voltage plots and the corresponding phasor diagram are shown in the Figure 12.2.6 below.

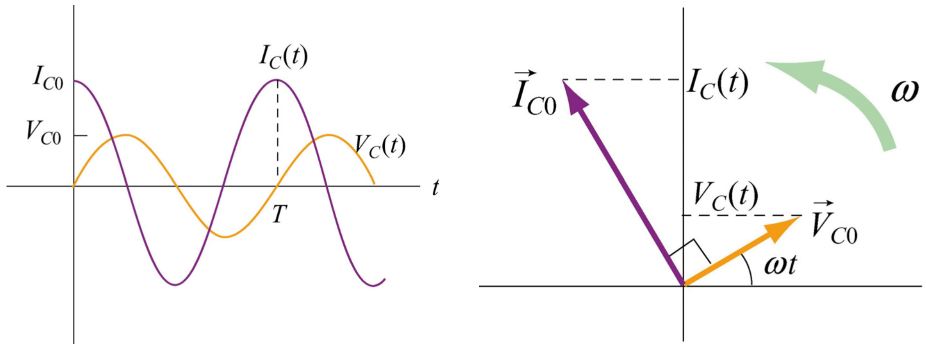


Figure 12.2.6 (a) Time dependence of $I_C(t)$ and $V_C(t)$ across the capacitor. (b) Phasor diagram for the capacitive circuit.

Notice that at $t = 0$, the voltage across the capacitor is zero while the current in the circuit is at a maximum. In fact, $I_C(t)$ reaches its maximum before $V_C(t)$ by one quarter of a cycle ($\phi = \pi / 2$). Thus, we say that

The current leads the voltage by $\pi/2$ in a capacitive circuit

12.3 The *RLC* Series Circuit

Consider now the driven series *RLC* circuit shown in Figure 12.3.1.

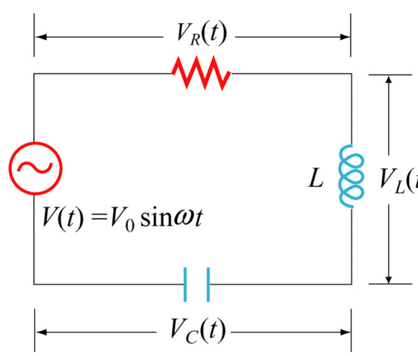


Figure 12.3.1 Driven series *RLC* Circuit

Applying Kirchhoff's loop rule, we obtain

$$V(t) - V_R(t) - V_L(t) - V_C(t) = V(t) - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0 \qquad (12.3.1)$$

which leads to the following differential equation:

$$L\frac{dI}{dt}+IR+\frac{Q}{C}=V_0\sin\omega t \tag{12.3.2}$$

Assuming that the capacitor is initially uncharged so that $I=+dQ/dt$ is proportional to the *increase* of charge in the capacitor, the above equation can be rewritten as

$$\boxed{L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{Q}{C}=V_0\sin\omega t} \tag{12.3.3}$$

One possible solution to Eq. (12.3.3) is

$$Q(t)=Q_0\cos(\omega t-\phi) \tag{12.3.4}$$

where the amplitude and the phase are, respectively,

$$\begin{aligned} Q_0 &= \frac{V_0/L}{\sqrt{(R\omega/L)^2+(\omega^2-1/LC)^2}}=\frac{V_0}{\omega\sqrt{R^2+(\omega L-1/\omega C)^2}} \\ &= \frac{V_0}{\omega\sqrt{R^2+(X_L-X_C)^2}} \end{aligned} \tag{12.3.5}$$

and

$$\tan\phi=\frac{1}{R}\left(\omega L-\frac{1}{\omega C}\right)=\frac{X_L-X_C}{R} \tag{12.3.6}$$

The corresponding current is

$$I(t)=+\frac{dQ}{dt}=I_0\sin(\omega t-\phi) \tag{12.3.7}$$

with an amplitude

$$I_0=-Q_0\omega=-\frac{V_0}{\sqrt{R^2+(X_L-X_C)^2}} \tag{12.3.8}$$

Notice that the current has the same amplitude and phase at all points in the series *RLC* circuit. On the other hand, the instantaneous voltage across each of the three circuit elements *R*, *L* and *C* has a different amplitude and phase relationship with the current, as can be seen from the phasor diagrams shown in Figure 12.3.2.

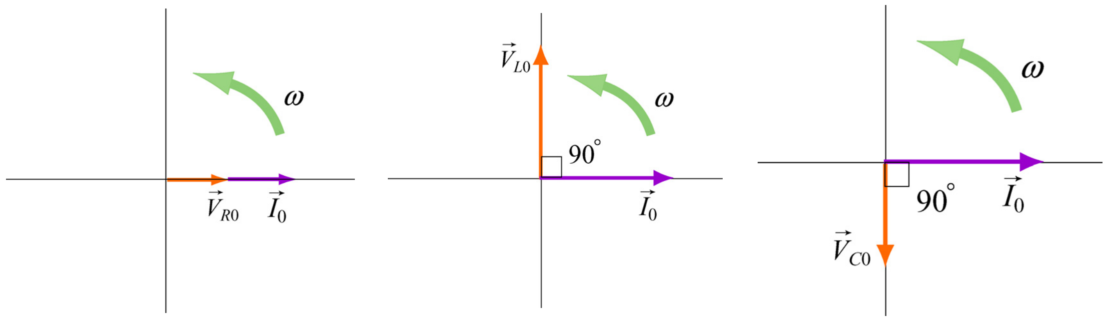


Figure 12.3.2 Phasor diagrams for the relationships between current and voltage in (a) the resistor, (b) the inductor, and (c) the capacitor, of a series *RLC* circuit.

From Figure 12.3.2, the instantaneous voltages can be obtained as:

$$\begin{aligned} V_R(t) &= I_0 R \sin \omega t = V_{R0} \sin \omega t \\ V_L(t) &= I_0 X_L \sin\left(\omega t + \frac{\pi}{2}\right) = V_{L0} \cos \omega t \\ V_C(t) &= I_0 X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -V_{C0} \cos \omega t \end{aligned} \tag{12.3.9}$$

where

$$V_{R0} = I_0 R, \quad V_{L0} = I_0 X_L, \quad V_{C0} = I_0 X_C \tag{12.3.10}$$

are the amplitudes of the voltages across the circuit elements. The sum of all three voltages is equal to the instantaneous voltage supplied by the AC source:

$$V(t) = V_R(t) + V_L(t) + V_C(t) \tag{12.3.11}$$

Using the phasor representation, the above expression can also be written as

$$\vec{V}_0 = \vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0} \tag{12.3.12}$$

as shown in Figure 12.3.3 (a). Again we see that current phasor $\vec{I}_0$ leads the capacitive voltage phasor $\vec{V}_{C0}$ by $\pi/2$ but lags the inductive voltage phasor $\vec{V}_{L0}$ by $\pi/2$. The three voltage phasors rotate counterclockwise as time passes, with their relative positions fixed.

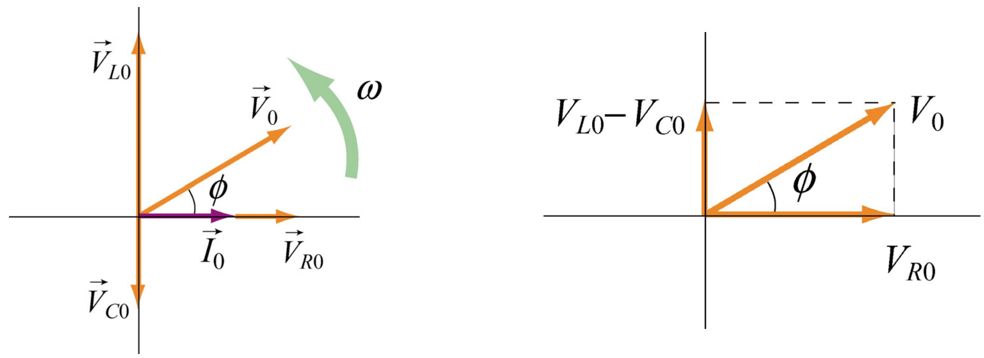


Figure 12.3.3 (a) Phasor diagram for the series *RLC* circuit. (b) voltage relationship

The relationship between different voltage amplitudes is depicted in Figure 12.3.3(b). From the Figure, we see that

$$\begin{aligned} V_0 &= |\vec{V}_0| = |\vec{V}_{R0} + \vec{V}_{L0} + \vec{V}_{C0}| = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} \\ &= \sqrt{(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2} \\ &= I_0 \sqrt{R^2 + (X_L - X_C)^2} \end{aligned} \tag{12.3.13}$$

which leads to the same expression for I_0 as that obtained in Eq. (12.3.7).

It is crucial to note that the maximum amplitude of the AC voltage source V_0 is not equal to the sum of the maximum voltage amplitudes across the three circuit elements:

$$V_0 \neq V_{R0} + V_{L0} + V_{C0} \tag{12.3.14}$$

This is due to the fact that the voltages are not in phase with one another, and they reach their maxima at different times.

12.3.1 Impedance

We have already seen that the inductive reactance $X_L = \omega L$ and capacitance reactance $X_C = 1/\omega C$ play the role of an effective resistance in the purely inductive and capacitive circuits, respectively. In the series *RLC* circuit, the effective resistance is the *impedance*, defined as

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{12.3.15}$$

The relationship between Z, X_L and X_C can be represented by the diagram shown in Figure 12.3.4:

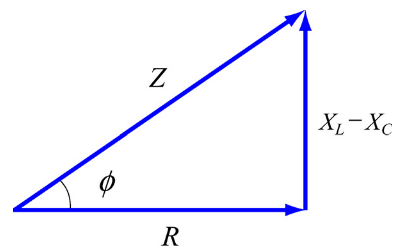


Figure 12.3.4 Diagrammatic representation of the relationship between Z, X_L and X_C.

The impedance also has SI units of ohms. In terms of Z, the current may be rewritten as

$$I(t) = \frac{V_0}{Z}\sin(\omega t - \phi) \tag{12.3.16}$$

Notice that the impedance Z also depends on the angular frequency ω, as do X_L and X_C.

Using Eq. (12.3.6) for the phase ϕ and Eq. (12.3.15) for Z, we may readily recover the limits for simple circuit (with only one element). A summary is provided in Table 12.1 below:

Simple Circuit	R	L	C	$X_L = \omega L$	$X_C = \frac{1}{\omega C}$	$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$	$Z = \sqrt{R^2 + (X_L - X_C)^2}$
purely resistive	R	0	∞	0	0	0	R
purely inductive	0	L	∞	X_L	0	$\pi/2$	X_L
purely capacitive	0	0	C	0	X_C	$-\pi/2$	X_C

Table 12.1 Simple-circuit limits of the series *RLC* circuit

12.3.2 Resonance

Eq. (12.3.15) indicates that the amplitude of the current $I_0 = V_0 / Z$ reaches a maximum when Z is at a minimum. This occurs when $X_L = X_C$, or $\omega L = 1/\omega C$, leading to

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{12.3.17}$$

The phenomenon at which I_0 reaches a maximum is called a resonance, and the frequency ω_0 is called the resonant frequency. At resonance, the impedance becomes $Z = R$, the amplitude of the current is

$$I_0 = \frac{V_0}{R} \tag{12.3.18}$$

and the phase is

$$\phi = 0 \tag{12.3.19}$$

as can be seen from Eq. (12.3.5). The qualitative behavior is illustrated in Figure 12.3.5.

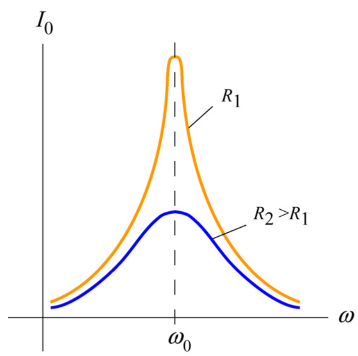


Figure 12.3.5 The amplitude of the current as a function of ω in the driven *RLC* circuit.

12.4 Power in an AC circuit

In the series *RLC* circuit, the instantaneous power delivered by the AC generator is given by

$$\begin{aligned} P(t) = I(t)V(t) &= \frac{V_0}{Z}\sin(\omega t - \phi)\cdot V_0 \sin \omega t = \frac{V_0^2}{Z}\sin(\omega t - \phi)\sin \omega t \\ &= \frac{V_0^2}{Z}\left(\sin^2 \omega t \cos \phi - \sin \omega t \cos \omega t \sin \phi\right) \end{aligned} \tag{12.4.1}$$

where we have used the trigonometric identity

$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi \tag{12.4.2}$$

The time average of the power is

$$\begin{aligned}\langle P(t)\rangle &= \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin^2\omega t\cos\phi\, dt - \frac{1}{T}\int_0^T \frac{V_0^2}{Z}\sin\omega t\cos\omega t\sin\phi\, dt \\ &= \frac{V_0^2}{Z}\cos\phi\langle\sin^2\omega t\rangle - \frac{V_0^2}{Z}\sin\phi\langle\sin\omega t\cos\omega t\rangle \\ &= \frac{1}{2}\frac{V_0^2}{Z}\cos\phi\end{aligned} \tag{12.4.3}$$

where Eqs. (12.2.5) and (12.2.7) have been used. In terms of the rms quantities, the average power can be rewritten as

$$\langle P(t)\rangle = \frac{1}{2}\frac{V_0^2}{Z}\cos\phi = \frac{V_{\text{rms}}^2}{Z}\cos\phi = I_{\text{rms}}V_{\text{rms}}\cos\phi \tag{12.4.4}$$

The quantity $\cos\phi$ is called the *power factor*. From Figure 12.3.4, one can readily show that

$$\cos\phi = \frac{R}{Z} \tag{12.4.5}$$

Thus, we may rewrite $\langle P(t)\rangle$ as

$$\langle P(t)\rangle = I_{\text{rms}}V_{\text{rms}}\left(\frac{R}{Z}\right) = I_{\text{rms}}\left(\frac{V_{\text{rms}}}{Z}\right)R = I_{\text{rms}}^2 R \tag{12.4.6}$$

In Figure 12.4.1, we plot the average power as a function of the driving angular frequency ω.

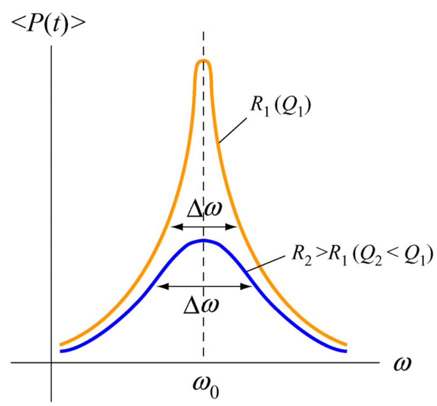


Figure 12.4.1 Average power as a function of frequency in a driven series *RLC* circuit.

We see that $\langle P(t)\rangle$ attains the maximum when $\cos\phi = 1$, or $Z = R$, which is the resonance condition. At resonance, we have

$$\langle P \rangle_{\max} = I_{\text{rms}} V_{\text{rms}} = \frac{V_{\text{rms}}^2}{R} \tag{12.4.7}$$

12.4.1 Width of the Peak

The peak has a line width. One way to characterize the width is to define $\Delta\omega = \omega_+ - \omega_-$, where $\omega_\pm$ are the values of the driving angular frequency such that the power is equal to half its maximum power at resonance. This is called *full width at half maximum*, as illustrated in Figure 12.4.2. The width $\Delta\omega$ increases with resistance *R*.

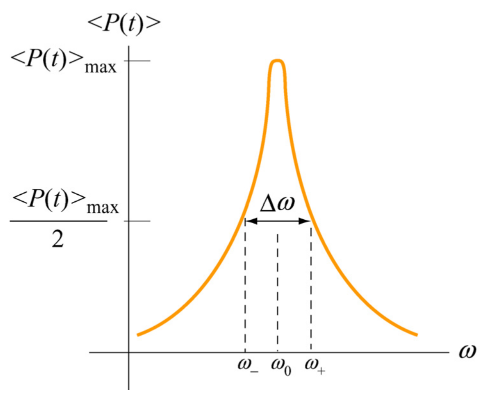


Figure 12.4.2 Width of the peak

To find $\Delta\omega$, it is instructive to first rewrite the average power $\langle P(t) \rangle$ as

$$\langle P(t) \rangle = \frac{1}{2} \frac{V_0{}^2 R}{R^2 + (\omega L - 1/\omega C)^2} = \frac{1}{2} \frac{V_0{}^2 R \omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2} \tag{12.4.8}$$

with $\langle P(t) \rangle_{\max} = V_0^2 / 2R$. The condition for finding $\omega_\pm$ is

$$\frac{1}{2} \langle P(t) \rangle_{\max} = \langle P(t) \rangle \Big|_{\omega_\pm} \quad \Rightarrow \quad = \frac{V_0{}^2}{4R} = \frac{1}{2} \frac{V_0{}^2 R \omega^2}{\omega^2 R^2 + L^2(\omega^2 - \omega_0^2)^2} \Bigg|_{\omega_\pm} \tag{12.4.9}$$

which gives

$$(\omega^2 - \omega_0^2)^2 = \left(\frac{R\omega}{L} \right)^2 \tag{12.4.10}$$

Taking square roots yields two solutions, which we analyze separately.

case 1: Taking the positive root leads to

$$\omega_+^2 - \omega_0^2 = +\frac{R\omega_+}{L} \tag{12.4.11}$$

Solving the quadratic equation, the solution with positive root is

$$\omega_+ = \frac{R}{2L} + \sqrt{\left(\frac{R}{4L}\right)^2 + \omega_0{}^2} \tag{12.4.12}$$

Case 2: Taking the negative root of Eq. (12.4.10) gives

$$\omega_-^2 - \omega_0^2 = -\frac{R\omega_-}{L} \tag{12.4.13}$$

The solution to this quadratic equation with positive root is

$$\omega_- = -\frac{R}{2L} + \sqrt{\left(\frac{R}{4L}\right)^2 + \omega_0{}^2} \tag{12.4.14}$$

The width at half maximum is then

$$\Delta\omega = \omega_+ - \omega_- = \frac{R}{L} \tag{12.4.15}$$

Once the width $\Delta\omega$ is known, the quality factor Q (not to be confused with charge) can be obtained as

$$\boxed{Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0 L}{R}} \tag{12.4.16}$$

Comparing the above equation with Eq. (11.8.17), we see that both expressions agree with each other in the limit where the resistance is small, and $\omega' = \sqrt{\omega_0^2 - (R/2L)^2} \approx \omega_0$.

12.5 Transformer

A transformer is a device used to increase or decrease the AC voltage in a circuit. A typical device consists of two coils of wire, a primary and a secondary, wound around an iron core, as illustrated in Figure 12.5.1. The primary coil, with N_1 turns, is connected to alternating voltage source $V_1(t)$. The secondary coil has N_2 turns and is connected to a "load resistance" R_2. The way transformers operate is based on the principle that an alternating current in the primary coil will induce an alternating emf on the secondary coil due to their mutual inductance.

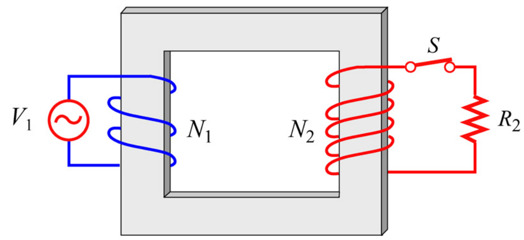


Figure 12.5.1 A transformer

In the primary circuit, neglecting the small resistance in the coil, Faraday's law of induction implies

$$V_1 = -N_1 \frac{d\Phi_B}{dt} \tag{12.5.1}$$

where Φ_B is the magnetic flux through one turn of the primary coil. The iron core, which extends from the primary to the secondary coils, serves to increase the magnetic field produced by the current in the primary coil and ensure that nearly all the magnetic flux through the primary coil also passes through each turn of the secondary coil. Thus, the voltage (or induced emf) across the secondary coil is

$$V_2 = -N_2 \frac{d\Phi_B}{dt} \tag{12.5.2}$$

In the case of an ideal transformer, power loss due to Joule heating can be ignored, so that the power supplied by the primary coil is completely transferred to the secondary coil:

$$I_1 V_1 = I_2 V_2 \tag{12.5.3}$$

In addition, no magnetic flux leaks out from the iron core, and the flux Φ_B through each turn is the same in both the primary and the secondary coils. Combining the two expressions, we are lead to the transformer equation:

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}} \tag{12.5.4}$$

By combining the two equations above, the transformation of currents in the two coils may be obtained as:

$$I_1 = \left(\frac{V_2}{V_1}\right) I_2 = \left(\frac{N_2}{N_1}\right) I_2 \qquad (12.5.5)$$

Thus, we see that the ratio of the output voltage to the input voltage is determined by the *turn ratio* N_2 / N_1. If $N_2 > N_1$, then $V_2 > V_1$, which means that the output voltage in the second coil is greater than the input voltage in the primary coil. A transformer with $N_2 > N_1$ is called a *step-up* transformer. On the other hand, if $N_2 < N_1$, then $V_2 < V_1$, and the output voltage is smaller than the input. A transformer with $N_2 < N_1$ is called a *step-down* transformer.

12.6 Parallel *RLC* Circuit

Consider the parallel *RLC* circuit illustrated in Figure 12.6.1. The AC voltage source is $V(t) = V_0 \sin \omega t$.

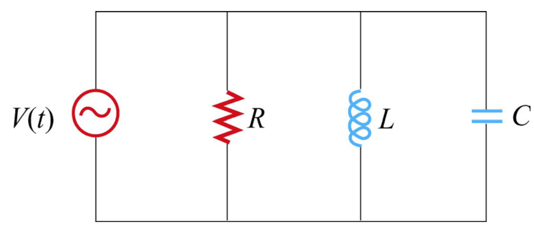


Figure 12.6.1 Parallel *RLC* circuit.

Unlike the series *RLC* circuit, the instantaneous voltages across all three circuit elements *R*, *L*, and *C* are the same, and each voltage is in phase with the current through the resistor. However, the currents through each element will be different.

In analyzing this circuit, we make use of the results discussed in Sections 12.2 – 12.4. The current in the resistor is

$$I_R(t) = \frac{V(t)}{R} = \frac{V_0}{R} \sin \omega t = I_{R0} \sin \omega t \qquad (12.6.1)$$

where $I_{R0} = V_0 / R$. The voltage across the inductor is

$$V_L(t) = V(t) = V_0 \sin \omega t = L \frac{dI_L}{dt} \qquad (12.6.2)$$

which gives

$$I_L(t) = \int_0^t \frac{V_0}{L} \sin \omega t' dt' = -\frac{V_0}{\omega L} \cos \omega t = \frac{V_0}{X_L} \sin\left(\omega t - \frac{\pi}{2}\right) = I_{L0} \sin\left(\omega t - \frac{\pi}{2}\right) \qquad (12.6.3)$$

where $I_{L0} = V_0 / X_L$ and $X_L = \omega L$ is the inductive reactance.

Similarly, the voltage across the capacitor is $V_C(t) = V_0 \sin\omega t = Q(t)/C$, which implies

$$I_C(t) = \frac{dQ}{dt} = \omega C V_0 \cos\omega t = \frac{V_0}{X_C}\sin\left(\omega t + \frac{\pi}{2}\right) = I_{C0}\sin\left(\omega t + \frac{\pi}{2}\right) \quad (12.6.4)$$

where $I_{C0} = V_0 / X_C$ and $X_C = 1/\omega C$ is the capacitive reactance.

Using Kirchhoff's junction rule, the total current in the circuit is simply the sum of all three currents.

$$\begin{aligned} I(t) &= I_R(t) + I_L(t) + I_C(t) \\ &= I_{R0}\sin\omega t + I_{L0}\sin\left(\omega t - \frac{\pi}{2}\right) + I_{C0}\sin\left(\omega t + \frac{\pi}{2}\right) \end{aligned} \quad (12.6.5)$$

The currents can be represented with the phasor diagram shown in Figure 12.6.2.

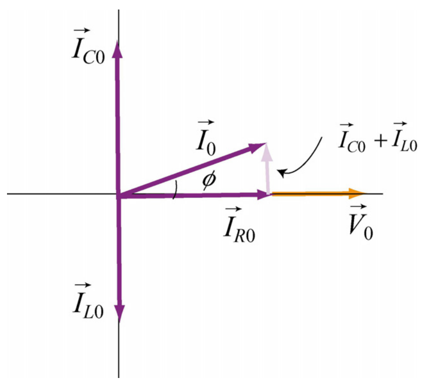


Figure 12.6.2 Phasor diagram for the parallel *RLC* circuit

From the phasor diagram, we see that

$$\vec{I}_0 = \vec{I}_{R0} + \vec{I}_{L0} + \vec{I}_{C0} \quad (12.6.6)$$

and the maximum amplitude of the total current, I_0, can be obtained as

$$\begin{aligned} I_0 &= |\vec{I}_0| = |\vec{I}_{R0} + \vec{I}_{L0} + \vec{I}_{C0}| = \sqrt{I_{R0}^2 + (I_{C0} - I_{L0})^2} \\ &= V_0\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = V_0\sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2} \end{aligned} \quad (12.6.7)$$

Note however, since $I_R(t)$, $I_L(t)$ and $I_C(t)$ are not in phase with one another, I_0 is not equal to the sum of the maximum amplitudes of the three currents:

$$I_0 \neq I_{R0} + I_{L0} + I_{C0} \tag{12.6.8}$$

With $I_0 = V_0 / Z$, the (inverse) impedance of the circuit is given by

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2} \tag{12.6.9}$$

The relationship between Z, R, X_L and X_C is shown in Figure 12.6.3.

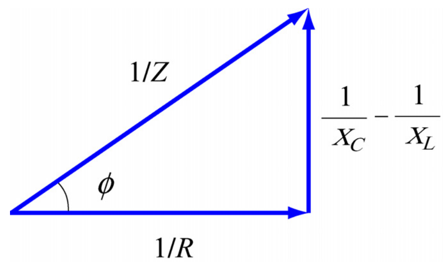


Figure 12.6.3 Relationship between Z, R, X_L and X_C in a parallel *RLC* circuit.

From the figure or the phasor diagram shown in Figure 12.6.2, we see that the phase can be obtained as

$$\tan\phi = \left(\frac{I_{C0} - I_{L0}}{I_{R0}}\right) = \frac{\dfrac{V_0}{X_C} - \dfrac{V_0}{X_L}}{\dfrac{V_0}{R}} = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right) = R\left(\omega C - \frac{1}{\omega L}\right) \tag{12.6.10}$$

The resonance condition for the parallel *RLC* circuit is given by $\phi = 0$, which implies

$$\frac{1}{X_C} = \frac{1}{X_L} \tag{12.6.11}$$

The resonant frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{12.6.12}$$

which is the same as for the series *RLC* circuit. From Eq. (12.6.9), we readily see that $1/Z$ is minimum (or Z is maximum) at resonance. The current in the inductor exactly cancels out the current in the capacitor, so that the total current in the circuit reaches a minimum, and is equal to the current in the resistor:

$$I_0 = \frac{V_0}{R} \tag{12.6.13}$$

As in the series *RLC* circuit, power is dissipated only through the resistor. The average power is

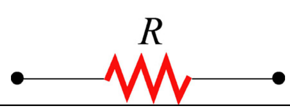
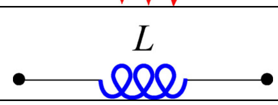
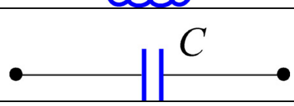
$$\langle P(t) \rangle = \langle I_R(t)V(t) \rangle = \langle I_R^2(t)R \rangle = \frac{V_0^2}{R}\langle \sin^2 \omega t \rangle = \frac{V_0^2}{2R} = \frac{V_0^2}{2Z}\left(\frac{Z}{R}\right) \tag{12.6.14}$$

Thus, the power factor in this case is

$$\text{power factor} = \frac{\langle P(t) \rangle}{V_0^2 / 2Z} = \frac{Z}{R} = \frac{1}{\sqrt{1+\left(R\omega C - \dfrac{R}{\omega L}\right)^2}} = \cos\phi \tag{12.6.15}$$

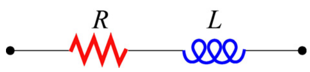
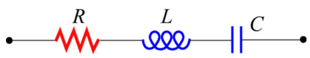
12.7 Summary

- In an AC circuit with a sinusoidal voltage source $V(t) = V_0 \sin \omega t$, the current is given by $I(t) = I_0 \sin(\omega t - \phi)$, where I_0 is the amplitude and ϕ is the phase constant. For simple circuit with only one element (a resistor, a capacitor or an inductor) connected to the voltage source, the results are as follows:

Circuit Elements	Resistance /Reactance	Current Amplitude	Phase angle ϕ
	R	$I_{R0} = \frac{V_0}{R}$	0
	$X_L = \omega L$	$I_{L0} = \frac{V_0}{X_L}$	$\pi / 2$ current lags voltage by $90°$
	$X_C = \frac{1}{\omega C}$	$I_{C0} = \frac{V_0}{X_C}$	$-\pi / 2$ current leads voltage by $90°$

where X_L is the **inductive reactance** and X_C is the **capacitive reactance**.

- For circuits which have more than one circuit element connected in series, the results are

Circuit Elements	Impedance Z	Current Amplitude	Phase angle ϕ
	$\sqrt{R^2+X_L^2}$	$I_0=\dfrac{V_0}{\sqrt{R^2+X_L^2}}$	$0<\phi<\dfrac{\pi}{2}$
	$\sqrt{R^2+X_C^2}$	$I_0=\dfrac{V_0}{\sqrt{R^2+X_C^2}}$	$-\dfrac{\pi}{2}<\phi<0$
	$\sqrt{R^2+(X_L-X_C)^2}$	$I_0=\dfrac{V_0}{\sqrt{R^2+(X_L-X_C)^2}}$	$\phi>0$ if $X_L>X_C$ $\phi<0$ if $X_L<X_C$

where Z is the **impedance** Z of the circuit. For a series RLC circuit, we have

$$Z=\sqrt{R^2+\left(X_L-X_C\right)^2}$$

The phase angle between the voltage and the current in an AC circuit is

$$\phi=\tan^{-1}\left(\frac{X_L-X_C}{R}\right)$$

- In the parallel RLC circuit, the impedance is given by

$$\frac{1}{Z}=\sqrt{\frac{1}{R^2}+\left(\omega C-\frac{1}{\omega L}\right)^2}=\sqrt{\frac{1}{R^2}+\left(\frac{1}{X_C}-\frac{1}{X_L}\right)^2}$$

and the phase is

$$\phi=\tan^{-1}\left[R\left(\frac{1}{X_C}-\frac{1}{X_L}\right)\right]=\tan^{-1}\left[R\left(\omega C-\frac{1}{\omega L}\right)\right]$$

- The **rms** (root mean square) voltage and current in an AC circuit are given by

$$V_{\text{rms}}=\frac{V_0}{\sqrt{2}},\qquad I_{\text{rms}}=\frac{I_0}{\sqrt{2}}$$

- The average power of an AC circuit is

$$\langle P(t)\rangle=I_{\text{rms}}V_{\text{rms}}\cos\phi$$

where $\cos\phi$ is known as the **power factor**.

- The **resonant frequency** ω_0 is

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

At **resonance**, the current in the series *RLC* circuit reaches the maximum, but the current in the parallel *RLC* circuit is at a minimum.

- The **transformer equation** is

$$\boxed{\frac{V_2}{V_1} = \frac{N_2}{N_1}}$$

where V_1 is the voltage source in the primary coil with N_1 turns, and V_2 is the output voltage in the secondary coil with N_2 turns. A transformer with $N_2 > N_1$ is called a *step-up* transformer, and a transformer with $N_2 < N_1$ is called a *step-down* transformer.

12.8 Problem-Solving Tips

In this chapter, we have seen how phasors provide a powerful tool for analyzing the AC circuits. Below are some important tips:

1. Keep in mind the phase relationships for simple circuits

   (1) For a resistor, the voltage and the phase are always in phase.
   (2) For an inductor, the current lags the voltage by $90°$.
   (3) For a capacitor, the current leads to voltage by $90°$.

2. When circuit elements are connected in *series*, the instantaneous current is the same for all elements, and the instantaneous voltages across the elements are out of phase. On the other hand, when circuit elements are connected in *parallel*, the instantaneous voltage is the same for all elements, and the instantaneous currents across the elements are out of phase.

3. For series connection, draw a phasor diagram for the voltages. The amplitudes of the voltage drop across all the circuit elements involved should be represented with phasors. In Figure 12.8.1 the phasor diagram for a series *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

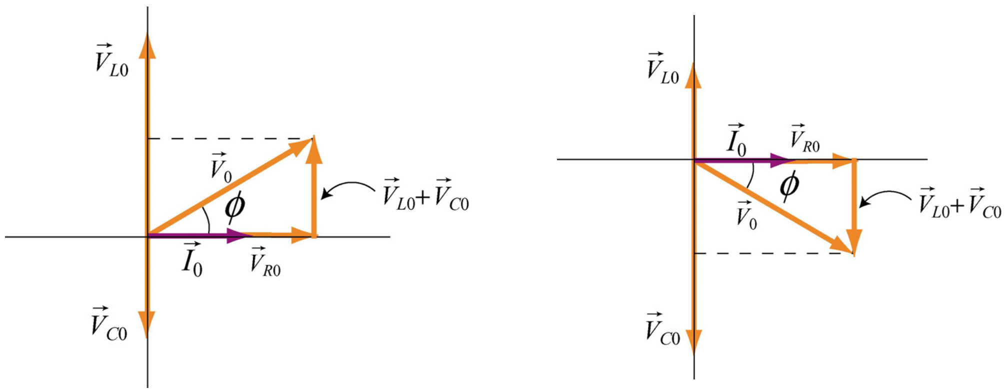


Figure 12.8.1 Phasor diagram for the series *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.1(a), we see that $V_{L0} > V_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. On the other hand, in the capacitive case shown in Figure 12.8.1(b), $V_{C0} > V_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ.

4. When $V_{L0} = V_{C0}$, or $\phi = 0$, the circuit is at resonance. The corresponding resonant frequency is $\omega_0 = 1/\sqrt{LC}$, and the power delivered to the resistor is a maximum.

5. For parallel connection, draw a phasor diagram for the currents. The amplitudes of the currents across all the circuit elements involved should be represented with phasors. In Figure 12.8.2 the phasor diagram for a parallel *RLC* circuit is shown for both the inductive case $X_L > X_C$ and the capacitive case $X_L < X_C$.

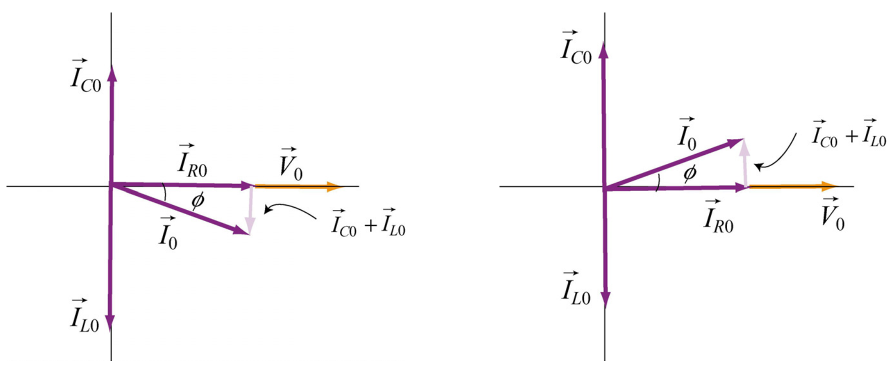


Figure 12.8.2 Phasor diagram for the parallel *RLC* circuit for (a) $X_L > X_C$ and (b) $X_L < X_C$.

From Figure 12.8.2(a), we see that $I_{L0} > I_{C0}$ in the inductive case and $\vec{V}_0$ leads $\vec{I}_0$ by a phase ϕ. On the other hand, in the capacitive case shown in Figure 12.8.2(b), $I_{C0} > I_{L0}$ and $\vec{I}_0$ leads $\vec{V}_0$ by a phase ϕ.

12.9 Solved Problems

12.9.1 *RLC* Series Circuit

A series *RLC* circuit with $L = 160\ \text{mH}$, $C = 100\ \mu\text{F}$, and $R = 40.0\,\Omega$ is connected to a sinusoidal voltage $V(t) = (40.0\,\text{V})\sin\omega t$, with $\omega = 200\ \text{rad/s}$.

(a) What is the impedance of the circuit?

(b) Let the current at any instant in the circuit be $I(t) = I_0 \sin(\omega t - \phi)$. Find I_0.

(c) What is the phase ϕ?

Solution:

(a) The impedance of a series *RLC* circuit is given by

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{12.9.1}$$

where

$$X_L = \omega L \tag{12.9.2}$$

and

$$X_C = \frac{1}{\omega C} \tag{12.9.3}$$

are the inductive reactance and the capacitive reactance, respectively. Since the general expression of the voltage source is $V(t) = V_0 \sin(\omega t)$, where V_0 is the maximum output voltage and ω is the angular frequency, we have $V_0 = 40\ \text{V}$ and $\omega = 200\ \text{rad/s}$. Thus, the impedance Z becomes

$$\begin{aligned} Z &= \sqrt{(40.0\ \Omega)^2 + \left((200\ \text{rad/s})(0.160\ \text{H}) - \frac{1}{(200\ \text{rad/s})(100\times 10^{-6}\ \text{F})}\right)^2} \\ &= 43.9\,\Omega \end{aligned} \tag{12.9.4}$$

(b) With $V_0 = 40.0\,\text{V}$, the amplitude of the current is given by

$$I_0 = \frac{V_0}{Z} = \frac{40.0\,\text{V}}{43.9\,\Omega} = 0.911\,\text{A} \tag{12.9.5}$$

(c) The phase between the current and the voltage is determined by

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{\omega L - \dfrac{1}{\omega C}}{R}\right)$$

$$= \tan^{-1}\left(\frac{(200 \text{ rad/s})(0.160 \text{ H}) - \dfrac{1}{(200 \text{ rad/s})(100\times 10^{-6} \text{ F})}}{40.0\ \Omega}\right) = -24.2° \qquad (12.9.6)$$

12.9.2 *RLC* Series Circuit

Suppose an AC generator with $V(t) = (150\,\text{V})\sin(100t)$ is connected to a series *RLC* circuit with $R = 40.0\ \Omega$, $L = 80.0\text{ mH}$, and $C = 50.0\ \mu\text{F}$, as shown in Figure 12.9.1.

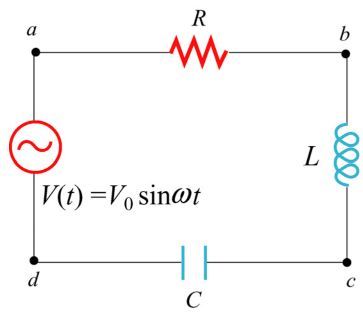


Figure 12.9.1 *RLC* series circuit

(a) Calculate V_{R0}, V_{L0} and V_{C0}, the maximum of the voltage drops across each circuit element.

(b) Calculate the maximum potential difference across the inductor and the capacitor between points *b* and *d* shown in Figure 12.9.1.

Solutions:

(a) The inductive reactance, capacitive reactance and the impedance of the circuit are given by

$$X_C = \frac{1}{\omega C} = \frac{1}{(100 \text{ rad/s})(50.0\times 10^{-6} \text{ F})} = 200\ \Omega \qquad (12.9.7)$$

$$X_L = \omega L = (100 \text{ rad/s})(80.0\times 10^{-3} \text{ H}) = 8.00\ \Omega \qquad (12.9.8)$$

and

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(40.0\ \Omega)^2 + (8.00\ \Omega - 200\ \Omega)^2} = 196\ \Omega \qquad (12.9.9)$$

respectively. Therefore, the corresponding maximum current amplitude is

$$I_0 = \frac{V_0}{Z} = \frac{150\ \text{V}}{196\ \Omega} = 0.765\,\text{A} \qquad (12.9.10)$$

The maximum voltage across the resistance would be just the product of maximum current and the resistance:

$$V_{R0} = I_0 R = (0.765\ \text{A})(40.0\ \Omega) = 30.6\,\text{V} \qquad (12.9.11)$$

Similarly, the maximum voltage across the inductor is

$$V_{L0} = I_0 X_L = (0.765\ \text{A})(8.00\ \Omega) = 6.12\,\text{V} \qquad (12.9.12)$$

and the maximum voltage across the capacitor is

$$V_{C0} = I_0 X_C = (0.765\ \text{A})(200\ \Omega) = 153\ \text{V} \qquad (12.9.13)$$

Note that the maximum input voltage V_0 is related to V_{R0}, V_{L0} and V_{C0} by

$$V_0 = \sqrt{V_{R0}{}^2 + (V_{L0} - V_{C0})^2} \qquad (12.9.14)$$

(b) From b to d, the maximum voltage would be the difference between V_{L0} and V_{C0}:

$$|V_{bd}| = |\vec{V}_{L0} + \vec{V}_{C0}| = |V_{L0} - V_{C0}| = |6.12\ \text{V} - 153\ \text{V}| = 147\,\text{V} \qquad (12.9.15)$$

12.9.3 Resonance

A sinusoidal voltage $V(t) = (200\,\text{V})\sin\omega t$ is applied to a series *RLC* circuit with $L = 10.0\ \text{mH}$, $C = 100\ \text{nF}$ and $R = 20.0\ \Omega$. Find the following quantities:

(a) the resonant frequency,

(b) the amplitude of the current at resonance,

(c) the quality factor Q of the circuit, and

(d) the amplitude of the voltage across the inductor at the resonant frequency.

Solution:

(a) The resonant frequency for the circuit is given by

$$f = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = \frac{1}{2\pi}\sqrt{\frac{1}{\left(10.0\times 10^{-3}\text{ H}\right)\left(100\times 10^{-9}\text{ F}\right)}} = 5033\,\text{Hz} \quad (12.9.16)$$

(b) At resonance, the current is

$$I_0 = \frac{V_0}{R} = \frac{200\text{ V}}{20.0\ \Omega} = 10.0\,\text{A} \quad (12.9.17)$$

(c) The quality factor Q of the circuit is given by

$$Q = \frac{\omega_0 L}{R} = \frac{2\pi\left(5033\text{ s}^{-1}\right)\left(10.0\times 10^{-3}\text{ H}\right)}{\left(20.0\ \Omega\right)} = 15.8 \quad (12.9.18)$$

(d) At resonance, the amplitude of the voltage across the inductor is

$$V_{L0} = I_0 X_L = I_0\omega_0 L = \left(10.0\text{ A}\right)2\pi\left(5033\text{ s}^{-1}\right)\left(10.0\times 10^{-3}\text{ H}\right) = 3.16\times 10^{3}\,\text{V} \quad (12.9.19)$$

12.9.4 *RL* High-Pass Filter

An *RL* high-pass filter (circuit that filters out low-frequency AC currents) can be represented by the circuit in Figure 12.9.2, where R is the internal resistance of the inductor.

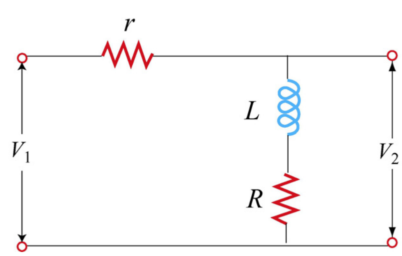


Figure 12.9.2 *RL* filter

(a) Find V_{20}/V_{10}, the ratio of the maximum output voltage V_{20} to the maximum input voltage V_{10}.

(b) Suppose $r = 15.0\ \Omega$, $R = 10\Omega$ and $L = 250$ mH. Find the frequency at which $V_{20}/V_{10} = 1/2$.

Solution:

(a) The impedance for the input circuit is $Z_1 = \sqrt{(R+r)^2 + X_L^2}$ where $X_L = \omega L$ and $Z_2 = \sqrt{R^2 + X_L^2}$ for the output circuit. The maximum current is given by

$$I_0 = \frac{V_{10}}{Z_1} = \frac{V_0}{\sqrt{(R+r)^2 + X_L^2}} \qquad (12.9.20)$$

Similarly, the maximum output voltage is related to the output impedance by

$$V_{20} = I_0 Z_2 = I_0 \sqrt{R^2 + X_L^2} \qquad (12.9.21)$$

This implies

$$\frac{V_{20}}{V_{10}} = \frac{\sqrt{R^2 + X_L^2}}{\sqrt{(R+r)^2 + X_L^2}} \qquad (12.9.22)$$

(b) For $V_{20}/V_{10} = 1/2$, we have

$$\frac{R^2 + X_L^2}{(R+r)^2 + X_L^2} = \frac{1}{4} \quad \Rightarrow \quad X_L = \sqrt{\frac{(R+r)^2 - 4R^2}{3}} \qquad (12.9.23)$$

Since $X_L = \omega L = 2\pi f L$, the frequency which yields this ratio is

$$f = \frac{X_L}{2\pi L} = \frac{1}{2\pi(0.250\ \mathrm{H})}\sqrt{\frac{\left(10.0\ \Omega + 15.0\ \Omega\right)^2 - 4\left(10.0\ \Omega\right)^2}{3}} = 5.51\,\mathrm{Hz} \qquad (12.9.24)$$

12.9.5 *RLC* Circuit

Consider the circuit shown in Figure 12.9.3. The sinusoidal voltage source is $V(t) = V_0 \sin\omega t$. If both switches S_1 and S_2 are closed initially, find the following quantities, ignoring the transient effect and assuming that R, L, V_0 and ω are known:

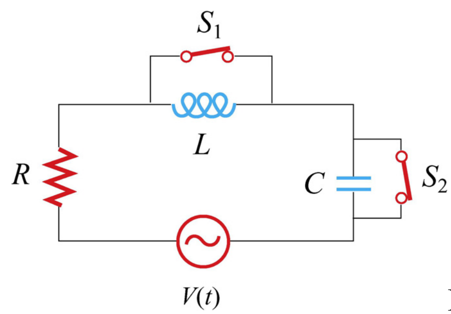


Figure 12.9.3

(a) the current $I(t)$ as a function of time,

(b) the average power delivered to the circuit,

(c) the current as a function of time a long time after only S_1 is opened.

(d) the capacitance C if both S_1 and S_2 are opened for a long time, with the current and voltage in phase.

(e) the impedance of the circuit when both S_1 and S_2 are opened.

(f) the maximum energy stored in the capacitor during oscillations.

(g) the maximum energy stored in the inductor during oscillations.

(h) the phase difference between the current and the voltage if the frequency of $V(t)$ is doubled.

(i) the frequency at which the inductive reactance X_L is equal to half the capacitive reactance X_C.

Solutions:

(a) When both switches S_1 and S_2 are closed, the current goes through only the generator and the resistor, so the total impedance of the circuit is R and the current is

$$I_R(t) = \frac{V_0}{R}\sin\omega t \tag{12.9.25}$$

(b) The average power is given by

$$\langle P(t)\rangle = \langle I_R(t)V(t)\rangle = \frac{V_0^2}{R}\langle \sin^2\omega t\rangle = \frac{V_0^2}{2R} \tag{12.9.26}$$

(c) If only S_1 is opened, after a long time the current will pass through the generator, the resistor and the inductor. For this *RL* circuit, the impedance becomes

$$Z = \frac{1}{\sqrt{R^2 + X_L^2}} = \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \tag{12.9.27}$$

and the phase angle ϕ is

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) \tag{12.9.28}$$

Thus, the current as a function of time is

$$I(t) = I_0 \sin(\omega t - \phi) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right) \tag{12.9.29}$$

Note that in the limit of vanishing resistance $R = 0$, $\phi = \pi/2$, and we recover the expected result for a purely inductive circuit.

(d) If both switches are opened, then this would be a driven *RLC* circuit, with the phase angle ϕ given by

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - \dfrac{1}{\omega C}}{R} \tag{12.9.30}$$

If the current and the voltage are in phase, then $\phi = 0$, implying $\tan\phi = 0$. Let the corresponding angular frequency be ω_0; we then obtain

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{12.9.31}$$

and the capacitance is

$$C = \frac{1}{\omega_0^2 L} \tag{12.9.32}$$

(e) From (d), we see that when both switches are opened, the circuit is at resonance with $X_L = X_C$. Thus, the impedance of the circuit becomes

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = R \tag{12.9.33}$$

(f) The electric energy stored in the capacitor is

$$U_E = \frac{1}{2}CV_C^2 = \frac{1}{2}C(IX_C)^2 \tag{12.9.34}$$

It attains maximum when the current is at its maximum I_0:

$$U_{C,\max} = \frac{1}{2}CI_0^2 X_C^2 = \frac{1}{2}C\left(\frac{V_0}{R}\right)^2 \frac{1}{\omega_0{}^2 C^2} = \frac{V_0^2 L}{2R^2} \tag{12.9.35}$$

where we have used $\omega_0^2 = 1/LC$.

(g) The maximum energy stored in the inductor is given by

$$U_{L,\max} = \frac{1}{2}LI_0^2 = \frac{LV_0^2}{2R^2} \tag{12.9.36}$$

(h) If the frequency of the voltage source is doubled, i.e., $\omega = 2\omega_0 = 1/\sqrt{LC}$, then the phase becomes

$$\phi = \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) = \tan^{-1}\left(\frac{\left(2/\sqrt{LC}\right)L - \left(\sqrt{LC}/2C\right)}{R}\right) = \tan^{-1}\left(\frac{3}{2R}\sqrt{\frac{L}{C}}\right) \tag{12.9.37}$$

(i) If the inductive reactance is one-half the capacitive reactance,

$$X_L = \frac{1}{2}X_C \quad \Rightarrow \quad \omega L = \frac{1}{2}\left(\frac{1}{\omega C}\right) \tag{12.9.38}$$

then

$$\omega = \frac{1}{\sqrt{2LC}} = \frac{\omega_0}{\sqrt{2}} \tag{12.9.39}$$

12.9.6 *RL* Filter

The circuit shown in Figure 12.9.4 represents an *RL* filter.

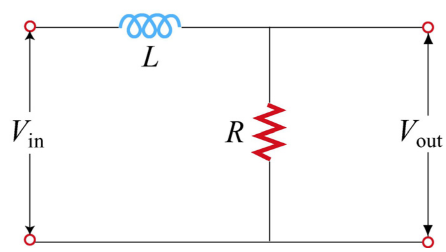


Figure 12.9.4

Let the inductance be L = 400 mH, and the input voltage $V_{\text{in}} = (20.0\,\text{V})\sin\omega t$, where $\omega = 200$ rad/s .

(a) What is the value of R such that the output voltage lags behind the input voltage by $30.0°$?

(b) Find the ratio of the amplitude of the output and the input voltages. What type of filter is this circuit, high-pass or low-pass?

(c) If the positions of the resistor and the inductor are switched, would the circuit be a high-pass or a low-pass filter?

Solutions:

(a) The phase relationship between V_L and V_R is given by

$$\tan\phi = \frac{V_L}{V_R} = \frac{IX_L}{IX_R} = \frac{\omega L}{R} \tag{12.9.40}$$

Thus, we have

$$R = \frac{\omega L}{\tan\phi} = \frac{(200\,\text{rad/s})(0.400\,\text{H})}{\tan 30.0°} = 139\,\Omega \tag{12.9.41}$$

(b) The ratio is given by

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{V_R}{V_{\text{in}}} = \frac{R}{\sqrt{R^2 + X_L^2}} = \cos\phi = \cos 30.0° = 0.866. \tag{12.9.42}$$

The circuit is a low-pass filter, since the ratio $V_{\text{out}} / V_{\text{in}}$ decreases with increasing ω.

(c) In this case, the circuit diagram is

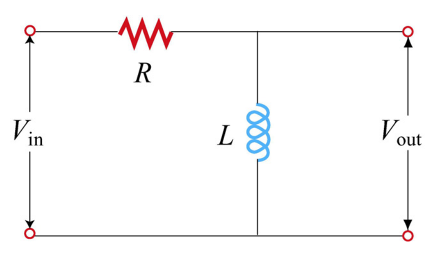


Figure 12.9.5 *RL* high-pass filter

The ratio of the output voltage to the input voltage would be

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{V_L}{V_{\text{in}}} = \frac{X_L}{\sqrt{R^2 + X_L^2}} = \frac{\omega^2 L^2}{\sqrt{R^2 + \omega^2 L^2}} = \left[1 + \left(\frac{R}{\omega L}\right)^2\right]^{-1/2}$$

The circuit is a high-pass filter, since the ratio $V_{\text{out}} / V_{\text{in}}$ approaches one in the large-ω limit.

12.10 Conceptual Questions

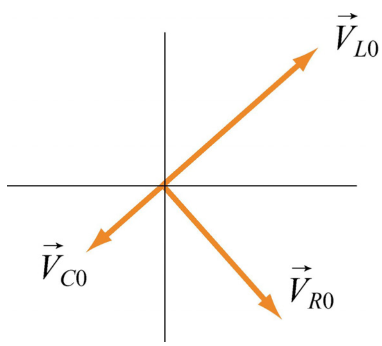
1. Consider a purely capacitive circuit (a capacitor connected to an AC source).

(a) How does the capacitive reactance change if the driving frequency is doubled? halved?

(b) Are there any times when the capacitor is supplying power to the AC source?

2. If the applied voltage leads the current in a series *RLC* circuit, is the frequency above or below resonance?

3. Consider the phasor diagram shown in Figure 12.10.1 for an *RLC* circuit.



(a) Is the driving frequency above or below the resonant frequency?

(b) Draw the phasor $\vec{V}_0$ associated with the amplitude of the applied voltage.

(c) Give an estimate of the phase ϕ between the applied AC voltage and the current.

4. How does the power factor in an *RLC* circuit change with resistance *R*, inductance *L* and capacitance *C*?

5. Can a battery be used as the primary voltage source in a transformer?

6. If the power factor in an *RLC* circuit is $\cos\phi = 1/2$, can you tell whether the current leading or lagging the voltage? Explain.

12.11 Additional Problems

12.11.1 Reactance of a Capacitor and an Inductor

(a) A $C = 0.5-\mu\text{F}$ capacitor is connected, as shown in Figure 12.11.1(a), to an AC generator with $V_0 = 300\text{ V}$. What is the amplitude I_0 of the resulting alternating current if the angular frequency ω is (i) 100 rad/s, and (ii) 1000 rad/s?

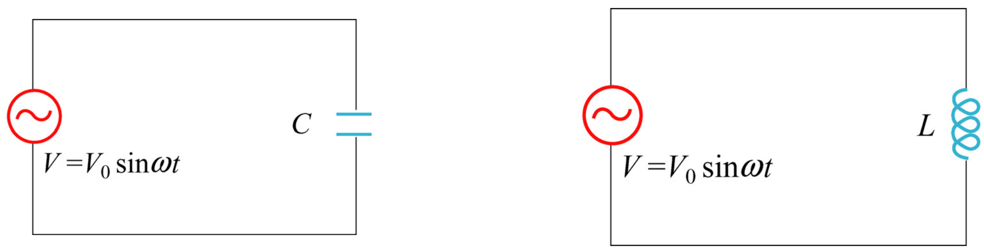


Figure 12.11.1 (a) A purely capacitive circuit, and (b) a purely inductive circuit.

(b) A 45-mH inductor is connected, as shown in Figure 12.10.1(b), to an AC generator with $V_0 = 300\text{ V}$. The inductor has a reactance $X_L = 1300\ \Omega$. What must be

(i) the applied angular frequency ω and

(ii) the applied frequency f for this to be true?

(iii) What is the amplitude I_0 of the resulting alternating current?

(c) At what frequency f would our 0.5-μF capacitor and our 45-mH inductor have the same reactance? What would this reactance be? How would this frequency compare to the natural resonant frequency of free oscillations if the components were connected as an *LC* oscillator with zero resistance?

12.11.2 Driven *RLC* Circuit Near Resonance

The circuit shown in Figure 12.11.2 contains an inductor *L*, a capacitor *C*, and a resistor *R* in series with an AC generator which provides a source of sinusoidally varying emf $V(t) = V_0 \sin\omega t$.

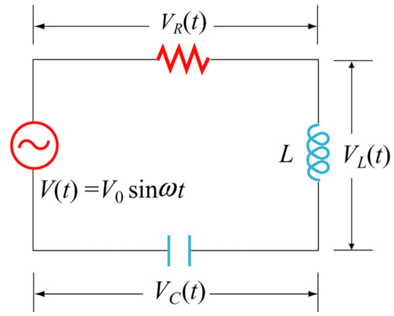


Figure 12.11.2

This emf drives current $I(t) = I_0 \sin(\omega t - \phi)$ through the circuit at angular frequency ω.

(a) At what angular frequency ω will the circuit resonate with maximum response, as measured by the amplitude I_0 of the current in the circuit? What is the value of the maximum current amplitude $I_{\max}$?

(b) What is the value of the phase angle ϕ between $V(t)$ and $I(t)$ at this resonant frequency?

(c) Suppose the frequency ω is *increased* from the resonance value until the amplitude I_0 of the current decreases from $I_{\max}$ to $I_{\max}/\sqrt{2}$. Now what is the phase difference ϕ between the emf and the current? Does the current lead or lag the emf?

12.11.3 *RC* Circuit

A series *RC* circuit with $R = 4.0\times10^3\,\Omega$ and $C = 0.40\ \mu\text{F}$ is connected to an AC voltage source $V(t) = (100\text{ V})\sin\omega t$, with $\omega = 200\text{ rad/s}$.

(a) What is the rms current in the circuit?

(b) What is the phase between the voltage and the current?

(c) Find the power dissipated in the circuit.

(d) Find the voltage drop both across the resistor and the capacitor.

12.11.4 Black Box

An AC voltage source is connected to a "black box" which contains a circuit, as shown in Figure 12.11.3.

Figure 12.11.3 A "black box" connected to an AC voltage source.

The elements in the circuit and their arrangement, however, are unknown. Measurements outside the black box provide the following information:

$$V(t) = (80 \text{ V})\sin\omega t$$
$$I(t) = (1.6 \text{ A})\sin(\omega t + 45°)$$

(a) Does the current lead or lag the voltage?

(b) Is the circuit in the black box largely capacitive or inductive?

(c) Is the circuit in the black box at resonance?

(d) What is the power factor?

(e) Does the box contain a resistor? A capacitor? An inductor?

(f) Compute the average power delivered to the black box by the AC source.

12.11.5 Parallel *RL* Circuit

Consider the parallel *RL* circuit shown in Figure 12.11.4.

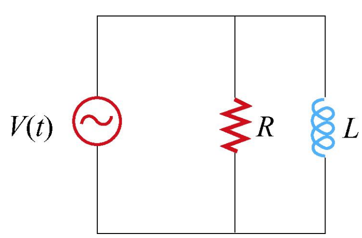


Figure 12.11.4 Parallel *RL* circuit

The AC voltage source is $V(t) = V_0 \sin\omega t$.

(a) Find the current across the resistor.

(b) Find the current across the inductor.

(c) What is the magnitude of the total current?

(d) Find the impedance of the circuit.

(e) What is the phase angle between the current and the voltage?

12.11.6 *LC* Circuit

Suppose at $t = 0$ the capacitor in the *LC* circuit is fully charged to Q_0. At a later time $t = T/6$, where T is the period of the *LC* oscillation, find the ratio of each of the following quantities to its maximum value:

(a) charge on the capacitor,

(b) energy stored in the capacitor,

(c) current in the inductor, and

(d) energy in the inductor.

12.11.7 Parallel *RC* Circuit

Consider the parallel *RC* circuit shown in Figure 12.11.5.

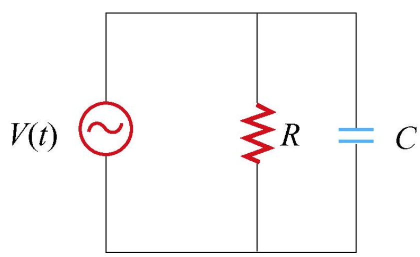


Figure 12.11.5 Parallel *RC* circuit

The AC voltage source is $V(t) = V_0 \sin \omega t$.

(a) Find the current across the resistor.

(b) Find the current across the capacitor.

(c) What is the magnitude of the total current?

(d) Find the impedance of the circuit.

(e) What is the phase angle between the current and the voltage?

12.11.8 Power Dissipation

A series *RLC* circuit with $R = 10.0\ \Omega$, $L = 400$ mH and $C = 2.0\ \mu$F is connected to an AC voltage source which has a maximum amplitude $V_0 = 100$ V.

(a) What is the resonant frequency ω_0?

(b) Find the rms current at resonance.

(c) Let the driving frequency be $\omega = 4000$ rad/s. Compute X_C, X_L, Z and ϕ.

12.11.9 FM Antenna

An FM antenna circuit (shown in Figure 12.11.6) has an inductance $L = 10^{-6}$ H, a capacitance $C = 10^{-12}$ F and a resistance $R = 100\Omega$. A radio signal induces a sinusoidally alternating emf in the antenna with an amplitude of 10^{-5} V.

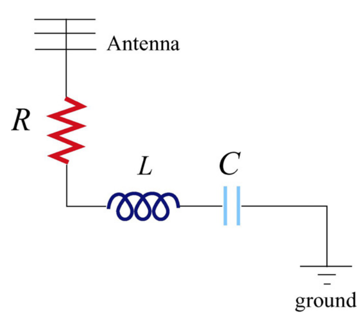


Figure 12.11.6

(a) For what angular frequency ω_0 (radians/sec) of the incoming waves will the circuit be "in tune"-- that is, for what ω_0 will the current in the circuit be a maximum.

(b) What is the quality factor Q of the resonance?

(c) Assuming that the incoming wave is "in tune," what will be the amplitude of the current in the circuit at this "in tune" frequency.

(d) What is the amplitude of the potential difference across the capacitor at this "in tune" frequency?

12.11.10 Driven *RLC* Circuit

Suppose you want a series *RLC* circuit to tune to your favorite FM radio station that broadcasts at a frequency of $89.7\,\text{MHz}$. You would like to avoid the obnoxious station that broadcasts at $89.5\,\text{MHz}$. In order to achieve this, for a given input voltage signal from your antenna, you want the width of your resonance to be narrow enough at $89.7\,\text{MHz}$ such that the current flowing in your circuit will be 10^{-2} times less at $89.5\,\text{MHz}$ than at $89.7\,\text{MHz}$. You cannot avoid having a resistance of $R = 0.1\,\Omega$, and practical considerations also dictate that you use the minimum L possible.

(a) In terms of your circuit parameters, L, R and C, what is the amplitude of your current in your circuit as a function of the angular frequency of the input signal?

(b) What is the angular frequency of the input signal at the desired resonance?

(c) What values of L and C must you use?

(d) What is the quality factor for this resonance?

(e) Show that at resonance, the ratio of the amplitude of the voltage across the inductor with the driving signal amplitude is the quality of the resonance.

(f) Show that at resonance the ratio of the amplitude of the voltage across the capacitor with the driving signal amplitude is the quality of the resonance.

(g) What is the time averaged power at resonance that the signal delivers to the circuit?

(h) What is the phase shift for the input signal at $89.5\,\text{MHz}$?

(i) What is the time averaged power for the input signal at $89.5\,\text{MHz}$?

(j) Is the circuit capacitive or inductive at $89.5\,\text{MHz}$?

Chapter two

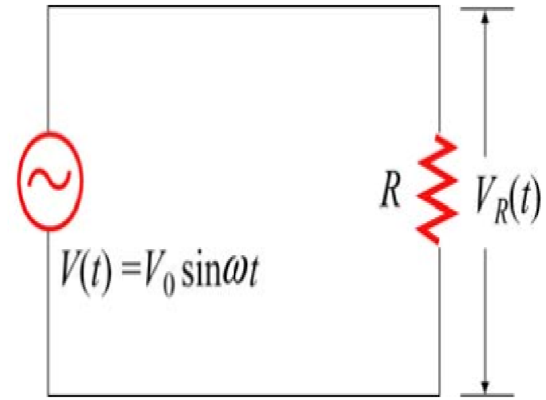
Alternating Current Circuits

- **Introduction**

Every time we turn on a television, a stereo system, or any of a other electric devices , we call on alternating currents (AC) to provide the power to operate them. We begin our study of AC circuits by examining the characteristics of a circuit containing a source of emf and one other circuit element: a resistor, a capacitor, or an inductor. Then we examine what happens when these elements are connected in combination with each other.

AC Circuit with a Pure Resistor:

We first consider a simple circuit consisting of a resistor and an AC source as in the following figure .



If the potential drop across the resistor is V_R and the current the resistor is I. Applying Kirchhoff's loop rule yields

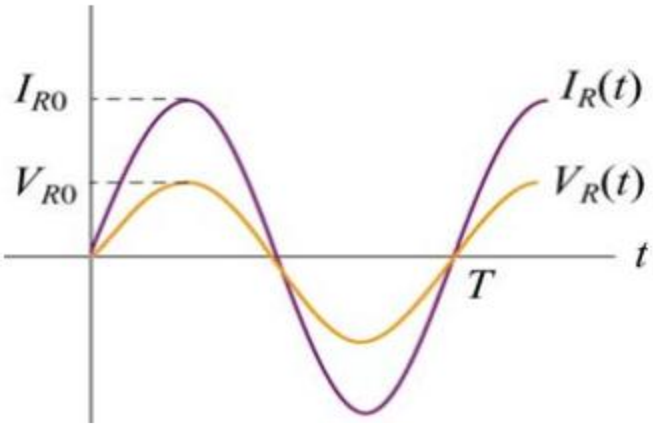
$$V(t) - V_R(t) = V(t) - I_R(t)R = 0$$

where $V_R(t) = I_R(t)R$ is the instantaneous voltage drop across the resistor. The instantaneous current in the resistor is given by

$$I_R(t) = \frac{V_R(t)}{R} = \frac{V_{R0}\sin\omega t}{R} = I_{R0}\sin\omega t$$

where $V_{R0} = V_0$, and $I_{R0} = V_{R0}/R$ is the maximum current.

Comparing Eq. (2) of the current I with Eq. (2) of the voltage , we find that *I* and V are **in phase** with each other, **meaning that they reach their maximum or minimum values at the same time.**

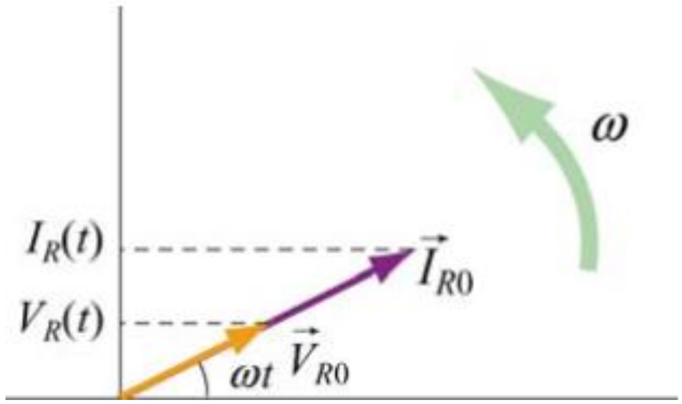


The time dependence of the current and the voltage across the resistor is shown in the Figure

The behavior of $I_R(t)$ and $V_R(t)$ can also be represented with **a phasor diagram**, as shown in the figure .

(i) length: the length corresponds to the amplitude.

(ii) angular speed: the vector rotates counterclockwise with an angular speed ω.



(iii) projection: the projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time *t*.

From the phasor diagram, we readily see that both the current and the voltage are in phase with each other.

The resistance of the circuit is not dependent on the source frequency and is calculated from the relationship:

$$R = \frac{V}{I} = \frac{V_{\max}}{I_{\max}} = \frac{V_{eff}}{I_{eff}}$$

Problem

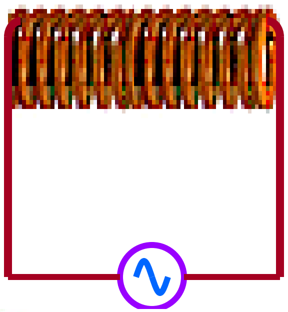
An AC voltage source has an output of V= 2 × 10^2 sin wt . This source is connected to a 100 Ω resistor. Find the rms voltage and rms current in the resistor.

$$V_{\max} = 2.00 \times 10^2 \text{ V}$$

$$V_{\text{rms}} = \frac{V_{\max}}{\sqrt{2}} = \frac{2.00 \times 10^2 \text{ V}}{\sqrt{2}} = 141 \text{ V}$$

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{R} = \frac{141 \text{ V}}{1.00 \times 10^2 \, \Omega} = 1.41 \text{ A}$$

AC Circuit with a Pure Inductor:



$V(t) = V_0 \sin \omega t$

Consider now a purely inductive circuit with an inductor connected to an AC generator, as shown in Figure.

Applying the Kirchhoff's rule for inductors, the circuit equation can be written in the following form.

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0$$

which implies

$$\frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{L0}}{L}\sin\omega t$$

where $V_{L0} = V_0$. Integrating over the above equation, we find

$$I_L(t) = \int dI_L = \frac{V_{L0}}{L}\int \sin\omega t \; dt = -\left(\frac{V_{L0}}{\omega L}\right)\cos\omega t = \left(\frac{V_{L0}}{\omega L}\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$

where we have used the trigonometric identity

$$-\cos\omega t = \sin\left(\omega t - \frac{\pi}{2}\right)$$

From the equation the amplitude of the current through the inductor is

$$I_{L0} = \frac{V_{L0}}{\omega L} = \frac{V_{L0}}{X_L}$$

where

$$X_L = \omega L$$

is called the ***inductive reactance***. It has SI units of ohms (Ω), just like resistance. However, unlike resistance, X_L depends linearly on the angular frequency ω. Thus, the resistance to current flow increases with frequency. This is due to the fact that at higher frequencies the current changes more rapidly than it does at lower frequencies. On the other hand, the inductive reactance vanishes as ω approaches zero.

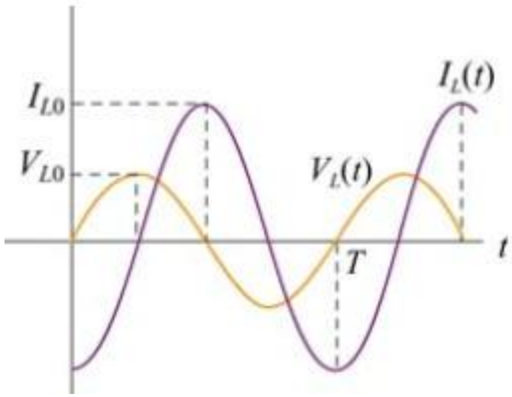
By comparing Eqs. of the current and voltage

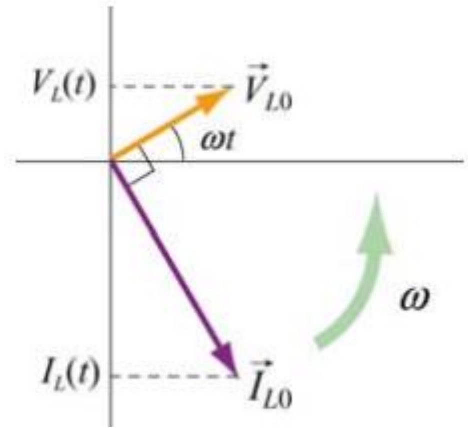
$$V(t) = V_0 \sin \omega t \qquad\qquad I_L(t) = I_{L0} \sin\left(\omega t - \frac{\pi}{2}\right)$$

We can be seen that, the current is out of phase with *Voltage* by ϕ = π / 2; it reaches its maximum value after *V* does by one quarter of a cycle. Thus, we say that

The current lags voltage by $\pi / 2$ in a purely inductive circuit

The current and voltage plots and the corresponding phasor diagram are shown in the figure below.





Problem

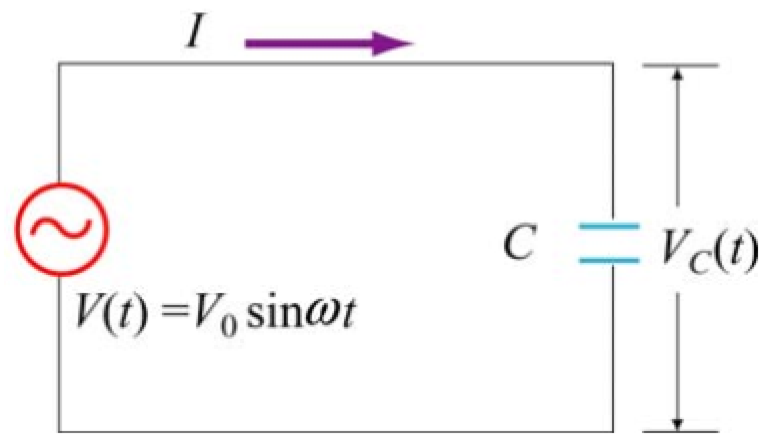
In a purely inductive AC circuit L = 25 mH and the rms voltage is 1.50 × 10^2 V. Find the inductive reactance and rms current in the circuit if the frequency is 60.0 Hz .

Solution

$$X_L = 2\pi f L = 2\pi(60.0\ \text{s}^{-1})(25.0 \times 10^{-3}\ \text{H}) = 9.42\ \Omega$$

$$I_{\text{rms}} = \frac{V_{L,\text{rms}}}{X_L} = \frac{1.50 \times 10^2\ \text{V}}{9.42\ \Omega} = 15.9\ \text{A}$$

AC Circuit with a Capacitor:



Now consider the simple series circuit in the figure, consisting of a capacitor connected to an AC generator.

Again, Kirchhoff's voltage rule implies

$$V(t) - V_C(t) = V(t) - \frac{Q(t)}{C} = 0$$

which yields

$$Q(t) = CV(t) = CV_C(t) = CV_{C0}\sin\omega t$$

where $V_{C0} = V_0$. On the other hand, the current is

$$I_C(t) = +\frac{dQ}{dt} = \omega C V_{C0}\cos\omega t = \omega C V_{C0}\sin\left(\omega t + \frac{\pi}{2}\right)$$

where we have used the trigonometric identity $\cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$

The above equation indicates that the maximum value of the current is

$$I_{C0} = \omega C V_{C0} = \frac{V_{C0}}{X_C}$$

where

$$X_C = \frac{1}{\omega C}$$

is called the ***capacitance reactance***. It also has SI units of **ohms** and represents the effective resistance for a purely capacitive circuit. Note that *XC* is **inversely proportional to both *C* and ω , and diverges as ω approaches zero**.

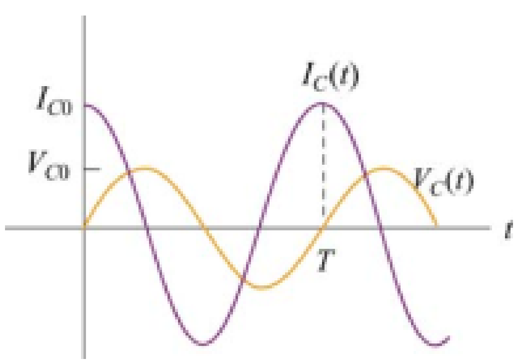
By comparing Eqs. of the current and voltage

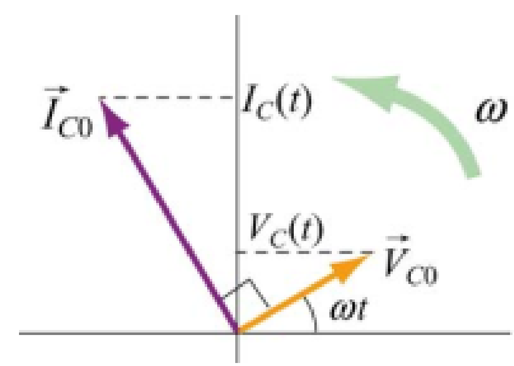
$$V(t) = V_0 \sin\omega t$$

$$I_C(t) = I_{C0} \sin\left(\omega t + \frac{\pi}{2}\right)$$

We can say that, **the voltage across a capacitor always lags the current by 90°**

The current and voltage plots and the corresponding phasor diagram are shown in the figure below.





Notice that at $t=0$, the voltage across the capacitor is zero while the current in the circuit is at a maximum. In fact, $I_C(t)$ reaches its maximum before $V_C(t)$ by one quarter of a cycle ($\phi=\pi/2$). Thus, we say that

The current leads the voltage by π/2 in a capacitive circuit or the voltage across a capacitor always lags the current by π/2 .

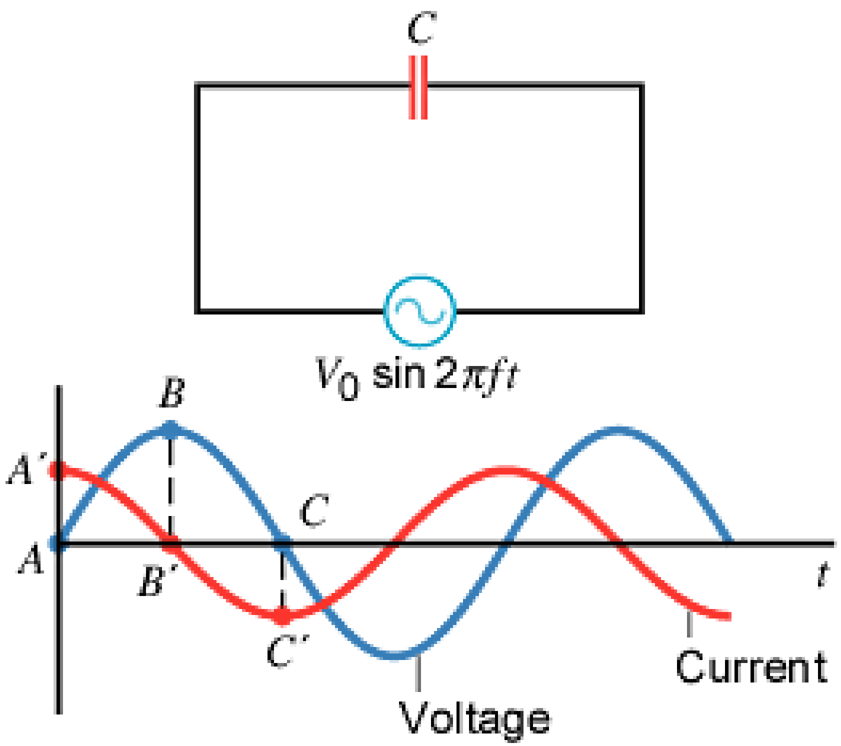
Problem

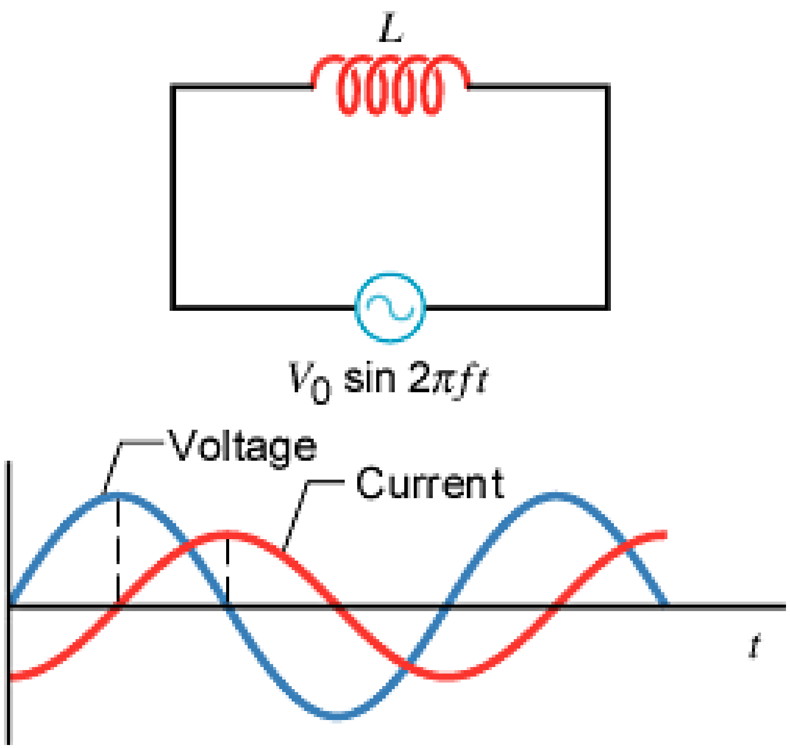
An 8μF capacitor is connected to the terminals of an AC generator with an rms voltage of 150 V and a frequency of 60 Hz. Find the capacitive reactance and the rms current in the circuit.

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi\,(60.0\text{ Hz})(8.00\times10^{-6}\text{ F})} = \boxed{332\ \Omega}$$

$$I_{\text{rms}} = \frac{\Delta V_{C,\text{rms}}}{X_C} = \frac{1.50\times10^{2}\text{ V}}{332\ \Omega} = \boxed{0.452\text{ A}}$$

A Comparison:Capacitive versus Inductive





The current *leads* the voltage by one-quarter of a cycle or by a phase angle of 90°.

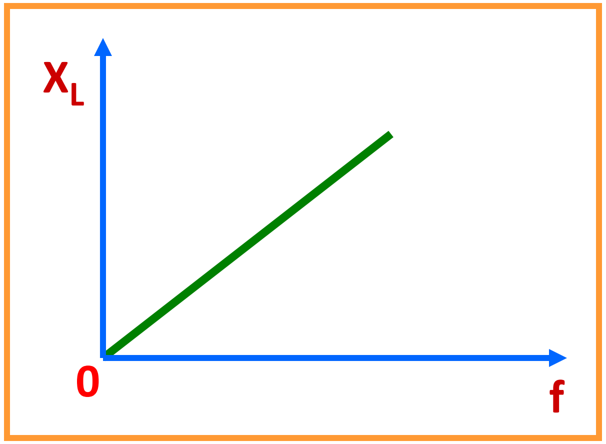
The current *lags behind* the voltage by one-quarter of a cycle or by a phase angle of 90°.

Variation of X_L with Frequency:

$X_L = \omega L$

X_L is Inductive Reactance and $\omega = 2\pi f$

$X_L = 2\pi f L$ i.e. $X_L \alpha f$

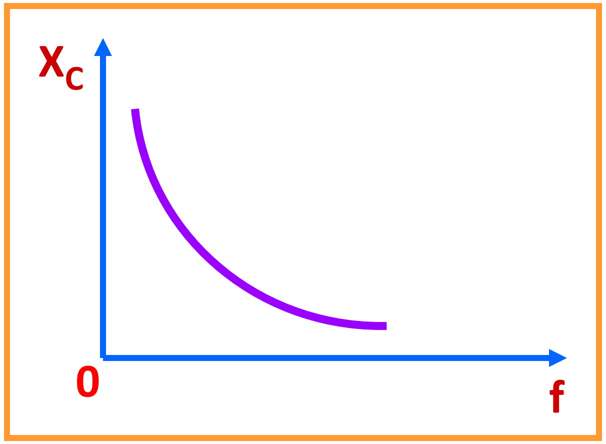


Variation of X_C with Frequency:

$X_C = 1 / \omega C$

X_C is Inductive Reactance and $\omega = 2\pi f$

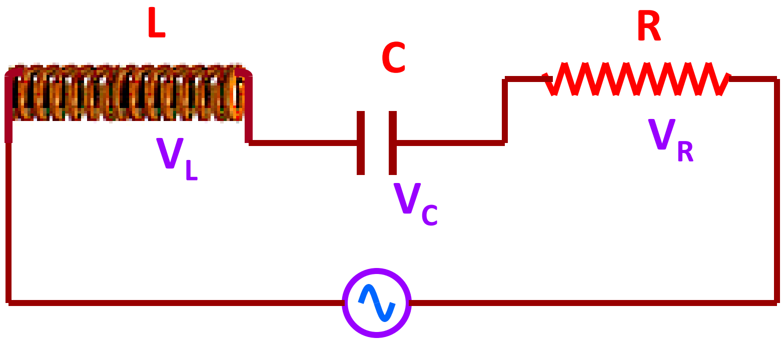
$X_C = 1 / 2\pi f C$ i.e. $X_C \alpha 1 / f$



TIPS:

1) Inductance (L) can not decrease Direct Current. It can only decrease Alternating Current.
2) Capacitance (C) allows AC to flow through it but blocks DC.

AC Circuit with L, C, R in Series Combination:



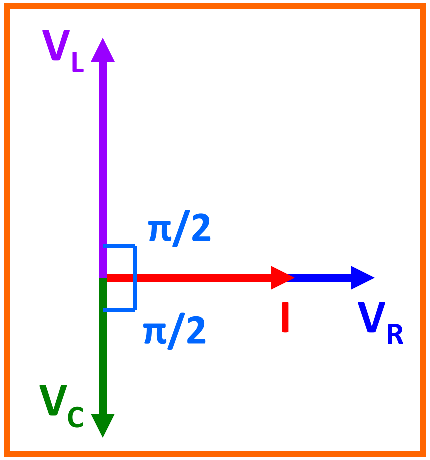
$V = V_0 \sin \omega t$

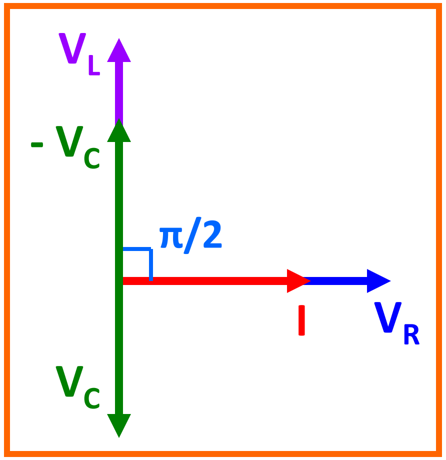
The applied AC appears as Voltage drops V_R, V_L and V_C across R, L and C respectively.

1) In R, current and voltage are in phase.

2) In L, current lags behind voltage by π/2

3) In C, current leads the voltage by π/2



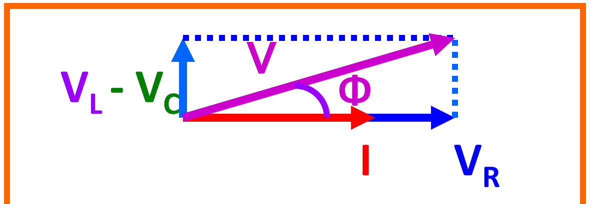


$$V = \sqrt{[V_R^2 + (V_L - V_C)^2]}$$

$$I = \frac{V}{\sqrt{[R^2 + (X_L - X_C)^2]}}$$

$$Z = \sqrt{[R^2 + (X_L - X_C)^2]}$$

$$Z = \sqrt{[R^2 + (\omega L - 1/\omega C)^2]}$$



$$V = \sqrt{[V_R^2 + (V_L - V_C)^2]}$$

$$\tan \Phi = \frac{X_L - X_C}{R} \quad \text{or} \quad \tan \Phi = \frac{\omega L - 1/\omega C}{R}$$

$$\tan \Phi = \frac{X_L - X_C}{R}$$ or $$\tan \Phi = \frac{\omega L - 1/\omega C}{R}$$

Special Cases:

Case I: When $X_L > X_C$ i.e. $\omega L > 1/\omega C$,

$\tan \Phi = +ve$ or Φ is $+ve$

The current lags behind the emf by phase angle Φ and the LCR circuit is inductance - dominated circuit.

Case II: When $X_L < X_C$ i.e. $\omega L < 1/\omega C$,

$\tan \Phi = -ve$ or Φ is $-ve$

The current leads the emf by phase angle Φ and the LCR circuit is capacitance - dominated circuit.

Case III: When $X_L = X_C$ i.e. $\omega L = 1/\omega C$,

$\tan \Phi = 0$ or Φ is $0°$

The current and the emf are in same phase. The impedance does not depend on the frequency of the applied emf. LCR circuit behaves like a purely resistive circuit.

Resonance in AC Circuit with L, C, R:

When $X_L = X_C$ i.e. $\omega L = 1/\omega C$, $\tan \Phi = 0$ or Φ is $0°$

$Z = \sqrt{[R^2 + (\omega L - 1/\omega C)^2]}$ becomes $Z_{min} = R$ and $I_{0max} = E / R$

i.e. The impedance offered by the circuit is minimum and the current is maximum. This condition is called resonant condition of LCR circuit and the frequency is called resonant frequency.

At resonant angular frequency ω_r,

$\omega_r L = 1/\omega_r C$ or $\omega_r = 1 / \sqrt{LC}$ or $f_r = 1 / (2\pi \sqrt{LC})$

Resonant Curve & Q - Factor:

Band width = $2\, \Delta \omega$

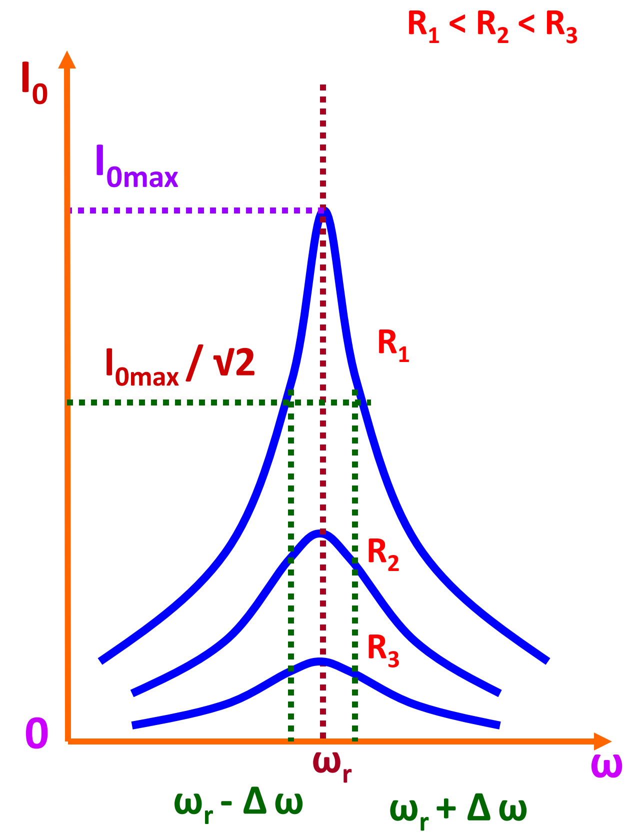
Quality factor (Q – factor) is defined as the ratio of resonant frequency to band width.

$$Q = \omega_r / 2\, \Delta \omega$$

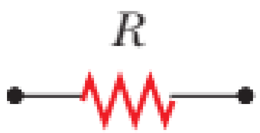
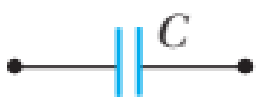
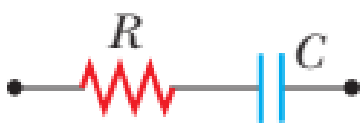
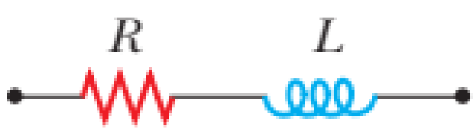
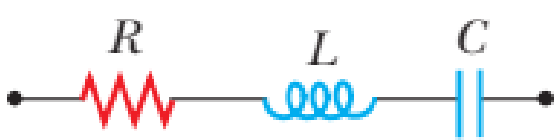
It can also be defined as the ratio of potential drop across either the inductance or the capacitance to the potential drop across the resistance.

$Q = V_L / V_R$ or $Q = V_C / V_R$

or $Q = \omega_r L / R$ or $Q = 1 / \omega_r CR$



Impedance Values and Phase Angles for Various Combinations of Circuit Elements[a]

Circuit Elements	Impedance Z	Phase Angle ϕ
	R	0°
	X_C	–90°
	X_L	+90°
	$\sqrt{R^2 + X_C^2}$	Negative, between –90° and 0°
	$\sqrt{R^2 + X_L^2}$	Positive, between 0° and 90°
	$\sqrt{R^2 + (X_L - X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

[a] In each case, an AC voltage (not shown) is applied across the combination of elements (that is, across the dots).

Problem A series *RLC* AC circuit has resistance $R = 2.50 \times 10^2\ \Omega$, inductance $L = 0.600$ H, capacitance $C = 3.50\ \mu$F, frequency $f = 60.0$ Hz, and maximum voltage $\Delta V_{max} = 1.50 \times 10^2$ V. Find **(a)** the impedance, **(b)** the maximum current in the circuit, **(c)** the phase angle, and **(d)** the maximum voltages across the elements.

$$X_L = 2\pi f L = 226\ \Omega \qquad X_C = 1/2\pi f C = 758\ \Omega$$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{(2.50 \times 10^2\ \Omega)^2 + (226\ \Omega - 758\ \Omega)^2} = \boxed{588\ \Omega}$$

$$I_{max} = \frac{\Delta V_{max}}{Z} = \frac{1.50 \times 10^2\ \text{V}}{588\ \Omega} = \boxed{0.255\ \text{A}}$$

$$\phi = \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\left(\frac{226\ \Omega - 758\ \Omega}{2.50 \times 10^2\ \Omega}\right) = \boxed{-64.8^\circ}$$

$$\Delta V_{R,max} = I_{max} R = (0.255\ \text{A})(2.50 \times 10^2\ \Omega) = \boxed{63.8\ \text{V}}$$

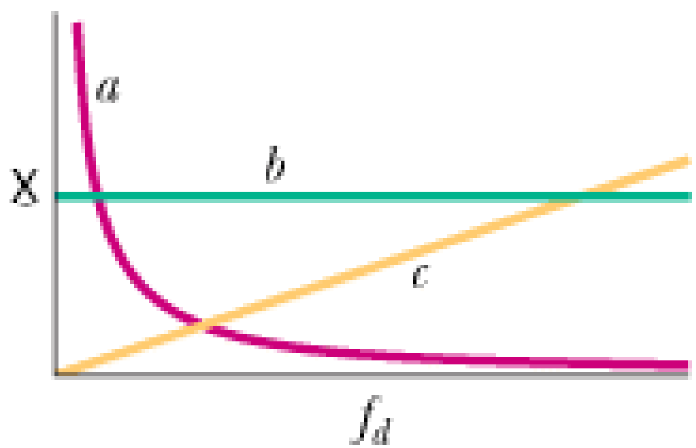
$$\Delta V_{L,max} = I_{max} X_L = (0.255\ \text{A})(2.26 \times 10^2\ \Omega) = \boxed{57.6\ \text{V}}$$

$$\Delta V_{C,max} = I_{max} X_C = (0.255\ \text{A})(7.58 \times 10^2\ \Omega) = \boxed{193\ \text{V}}$$

Exercise

Analyze a series *RLC* AC circuit for which $R = 175\ \Omega$, $L = 0.500$ H, $C = 22.5\ \mu F$, $f = 60.0$ Hz, and $\Delta V_{max} = 325$ V. Find (a) the impedance, (b) the maximum current, (c) the phase angle, and (d) the maximum voltages across the elements.

Exercise



In the above Reactance VS. frequency diagram, identify them as that of a capacitor, inductor, and resistor.