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English Section

Operations Research

An Introduction

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1st Edition

Chapter (1)

Basics of Operations Research

(1-1): Introduction:

The main objective of this chapter is to provide an overall view of operations research (abbreviated OR) and its origin as well as the fundamental phases in an OR study. Although it is not possible to detail all the phases of OR here, the objective is to present a unified treatment of the subject that can be used as a general guideline for solving OR problems. Therefore, this chapter provides an overview of OR applied to problem formulation and its solution.

(1-2): Development of Operations Research:

During the World War II, the military management in U.K. (or England) called on a teamwork of scientists to study the strategic and tactical problems associated with air and land defense of the country. Their objective was to determine the most effective utilization of limited military resources. The applications included among others, studies of the way to use the newly invented radar and of effectiveness of new types of bombers. The establishment of this scientific teamwork marked the first formal operations research activity.

The name of operation research was apparently coined because the teamwork was dealing with research on military

operations. Since its birth, this new decision-making field has been characterized by the use of scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.

The encouraging results achieved by the British operations research teams motivated the United States military management to start similar activities. Many successful application of the U.S. teamwork's included the study of complex logistical problems, the invention of new flight patterns, the planning of sea mining, and the effective utilization of electronic equipment.

Following the war, the success of this military teamwork's attracted the attention of industrial managers who were seeking solutions to their problems, which were becoming more acute because of the introduction of functional specialization into business organizations. Despite the fact that specialized functions are established primarily to serve the overall objective of the organization, the individual objectives of these functions may not always be consistent with the goals of the organization. This has resulted in complex decision problems that ultimately have forced business organizations to seek the utilization of the effective tools of OR.

Although Great Britain is credited with the initiation of OR as a new discipline, the leadership in the rapidly growing field was soon

taken over by the United States. The first widely accepted mathematical technique in the field, called the simplex method of linear programming, was developed in 1947 by the American mathematician George B. Dantzig. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in both academic institutions and industry.

Today, the impact of OR can be felt in many areas. This is indicated by the number of academic institutions offering courses in this subject at all degree levels.

Many management consulting firms are currently engaged in OR activities. These activities have gone beyond military and business applications to include hospitals, financial institutions, libraries, city planning, transportation systems and even crime investigation studies.

(1.3): The Concept of OR:

An OR study consists in building a model of the physical situation. An OR model is defined as simplified representation of a real-life system. This system may already be in existence or may still be an idea awaiting execution. In the first case, the model's objective is to analyze the behavior of the system in order to improve its performance. While, in the second case, the objective is to identify

the best structure of the future system. The complexity of a real system results from the very large number of elements or variables controlling the behavior of the system.

A teamwork of scientists in the branch of OR defined OR as that scientific branch which concerned with the model construction, its formulation, how it can be found its optimum technique for its solution and how can use these techniques in order to answer for the quarry about what will happen if there is an specific condition (or a set of conditions) will be hold (or not), which is called the sensitivity analysis. Others defined that OR is the branch which concerned with the cases studies or the applied statistics.

From the preceding definitions, the branch of OR depends on a set of basic characteristics shown as follows:

1- **Applied the scientific methods for treating the different problems.**
2- **Construct a mathematical model for solving the problems and take the optimal decision from the feasible solutions.**
3- **Determine the model which can be used in the sensitivity analysis.**

(1.3.1): Types of OR Models:

Although it is not possible to present fixed rules about how a model is constructed, it may be helpful to present ideas about

possible OR model types, their general structures, and their general characteristics.

The most important type of OR model is the symbolic or mathematical model. In formulating this type, one assumes that all relevant variables are quantifiable. Thus mathematical symbols are used to represent variables, which are then related by the appropriate mathematical functions to describe the behavior of the system. The solution of the model is then achieved by appropriate mathematical manipulation as will be shown in the succeeding chapters.

Most of OR analysts identify the name operations research primarily with mathematical models. The reason may be that such models are amenable to mathematical analysis, which usually makes it possible to find the "best" solution by means of convenient mathematical tools. It is not surprising then that most of the attention in OR has been directed toward the development of mathematical models.

In addition to mathematical models, simulation and heuristic models are used. Simulation models "imitate" the behavior of a specific system over a period of time. This is achieved by specifying a number of events which are point in time whose occurrence signifies that important information pertaining to the behavior of the system can be gathered. Once such events are defined, it is necessary to pay

attention to the system only when an event occurs. The information yielding measures of performance for the system is accumulated as statistical observations, which are updated as each event takes place. Naturally, the simulation model is not as convenient as the (successful) mathematical models which yield a general solution to the problem.

While mathematical seek the determination of the best or the optimum solution, sometimes the mathematical formulation may be too complex to allow an exact solution. Even, if the optimum solution can be attained eventually, the required computation may be impractically long. In this case, heuristics can be rules that given a current solution to the model, allow the determination of an improved solution.

(1.3.2.): Structure of Mathematical Models:

Mathematical model includes three basic sets of elements stated as follows:

1- **Decision variables and parameters: The decision variables are the unknowns to be determined from the solution of the model. The parameters represent the controlled variables of the system. For example, the production level represents a decision variable. Example of the parameters in this case include the production and consumption rates of the stocked** item. In general, the parameters of the model may be deterministic or probabilistic.

2- <u>Constraints or restrictions:</u> To account for the physical limitations of the system, the model must include constraints which limit the decision variables to their feasible (or permissible) values. This is usually expressed in the form of constraining mathematical functions. For example, let x_1 and x_2 be the number of units to be produced of two products (decision variables) and let a_1 and a_2 be their respective per unit requirements of raw material (parameters). If the total amount available of this raw material is A, the corresponding constraint function is given by $a_1x_1 + a_2x_2 \leq A$.

3- <u>Objective Function:</u> This defines the measure of effective-eness of the system as a mathematical function of its decision variables. For example, if the objective of the system is to maximize the total profit, the objective function must specify the profit in terms of the decision variables. In general, the *optimum* solution to the model is obtained when the corresponding values of the decision variables yield the best value of the objective function while satisfying all the constraints. This means that the objective function acts as an indicator for the achievement of the optimum solution.

Mathematical models in operations research may be viewed generally as determining the values of the decision variables x_j for $j = 1, 2, \ldots, n$, which will:

$$\text{Optimize} \quad x_0 = f(x_1, \ldots, x_n)$$

Subject to

$$g_i(x_1, \ldots, x_n) \leq b_i \;, \quad i = 1,2,\ldots,m$$

$$x_j \geq 0, \quad j = 1,2,\ldots,n$$

The function f is the objective function, while $g_i \leq b_i$ represents the i^{th} constraint, where b_i is a known constant. The constraints $x_i \geq 0$ are called the nonnegativity constraints, which restrict the variables to zero or positive values only. In most real-life systems, the non-negativity constraints appear to be a natural requirement.

*Model Optimization:

The above discussion indicates that a mathematical model seeks to "optimize" a given objective function subject to a set of constraints. Optimization is generally taken to signify the "maximization" or "minimization" of the objective function. But this is about the extent to which the word "optimization" goes in unifying mathematical models. By this it is meant that two analysts working on the same problem independently may yield two different models with different objective criteria. For example, analyst A may

prefer to maximize profit, while Analyst B may rightly prefer to minimize cost. The two criteria are not equivalent in the sense that with the same constraints the two models may not produce the same optimum solution. This can be made clear by realizing that, although cost may be under the immediate control of the organization in which the study is made, profit could be effected by uncontrollable factors such as the market situation dictated by competitors.

The main conclusion at this point is that "the" optimum solution of a model is the best only relative to that model. In other words, one must not think that this optimum is *the* best for the problem under consideration. Rather, it is the *best* only if the specified criterion can be justified as a true representation of the goals of the entire organization in which the problem exists.

(1.4): Phases of Operations Research Study:

An OR study cannot be conducted and controlled by the OR analyst alone. Although he may be the expert on modeling and model solution techniques, he cannot possibly be an expert in al the areas where OR problems arise. Consequently, an OR team should include members of the organization directly responsible for the functions in which the problem exists as well as for the execution and implementation of the recommended solution. In other words, an OR analyst commits a grave mistake by assuming that he can

solve problems without the cooperation of those who will implement his recommendations.

The major phases through which the OR team would proceed in order to effect an OR study include

1-Definition of the problem.

2-Construction of the model.

3-Solution of the model.

4-Validation of the model.

5-Implementation of the final results.

Although the above sequence is by no means standard, it seems generally acceptable. Except for the "model solution" phase, which is based generally on well-developed techniques, the remaining phases do not seem to follow fixed rules. This stems from the fact that the procedures for these phases depend on the type of problem under investigation, and the operating environment in which it exists. In this respect, an operations research team would be guided in the study principally by the different professional experiences of its members rather than by fixed rules.

(1.5): Linear Programming Models and its Applications:

Linear programming is a class of mathematical program-ming models concerned with the efficient allocation of limited resources

to known activities with the objective of meeting a desired goal (such as maximizing profit or minimizing cost). The distinct characteristic of linear programming models is that the functions representing the objective and the constraints are linear.

This chapter introduces the reader to some of the applications of linear programming. The examples are taken from actual applications in different fields in order to illustrate the diverse uses of this type of model. As stated in Chapter 1, a presentation of the procedure for gathering data for the model will take the discussion far afield. Instead, the analysis will concentrate on how the assumption of linearity can be justified.

The linearity of some models can be justified based on the physical properties of the problem; other models, which in the direct sense are nonlinear, can be linearized by the proper use of mathematical transformations. Examples of these types will be presented in the next section. Formulation of the theoretical problem in a linear programming models means that we have to determine the following basics:

1-Decision variables and parameters.

2-Objective function.

3-Constraints and non- negativity constraints.

These basic components may be stated in matrix form as follows: Determine the decision variable $x_1, x_2, \ldots, x_n$ in which will optimize

$$f(x) = f(x_1, x_2, \ldots, x_n) \text{ subject to: } \underline{A}\,\underline{x} \leq \underline{b},\ \underline{x} \geq 0$$

Examples of Linear Programming Applications:

The applications in this chapter are excerpted from the following areas:

1-Production planning.

2-Feed mix.

3-Stock cutting or slitting.

4-Water-quality management.

5-Oil drilling and production.

6-Assembly balancing.

7-Inventory.

*Example (1): (Production Planning):

Three products are processed through three different operations. The times (in minutes) required per unit of each product, the daily capacity of the operations (in minutes per day) and the profit per unit sold of each product (in dollars) are as follows:

Operation	Time per unit (minutes)			Operation capacity (minutes/day)
	Product 1	Product 2	Product 3	
1	1	2	1	430
2	3	0	2	460
3	1	4	0	420
Profit/unit ($)	3	2	5	

The zero times indicate that the product does not require the given operation. It is assumed that all units produced are sold. Moreover, the given profits per unit are net values that result after all pertinent expenses are deducted. The goal of the model is to determine the optimum daily production for the three products that maximizes profit.

As mentioned in Chapter 1, the main elements of a mathematical model are (1) the variables or unknowns, (2) the objective function, and (3) the constraints. The variables are immediately identified as the daily number of units to be manufactured of each product. Let x_1, x_2, and x_3 be the number of daily units produced of products 1, 2, and 3. Because of the assumption that all units produced are sold, the total profit x_0 (in dollars) for the three products is $x_0 = 3x_1 + 2x_2 + 5x_3$.

The constraints of the problem must ensure that the total processing time required by all produced units does not exceed the daily capacity of each operation. These are expressed as:

Operation 1: $1x_1 + 2x_2 + 1x_3 \leq 430$

Operation 2: $3x_1 + 0x_2 + 2x_3 \leq 460$

Operation 3: $1x_1 + 4x_2 + 0x_3 \leq 420$

Because it is nonsensical to produce negative quantities, the additional *non-negativity constraints* $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ must be added.

Some operations research users have a tendency to replace the inequality ($\leq$) in the "operations constraints" by a strict equation (=). The justification is that it is better to use all available resources than to "waste" part of it. This reasoning does not hold since the use of ($\leq$) automatically implies (=). Thus, if the *optimum* solution requires that all constraints be satisfied exactly, the inequalities ($\leq$) still allow this. In other words, the strict equalities should not be imposed unless the problem requires that all operations must work to full capacity. This is completely different from simply stipulating that the capacity of each operation should not be exceeded.

The linear programming model is now summarized as follows.

Maximize $x_0 = 3\,x_1 + 2\,x_2 + 5\,x_3$

Subject to :

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &\le 430 \\
2x_1 \qquad + 2x_3 &\le 460 \\
x_1 + 4x_2 \qquad &\le 420 \\
x_1,\ x_2,\ x_3 &\ge 0
\end{aligned}
$$

What makes the above problem fit a linear programming model? Several implicit assumptions allow (1) imposing constant proportionality between the number of units of a product and its total contribution to the objective function (or its usage of each operation's time), and (2) adding directly the profit contributions (or the time requirements) of each product to obtain the total profit of the system (or the total usage of a given operation's time). Suppose, for example, a price break is allowed so that if the size of an order exceeds a certain quantity, the sale price (and hence profit) per unit decreases by a fixed amount. In this case, the constant proportionality assumption built in the objective function is invalid. Another example is that if defective pieces are reworked on the same operation, it is no longer true that the time requirement per unit is constant for each operation. A third example is that the volume of sales for the three products may be interdependent. Unless the relationships between volumes of sales are linear, the direct addition of the individual profit contributions as given in the above objective function will be unacceptable. A specific illustration is as follows. Let *y* and *z* be the sales volumes of two competing

products where an increase in the sales volume of one product adversely affects the sales volume of the other. Mathematically, this means that y is proportional to $1/z$. If b is the proportionality constant, then $y = b/z$, or $yz = b$, which is not a linear constraint.

The above discussion suggests situations where the linearity assumption is not justified. Some nonlinearities, however, may be "approximated" by linear functions. For example, the nonlinearities created by the quantity discount may be approximated by a linear function.

Example (2):

A company produce three products 1, 2 and 3 by using three different raw materials A, B and C. The following table represents the required per unit of each product, the daily capacity of each material (A and B in kilograms, C in hours) and price per unit sold of each product (in dollars) are as follows:

Product / Raw Material	K.G and hours per unit produced			Available capacity
	1	2	3	
A	5	3	7	2100 K.G
B	4	2	5	1600 K.G
C	3	2	4	1700 K.G
Sold price per unit produced	51	32	52	

In addition you have the following data:

- **The unit cost for each of the raw materials A and B are 4 and 3$ respectively, and the wage rate for each hour in the C operation is one dollar. Besides that, each unit produced needs 5, 6 and 3$ respectively as a tips.**
- **The demand units for marketing the three products are 100, 150 and 200 units respectively.**
- **The number of units produced from the product A must be twice of B.**
- **The fixed cost of this company is 2500$.**

Required: Formulate the problem as a linear programming problem.

Solution:

***Decision variable and parameters:**

Assume that x_1 , x_2 and x_3 are the three decision variables that denote the number of unit must be produced from the three different products respectively A, B and C, then the following table represents the different parameters:

Product Raw mat.	1	2	3	Cost per unit of raw mat.	Capacity
A B C	5 4 3	3 2 2	7 5 4	4$ / 1 K.G 3$ / 1 K.G 1$ / 1 H	2100 1600 1700
Var. cost (tips)	5	6	3		
Var. cost (raw mat.)	4(5)+3(4) +1(3) =35	4(3)+3(2) +1(2) =20	4(7)+3(5) +1(4) =47		
Total var. cost	35+5= 40	26	50		
Price unite	51	32	52		
Profit unite	11	6	2		

Then, we have to find x_1 , x_2 and x_3 by which make the total net profit function:

$$f(x) = 11x_1 + 6x_2 + 2x_3 - 2500 \qquad \textbf{Maximization}$$

Subject to:

$$
\begin{array}{lllll}
1) & 5x_1 + 3x_2 + 7x_3 & & & \le 2100 \\
2) & 4x_1 + 2x_2 + 5x_3 & & & \le 1600 \\
3) & 3x_1 + 2x_2 + 4x_3 & & & \le 1700 \\
4) & x_1 & & & \ge 100 \\
 & & x_2 & & \ge 150 \\
 & & & x_3 & \ge 200 \\
5) & x_1 - 2x_2 & & & = 0 \\
6) & x_1 & & & \ge 0 \\
 & & x_2 & & \ge 0 \\
 & & & x_3 & \ge 0
\end{array}
$$

(1.6): Definitions of linear programming forms:

The real-life examples in the preceding section show that a linear program may be of the maximization or minimization type. The constraints may be of the type ($\le$) , (=), or ($\ge$) and the variables may be nonnegative or unrestricted in sign. A general linear programming model thus is usually defined as follows.

Maximize or minimize $x_0 = c_1x_1 + c_2x_2 + \ldots + c_n x_n$

Subject to :

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \quad (\leq, =, \text{ or } \geq)\ b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \quad (\leq, =, \text{ or } \geq)\ b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \ (\leq, =, \text{ or } \geq)\ b_m$$

$$x_1, \quad x_2, \ldots \quad, x_n \ \geq 0$$

Where c_j, b_i , and a_{ij} (i= 1,2, …, m; j = 1,2, …, n) are constants determined from the technology or the problem, and x_j are the decision variables. Only one sign ($\leq$, = , or $\geq$) holds for each constraint. Although all variables are declared nonnegative, the preceding section shows that every unrestricted variable can be converted equivalently to nonnegative variables. The non-negative restriction is essential for the development of the solution method for linear programming.

Linear programming models often represent "allocation" problems in which limited resources are allocated to a number of activities. In terms of the above formulation, the coefficients c_i, a_{ij}, and b_i are interpreted physically as follows.

If b_i is the available amount of resource *i* , then a_{ij} is the amount of resource *i* that must be allocated to each unit of activity *j*. The "worth" per unit of activity *j* is equal to c_j.

After formulating a linear programming model, the analyst's next step is to solve the model. Because linear programming models are presented in a variety of forms [maximization or minimization for the objective function and ($\leq$, = , and/or $\geq$) for the constraints], it is necessary to modify these forms to fit the solution procedure that will be presented in the next chapter. Two forms are introduced for this purpose: the *canonical* form and the *standard* form. The standard form is used directly for solving the model; the canonical form is particularly useful in presenting duality theory. The details of the two forms are now presented.

(I):The Canonical Form:

The general linear programming problem defined above can always be put in the following form, which will be referred to as the *canonical* form:

$$\text{Maximize} \quad x_0 = \sum c_j x_j$$

Subject to

$$\sum a_{ij}x_j \leq b_i, \quad i= 1,2,\dots,m$$

$$X_j \geq 0, \; j = 1,2, \dots, n$$

The characteristics of this form are:

1-All decision variables are nonnegative.

2-All constraints are of the ($\leq$) type.

3-The objective function is of the maximization type.

A linear programming problem can be put in the canonical form by the use of five elementary transformations.

1-The minimization of a function, *f* (x), is mathematically equivalent to the maximization of the negative expression of this function, -*f*(x). For example, the linear objective function

$$\text{Minimize } x_0 = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

is equivalent to

$$\text{maximize } g_0 = -x_0 = -c_1x_1 - c_2x_2 - \dots - c_nx_n$$

With $x_0 = -g_0$. Consequently, in any linear programming problem, the objective function can be put in the maximization form.

2-An inequality in one direction ($\leq$ or $\geq$) may be changed to an inequality in the opposite direction ($\geq$ or $\leq$) by multiplying both sides of the inequality by (-1). For example, the linear constraint.

$$a_1x_1 + a_2x_2 \geq b$$

is equivalent to

$$-a_1x_1 - a_2x_2 \leq -b$$

Also,

$$P_1x_1 + p_2x_2 \leq q$$

Is equivalent to

$$-p_1x_1 - p_2x_2 \geq -q$$

3-An equation may be replaced by two (weak) inequalities in opposite directions. For example,

$$a_1x_1 + a_2x_2 = b$$

is equivalent to the two simultaneous constraints

$$a_1x_1 + a_2x_2 \leq b \quad \text{and} \quad a_1x_1 + a_2x_2 \geq b$$

or $$a_1x_1 + a_2x_2 \leq b \quad \text{and} \quad -a_1x_1 - a_2x_2 \leq -b$$

4-An inequality constraint with its left-hand side in the absolute form can be changed into two regular inequalities. Thus, for $b \geq 0$,

$$|\, a_1x_1 + a_2x_2 \,| \leq b$$

is equivalent to

$$a_1x_1 + a_2x_2 \geq -b \quad \text{and} \quad a_1x_1 + a_2x_2 \leq b$$

Similarly, for $q \geq 0$,

$$|\, p_1x_1 + p_2x_2 \,| \geq q$$

is equivalent to either

$$p_1x_1 + p_2x_2 \geq q \quad \text{or} \quad p_1x_1 + p_2x_2 \leq -q$$

4- **A variable which is unconstrained in sign (that is, positive, negative, or zero) is equivalent to the difference between two nonnegative variables. Thus, if x is unconstrained in sign, it can be replaced by ($x^+ - x^-$) where $x^+ \geq 0$ and $x^- \geq 0$.**

<u>-Example (3):</u>

Consider the linear programming problem

$$\text{Minimize } x_0 = 3x_1 - 3x_2 + 7x_3$$

Subject to

$$X_1 + x_2 + 3x_3 \leq 40$$

$$x_1 + 9x_2 - 7x_3 \geq 50$$

$$5x_1 + 3x_2 = 20$$

$$|5x_2 + 8x_3| \leq 100$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0$$

x_3 is unconstrained in sign

The problem can be put in the canonical form as follows. By the fourth transformation,

$$|5x_2 + 8x_3| \leq 100$$

Is equivalent to :

$$5x_2 + 8x_3 \leq 100 \quad \text{and } 5x_2 + 8x_3 \geq -100$$

By the fifth transformation,

$$x_3 = x_3^+ - x_3^-$$

Where $x_3^+ \geq 0$ and $x_3^- \geq 0$. Finally, if the objective function is transformed to maximization, the canonical form becomes.

$$\text{Maximize } g_0 = (-x_0) = -3x_1 + 3x_2 - 7(x_3^+ - x_3^-)$$

Subject to

$$X_1 + x_2 + 3(x_3^+ - x_3^-) \leq 40$$

$$-x_1 - 9x_2 + 7(x_3^+ - x_3^-) \leq -50$$

$$5x_1 + 3x_2 \leq 20$$

$$-5x_1 - 3x_2 \leq -20$$

$$5x_2 + 8(x_3^+ - x_3^-) \leq 100$$

$$-5x_2 - 8(x_3^+ - x_3^-) \leq 100$$

$$X_1 \geq 0,\; x_2 \geq 0,\; x_3^+ \geq 0,\; x_3^- \geq 0$$

The only difference between the original and the canonical forms in the above example occurs in the objective function where x_0 in the original problem becomes equal to $(-g_0)$ in the canonical form. The values of the variables are the same in both cases, however, since the constraints are mathematically equivalent.

(2):The Standard Form:

The characteristics of the standard form are:

1-All constraints are equations except for the non-negativity constraints which remain inequalities ($\geq$ 0).

2-The right-hand side element of each constraint equation is nonnegative.

3-All variables are nonnegative.

4-The objective function is of the maximization or the minimization type.

Inequality constraints can be changed to equations by augmenting (adding or subtracting) the left-hand side of each such constraint by a nonnegative variable. These new variables are called slack variables and are added if the constraint is ($\leq$) or subtracted if the constraint is ($\geq$). The right-hand side can be made always positive by multiplying both sides of the resulting equation by (-1) whenever necessary. The remaining characteristics can be realized by using the elementary transformations introduced with the canonical form.

To illustrate the concept of the slack variables, the constraint

$$A_1x_1 + a_2x_2 \geq b\ ,\quad b \geq 0$$

Is changed in the standard form to

$$A_1x_1 + a_2x_2 - S_1 = b$$

Where $S_1 \geq 0$. Also, the constraint

$$P_1x_1 + p_2x_2 \leq q, \quad q \geq 0$$

Is changed to

$$P_1x_1 + p_2x_2 + S_2 = q$$

Where $S_2 \geq 0$

The standard form plays an important role in the solution of the linear programming problem. For example, consider the following linear program with all ($\leq$) constraints:

$$\text{Maximize} \quad x_0 = \sum c_jx_j$$

Subject to :

$$\sum a_{ij}x_j \leq b_i \;, \quad (b_i \geq 0)\;, \; i = 1,2, \ldots, m$$

$$x_i \geq 0 \quad , \qquad\qquad j = 1, 2, \ldots, n$$

This is expressed in the standard form as

$$\text{Maximize} \quad x_0 = \sum c_jx_j$$

Subject to:

$$\sum a_{ij}x_j + S_i = b_i, \; i = 1,2,\ldots, m$$

$$x_j \geq 0, \; j = 1,2, \ldots, n$$

$$S_i \geq 0, \; i = 1,2,\ldots, m$$

The standard form basically reduces the linear program to a set of (m) equations in (m + n) unknowns. A solution to these equations is of interest only if it is feasible, that is, if it satisfies the non-negativity constraints $x_j \geq 0$ and $S_i \geq 0$, for all i and j. The optimum solution is then given by the feasible solution that maximizes x_0.

The idea given above is simple, but the difficulty is that a set of (m) equations in (m + n) unknowns usually yields infinity of solutions. Consequently, since it is computationally impossible to determine every feasible point, a procedure is needed which locates the optimum after checking a finite number of solution points. Chapter 3 presents the simplex method, which is an interactive algorithm proven to converge to the optimum (when it exists) in a finite number of iterations.

In summary, the following table represents the different characteristics between each of the canonical and the standard form as follows:

Canonical form	Standard form
1-The objective function f(x) must be in the maximization form.	**1-The objective function may be either in Maximization or in minimization form.**
2-Each constraint must be in the form ($\leq$)	**2-The right hand side for each constraint (constant) must be positive**
3-Each decision variable must be nonnegative.	**3-Each constraint must be in the equation form (=).**
	4-Each decision variable must be nonnegative.

Note that: If there is a constraint in the absolute value form, i.e., we have as the following form:

$$| ax_1 \pm bx_2 | < C$$

Then we have the two forms:

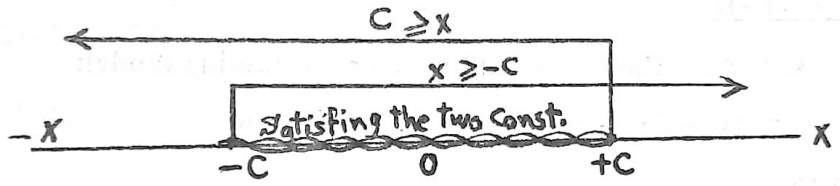
I- If $| ax_1 \pm bx_2 | \leq c$

i.e., $-c \leq ax_1 \pm bx_2 \leq c$

Put $y = a_1x \pm bx_2$

Then: $-c \leq y \leq c$

Graphically (As we will show in the following chapter) we can imagine the follow sketch:



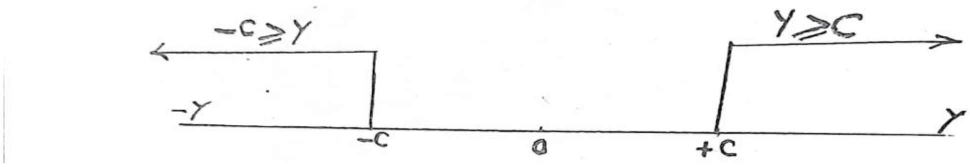
The intersection between the two arrows which are represent the area by which satisfied the two sub constraints $y = ax_1 \pm bx_2 \geq -c$ and $y = ax_1 \pm bx_2 \leq c$ means that we must take the two constraints resulted from the constraint of the absolute value in the solution.

II-If $| ax_1 \pm bx_2 | \geq c$ meaning that

$-c \geq ax_1 \pm bx_2 \geq c$

i.e., $-c \geq y \geq c$, where $y = ax_1 \pm bx_2$

Graphically:



Since there is no intersection between the two arrows, (i.e., there is no feasible solution between the two arrows), then, we have to take only one of the two sub-constraints: $y = ax_1 \pm bx_2 \geq c$ or

$y = ax_1 \pm bx_2 \geq c$ in the solution.

Example (4):

If you have the following linear programming model:

$f(x) = 2x_1 - 3x_2 + 5x_3$ **(minimization)**

Subject to:

(1)	$x_1 + 2x_2 + x_3$	$\leq$	35
(2)	$2x_1 + 7x_2 - 5x_3$	$\geq$	70
(3)	$3x_1 - 2x_2$	=	5
(4)	$\lvert 5x_1 + 12x_2 - 7x_3 \rvert$	$\leq$	90
(5)	$\lvert 7x_1 - 8x_2 - x_3 \rvert$	$\geq$	12
	x_1 , x_3	$\geq$	0

x_2 **is unrestricted in sign.**

Required:

Put the linear programming model in the canonical form and the standard form.

Solution:

(I)The canonical form:

Since x_2 **is unrestricted in sign, then, we have to substitute that:**

$x_2 = x_2^+ - x_2^-$ **where** x_2^+ **and** x_2^- **are nonnegative.**

i.e., $x_2^+ \geq 0$ and $x_2^- \geq 0$, then we have:

$$x_0 = -f(x) = -2x_1 + 3(x_2^+ - x_2^-) - 5x_3 \qquad \text{maximization}$$

subject to:

(1) $x_1 + 2(x_2^+ - x_2^-) + x_3 \leq 35$

(2) $-2x_1 - 7(x_2^+ - x_2^-) + 5x_3 \leq -70$

(3) $3x_1 - 3(x_2^+ - x_2^-) \leq 5$ **and**

$-3x_1 + 2(x_2^+ - x_2^-) \leq -5$

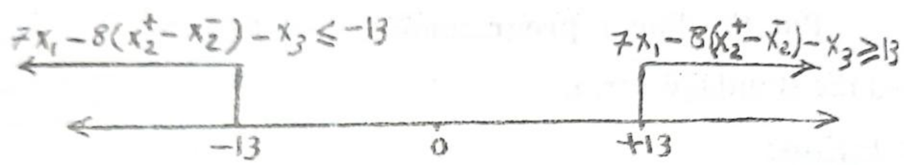
(4) $90 \leq 5x_1 + 12(x_2^+ - x_2^-) - 7x_3 \leq 90$

Then, we have the following two constraints in the solution:

$5x_1 + 12(x_2^+ - x_2^-) - 7x_3 \leq 90$ **and**

$-5x_1 - 12(x_2^+ - x_2^-) + 7x_3 \leq -90$

(5) $|\,7x_1 - 8x_2 - x_3\,| \geq 13$



Then we have to take only one of the following two constraints in the solution:

$-7x_1\ \ 8(x_2^+ - x_2^-) + x_3 \leq -13$

Or:

$7x_1 \;\; 8(x_2^+ - x_2^-) - x_3 \qquad \leq -13$

Where: $x_1 ,\; x_2^+ ,\; x_2^- ,\; x_3 \qquad \geq 0$

(II): The standard form:

Put $x_2 = x_2^+ - x_2^-$, then the standard form can be summarized as:

$F(x) = 2x_1 - 3(x_2^+ - x_2^-) + 5x_3$ Minimization.

Subject to:

(1) $x_1 + 2(x_2^+ - x_2^-) + x_3 + x_4 = 35$

(2) $2x_1 + 7(x_2^+ - x_2^-) - 5x_3 - x_5 = 70$

(3) $3x_1 - 3(x_2^+ - x_2^-) = 50$

(4) $-90 \leq 5x_1 + 12(x_2^+ - x_2^-) - 7x_3 \leq 90$

Then we have the following two constraints in the solution

$$5x_1 + 12(x_2^+ - x_2^-) - 7x_3 + x_6 = 90$$

And

$$-5x_1 - 12(x_2^+ - x_2^-) + 7x_3 + x_7 = 90$$

(5) $-12 \geq 7x_1 - 8(x_2^+ - x_2^-) - x_3 \geq 12$

then we have to take only one of the following two constraints in the solution

$$7x_1 - 8(x_2^+ - x_2^-) - x_3 - x_8 = 12$$

Or

$$-7x_1 + 8(x_2^+ - x_2^-) - x_3 - x_9 = 12$$

Where : $x_1, x_2^+, x_2^-, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$

Problems:

1-Four products are processed successively on two machines. The manufacturing times in hours per unit of each product are tabulated below for the two machines.

Machine	Time per unit (hours)			
	Product 1	Product 2	Product 3	Product 4
1	2	3	4	2
2	3	2	1	2

The total cost of producing a unit of each product is based directly on the machine time. Assume the cost per hour for machines 1 and 2 are \$10 and \$15. The total hours budgeted for all the products on machines 1 and 2 are 500 and 380. If the sales price per unit for products 1,2,3, and 4 are \$65, \$70, \$55 and \$45, formulate the problem as a linear programming model to maximize total net profit.

2-A company produces two types of cowboy hats. Each hat of the first type requires twice as much labor time as the second type. If all hats are of the second type only, the company can produce a total of

500 hats a day. The market limits daily sales of the first and second types to 150 and 250 hats. Assume the profits per hat are $8 for type 1 and $5 for type 2. Determine the number of hats to be produced of each type in order to maximize profit.

3-A manufacturer produces three models. (I, II, and III) of a certain product. He uses two types of raw material (A and B) of which 2000 and 3000 units are available, respectively. The raw material requirements per unit of the three models are given below.

Raw material	Requirements per unit of given model		
A	2	3	5
B	4	2	7

The labor time for each unit of Model I is twice that of Model II and three times that of Model III. The entire labor force of the factory can produce the equivalent of 700 units of Model 1. A market survey indicates that the minimum demand of the three models are 200, 200, and 150 units, respectively. However, the ratios of the number of units produced must be equal to 3: 2: 5. Assume that the profit per unit of Models I, II, and III are 30, 20, and 50 dollars. Formulate the problem as a linear programming model in order to determine the number of units of each product which will maximize profit.

Chapter (2)

Solving the Linear Programming Models

The purpose of this chapter is to present the procedure for solving the linear programming models. The presentation starts with a graphical solution of two-decision variables problem, which is subsequently used to develop an understanding of the algebraic procedure for solving linear programs which called the simplex procedure.

(2-1): Graphical Solution of Two-Variable Linear Programs:

The purpose of the graphical solution is not to provide a practical method for solving the linear programming models, since practical problems usually include a large number of decision variables. Instead, the graphical method demonstrates the basic concepts for developing the general algebraic technique (simplex methods) for linear programs with more than two variables.

The graphical solution is based on how can we graph of either linear equations or inequalities, and determine the solution space for each of them. In summary, the graphical solution passes through the following steps: -

1-Skech graphically the coordinates of any two points for each constraint, then plot the feasible solution space by which enclosed by all constraints in the (x_1, x_2) plane. The non-negativity constraints

specify that the feasible solutions must lie in the first quadrant defined by $x_1 \geq 0$ and $x_2 \geq 0$. Note that each of the constraints which will be plotted with ($\leq$ or $\geq$) replaced by (=), thus yielding simple straight-line equations. The region in which each constraint holds is indicated by an arrow on its associated straight line. After, determining the resulting feasible solution space (area), if any constraint can be deleted without effecting the solution space, then it is called a redundant constraint. Every point within or on the boundaries of the solution space satisfies all the constraints is called a feasible solution. And every corner in the solution space is called a basic feasible solution. (Extreme point).

2-Skech graphically the coordinates of any two point for the linear function of the objective function f(x) or x_0 = 0, then the optimum solution is that point in the solution space which yields the largest value (in case of maximum) or the lowest value (in case of minimum) of *f*(*x*) or x_0. The optimum solution can be determined by moving the line of *f*(*x*) or x_0 = 0 parallel to itself in the direction of the solution space, then it will be passes through the first or the latest point in the solution space which determined the minimum or maximum value of *f*(*x*) or x_0, then we have to determine the coordinates of these points. Substituting these values into the objective function gives the optimum solution.

<u>*Revision of graphical representation for the linear equations and inequalities:</u>

In this section we will represent how can we graph the linear equations and inequalities?

<u>a)Linear equation:</u>

A linear equation in two unknowns x_1 and x_2 has the standard form:

$$Ax_1 + bx_2 = C$$

Where a, b and c are real numbers.

Linear equations are first-degree equations. It is better to discuss first how we can represent graphically the linear equations. Graphing straight line of a linear equation of the form:

$ax_1 + bx_2 = c$ where a, b and $C \neq 0$ intersects the x_1 axis in (c/a) units from the origin, and similarly intersects the x_2 axis in (c/b) units from the origin. When the linear equation is in the form $ax_1 + bx_2 = 0$, then the straight line for this equation passes through the origin point (0,0) and has no intercepts at all x_1 and x_2 axis.

<u>Example(1):</u>

If you have the linear equation:

$$2x_1 + 3x_3 = 6$$

Then it can be put in the two intersects by dividing both sides by 6 then we have the following form:

$(2x_1) / 6 + (3x_2) / 6 = 1$ or $x_1 / (6/2) + x_2 / (6/3) = 1$

Then $x_1 /3 + x_2/2 = 1$

Which mean that the linear equation $2x_1 + 3x_2 = 6$ is intersects the two axis x_1 and x_2 in 3 and 2 units from the origin point as it be shown in figure (1):

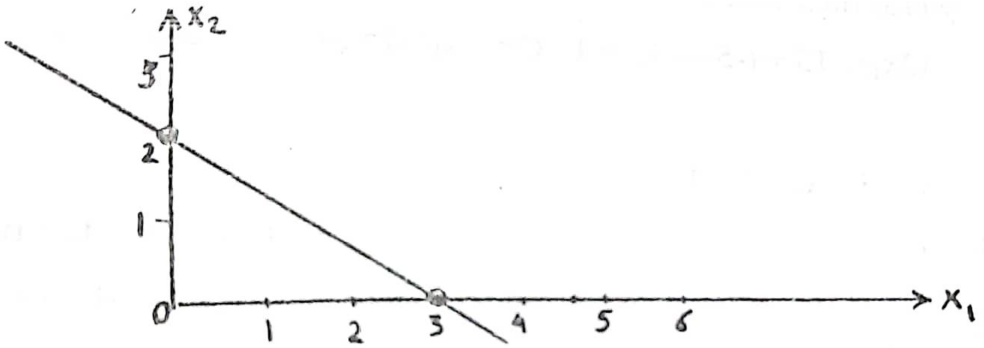


Figure (1)

Figure (1)

The preceding method for graphing the straight line for the linear equation can be achieved in another way by determining the intercepts of an equation simply, we first setting $x_1 = 0$ and solve the equation for x_2 and second setting $x_2 = 0$ and solve the equation for x_1.

In the preceding example, the following table summarizes the two intercepts from the origin point.

X_1	**0**	**3**
X_2	**2**	**0**

<u>Example (2):</u>

Graph a straight line using the two distinct intercepts of the equation:

$3x_1 – 5x = 15$ by two different methods.

<u>Solution:</u>

We can graph the straight line by using the two methods of intercepts as follows:

1-Since $3x_1 – 5x_2 = 15$, then dividing the two sides of this equation by (15) then we have:

$$(3x_1) / 15 + (-5x_2)/15 = 1 \quad \text{Or} \quad x_1/ (15/3) + x_2 (15/(-5)) = 1$$

i.e.,

$$x_1 / 5 + x_2 /-3 = 1$$

Which means that the equation $3x_1 – 5x_2 = 15$ intercepts the two axis x_1 and x_2 in 5 and (-3) units respectively as it be shown in figure (2).

2-The two intercepts can be achieved in another way by setting x_1 = 0 and solving for x_2, the: $-5x_2 = 15$

i.e., $x_2 = -3$

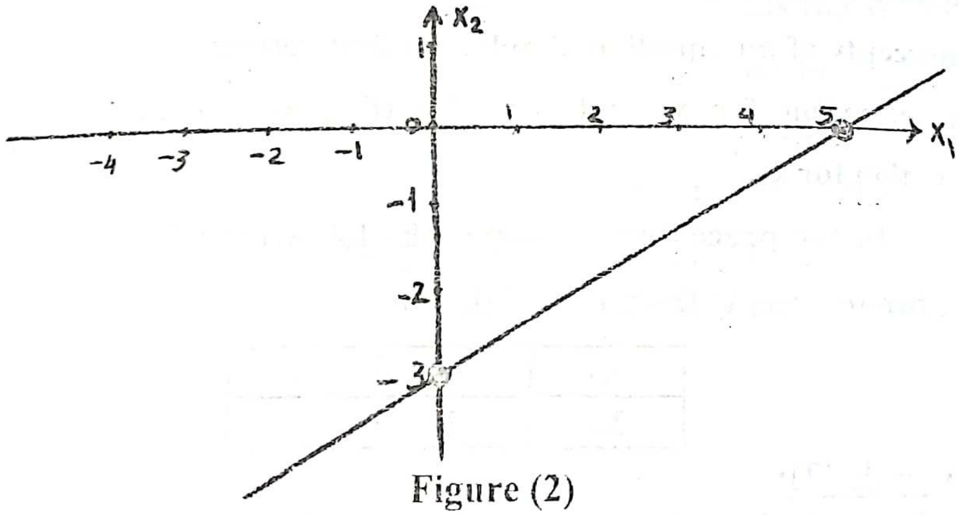


Figure (2)

Also, by setting $x_2 = 0$ in the equation and solving for x_1 , then we have:

$3x_1 = 15$, i.e., $x_1 = 5$

The following table summarizes the preceding results

X_1	**0**	**5**
X_2	**-3**	**0**

Finally, the linear equation (=) means that each point on the graph line only will satisfy the equation.

<u>Example (3):</u>

Use the two intercepts point to graph $2x_1 + 5x_2 = 10$

<u>Solution:</u> We find the two intercepts by first setting $x_1 = 0$ and solving for x_2, then setting $x_2 = 0$ and solving for x_1. When $x_1 = 0$, we get $x_2 = 2$ and when $x_2 = 0$ we get $x_1 = 5$, the following table summarizes these two points:

X_1	**0**	**5**
X_2	**2**	**0**

Or,

Since $2x_1 + 5x_2 = 10$

Then $(2x_1) / 10 + (5x_2)/ 10 = 1$

i.e.,

$$x_1 / (10/2) + x_2 (10/5) = 1$$

i.e.,

$$(x_1) / 5 + (x_2) / 2 = 1$$

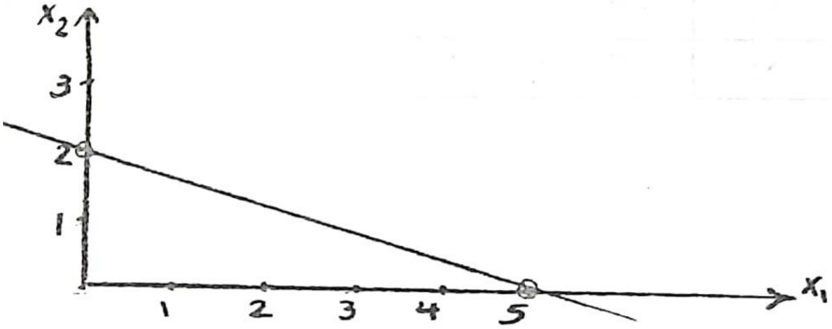


Figure (3)

i.e., the graph line for this equation intercepts the two axis x_1 and x_2

in 2 and 5 units from the origin respectively as it be shown in figure (3)

Note that each point on the graph line for the equation only satisfies the equation.

<u>Remark:</u>

If you have the following equation $ax_1 + bx_2 = 0$, then the graph line will passes through the origin point (0 , 0).

Also this line passes through the coefficients for the two variable with simple different, i.e., if you set that x_1 will equal to (b) the coefficient of the second unknown variable (x_2) then, the value of x_2 will be equal the same coefficient of x_1 but with opposite sign. i.e., if you set that $x_1 = b$ then the value of x_2 must be equal to (-a).

<u>Example (4):</u>

Graph the line for the equation: $2x_1 + 3x_2 = 0$

<u>Solution:</u>

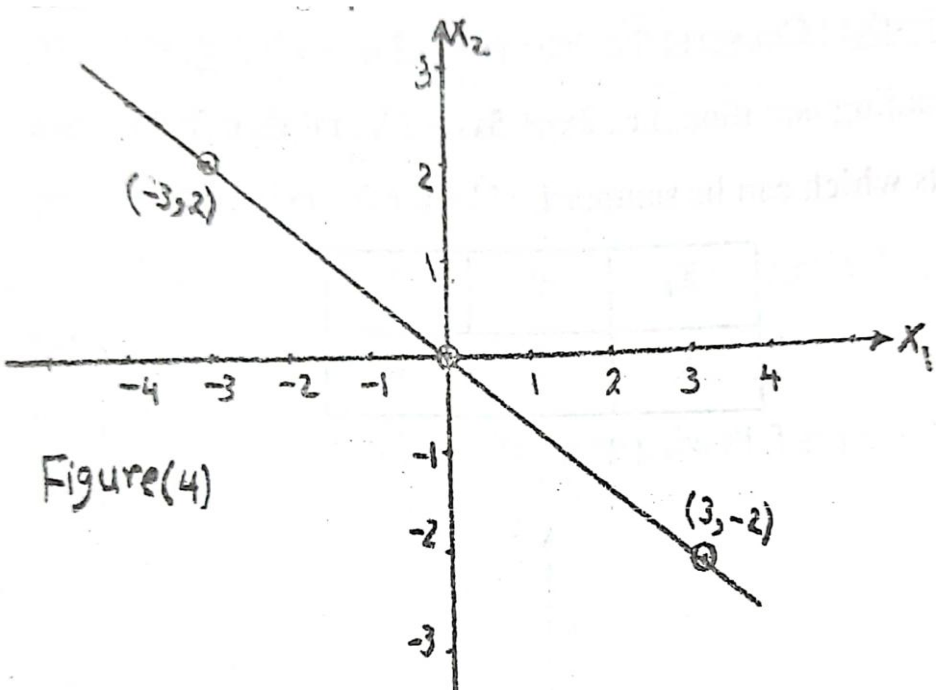
Since the equation in the form ax + bx = 0, then it will passes through the origin point (0 , 0) and (3 , -2) i.e., we have the following table :

X_1	0	3
X_2	0	-2

Or

X_1	0	-3
X_2	0	2

Then we have the graph line as it be shown in figure (4)



(b): Inequalities:

Graphing the linear inequalities in the form:

$Ax_1 + bx_2 \quad C$, where a, b and $C \neq 0$ are real numbers by converting the inequality into a linear equation in the form $ax_1 + 6x_2 = C$ and then we have to determine the two intercepts from the origin as it be shown in the preceding examples.

After plotting the graph line for the inequality, then we can determine the direction which will satisfy this inequality by using the origin point for determining its direction. The following examples will represent this procedure.

Example (5):

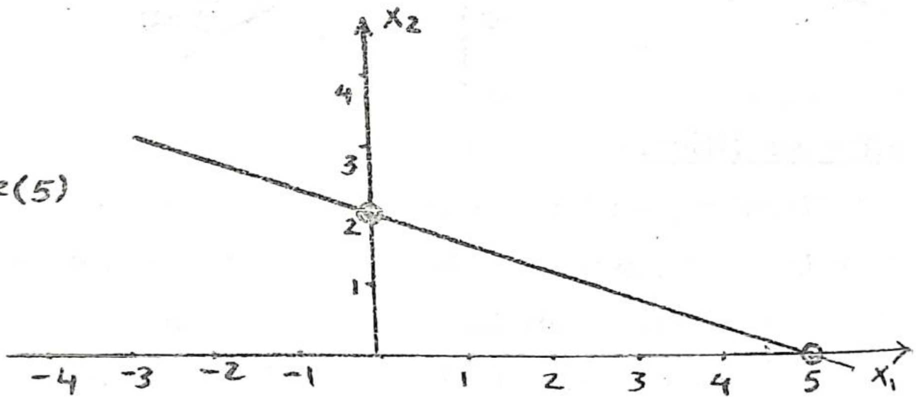
Determine graphically the region in which satisfy the following inequality: $2x_1 + 5x_2 \leq 10$

Solution:

Firstly: Convert the inequality $2x_1 + 5x_2 \leq 10$ to its corresponding equation. i.e., $2x_1 + 5x_2 = 10$, and then find the two intercepts which can be summarized by the following table:

X_1	0	5
X_2	2	0

Then we have the following graph (figure 5):



Figure(5)

Secondly: in order to determine the area which satisfy the inequality $2x_1 + 5x_2 \leq 10$, we can use any point in the two dimension (x_1, x_2) ~ plane. For simplicity, we use the origin point (0 , 0) for achieving this task. Then, we have to put $x_1 = 0$ and $x_2 = 0$ in both sides for this inequality.

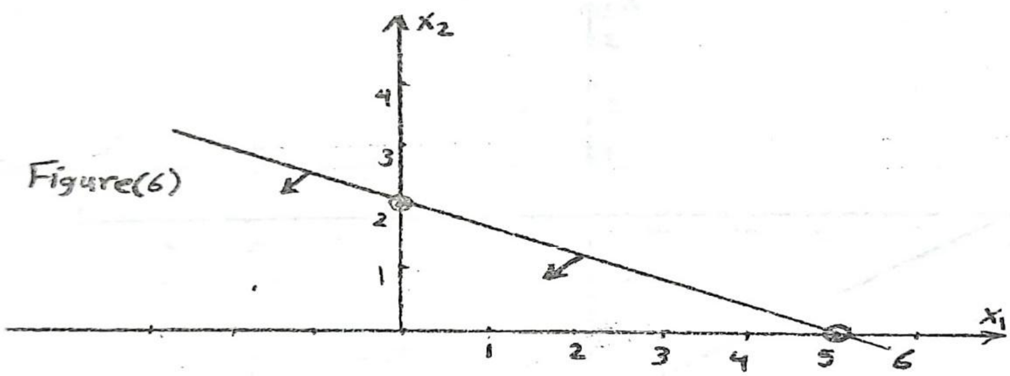
i.e.,

$$2(0) + 5(0) \quad [\ \ ] \quad 10$$

i.e., $\quad 0 \quad [\ \ ] \quad 10$

then put the suitable relation between the two sides. Then, we have $0 \leq 10$, i.e., the origin point lies in the area which satisfy the relation less than (<), i.e., any point in the area which the origin point exist will satisfy this inequalities.

Then we can determine the region in which this inequality holds is indicated by an arrow on its associated straight line as it be shown in Figure (6).



Figure(6)

Remark: If the relation of the inequality ($\leq$ or $\geq$) have the same relation for substituting by the origin point (0 , 0) in this inequality, then the origin point in the same region for the inequality. Conversely; if the relation is opposite, then the origin point is in the converse region for the region which satisfy the inequality.

Example (6):

Determine graphically the region in which satisfy the following inequality: $3x_1 + 5x_2 \leq -15$

Solution:

Firstly convert the inequality $3x_1 + 5x_2 \leq -15$ to the equation $3x_1 + 5x_2 = -15$, then we have the following two points (intercepts)

X_1	0	-5
X_2	3	0

Graphing the two points, then we have the following figure (7).

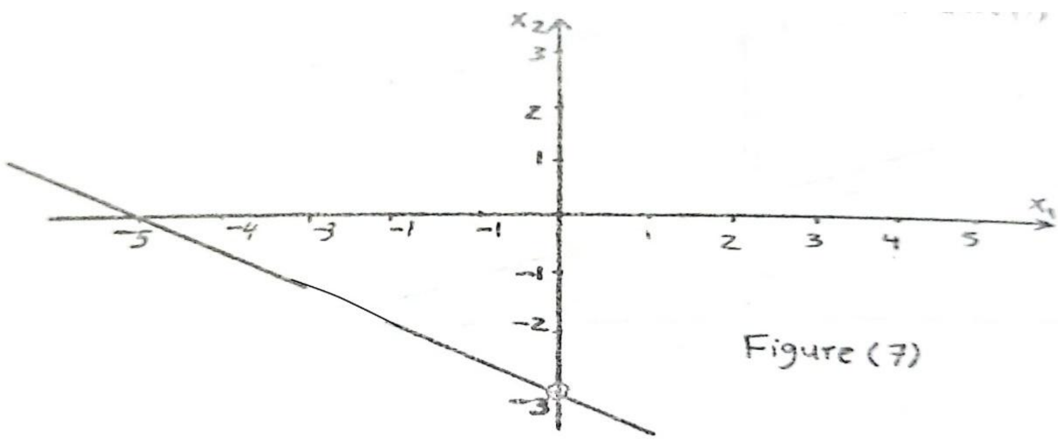


Figure (7)

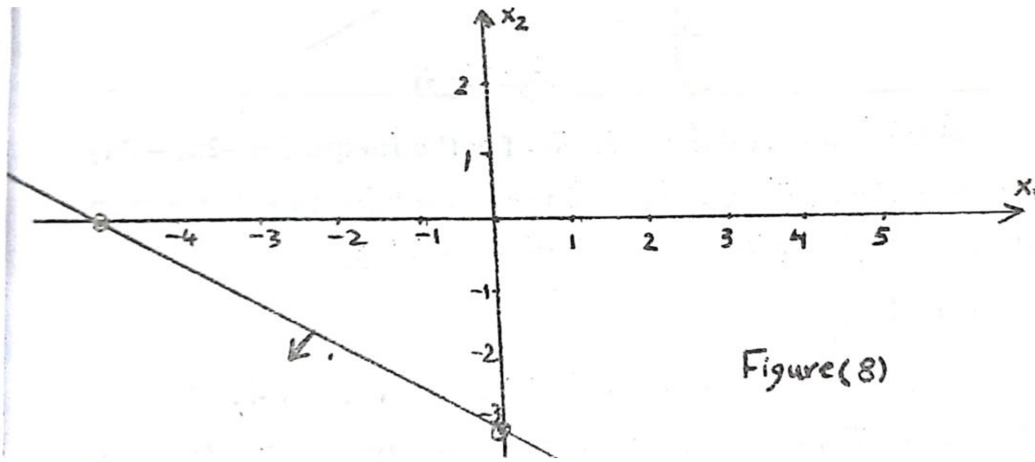
Now, in order to determine the region by which the inequality $3x_1 + 5x_2 \leq -15$, then we have to substitute the origin point (0 , 0) in the inequality, then we have : 3 (0) + 5 (0) [ ] -15

i.e., 0 [ ] -15

Then the correct relation between the two sides of the last relation is (>), i.e., the origin point (0 , 0) lies in the opposite region of this inequality.

In other meaning, since each of the two inequalities $3x_1 + 5x_2 \leq -15$ and 0 (>) -15 are conversely, then any point in the line graph in Figure (7) or in the opposite region for the origin point exist is satisfied the inequality $3x_1 + 5x_2 \leq -15$.

Therefore, the region in which holds the inequality is indicated by an arrow on its associated straight line as it be shown in figure (8)



Figure(8)

Example (7):

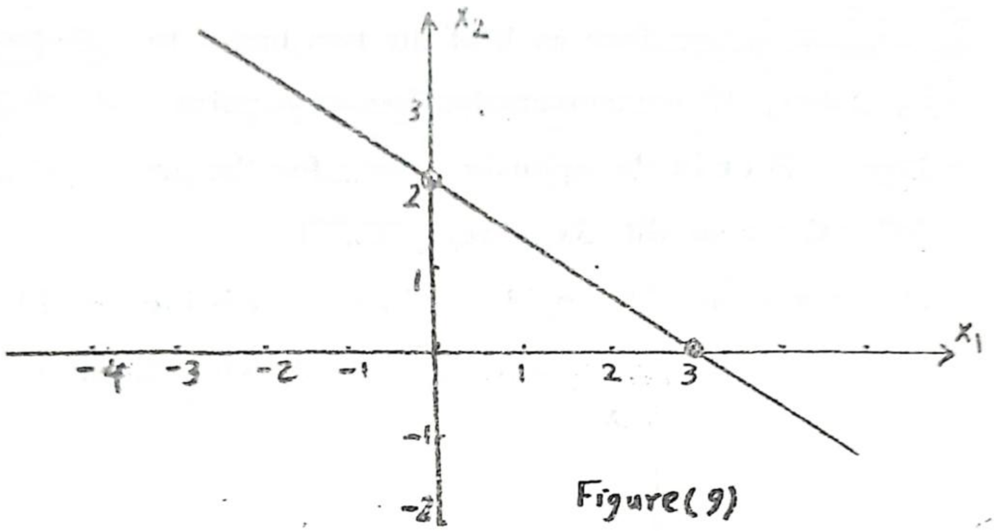
Determine graphically the region in which satisfy the following inequality: $-2x_1 - 3x_2 \leq -6$

Solution:

Firstly, convert the inequality into the following equation then, we have: $-2x_1 - 3x_2 = -6$**, and the following table represents the two intercepts:**

X_1	0	3
X_2	2	0

Graphing the two points then we have the following figure (9)



Figure(9)

Now, in order to determine the region for the inequality $-2x_1 - 3x_2 \leq -6$ **by using the origin point (0 , 0) then we have to substitute in both the two sides of this inequality, then we have:**

$-2\ (0) - 3\ (0)$ [ ] -6

0 [ ] -6

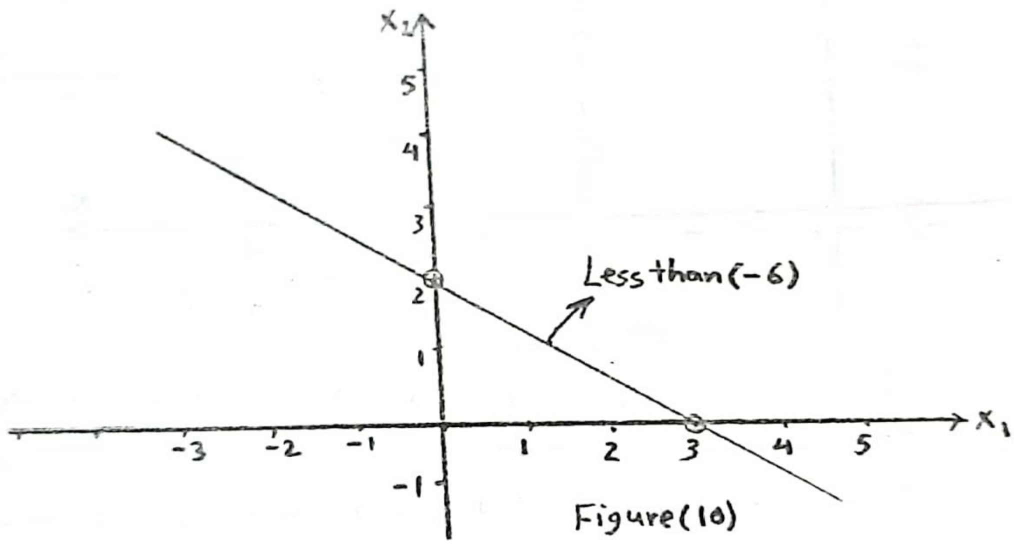
Then the correct relation between the two sides is 0 (>) -6.

Now, since the relation for each of the inequality $-2x_1 - 3x_2 \leq -6$ and the relation resulted from substituting with the origin point 0 (>) are conversely, then the region by which hold the inequality is conversely with the origin point.

i.e., any point in the graph line $-2x_1 - 3x_2 = -6$ or in the opposite region by which the origin point exist is the region which satisfy this inequality.

Then the region in which holds the inequality:

$-2x_1 - 3x_2 \leq -6$ is indicated by an arrow on its associated straight line as in figure (10)



Figure(10)

Example (8):

Determine graphically the direction which satisfy each of the following

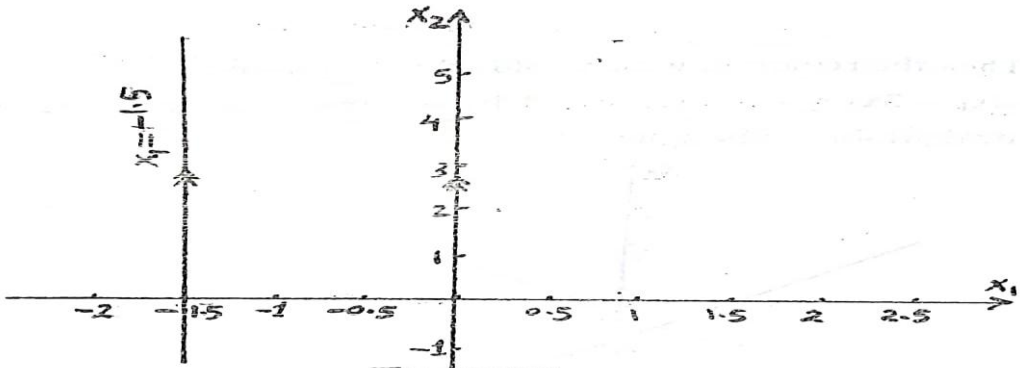
(a) $2x_1 \leq -3$

(b) $3x_1 \geq -9$

Solution :

(a)In order to find the direction which hold the inequality $2x_1 \leq -3$ convert the inequality into an equation: $2x_1 = -3$, i.e., $x_1 = -1.5$

Then: $x_1 = -1.5$ can be represented graphically through a line by which intercept x_1 axis in (-1.5) unit and parallel with the (x_2) axis as it be shown in figure (11).



Figure(11)

Now in order to determine the direction by which satisfy the inequality $2x_1 \leq -3$ by using the origin point, then 2 (0) [ ] -3 i.e., 0 [ ] -3 the right relation between the two sides is 0 (>) -3 i.e., the region for the inequality $2x_1 \leq -6$ is conversely with the region which the origin point is exist. i.e., the direction of the inequality $2x_1 \leq -3$

can be represented by an arrow on its straight line as it be shown in figure (12)

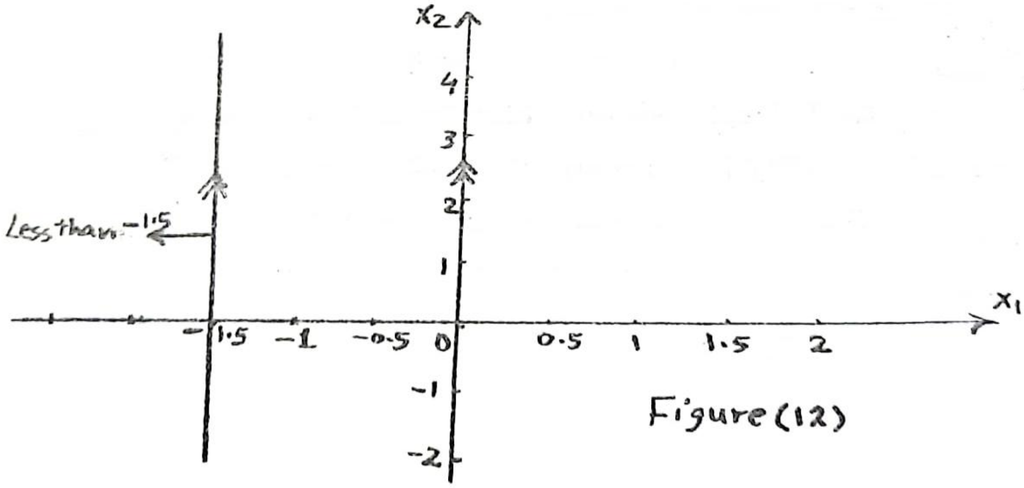
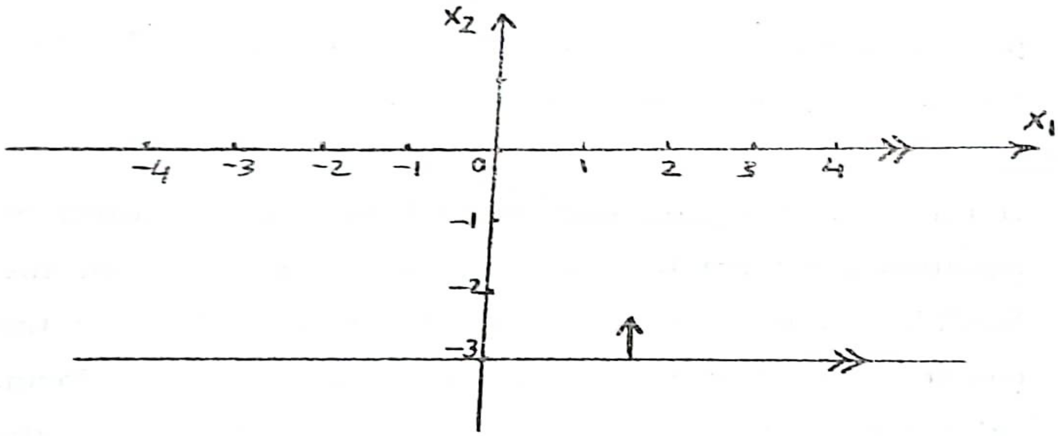


Figure (12)

(b)Similarly, it can be show that figure (13) represents the direction by which the inequality $3x_1 \geq -9$ is satisfied or hold.



<u>Graphical solution steps for the linear programming models:</u>

We can summarize the steps for solving the linear programming models graphically as follows:

1- **Each constraint will be plotted first with ($\leq$ or $\geq$) replaced by (=), thus yielding simple straight line equation, the region in which each constraint holds is indicated by an arrow on its associated straight line, then the resulting feasible space solution is given [feasible solution].**
2- **The optimum solution is the point in the feasible solution which yields the largest (maximum) value or the lowest (minimum) value for the equation of the objective function x_0 or $f(x)$. This optimum point can be determined from moving the graph line for the objective function $f(x)$ or $x_0 = 0$ parallel to itself in the direction of the feasible solution even so it passes through the first or the latest point in the feasible solution, then, we can determine the optimum point in the feasible solution.**

<u>Remarks:</u>

1- **If the linear programming model have one constraint in equation form (=) from the set of constraints, then the feasible solution will be a part of the graph line for the constraint in which it be found in the equation form, subject to it must be** lie in the feasible solution for the remaining set of constraints for the linear programming model.

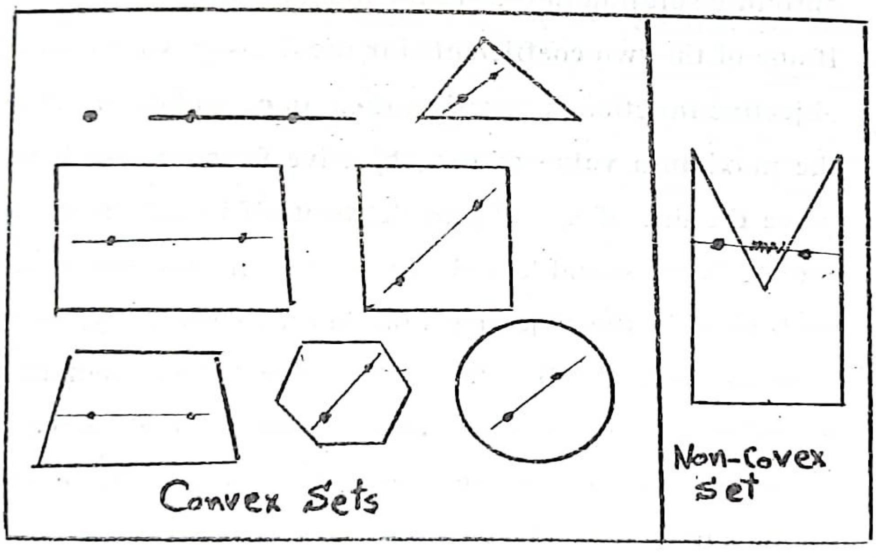
2- If the linear programming model have at least two constraints in equations form (=) from the set of constraints, then the feasible solution will be at most one point. This point is the intersection between the two constraints in which they are found in the equation form (=) subject to this intersection lies in the feasible solution for the remaining set of constraints for the linear programming model. Furthermore, this intersection point is considered the optimum solution in either determination the maximum or the minimum value of the objective function.

3- If the two coefficients for the two decision variables x_1 and x_2 are positive numbers, then the optimum solution for the LPM is the first point that the line of the objective function $x_0 = 0$ passes through the feasible solution in case of minimization for x_0. Conversely the latest point that the line $x_0 = 0$ passes through the feasible solution is considered the optimum solution in case of the maximization of x_0.

4- If one of the two coefficients for the decision variable in the objective function is negative, then, in case of determination the maximum value of the objective function, we have to move the line of $x_0 = 0$ parallel to itself in the direction of the decision variable axis by which it has the positive coefficient in the objective function x_0. conversely, we will have to move the line for $x_0 = 0$ parallel to itself in the direction of the decision variable axis which have the negative coefficient in the objective function in case of minimization value of x_0.

5- If the two coefficient for the decision variables x_1 and x_2 are negative values, then we have to move the line x_0 parallerly to itself in the direction for the decision variable which have the lowest negative coefficient in case of maximization x_0, and conversely in case of the minimization value of x_0.

6- The feasible space solution resulted from the graphical solution for the linear programming model is a convex set, where the convex set is a solution space by which the line passes between any two arbitrary points must be lies in the solution space.



7- After we have determined the feasible solution space, if there is line graph for any constraint can be deleted without affecting the solution space, then the constraint for this line is considered a redundant constraint.

8- The types of solutions resulted from the graphical solution for any linear programming model: there are four types of solutions:

(a)Feasible Solutions:

After determining the feasible solution space, then any point in this space is considered a feasible solution.

(b)Basic Solutions:

Any intersection between two arbitrary constraint lines is considered a basic solution.

(c)Basic Feasible Solution: (or the Extreme Points):

The corners points for the feasible solution is considered the basic feasible solutions.

(d)Optimal Solution:

If the feasible solution is existing, then the optimal solution is at least one point of these basic feasible solutions by which it makes the value of the objective function in its maximum value in case of determination the value of x_i's that make x_0 maximization, and vice versa.

Example (9):

Determine the feasible solution space for the following constraints and determine the redundant constraints if there are exist:

$$\begin{array}{lllll}
(1) & x_1 + x_2 & \le & 4 \\
(2) & 4x_1 + 3x_2 & \le & 12 \\
(3) & -x_1 + x_2 & \le & 1 \\
(4) & x_1 + x_2 & \le & 6 \\
(5) & x_1 & \ge & 0 \\
(6) & \quad\quad x_2 & \ge & 0
\end{array}$$

Solution:

In order to determine the feasible solution space, we have to graph the line for each constraint and determine the region by which satisfied this constraint as follows:

(1)The non-negativity constraints:

$x_1 \ge 0$, $x_2 \ge 0$ specify that the feasible solution must lie in the first quadrant.

***The 1st constraint:** $x_1 + x_2 \leq 4$

Convert into an equation: $x_1 + x_2 = 4$, then the following table represents the two intercepts.

X_1	0	4
X_2	4	0

and since all the coefficients of x_1 , x_2 and the right hand side of the constraint is positive, then the region in witch satisfies the 1st constraint lie down its line graph.

***The 2nd constraint:** $4x_1 + 3x_2 \leq 12$

Convert into an equation: $4x_1 + 3x_2 = 12$,

Then the following table represents the two intercepts:

X_1	0	3
X_2	4	0

and since, all the coefficient of x_1 , x_2 and the R.H.S of the constraint is positive, then the region in witch satisfies the 2nd constraint lie down its line graph.

***The 3rd constraint:** $-x_1 + x_2 \leq 1$

Convert into an equation: $-x_1 + x_2 = 1$,

then the following table represents the two intercepts.

X_1	0	-1
X_2	1	0

Now, in order to determine the direction for this constraint, we will use the origin point as follow:

-(0) + 0 [ ] 1

0 [<] 1

Since, we have the same relation for the inequality resulted from substituting the origin point and the constraint, then the region for the line constraint can be indicated with an arrow at the same direction of the origin point.

<u>*The 4th constraint:</u> $x_1 + x_2 \leq 6$

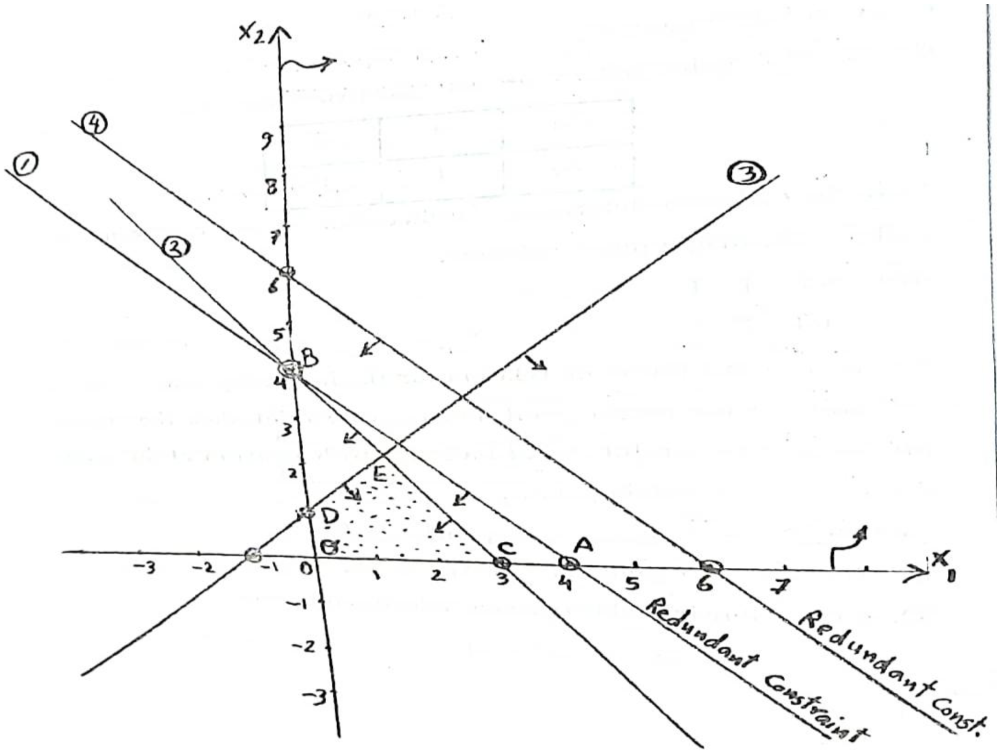
Convert into an equation : $x_1 + x_2 = 6$,

Then the following table represents the two intercepts.

X_1	0	6
X_2	6	0

and the region by which satisfy this constraint is in the direction of the origin point.

<u>(2)Determination the feasible solution space graphically:</u>



Reduction the feasible solution

-Non-negativity constraints	**:**	**$x_1 O x_2$**
-1st constraint	**:**	**AOB**
-2nd constraint	**:**	**COB**
-3rd constraint	**:**	**CODE**
-4th constraint	**:**	**CODE (Feasible Solution)**

Therefore, the feasible solution space is CODE. And the two constraints enumerated with (1) and (4) are redundant constraints, since their lines can be deleted without affecting in the solution space CODE.

Example (10):

If you have the following model:

$x_0 = x_1 + 2x_2$

Subject to:

(1)	$-3x_1 + 3x_2$	$\leq$	9
(2)	$x_1 + x_2$	$\leq$	2
(3)	$x_1 + x_2$	$\leq$	6
(4)	$x_1 + 3x_2$	$\geq$	6
(5)	x_1	$\geq$	0
	x_2	$\geq$	0

Required:

1- **Determine the optimum solution in either maximization or minimization for x_0.**
2- **Determine the different types of solution.**
3- **Determine the redundant constraints if there are exist.**

Solution:

1-The idea of the graphical solution is to plot the feasible solution space, which is defined as the space enclosed by constraints (1) through (5). The optimum solution is the point in the solution space which maximize or minimize the value of the objective function x_0.

***Non-negativity restrictions: $x_1 \geq 0$ and $x_2 \geq 0$ implies that the feasible solution must be lie in the 1st quadrant.**

<u>-1st constraint:</u> $-3x_1 + 3x_2 \leq 9$

Convert into an equation $-3x_1 + x_2 = 9$,

Then the following table represent the two intercepts:

X_1	0	-3
X_2	3	0

And by using the origin point (0 , 0) for determining the direction for this inequality

-3 (0)+ 3 (0) [ ] 9

0 [<] 9

i.e., each of the direction of the origin point and the direction of this inequality are the same.

<u>-2nd constraint:</u> $x_1 - x_2 \leq 2$

Convert into an equation, then we have $x_1 - x_2 = 2$, and the following table represents the two intercepts:

X_1	0	2
X_2	-2	0

And by using the origin point,

Then: $0 - 0$ [ ] 2

0 [<] 2

i.e., both the direction for the inequality and the origin point are the same.

<u>3rd constraint:</u> $x_1 + x_2 \leq 6$

Convert into an equation, then $x_1 + x_2 = 6$, and the following table represents the two intercepts:

X_1	0	6
X_2	6	0

since the two coefficients for the decision variables x_1 , x_2 and the R.H.S for the constraint is positive number, then the direction for this inequality and the origin point are the same direction.

<u>4th constraint:</u> $x_1 + 3x_2 \geq 6$

Convert into an equation $x_1 + 3x_2 = 6$,

Then, the following table represents the two intercepts:

X_1	0	6
X_2	2	0

Since the two coefficients for x_1 , x_2 and the R.H.S are positive numbers, then the direction for this inequality ($\geq$) is in the opposite direction which implies the origin point.

<u>*Graphing the objective function line:</u>

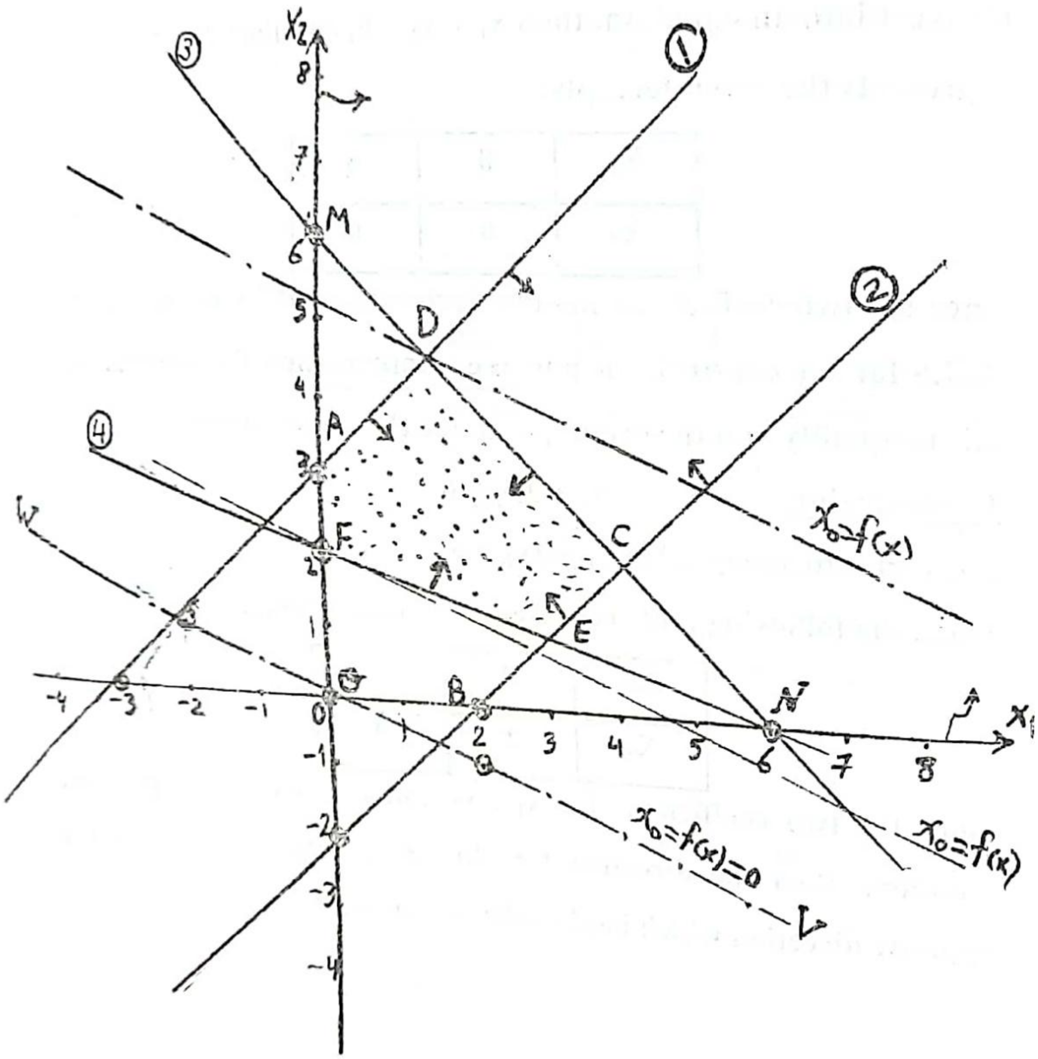
Suppose that $x_0 = 0$, then we have the following equation:

$$x_1 + 2x_2 = 0$$

The following table represents the points that passes through that line:

X_1	0	2	-2
X_2	0	-1	1

Then, we have the following graph:



Reduction for the feasible solution:

***Non-negativity constraints:** **x_1 OX$_2$**

***1st constraints = x_1 OA (1)**

***2nd constraints = (2) BOA (1)**

***3rd constraints = OBCDA**

***4th constraints = ECDAF (Feasible solution space)**

-The line for the objective function $x_0 = 0$ is VOW

Now, for determining the optimal solution in either the minimization or the maximization cases, we have to move the line graph for the objective function $x_1 + 2x_2 = 0$ (VOW) parallel to itself in the direction of the feasible solution ECDAF even it passes through the first point (since the two coefficients are positive value) in the feasible solution space in case of minimization for x_0 or even it passes through the latest point in the feasible solution in case of the maximization for x_0. one can see that the minimum value of x_0 occurs where the line of the objective function passes through point F whose coordinates are $x_1 = 0$ and $x_2 = 2$ units. Substituting these values into the objective function gives $x_0 = x_1 + 2x_2 = 0 + 2\ (2) = 4$ units. Also, one can see that the maximum value of x_0 occurs where the line of the objective function passes through the point D whose coordinates are $x_1 = {}^3/_2$ and $x_2 = {}^9/_2$. Substituting these values in the objective function gives $x_0 = x_1 + 2x_2 = ({}^3/_2) + 2({}^9/_2) = ({}^{21}/_2) = 10.5$ unit.

An interesting observation is that either the minimum value or the maximum value of the objective function x_0 always occurs at one of the corner points (Extreme points) E, C, D, A and F of the solution space. The choice of a specific corner point as the optimum depends on the slope of the objective function. As an illustration, the reader can verify graphically that the changes in the objective function

given in the table blow produce the optimum solution in the two cases:

Coordinates for the corner points	**$f(x) = x_0 = x_1 + 2x_2$**	**Remarks**
F (0 , 2)	**$f_F(x) = 0 + 2(2) = 4$**	**Minimum value**
E (3 , 1)	**$f_E(x) = 3 + 2(1) = 5$**	
C (4 , 2)	**$f_C(x) = 4 + 2(2) = 8$**	
D ($^3/_2$, $^9/_2$)	**$f_D(x) = 3/2 + 2(^9/_2) = 10.5$**	
A (0 , 3)	**$f_A(x) = 0 + 2(3) = 6$**	**Maximum value**

<u>2-The Different Types of Solutions are:</u>

<u>(a): Feasible solutions:</u>

Any point lie in the feasible solution space ECDAF is considered a feasible solution. Henceforth, there are infinite numbers of solutions in this problem.

<u>(b): Basic solutions:</u>

The set of Basic solutions were the set of points O, B, N, F, E, C, D, A, and M. therefore, any point resulted from the intersection between the lines of two constraint in the 1st quadrant is defined as a basic solution.

(c): Basic feasible solutions (Extreme points):

The Basic Feasible solutions in this problem were E, C, D, A and F.

(d): The optimal solution(s):

The point F(0 , 2) is considered the optimum solution in case of minimization the value of x_0. And the point D ($^3/_2$, $^9/_2$) is considered the optimum solution in case of maximization the value of x_0.

3-Since all the lines graph for the set of all constraints implies the feasible solution except for the 1st non-negativity constraint $x_1 \geq 0$, then all constraints for the problem considered basically (non-redundant) except for the 1st non-negativity constraint $x_1 \geq 0$ which is considered a redundant constraint , i.e., there is only one redundant constraint in this problem.

Example (11):

Suppose that you have the following (LPM):

$f(y) = 5y_1 + 2y_2$ Max (Min)

subject to:

$$
\begin{aligned}
y_1 + y_2 &\leq 10 \\
y_1 &= 5 \\
y_1 \, , \, y_2 &\geq 0
\end{aligned}
$$

find the optimum solution and the types of solutions.

Solution:

In order to determine the optimal solution for the LPM in either maximization or minimization form graphically we have to determine the feasible solution space as the following:

*The non-negativity constraints: $x_1 \geq 0$ and $x_2 \geq 0$ implies that the feasible solution space must be lie in the 1st quadrant.

-1st constraint: $y_1 + y_2 \leq 10$

Convert it into an equation yields to $y_1 + y_2 = 10$,

The following table represents the two intercepts:

X_1	0	-10
X_2	10	0

The region by which it is satisfy the inequality for this constraint is the same region that the origin point is exist.

-2nd constraint: $y_1 = 5$

The line graph for this equation is vertical on y_1 axis and intercept the y_1 axis in 5 units vertically and parallel to the y_2 axis. Each point in the line graph for this constraint is only satisfy this equation.

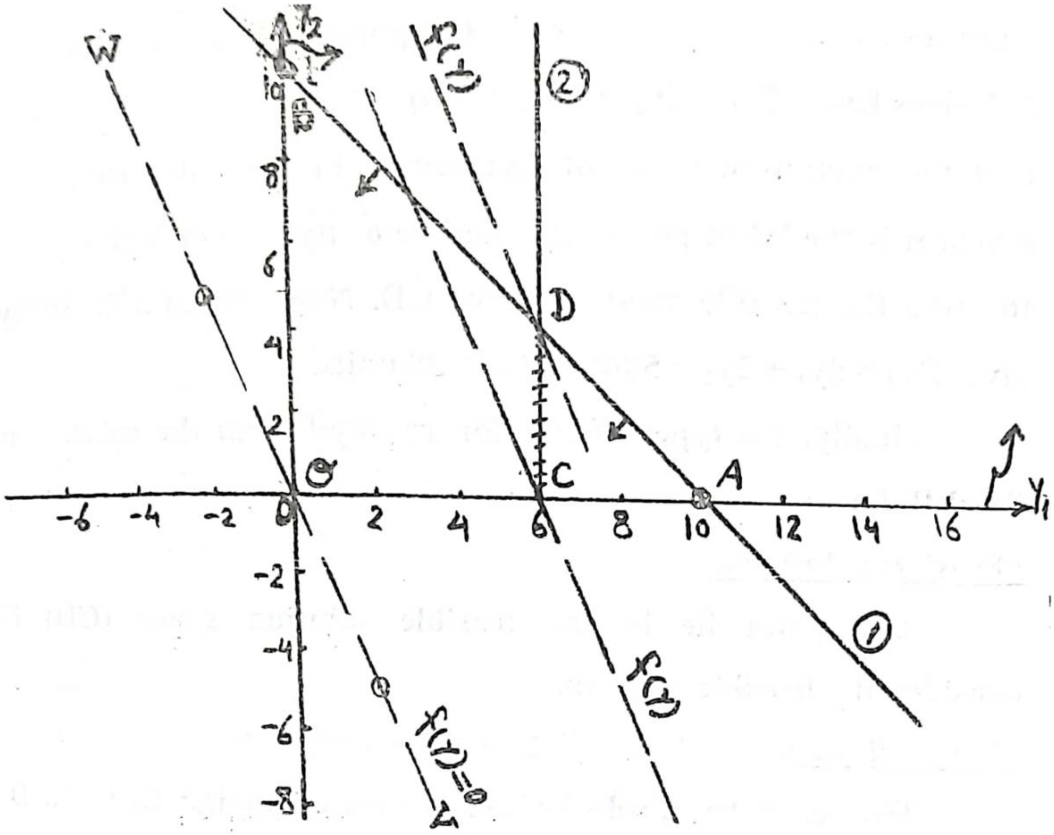
-Graphing the objective function:

Suppose that $f(y) = 0$, then we have the following;

$5y_1 + 2y_2 = 0$,

X_1	0	2	-2
X_2	0	-5	5

The following table represents the point that passes through the line for this equation then, we have the following graph:



Reduction of the feasible solution space:

***Non-negativity constraints : x_1 O x_2**

***1st constraint : AOB**

***2nd constraint : CD (feasible solution)**

-The line graph for the objective function *f*(*y*) = 0 is VOW.

Then, in order to determine the optimal solution, we have to move the line VOW parallel to itself in the direction of the feasible solution space (CD) even it passes through the first point in the line CD (since the two coefficients of y_1 and y_2 are positive numbers) in case of minimization of *f*(*y*). One can see that the minimum value of f(y) occurs in the point C(6,0). Substituting in f(y) gives f(y) = $5y_1 + 2y_2$ = 5 (6) + 2(0) = 30 units. Also, one can see that the maximum value of f(y) occurs in the point D(6, 4) by which it is the latest point that the line of f(y) = 0 or VOW passes through the feasible solution space CD. Now substituting in f(y) gives f(y) = $5y_1 + 2y_2$ = 5 (6) + 2(4) = 38 units.

Finally, the types of solution resulted from the solution of this LPM are:

<u>*Feasible solutions:</u>

Any point lie in the feasible solution space (CD) is considered a feasible solution.

<u>*Basic solution:</u>

The set of basic solution are the set of points: O, C, A, D and B

<u>*Basic feasible solutions:</u> **The two corners of the feasible solution space (CD) are the basic feasible solution space, i.e., the two points C and D are the basic feasible solution (Extreme Points).**

<u>*Optimal solutions:</u>

The point C (6, 0) is considered the optimal solution in case of minimization the value of the objective function f(y), and the point D (6 , 4) is considered the optimal solution in case of maximization the value of the objective function f(y).

<u>Example (12):</u>

Determine the value of x_1 and x_2 by which:

$f(x) = -x_2$ **Max (Min)**

Subject to:

(1)	$x_1 + x_2$	$\geq$	1
(2)	$x_1 + x_2$	$\leq$	2
(3)	$x_1 - x_2$	$\leq$	1
(4)	$x_1 - x_2$	$\geq$	-1
(5)	$x_1 \; , \; x_2$	$\geq$	0

<u>Solution:</u>

***Non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ implies that the feasible solution space must lie in the 1^{st} quadrant.**

-1st constraint:

$x_1 + x_2 \geq 1$, convert into an equation, $x_1 + x_2 = 1$,

Then the following table represent the two intercepts:

X_1	0	1
X_2	1	0

And the direction of the inequality of this constraint is opposite to the region in which the origin point is exist.

-2nd constraint:

$x_1 + x_2 \leq 2$, convert into an equation: $x_1 + x_2 = 2$,

Then the following table represents the two intercepts:

X_1	0	2
X_2	2	0

And the direction of the inequality of this constraint is the same direction by which the origin point is exist.

-3rd constraint: $x_1 - x_1 \leq 1$, convert into an equation, $x_1 - x_2 = 1$,

X_1	0	1
X_2	-1	0

then the following table represents the two intercepts.

Now, in order to determine the region by which satisfies the inequality of this constraint, let us used the origin point for this determination:

$0 - 0 \quad [\quad] \quad 1$

$0 \quad [<] \quad 1$

i.e., the direction of this inequality is the same region by which the origin point is exist, since the two relation have the same (<).

<u>-4th constraint:</u>

$x_1 - x_2 \geq -1$, convert into an equation: $x_1 - x_2 = -1$,

The following table represents the two intercepts:

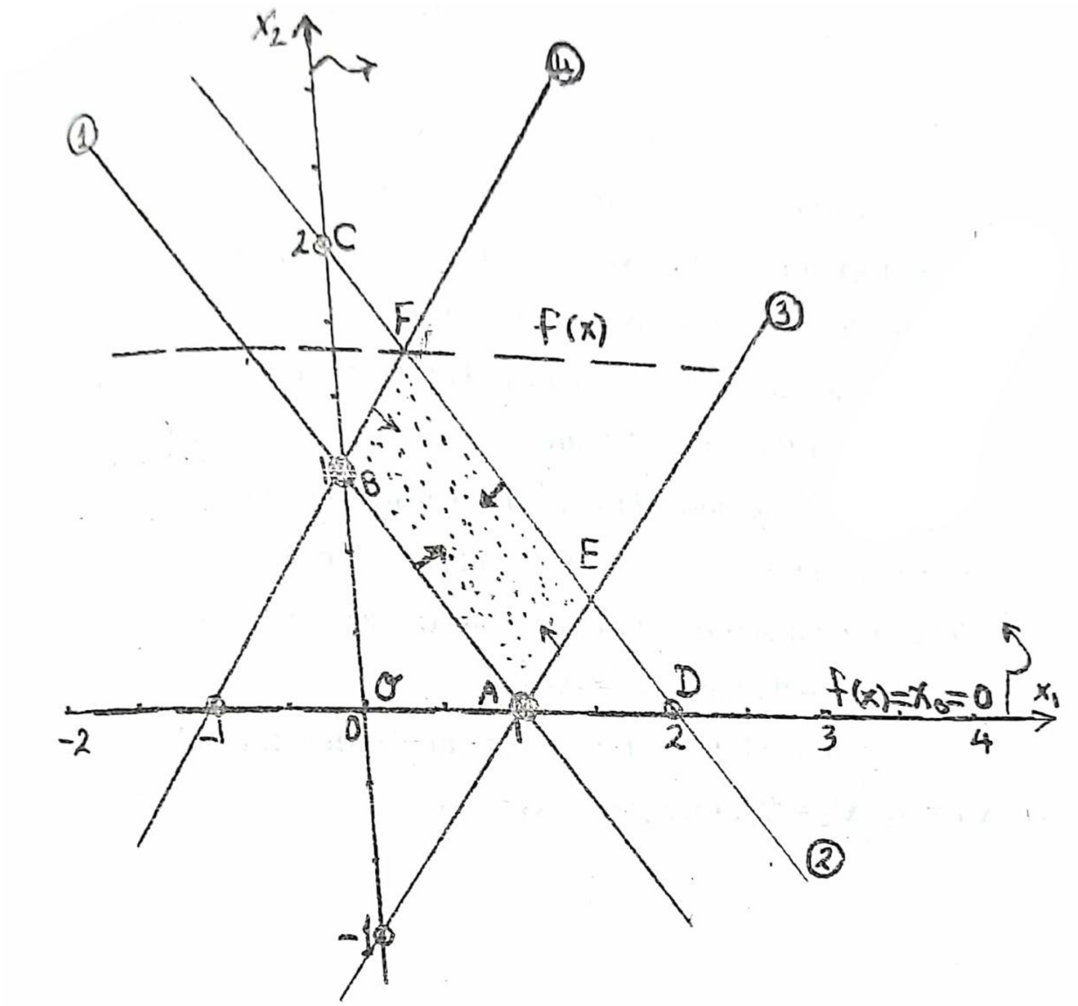
X_1	**0**	**1**
X_2	**1**	**0**

And, $0 - 0 \quad [\quad] \quad -1$

$0 \quad [>] \quad -1$

i.e., each of the region satisfying this inequality and the region by which the origin point is exist are the same.

<u>*Graphing the objective function:</u> Suppose that $f(x) = 0$, then we have the following: $-x_2 = 0$, this equation implies that the line for $f(x) = -x_2 = 0$ is the same line for the x_1 axis. Then, we have the following graph:



<u>**Reduction of the feasible solution**</u>**:**

-Non-negativity constraints : $x_1 Ox_2$

-1st constraint : $x_1 ABX_2$

-2nd constraint : ABCD

-3rd constraint : ABCE

-4th constraint : ABFE (Feasible Solution)

***The line graph for the objective function is ($-x_1$ O x_1).**

Since the coefficient of the decision variable x_2 is negative, therefore the decision is conversely, i.e., in order to determine the optimal solution, we have to move the line graph for the objective function f(x) = 0 or the line ($-x_1$ O x_1) parallel to itself even it passes through the first point in the feasible solution space (ABFE) in case of maximization the value of *f*(*x*), or even it passes through the latest point in the feasible solution in case of minimization the value of *f*(*x*). Therefore, the point A (1 , 0) is considered the optimal solution in case of maximization the value of *f*(*x*), and the point *F*($^1/_2$, $^3/_2$) is considered the optimal solution in case of minimization the value of *f*(*x*). Hence:

-The optimal solution in case of maximization the value of *f*(*x*) is $x_1^* = 1$, $x^*_2 = 0$, then $f(x) = -x_2 = 0$

-The optimal solution in case of minimization the value of *f*(*x*) is $x^*_1 = {}^1/_2$, $x^*_2 = {}^3/_2$, then $f(x) = -x_2 = -{}^3/_2$.

Note that, one can see that the optimal solution is correct from substitution in the objective function by the corners coordinates (extreme points) for the feasible solution space, as it be shown in the following table:

Extreme points	$f(x) = -x_2$	Remarks
A (1 , 0)	$f_A(x) = 0$	Maximum value
E ($^3/_2$, $^1/_2$)	$f_E(x) = -^1/_2$	
F ($^1/_2$, $^3/_2$)	$f_F(x) = -^3/_2$	Minimum value
B (0 , 1)	$f_B(x) = -1$	

Note that if $f(x) = -x_1$, then the optimal solution is the point B (0 , 1) in case of maximization the value of $f(x)$, and the point E ($^3/_2$, $^1/_2$) in case of minimization the value of $f(x)$, since the line graph for the objective function $f(x) = -x_1 = 0$ will be the same line for the x_2 axis in this case.

Development of the simplex Methods:

In the preceding section, we present the graphical solution for the LPM which have only two decision variables say x_1 and x_2 in which the solution space plotted in the (x_1 , x_2) ~ plane or in other meaning in two dimension. The graphical solution will be more complexity when the number of decision variables are more than two variables. In this section, we will present the simplex methods which are based on determining some of extreme points in a selective manner.

In order to determine the optimal solution for a linear programming model, firstly, put the right hand side for all the constraint are positive values. Then, we have the following results:

(1)If all the constraints of the LPM are in the relation form less than or equal to, i.e., ($\leq$) , then we must use the ordinary simplex method.

(2)If there is at least one constraint in an equation form (=) or in the form ($\geq$), then, we must use one of the following two methods:

a- The M-technique

b- The Two Phase technique.

All algorithms for the simplex methods started from an initial solution (or table) which is called a basic solution. If the solution yields all nonnegative basic variables, it is called a basic feasible

solution; otherwise, it is infeasible. A feasible extreme point is thus defined by a basic feasible solution. This fundamental property shows how the extreme point of the solution space is translated algebraically as the basic solutions of the equations representing in the linear program.

<u>*Optimality and feasibility conditions of the simplex methods:</u>

The simplex methods are considered as an extension for the algebra solution which is search about the optimum solution to a general linear program in (m) equations and (n) unknowns may be obtained by solving $C^n_m = n!/[m!(n-m)!]$ sets of simultaneous equations. This procedure is inefficient. First, the number of possible basic solution may be too long. Second, many of these solutions may be infeasible or nonexistent. Third, the object function plays a passive role in the calculation, since it is used only after all the basic feasible solutions have been determined.

The simplex method is designed specifically to avoid these inefficiencies. The overall approach is to start from a basic feasible solution (that is a feasible extreme point) and then move successively through a sequence of (non-redundant) basic feasible solutions such that each new solution has the potential to improve the value of the objective function. The basis of the simplex method, which guarantees generating such a sequence of basic solutions is two fundamental conditions:

(1): The Feasibility condition:

The feasibility condition guarantees, that starting with a basic feasible solution, only basic feasible solutions are encountered during computation. This condition implies the values for the basic variables in any solution must verify the non-negativity constraints.

(2): The Optimality Condition:

The optimality condition ensures that no inferior solutions (relative to the current solution point) are even encountered. This condition implies that row of the test of optimality must be zero or positive coefficients in case of maximization the value of the objective function, and vice versa, these coefficients must be zero of negative values in case of minimization the value of the objective function.

***The Ordinary simplex method:**

If the general linear model have positive values in the R.H.S of the constraints, and all these constraints are inequalities in the form ($\leq$) except for the non-negativity constraint, then we have to use the ordinary simplex method passes through the following steps:

1-Put the LPM in the standard form, then we have a set of slack variables equal to the number of constraints.

2-The set of the slack variables are considered as the basic variable for the set of constraints. Then construct the initial solution tableau as the following table:

Coef. Of f(x) all Var. / Basic variable (B.V)	C_J	$C_1\ C_2\ ..\ C_n\ \ 0\ \ 0\ ....\ 0$ $x_1\ x_2\ ....\ x_n\ \ x_{n+1}\ x_{n+2}\ ..x_{n+m}$	Solution (R.H.S)	Ratio
X_{n+1} X_{n+2} : : X_{n+m}	0 0 : : 0	Coefficient of all the Decision and slack variable in the set of constraints	The feasibility condition	
E_J				Max (Min)
$E_J - C_J$		Optimality condition		

Note that, the special arrangement of the above tableau provides useful information. The columns associated with the starting basic variable always appear immediately to the left of the solution column, and their constraints coefficients will constitute an identity matrix.

The next step is to determine a new basic feasible solution (extreme point) with an improved value of the objective function. The simplex method does this by selecting a current non-basic variable to be increased above zero providing its coefficient in the objective function has potential to improve the value of the objective function x_0 or $f(x)$.

<u>3-Solution improvement:</u>

If the optimality condition is not satisfied, then one can improve to solution according the following steps:

<u>(a): Determine the pivot column (Entering variable):</u>

The non-basic variable which have the most negative coefficient in the row of the optimality condition ($E_J - C_J$) in case of maximization the value of the objective function $f(x)$ or x_0 is called the entering variable and its column is called the pivot column. Vice versa, the non-basic variable which have the most positive coefficient in the row of the optimality condition ($E_J - C_J$) in case of minimization f(x) or x_0 is called the entering variable.

<u>(b): Determine the pivot row (Leaving variable):</u>

In a geometric interpretation for the leaving vector (leaving variable from the set of basic variables), then we have to determine the intercepts of all the constraints with the nonnegative direction of the entering variable. Then, the constraint which have the smallest

intercept defines the leaving variable. Therefore, the general conclusion then is that if the constraint coefficient under the entering variable is negative or zero, the corresponding constraint does not intersect the nonnegative direction graphically of the axis defining the entering variable and hence will have no effect on feasibility. The same information may be secured directly from the preceding tableau by taking the ratios of the value of solution column to the positive constraint coefficients under the entering variable. The basic variable associated with the minimum ratio (smallest intercept) is the leaving variable and its row is the pivot row.

<u>(C): Determine the Pivot Element:</u>

The intersection between the pivot column and the pivot row determine the pivot element in any interaction for the solution.

- **The improvement of the simplex tableau according to the Gauss-Jordan alimentation passes through the following steps:**
- **The first step is to divide the pivot row by the pivot element and replace the basic leaving variable by the entering variable.**
- **The detailed row operations are given as follows:**

New Row = Old Row – (Intercept for the old row with the pivot column) × the new pivot equation (or row).

The result of these calculation gives the second tableau, which is further improve since all the two conditions (Feasibility and optimality) are satisfied.

Note that, the only difference between maximization and minimization occurs in the optimality condition: In minimization, the entering variable is the one with the most positive coefficient in the optimality condition (E_J – C_J). The feasibility condition remains the same since it depends on the constraints and not the objective function.

To conclude the preceding section, the following is a summary of the optimality and feasibility conditions.

Optimality condition. Given the coefficient of (E_J – C_J) expressed in terms of the non-basic variables only, one selects the entering variable in maximization (minimization) as the non-basic variable having the most negative (most positive) coefficient in the (E_J – C_J) row. A tie between two non-basic variables may be broken arbitrarily, when all the left hand side coefficients of the (E_J – C_J) are nonnegative (non-positive), the optimum is reached.

Feasibility condition. The leaving variable is the basic variable corresponding to the smallest ratio of the current value of the basic variables to the positive constraint coefficient of the entering variable.

Finally, a summary of the computational procedure of the simplex method is shown as in the following steps:

Step (1): **Express the standard from of the linear program in a tableau as we show in the preceding section.**

Step (2): **Select a starting basic feasible solution. This step involves two cases:**

1- **If all the constraints in the original problem are ($\leq$), the slack variables are used for a starting solution.**
2- **If the constraints in the original problem include ($\geq$) or (=), then, the two techniques [M-technique or two phase technique] are used the artificial technique to give a starting basis.**

Step (3): **Generate new basic feasible solution using the optimality and feasibility conditions until the optimal solution is attained. This step assumes that the optimal solution exists and is bounded. The cases of nonexistent and unbounded solutions are discussed in the succeeding section.**

The following examples are now introduced to sum up the basic features of the ordinary simplex method.

Example (13):

Find the value of x_1 , x_2 and x_3 by which:

$x_0 = 5x_1 + 4x_2 + 6x_3$ **(Maximize)**

Subject to :

$X_1 + x_2 + x_3$	$\leq$	100
$3x_1 + 2x_2 + 4x_3$	$\leq$	200
$3x_1 + 2x_2$	$\leq$	150
X_1 , x_2 , x_2	$\geq$	0

Solution:

Since all the R.H.S in the set of constraints in the model are positive numbers, and all the constraints for the problem are in the type ($\leq$), then we will use the ordinary simplex method passes through the following steps:

Step (1): **Express the LPM in its standard form;**

$X_0 = 5x_1 + 4x_2 + 6x_3 + (0)x_4 + (0)x_5 + (0)x_6$ **(Maximize)**

Subject to:

$X_1 + x_2 + x_3 + (x_4) = 100$

$3x_1 + 2x_2 + 4x_3 + (x_5) = 200$

$3x_1 + 2x_2 \qquad\qquad + (x_6) = 150$

$X_i \geq 0$, for i = 1,2, …., 6

<u>Step (2):</u> Construct the initial solution tableau, then select a starting basic feasible solution using the optimality and feasibility conditions until the optimal solution is obtained. The following tables and computations summarized these steps:

<u>Step (3):</u> The preceding form is expressed in the following starting tableau by considering the slack variables are the basic variables:

1st tableau

Basic var. (B.V)	C_J Coef. B.V	5	4	6	0	0	0	Solution (R.H.S)	Ratio
		x_1	x_2	x_3	x_4	x_5	x_6		
X_4	0	1	1	1	1	0	0	100	100/1=100
X_5	0	3	2	(4)	0	1	0	200	200/4=50 ←
X_6	0	3	2	0	0	0	1	150	-
E_J		0	0	0	0	0	0	0	Maximize
E_J-C_J		-5	-4	-6	0	0	0		
				↑					

*Now, since all the solution values (100, 200, 150) are positive numbers, then, the starting solution tableau is considered a feasible solution.

*And, since the optimality condition ($E_J - C_J$) implies a negative coefficients (-5 , -4 , -6) corresponding to the non-basic variables x_1 , x_2 , x_3, then, the starting tableau is not an optimal solution since we search for the optimum values for the set of variable by which maximize the value of the objective function x_0 or f(x). Henceforth, we have to improve the preceding tableau as follows:

<u>First:</u> Determine the entering variable which have the most negative coefficient in the row ($E_J - C_J$), then the non-basic variable x_3 is considered as the entering variable in the solution to be one of the basic variable in the succeeding tableau.

<u>Second</u>: By taking the ratios (corresponding to the positive coefficient under the entering variable) by which are shown in the latest column (100/1 = 100, 200/4 = 50). Then, the basic variable x_5 associated with the minimum ration is the leaving variable.

<u>Third</u>: The intersection between the entering variable (x_3) with the leaving variable (x_5) is (4), then the pivot element is (4). Therefore, introduce x_3 and drop x_5, then the basic variables in the succeeding tableau will be x_4 , x_3 and x_6 and then, we have to determine the following rows as follows:

***The 2^{nd} row (x_3) is obtained by dividing the preceding leaving variable row x_5, i.e., (3, 2, 4, 0, 1, 0, 200) on the pivot element (4), then the x_3 row implies the following coefficient:**

($^3/_4$, $^2/_4$, 1 , 0 , $^1/_4$, 0 , 50).

***The New 1^{st} row, i.e., the new row for the basic variable x_4 can be accomplished by using the following row operation:**

New row of x_4 = Old row of x_4 –(4x the new row of the pivot row or x_3) then we have the following:

1	–	1	× ($^3/_4$)	$= 1 - {}^3/_4 = {}^4/_4 - {}^3/_4$	$= {}^1/_4$
1	–	1	× ($^2/_4$)	$= 1 - {}^1/_2$	$= {}_1/_1$
1	–	1	× 1	= 1 – 1	= 0
1	–	1	× 0	= 1 – 0	= 1
0	–	1	× ($^1/_4$)	$= 0 - {}^1/_4$	$= {}^{-1}/_4$
0	–	1	× 0	= 0 – 0	= 0
100	–	1	× 50	= 100 – 50	= 50

***Also, the new 3^{rd} row, i.e., the new row for the variable x_6 can be accomplished by using the following row operation:**

New row of x_6 = old row of x_6 – (zero x the new row of x_3) then, we have the following:

New row of x_6 = old row of x_6

= (3 , 2 , 0 , 0 , 0 , 1 , 150)

*The E_J row can be accomplished by finding the summation of multiplying the column of the coefficient for the basic variable by the coefficient under each variable in the succeeding tableau, then we can get the row of optimality ($E_J - C_J$) as it be shown in the 1st improvement

<u>First Improvement:</u> (or 1st iteration):

2nd tableau

Basic var. (B.V)	C_J Coef. B.V	5 x_1	4 x_2	6 x_3	0 x_4	0 x_5	0 x_6	Solution (R.H.S)	Ratio
x_4	0	$^1/_4$	$^1/_2$	0	1	$^{-1}/_4$	0	50	100
x_3	6	$^3/_4$	$^2/_4$	1	0	$^1/_4$	0	50	100
x_6	0	3	(2)	0	0	0	1	150	75 ←
E_J		$^9/_2$	3	6	0	$^3/_2$	0	300	Maximize
E_J-C_J		$^{-1}/_2$	-1	0	0	$^3/_2$	0		
			↑						

*Now, since all the solution values (50, 50, 150) are positive numbers, then the 2nd tableau is considered a feasible solution.

***And, since the row of the optimality condition ($E_J - C_J$) implies a negative coefficients ($-^1/_2$, -1) corresponding to the nonbasic variables (x_1 , x_2), then the 2nd tableau is not an optimal solution, since we search for the optimum values for the set of x_J's by which maximize the value of the objective function f(x) or x_0. Henceforth, we have to improve the 2nd tableau as follows:**

<u>First</u>: Determine the entering variable which have the most negative coefficient in the ($E_J - C_J$) row, then the non-basic variable x_2 is considered as the entering variable in the solution to be one of the basic variable in the succeeding tableau.

<u>Second</u>: By taking the ratios (corresponding to the positive coefficient under the entering variable) ➔ ($50/(^1/_2) = 100$, $50/(^1/_2) = 100$, $150/2=75$), then the basic variable x_6 associated with the minimum ratio is the leaving variable.

<u>Third</u>: The intersection between the entering variable (x_2) and the leaving variable (x_6) is (2), then the pivot element is (2). Therefore, introduce x_2 and drop x_6, then the basic variables in the succeeding tableau will be x_4, x_3, x_2, and then we have to determine the following rows as follows:

***The 3rd row (x_2) is obtained by dividing the preceding leaving variable row (x_6), i.e., (3, 2, 0, 0, 0, 1, 150) on the pivot element (2), then the x_2 row implies the following coefficient:**

$(^3/_2\ ,\ 1\ ,\ 0\ ,\ 0\ ,\ 0\ ,\ ^1/_2\ ,\ 75)$.

***The new 1st row, i.e., the new row for the basic variable x_4 can be accomplished by using the following row operations:**

New row of x_4 = old row of x_4 – $(^1/_2)$ × the new row of the pivot row or x_2, then we have the following computation:

$$^1/_4 - (^1/_2) \times (^3/_2) = (^1/_4) - (^3/_4) = {}^{-2}/_4 = {}^{-1}/_2$$

$$^1/_2 - (^1/_2) \times 1 = (^1/_2) - (^1/_2) = 0$$

$$0 - (^1/_2) \times 0 = 0 - 0 = 0$$

$$1 - (^1/_2) \times 0 = 1 - 0 = 1$$

$$^{-1}/_4 - (^1/_2) \times 0 = -(^1/_2) - 0 = {}^{-1}/_4$$

$$0 - (^1/_2) \times (^1/_2) = 0 - (^1/_4) = {}^{-1}/_4$$

$$50 - (^1/_2) \times (75) = 100/2 - 75/2 = 25/2$$

***Also, the new 2nd row, i.e., the new row for the basic variable x_3 can be accomplished by using the following row operation:**

New row of x_3 = old row of x_3 – (2/4) × the new row of the pivot row or x_2, then we have:

$$^3/_4 - {}^2/_4 \times {}^3/_2 = 0$$

$$^2/_4 - {}^2/_4 \times 1 = 0$$

$$1 - {}^2/_4 \times 0 = 1$$

$0 \quad - {}^{2}/_{4} \times 0 \quad = 0$

${}^{1}/_{4} \quad - {}^{2}/_{4} \times 0 \quad = {}^{1}/_{4}$

$0 \quad - {}^{2}/_{4} \times {}^{1}/_{2} \quad = -{}^{1}/_{4}$

$50 \quad - {}^{2}/_{4} \times 75 \quad = 25/2$

***The E_J row and the optimality condition ($E_J - C_J$) can be accomplished in the second improvement as it be shown in the following 3rd tableau:**

<u>Second Improvement:</u> (or the 2nd iteration)

3rd tableau

Basic var. (B.V)	C_J Coef. B.V	5	4	6	0	0	0	Solution (R.H.S)	Ratio
		x_1	x_2	x_3	x_4	x_5	x_6		
x_4	0	$^{-1}/_2$	0	0	1	$^{-1}/_4$	$^{-1}/_4$	25/2	
x_3	6	0	0	1	0	$^{1}/_4$	$^{-1}/_4$	25/2	
x_2	4	$^{3}/_2$	1	0	0	0	$^{1}/_2$	75	
E_J		6	4	6	0	$^{3}/_2$	$^{1}/_2$	375	Maximize
E_J-C_J		1	0	0	0	$^{3}/_2$	$^{1}/_2$		

Now, in the 3rd tableau, since:

- **All the elements in the column of solution variable are positive values corresponding to the basic variable (x_2 , x_3 and x_4), then the 3rd tableau is considered a feasible solution.**
- **<u>All</u> the coefficient for the basic variable (x_2 , x_3 , x_4) in the optimality condition row ($E_J - C_J$) are zeros, and <u>all</u> the coefficient for the non-basic variable (x_1 , x_5, x_6) are nonnegative or positive values and we search about maximization the value of x_0 or f(x), therefore the 3rd tableau is considered a unique optimal solution. Henceforth, the optimal solution is:**

$x^*_1 = 0$, $x^*_2 = 75$, $x^*_3 = {}^{25}/_2$, $x^*_4 = {}^{25}/_2$, $x^*_5 = x^*_6 = 0$, $f(x^*) = x^*_0 = 375$

<u>Remarks:</u>

For each iteration in the ordinary simple method and the succeeding simplex methods (M-technique and two phase method) note that:

1. **The associated column coefficients of the basic variables for any simplex tableau give an identity matrix in which the diagonal elements are ones and all the others are zeros.**
2. **The associated coefficient in the optimality conditions ($E_J - C_J$) for any basic variable is zero in each solution tableau.**
3. **During the improvement iterations note that:**
   a. **If the old row intercepts the pivot column with zero element, then: the new row = the old row.**

b. If the old row intercepts the pivot column with the identity element (one), then: the new row = the old row – the new pivot row, and vice versa:

If the old row intercepts the pivot column with the negative identity element, i.e., (-1) then:

The new row = the old row + the new pivot row.

Example (14):

Find x_1 , x_2 and x_3 in which:

$F(x) = 3x_1 + 2x_2 + 5x_3$ maximize

Subject to:

$$x_1 + 2x_2 + x_3 \le 430$$

$$-3x_1 \quad -2x_3 \ge -460$$

$$-x_1 \; -4x_2 \ge -420$$

$$x_1 , x_2 , x_3 \ge 0$$

Solution: Note that, before determining the suitable simplex method, you have to make all the constants (R.H.S) for all constraints are positive values, so that, we have to multiply each of the 2nd and the 3rd constraints with (-1) to make each constant in its positive value.

Then we have the following general form

$f(x) = 3x_1 + 2x_2 + 5x_3$ (Max.)

Subject to:

$$x_1 + 2x_2 + x_3 \leq 430$$

$$3x_1 + 2x_3 \leq 460$$

$$x_1 + 4x_2 \leq 420$$

$$x_1 , x_2 , x_3 \geq 0$$

Now, since, all the constraints are inequalities in the form ($\leq$), after we make all the R.H.S are positive values, therefore, we have to use the ordinary simplex method as follows:

***Put the model in its standard form:**

$$f(x) = 3x_1 + 2x_2 + 5x_3 + (0)x_4 + (0)\, x_5 + (0)\, x_5 \quad \text{(Max.)}$$

Subject to:

$$x_1 + 2x_2 + x_3 + (x_4) = 430$$

$$3x_1 + 2x_3 + (x_5) = 460$$

$$x_1 + 4x_2 + (x_6) = 420$$

$$x_J \geq 0 \text{ , for; } J = 1, 2, \ldots, 6$$

A convenient way for recording the information about the starting solution and its improvement are in the following tableaus:

1-Starting Tableau (Initial Solution):

1^{st} Tableau

Basic var. (B.V)	C_J Coef. B.V	3 X_1	2 X_2	5 X_3	0 X_4	0 X_5	0 X_6	Solution (R.H.S)	Ratio
X_4	0	1	2	1	1	0	0	430	430/1 = 430
X_5	0	3	0	(2)	0	1	0	460	460/2=230 ←
X_6	0	1	4	0	0	0	1	420	-
E_J		0	0	0	0	0	0	0	Maximize
E_J-C_J		-3	-2	-5	0	0	0		
				↑					

In the preceding tableau not that:

- **All the solution values for the basic variables are nonnegative (positive values), then this tableau is considered a basic feasible solution (Feasibility condition).**
- **All the coefficients for the optimality condition ($E_J - C_J$) are not positive for all the non-basic variable (x_1, x_2 , x_3), and the objective is to maximize x_0 or $f(x)$, then the solution is not optimal, so that the first iteration for improve the solution is to introduce x_3 as the entering variable, then by taking the ratios by which are denoted in the last column in the preceding table, then x_5 becomes the leaving variable, i.e.,**

introduce x_3 and drop x_5, then the pivot element is (2). Henceforth, the new pivot row is:

($^3/_2$, 0 , 1 , 0 , $^1/_2$, 0 , 230), and then,

the new row for x_4 = the old row of x_4 – 1 × the new pivot row

= the old row for x_4 – the new pivot row.

And;

The new row for x_6 = the old row for x_6 – 0 x the new pivot row

∴ The new row for x_6 = the old row for x_6

Then the new tableau is thus given by:

First iteration: (Second Tableau):

2nd Tableau

Basic var. (B.V)	C_J	3	2	5	0	0	0	Solution (R.H.S)	Ratio
	Coef. B.V	X_1	X_2	X_3	X_4	X_5	X_6		
X_4	0	$^{-1}/_2$	(2)	0	1	$^{-1}/_2$	0	200	200/2=100 ←
X_3	5	$^3/_2$	0	1	0	$^1/_2$	0	230	-
X_6	0	1	4	0	0	0	1	420	420/4=105
E_J		$^{15}/_2$	0	5	0	$^5/_2$	0	1150	Maximize
E_J-C_J		$^9/_2$	-2	0	0	$^5/_2$	0		

↑

Second iteration: (x_2) is the entering variable. By taking ratios as it be shown in the last column in the preceding tableau, then (x_4) is the leaving variable, i.e., introduce x_2 and drop x_4, then the pivot element is (2). Therefore, the detailed rows operations are given as follows:

The new pivot row is ($-^1/_4$, 1 , 0 , $^1/_2$, $-^1/_4$, 0 , 100).

And;

The new row for x_6 = the old row for x_6 – 4 the new pivot row, i.e.,

1	–	4	× $(-^1/_4)$	= 1 – 1	= 2
4	–	4	× 1	= 4 – 4	= 0
0	–	4	× 0	= 0 – 0	= 0
0	–	4	× $(^1/_2)$	= 0 – 2	= -2
0	–	4	× $(-^1/_4)$	= 0 + 1	= 1
1	–	4	× 0	= 1 – 0	= 1
420	–	4	× 100	= 420 – 400	= 20

Therefore, the tableau for the second iteration can be shown as follows:

<u>Second iteration:</u>

3rd Tableau

Basic var. (B.V)	C_J	3	2	5	0	0	0	Solution (R.H.S)	Ratio
	Coef. B.V	X_1	X_2	X_3	X_4	X_5	X_6		
X_2	2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100	
X_3	5	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
X_6	0	2	0	0	-2	1	1	20	
E_J		7	2	5	1	2	0	1350	Maximize
E_J-C_J		4	0	0	1	2	0		

<u>Note that in the latest tableau:</u> Since all the values for the basic variables (x_2 , x_3 , x_6) are nonnegative values (100, 230, 20), then this tableau is considered a basic feasible solution.

- **Since *<u>all</u>* the coefficients in the optimality row ($E_J - C_J$) implies:**
- **Zeros coefficients only corresponding to the basic variables (x_2 , x_3 , x_6).**
- **Nonnegative coefficients corresponding to all the non-basic variables (x_1, x_4, x_5), then this tableau is considered a unique optimum solution. This yields the optimal solution is:**
- **$X^*_1 = 0$, $x^*_2 = 100$, $x^*_3 = 230$, $x^*_4 = 0$, $x^*_5 = 0$, $x^*_6 = 20$, $f(x^*) = 1350$.**

The Economic Importance for the solution of the LPM:

(Shadow Prices):

There is no doubt that the best outcomes resulted from the solution of the linear programming models is to give an analytical tool for the economic, management and accounting models for the decision maker. Where the linear programming models is considered an effective tool for determining the cost of an alternative opportunity chance or in other meaning duality (As we see in the following section) or shadow prices.

The shadow price for any resource (constraint) in the LPM is considered the increasing (or decreasing) value of the objective function x_0 in the primal resulted from increasing (or decreasing) the value of the available amount or the right hand side from this resource (constraint) with only one unit in the problem. The shadow prices can be determined from the optimal solution tableau from the coefficients under the slack variables which are in the optimality condition row ($E_J - C_J$). If the value of the shadow price for any resource (constraint) is equal to zero, then, increasing any additional units on the available amount for this resource will not increase the value of optimum value for the objective function $f(x^*)$. In this case, one can verify that there were some units from this available resource are not completely used. Otherwise, if the shadow price for any constraint is different from zero, then, it means that the

available resource for this constraint will be completely used, and hence, if we increase the available resource with some amount, it will be increase the value of *f*(x^*).

<u>Application:</u>

In the preceding example the shadow prices are the three coefficient under the slack variable x_4 , x_5, and x_6 in the optimality condition (E_J – C_J), i.e., the shadow prices are (1, 2 and zero) for the three constraints respectively.

***The shadow price for the first constraint is (1) unit. This means that all the value for the available resource amount (R.H.S for the 1st constraint) is completely used (Full used). Hence, one can verify from that by substituting in the first constraint by the optimal values resulted from the optimal solution as follows:**

$$x_1 + 2x_2 + x_3 \leq \quad 43 \text{ , then}$$

$$\text{L.H.S} = x^*_1 + 2x^*_2 + x^*_3 = 0 + 2\,(100) + 230 = 430 = \text{R.H.S}$$

Also;

***The shadow price for the second constraint is (2). This means that all the value for the available resource amount (R.H.S for the 2nd constraint) is completely used (Full used). And hence, one can verify by substituting in the 2nd constraint as follows:**

$$3x_1 + 2x_3 \quad \leq \quad 460$$

L.H.S = $3x^*_1 + 2x^*_3$ = 3 (0) + 2(230) = 460 = R.H.S

But,

***The shadow price for the third constraint is equal to zero, i.e., there are some amount of the available resource for the 3rd constraint are not used. Hence, if, one can verify from this remark, by substituting in the 3rd constraint by the value of the optimal variables: i.e.,**

$$x_1 + 4x_2 \quad \leq \quad 420 \text{ , then}$$

L.H.S = $x^*_1 + 4x^*_2 + x^*_3$ = 0 + 4 (100) = 400 < 420 (R.H.S)

Note that, besides the mathematically correct for the inequality of the 3rd constraint, the difference between the two sides = R.H.S – L.H.S = 420 – 400 = 20 which is the same optimum value by which the slack variable for the 3rd constraint in the primal problem [i.e., variable x^*_6] appear with the same value for the difference between the two sides in the basic variables the optimum solution.

<u>In summary one can show that:</u>

The value of the *i*th shadow price for the *i*th constraint in the primal linear programming problem is equal to the *i*th slack variable coefficient in the optimality row condition ($E_J - C_J$), i.e.,

the value of the 1st shadow price = the coefficient of the 1st slack variable in the row ($E_J - C_J$)

, the value of the 2nd shadow price = the coefficient of the 2nd slack variable in the row ($E_J - C_J$)

And so on .....,

<u>2-The simplex methods when the R.H.S are positive value and at least one constraint is of the types (= or >):</u>

<u>[Artificial variables techniques]:</u>

This section shows how a starting basic feasible solution can be secured when the slack variables do not readily provide such a solution. In general, this will be the case when at least one of the constraints is of the type (=) or ($\geq$). Two (closely related) methods based on the use of the "artificial" variables are devised for this purpose:

a. The "M-technique" or the "method of penalty"

b. The "two-phase" technique.

<u>(A): The "M-technique" or the "Method of penalty":</u>

This technique is named by the M-technique because it is achieved by assigning a very large per-unit penalty to a set of variables called by the artificial variables added to the set of constraints of the types (= or $\geq$). The basic steps for the M-technique are as follows;

<u>Step</u> 1: Express the LPM in its standard form.

Step 2: In order to create the identity matrix and the basic variable, we must add nonnegative variables to the left-hand side of each of the equation corresponding to any constraint of the types ($\geq$) or (=). These variables are called by "artificial variables" and their addition causes violation of the corresponding constraints. This difficulty is overcome by ensuring that the artificial variables will be zero (=0) in the final or optimal solution. This is achieved by assigning a very large per-unit penalty to these variables in the objective function. Such a penalty will be designated by (-M) for maximization the objective function f(x) or x_0 and (+M) for minimization problem, where M > 0.

Step 3: Use the artificial variables for the starting basic solution. However, in order for the starting tableau to be prepared properly, the objective function must be expressed in terms of the non-basic variables only. This means that the objective coefficients of the artificial variables must equal zero, a result which can be achieved by adding proper multiples of the constraint equations to the objective row.

Step 4: Proceed with the regular steps of the simplex method.

Note that, the artificial variables only provide a mathematical trick for obtaining a starting solution. The effect of these variables on the final solution is cancelled by the high penalty in the objective

function. Perhaps this clarifies the use of the name "artificial" since these variables are fictitious and do not have a direct physical interpretation in terms of the original problem.

Example (15):

Determine the value of x_1 and x_2 by which:

$f(x) = 4x_1 + x_2$ minimize

Subject to:

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$X_1 + 2x_2 \leq 4$

$x_1 , x_2 \geq 0$

Required: 1-Solve the LPM graphically and determine the different types of solutions.

2-By using the suitable simplex technique verify from the preceding results in (1).

Solution:

Note that, as we mentioned above in the graphical solution for the linear programming model, since there is one constraint in the equation form (=), then the feasible solution space will be a part

from the corresponding line graph to this constraint as it will be shown in this example:

<u>(1):Graphical solution:</u>

***Non-negativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ implies that the feasible solution space must be lie in the first quadrant.**

<u>-The 1st constraint:</u> $3x_1 + x_2 = 3$

The following table represents the two intercepts from the two axis.

X_1	0	1
X_2	3	0

And since the constraint is in the equation form (=), then each point in the line graph for this equation only satisfies this equation.

<u>-The 2nd constraint:</u> $4x_1 + 3x_2 \geq 6$, convert the inequality into an equation, then we have:

$4x_1 + 3x_2 = 6$, and hence the following table represents the two intercepts.

X_1	0	1.5
X_2	2	0

And since all the variable coefficients and the constant (R.H.S) in this constraint are positive values, then any point in the line graph

corresponding to this constraint or up to this line satisfies this constraint ($\geq$) .

<u>-The 3rd constraint:</u> $x_1 + 2x_2 \leq 4$, convert the inequality into an equation, then, we have:

$X_1 + 2x_2 = 4$, and hence the following table represents the two intercepts.

X_1	0	4
X_2	2	0

And since all the variable coefficients and the constant (R.H.S) for this constraint are positive values, then any point in the line graph corresponding to this constraint or under this line satisfies this constraint ($\leq$).

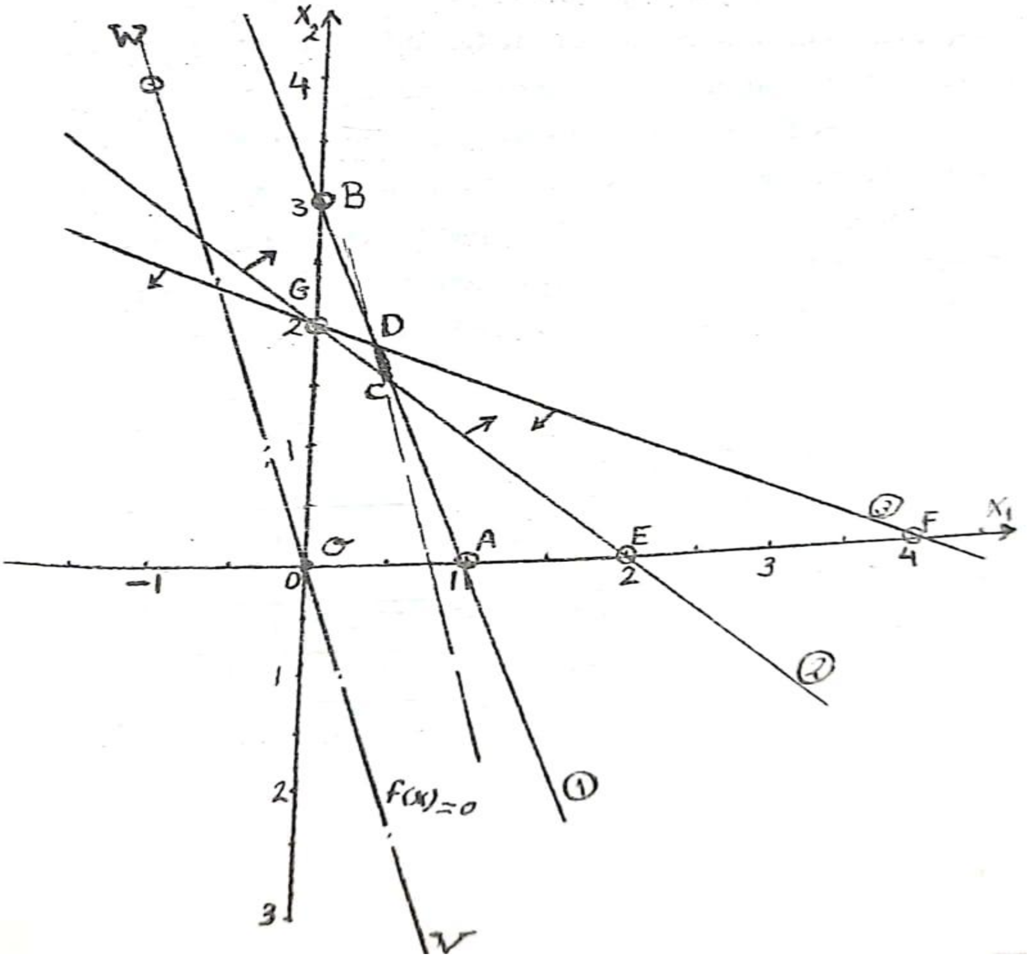
<u>-Graph the line for the objective function</u>:

Suppose that f(x) = 0 , then, we have:

$4x_1 + x_2 = 0$, the following table represent the points by which phases with f(x) = 0

X_1	0	1	-1
X_2	0	-4	4

-Determining the feasible solution graphically and hence the types of solution:



Reduction for the solution space:

-Non-negativity constraints: x_1 O x_2

*** 1st constraint: AB**

*** 2nd constraint: CB**

*** 3rd constraint: CD** **(Feasible Solution Space)**

Now, by moving the line (WOV) for the objective function parallel to itself in the direction of the feasible solution space (CD) to pass the first point in this line since we have to search about the optimum values for x_1 and x_2 by which minimize the objective function. Therefore, one sees that the minimum value for f(x) occurs where the line of the objective function (WOV) passes through point D whose coordinates are $x^*_1 = {}^2/_5$ and $x^*_2 = {}^9/_5$. Substituting these values into the objective function gives:

$$f(x^*) = 4x^*_1 + x^*_2 = 4\,({}^2/_5) + ({}^9/_5) = {}^{17}/_5$$

As an illustration, the reader can verify graphically from the following table about this correct answer:

Corners of the solution space	$F(x) = 4x_1 + x_2$	Remarks
D(2/5 , 9/5)	$F_D(x) = 4(2/5) + 9/5 = 17/5$	Minimum
C(3/5 , 6/5)	$F_C(x) = 4(3/5) + 6/5) = 18/5$	Maximum

Finally the types of solutions resulted from the graphical solution are:

1-<u>Basic solutions:</u> the points O, E, F, C, D, G and B are the basic solutions.

2-<u>Feasible solutions</u>: Any point in the feasible solution space CD is considered a feasible solution.

3-<u>Basic feasible solutions</u>: (Extreme Points) the two points (C) and (D) are the basic feasible solutions.

4-Optimum solution: D [($x^*_1 = 2/5$, $x^*_2 = 9/5$) , $f(x^*) = 17/5$]

<u>2-The solution with the suitable simplex technique:</u>

Since all the constant (R.H.S) for all constraints are positive numbers, and there are at least one constraint is in the type ($\geq$ or =), therefore, either the M-technique or the two-phase technique are suitable for the solution of this LPM.

Now, we will use the M-technique:

<u>Step 1</u>: Express the model in its standard form:

$$F(x) = 4x_1 + x_2 \qquad \text{(Min)}$$

<u>Subject to</u>:

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 - x_3 = 6$$

$$x_1 + 2x_2 + (x_4) = 4$$

$$x_1 , x_2 , x_3 , x_4 \geq 0$$

<u>Step 2</u>: Add an artificial variable corresponding to the constraint of the type (= or $\geq$) and express these artificial variable with (-M) coefficient in the equation of the objective function in case of maximization the value of f(x) or with (M) coefficient in case of

minimization value of f(x), where M > 0. Then, we have the following model:

$$f(x) = 4x_1 + x_2 + 0\,(x_3 + x_4) + M\,(x_5 + x_6) \qquad \text{(Min)}$$

<u>Subject to</u>:

$$3x_1 + x_2 \qquad\qquad + (x_5) \qquad = 3$$

$$4x_1 + 3x_2 - x_3 \qquad\qquad + (x_6) = 6$$

$$x_1 + 2x_2 \qquad + (x_4) \qquad\qquad = 4$$

$$x_i \geq 0 \text{ for } i = 1 : 6$$

<u>Step 3</u>: construct the initial solution tableau by considering that the slack variable corresponding to the constraint in the form ($\leq$) is the basic variables, and the artificial variable corresponding to the constraint are in either the form (= or $\geq$) is the basic variable for these constraints. Then, we have the following tableau:

<u>Starting (or initial) solution tableau:</u>

1^{st} tableau

(B.V)	C_J / Coef. B.V	X_1 (4)	X_2 (1)	X_3 (0)	X_4 (0)	X_5 (M)	X_6 (M)	Solution	Ration
X_5	M	(3)	1	3	3	1	3	3	3/3 = 1 ←
X_6	M	4	3	-1	0	0	1	6	6/4 = 1.5
X_4	0	1	2	0	1	0	0	4	4/1 = 4
E_J		7M	4M	-M	0	M	M	9M	Minimize
E_J-C_J		7M-4 ↑	4M-1	-M	0	0	0		

<u>1^{st} Iteration:</u> Since, this is a minimization problem, so that, the entering variable must be the variable which has the largest positive coefficient in the optimality row condition ($E_J - C_J$), then we have to introduce the variable x_1 (entering variable), and drop the variable x_5 (leaving variable). Hence the pivot element is (3). Then,

-The new pivot row elements are (1, 1/3 , 0 , 0 , 1/3 , 0 , 1).

-And:

the new row for x_6 = the old row for x_6 – 4 × the new pivot row.

Then we have the following computations:

4	**–**	**4**	**×**	**1**	**= 4 – 4**	**= 0**
3	**–**	**4**	**×**	**$^1/_3$**	**= 3 – $^4/_3$**	**= $^5/_3$**
-1	**–**	**4**	**×**	**0**	**= -1 – 0**	**= -1**
0	**–**	**4**	**×**	**0**	**= 0 – 0**	**= 0**
0	**–**	**4**	**×**	**($^1/_3$)**	**= 0 – $^4/_3$**	**= -$^4/_3$**
1	**–**	**4**	**×**	**0**	**= 1 – 0**	**= 1**
6	**–**	**4**	**×**	**1**	**= 6 – 4**	**= 2**

-And the new row for x_4 = the old row for x_4 – 1× the new pivot row

i.e.,

the new row for x_4 = the old row for x_4 – the new pivot row as it be shown in the succeeding tableau, besides the row of $E_J^{'}$ and the row of optimality condition ($E_J – C_J$).

<u>1st improvement:</u>

2^{nd} tableau

(B.V)	C_J / Coef. B.V	4 X_1	1 X_2	0 X_3	0 X_4	M X_5	M X_6	Solution	Ratio
X_1	4	1	$^1/_3$	0	0	$^1/_3$	0	1	3
X_6	M	0	$(^5/_3)$	-1	0	$-^4/_3$	1	2	$^6/_5$ ←
X_4	0	0	$^5/_3$	0	1	$-^1/_3$	0	3	$^9/_5$
E_J		4	$^4/_3+(^5/_3)M$	-M	0	$^4/_3-(^4/_3)M$	M	4 + 2M	(Min.)
E_J-C_J		0	$^1/_3+^5/_3M$ ↑	-M	0	$^4/_3-^7/_3M$	0		

2^{nd} iteration:

Introduce x_2 and drop x_6.

Then the pivot element is $(^5/_3)$

***Hence: the new pivot row = the old pivot row ÷ (5/3),**

Then, the new pivot row = (0 , 1 , -3/5 , 0 , -4/5 , 3/5 , 6/5)

And;

The new row for x_1 = the old row of x_1 – (1/3) × the new pivot row

And;

The new row for x_4 = the old row for x_4 – (5/3) × the new pivot row

Hence, we have the following tableau: " <u>2^{nd} improvement":</u>

3^{rd} tableau

(B.V)	C_J / Coef. B.V	X_1 (4)	X_2 (1)	X_3 (0)	X_4 (0)	X_5 (M)	X_6 (M)	Solution	Ratio
X_1	4	1	0	1/5	0	3/5	-1/5	3/5	3
X_2	1	0	1	-3/5	0	-4/5	3/5	6/5	~
X_4	0	0	0	1	1	1	-1	1	1 ←
E_J		4	1	1/5	0	8/5	-1/5	18/5	(Min)
E_J-C_J		0	0	1/5	0	8/5 –M	-1/5-M		
				↑					

<u>3^{rd} Iteration</u>: Introduce x_3 and drop x_4. So that, the pivot element is (1). Then:

The new pivot row = (0 , 0 , 1 , 1 , 1 , -1 , 1)

And;

The new row for x_1 = the old row for x_1 – (1/5) × the new pivot row.

And;

The new row for x_2 = the old row for x_2 – (-3/5) × the new pivot row

= the hold row for x_2 + 3/5 × the new pivot row.

Hence, the following tableau represents the preceding computations besides the row of E_J and the optimality condition row ($E_J - C_J$):

<u>3rd improvement:</u> **4th tableau**

(B.V)	C_J	4	1	0	0	M	M	Solution	Ratio.
	Coef. B.V	X_1	X_2	X_3	X_4	X_5	X_6		
X_1	4	1	0	0	-1/5	2/5	0	2/5	
X_2	1	0	1	0	3/5	-1/5	0	9/5	
X_3	0	0	0	1	1	1	-1	1	
E_J		4	1	0	-1/5	7/5	0	17/5	(Min)
E_J-C_J		0	0	0	-1/5	7/5-M	-M		

<u>Now, from the preceding tableau note that:</u>

- **Since, all the values for the basic variables (solution column) are positive values, so that this tableau is considered a basic feasible solution (feasibility condition),**
- **Since all the coefficients corresponding to the basic variables x_1 , x_2 and x_3 are zeros <u>only</u>, and negative coefficient corresponding to <u>all</u> the non-basic variables (x_4 , x_5 and x_6), so that the 4^{th} tableau is considered a unique optimal solution (optimality condition).**

Henceforth, the optimal solution for this problem is the same graphical solution where:

$x^*_1 = {}^2/_5$, $x^*_2 = {}^9/_5$, $x^*_3 = 1$, $x^*_4 = x^*_5 = x^*_6 = 0$ **and** $f(x^*) = {}^{17}/_5$

<u>Remark:</u>

In the preceding example, if the objective function is to maximize f(x), in this case the only difference between the preceding solution and the case of maximize f(x) is to modifying the objective function accordingly to become in the form.

$F(x) = 4x_1 + x_2 + 0(x_3 + x_4) - M\,(x_5 + x_6)$ **maximize.**

But the set of constraints still in the same form as in the preceding solution.

One can solve that problem, then the optimal solution resulted from the M-technique gives the following solution as it stated in the preceding graphical solution:

$x^*_1 = 3/5$, $x^*_2 = 6/5$, $f(x^*) = 18/5$.

<u>(B):The two-phase technique:</u>

A drawback of the M-technique is the possible computational error that could result from assigning a very large value to the constant M. To illustrate this point, suppose that: M = 10,000 in the preceding example. Then, in the starting tableau, the objective coefficients of x_1 and x_2 are (-4 + 70,000) and (-1 + 40,000). Then, the effect of the original coefficient of x_1 and x_2 (=4 and 1 respectively) is now too small compared with the large numbers created by the multiples of M. Due to the round off error, which is inherent in any digital computer, the solution may become insensitive to the relative values of the original objective coefficients of x_1 and x_2. The dangerous outcome is that x_1 and x_2 may be treated as is they have equal coefficients in the objective function. To alleviate this difficulty, the new method eliminates the use of constant M by solving the problem in two phases (hence, the name "Two-phase method). These two phases are outlined as follows:

<u>Phase I :</u> Formulate a new problem by replacing the original or the primal objective function by the sum of the artificial variables. Then, the new objective function is then minimized (if the primal or the original objective function is either maximize or minimize) subject to the constraints of the original problem.

If the problem has a feasible solution space, then the minimum value of the new objective function will be reached to zero (which indicates that all the artificial variables are leaving from the basic variables, i.e., equal to zero). Then, we have to go to phase II. Otherwise if the minimum value is greater than zero, then the problem is terminated with the information that no feasible solution exists.

Phase II:

Use the optimum basic feasible solution resulted from phase I as a starting solution for the original problem after deleting the row of the new objective and the artificial variables columns from the solution tableau. In this case, the original objective function is expressed in terms of the non-basic variables by using the Gauss-Jordan eliminations.

Example (15) : Solve the preceding LPM stated in the preceding example by using the two-phase method.

Solution: Phase I :

Step 1 : Put the LPM in its standard form, added the artificial variable to the constraints corresponding to the form (= or $\geq$), find the sum of the set of constraints which have the artificial variable, then replace the sum of the set of the artificial variable with the new objective function F_0 (Notice that, the new objective function F_0 is

always of the minimization type regardless of whether the original problem is maximization or minimization). Then we have the following:

$F(x) = 4x_1 + x_2 + 0\,(x_3 + x_4) + x_5 + x_6$ (Minimize)

Subject to:

$$3x_1 + x_2 \qquad + (x_5)\ \text{artif.} \qquad = 3$$

$$4x_1 + 3x_2 - x_3 \qquad + (x_6)\ \text{artif.} \quad = 6$$

$$X_1 + 2x_2 \qquad + (x_4) \qquad = 4$$

$$X_i \geq 0 \text{ for } i = 1 : 6$$

The new objective function F_0 resulted from determining the sum of the set of constraints in which have an artificial variable, then we have to find the sum of the 1st and the 2nd constraints, then we have:

$$7x_1 + 4x_2 - x_3 + (x_5 + x_6) = 9$$

i.e.,

$7x_1 + 4x_2 - x_3 + F_0 = 9$ ➔ the new objective function (minimize)

<u>Step 2</u>: Put the new objective function F_0 and the set of constraints for the original problem in the following starting tableau:

<u>Starting tableau: </u>Initial solution for the 1st phase :

(1st tableau)

B.V	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio.
x_5	(3)	1	0	0	1	0	3	$^3/_3=1$ ←
x_6	4	3	-1	0	0	1	6	$^6/_4=1.5$
x_4	1	2	0	1	0	0	4	$^4/_1=4$
F_0	7	4	-1	0	0	0	9	minimize

↑

<u>1st iteration</u>: Introduce (x_1) (entering variable), and drop (x_5) (leaving variable), then, the pivot element is (3),

Hence, we have the following computations:

The new pivot row = (1, 1/3 , 0 , 0 , 1/3 , 0 , 1) , and;

The new row for x_6 = the old row for x_6 – 4 × the new pivot row, i.e.:

$$4 - 4 \times 1 = 0$$

$$3 - 4 \times {}^1/_3 = {}^5/_3$$

$$-1 - 4 \times 0 = -1$$

$$0 - 4 \times 0 = 0$$

$$0 - 4 \times {}^1/_3 = -{}^4/_3$$

$$1 - 4 \times 0 = 1$$

6 – 4 × 1 = 2

And, The new row for x_4 = the old row for x_4 – 1 × the new pivot row

And, The new row for F_0 = the old row for F_0 – 7 × the new pivot row.

7 – 7 × 1 = 0

4 – 7 × $^1/_3$ = $^5/_3$

-1 – 7 × 0 = -1

0 – 7 × 0 = 0

0 – 7 × $^1/_3$ = $-^7/_3$

0 – 7 × 0 = 0

9 – 7 × 1 = 2

Then, we have the following 1^{st} iteration tableau:

2^{nd} tableau for the 1^{st} phase

B.V	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio.
X_1	1	$^1/_3$	0	0	$^1/_3$	0	1	3
X_6	0	$^5/_3$	-1	0	$-^4/_3$	1	2	1.2 ←
x_4	0	$^5/_3$	0	1	$-^1/_3$	0	3	1.5
F_0	0	$^5/_3$	-1	0	$-^7/_3$	0	2	minimize

The 2^{nd} iteration: Introduce x_2 (entering variable), and drop x_6 (leaving variable), then, the pivot element is $^5/_3$, hence, we have the following computations:

Hence, the new pivot row = the old pivot row ÷ ($^5/_3$), i.e.,

The new pivot row = (0 , 1 , $-^3/_5$, 0 , $-^4/_5$, $^3/_5$, $^6/_5$) ,

and the new row for x_1 = the old row of x_1 – $^1/_3$ × the new pivot row,

i.e.,

$$1 - \tfrac{1}{3} \times 0 = 1$$

$$\tfrac{1}{3} - \tfrac{1}{3} \times \tfrac{1}{3} = 0$$

$$0 - \tfrac{1}{3} \times (-\tfrac{3}{5}) = \tfrac{1}{5}$$

$$0 - \tfrac{1}{3} \times 0 = 0$$

$$\tfrac{1}{3} - \tfrac{1}{3} \times (-\tfrac{4}{5}) = \tfrac{3}{5}$$

$$0 - \tfrac{1}{3} \times (\tfrac{3}{5}) = -\tfrac{1}{5}$$

$$1 - \tfrac{1}{3} \times (\tfrac{6}{5}) = \tfrac{3}{5}$$

And,

The new row for x_4 = the old row for x_4 - $^5/_3$ × the new pivot row.

And,

The new row for F_0 = the old row for F_0 - $^5/_3$ × the new pivot row.

<u>Then, we have the following 2nd iteration tableau:</u>

3rd tableau

B.V	x_1	x_2	x_3	x_4	x_5	x_6	Solution	Ratio.
X_1	1	0	1/5	0	3/5	-1/5	$^3/_5$	
X_2	0	1	-3/5	0	-4/5	3/5	$^6/_5$	
x_4	0	0	1	1	1	-1	1	
F_0	0	0	0	0	-1	-1	0	

Now, since $F_0 = 0$, then the problem has a feasible solution and phase II can be carried out.

Phase II: The row of F_0 and the columns for the artificial variables are eliminated from the last tableau [3rd tableau for the 1st phase] since they are non-basic variables. Then, the initial solution for phase II is obtained by replacing the row of F_0 equation by the original f(x) equation [i.e., E_J and the optimality row condition ($E_J - C_J$)].

This gives: The Starting tableau for the 2nd phase (II):

4^{th} tableau

(B.V)	C_J / Coef. B.V	4 X_1	1 X_2	0 X_3	0 X_4	Solution	Ratio.
X_1	4	1	0	$^1/_5$	0	$^3/_5$	$^3/_5 \div {}^1/_5 = 3$
X_2	1	0	1	$-^3/_5$	0	6/5	~
X_4	0	0	0	(1)	1	1	$1 \div 1 = 1$ 🡐
E_J		4	1	$^1/_5$	0	$^{18}/_5$	(Min)
$E_J - C_J$		0	0	$^1/_5$	0		

↑

This tableau is not optimal since x_3 has a positive coefficient in the optimality condition row. Then, by carrying out an additional iteration with x_3 as the entering variable and x_4 as the leaving variable. Then, we have the 1^{st} iteration.

<u>1^{st} iteration:</u> Introduce (x_3) (entering variable), and drop (x_4) (leaving variable), then the pivot element is (1), hence, we have the following computations:

The new pivot row = the old pivot row, since the pivot element is equal to (1).

And;

The new row for x_1 = the old row for x_1 – $^1/_5$ × the new pivot row, then we have the following results:

$1 \quad - \quad {}^{1}/_{5} \quad \times \quad 0 \quad = 1$

$0 \quad - \quad {}^{1}/_{5} \quad \times \quad 0 \quad = 0$

${}^{1}/_{5} \quad - \quad {}^{1}/_{5} \quad \times \quad 1 \quad = 0$

$0 \quad - \quad {}^{1}/_{5} \quad \times \quad 1 \quad = -{}^{1}/_{5}$

${}^{3}/_{5} \quad - \quad {}^{1}/_{5} \quad \times \quad 1 \quad = {}^{2}/_{5}$

And;

The new row for x_2 = the old row for x_2 – $(-{}^{3}/_{5})$ × the new pivot row, then we have:

$0 \quad - \quad (-{}^{3}/_{5}) \quad \times \quad 0 \quad = 0$

$1 \quad - \quad (-{}^{3}/_{5}) \quad \times \quad 0 \quad = 1$

$-{}^{3}/_{5} \quad - \quad (-{}^{3}/_{5}) \quad \times \quad 1 \quad = 0$

$0 \quad - \quad (-{}^{3}/_{5}) \quad \times \quad 1 \quad = {}^{3}/_{5}$

${}^{6}/_{5} \quad - \quad (-{}^{3}/_{5}) \quad \times \quad 1 \quad = {}^{9}/_{5}$

Therefore, we have the following 1st improvement iteration for the 2nd phase:

5^{th} tableau

(B.V)	C_J / Coef. B.V	4	1	0	0	Solution	Ratio.
		X_1	X_2	X_3	X_4		
X_1	4	1	0	0	$-^1/_5$	$^2/_5$	
X_2	1	0	1	0	$^3/_5$	$^9/_5$	
X_3	0	0	0	1	1	1	
E_J		4	1	0	$-^1/_5$	$^{17}/_5$	(Min)
E_J-C_J		0	0	0	$-^1/_5$		

***Now, since all the solution values corresponding to the basic variables (x_1, x_2 , x_3) are positive values, then this tableau is considered a basic feasible solution and;**

Since <u>all</u> the coefficients corresponding to the basic variables are only zeroes and negative coefficient corresponding to the nonbasic variable (x_4), and since f(x) is in minimization type, then this tableau is a unique optimal solution. Then, the optimal solution is:

$$x^*_1 = {}^2/_5 , x^*_2 = {}^9/_5 , x^*_3 = 1, x^*_4 = x^*_5 = x^*_6 = 0, f(x^*) = {}^{17}/_5$$

Example (16) :

Find x_1 , x_2 and x_3 by which:

$F(x) = 5x_1 – 6x_2 – 7x_3$ maximize

Subject to:

$$X_1 + 5x_2 - 3x_3 \geq 15$$

$$5x_1 - 6x_2 + 10x_3 \leq 20$$

$$X_1 + x_2 + x_3 = 5$$

$$X_1 , x_2 , x_3 \geq 0$$

Solution:

Since, the constants (R.H.S) for the set of constraints are positive umbers, and there is at least one constraint is not in the type ($\leq$), then, we have to use either the M-technique or the two-phase method. We will solve this problem by using the two-phase method.

Phase (I):

Put the LPM in its standard form, added the artificial variable to the constraint in which they are in the type (= or $\geq$), and find the objective function F_0 from the summation for the set of constraints of the type (= and $\geq$), then we have the following:

$f(x) = 5x_1 – 6x_2 – 7x_3$ (Maximize)

Subject to:

$x_1 + 5x_2 - 3x_3 - x_4 \quad + \qquad (x_6) \text{ artif.} \qquad = 15$

$5x_1 - 6x_2 + 10x_3 \qquad + (x_5) \qquad\qquad = 20$

$x_1 + x_2 + x_3 \qquad\qquad + (x_7) \text{ artif.} = 5$

$x_i \geq 0$ **for** $i = 1 : 6$

The sum for the 1st and the 3rd constraints ($\geq$ **and =), then we have:**

$2x_1 + 6x_2 - 2x_3 - x_4 + (x_6 + x_7) = 20$

Substituting $F_0 = x_6 + x_7$ **then:**

$2x_1 + 6x_2 - 2x_3 - x_4 + F_0 = 20 \qquad\qquad$ **(minimize)**

Then, we have the following starting solution tableau for phase I:

Starting or initial tableau

1st tableau

B.V	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution	Ratio.
x_6	1	5	-3	-1	0	1	0	15	3 ←
x_5	5	-6	10	0	1	0	0	20	-
x_7	1	1	1	0	0	0	1	5	5
F_0	2	6	-2	-1	0	0	0	20	Minimize
		↑							

<u>1st iteration:</u> Introduce x_2 (entering variable), and drop x_2, then the pivot element is (5), hence we have the following computations:

- **The new pivot row = the old pivot row ÷ 5**

  $= (^1/_5\ , 1, -^3/_5\ , -^1/_5\ , 0, ^1/_5, 0, 3)$

- **The new row for x_5 = the old row for x_5 – (-6) × the new pivot row.**

  = the old row for x_5 + 6 × the new pivot row.

- **The new row for x_7 = the old row for x_7 – 1 × the new pivot row.**

- **The new row for F_0 = the old row for F_0 – 6 × the new pivot row.**

<u>Hence we have the 1st iteration tableau as follows:</u>

2nd tableau

B.V	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution	Ratio.
x_2	$^1/_5$	1	$-^3/_5$	$-^1/_5$	0	$^1/_5$	0	3	~
x_5	$^{31}/_5$	0	$^{32}/^5$	$-^6/_5$	1	$^6/_5$	0	38	-5.9
x_7	$^4/_5$	0	$^8/_5$	$^1/_5$	0	$-^1/_5$	1	2	1.25 ←
F_0	$^4/_5$	0	$^8/_5$	$^1/_5$	0	$-^6/_5$	0	2	Minimize

↑

<u>2nd iteration</u>: Introduce x_3 (entering variable) and drop x_7 (leaving variable), then the pivot element is ($^8/_5$), and hence, we have the following computations:

- **The new pivot row = the old pivot row ÷ $^8/_5$**

  = the old pivot row × $^5/_8$

  = ($^1/_2$, 0 , 1 , $^1/_8$, 0 , -$^1/_8$, $^5/_8$, $^5/_4$),
- **The new row for x_2 = the old row for x_2 – (-$^3/_5$) × the new pivot =the old row for x_2 + $^3/_5$ × the new pivot row,**
- **The new row for x_5 = the old row for x_5 – $^{32}/_5$ × the new pivot row,**
- **The new row for F_0 = the old row for F_0 – $^8/_5$ × the new pivot row.**

<u>Hence, we have the 2nd iteration tableau as follows:</u>

3rd tableau

B.V	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Solution	Ratio.
x_2	$^1/_2$	1	0	-$^1/_8$	0	$^1/_8$	$^3/_8$	$^{15}/_4$	
x_5	3	0	0	-2	1	2	-4	30	
X_3	$^1/_2$	0	1	$^1/_8$	0	-$^1/_8$	$^5/_8$	$^5/_4$	
F_0	0	0	0	0	0	-1	-1	0	Minimize

Since, F_0 = 0, then, the problem has a basic feasible solution and phase II can be carried out.

Phase II: The artificial variables and the row for F_0 are eliminated from the last tableau since these artificial variables are non-basic. And then, the initial tableau for phase II is obtained by replacing the F_0 equation by the original f(x) equation. This gives the following tableau:

Initial solution for phase II

4^{th} tableau

(B.V)	C_J	5	-6	-7	0	0	Solution	Ratio.
	Coef. B.V	x_1	x_2	x_3	x_4	x_5		
X_2	-6	$^1/_2$	1	0	$-^1/_8$	0	$^{15}/_4$	7.5
X_5	0	3	0	0	-2	1	30	10
X_3	-7	$^1/_2$	0	1	$^1/_8$	0	$^5/_4$	2.5 ←
E_J		$-^{13}/_2$	-6	-7	$-^1/_8$	0	$-^{125}/_4$	(Max.)
E_J-C_J		$-^{23}/_2$ ↑	0	0	$-^1/_8$	0		

1st iteration: Introduce x_1 (entering variable), and drop x_3 (leaving variable), then the pivot element is ($^1/_2$), and hence we have the following computations.

*The new pivot row = the old pivot row ÷ ($^1/_2$)

= the old pivot row × 2

= (1, 0 , 2, $^{1}/_{4}$, 0 , $^{5}/_{2}$),

*The new row for x_2 = the old row for x_2 – ($^{1}/_{2}$) × the new pivot row,

*The new row for x_5 = the old row for x_5 – 3 × the new pivot row,

*The new element for each of the E_J and (E_J – C_J) rows are computed in the following tableau:

5th tableau

(B.V)	C_J	5	-6	-7	0	0	Solution	Ratio.
	Coef. B.V	x_1	x_2	x_3	x_4	x_5		
X_2	-6	0	1	-1	$-^{1}/_{4}$	0	$^{5}/_{2}$	
X_5	0	0	0	-6	$-^{11}/_{4}$	1	$^{45}/_{2}$	
X_1	5	1	0	2	$^{1}/_{4}$	0	$^{5}/_{2}$	
E_J		5	-6	16	$^{11}/_{4}$	0	$-^{5}/_{2}$	(Max.)
E_J-C_J		0	0	23	$^{11}/_{4}$	0		

<u>In the last tableau note that:</u>

Since, all the solution values corresponding to the basic variables (x_1 , x_2, x_5) are positive values, then this tableau is considered a basic feasible solution. And since, <u>all</u> the coefficients corresponding to the

basic variables are <u>only</u> zeros and nonnegative (positive) coefficients corresponding to <u>all</u> the non-basic variables (x_3 and x_4), then this solution tableau is considered a unique optimal solution. Hence the optimal solution is:

$$x^*_1 = {}^5/_2\ ,\ x^*_2 = {}^5/_2\ ,\ x^*_3 = x^*_4 = 0\ ,\ x^*_5 = {}^{45}/_2\ ,\ x^*_6 = x^*_7 = 0\ ,\ f(x^*) = -{}^5/_2$$

<u>Remarks:</u>

When the LPM is solved by either the M-technique or the two-phase method, one of the following three cases may be hold:

1- **All the artificial variables are removed (leave) from the basic variables, in this case, the value of F_0 will be equal to zero and hence there is a feasible solution space for the LPM and there is an optimal solution for the problem.**
2- **Situations could exist where an artificial variable is still a basic but at zero level in the solution, then the value of the objective function F_0 also will be equal to zero. Henceforth, there is an optimum solution and the constraint in which its corresponding artificial variable is in zero level consider a redundant constraint.**
3- **If the solution is feasible and at least one artificial variable still in the basic variable with a value greater than zero, then F_0 will not reached to zero and the problem is terminated with the information that no feasible solution space exists.**

The Dual Problem "Duality":

Every linear programming problem has a second problem associated with it. One problem is called "The primal or the original" problem and the other is called "the dual problem". The two problems possess very closely related properties, so that the optimal solution to one problem yields complete information about the optimal solution to the other. In this section, illustrative examples are used to point out the relationships between the two problems. In certain cases, these relationships are useful in reducing the computational effort associated with solving linear programming problems.

In this section, we will define the dual problem when its primal is given in one of the two forms:

1-Canonical form.

2-Standard form.

Therefore, the dual problem for each case of the primal form will be considered separately.

The dual problem when the primal is in its canonical form:

Consider the following linear programming model in its canonical form as it is stated in the preceding chapter as follows:

$$\text{Maximize } f(x) = x_0 = \sum^{n} C_J\, x_J$$

Subject to :

$$\sum^{n} a_{iJ}\; x_J \le b_i \qquad , \text{where } i = 1, 2, 3, \ldots, m \;,$$

$$x_J \ge 0 \qquad\qquad J = 1,2,3, \ldots, n$$

then, the dual problem is constructed from the canonical form for the primal problem (and vice versa) as follows:

a) **Each constraint in one problem corresponds to a decision variable in the other problem.**
b) **The constants or the elements of the right-hand side of the set of constraints in one problem are equal to the respective coefficients for the decision variables of the objective function in the other problem.**
c) **One problem search or seeks maximization and the other problem search minimization.**
d) **The maximization problem has a set of constraint in the type ($\le$), and the minimization problem has a set of constraints in the type ($\ge$).**
e) **The set of the decision variables in both of the two problems are nonnegative variables.**

The following table represents the preceding relationships between the two problems:

Primal problem in the canonical form	Dual problem
-The number of decision variable	↔The number of constraints
-The number of constraints	↔The number of decision variables
-R.H.S for the J^{th} constraint	↔The coefficient for the J^{th} decision variable in the objective function
-The coefficient for the J^{th} decision variable in the objective function	↔The R.H.S for the J^{th} constraint.
-If the coefficient matrix for the set of decision variables in the set of constraints is denoted by $A_{n\times m}$.	↔Then the coefficient matrix for the set of the decision variable in the set of constraints is denoted by $A_{n\times m}$.
-Non-negativity constraints	↔Non-negativity constraints
-The optimum value for the objective function	↔The optimum value for the objective function

Therefore, in order to determine the associated dual problem for the preceding primal problem, we have to suppose the set of the decision variables. Then, suppose y_1, y_2 , y_3 , ….. and y_m are the decision variables for the dual problem. Henceforth, the dual problem becomes: Find y_1, y_2 , … y_m by which:

Minimize : $f(y) = y_0 = \sum^m b_i y_i$

Subject to :

$$\sum^{m} a_{iJ}\, y_i \geq C_J \quad , \text{where,} \quad J = 1, 2, 3, \ldots, n ,$$

$$y_i \geq 0 \qquad\qquad i = 1, 2, 3, \ldots\ldots m$$

Note that, according to the definition for the dual problem, the dual problem for the dual problem gives the primal problem.

Example (16):

Consider the problem

$$\text{Maximize} \quad x_0 = 5x_1 + 6x_2$$

Subject to:

$x_1 + 9x_2 \leq 60$ $\quad (y_1)$ **corresponding decision**

$2x_1 + 3x_2 \leq 45$ $\quad (y_2)$ **variables for the**

$5x_1 - 2x_2 \leq 20$ $\quad (y_3)$ **dual problem**

$x_2 \leq 30$ $\quad (y_4)$

$x_1 , x_2 \geq 0$

Then, let y_1, y_2 , y_3 and y_4 by the dual decision variables associated with the 1st , 2nd , 3rd and 4th primal constraints, then, the dual problem is given by: find y_1, y_2 , y_3 and y_4 by which:

$$\text{Minimize} \quad f(y) = y_0 = 60y_1 + 45y_2 + 20y_3 + 30y_4$$

Subject to:

$$y_1 + 2y_2 + 5y_3 \qquad \geq \qquad 5$$

$$9y_1 + 3y_2 - 2y_3 + y_4 \geq 6$$

$$y_1 , y_2 , y_3 , y_4 \geq 0$$

Note that, the dual problem in this case has fewer constraints. And, since, the optimal solution of one problem can be obtained from the optimal solution of the other, it is computationally more efficient to solve the dual problem in this case. This follows because computational difficulty in linear programming mainly depends on the number of constraints rather than the number of decision variables. This point indicates one of the main advantages of the dual problem.

<u>Example (17):</u> Consider the following LPM:

$$F(x) = 2x_1 + 3x_2 \qquad \text{maximize}$$

<u>Subject to:</u>

$$x_1 - 2x_2 \leq 7$$

$$3x_1 - 5x_2 \geq -9$$

$$2x_1 \leq 15$$

$$3x_2 \leq 11$$

$$|4x_1 - 6x_2| \leq 23$$

$$X_1 , x_2 \geq 0$$

<u>Required</u>:

1-Put the LPM in its canonical form.

2-From your results in (1) determine the dual problem.

Solution:

1-Determination the canonical form for the given LPM:

$f(x) = 2x_1 + 3x_2$ (maximize)

subject to:

(1) $x_1 - 2x_2 \leq 7$

(2) $-3x_1 + 5x_2 \leq 9$

(3) $2x_1 \leq 15$

(4) $3x_2 \leq 11$

(5) $-23 \leq 4x_1 - 6x_2 \leq 23$

Then:

$4x_1 - 6x_2 \leq 23$ and $4x_1 - 6x_1 \geq -23$

Then the two constraints associated to the 5th constraint in the primal problem in the canonical form are:

$4x_1 - 6x_2 \leq 23$

And,

$-4x_1 + 6x_2 \leq 23$

Henceforth, the primal problem in its canonical form can be summarized as follows;

Find x_1 , x_2 by which:

$$f(x) = 2x_1 + 3x_2 \qquad \textbf{(Maximization)}$$

<u>subject to:</u>

$$\begin{array}{lcll}
x_1 - 2x_2 & \le & 7 & (y_1) \\
-3x_1 + 5x_2 & \le & 9 & (y_2) \\
2x_1 & \le & 15 & (y_3) \\
\quad\;\; 3x_2 & \le & 11 & (y_4) \\
4x_1 - 6x_2 & \le & 23 & (y_5)
\end{array}$$

And, $$\begin{array}{lcll}
-4x_1 + 6x_2 & \le & 23 & (y_6) \\
x_1 \;\;,\;\; x_2 & \ge & 0 &
\end{array}$$

2-From the canonical form for the preceding LPM, then the dual problem is to find the decision variable y_1, y_2 , y_3 , y_4 , y_5 and y_6 by which:

$$F(y) = y_0 = 7y_1 + 9y_2 + 15y_3 + 11y_4 + 23y_5 + 23y_6 \quad \textbf{(minimization)}$$

Subject to:

$$\begin{array}{lcl}
y_1 - 3y_2 + 2y_3 + (0)\,y_4 + 4y_5 - 4y_6 & \ge & 2 \\
-2y_1 + 5y_2 + (0)y_3 + 3y_4 + 6y_5 + 6y_6 & \ge & 3
\end{array}$$

yi $\geq$ **0 , for ,** ***i*** **= 1 : 6**

Variants of the Simplex Method Applications

(Special Cases in the LPM):

This section introduces some important cases often encountered in the simplex method applications. The vehicles of explanation are numerical example illustrating the different cases. These examples are depicted graphically to allow the reader to visualize the properties of the different cases. The cases discussed here include how can it be discovered in either the graphical solutions or in the successfully iterations for the simplex methods. Furthermore, how can the solution for the model be completed to reach the optimal solution. These cases are represented as follows:

1- Multiple (Alternative) optimal solutions.

2- Multiple qualified (candidate) entering basic variables.

3- Multiple qualified (candidate) leaving from the basic variables.

4- Unbounded solutions.

5- Nonexistent feasible solution.

5- Minimum value constraint for a decision variable.

(1): Multiple Optimal Solutions:

Graphically, this case occurs when the line graph for the objective function is parallel to a binding constraint (that is a constraint which is satisfied in equality sense by the optimal solution). In such cases, the objective function may assume the same optimal value at more than one basic solution. These are called multiple optimum basic solutions. And, any weighted average of the optimal basic solutions should also yield an alternative basic (or non-basic) solution, which implies that the problem has infinite number of solutions with each solution yielding the same value of the objective function. This situation also is also occurring in the optimum solution tableau when the coefficient for at least one non-basic variable in the row by which represents their optimality condition ($E_J – C_J$) is equal to zero. Then, one can see that, if a succeeding iteration after the optimum solution is reached will performed by selecting this non-basic variable (which have the zero coefficient in the ($E_J – C_J$) row as an entering variable, then an alternative solution will yield a second basic (or non-basic) optimum (or no optimum) solution with the same value for the objective function $f(x^*)$ or x^*_0. Note that, from any two alternative optimum solutions, one can derive a set of other different infinite numbers of alternative optimum basic solutions. These different alternative optimum solutions can be derived as follows:

Suppose that the 1st alternative optimum solution have the following: x^*_1 , x^*_2 , $f(x^*)$, and Suppose that the 2nd alternative optimum solution have the following: x^*_1 , x^*_2 , $f(x^*)$. Then, one can drive a set of

different alternative optimum solutions have the same optimum value of the objective function $f(x^*)$ from the following two equations:

$x^*_1 = Px^*_1 + (1-P)\, x^*_1$ **and**

$x^*_2 = Px^*_2 + (1-P)\, x^*_2$ **were P any arbitrary positive fraction in the closed interval between zero and one, i.e., $0 \le P \le 1$. This situation is illustrated by the following example.**

Example (19):

Find the value of x_1 and x_2 by which:

$f(x) = (5/2)\, x_1 + x_2$ **(Maximization)**

Subject to:

$$3x_1 + 6x_2 \le 18$$

$$5x_1 + 2x_2 \le 10$$

$$x_1 , x_2 \ge 0$$

Solution:

Firstly: Not that the 2nd constraint, one can divide the two sides by two, then we have the following form:

$$5x_1 + 2x_2 \le 10 \rightarrow (5/2)\, x_1 + x_2 \ge 5.$$

Note that the L.H.S for the 2nd constraint form becomes the same form for the objective function *f(x)*, i.e., the slope for the objective function is the same slop for the equation for the 2nd constraint. Henceforth, one can predict that we will have a multiple optimum solution.

Now, let us solve this problem graphically. Then, one can see that the feasible solution space and the line graph for the objective function are as follows:

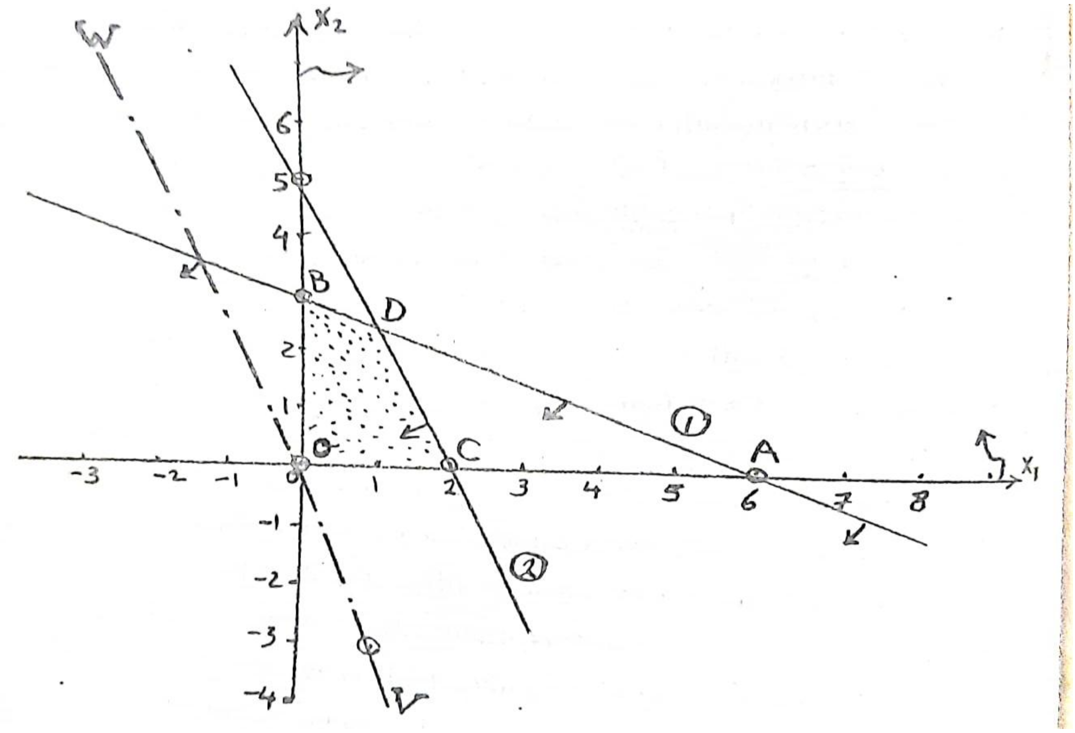


Figure ( )

<u>Reduction for the feasible solution space:</u>

*Non-negativity constraints : x_1Ox_2

*1st constraint : AOB

*2nd constraint : COBD (feasible solution)

The line graph for *f(x)* is VOW. Then, if the line graph for the objective function VOW is moved in parallel to itself in the direction of the feasible

solution space (COBD) to pass with the last point in the feasible solution space, then, the line VOW will be binding with the 2nd constraint line graph. In this case, any point lies on the line (CD) will give an optimum solution with the same value of $f(x^*)$. One can see that from substituting in *f(x)* by the coordinates of the two point (C) (2, 0) and (D) (1 , 5/2) as the following:

Since : $f(x) = (5/2)x_1 + x_2$, then:

$f_C(x) = (5/2)(2) + 0 = 5$

and

$f_D(x) = (5/2)x_1 + (5/2) = 5$

Therefore, any point between the two points C and D will have the same value for $f(x^*) = 5$.

Now, if one can solve the LPM by using the simplex method, the suitable method in this case is the ordinary simplex method (since, all the R.H.S for the set of constraints are positive numbers and all the constraints are in the ($\leq$) type), then we have the following solution:

<u>*The standard form for the LPM is:</u>

$f(x) = 2.5x_1 + x_2 + 0(x_3 + x_4)$ (Max.)

<u>Subject to:</u>

$$x_1 + 2x_2 + (x_3) = 6$$

$$5x_1 + 2x_2 + (x_4) = 10$$

$$x_i \geq 0 \text{ , for } i = 1:4$$

And hence we have the following tableaus for the initial solution and its improvement successively iterations:

(B.V)	C_J	5/2	1	0	0	Solution	Ratio
	Coef. B.V	X_1	X_2	X_3	X_4		
X_3	0	1	2	1	0	6	6
X_4	0	(5)	2	0	1	10	2🡐
E_J		0	0	0	0	0	(Max)
E_J-C_J		-5/2	-1	0	0		
X_3	0	0	(8/5)	1	-1/5	4	5/2🡐
X_1	5/2	1	2/5	0	1/5	2	5
E_J		5/2	1	0	1/2	5	(Max)
E_J-C_J		0	(0)	0	1/2		

<u>In the last iteration tableau, note that:</u>

- **Since all the basic values (solution) are nonnegative values, hence this tableau is considered a basic feasible solution.**
- **Since the coefficient of the optimality condition are zero for the basic variable (x_1 , x_3) and the <u>non-basic variable (x_2)</u>, and nonnegative coefficient for the remaining non-basic variable (x_4) for the Maximization type of *f(x)*. Hence the last tableau is one of a set of multiple optimum solutions. The 1st optimum solution is $x^*_1 = 2$, $x^*_2 = 0$, $x^*_3 = 4$, $x^*_4 = 0$, $f(x^*) = 5$.**

Now, however, inspection of the optimum tableau shows that, the non-basic variable (x_2) has a zero coefficient in the optimality row condition ($E_J - C_J$). This is an indication that an alternative solution exists. Then, the alternative solution for the last optimal solution tableau can be derived by introducing x_2 as the entering variable, and hence the ratio values are [(4 ÷ 8/5 = 5/2) and (2÷2/5=5)] corresponding to the basic

variable (x2) then, drop x3 from the last tableau as a leaving variable, then, we have the following iteration:

An Alternative Optimal Solution

(B.V)	C_J Coef. B.V	5/2	1	0	0	Solution	Ratio
		X_1	X_2	X_3	X_4		
X_2	1	0	1	5/8	-1/8	5/2	
X_1	5/2	1	0	-1/4	¼	1	
E_J		5/2	1	0	½	5	
E_J-C_J		0	0	(0)	½		

Note that the last tableau satisfies the two conditions (feasibility & optimality), and with inspection of the (EJ – CJ) row, one can see that the non-basic variable (x3) has a zero coefficient in the row of optimality. This is also an indication that an alternative optimum solution exists. Then, the 2nd alternative optimal solution is:

$x^*1 = 1$, $x^*2 = 5/2$, $x^*3 = x^*4 = 0$, $f(x^*) = 5$

Now, if we can see, how can to derive different set of alternative optimum solutions, then, from the preceding two optimum solutions, substitute in the two linear equations stated above as follows:

$x^*_1 = P\,x^*_1 + (1- P)\,x^*_1$ and $x^*_2 = P\,x^*_2 + (1- P)\,x^*_2$

where: $0 \le P \le 1$, $x^*_1 = 2$, $x^*_2 = 0$, $x^*_2 = 1$, $x^*_2 = 5/2$.

Now: if one substitute in the preceding two linear equations with P = 0 , then, it gives that : $x^*_1 = x^*_1 = 1$, and $x^*_2 = x^*_2 = 0$, which is the 1st alternative optimum solution. Also, if one substitute in the two linear equations with P = 1, then, the 2nd alternative optimum solution will be obtained. Another alternative optimum solution can be derived from the preceding two linear equations by substituting in these equations with any fraction value for the value of (P):

For example: let P = 1/2 , then, these two linear equations give:

$x^*_1 = Px_1 + (1–P)x_1 = 1/2(2) + (1–1/2)(1) = 1+1/2= 3/2$, and

$x^*_2 = Px_2 + (1–P)x_2 = 1/2(0) + (1–1/2)(5/2) = 5/4$, and

$f(x^*) = (5/2)x_1 + x_2 = (5/2) (3/2) + 5/4 = 20/4 = 5 = f(x^*) = f(x^*)$

Another example: Let P = 1/5, then it gives:

$x^*_1 = Px_1 + (1–P)x_1 = (1/5)(2) + (1–1/5)(1) = 6/2$, and

$x^*_2 = Px_2 + (1–P)x_2 = 1/5(0) + (1–1/5)(5/2) = 20/10= 2$, and

$f(x^*) = (5/2)x_1 + x_2 = (5/2) (6/5) + 2 = 3 +2 = 5$

Also: Let P = 1/3 , then it gives:

$x^*_1 = Px_1 + (1–P)x_1 = (1/3)(2) + (1–1/3)(1) = 4/3$, and

$x^*_2 = Px_2 + (1–P)x_2 = 1/3(0) + (1–1/3)(5/2) = 5/3$, and then

$f(x^*) = (5/2)x_1 + x_2 = (5/2) (4/3) + 5/3 = 5 = f(x^*) = f(x^*)$.

And so on, it is so easy to get a set of infinite alternative optimum solutions for the different fraction positive values of (P) lies between zero and one.

(2): Multiple Qualified (Candidate) Entering Basic Variables:

(Multiple Candidate Pivot Columns)

For any improvement to the solution, during the inspection process for the optimality condition row ($E_J – C_J$) , one can see that, at least two non-basic variables have the same greatest negative coefficients in the

$(E_J - C_J)^{th}$ row in case of maximization the value of the objective function, or at least two non-basic variable have the same greatest positive coefficients in this row in case of minimization the value of the objective value *f(x)*. Hence, there is no rule for determining which non-basic variable is selected to be the entering variable from these sets of candidate entering variables (multiple pivot columns), but, it is so easy to use the logical mathematics in order to improve the solution for the available iteration. In this case, let us assume that this iteration for improvement the solution is either in the initial (starting) solution tableau or not. Then, we will present the two cases respectively as follows:

1- **If the multiple candidate entering variable occurs in the 2nd or, 3rd or, ,…. Or, .nth iteration not in the 1st tableau (Initial or starting tableau), then, one can select the non-basic variable by which lead to the best improvement on the value of the objective function *f(x)*. This best value of *f(x)* will be achieved when one select one of these non-basic candidate variables to be basic variable which have the most positive coefficient for the objective function equation *f(x)* in case of maximizations the value of *f(x)*. On contrary, select the non-basic variable by which have the lowest positive coefficient in case of minimization the value of *f(x)* supposing that all these devoted entering variable have positive coefficient on the equation of *f(x)*. Furthermore, if all these candidate entering variable have negative coefficients in *f(x)*, then in case of maximization the value of *f(x)*, then by using the logical mathematics, one can select the non-basic variable by which has the lowest negative coefficient on the** objective function equation in case of maximize the value for ***f(x)*,** i.e., if there are non-basic decision variable x_1, x_2 and x_3 are candidate to be one of them is selected as an entering variable in the 2nd or 3rd, …or, nth improvement iteration with -3, -5, -9 coefficients in the equation of the objective function ***f(x)*** respectively, i.e., $f(x) = -3x_1 - 5x_2 - 9x_3$ (Max.), then, the logical mathematics asserts that one select the non-basic variable (x_1) to be the entering variable in the succeeding tableau in case of maximize ***f(x)*,** since x_1 have the lowest negative coefficient in ***f(x)*,** and vice versa, select the non-basic variable x_3 to be the entering variable in the succeeding tableau in case of minimize ***f(x)*.**

2- If, the multiple candidate entering variables occurs in the initial solution tableau, then the coefficients of these non-basic variables will be have the same coefficient in ***f(x)*.** Then, one can select the best by which it makes the value of ***f(x)*** in the succeeding tableau is the best value, i.e., if x_1 and x_2 are two non-basic variables candidate to be one of them to be an entering variable and if the selection of x_1 as an entering variable make the value of ***f(x)*** to be equal to 1000$, and the selection of x_2 as an entering variable make the value of ***f(x)*** to be 900$, then, the logical mathematics asserts that the variable x_1 must be selected as an entering variable in case of maximize the value of ***f(x)*,** on the contrary select x_2 to be an entering variable in case of minimize the value of ***f(x)*.** in order to determine the value of ***f(x)*** in the succeeding tableau, it is so easy to compute the new value for ***f(x)*** as follows:

The new value for *f(x)* =[the old value of *f(x)* – (the value of multiplication of the corresponding in two elements in either the pivot row or column) ÷ pivot element].

(3): Multiple Qualified (or Candidate) Leaving Variables:

(Multiple Candidate Pivot Rows):

For any simplex tableaus in either the starting (or the initial tableau) or an improvement iteration tableau, after determining the entering variable, and calculate the values of ratios, then one can see that at least two basic variables have the same minimum ratio, then, we have the multiple candidate pivot rows (multiple candidate leaving variables). In this case, it is so easy to apply the rule by which it is achieved by Charnize & Cooper. To handle that rule, one can have divided the element for the 1st column of the identity matrix (basic variables columns) lies in a right position for the pivot column on the positive coefficients for the pivot column elements and select the leaving variable which have the zero ratio. If the ratios are the same zero values, one can transport to the second Column in the identity on the right position for the preceding column in the identity matrix even the ratios to be different, hence select the leaving variable corresponds to the zero ratio. The following example illustrates this case.

Example (20):

Find x_1 and x_2 by which:

$f(x) = 3x_1 + 5x_2$ (Maximization)

Subject to:

$x_1 + x_2 \leq 8$

$x_2 \leq 6$

$3x_1 + 2x_2 \leq 12$

$x_1 , x_2 \geq 0$

Solution:

The following tableaus represent the solution for this LPM by the ordinary simplex method (since all the R.H.S are positive numbers and all the constraint are in the ($\leq$) type).

(B.V)	C_J Coef. B.V	3 X_1	5 X_2	0 X_3	0 X_4	0 X_5	Solution	Ratio 1st	2nd	3rd
X_3	0	1	1	1	0	0	8	8	~	~
X_4	0	0	1	0	1	0	6	6	$^0/_1=0$	$\leftarrow {}^1/_1=1$
X_5	0	3	(2)	0	0	1	12	6	$^0/_2=0$	$\leftarrow {}^0/_2=0$
E_J		0	0	0	0	0			(Max.)	
$E_J - C_J$		-3	-5	0	0	0				

X_3	0	$-^1/_2$	0	1	0	$-^1/_2$	2		~	~
X_4	0	$-^3/_2$	0	0	1	$-^1/_2$	0		$\leftarrow {}^0/_1=0$	$^1/_1=1$
X_2	5	$^3/_2$	1	0	0	$^1/_2$	6		$\leftarrow {}^0/_2=0$	$^0/_2=0 \leftarrow$
E_J		$^{15}/_2$	5	0	0	$^5/_2$	30		(Max.)	
$E_J - C_J$		$^9/_2$	0	0	0	$^5/_2$				

Note that in the initial solution, the two candidate leaving variables are x_4 and x_5 which has the minimum 1st ratio which is equal to (6). Then, the

2^{nd} ratios must be computed dividing the corresponding elements for the 1^{st} identity matrix in the right position of the entering variable (x_2) on the pivot column element, then we have the same 2^{nd} ratio (0/1 = 0 and 0/2 = 0). Hence, transport to the second column for the identity matrix (on the right) and compute the ratios (1/1 = 1 and 0/2 = 0).

Therefore, the ratio corresponding to the basic variable (x_5) is equal to zero. Then, drop x_5 which is considered the leaving variable in the initial solution. Henceforth, the resulted optimal solution is:

$x^*_1 = 0, x^*_2 = 6 , x^*_3 = 4 , x^*_4 = 0 , x^*_5 = 0 , f(x^*) = 30$

Note that, in this example, if the LPM is solved graphically, one can see that the 2^{nd} constraint is redundant constraint, since its slack variable (x_4) is equal to zero in the solution column. Besides, the 2^{nd} tableau named by degeneracy solution in this case.

(4): Unbounded Feasible Solution:

This case occurs when the feasible solution space is unbounded, so that the value of the objective function can be increased indefinitely. It is not necessary, however, that an unbounded feasible solution space yield an unbounded value for the objective function. In this case, graphically, the feasible solution space will be unbounded (i.e., not closed from any direction). But, in the simplex techniques, there is at last one column vector corresponding to non-basic variable implies only either zero or

negative values in this column. In this case, two different sub-cases may be occurring as the following:

1-Unbounded feasible solution and there is an optimal solution for the LPM.

2-Unbounded feasible solution and unbounded optimal solution.

In the 1st sub-case, it will be occurring when the optimal solution will be achieved or satisfied, but one at least of the columns for the non-basic variables contains only zero or negative numbers, but it will not effect on the solution since the optimal solution is reachable. But, the 2nd sub-case will be occurring when the pivot column for any iteration contains only either zeros or negative numbers. Hence, the solution will be stopped without any successfully improvement iterations. The following examples illustrate the two sub-cases for unbounded feasible solution as follow:

<u>Firstly</u>: The following example illustrate how can the feasible solution may be unbounded and there is bounded optimal solution.

<u>Example (21):</u>

If you have the following LPM:

$f(x) = x_1 + 4x_2$ **(Minimization)**

subject to:

$x_1 + x_2 \geq 4$

$x_1 - 2x_2 \leq 2$

$2x_1 - x_2 \quad \geq \quad 2$

$x_1 , \; x_2 \quad \geq \quad 0$

Required:

1-Solve the LPM graphically. Comment?

2-By using the suitable simplex method, solve the LPM?

Solution:

1-Graphical Solution:

***Non-negativity constraints** $x_1 \geq 0$, $x_2 \geq 0$ **specify that the feasible solution space must be lie in the 1st quadrant.**

-1st Constraint: $x_1 + x_2 \geq 4 \rightarrow x_1 + x_2 = 4$

The following table represents the two intercepts

X_1	0	4
X_2	4	0

And, the region in which the origin point is not exist holds this constraint since the coefficients for x_1 and x_2 are positive and the R.H.S is positive value.

-2nd Constraint: $x_1 - 2x_2 \leq 2 \rightarrow x_1 - 2x_2 = 2$

The following table represents the two intercepts,

X_1	0	2
X_2	-1	0

Now, in order to determine the region by which holds this constraint, we use the origin point:

$$0 - 2\,(0) \;[\quad]\; 2$$

$$0 \;[\,<\,]\; 2$$

Since the inequality for this constraint have the same direction with the substituting in the constraint with the coordinate for the origin point, so that the direction by which the origin point is exist holds the constraint inequality.

<u>-3rd constraint:</u> $2x_1 - x_2 \geq 2 \rightarrow 2x_1 - x_2 = 2$

The following table represents the two intercepts:

X_1	**0**	**-1**
X_2	**-2**	**0**

$$2(0) - (0) \;[\quad]\; 2$$

$$0 \;[\,<\,]\; 2$$

i.e., the region by which the origin point <u>is not</u> exist holds the constraint inequality.

<u>*Graphing the line for the objective function equation:</u>

Suppose that *f(x)* = 0, then we have: $x_1 + 4x_2 = 0$, the following table represents the points in which passes through this equation.

X_1	**0**	**4**
X_2	**0**	**-1**

***Determination the feasible solution, and hence the optimal solution if there is exist:**

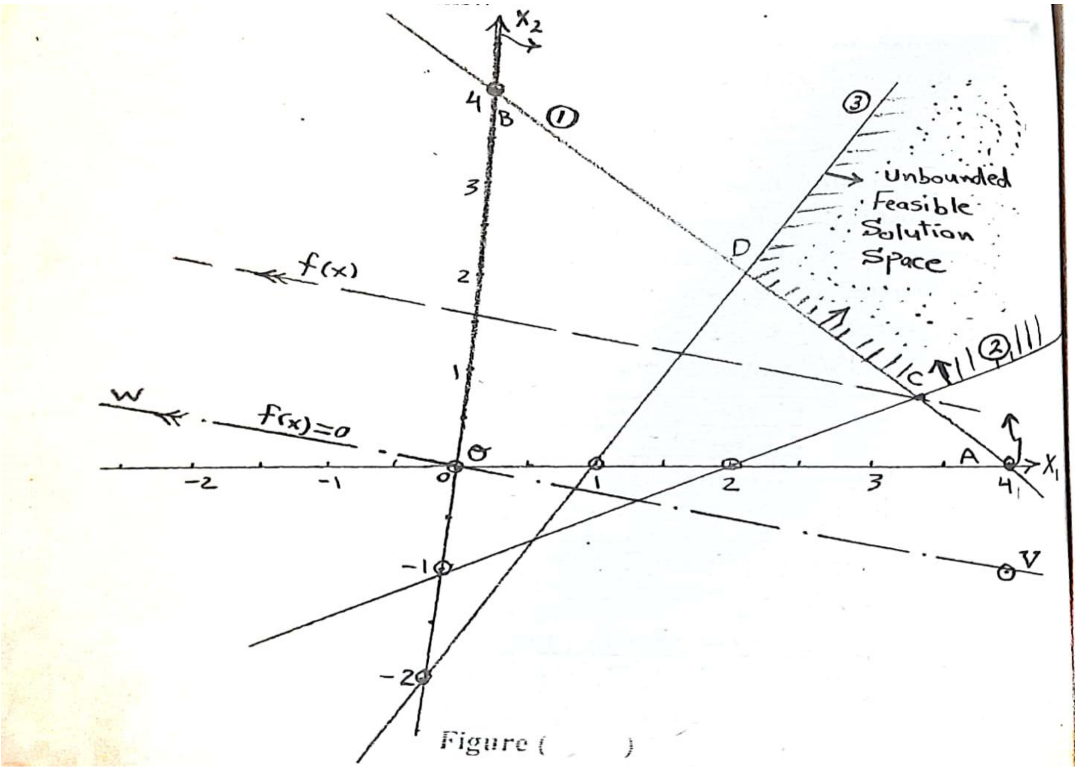


Figure ( )

Reduction of the feasible solution space:

***Non-negativity constraint : x_1Ox_2**

***1st constraint : x_1ABx_2**

***2nd constraint :(2)CBx_2**

***3rd constraint :(2)CD(1)**

***Line graph for the objective function *f(x)* = 0 is ←VOW→**

Now, since the coefficients for the two decision variables x_1 and x_2 in the objective function are positive values, so that, if one move the line VOW parallel to itself in the direction of the feasible solution space which

is considered unbounded feasible solution space [←(1) CD (2) →], then, one sees that the line VOW for *f(x)*= 0 will passes the unbounded feasible solution space in its 1st point (since we have to minimize *f(x)*), through the point C, whose coordinates are x_1 = 10/3 and x_2 = 2/3. Substituting these values into the objective function gives, *f(x)*=$x_1 + 4x_2$= 10/3 + 2/3= 12/3=4. Consequently, a problem may have unbounded solution space, but still the optimal solution is bounded.

<u>Note that</u>: Although the feasible solution is unbounded feasible solution, there is an optimal solution for the linear programming problem:

$x^*_1 = 10/3$, $x^*_2 = 2/3$, $f(x^*) = 4$. (1st sub-case)

2-The reader can solve this problem by using either the M-technique or the two phase method, then the optimal solution is found and in this case one of the non-basic variable implies either zeros or negative values.

Note that, if the objective is to maximize the value of the objective function *f(x)*, in the case, one can see that this yield to a problem which have unbounded feasible solution space and unbounded optimal solution. In this case, the two decision variables x_1 and x_2 can be reach to infinity (∞) and hence the value of *f(x)* also can be reached to (∞). (2nd sub-case)

<u>Secondly</u>: The following example represents either the unbounded feasible solution or unbounded optimal solution.

<u>Example (22):</u>

Find x_1 and x_2 by which:

$F(x) = 6x_1 - 2x_2$ (maximize)

Subject to:

$$2x_1 - x_2 \leq 2$$

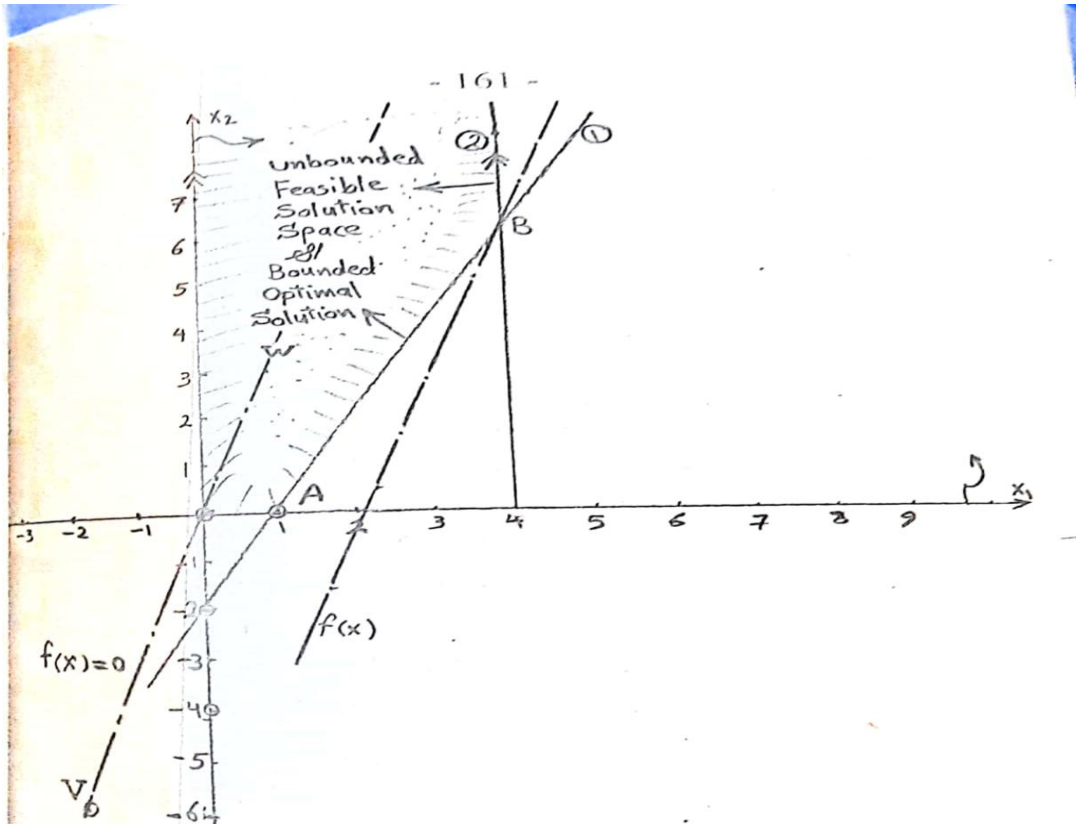
$$x_1 \leq 4$$

$$x_1 , x_2 \geq 0$$

Graphically and with the suitable simplex method.

<u>Solution</u>:

It is so easy for the reader to verify that the feasible solution space and the graph line for the objective function are as in the following figure:



<u>Reduction the solution space:</u>

***Non-negativity constraint : x_1Ox_2**

***1st constraint :🡐(1) AOx_2**

***2nd constraint :🡐(3) $BAOx_3$**

<u>Graph line for *f(x)* = 0</u> is VOW (feasible solution)

Then, in order to determine the optimum solution, one can move the line VOW for the objective function *f(x)* = 0 parallel to itself in the direction of the feasible solution space (3)B A O x_2 to increase the value of *f(x)* even so it passes with the latest point. One can show that, although the feasible solution space is unbounded, the optimal solution is bounded by the point B, since the latest point that the line VOW passes through the feasible solution in B(4, 6), i.e., $x^*_1 = 4$, $x^*_2 = 6$ and $f(x^*) = 6x^*_1 - 2x^*_2 = 6(4) - 2(6) = 12$.

(Unbounded feasible solution space, but bounded optimal solution).

And, by using he ordinary simplex method for solving this LPM, since all the R.H.S for all constraints are positive numbers and all the constraints in the ($\leq$) type. Then, in summary we have the following standard form and the required tableaus for the solution:

$$F(x) = 6x_1 - 2x_2$$

Subject to:

$$2x_1 - x_2 + (x_3) = 2$$

$$x_1 + (x_4) = 4$$

x_1 , x_2 , x_3 , $_4 \geq 0$

Simplex tableaus

(B.V)	C_J Coef. B.V	6	-2	0	0	Solution	Ratio
		X_1	X_2	X_3	X_4		
X_3	0	(2)	-1	1	0	2	2/2 = 1 ←
X_4	0	1	0	0	1	4	4/1 = 4
E_J		0	0	0	0	0	(Max.)
E_J-C_J		-6	2	0	0		

X1	6	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	
X	0	0	$(\frac{1}{2})$	$-\frac{1}{2}$	1	3	←
E_J		6	-3	1	0	6	(Max.)
$E_J - C_J$		0	-1	1	0		

X_1	6	1	0	0	1	4	
X_2	-2	0	1	-1	2	6	
E_J		6	-2	2	4	12	(Max.)
$E_J - C_J$		0	0	2	4		

Since the feasibility and optimality conditions are satisfied, so that the optimal solution is $x^*_1 = 4$, $x^*_2 = 6$ and $f(x^*) = 12$, which is the same as in he graphical solution.

Note that, each of the column for the non-basic variable x_2 in the starting tableau (initial solution), and the column for the non-basic variable x_3 in the 2nd iteration (or the 3rd tableau) contains zero and negative value, this indicates that the feasible solution space is unbounded. Therefore, in this case there is unbounded solution space, but bounded optimal solution.

Example (23):

Determine the value of x_1 and x_2 by which:

F(x) **= $x_1 + 2x_2$ (Maximize)**

Subject to:

$$x_1 - x_2 \leq 10$$

$$2x_1 - x_2 \leq 40$$

$$x_1 \ , \ x_2 \geq 0$$

Solution:

In summary, one can see that the feasible solution space and the graph line for *f(x)* = $x_1 + 2x_2 = 0$ as in the following graph) [the reader can verify as we mentioned above]:

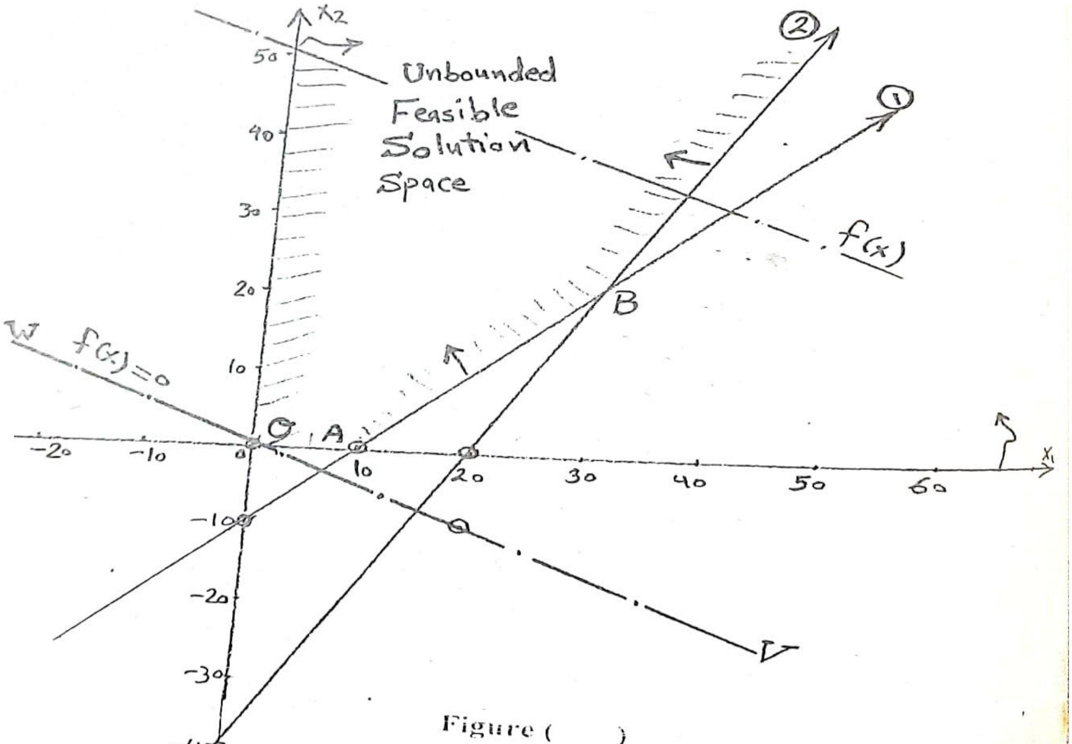


Figure ( )

Reduction of the feasible solution space:

***Non-negativity constraints : x_1Ox_2**

***1st constraint : (1)AOx_2**

***2nd constraint : (2) $BAOx_2$**

***Line graph for *f(x)* = 0 is VOW**

Now, since the coefficients for the two decision variables x_1 and x_2 in the objective function are positive values, so that, if one move the line VOW for the objective function equation *f(x)* = 0 parallel to itself in the direction of increasing *f(x)*, one sees that, the unbounded feasible solution space permits to increase each of either the value of the decision variables or the value of the objective function *f(x)* to infinity (∞), since the feasible solution space is not have any line graph closed it in its upper direction, hence x_1 , x_2 and *f(x)* can be increased indefinitely without effecting the feasibility of the problem in this case.

The reader can carry out the solution by using the ordinary simplex method and verify that the pivot column for a specific iteration to improve the solution contains either zeros or negative values, then it could not to be able to determine the leaving variable, and hence, the solution will be stopped without determining an bounded optimal solution.

(5): Non-existing Feasible Solution Space:

This case occurs when the problem is such that no at least one point can be satisfied by all the constraint. In this case, the solution space

is empty and the problem has no feasible solution space. Graphically, the set of constraints have not any point satisfy all these constraints. Furthermore, in the simplex methods, the optimal basic solution will include at least one of the artificial variable at a positive level in the column of the solution. This indicates that the problem has no feasible solution, since a positive value of any artificial variables in the solution column means that its constraint is not satisfied and hence the solution actually represents a different problem. Finally, it must be stated that, when an artificial variable appears in the optimal basis at zero level in the solution column, then the corresponding constraint to this artificial variable is not violated and hence the problem has a feasible solution (redundant constraint). The following example shows how the non-existing feasible solution space can be detected by either the graphical solution or the simplex method.

Example (24): **(Problem with no feasible solution)**

Find the value of x_1 and x_2 by which:

$F(x) = 3x_1 + 2x_2$ **(Maximize)**

Subject to:

$2x_1 + x_2 \leq 2$

$3x_1 + 4x_2 \geq 12$

$x_1 , x_2 \geq 0$

Solution:

For solving this problem graphically, one can see that there is no feasible solution as we state in the following graph:

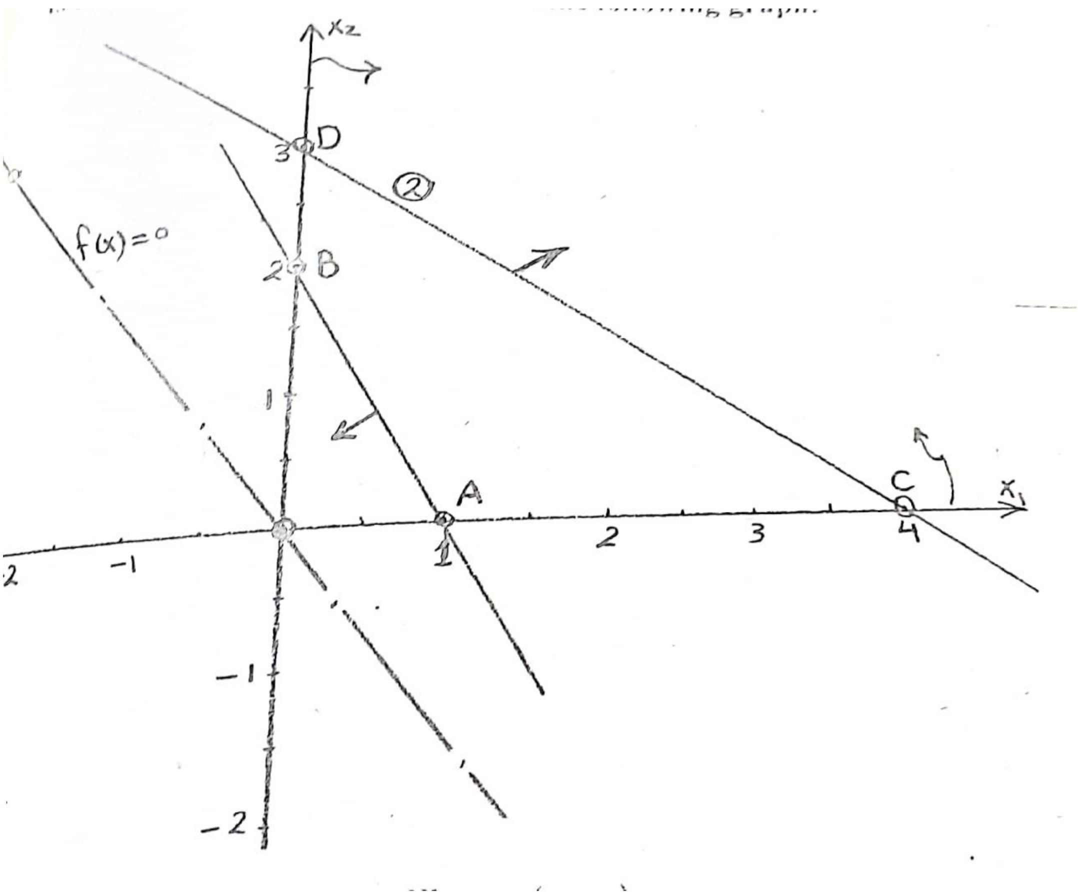


Figure (   )

__**Reduction for the feasible solution space:**__

***Non-negativity constraints : x_1Ox_2**

***1st constraint : AOB**

***2nd constraint : No feasible solution space exists**

From the preceding figure, there is no point at least satisfies the two constraint for this problem, i.e., there is no feasible solution in this case.

In order to comparison the graphical results with the algebraic procedure of the simplex method, the suitable simplex method for this problem is either the M-technique or the two phase method. We will illustrate the solution by using the M-technique. Then, we have to put the problem in the standard form, added the artificial variable for the constraint in the ($\geq$ or=) type, and put the artificial variable in the objective function with (-M) coefficient since we have to maximize *f(x)*. Then, we have the following:

$F(x) = 3x_1 + 2x_2 + 0(x_3 + x_4) - Mx_5$ **(Maximize)**

<u>Subject to:</u>

$2x_1 + x_2 + (x_3) \qquad = 2$

$3x_1 + 4x_2 \quad - x_4 + (x_5)$ **arti.** $\quad = 12$

$x_i \geq 0$ **for** $i = 1 : 5$

The starting (initial) tableau and its interactions are as follows:

(B.V)	C_J Coef. B.V	X_1	X_2	X_3	X_4	x_5	Solution	Ratio
	C_J	3	2	0	0	-M		
X_3	0	2	(1)	1	0	0	2	2/1=2 ←
X_5	-M	3	4	0	-1	1	12	12/4=3
E_J		-3M	-4M	0	-M	-M	12M	(Max.)
E_J-C_J		-M-3	-4M-2	0	-M	0		

X_2	2	2	1	1	0	0	2	
X_5	-M	-5	0	-4	-1	1	4	
E_J		4+5M	2	2+4M	M	-M	4-4M	(Max.)
E_J-C_J		1+5M	0	2+4M	M	0		
		(+)		(+)	(+)			

Now, according to the optimality condition, the last solution tableau is considered optimal. Note that, however, that the optimal (basic) solution includes the artificial variable (x_5) at a positive level (=4) as a basic variable. This indicates that the problem has no feasible solution since a positive value of x_5 means that the 2^{nd} constraint is not satisfied and hence the solution actually represents a different problem.

Note that, it must be stated that when an artificial variable appears in the optimal basis at zero level, the corresponding constraint is not violated and hence the problem has a feasible solution.

(6): Minimum Value Constraints for a Decision Variable:

In this case, one sees after the formulation process is achieved, that at least one of the decision variable x_i had a constraint in the following general form:

$x_i \geq b_i$ where $i = 1$, or 2, or 3, …. Or n

and b_i is a positive value.

In this case, the value (b_i) is called the minimum value of the decision variable x_i. In this case, it is so easy to reduce the number of constraints for the problem by the set of decision variable in which have a constraint in its minimum value after substituting each variable has the minimum value constraint by the following linear equation $x_i = x_i + b_i$

This situation is illustrated by the following example.

Example (25):

Find x_1 , x_2 and x_3 by which:

$$F(x) = 2x_1 + x_2 + 4x_3 \qquad \text{(Maximize)}$$

Subject to:

$$X_1 + 2x_2 - x_3 \leq 5$$

$$2x_1 + 3x_2 + x_3 \leq 8$$

$$x_1 \geq 2$$

$$2x_2 \geq 2$$

$$x_3 \geq 0$$

Solution:

Not that, each of the two decision variable x_1 and x_2 has a constraint for its minimum value: $x_1 \geq 2$ and $x_2 \geq 1$, then it is so easy to

delete each of the 3rd and the 4th constraints from this LPM after substituting with: $x_1 = x_1 + 2$ and $x_2 = x_2 + 1$

Then, we have the following LPM:

$$F(x) = 2(x_1 + 2) + (x_2 + 1) + 4x_3$$

$$= 2x_1 + x_2 + 4x_3 + 5 \qquad \textbf{(Max.)}$$

<u>Subject to:</u>

<u>*The 1st constraint:</u>

$(x_1 + 2) + 2(x_2 + 1) - x_3 \leq 5$, **then**

.. $x_1 + 2x_2 - x_3 \leq 1$

<u>*the 2nd constraint:</u>

$2(x_1 + 2) + 3\ (x_2 + 1) + x_3 \leq 8$

.. $2x_1 + 3x_2 + x_3 \leq 1$

, $x_1 , x_2 , x_3 \geq 0$

<u>In summary, the problem becomes:</u>

We have to find the value of x_1 , x_2 and x_3 by which:

$F(x) = 2x_1 + x_2 + 4x_3 + 5$ **(Maximize)**

<u>Subject to:</u>

$X_1 + 2x_2 - x_3 \leq 1$

$2x_1 + 3x_2 + x_3 \leq 1$

X_1 , x_2 , x_3 $\geq$ 0

Note that the preceding procedure for the minimum value constraint for the decision variable will reduces the set of constraints in its minimum number of constraints, in which reflects on the computation in the simplex tableau. This notation represented in this example, since the original or primal problem have four constraints, and if we solve this problem in this case, we will use any of either M~ Technique or the two phase method. However, when, we reduce the number of constraints to become two constraints only instead of four constraints, furthermore, it is so easy to use the ordinary simplex method, hence this reduction will present some facility in the computation.

Now let us present the solution after the process of reduction for the number of constraints:

$f(x) = 2x_1 = x_2 + 4x_3$ **(Max.)**

Subject to:

$$x_1 + 2x_2 - x_3 + (x_4) = 1$$

$$-2x_1 + 3x_2 + x_3 + (x_5) = 9$$

$$X_1 , x_2 , x_3 , x_4 , x_5 \geq 0$$

Note that, the constant (5) in the objective function *f(x)* will be under taken at the end of solution. Then, we have the following tableaus:

(B.V)	C_J Coef. B.V	2 X_1	1 X_2	4 X_3	0 X_4	0 x_5	Solution	Ratio
X_4	0	1	2	-1	1	0	1	~
X_5	0	2	3	(1)	0	1	1	و←
E_J		0	0	0	0	0	0	(Max.)
E_J-C_J		-2	-1	-4	0	0		

X_4	0	3	5	0	1	1	2	
X_3	4	2	3	1	0	1	1	
E_J		8	12	4	0	4	4	(Max.)
E_J-C_J		6	11	0	0	4		

Now, since, the latest tableau is considered an optimal basic feasible solution, where the two optimality and feasibility conditions are hold. So that, we have the following results:

$x^*_1 = 0$, $x^*_2 = 0$, $x^*_3 = 1$, $x_4 = 2$, $x^*_5 = 0$, $f(x^*) = 4$

then:

$..x^*_1 = x^*_1 + 2 \rightarrow \quad \therefore\ x^*_1 = 0 + 2 = 2$

, $..x^*_2 = x^*_2 + 1 \rightarrow \quad \therefore\ x^*_2 = 0 + 1 = 1$

And $f(x) = f(x^*) + 5 = 4 + 5 = 9$

Problems

(1): Two product A and, B passes through two machines (1) and (2). The unit produced from the product A needs four hours in the 1^{st} machine and three hours in the 2^{nd} machine, and the unit produced from the product B needs two hours in the 1^{st} machine and only one hour in the 2^{nd} machine. If the available capacity for the two machines are 18 and 12 hours respectively, and the unit profit for each product are 4 (L.E) and 2 (L.E) respectively.

Required:

- **(A)Formulate the problem in a Linear programming model.**
- **(B)Determine the optimal solution for the LPM.**
- **(C)Determine the Dual problem.**
- **(D)Put each of the primal and dual problem in its canonical and standard form.**
- **(E)From your preceding results in A and B, determine the shadow prices for the set of resources.**

(2): Determine the solution space graphically for the following inequalities:

$$x_1 + x_2 \leq 4$$

$$4x_1 + 3x_2 \leq 12$$

$$-x_1 + x_2 \geq 1$$

$$x_1 + x_2 \leq 6$$

$$x_1 \ , x_2 \geq 0$$

Which constraints implies by others? Reduce the system to the smallest number of constraints which will define the same solution space.

(3): Solve the following problem graphically:

$$f(x) = 5x_1 + 2x_2$$

Subject to:

$$x_1 + x_2 \leq 10$$

$$x_1 = 5$$

$$x_1 , x_2 \geq 0$$

(4): Consider the graphical representation of the following LPM:

$F(x) = 5x_1 + 3x_2$ **Maximize (or Minimize)**

$$x_1 + x_2 \leq 6$$

$$x_1 \geq 3$$

$$x_2 \geq 3$$

$$2x_1 + 3x_2 \geq 3$$

$$x_1 , x_2 \geq 0$$

A-In each of the following cases indicate if the solution space has one point, infinite number of points, or no points:

(i)The constraints are as given above.

(ii)The constraint $x_1 + x_2 \leq 6$ is changed to $x_1 + x_2 \leq 5$

(iii)The constraint $x_1 + x_2 \leq 6$ is changed to $x_1 + x_2 \leq 7$.

B-For all cases in (A), determine the number of feasible extreme points if any.

C-For the cases in (A), in which a feasible solution space exists, determine the maximum and minimum value of *f(x)* and their associated extreme points.

(5): Consider the following LPM:

$F(x) = x_1 - x_2 + 3x_3$ **(Maximize)**

<u>Subject to:</u>

$$x_1 + x_2 + x_3 \leq 10$$

$$x_1 - x_3 \leq 5$$

$$2x_1 - 2x_2 + 3x_3 \leq 0$$

$$X_1 , x_2 , x_3 \geq 0$$

<u>Required</u>:

1-Put the LPM in its canonical and standard from.

2-Determine the optimal solution.

3-Determine the dual problem.

4-From you preceding results in (2), determine the optimal solution tableau for the dual problem.

(6): Consider the following LPM:

$$f(x) = 6x_1 - 2x_2 \qquad \text{(Maximize)}$$

Subject to:

$$x_1 - x_2 \leq 1$$

$$3x_1 - x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Show graphically that at the optimal solution, the two decision variable x_1 and x_2 can be increased indefinitely while the value objective functions remain constant.

(7): Consider the following LPM:

$$F(x) = 3x_1 + 2x_2 \qquad \text{(Maximize)}$$

Subject to:

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$X_1, x_2 \geq 0$$

Show graphically that the problem has no feasible extreme point. What can conclude concerning the solution to the problem?

(8): Consider the following LPM:

$$f(x) = x_1 - 3x_2 - 2x_3 \qquad \text{(Minimize)}$$

Subject to:

$$2x_1 - x_2 + 2x_3 \leq 7$$

$-2x_1 + 4x_2 \leq 12$

$X_1, x_2, x_3 \geq 0$

Required:

(i)Put the model in its canonical and standard form.

(ii)Determine the optimal solution.

(iii)Determine the dual problem. And find the optimal solution tableau for the dual problem by using your preceding results in (ii). Finally determine the shadow prices by using two different methods.

(9): Solve the following linear programming models graphically and with the suitable simplex method:

(A): $f(x) = x_1 + 2x_2$ **(Maximize)**

Subject to:

$-3x_1 + 3x_2 \leq 9$

$x_1 - x_2 \leq 2$

$x_1 + x_2 \leq 6$

$x_1 + 3x_2 \leq 6$

$x_1, x_2 \geq 0$

(B): $f(x) = -x_1 + x_2$ **(Maximize)**

Subject t to:

$x_1 + x_2 \leq 2$

$x_1 \leq 2$

$x_1 , x_2 \geq 0$

(C): $f(x) = -x_1 + x_2$ **(Maximize)**

Subject to:

$x_1 + x_2 \leq 4$

$2x_1 + 5x_2 \leq 10$

$x_1 , x_2 \geq 0$

(10): Determine x_1 , x_2 and x_3 by which:

$f(x) = 2x_1 + 3x_2 - 5x_3$ **(Maximize)**

Subject to:

$x_1 + x_2 + x_3 = 7$

$2x_1 + 5x_2 + x_3 \geq 10$

$x_1 , x_2 , x_3 \geq 0$

(11): Consider the following LPM:

$f(x) = 5x_1 - 6x_2 - 7x_3$ **(Minimize)**

Subject to:

$x_1 + 5x_2 - 3x_3 \geq \quad 15$

$5x_1 - 6x_2 + 10x_3 \quad \leq \quad 20$

$x_1 + x_2 + x_3 \quad = \quad 5$

$x_1 \ , \ x_2 \ , x_3 \quad \geq \quad 0$

Required:

A-Put the LPM in either the canonical form or the standard form.

B-Determine the dual problem.

C-Solve the primal problem.

(12): Solve the following LPM graphically and determine the different types of solutions:

$f(x) = -x_2$ **(Minimize)**

Subject to:

$x_1 + x_2 \geq \quad 1$

$x_1 + x_2 \leq \quad 2$

$x_1 - x_2 \quad \leq \quad 1$

$x_1 - x_2 \quad \geq \quad -1$

$x_1 \ , \ x_2 \quad \geq \quad 0$

And, solve the same LPM considering that = = f(x) = - x_1 and in maximization form.

(13): Consider the following LPM:

$f(x) = 2x_1 + x_2$ (Maximize)

Subject to:

$$x_1 + 2x_2 \leq 10$$

$$x_1 + x_2 \leq 6$$

$$x_1 - x_2 \leq 2$$

$$x_1 - x_2 \geq 0$$

Required:

(a): Solve the LPM graphically and determine the types of solutions.

(b): Solve the problem by using the simplex method.

(c): Find the dual problem.

(d): From your preceding results in (b), find the optimal solution tableau for the dual problem.

(e): Determine the shadow prices by using two different methods.

(14): Solve the following LPM graphically and by the simplex method:

$f(x) = -2x_1 - x_2$ (Minimize)

Subject to:

$x_1 + x_2 \leq \quad 2$

$x_1 + 3x_2 \quad \geq \quad 3$

$x_1 \ , x_2 \geq \quad 0$

(15): Solve the following LPM graphically and by the simplex method:

$f(x) = 4x_1 + 8x_2$ **(Minimize)**

Subject to:

$x_1 - x_2 \quad \geq \quad 2$

$2x_1 + x_2 \quad \geq \quad 5$

$x_1 \ , \ x_2 \quad \geq \quad 0$

(16): Consider the following LPM:

$f(x) = - x_2$ **(Minimize)**

Subject to:

$x_1 + 2x_2 \quad \geq \quad 6$

$x_1 - x_2 \quad \leq \quad 2$

$x_2 \quad \leq \quad 2$

$x_1 \ , \ x_2 \quad \geq \quad 0$

Required: Determine the optimal solution for the LPM

(17): Consider the following LPM:

$f(x) = - x_1$ (Minimize)

<u>Subject to:</u>

$$x_1 + x_2 \leq 2$$

$$| x_1 - x_2 | \leq 1$$

$$x_1 \ , \ x_2 \geq 0$$

<u>Required:</u>

A-Put the model in each of the canonical and the standard form.

B- Determine the dual problem.

C- Find the optimal solution for the primal problem.

D- From you preceding result in (c), determine the optimal solution tableau for the dual problem.

E- Determine the shadow prices.

Chapter (2)

Games Theory

In this chapter we will deal with decisions under uncertainty involving two or more intelligent opponents in which each opponent aspires to optimize his own decision but at the expense of the other opponents. The basics governing the solution of such decision problem is called the theory of game.

In Game Theory an opponent is referred to as a player (or team). Each player or team has a number of choices, finite or infinite, called strategies. In other meaning the game is a comparative situation between two or some peoples called players. And the outcomes or payoffs of a game are summarized as functions of the different strategies for each player. Each player wants to wine in this game. Note that, the game with two players, where a gain of one player equals a loss to the other is known as Two-Person Zero-Sum Game. Also the game is being according to rules previously stated. Each player have his private strategies to wine which is known by the other player, but the other player cannot know which strategy between them will use in the game by his comparative player. So, a win or gain for one player is considered a loss for the other one. So the sum of win and loss for the players will equal to zero. Such as, if the player A loss (-2) points, the player B will win (+ 2) points, so that:

-2 + 2 = zero which is known as Two-Person Zero-Sum Game.

The game between two players is expressed in a matrix form for showing the comparative situation between the two players as follows:

1- Rows player: the number of his strategies will equal to the number of the rows in the payoffs matrix.

2- Columns player: the number of his strategies will equal to the number of the columns in the payoffs matrix.

Optimal Solution of Two-Pearson Sum Game:

The following example illustrates the solution for the game in case of existing a saddle point and using **pure strategy** for each player.

Example (1): To illustrate the definitions of a two-person zero-sum game consider the following game:

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \end{array}$$

Note that: when the game is expressed in matrix form, it means that:

(a): The positive numbers: means gain or win for the row player.

(b): The negative numbers: means gain or win the columns player.

Therefore, from the preceding payoffs matrix, in this game each player has two strategies. This yields that the dimension (or the degree) for the game is in 2×2 game matrix. The elements for this matrix game means that, if the row player (A) plays or uses his first (1^{st}) strategy**:** Then he will gain (or win) from the columns player (B) 2 or 3 points respectively. And, if the player (A) plays or uses his second (2^{nd}) strategy, then the rows player (A) will loss 1 or 2 points respectively by which are considered a gains for the columns player (B) by the same values. The optimal solution to such a game may require each player to play a pure strategy or some mixture strategies.

In this situation the player (A) will play by his first strategy to maximize his gain points or profits. Henceforth, the player (B) will tries to minimize these losses, so that the player (B) will play by his 1^{st} strategy also to reduce his losses. So, the saddle point will be the 1^{st} strategy for each of the two players (A) and (B) respectively. And then, the player (A) will win (2) points and the player (B) will lose (2) points, i.e., the value for the game is equal to (2). And therefore, the sum for the game is equal to 2 – 2 = zero, i.e., we have a Two-Person Zero-Sum Game.

The result is that a very conservative criterion is usually proposed for solving two-person zero-sum games. This is the minimax (maximin) criterion, where each player is intelligent and hence actively tries to defeat his opponent. To accommodate the fact that each opponent or player is working against the other interest, the minimax criterion selects each player (mixed or pure) strategy which yields the best possible outcomes. An optimal solution is said to be reached if neither player finds it benefit to alter his strategy. In this case, the game is said to be stable or in a state of equilibrium or a saddle point. Since the game matrix is usually expressed in terms of the payoff to player A (whose strategies are represented by the rows), the (conservative) criterion calls of A to select the strategy (pure or mixed) which maximizes his minimum gain, the minimum being taken over all the strategies of player B. Then, by the same reasoning player B selects his strategy which minimizes his maximum losses. Again, the maximum is taken over all A's strategies. Therefore, in order to determine the saddle point for any matrix game, we will add a column to the matrix and put in this column the minimum values in each row, and we add a row to the matrix and put in this row the maximum values in each column. Then, determine the maximum value in the additional column and the minimum value in the additional row. If the maximum value in the additional column is equal to the minimum value in the additional row, then we have a saddle point, then, we can say this game have a saddle point and then we have pure strategies according to the minimax or maximin criterion. The intersection between the two points of the saddle point represents the optimal strategy for the two players A and B which are called pure strategies and then, we can determine the value of game. The following example represents this criterion.

Example(2): If you have the following payoff matrix:

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -1 \\ 4 & 3 \end{bmatrix} \end{array}$$

Required: determine the saddle point for the game if there is exist, and determine the optimum strategy (pure or mixed), and the value for the game.

Solution:

When player A plays his first strategy, he may gain 3 points or lose one point depending on player B's selected strategy. He can guarantee, however, a gain of at least min (3 , -1) = -1 regardless of B,s selected strategy. Similarly, if A plays his second strategy, he guarantees an income of at least min (4 , 3) = 3 . Thus, the minimum value in each row represents the minimum gain guaranteed A if he plays his pure strategies. These are indicated in the above matrix by "Row minimum ". Now player A by selecting his 1st strategy, is maximizing his minimum gains. This gain is given and his corresponding gain is called the maximin (or lower) value of the game, as it be shown in the following result:

$$\begin{array}{c c c}
 & B & \text{Row minimum} \\
A & \begin{bmatrix} 3 & -1 \\ 4 & 3 \end{bmatrix} & \begin{array}{l} -1 \\ (3) \rightarrow \text{max-min.} \end{array} \\
\text{Column Maximum} & \begin{array}{cc} 4 & \begin{array}{c} (3) \\ \downarrow \\ \text{mini-max} \end{array} \end{array} &
\end{array}$$

Vice versa, on the other hand for player B, he wants to minimize his losses. He realizes that, if he plays his 1^{st} pure strategy, he can lose no more than max(3 , 4) = 4 regardless of A's selections. A similar argument can also be applied to the 2^{nd} strategy. The corresponding results are thus indicated in the above matrix by "column maximum". Player B will then select the strategy and his corresponding loss is given by min (4 , 3) = 3 . player B,s selection is called the minimax strategy and his corresponding loss is called the minimax (or upper) value of the game.

Now, in the above example, since the value for the maxmin = minimax value = 3. This implies that the game has saddle point which is given by the entry (2 , 2) of the game matrix, and then; the game has a saddle point. Therefore; the value of the game is equal to 3, which means that, the player (A) will use his second strategy, and the player

(B) will use also his second strategy. And then the player A will win 3 points, and the player B will lose 3 points. And then, the value of game = 3 – 3 = zero. Henceforth; the game is called a Two-Person Zero-Sum Game.

Then we conclude that there is a saddle point, i.e., we have a pure strategy. And the optimum strategy is the 2nd strategy for each player A and B, which means that the player A gains 3 points and the player B loses 3 points.

Determination the value of game and the optimum strategies when there is no saddle point [mixed strategies]:

The above section shows that the existence of a saddle point immediately yields to the optimal pure strategies for the game. But, some games do not have saddle points, henceforth, we have some mixed strategies. The following example represents this concept.

Firstly: if the payoffs game matrix is in the dimension 2 x 2 and there is no saddle point (mixed strategies):

If there is no saddle point, in this case the game is said to be unstable. And hence each player can improve his payoff by selecting a different strategy. Henceforth, to give an optimal solution to the game has led to the idea of using mixed strategies. Each player, instead of selecting a pure strategy only, may play all his strategies according to a predetermined set of probabilities. Then, we can find the solution for the game by using each of the following two different ways. For simplicity, the two different ways suppose that the dimension for the game matrix is 2×2 :

Firstly: ***Algebra method or technique***:

We assumed that the rows player A will played his 1st and 2nd strategies by the two probabilities (p) and (1 – p) respectively. Also, We assumed that the columns player B will played his 1st and 2nd strategies by the two probabilities (q) and (1 – q) respectively. Then we have the following form for the payoff game matrix:

$$
\begin{array}{cccc}
 & q & 1-q & \\
A & \left[\begin{array}{c} X_{11} \\ X_{21} \end{array}\right. & \left.\begin{array}{c} X_{12} \\ X_{22} \end{array}\right] & \begin{array}{c} p \\ 1-P \end{array}
\end{array}
$$

→ Then, to determine the values for (p) and (*q*) , mathematically we have to make the maximin and the minimax expected payoffs must be equal. Assuming that the wining equality in chances of player A in cases of player B uses his first or second strategies, then, we have the following equation:

$$X_{11}\,(p) + X_{21}\,(1 - p)\,) = X_{12}\,(p) + X_{22}\,(1- p)$$

Then, we can determine each of the two probabilities (p) and its complement (1- p).

Also, to determine the values for (q) and its complement (1 – q), assuming that the wining equality in chances for the player B in the two cases for the player A uses his 1st or 2nd strategies, then, we have the following equation:

$$X_{11}(q) + X_{12}(1-q) = X_{21}(q) + X_{22}(1-q)$$

Then, we can determine each of the two probabilities (q) and its complement (1- q).

→And the optimal strategies for the two players are the strategy by which have the highest probability for each of the two players A and B

→And thus, the optimal expected value of the game is given by the following equation:

$V^* = \sum \sum a_{ij} x_i^* y_j^*$, where (a_{ij}) represents the elements for the payoff matrix, and x_i^* y_j^* represents the different resulted probabilities corresponding to each element in the game matrix. If the result value of the game is positive, it means a gain for the rows player A , vice versa, if the result value of the game is negative , it means a gain for the columns player B .

<u>Secondly: Matrix method or technique</u>:

If we have a matrix in 2 x 2 : →

$$G = \begin{bmatrix} X11 & X12 \\ X21 & X22 \end{bmatrix}$$

Then, find each of the following calculation:

* | G | or Δ_G: the determinant value for the payoff game matrix.

* $\overline{G}$ or Gc : the cofactor matrix for the payoff game matrix G.

G_c^T : the transpose of cofactor matrix for the payoff matrix game G.

Then, in order to find the probabilities of playing each strategy for each player and the value for the game, we have to find the following results:

* The probabilities for the rows player A is equal to:

$$= [\,(1 \quad 1)G_c^T\,] / [(1 \quad 1)\, G_c \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

* The probabilities for the columns player B is equal to:

$$= [\,(1 \quad 1)\, G_c\,] / [(1 \quad 1)\, G_c \begin{pmatrix} 1 \\ 1 \end{pmatrix}\;]$$

* The value for the game is equal to:

$$V^* = |G| \text{ or } \Delta_{\mathbf{G}} \Big/ \left[(1 \quad 1)\, G_{\mathbf{c}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

Not that, if the payoff matrix game is in a dimension ($2 \times n$) or

($m \times 2$),where (n) or (m) is greater than 2, i.e., (n) or (m) is equal to 3 or 4 or …….., then, we can solve the payoff matrix game as follows:

1- Test about if there is (or is not) a saddle point.

2- If there is a saddle point, then, we have a pure strategy, and then we determine the optimal pure strategy for each player which have the most or largest probability and then we have to find the value for the game.

3- If there is no saddle point, then, we must have partitioned the payoff matrix game into a set of square submatrices each one have the dimension of 2×2 . For each submatrix, we have to test about if there is (or is not) a saddle point, i .e., a pure or mixed strategy. And then, we can determine the optimal strategy in case of pure strategy which have the most (or largest) probability and then we have to find the value for the game. Or, we have to find the set of probabilities for each player in case of mixed strategies by using either algebra technique or the matrix technique.

*** Note that if (n) or (m) > 4 , we can applied the dominance principal according to the viewpoint of the rows player A or the column player B. If we applied the dominance principal from the viewpoint of the rows player A, i.e., (n ≥ 4) then, we have to delete the rows in the payoff matrix game by which contains the largest negative elements. On the contrary, if we applied the dominance principal from the viewpoint of the columns player B, i.e., (m ≥ 4) then, we have to delete the columns in the payoff matrix game by which contains the largest positive elements.

Example (3): solve the following payoff matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Firstly, we have to test if there is (or is not) a saddle point by using the maximin (or minimax) criterion as follows:

$$\begin{array}{ccc} & B & \text{Row minimum} \\ A & \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ (1) \rightarrow \text{maximin.} \end{array} \\ \text{Column Maximum} & \begin{array}{cc} 3 & (2) \\ & \downarrow \\ & \text{minimax} \end{array} & \neq \end{array}$$

Now, since the value for the maximin ≠ the value for the minimax. So that, there is no saddle point, i.e., there is no pure strategies and then we have mixed strategies. Then, we have to find the set of probabilities for the set of strategies and the value for the game as follows:

* | G | or Δ_G: the determinant value for the payoff game matrix.

$$\Delta_G = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 \times 1 - 2 \times 3 = 1 - 6 = -5 \neq 0 \text{ . And,}$$

* $\overline{G}$ or Gc : the cofactor matrix for the payoff game matrix G is:

$$\overline{G} \text{ or } Gc = \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} \text{ . And,}$$

* $\overline{G}^{T}$ or G_c^{T} : the transpose of cofactor matrix for the payoff matrix game $G = Gc^{T} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix}$. And,

Then, in order to find the probabilities of playing each strategy for each player and the value for the game, we have to find the following calculation:

* The probabilities for the rows player A is equal to:

$$= [\, (1 \quad 1) G_c^{T} \,] \,/\, [\, (1 \quad 1)\, G_c \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

$$= [(1 \quad 1) \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}] \; / \; [(1 \quad 1) \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= (-2 \quad -1) \, / \, (-3) = [\, 2/3 \quad 1/3 \,]$$

i.e., the probabilities for playing the rows player A his 1st and 2nd strategies are equal to 2/3 and 1/3 respectively.

* The probabilities for the columns player B is equal to:

$$= [\,(1 \quad 1)\, G_{\mathbf{c}}\,] \, / \, [(1 \quad 1)\, G_{\mathbf{c}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

$$= [(1 \quad 1) \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}] \; / \; [(1 \quad 1) \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= (-1 \quad -2) \, / \, (-3) = [\, 1/3 \quad 2/3 \,]$$

i.e., the probabilities for playing the columns player B his 1st and 2nd strategies are equal to 1/3 and 2/3 respectively. And, the optimum strategy for each player is that strategy which have the most probabilities. Therefore, the optimum strategy for each player is the 1st strategy for the rows player A and the 2nd strategy for the rows player B.

* And finally, the value for the game is equal to:

$$V^* = \mid G \mid \text{ or } \Delta_G \; / \; [(1 \quad 1)\, G_c \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

Where, $\Delta_G = (1 \times 1) - (2 \times 3)$

$$= 1 - 6 = -5 \neq \text{zero}$$

So that the value for the game is = (-5)/ (- 3) = 5/3 . And, since this value is positive, then the rows player A gains 5/3 points and the columns player B loses the same value 5/3 .

→ * **<u>If the dimension for the game matrix payoff is different from 2 x 2, then, we have two methods for solution:</u>**

[1]: <u>Appling Dominance Principal</u>:

Here we can reduce the game matrix payoff G to be 2×2 or to be $n \times 2$ or to be $2 \times m$ by using the dominance principal(where n or m ≥ 3) and partitioned the matrix game into a set of submatrices each one is in the dimension 2×2 , and solve each submatrix by using the previous two method (algebra method or matrices) as follows :

(a) : If the game matrix payoff is in the dimension ($n \times 2$):

In this case the game matrix has a set of (n) rows (where

$n > 2$ rows) and 2 columns. Then, we applied the dominance principal from the viewpoint for the rows player A to reduce the dimension for the game matrix to be 4 or 3 $\times$ 2 or 2 $\times$ 2 ,by keeping the strategies which have the most positive numbers

(Highest gain) and deleting the strategies which have the most negative numbers (highest loss). And then, solving the matrix (2$\times$2) by using either the algebra or matrix method, and solving the matrix (4 or 3$\times$2) by partitioned this game matrix into a set of a square submatrices each one is in the dimension (2$\times$2), and solving each submatrix by either algebra or matrix method.

Vice versa, in this case the game matrix has a set of (m) columns (where, $m > 2$ columns) and 2 rows. Then, we applied the dominance principal from the viewpoint for the columns player B to reduce the dimension for the game matrix to be 2 $\times$ 4 or 3 or 2 $\times$ 2 ,by keeping the strategies which have the most negative numbers (highest gain for the player B) and deleting the strategies which have the most positive numbers

(Highest loss). And then, solving the matrix (2$\times$2) by using either the algebra or matrix method, and solving the matrix (2$\times$4 or 3) by partitioned this game matrix into a set of a square submatrices each one is in the dimension (2$\times$2), and solving each submatrix by either algebra or matrix method.

Example:

Solve the following game:

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & 2 \\ -1 & -4 \\ 3 & 1 \end{bmatrix} \end{array}$$

Solution:

Firstly, we have to test about if there is (or is not) a saddle point by using the maximin (or minimax) criterion as follows:

$$\begin{array}{lccc l} & & B & \text{Row-min} & \\ & 1 & 2 & 1 & \\ A & -1 & -4 & -4 & \\ & 3 & 1 & 1 & \\ & 3 & 2 & (1) & \to \text{maximin.} \\ \text{Column} & 3 & (2) & \neq & \\ \text{Maximum} & & \text{minimax} & & \end{array}$$

Now, since the value for the maximin ≠ the value for the minimax. So that, there is no saddle point, i.e., there is no pure strategies, and then we have mixed strategies. Note that, we can have applied the dominance principal from the viewpoint of the rows player A to reduce the dimension of the game matrix to become in the dimension of 2×2 by keeping his 1st and 3rd strategies (highest gains or profits) and deleting his 2nd strategy (highest loss), or we have to partitioned the game matrix into three submatrices and test for each submatrix however there is (or is not) a saddle point, hence solve each submatrix to determine if there is (are) a pure or mixed strategies.

In this case, we prefer to latest appointment, i.e., we have to partition the game matrix into three submatrices and test for each submatrix however there is (or is not) a saddle point, hence solve each submatrix to determine if there is (are) a pure or mixed strategies as follows:

* **1st submatrix:** G1, where:

$$G1 = A \begin{array}{c} B \\ \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix} \end{array}$$

In the above submatrix G1, firstly we have to test wherever or no there is a saddle point as follows:

$$
\begin{array}{cc}
 & \text{B} \\
G1 = A & \begin{bmatrix} 1 & 2 \\ -1 & -4 \end{bmatrix} \\
\text{Column} & \begin{matrix} (1) & 2 \end{matrix}
\end{array}
\begin{array}{l}
\text{Row-min} \\
(1) \rightarrow \text{maximin.} \\
-4 \\
\\
\end{array}
$$

$$\text{Maxmini} = \text{Minimax}$$

Now, in the above submatrix G1, since the value for the maxmini = minimax value = 1. This implies that the game has saddle point which is given by the entry (1 , 1) of the game matrix, and then; the game has a saddle point. Therefore; the value of the game is equal to 1, which means that, the player (A) will use his 1st strategy, and the player (B) will use also his 1st strategy. And then the player A will win 1 point, and the player B will lose 1 point. And then, the value of game = 1 – 1 = zero. Henceforth; the game is called a Two-Person Zero-Sum Game.

Then we conclude that there is a saddle point, i.e., we have a pure strategy. And the optimum strategy is the 1st strategy for each player A and B, which means that the player A gains 1 point and the player B loses 1 point.

* **2nd submatrix:** G2, where :

$$G2 = \begin{array}{c} B \\ \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \end{array} A$$

Also, In the above submatrix G2, firstly we have to test wherever or no there is a saddle point as follows:

$$\begin{array}{cccc} & B & & \text{Row-min} \\ A & \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ (1) \end{array} & \rightarrow \text{maximin.} \\ \text{Column Maximum} & 3 \quad (2) & \neq & \\ & \downarrow & & \\ & \text{minimax} & & \end{array}$$

Now, since the value for the maximin ≠ the value for the minimax. So that, there is no saddle point, i.e., there is no pure strategies and then we have mixed strategies. Then, we have to find the set of probabilities for the set of strategies and the value for the game as follows:

* | G2 | or Δ_G: the determinant value for the payoff game matrix.

$$\Delta_{G2} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 \times 1 - 2 \times 3 = 1 - 6 = -5 \neq 0 \text{ . And,}$$

* $\overline{G2}$ or $G2_C$: the cofactor matrix for the payoff game matrix $\overline{G2}$ is:

$$\overline{G2} \text{ or } G2_C = \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} \text{. And,}$$

* $\overline{G2}^{\mathbf{T}}$ or $G2_{\mathbf{c}}^{\mathbf{T}}$: the transpose of cofactor matrix for the payoff matrix game G2 = $G2c^{\mathbf{T}} = \begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix}$. And,

Then, in order to find the probabilities of playing each strategy for each player and the value for the game, we have to find the following calculation:

* The probabilities for the rows player A is equal to:

$$= [\,(1 \quad 1)G2_{\mathbf{c}}^{\mathbf{T}}\,] \,/\, [(1 \quad 1)\, G2_{\mathbf{c}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

$$= [(1 \quad 1) \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}] \,/\, [\,(1 \quad 1) \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= (-2 \quad -1) \,/\, (-3) = [\,2/3 \quad 1/3\,]$$

i.e., the probabilities for playing the rows player A his 1st and 2nd strategies are equal to 2/3 and 1/3 respectively.

* The probabilities for the columns player B is equal to:

$$= [\,(1 \quad 1)\, G2_{\mathbf{c}}\,] \,/\, [(1 \quad 1)\, G2_{\mathbf{c}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

$$= [(1 \quad 1) \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}] \,/\, [\,(1 \quad 1) \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= (-1 \quad -2)\,/\,(-3) = [\,1/3 \quad 2/3\,]$$

i.e., the probabilities for playing the columns player B his 1st and 2nd strategies are equal to 1/3 and 2/3 respectively. And, the optimum strategy for each player is that strategy which have the most probabilities. Therefore, the optimum strategy for each player is the 1st strategy for the rows player A and the 2nd strategy for the rows player B.

And finally, the value for the game is equal to:

$$V^* = |\,G2\,| \text{ or } \Delta_{\mathbf{G2}} \,/\, [(1 \quad 1)\, G_{\mathbf{c}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}]$$

Where, $\Delta_{\mathbf{G2}} = (1 \times 1) - (2 \times 3) = 1 - 6 = -5 \neq$ zero

So that, the value for the game is = (-5)/ (- 3) = 5/3 . And, since this value is positive, then the rows player A gains 5/3 points and the columns player B loses with the same value 5/3 .

* **The 3rd submatrix:** G3, where:

$$G2 = \begin{array}{c} B \\ \begin{bmatrix} -1 & -4 \\ 3 & 1 \end{bmatrix} \end{array} A$$

Also, In the above submatrix G3, firstly we have to test wherever or no there is a saddle point as follows:

$$\begin{array}{ll} & \qquad B \qquad\qquad \text{Row-min} \\ G3 = A & \begin{bmatrix} -1 & -4 \\ 3 & 1 \end{bmatrix} \begin{matrix} -4 \\ (1) \end{matrix} \rightarrow \text{maximin.} \\ \text{Column Maximum} & \quad 3 \qquad \underset{\substack{\downarrow \\ \text{minimax}}}{(1)} \quad = \end{array}$$

Now, in the above submatrix G3, since the value for the maxmin = minimax value = 1. This implies that the game has a saddle point which is given by the entry (1 , 1) of the game matrix, and then; the game has a saddle point. Therefore; the value of the game is equal to 1, which means that, the player (A) will use his 2^{nd} strategy, and the player (B) will use also his 2^{nd} strategy. And then, the player A will win 1 point, and the player B will lose 1 point. And then, the value of game = 1 – 1 = zero. Henceforth; the game is called a Two-Person Zero-Sum Game.

Then we conclude that there is a saddle point, i.e., we have a pure strategy. And the optimum strategy is the 1st strategy for each player A and B, which means that the player A gains 1 point and the player B loses 1 point.

<u>Example (4)</u>: Solve the following game:

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -3 & 7 & 5 \\ -2 & 4 & 5 & 2 \end{bmatrix} \end{array}$$

<u>Solution:</u>

In this game, we have three different ways for solutions. Firstly, without applying the dominance principal for reducing the set of strategies for any player (the best solution). In this case, we have to test firstly wherever or no there is a saddle point. If there is a saddle point, then we have pure strategies, but if there is no saddle point, then we have to use the submatrices by solving a set of 6 (the number of combination = $C^4{}_2$ = 6) game submatrix and determining the optimal strategy for each player in the 6 submatrix. We are left this direction for solution to each student.

Secondly, by applying the dominance principal from the viewpoint of the columns player B to reduced his four strategies to become 3 or 2 strategies if there is no saddle point in the given game payoff matrix.

In case of applying the dominance principal from the viewpoint of the columns player B to reduced his four strategies to become 3 strategies, i.e., to reduce the game matrix to become 2×3 , then , each of the 1st , 2nd and the 4th strategy for the player B will dominate his 3rd strategy, and then the game matrix become :

$$G = \begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -3 & 5 \\ -2 & 4 & 2 \end{bmatrix} \end{array}$$

We are left the completion this direction for solution to each student.

Thirdly, in case of applying the dominance principal from the viewpoint of the columns player B to reduced his four strategies to become only 2 strategies, i.e., to reduce the game matrix to become

2×2 , then , each of the 1st , 2nd are dominate the 3rd and the 4th strategy for the player B, and then the game matrix become in the form:

$$G = \begin{array}{cc} B & \\ \begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix} & A \end{array}$$

We are left the completion of this direction for solution to each student.

→**Note that**: In case of the mixed strategies, we will reject the results for any submatrix is not logical in its results, i.e., we will reject the results for any submatrix gave the probabilities for playing any set of strategies for any player are either negative values or the sum of the set of probabilities for playing the corresponding strategies greater than one.

Example (5):

Determine the optimal strategies and the value for the game for the following game payoff matrix:

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 9 & -3 & -6 \\ 5 & 6 & 7 \end{bmatrix} \end{array}$$

Solution:

To determine the optimal strategies and the value for the game, we have to test however there is a saddle point or not as follows:

$$\begin{array}{ccccc} & & B & & \text{Row-min.} \\ A & \left[\begin{array}{c} 9 \\ 5 \end{array}\right. & \begin{array}{c} -3 \\ 6 \end{array} & \left.\begin{array}{c} -6 \\ 7 \end{array}\right] & \begin{array}{l} -6 \\ (5) \quad \text{maximin. } 5 \neq 6 \end{array} \\ \text{Col-Max.} & 9 & (6) & 7 & \\ & & \text{minmax} & & \end{array}$$

Since, the value for the row maximin (5) ≠ the value for the column minimax (6), So, there is no saddle point. Then, we will partitioned the matrix into a set of submatrices as follows:

[1]First submatrix: is $\begin{bmatrix} 9 & -3 \\ 5 & 6 \end{bmatrix}$

[1]second sub-matrix: is $\begin{bmatrix} 9 & -6 \\ 5 & 7 \end{bmatrix}$

[1]third sub-matrix: is $\begin{bmatrix} -3 & -6 \\ 6 & 7 \end{bmatrix}$

Then, each one of the three submatrix will be solved as follows:

[1]: First submatrix:

We have to know however there is (or is not) a saddle point.

$$\begin{array}{ccc} & & \text{min.} \\ \left[\begin{matrix} 9 & -3 \\ 5 & 6 \end{matrix}\right. & & \begin{matrix} -3 \\ (5)\ \text{max.min} \end{matrix} \\ \text{Max. } 9 & (6) \rightarrow & \text{min.max },\ 5 \neq 6 \end{array}$$

So, there is no saddle point, and then we will have mixed strategies:

[1] $|G| = \begin{vmatrix} 9 & -3 \\ 5 & 6 \end{vmatrix} = (9 \times 6) - ((-3) \times 5) = 69$

[2] $G^T{}_C = \begin{bmatrix} 6 & 3 \\ -5 & 9 \end{bmatrix}$

[3] $G_C = \begin{bmatrix} 6 & -5 \\ 3 & 9 \end{bmatrix}$, and then,

We are left the completion of this direction for solution to each student.

Then, one can see that the probabilities for playing the rows player A his 1st and 2nd strategies are equal to **(1/13 , 12/13) ,** the probabilities for playing the column player B his 1st and 2nd strategies are equal to **(……. , ……..) , and the value for the game is equal to (69/13) , i.e., the player A will gain (69/13) points and the player B will lose the same value .** Now, Since the value for the game is positive value, then the rows player A will gain 69/13 points and the columns player B will lose 69/13 and the optimal strategies for the two players are the strategies which have the highest probabilities, i.e., the optimum strategy for the player A is his 2nd strategy and the optimum strategy for the player B is his 1st strategy.

[2]: The Second submatrix:

We have to know however there is (or is not) a saddle point as follows:

$$
\begin{array}{c c c}
 & B & \\
 & & \text{min.} \\
A & \begin{bmatrix} 9 & -6 \\ 5 & 7 \end{bmatrix} & \begin{array}{c} -6 \\ (5) \rightarrow \text{max.min} \end{array} \\
\text{Max.} & 9 \qquad (7) \rightarrow \text{min.max} , 7 \neq 5 &
\end{array}
$$

So, there is no saddle point, and then we will have mixed strategies:

$$* \quad |\mathbf{G}| = \begin{vmatrix} 9 & -6 \\ 5 & 7 \end{vmatrix} = (9 \times 7) - ((-6) \times 5) = 93$$

$$* \quad \overline{\mathbf{G}}^{\backslash} = \begin{bmatrix} 7 & 6 \\ -5 & 9 \end{bmatrix}$$

$$* \quad \overline{\mathbf{G}} = \begin{bmatrix} 7 & -5 \\ 6 & 9 \end{bmatrix}$$, and then,

One can show the following results: that the probabilities for playing the rows player A his 1st and 2nd strategies are equal to (2/17 , 15/17)

, the probabilities for playing the column player B his 1st and 2nd strategies are equal to **(13/17 , 4/17) , and the value for the game is equal to (93/17) , i.e., the player A will gain (93/17) points and the player B will lose the same value .** Now, Since the value for the game is positive value, then the rows player A will gain (93/17) points and the columns player B will lose (93/17) and the optimal strategies for the two players are the strategies which have the highest probabilities, i.e., the optimum strategy for the player A is his 2nd strategy and the optimum strategy for the player B is his 1st strategy.

[3]: The Third submatrix:

We have to know however there is (or is not) a saddle point as follows:

$$
\begin{array}{ccccc}
 & & B & & \text{Row-min.} \\
 & & & & \\
|G| & = & \left[\begin{array}{cc} -3 & -6 \\ 6 & 7 \end{array}\right] & \begin{array}{c} -6 \\ (6) \rightarrow \text{max.min} \end{array} & A \\
 & \text{Max.} & 6 \quad\quad 7 \rightarrow & & \\
\end{array}
$$

$\hookrightarrow$ min.max , 6 = 6

since, the rows min.max = the columns max.min = 6 , so that, there is a saddle point and then we have a pure strategies. Hence, the optimal strategies for the rows player A is his 2nd strategy and the optimal strategies for the columns player B is his 1st strategy. And then, the player A will win 6 points, and the player B will lose 6 points. And then, the value of game = 6 – 6 = zero. Henceforth; the game is called a Two-Person Zero-Sum Game.

Example (6):

Find the optimal strategies for the two players A and B and determine the value of the game:

$$A\begin{bmatrix}2 & 4\\ 1 & -3\end{bmatrix}$$

*__Solution__

$$G = \begin{array}{cc|l} & & \text{min.}\\ \hline \multicolumn{2}{c|}{\begin{bmatrix}2 & 4\\ 1 & -3\end{bmatrix}} & \begin{array}{l}2 \rightarrow \text{max.min}\\ -3\end{array}\\ \hline \text{Max. } 2 & 4 & \end{array}$$

$$\text{Max. } 2 \rightarrow \text{min.max},\ 2 = 2$$

Since, the rows min.max = the columns max.min = 2 , so that, there is a saddle point and then we have a pure strategies. Hence, the optimal strategies for the rows player A is his 1^{st} strategy and the optimal strategy for the columns player B is also his 1^{st} strategy. And then, the player A will win 2 points, and the player B will lose 2 points. And then, the value of game = 2 – 2 = zero. Henceforth; the game is called a Two-Person Zero-Sum Game.

Example (7) [May 2005]

Determine the optimal strategies for the two players A and B and the value for the game For the following game payoff matrix:

$$A\begin{bmatrix} 6 & 4 & -1 & 0 & -3 \\ 3 & 2 & -4 & 5 & -1 \end{bmatrix}$$

Solution:

To determine the optimal strategies and the value for the game, we have to firstly search about wherever there is(or is not) a saddle point as follows:

$$\begin{array}{cc} & \text{Row-min} \\ A\begin{bmatrix} 6 & 4 & -1 & 0 & -3 \\ 3 & 2 & -4 & 5 & -1 \end{bmatrix} & \begin{array}{l} -3 \rightarrow \text{max.min} \\ -4 \end{array} \end{array}$$

$$\text{Colu.-Max.} \quad 6 \quad 4 \quad -1 \quad 5 \quad (-1) \rightarrow \text{min.max} \qquad -1 \neq -3$$

Since, the value for the row maximin (-3) ≠ the value for the column minimax (-1), So, there is no saddle point. Then, we will have partitioned the matrix into a set of submatrices. But, since the dimension for the matrix game is (2 x 5) , so that, we have to applied the dominance principal from the viewpoint of the columns player. Here, we will apply the dominance principal for the favor of player (B) to delete for the player (B) or reduced the game matrix to be in the 2×3 dimension. According to the dominance principal from the viewpoint of the columns player B, his 3^{rd} , 4^{th} and 5^{th} will dominate his 1^{st} and 2^{nd} strategies. Henceforth, we have the following game payoff matrix:

$$\begin{bmatrix} -1 & 0 & -3 \\ -4 & 5 & -1 \end{bmatrix}$$

To determine the optimal strategies and the value for the game, we have to firstly search about wherever there is(or is not) a saddle point as follows:

$$\begin{array}{lcl} & & \text{row-min.} \\ A & \begin{bmatrix} -1 & 0 & -3 \\ -4 & 5 & -1 \end{bmatrix} & \begin{array}{l} -3 \rightarrow \text{max.min} \\ -4 \end{array} \end{array}$$

$$\text{Col.-Max.} \quad -1 \quad 5 \quad -1 \rightarrow \text{min.max} \quad -1 \neq -3$$

Since, the value for the rows maximin (-3) $\neq$ the value for the columns minimax (-1), So, there is no saddle point. Then, we will have to partition the matrix game into a set of (3) submatrices. And then, determine the optimal strategies and the value for the game for each one of the three submatrices by using either Algebra method or the matrix method. Each student tries for completing the solution for these submatrices.

Exercises

[1]: Assuming that, you have the following payoff matrix:

$$
\begin{array}{cc}
 & B \\
A & \begin{bmatrix} 8 & 2 & 9 & 5 \\ 6 & 5 & 7 & 18 \\ 7 & 3 & -4 & 10 \end{bmatrix}
\end{array}
$$

Required: Determine the optimal strategies and the value of the game and then put the game matrix in its LPM.

[2]: Assuming that, you have the following game between the two players A and B:

$$
\begin{array}{cc}
 & B \\
A & \begin{bmatrix} 2 & 2 & 3 & -1 \\ 4 & 3 & 2 & 6 \end{bmatrix}
\end{array}
$$

Required: Determine the optimal strategies and the value of the game.

[3]: Determine the saddle point and optimal strategies for the following games:

(1): $\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$ (2): $\begin{bmatrix} 2 & 0 & 4 \\ 1 & -3 & 2 \end{bmatrix}$

(3): $\begin{bmatrix} 1 & -3 \\ 2 & 4 \\ -1 & 5 \end{bmatrix}$ (4): $\begin{bmatrix} 3 & 2 & -2 \\ 1 & -3 & -4 \\ 0 & 1 & -3 \end{bmatrix}$

(5): $\begin{bmatrix} 1 & -3 \\ 4 & 0 \\ 3 & -1 \end{bmatrix}$

[4]: In a game between two players A and B, if you know the following:

(a)If the rows player (A) play with his 1st strategy and his comparative (B) will play by his 2nd strategy, then the player (A) will win one point.

(b)If the rows player (A) play by his 2nd strategy and his comparative (B) will play by his second strategy, also player (A) will win 4 points.

Required:

[1]: Determine the elements for the game matrix between A and B.

[2]: Determine the optimal strategies for A and B and find the value of the game.

(5): Solve the following games:

[1]:
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

[2]:
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 3 & -6 \end{bmatrix} \end{array}$$

(6): Formulate the following games in its LPM:

[1] $\begin{bmatrix} 1 & 2 & 5 \\ 8 & 4 & 7 \\ -1 & 5 & -6 \end{bmatrix}$

[2] $\begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{bmatrix}$

[3] $\begin{bmatrix} 1 & 2 & 5 & 3 \\ -1 & 4 & 7 & 2 \\ 5 & -1 & 1 & 9 \end{bmatrix}$

and determine if there are mixed strategies or pure strategies.

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4th year Student

English Section

Operations Research

An Introduction

Prepared By

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2nd Edition

Chapter (3)

"**Queuing Theory**"

Introduction :

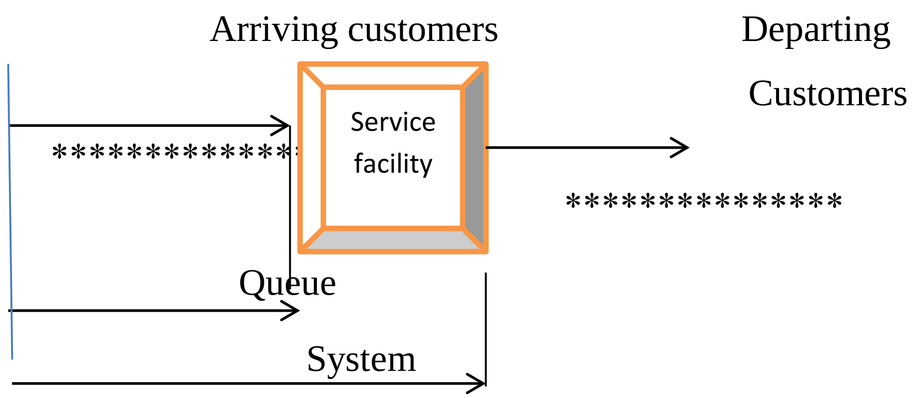
The queuing theory is basically discussed the flow of customers arriving at one or more service facilities. On arrival at the facility, the customer may be serviced immediately, or if willing may have to wait until the service facility is available. The flow of customers from finite or infinite population towards the service facility forms a queue or waiting line an account of lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers waiting time is required either for the service facilities or for the customer’s arrival.

In general, the queuing system consists of one or more queues, one or more servers and operates under a set of procedures. The service time allocated to each customer may be fixed or random variable depending on the type of service and having a specific probability density function. Depending upon the sever status, the incoming customer either waits at the queue or gets the turn to be served. If the server is free at the time of arrival of a customer, he can directly enter into the server for getting service and then leave the system.

Many of different situations in every-day life appear the queuing system or waiting lines. For example, it occurs in a barbershop, where the arriving individuals are the customers and the barbers are the servers. Another situation is illustrated by a machine breakdown, where the broken machine represents the customer calling for the service of a repairman. The preceding examples show that the term of customers may be interpreted in a variety of ways.

The Characteristics of Queuing System:

The queuing system is defined the waiting line or queue and the service channel(s). At any time, the number of customers in the system is equal to the number in the queue plus the number in the service. For simplicity the following figure represents the basic elements of the queuing system which have one service channel:



From this figure, the queuing system can be completely described by the following six main characteristics:

(1): **The input or arrival (inter-arrival) pattern or distribution**:

The arrival distribution determines the pattern by which the number of customers arrive at the system. From the theory of probability, the rate of arrival is a discrete random variable have a Poisson distribution (Markovian (M)) with parameter{ (λ) customer per time unit)} represents the mean or average number of customers by which they are arrive at the time unit, in other words which means that the inter-arrival time between two successive arrivals is a continuous random variable having an exponential distribution with time mean parameter{ ($1/\lambda$) time unit}

(2): **<u>The output or departure (inter-departure or service) pattern or distribution</u>**: By the same way, the departure or service time distribution determines the pattern by which the number of customers depart from the system. Also, from the theory of probability, the rate of departure is a discrete random variable have a Poisson (Markovian (M)) distribution with parameter{ (μ) customer per time unit)} represents the mean or average number of customers by which they are depart from the system at the time unit, in other words which means that the inter-departure time between two successive departures or services is a continuous random variable having an exponential distribution with time mean parameter{ ($1/\mu$) time unit}. These distributions are usually determined by sampling from actual situations.

(3): **<u>The number of service channels:</u>**

The service channel may be arranged in parallel or in series or as a more complex combination of both depending on the design of the system's service mechanism . In the case of parallel channels, several customers may be served simultaneously, but in the case of series channels, a customer must passes through all the channels before the service is completed. The queuing model is called a one-server model when the system has one server only, and a multiple-server model when the system has a number of parallel channels each one with one server, i.e. ($c \geq 1$).

(4): **Service discipline:**

The meaning of service discipline is the operation or the rule for selecting customers from the queue to start service. In this case we have following service disciplines:

* First Come First Served discipline (FCFS), which is the common service discipline, where the customers are admitted to start service according the order of their arrivals.

* Last Come First Served (LCFS) .

* Service in Random Order (SIRO) .

* General service Discipline (GD).

* Service with a specific Priority: it occurs when an arriving customers is given a higher priority for service over some other customers already in the system.

(5): **The maximum number of customers allowed in the system**:

The maximum number in the system can be either Finite or Infinite depending on the design of the facility. Where, in some facilities, only a limited number of customers are allowed to wait in the system. In this case, any newly arriving of customers are not permitted to join in the queue since the maximum number of customers allowed in the system limit has been reached.

(6): **<u>Calling source or the size of population</u>**:

The calling source or population is considered an important factor in the queuing theory analysis since the arrival pattern is depending on the source from which customers are generated. The calling source generating the arrivals may be finite or infinite.

From the preceding characteristics for the queuing models D.G. Kendall introduced a useful notations for the multiple server queuing models which describes the first three characteristics; namely: arrival distribution, departure distribution, and the number of parallel service channels. Later, A.Lee added the fourth and fifth characteristics to the notation; that is, the service discipline and the maximum number in the system. Finally, the Kendall&Lee notations is augmented by the sixth characteristic describing the calling source. The complete notations thus are appear in the following symbolic form :

$$(a / b / c) : (d / e / f)$$

Where:

a : is the arrival (or inter-arrival) distribution.

b : is the departure (or inter-departure or service time) distribution.

c : represents the number of parallel service channels in the system.

d : is the service discipline.

e : is the maximum number allowed in the system (in service + Waiting).

f : represents the calling source.

To illustrate the use the preceding notations, consider the model (M / M / c) : (FCFS / N / ∞). This model denotes that it have a Poisson arrival (exponential inter-arrival) distribution, Poisson departure (exponential inter-departure or service time) distribution, with (c) parallel servers, " first come first served" discipline, the maximum allowed number (N) in the system, and finally infinite calling source.

Transient and steady states & symbols:

The analysis of queuing theory involves the study of the system's behavior over time. The system is said to be in transient state when its operating characteristics or behavior is varying with time. It is usually occurring at the early stages of the system's operation, where its behavior is still dependent on the initial conditions. However, since one is mostly interested in the long run behavior, most attention in the queuing theory analysis has been reached to steady state result. A steady state condition is said to prevail when the behavior of the system becomes independent of time. In this text, the steady state analysis will be considered, although transient state solutions are available for some models. For simplicity, the system when the rate of the customer's arrival (λ) is less than the service rate (μ), then the steady state will be occurred.

For each model for the queuing models the reader is reminded that the queuing system is defined to include both queue and service channels. The following symbols will be used in connection with queuing models:

n = the number of customers in the system.

$P_n(t)$ = transient state probabilities of exactly (n) customers in the system at the time (t) assuming that the system started its operation at time equal to zero .

P_n = the steady state probabilities of exactly (n) customers in the system.

λ = the mean arrival rate ,i.e., the number of customers arriving per unit time.

μ = the mean service rate per busy server , i.e., the number of customers served per unit time.

c = the number of parallel servers.

$\rho = (\lambda / \mu)$ = the traffic intensity .

ρ/c = the utilization factor for ($c \geq 1$) service facilities.

W(t) = the probability density function (p.d.f) of waiting time in the system.

W_s = the expected waiting time per customer in the system.

W_q = the expected waiting time per customer in the queue.

L_s = the expected number of customers in the system.

L_q = the expected number of customers in the queue.

It can be proved under rather general conditions of arrival, departure, and service discipline the correctness of the following formulas:

$L_s = \lambda W_s \quad , \quad L_q = \lambda W_q$

$W_q = W_s - 1/\mu$, and $L_q = L_s - 1$ in the case of $c = 1$.

The steady state probabilities & the measures of effectiveness for the Poisson queuing Models(M/M/1): (FCFS/ ∞/∞):

The results of this section indicates that the given axioms lead to Poisson arrival (exponential inter-arrival time), and Poisson departure (exponential service time). These are the distributions representing Poisson queues.

The model (M/M/1): (FCFS/ ∞/∞) is called birth and death model. This model assumed that arrivals and departure are allowed simultaneously and there is only one server with queue length and calling source being unlimited.

In this section, we will show the different measures (without prove) in the case of the mean arrival rate (λ) is less than the mean service rate (μ) ,.i.e., the utilization factor (c = 1) is equal to: $\rho = \rho/c = \rho/1 = \rho = (\lambda / \mu) < 1$, then we have the following relations :

1-<u>The steady state Probabilities</u> according to the geometric distribution, which mean that the probability of that there are (n) customers in the system is equal to:

$$P(n) = \rho^n (1-\rho) = (\lambda / \mu)^n (1 - \lambda/\mu),$$

Where $n = 0, 1, 2, 3, \ldots\ldots, \infty$

2- <u>The expected number of customers in the system is :</u>

$$L_s = (\lambda / (\mu - \lambda)) .$$

3- <u>The expected number of customers in the queue is</u>:

$$L_q = (\lambda 2 / (\mu(\mu - \lambda)))$$

Or $= L_s - 1$

Or $= \rho \times L_s$

4- The expected waiting time per customer in the system is:

$$W_s = (1 / (\mu - \lambda))$$

5- The expected waiting time per customer in the queue is:

$$w_q = (\lambda / \mu(\mu - \lambda))$$

Or $$= w_s - (1/\mu)$$

Example (1):

If the arrival rate of customers at a banking counter follows a Poisson distribution with a mean of 30 customer per hour . and the service rate of the clerk also follows a Poisson distribution with mean of 45 customer per hour. Required :

1-Determine the characteristics of this queuing system?

2- Determine the first four steady state probabilities?

3- What is the probability of having zero customer in the system, or What is the probability that the service is idle or available?

4- What is the probability of having 8 customers in the system?

5- What is the probability of having 12 customers in the system?

6- Determine the four measures of effectiveness L_s , L_q , W_s

and w_q respectively?

Solution:

Given that the each of the arrival rate and the service rate having a Poisson distribution with (λ = 30 customer / hour) and (μ = 45 customer / hour).

Since, the value for the relative (λ / μ) = 30/45 = 0.67 < 1,

i.e., we have a steady state. Then we have the following:

1- In one line the characteristics of this queuing system are:

(M(λ=30) / M(μ=45) / 1): (FCFS/ ∞/∞) .

2- Finding the first four steady state probabilities: since,

$P(n) = \rho^n (1-\rho) = (\lambda/\mu)^n (1 - \lambda/\mu)$,

Where n = 0 , 1 , 2 ,3 ,……, ∞ , then:

$P(0) = (0.67)^0 (1 - 0.67) = 0.33$

$P(1) = (0.67)^1 (1- 0.67) = (0.67)^1 (1 - 0.67)$
$= 0.67 \times 0.33 = 0.2211 \approx 0.221$,

$P(2) = (0.67)^2 (1- 0.67) = (0.67)^2 (0.33)$
$= 0.4489 \times 0.33 = 0.148137 \approx 0.148$,

$P(3) = (0.67)^3 (1- 0.67) = (0.67)^3 (0.33)$
$= 0.300763 \times 0.33 = 0.09925179 \approx 0.099$,

Note that: the probability that the system is surely busy is the probability that there are at least one customer in the system = $P(x \geq 1) = 1- P(x< 1)$

$= 1 - P(x = 0) = 1 - 0.33 = 0.67$.

i.e., the system in this bank (counter) is considered busy along 67 % from its total time, and the service channel is idle (available or empty)or in other word the percentage of time that the customer will take the service without waiting is 33% .

3- The probability of having zero customer in the system, or the probability that the service channel is idle or available is equal to:

$P(0) = 0.33$, i.e., the percentage of time that the channel for the system is ideal is 33% .

4- the probability of having 8 customers in the

$$\text{System} = P(8) = (0.67)^{8} \; (1 - 0.67)$$

$$= (0.67)^{8} \, (0.33)$$

$$= 0.01340023$$

$$\approx 0.013$$

5- The probability of having 12 customers in the

$$\text{System} = P(12) = (0.67)^{12} \; (1 - 0.67)$$

$$= (0.67)^{12} \, (0.33)$$

$$= 0.000270097239 \approx 0$$

6- The four measures of effectiveness L_s , L_q , W_s

and w_q respectively are:

* $L_s = (\lambda / (\mu - \lambda)) = (30 / (45 - 30))$

$= 30 / 15 = 2$ customers.

* $L_q = (\lambda^2 / (\mu(\mu - \lambda))) = ((30)^2 / (45(45 - 30)))$

$= 900 / (45 \times 15) = 1.25 \approx 1$ customer in the queue.

Note that: we can find $L_q = L_s - 1$ customer in the service channel.

* $W_s = (1 / (\mu - \lambda)) = 1 / (45 - 30) = 1 / 15$ hour .

$= 60 \times (1 / 15) = 4$ minutes.

* $w_q = (\lambda / \mu(\mu - \lambda)) = 30 / (45(45 - 30))$

$= 30 / (45 \times 15) = 2 / 45$ hour.

$= 60 \times (2 / 45) = 2.667$ minutes .

Note that: we can find $w_q = W_s - (1/\mu) = 4 - 60(1/45)$

$= 4 - 4/3 = (12 - 4) / 3$

$= 8 / 3 = 2.667$ minutes.

Example (2):

At one man barber shop, customers arrive according to Poisson distribution with mean arrival rate of 5 person per hour, and the hair cut taking time was exponentially distributed with an average hair cutting 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:

1- The average number of customer in the shop, and the average numbers waiting for a haircut.

2- The percentage of time arrival can walk in straight without having to wait.

3- The percentage of time who have to wait before getting into the barber's chair.

Solution :

Given that:

* Customers arrival have a Poisson dist. With mean rate of 5 person, i.e., $\lambda = 5$ customer / hour.

* The hair cut taking time was exponentially distributed with an average hair cutting 10 minutes, i.e.,

($(1/\mu)$ = 10 minute / customer = (10/60) hour / customer = (1/6) hour / customer), i.e., the service rate have a Poisson distribution with mean rate μ = 6 customer / hour.

Therefore, we have the following queuing model:

$$(M_{(\lambda=5)} / M_{(\mu=6)} / 1): (FCFS/ \infty/\infty) .$$

Then, we have the following:

1- The average number of customer in the shop, and the average numbers waiting for a haircut are:

* The average number of customer in the shop = L_s , where:

$$L_s = (\lambda / (\mu - \lambda)) = (5 / (6 - 5))$$

$$= (5 /1) = 5 \text{ customers.}$$

And, the average numbers waiting for a haircut = L q

where: $L\,q = (\lambda^{2} / (\mu(\mu - \lambda))) = ((5) 2 / (6(6 – 5)))$

$= 25 / (6\times 1) = 4.25 \approx 4$ customer in the queue.

2- The percentage of time arrival can walk in straight without having to wait, i.e., the service is available or idle $= P(n) = \rho^{n} (1- \rho)$ for n = 0.

$= (\lambda / \mu)^{n} (1 – \lambda/\mu)$, when n = 0. i.e.,

$P(0) = (5/6)0 (1 – 5/6) = (0.833)^{0} (1 – 0.833)$

$= 1 \times 0.167 = 0.167 = 16.7$ %

3- The percentage of time who have to wait before getting into the barber's chair $= P(n \geq 1) = 1 – P(n < 1) = 1 – P(n = 0)$.

$= 1 – 0.167 = 0.833 = 83.3$ %

Example(3) :

Vehicles are passing through a toll gate at the rate of 70 car or vehicle per hour. And , the average time for passing the car through the gate is 45 seconds. Each of the arrival rate and the service rate follow a Poisson distribution. There is a complaint that the vehicles wait for a long duration. Determine :

1- The characteristics of this queuing model in one line?

2- The first three steady state probabilities?

3- The measures of effectiveness?

Solution :

1- In order to determine the characteristics of this queuing model in one line, firstly we have to determine the mean rate for each of the arrival and departure (or service) distributions.

* The mean rate for the arrival distribution(Poisson) with parameter $\lambda = 70$ vehicle / hour.

* The mean rate for the service time distribution (Exponential distribution) with parameter $1/\mu = 45$ seconds / vehicle, i.e., the mean rate for the departure or service vehicles in one hour is:

$\mu = (1/45) \times 60 \times 60 = (3600/45) = 80$ vehicles / hour , i.e., the probability distribution for the departure or service vehicles in one hour is a Poisson distribution with mean rate $\mu = 80$ vehicles per hour. Then , the characteristics of this queuing model in one line is as follows :

$$(M_{(\lambda=70)} / M_{(\mu=80)} / 1): (FCFS/ \infty/\infty) .$$

2- Since, $\lambda = 70$, $\mu = 80$,i.e., $(\lambda = 70) < (\mu = 80)$. Then we have a steady state probability. Therefore, we have the following probabilities:

$$P(n) = \rho^{n} (1- \rho) = (\lambda / \mu)^{n} (1 – \lambda/\mu) ,$$

Where $n = 0 , 1 , 2 ,3 ,……, \infty$, then:

$$P(0) = (70/80)^{0} (1 – 70/80) = (0.875)^{0} (1 – 0.875) = 0.125$$

$$P(1) = (0.875)^{1} (1- 0.875) = (0.875)^{1} (0.125)$$

$$= 0.109375 \approx 0.109$$

$P(2) = (0.875)^2 \; (1- 0.875) = (0.875)^2 \; (0.125)$

$= 0.765625 \times 0.125 = 0.095703125 \approx 0.096$,

3- The measures of effectiveness L_s , L_q , W_s

and w_q respectively are:

* $L_s = (\lambda / (\mu - \lambda)) = (70 / (80 - 70)) = 70 / 10 = 7$ vehicle.

* $L_q = (\lambda^2 / (\mu(\mu - \lambda))) = ((70)^2 / (80(80 - 70)))$

$= 4900 / (80 \times 10) = 6.125 \approx 6$ vehicle in the queue.

Note that: we can find $L_q = L_s - 1$ vehicle in the queue ,i.e., waited in the service channel (toll gate), i.e.,

$L_q = L_s - 1 = 7 - 1 = 6$ vehicle in the queue or waited in the service channel (toll gate). Which is the same preceding result.

* $W_s = (1 / (\mu - \lambda)) = 1 / (80 - 70) = (1 / 10)$ hour .

$= 60 \times (1 / 10) = 6$ minutes, and,

* $w_q = (\lambda / \mu(\mu - \lambda)) = 70 / (80(80 - 70))$

$= 70 / (80 \times 10) = (7/80)$ hour.

$= 60 \times (7/80) = 5.25$ minutes .

Note that: we can find $w_q = W_s - (1/\mu) = 6 - 60(1/80)$

$= 6 - 0.75 = 5.25$ minutes.

Which is the same preceding result.

Chapter (4)

"Transportation Problem"

The transportation problem is considered a special case from the linear programming model(LPM). The transportation model seeks the minimization of the total transportation costs resulted from transporting a specific homogeneous commodity from a set sources to a set of several destinations. Where the supply of each source and the demand for each destination are known. For example, a product may be transported from factories(sources) to retail stores(destinations). Although the transportation problem can be solved by the regular simplex techniques , but its special properties offer a more convenient solution procedure. The new procedure may appear different, but it can be explained directly in terms of the simplex techniques.

In this chapter, we will deal with how can we formulate the transportation table or model as a Linear Programming Model(LPM), finding the initial solution tableau by using the different techniques, and finally how can we search about of the optimality of this techniques by using the test of optimality techniques.

We will introduce a set of examples by which served this

Therefore, we will have concerned about the following:

1- Formulation the transportation table or model as a

Linear Programming Model(LPM).

2- Finding the initial or starting Basic Feasible solution tableau by using the following different techniques:

* The Northwest Corner Method.
* The Least(or Ascending) Cost (Maximum Profit) Method.
* Vogel's Approximation Method(VAM).

3- Tests of optimality for any basic feasible solution by

Using either of the following two methods:

* The Stepping Stone Method.
* The Method of Multipliers (or The Modified Distribution Method).

Example (1):

The following transportation table represents the unit transportation cost C_{ij} where i = 1:3, and j = 1:3 in dollars for a specific homogeneous commodity from a set of three sources A , B and C to a set three customers (1) , (2) , and (3). The supply and demand units from this commodity are given in this table:

Destination / Sources	(1)	(2)	(3)	Supply(S_i)
A	24		4	(90)
B	8	16	8	(60)
C	4	4	16	(30)
Demand(D_j)	(45)	(75)	(60)	(180)

Required:

1- Formulate the transportation problem as a Linear Programming Model(LPM).

2- Compare between the total transported costs resulted from the three different methods for determining the starting basic feasible solution.

Solution :

1- Before formulating the transportation problem as a linear programming model, we have to be sure that the transportation problem is in its balance case, i.e., the summation for the supply of commodity units for the set of sources or stores (S_i's) must be equal to the summation for the demand of commodity units for the set of destinations or customers (D_j's) ,i.e.,

$\sum S_i = \sum D_j$ for i = 1,2,3,……,n and

j = 1,2,3, ….. , m . Now, since we have:

$\sum S_i = 90 + 60 + 30 = 180$,

$\sum D_j = 45 + 75 + 60 = 180$,

i.e., the transportation problem is in its balance case.

Therefore, in order to formulate the transportation problem as a linear programming model, we have to suppose a set of decision variables x_{ij} where

i = 1: 3 , and J = 1:3 .

Then, we have to find x_{ij} for i = 1: 3 , and J = 1:3 by which :

The total transported costs:

$$T(c) = 4x_{11} + 8x_{12} + 24x_{13}$$

$$+ 8x_{21} + 4x_{22} + 8x_{23}$$

$$+ 16x_{31} + 4x_{32} + 16x_{33} \qquad (\text{ Minimization})$$

<u>Subject to</u>:

1- <u>Balance Constraint</u>:

$$\sum S_i = \sum D_j = 180$$

2- <u>Row(Supply) constraints</u>:

$$x_{11} + x_{12} + x_{13} = 90$$

$$x_{21} + x_{22} + x_{23} = 60$$

$$x_{31} + x_{32} + x_{33} = 30$$

3- <u>Column(Demand) constraints</u>:

$$x_{11} + x_{21} + x_{31} = 45$$

$$x_{12} + x_{22} + x_{32} = 75$$

$$x_{13} + x_{23} + x_{33} = 60$$

4- <u>Non-negativity constraints</u>:

$$x_{ij} \geq 0 \quad \text{for } i= 1 : 3 \quad \text{and } j = 1 : 3$$

2- The following tables represent the calculation for the total transported costs resulted from the three different methods for determining the starting basic feasible solution:

* <u>The initial solution tableau for the Northwest Corner Method</u>:

Destination / Sources	(1)	(2)	(3)	S_i
A	(45) 24	(45) 8	4	(90)
B	8	(30) 16	(30) 8	(60)
C	4	4	(30) 16	(30)
D_j	(45)	(75)	(60)	180

The total transported cost for the preceding initial solution tableau is equal to :

$T(c) = 24 \times 45 + 8 \times 45 + 16 \times 30 + 8 \times 30 + 16 \times 3$

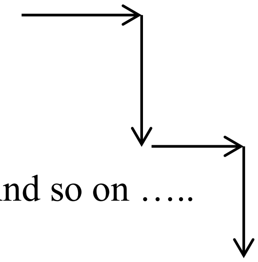
$= 2640$ dollar.

<u>Note that</u>:

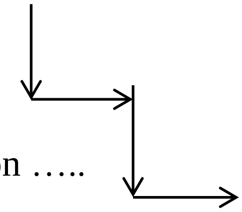
1- The solution in the preceding table is considered a basic feasible solution since the number of basic variable (occupied cells) include: (m + n – 1) =

3 + 3 – 1 = 5 basic variables, where (m) is the number of sources (stores A , B , C) and(n) is the number of destinations (customers (1) , (2) , (3)).

2- The occupied cells in the preceding tableau lies in a step function as follows:



And so on …..



And so on …..

3- If the preceding problem from the viewpoint of the transshipment company , then we have to maximize the total profits resulted from transporting the commodity units from the set of sources to the set of destinations. In this case the total profits which is resulted from the northwest corner is equal to the same total cost for that problem since this technique started by occupying the first cell which lies in the 1st row and 1 st column (i.e., the north west corner) , and so on…….., and then the total profits will equal to the total transported cost = 2640$ also.

* <u>The initial solution tableau for the Least (or Ascending) Cost Method</u> :

Destination / Sources	(1)	(2)	(3)	S_i
A	24	(30) 8	(60) 4	(90)
B	(15) 8	(45) 16	8	(60)
C	(30) 4	4	16	(30)
D_j	(45)	(75)	(60)	180

The total transported cost for the preceding initial solution tableau is equal to :

T(c) = 8 × 30 + 4 × 60 + 8 × 15 + 16 × 45 + 4 × 30

= 1440 dollar.

Also , Note that the solution in the preceding table is considered a basic feasible solution, since the number of basic variable (occupied cells) include: m + n – 1 = 3 + 3 – 1 = 5 basic variables, where (m) is the number of sources (stores A , B , C) and(n) is the number of destinations (customers (1) , (2) , (3)).

* <u>The initial solution tableau for the Vogel's Approximations Method</u> :

	(1)	(2)	(3)	Sup.	d1	d2	d3	T.C
A	24	8 (30)	4 (60)	(90)	4	16	-	480
B	8 (45)	16(15)	8	(60)	0	8	8	600
C	4	4 (30)	16	(30)	0	0	0	120
Dem.	(45)	(75)	(60)	(180)				1200
d1	4	4	4					
d2	4	4	-					
d3	4	12	-					

Note that the solution in the preceding table is considered a basic feasible solution since the number of basic variable

(occupied cells) include: m + n – 1 = 3 + 3 – 1 = 5 basic variables, where (m) is the number of sources (stores A , B , C) and(n) is the number of destinations (customers (1) , (2) , (3)).

And, the total transported cost for the preceding initial solution tableau as it mentioned above in the last column in the preceding tableau is equal to :

$T(c) = 8 \times 30 + 4 \times 60 + 8 \times 45 + 16 \times 15 + 4 \times 30$

$= 8(30) + 4(60) + 8(45) + 16(15) + 4(30)$

$= 1200$ dollar.

Therefore, by comparing the total transported cost (2640 $, 1440 $, 1200$ respectively), i.e., we find that the 3rd technique (Vogel's approximation technique) is the lowest total transported cost .

Example (2):

A company owns four different factories A_1, A_2, A_3 and A_4 producing a homogeneous commodity which is sold in three different destinations B_1 , B_2 and B_3 .The following table represents the number of unit produced in each factory , the number of demand units for each destination and the unit transported cost(L.E) for each one from the commodity units transported from the set of factories to the set of destinations :

Factory	Production units	Profit unit		
		B1	B2	B3
A1	(340)	10	7	8
A2	(550)	10	11	14
A3	(660)	9	12	4
A4	(230)	11	13	9
Demand Units		(320)	(660)	(250)

Required :

1) : Formulate the transportation problem as a linear programming model (LPM).
2) : Determine the total transported cost resulted from the initial solution for Vogel's Approximation Technique.

3) : From your preceding results in (2) determine the optimum solution for this transportation problem by using the stepping stone method.

Solution :

(1): Before formulating the transportation problem as a linear programming model, we have to be sure that the transportation problem is in its balance case, i.e., the summation for the supply of commodity units for the set of sources or stores (S_i's) must be equal to the summation for the demand of commodity units for the set of destinations or customers (D_j's) ,i.e.,

$\sum S_i = \sum D_j$ for i = 1,2,3,……,n and

j = 1,2,3, ….. , m . Now, since we have:

$\sum S_i = 340 + 550 + 660 + 230 = 1780$ commodity unit,

$\sum D_j = 320 + 660 + 250 = 1230$ commodity unit,

i.e., the transportation problem is not in its balance case. And there are a surplus of the total supply of the commodity units.

Then , we have to add a dummy destination(B_4) with zero cost unit for each cell in this dummy destination.

Therefore, in order to formulate the transportation problem as a linear programming model, we have to suppose a set of decision variables x_{ij}'s where i = 1: 3 , and J = 1:4. Then, the table for the transportation problem becomes as follows:

A_i \ B_j	B_1	B_2	B_3	B_4	S_i
A_1	10	7	8	0	(340)
A_2	10	11	14	0	(550)
A_3	9	12	4	0	(660)
A_4	11	13	9	0	(230)
D_j	(320)	(660)	(250)	(550)	(1780)

Then, we have to find x_{ij}'s for i = 1: 3 , and J = 1:4 by which :

The total transported costs:

$$
\begin{aligned}
T.C = {} & 10\,x_{11} + 7\,x_{12} + 8\,x_{13} + 0\,x_{14} \\
& +10\,x_{21} + 11\,x_{22} + 14\,x_{23} + 0\,x_{24} \\
& + 9\,x_{31} + 12\,x_{32} + 4\,x_{33} + 0\,x_{34} \\
& + 11\,x_{41} + 13\,x_{42} + 9\,x_{43} + 0\,x_{44} \quad \text{(Minimization)}
\end{aligned}
$$

Subject to :

(1): Balance constraint:

$\sum S_i = \sum D_j = 1780$ for i = 1 , 2 , 3 , 4, and

j = 1 , 2 , 3 , 4

(2): The set of supply(Row) constraints:

$$x_{11} + x_{12} + x_{13} + x_{14} = 340$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 550$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 660$$

$$x_{41} + x_{42} + x_{43} + x_{44} = 230$$

3- <u>The set of Demand(Column) constraints</u>:

$$x_{11} + x_{21} + x_{31} + x_{41} = 320$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 660$$

$$x_{13} + x_{23} + x_{33} + x_{43} = 250$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 250$$

4- <u>Non-negativity constraints</u>:

$x_{ij} \geq 0$ for i= 1 : 4 and j = 1 : 4

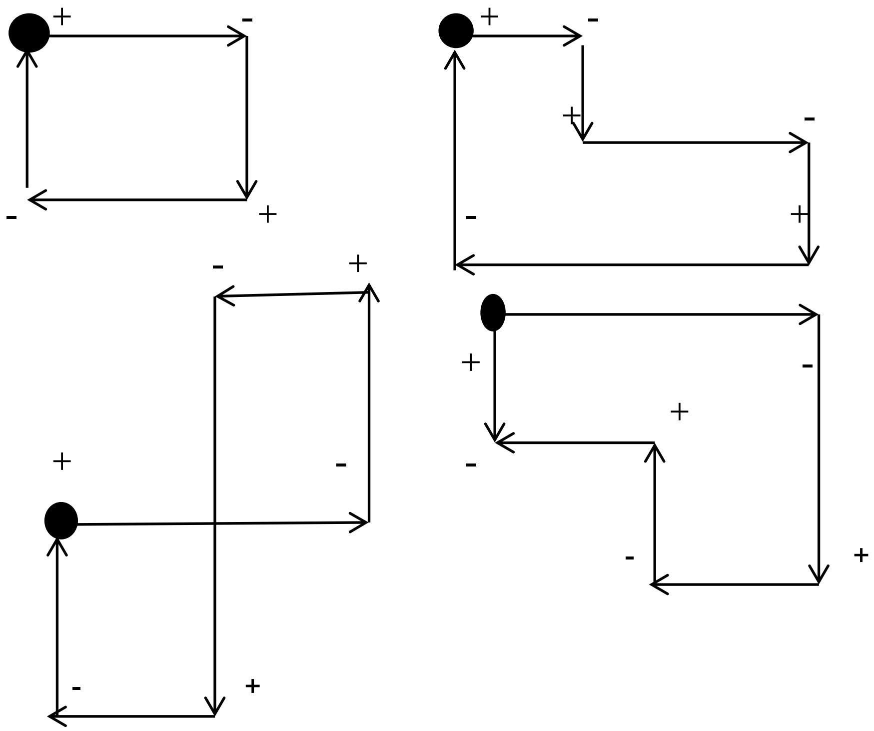
(2): : Determination the total transported cost resulted from the initial solution for Vogel's Approximation Technique:

	B_1	B_2	B_3	B_4	S_i	d_1	d_2	d_3	d_4	T.C
A_1	10	7 (340)	8	0	340	1	3	-	-	2380
A_2	10	11 (320)	14	0	550	1	1	1	-	3520
A_3	9 (320)	12	4 (250)	0 (90)	660	5	3	3	-	3880
A_4	11	13	9	0 (230)	230	2	2	2	-	0
D_j	(320)	(660)	(250)	(550)	1780					9780
d_1	1	4	4	-						
d_2	1	4	-	-						
d_3	1	1	-	-						
d_4	-	-	-	-						

Note that, the preceding initial solution tableau is considered a basic feasible solution since it contains a number of basic variable (occupied cells) is equal to:

(n + m – 1 = 4 + 4 – 1 = 7) basic variables with total transported cost is equal to (9780) L.E.

(3): Now, From the preceding results in (2), we will determine the optimum solution for this transportation problem by using the stepping stone method. The Stepping Stone technique search about optimality by evaluating the none-basic variables (empty cells) by calculating what is the effect in the transportation total cost if we thought to occupying any empty cell by one unit from the commodity units. This will be calculated by determining the closed path for each one of the empty cells. The closed path for each none-basic variable (empty cell) must contains a set of even number of basic variable, where the empty cell will have increased by one commodity unit and a half of these even basic cells will decreasing by one of commodity unit and the another half of these basic variable will be increasing by one commodity unit. The closed path for each none-basic variable may take one of the following diagrams:



The following table(1\ ) represents the evaluation for each none-basic variable(empty cell):

Table (1\)

Empty cell (non-basic cells)	The value of change in the total transportation costs for the closed path to the non-basic cell	Evidence Improvement.
A_1B_1	+10 – 9+ 0 – 0 + 11 – 7 = 5	-5
A_1B_3	+ 8 – 7 + 11 – 0 + 0 – 4 = 8	- 8
A_1B_4	+ 0 – 7 + 11 -0 = 4	- 4
A_2B_1	+ 10 – 9 + 0 – 0 = 1	- 1
A_2B_3	+ 14 – 4 + 0 – 0 = 10	- 10
A_3B_2	+ 12 – 0 + 0 – 11 = 1	- 1
A_4B_1	+ 11 – 0 + 0 – 9 = 2	- 2
A_4B_2	+ 13 – 0 + 0 – 11 = 2	- 2
A_4B_3	+ 9 – 0 + 0 – 4 = 5	- 5

Now, from the latest evaluation table (table (1\), note that each value of the evidence improvements (latest column) is negative coefficient, and since we desired to minimize the total transported costs (the same optimality conditions in the simplex techniques (negative coefficients in case of minimizing the value of the objective function in the LPM)). Therefore the

initial solution tableau (table (1) resulted from Vogel's Approximation Technique) is considered a unique optimal solution. Henceforth the optimum solution for the preceding transportation problem is:

$X^*_{11} = 0$, $x^*_{12} = 340$, $x^*_{13} = 0$, $x^*_{14} = 0$,

$X^*_{21} = 0$, $x^*_{23} = 0$, $x^*_{23} = 0$, $x^*_{24} = 230$,

$X^*_{31} = 320$, $x^*_{32} = 0$, $x^*_{33} = 250$, $x^*_{34} = 90$,

$X^*_{41} = 0$, $x^*_{42} = 0$, $x^*_{43} = 0$, $x^*_{44} = 230$.

And the minimum optimum transportation cost is equal to:

$(T.C^*) = 9780$ (L.E.)

Example (3):

The following table represents the initial solution tableau for transporting a specific commodity from three factories (A_1 , A_2 , A_3) to three stores(B_1 , B_2 ,B_3)by using the least cost technique :

S_i \ D_j	B_1	B_2	B_3
A_1	3	5 (200)	4 (1300)
A_2	2 (1800)	6	8 (700)
A_3	7	1 (1000)	9

Formulate the transportation problem as a LPM, and compare between the total transported costs for the Least Cost Technique and the Vogel's approximation technique for finding

the initial solution and then test about optimality for the initial solution tableau for the table of the Vogel's Approximation tableau by using the Modified Distribution Technique (Simplex Multipliers) as a test of optimality.

Solution:

* Formulation the transportation problem as a LPM:

Note that, since the units of transported commodity are distributed in the preceding table by using the Least Cost Technique, hence the equilibrium (balance) constraint is satisfied. Henceforth, in order to formulate the transportation problem as a LPM, we suppose the set of the decision variables x_{ij}'s represent the number of commodity units that transported from the i^{th} factory to the j^{th} store, where i = 1 : 3 and j = 1 : 3. Then, we have to find x_{ij}'s by which make the total transported cost (T.C) is equal to :

$$T.C = 3x_{11} + 5x_{12} + 4x_{13}$$
$$+2x_{21} + 6x_{22} + 8x_{23}$$
$$+7x_{31} + 1x_{32} + 9x_{33} \quad \text{(Minimization)}$$

Subject to :

(1): <u>Balance constraint</u>:

$\sum S_i = \sum D_j = 5000$ for i = 1 , 2 , 3 and

j = 1 , 2 , 3 .

(2): <u>The set of supply(Row) constraints:</u>

$$x_{11} + x_{12} + x_{13} = 1500$$

$$x_{21} + x_{22} + x_{23} = 2500$$

$$x_{31} + x_{32} + x_{33} = 1000$$

3- <u>The set of Demand(Column) constraints</u>:

$$x_{11} + x_{21} + x_{31} = 1800$$

$$x_{12} + x_{22} + x_{32} = 1200$$

$x_{13} + x_{23} + x_{33} = 2000$

4- Non-negativity constraints:

$x_{ij} \geq 0$ for $i = 1 : 3$ and $j = 1 : 3$

* In order to compare between the total transported costs for each of the Least cost and Vogel's approximation techniques , firstly , let us calculate the total transported cost for the least (or Ascending) cost technique as it be shown in the following table:

Table(1)

f \ st	B_1	B_2	B_3	S_i	T.C
A_1	3	5 (200)	4 (1300)	(1500)	6200
A_2	2 (1800)	6	8 (700)	(2500)	9200
A_3	7	1 (1000)	9	(1000)	1000
D_j	(1800)	(1200)	(2000)	(5000)	16400

* Determining the initial solution tableau and the total costs for the problem by using Vogel's Approximation Technique: the following table represents the initial solution tableau for the problem:

Table(2)

	B_1	B_2	B_3	S_i	d_1	d_2	d_3	T.C
A_1	3	5	4 (1500)	(1500)	1	1	1	6000
A_2	2 (1800)	6 (200)	8 (500)	(2500)	4	4	2	8800
A_3	7	1 (1000)	9	(1000)	6	-	-	1000
D_j	(1800)	(1200)	(2000)	(5000)				15800
d_1	1	4	4					
d_2	1	1	4					
d_3	-	1	4					

*Note that, each of the preceding two initial solution tableaus (1,2) are consider basic feasible solutions، since the number of basic variables (occupied cells) is equal to n + m – 1= 3 + 3 -1 = 5

* If we compare between the total transported costs for the two different techniques for the initial solution tableaus, we find that:

The total transported costs for Vogel's approximation technique (15800 L.E.) is less than the total transported costs for the Least cost technique (16400 L. E.).

Now, in order to test about the optimality for the initial solution tableau to the Vogel's approximation technique tableau by using the Modified distribution technique (Simplex multipliers), then, from the basic variables (occupied cells in table (2) = 5), we have the following five basic linear functions:

$R_1 + c_3 = 4$,

$R_2 + c_1 = 2$,

$R_2 + c_2 = 6$,

$R_3 + c_3 = 8$,

$R_3 + c_2 = 1$.

Suppose that $r_1 = 0$, then we can solve the preceding basic linear functions and determine the following multipliers as follows:

$R_1 = 0 \qquad c_3 = 4$,

$r_2 = 9 \qquad , c_2 = -3$,

$r_3 = 4 \qquad c_1 = -7$.

and then we can evaluate the coefficient for the non-basic variables (empty cells in table (2)) or in another meaning the simplex multipliers for the non-basic variable (Evidence Improvement $E_{ij} = r_i + c_j - c_{ij}$, where, ri and cj are the coefficients for the I $\underline{th}$ row and the j $\underline{th}$ column respectively, and the transported cost for the (ij)$\underline{th}$ cell.

The following table ($2^{\prime}$) represents the Simplex multipliers (Evidence Improvement) for the non-basic variable(empty cells) as follows:

Table($2^{\prime}$)

Empty cell	Evidence Improvement for the(ij)$^{\underline{th}}$ cell: $E_{ij} = r_i + c_j - c_{ij}$	Remark
A_1B_1	$E_{11}=r_1+c_1-c_{11}= 0 +(-7) - 3= -10$	-
A_1B_2	$E_{12}=r_1+c_2-c_{12}= 0 +(-3) - 5 = -8$	-
A_3B_1	$E_{31}=r_3 +c_1-c_{31}= 4 +(-7) – 7 = -14$	-
A_3B_3	$E_{33}= r_3+c_3-c_{33}= 4 +4 – 9 = -1$	-

Now, since all the coefficients for the evidence improvements (simplex multipliers (optimality conditions in the different methods for the simplex methods)) are negative, and we have to minimize the total transported costs, therefore, the initial solution tableau (Table(2) is a unique optimum solution, i.e., the number of commodity units transported from the i $^{\underline{th}}$ factory to the j $^{\underline{th}}$ store and the minimum transportation costs are as follows:

$x^*_{11} = x^*_{12} = 0$, $x^*_{13} = 1500$, $x^*_{21} = 1800$, $x^*_{22} = 200$, $x^*_{23} = 500$, $x^*_{31} = 0$, $x^*_{32} = 1000$, $x^*_{33} = 0$ and $T.C^* = 15800$ L.E.

<u>Remarks</u> :

1- In case of determining the optimum solution for the model of minimizing the total transported costs, if at least one value for the evidence improvement (Simplex Multipliers) is positive coefficient, then the solution by which we evaluate is considered non optimum. Then, in order to improve this solution, we select the non-basic variable (empty cell) which have the highest positive multiplier (or evidence improvement (E_{ij})) and determine the closed path for its non-basic cell. Then , we have to determine the minimum number of the commodity units which have a negative sign in the closed path(basic cells) for this non-basic cell , then, added and subtract this number from the basic variable in the closed path. Then, the total transported costs must decreased by the value of(Eij × the number of commodity units by which this cell will be occupied).

2- If we have at least two non-basic cells have the highest positive number for the values of the evidence improvement

(E_{ij}) are equal , then, we have to select the non-basic cell(entering variable in the LPM) which have the minimum transportation cost unite. Even though or additionally , if we have at least two non-basic cells have the highest positive number for the values of the evidence improvement (E_{ij}) are

equal , and they have the same minimum transportation cost unite , then, we have to select the non-basic cell which will be occupied with a high density of commodity units.

3- From the view point for the transshipment companies, then, in the formulation for the transportation problem as a LPM, then the objective function for the transportation problem becomes maximize the total transportation profits. In addition, to determine the initial solution tableaus by using the three different techniques note that:

*The results for the Northwest Corner Technique for finding the initial solution tableau must be the same results for either of the objective is to maximizing the total transported profits or

minimizing the total transportation costs, i.e., the results in the two cases will not be differ, i.e., the minimum transportation costs must be equal the maximum transportation profits without any difference in the two cases {(Max)or (Min) in the objective function}.

* On contrary, the results for the two another technique for finding the initial solution tableau will be different, since the objective will be differing, i.e., the Least Cost technique in case of minimizing the total transported costs is corresponding to the maximum profit technique in case of maximizing the total transportation profits. where, in the Maximum Profit Technique, for finding the initial solution tableau, we will search in the transportation table about the cell(variable) which have the maximum profit unit, and in the Vogel's Approximation Technique we will find the difference between the two highest profit unit.

Example (4):

A specific company had four factories A_1 , A_2, A_3 , A_4 in different areas for producing a homogeneous commodity. And the production for these factories was sailed in three marketing destinations B_1 , B_2 , B_3 respectively. The following table represents the number of supply units(S_i's) for the set of factories, the number of demand units($D_{j's}$) for the set of destinations, the price for each of commodity unit in each destination (P_j's in L.E.), the unit produced cost($C_{i's}$ in L.E.) for each commodity unit produced in the factory (i), and finally the cost of unit transported commodity(C_{ij}'s) which produced in the i $^{\underline{th}}$ factory and sailed in the j $^{\underline{th}}$ destination where i= 1: 4 and j=1:3.

Factory	Supply units(Si)	Cost of produced unit (ci)	Unit transported cost (T_{ij} 's)		
			B1	B2	B3
A1	2000	3	0.5	0.4	0
A2	3600	3.2	0	0.6	0.2
A3	1800	2.8	0.5	0	0.6
A4	4000	2.9	0.2	0.5	0.2
Demand units(Dj)			3000	4400	3800
Price unit(Pi)			2	6	4

If The company must be transport the demand of commodity units from the four factories to the three destinations.

Required:

1- Formulate the problem as a LPM.

2- Compare between the results for the three different techniques which determined the initial solution tableau for the transportation problem.

3- From your preceding results in (2), specifically from the initial solution tableau for the Descending profit unit, determine the optimum solution by using the stepping stone technique.

Solution:

Before to answer about any requirements, we have to be sure that the equilibrium constrained satisfied as follows:

Since, $\sum$ Si =2000 +3600 + 1800 + 4000 = 11400 commodity units, and $\sum$ Dj = 3000 + 4400 + 3800 = 11200 commodity units, i.e., there is a surplus from the supply units, i.e., in order to satisfied the equilibrium condition or constraint, we have to added a dummy column with zero L.E. loss or profit for transporting each of commodity unit from the set of different factories to this dummy marketing destination. Therefore, the dimension for the transportation table becomes 4 × 4 ,i.e., we will suppose a set of 16 decision variables. In addition, this

problem contains a mix of the price sold unit (as a revenue) and the two different costs (production & transportation) parameters. Then, we have to find the profit (or loss) for each of produced commodity unit in the $i^{\underline{th}}$ factory and transported to the $j^{\underline{th}}$ marketing destination. Then we have to calculate the profit (or loss) unit as follows:

Unit profit (or loss) = Price(P_j) – (the sum of produced and transported cost

($C_i + C_{ij}$). Then, we have the following:

* The Unit profit for the $(ij)^{th}$ cell is equal to:

{ (P_{ij}'s)(or loss L_{ij}'s) } = Price – (sum of the total production and

transportation costs).

i.e.,

$P_{12} = 6 – (3 + 0.4) = 2.6$,

$P_{13} = 4 – (3 + 0) = 1.0$,

$P_{21} = 6 - (3.2 + 0) = 1.8$,

$P_{22} = 6 – (3.2 + 0.6) = 2.2$,

$P_{23} = 4 – (3.2 + 0.2) = 0.6$,

$P_{31} = 5 – (2.8 + 0.5) = 1.7$,

$P_{32} = 6 – (2.8 + 0) = 3.2$,……. And so on for the residual cells.

And, since there are at least one cell achieve profit, therefore the main objective for this problem is to maximize the total transported profit resulted from the following transportation table:

	B_1	B_2	B_3	B_4	S_i
A_1	1.5	2.6	1.0	0.0	(2000)
A_2	1.8	2.2	0.6	0.0	(3600)
A_3	1.7	3.2	0.6	0.0	(1800)
A_4	1.9	2.6	0.9	0.0	(4000)
D_j	(3000)	(4400)	(3800)	(200)	(11400)

Then in order to formulate the preceding transportation table as a LPM, then we suppose that x_{ij} represent the number of commodity units transported from the source (i) to the

destination (j), then we have the following LPM where i= 1:4 and j= 1:4 :

Find the values of x_{ij} by which make the function of the total profits (T.P) as an objective function as follows:

$$T.P = 1.5\, x_{11} + 2.6\, x_{12} + 1\, x_{13} + 0.0\, x_{14}$$

$$+ 1.8\, x_{21} + 2.2\, x_{22} + 0.6\, x_{23} + 0.0\, x_{24}$$

$$+ 1.7\, x_{31} + 3.2\, x_{32} + 0.6\, x_{33} + 0.0\, x_{34}$$

$+ 1.9\, x_{41} + 2.6\, x_{42} + 0.9\, x_{43} + 0.0 x_{44}$ (Maximization)

Subject to:

(1): Equilibrium constraint:

$\sum Si = \sum Dj$ for i= 1:4 and j = 1:4

(2): The set of supply(Row) constraints:

$$x_{11} + x_{12} + x_{13} + x_{14} = 2000$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 3600$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 1800$$

$$x_{41} + x_{42} + x_{43} + x_{44} = 4000$$

3- The set of Demand(Column) constraints:

$$x_{11} + x_{21} + x_{31} + x_{41} = 3000$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 4400$$

$$x_{13} + x_{23} + x_{33} + x_{43} = 38000$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 200$$

4- Non-negativity constraints:

$x_{ij} \geq 0$ for i= 1 : 4 and j = 1 : 4

2- In order to Compare between the results for the three different techniques which determined the initial solution tableau for the transportation problem, let's compute the total transportation profits resulted from the three different techniques shown in the following three initial solution tableaus as follows:

* The initial solution tableau for the Northwest Corner Method:

The following table represents the calculation for the total transportation profits for the Northwest Corner Method:

Table (1)

	B_1	B_2	B_3	B_4	S_i
A_1	1.5 (2000)	2.6	1.0	0.0	(2000)
A_2	1.8 (1000)	2.2 (2600)	0.6	0.0	(3600)
A_3	1.7	3.2 (1800)	0.6	0.0	(1800)
A_4	1.9	2.6 (0)	0.9 (3800)	0.0 (200)	(4000)
D_j	(3000)	(4400)	(3800)	(200)	(11400)

Then, the total transportation profits resulted from the Northwest Corner is equal to:

T.P = 1.5 (2000) + 1.8 (1000) + 2.2 (2600) + 3.2 (1800) +

$+ 2.6 (0) + 0.9 (3800) + 0.0 (200) = 19700$ (L.E.)

In the preceding table, note that we have the case of Degeneracy. We introduce the treatment for the case of Degeneracy by occupied one of the two non-basic variables (x_{32}) or (x_{42}) by zero commodity unit in order to make the step function exists. It is preferable to put the zero commodity unit in the cell which have the most or higher or largest unit profit, i.e., we have to put (x_{42}) = zero of commodity unit in order to make the initial solution tableau in this case be a basic feasible solution with a basic variable (occupied cells) = n + m – 1 = 7 .

* <u>The initial solution tableau for the maximum or Descending) Profits Method:</u>

According to this method we will search in the transportation table about the cell which have the maximum profit unit, then we compare between the supply units and the demand unit for this basic cell and occupied it by the minimum value for the supply or demand unit. If there are at least two basic cells have the same maximum profit unit, then the logical mathematic asserts that we select the basic cell which will occupied with the high density from the commodity units.

The following table represents the calculation for the total transportation profits for the biggest or maximum Profit Method:

Table (2)

	B_1	B_2	B_3	B_4	S_i
A_1	1.5	2.6	1.0 (2000)	0.0	(2000)
A_2	1.8 + (1600)	2.2	0.6 - (1800)	0.0 (200)	(3600)
A_3	1.7	3.2 (1800)	0.6	0.0	(1800)
A_4	1.9 - (1400)	2.6 (2600)	0.9 +	0.0	(4000)
D_j	(3000)	(4400)	(3800)	(200)	(11400)

Then, the total transportation profits resulted from the Maximum or <u>Descending Profits</u> <u>Method</u> is equal to:

T.P = 1.0 (2000) + 1.8 (1600) + 0.6(1800) + 0.0(200) + 3.2 (1800)

+ 1.9 (1400) + 2.6 (2600) = 21140 (L.E.)

Note that, the preceding initial solution tableau is considered a basic feasible solution since it contains a number of basic variable (occupied cells) is equal to:

(n + m – 1 = 4 + 4 – 1 = 7) basic variables with total transported Profit is equal to (21140) L.E.

* <u>The initial solution tableau for the transportation problem by using Vogel's Approximation Method</u>:

According to this method we will calculate the difference between the two highest values for the unit profit in each row and each column in the transportation table. Then, we determine the penalty row or penalty column which have the most difference between the two highest values for the unit profit in each row and each column, and determine the basic cell which have the maximum profit unit, then we compare between the supply units and the demand unit for this basic cell and occupied it by the minimum value for the supply or demand unit. If there are at least two penalty rows or column have the same highest difference, then the logical mathematic asserts that we select the penalty row or column which have a relatively advantage, i.e., we select the penalty row or column which have the highest maximum profit unit, if it is still as a saddle then we will select the penalty row or column by which its basic cell will occupied with the high density from the commodity units and so on…..

The following table represents the calculation for the total transportation profits for the Vogel's Approximation Method:

Table (3)

	B_1	B_2	B_3	B_4	S_i	d_1	d_2	d_3	d_4
A_1	1.5	2.6 (2000)	1	0	(2000)	1.1	1.1	-	-
A_2	1.8 (3000)	2.2	0.6 (400)	0 (200)	(3600)	0.4	0.4	0.4	1.2
A_3	1.7	3.2 (1800)	0.6	0	(1800)	1.5	-	-	-
A_4	1.9	2.6 (600)	0.9 (3400)	0	(4000)	0.7	0.7	0.7	1
D_j	(3000)	(4400)	(3800)	(200)	11400				
d_1	0.1	0.6	0.1						
d_2	0.1	0	0.1						
d_3	0.1	0.4	0.3						
d_4	0.1	-	0.3						

Note that, the preceding initial solution tableau stated in table (1) is considered a basic feasible solution since it contains a number of basic variable(occupied cells) is equal to: $(n + m – 1) = 4 + 4 – 1 = 7$ basic variables with total transported Profit is equal to:

$$T.P = 2.5(2000) + 1.8(3000) + 0.6(400) + 0(200) + 3.2(1800) + 2.6(600) + 0.9(3400) = 21220 \text{ (L.E.)}.$$

Now, by comparing the total transportation profits for the three different techniques {(19700 L.E.) , (21140 L.E.) , (21220 L.E.) respectively} , then, we conclude that the Vogel's Approximation Method is the best from the three techniques, since it have the largest total profits.

3- Now, From the preceding results in (2), we find that is the Vogel's Approximation method have highest total transportation profits and is considered the best method for determining the initial tableau.

Then, we will test the initial solution for the Descending profits method by using the stepping stone technique as a test of optimality. Let us rewrite table(2)which represents the initial solution tableau for the descending profit units in order to evaluate the non-basic cells in table(2) by finding the closed path for each non-basic cell in this table:

Table (2)

	B_1	B_2	B_3	B_4	S_i
A_1	1.5	2.6	1.0 (2000)	0.0	(2000)
A_2	1.8 + (1600)	2.2	0.6 - (1800)	0.0 (200)	(3600)
A_3	1.7	3.2 (1800)	0.6	0.0	(1800)
A_4	1.9 - (1400)	2.6 (2600)	0.9 +	0.0	(4000)
D_j	(3000)	(4400)	(3800)	(200)	(11400)

Then, we will introduced the evaluation for the non-basic cells as it be shown in table $(2^{\backslash})$:

Table ($2^{\backslash}$)

Empty cells (non-basic)	The value of change in the total Profit for the closed path	Evidence Improve.
A_1B_1	+ 1.5 - 1.8 + 0.6 – 1 = - 0.7	0.7
A_1B_2	+2.6- 2.6+1.9- 1.8+0.6- 1 = - 0.3	0.3
A_1B_4	+0 – 1 + 0.6 – 0 = - 0.4	0.4
A_2B_2	+ 2.2 - 1.8 + 1.9 – 2.6 = - 0.3	0.3
A_3B_1	+1.7 – 1.9 + 2.6 – 3.2 = - 0.8	0.8
A_3B_3	+0.6 – 0.6+1.8–1.9+2.6–3.2=- 0.7	0.7
A_3B_4	+0- 0+1.8-1.9+2.6-3.2 = - 0.7	0.7
A_4B_3	+0.9 – 0.6 + 1.8 – 1.9 = (0.2)	(- 0.2) Entering variable.
A_4B_4	+ 0 – 0 +1.8 – 1.9 = - 0.1	0.1

Now, since, not all the coefficients for the evidence improvements (simplex multipliers (the optimality conditions in the different simplex methods)) are positive coefficients, and we have to maximize the total transported profits, therefore, the initial solution tableau (Table(2) is not an optimum solution. Hence, by returning to the closed path to the non-basic variable (empty cell) which have the most negative evidence improvement (the cell A_4B_3) in table(2). Now, from the closed path for the non-basic sell A_4B_3 as a simple closed path as it be shown in table (2). The closed path for this cell represent that there are two basic cells will decreased by the min(1400 , 1800) = 1400 commodity units. Hence , the total profits will increased by :

(0.2 × 1400 = 280 (L.E.)

And, according to the total changes in the overall cells on the closed path only, then, we have the following table:

Table (4)

	B_1	B_2	B_3	B_4	S_i
A_1	1.5	2.6	1.0 (2000)	0.0	(2000)
A_2	1.8 (3000)	2.2	0.6 (400)	0.0 (200)	(3600)
A_3	1.7	3.2 (1800)	0.6	0.0	(1800)
A_4	1.9	2.6 (2600)	0.9 (1400)	0.0	(4000)
D_j	(3000)	(4400)	(3800)	(200)	(11400)

Note that, the total transportation profit for the latest table is equal to:

$$T.P = 1.0(2000) + 1.8(3000) + 0.6(400) + 0.0(200) + 3.2(1800)$$

$$+ 2.6(2600) + 0.9(1400) = 21420 \quad (\text{L.E.})$$

Note that the value of increasing in the total transportation profits (after the improvement – before the improvement) must be equal to the inner product for the Evidence Improvement for the cell by which we improve the solution by using it × number of commodity units which is occupied to that cell, i.e., :

21420 – 21140 = 0.2 × 1400 = 280 (L.E.)

Then, we will introduced the evaluation for the non-basic cells stated in table (4) as it be shown in table (4\) :

Table(4\)

Empty cells (non-basic)	The value of change in the total Profit for the closed path	Evidence Improve.
A_1B_1	+ 1.5 - 1.8 + 0.6 – 1 = - 0.7	0.7
A_1B_2	+2.6- 2.6+0.9 – 1 = - 0.1	0.1
A_1B_4	+0 – 1 + 0.6 – 0 = - 0.4	0.4
A_2B_2	+ 2.2 - 2.6 + 0.9 – 0.6 = - 0.1	0.1
A_3B_1	+1.7–3.2+2.6 – 0.9 +0.6- 1.8 = - 1	1
A_3B_3	+0.6 – 3.2 +2.6– 0.9 = - 0.9	0.9
A_3B_4	+0- 0+ 0.6- 0.9+2.6-3.2 = - 0.9	0.9
A_4B_1	+1.9 – 0.9 + 0.6 – 1.8 = - 0.2	0.2
A_4B_4	+ 0 – 0 +0.6 – 0.9 = - 0.3	0.3

Now, since all the coefficients for the evidence improvements (simplex multipliers (optimality conditions in the different methods for the simplex methods)) are positive coefficients, and we have to maximize the total transported profits, therefore, the solution tableau stated in Table (2) is considered a unique optimum solution, i.e., the number of commodity units by which must transported from the i $\underline{th}$ factory to the j $\underline{th}$ store and the maximum transportation profits are as follows:

$x^*_{11} = x^*_{12} = 0$, $x^*_{13} = 2000$, $x^*_{14} = 0$, $x^*_{21} = 3000$, $x^*_{22} = 0$, $x^*_{23} = 400$,

$x^*_{24} = 200$, $x^*_{31} = 0$, $x^*_{32} = 1800$, $x^*_{33} = 0$,

$x^*_{34} = 0$, $x^*_{41} = 0$, $x^*_{42} = 2600$, $x^*_{43} = 1400$, $x^*_{44}=0$,

and then, the optimum maximum transportation profits is equal to: ($T.P^*$)= 21420 (L.E.).

Chapter (5)

"Network Analysis for the Projects"

In general, the project is a group of sequence activities, each one of these activities should be performed according to a specific priorities or a sequence for completing the project. The activity in any project is a work which needed time and resources for achieving it. The project needs some planning techniques, the main goal for these techniques is to maximize the degree of efficiency in achieving the different activities for any project [efficiency here means achieving such activities for these projects in its minimum time with minimum costs], such of these planning techniques called network analysis.

Network analysis are used for achieving the optimum management for the projects in the view of organizing and planning the time to avoid the deviations in the project performing that cases delay in performing, or such analysis can predict the deviation before it's happing, therefore it can be stopped or controlled.

The network analysis for the projects are a logical arrange and organizing a group of a sequence of activity. Activity in network is represent by arrow start from point which called starting event represent by O and ending by event called ending event as:

The main objective of network analysis is to determine or estimate the minimum time for achieving the project in its minimum total cost.

Network analysis is represented by a network graph consists of a set of nodes or events each one is represented by a circle and a set of activities each one is represented by an arrow.

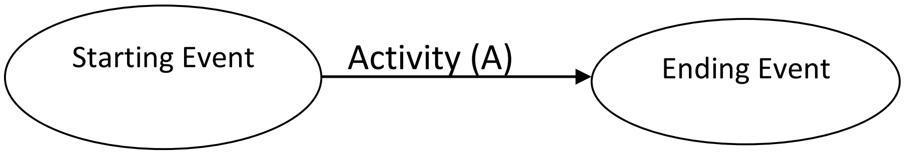
*<u>Basic Definition</u>:

[1]: <u>Event or Node</u> :

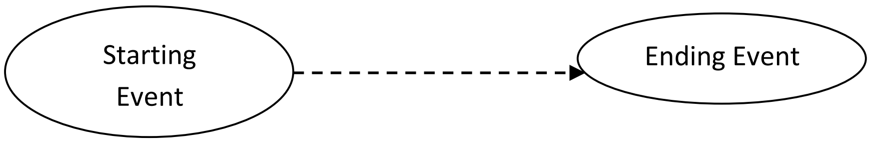
The event or node is the beginning of any activity in the project. It isn't requiring any time or costs. In the network graph the event or node is represented by circle included the number of this event as 1 , 2 , 3 , ….. , n ,where the event number (n) represents the ending event in the network graph.

[2]: <u>Activity</u>:

The activity is a specific work in the project requires a time and costs for being achieved. In the network graph each activity is represented by an arrow connected between two events, where the first event is the starting event for the activity and the other one is the ending event of the activity. The arrow for each activity is named by any capital letter A or B or C or D …. and so on as follows:



Sometimes, when we plotting the network graph, probably we need to use a dummy activity that represents an activity doesn't need any time or costs for achieving these activities.



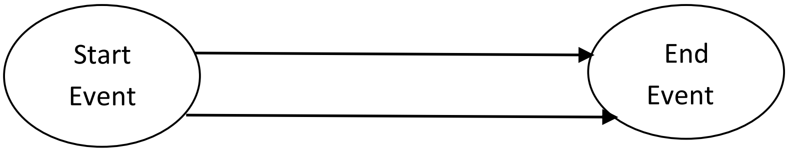
[3]: <u>Path</u>:

The path is a sequences of a group of arranged activities that start from the starting event of the project (1) and ending by the ending event of the project (n). Note that , the Path is not necessary to include all the activities in the network graph.

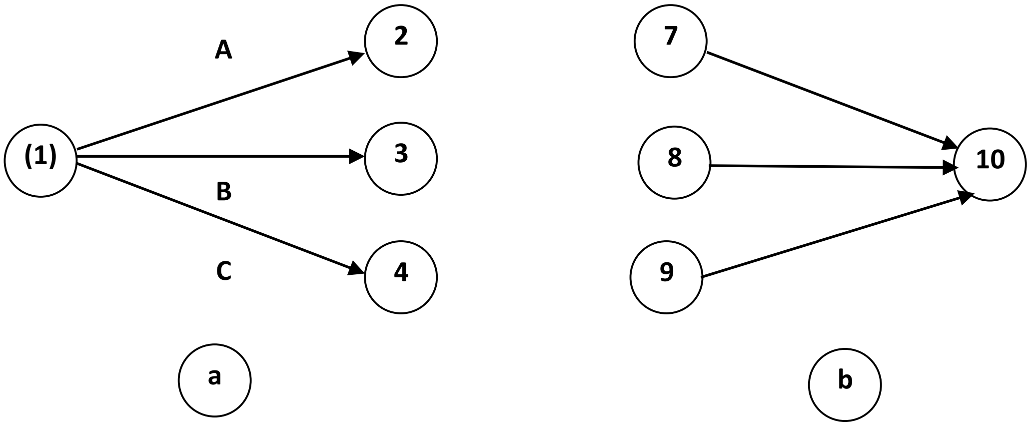
*** <u>Some Rules in the network diagram</u>:**

[1]: The network diagram must include only one starting event and also only one ending event.

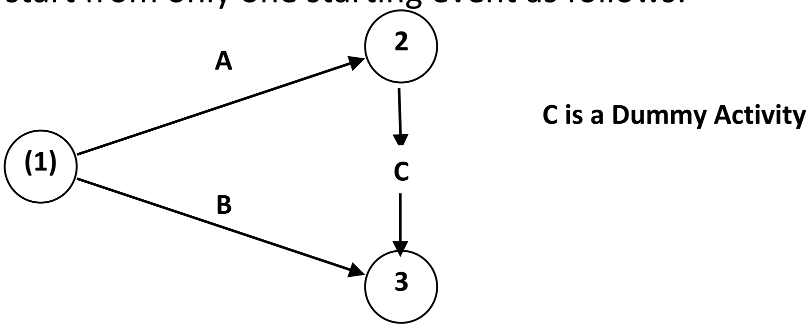
[2]: Any activity represented by only one arrow with only one starting event and only one ending event also. So the following diagram must not include in the network graph:



[3]: A group of activities can be beginning by only one starting event (a). And a group of activities can be ended by only one ending event (B), as follows:



[4]: The network graph cannot include two activities from only one starting event and one ending event. But it can have included two activities at least start from only one starting event as follows:



[5]: It is also preferred to draw the arrow diagram for the activities without any intersections between them as possible. Also it must be seeking to correcting the complex related and sequences of the activities of the project.

In addition, we must know for any new activity the following concepts:

(a): What is the activity or the set of activities that should being

performed before the new activity.

(b): What is the activity or the set of activities that can starting

with this new activity.

(c): What is the activity or the set of activities that should being performing after this new activity or should being following it.

Not that, we can use any number of dummy activities to achieving the sequences of activities.

Network Analysis Techniques :

Although we can be performing the Network Analysis as a Linear Programming Model (LPM), but it will be very hardly calculation because there are higher number of constraints corresponding to the number of activities as it will be seen later, so we can use the following simple techniques or methods:

[1]: Critical Path Method (CPM):

If the duration time for achieving the set of activities for the project are deterministic with high level of confidence, we can use the Critical Path Model (CPM).

[2]: Project Evaluation and Review Technique [PERT]:

If the duration time for achieving the set of activities for the project are probabilistic, then according to the Project Evaluation and Review Technique (PERT), we estimate the time for achieving the Set of activities for the project which are unknown times by using one of the probabilities distributions (Beta distribution),i.e., all times for achieving the set of activities are probabilities times.

[3]: PERT / Cost Technique:

The main objective of this technique is to determine the minimum duration time for achieving the set of activities for a specific project in its minimum cost. In the preceding two different techniques, the main defective for them is that each activity in the project determined by only the duration time ignoring a main side for achieving the projects activity which is the costs of the projects activity. So that, PERT / Cost technique includes the main two sides by which for each activity must be determined the time and the cost for achieving the project activities.

In Summary the main objective of PERT / Cost technique is to achieve the activities of the project in its minimum possible time in its minimum total costs.

[4]: Graphical Evaluation and Review Technique (GERT):

In GERT model the event (either the starting or ending nodes) in the network graph is probabilistic.

Finally, we conclude that the main objective of all of these techniques is to determine the duration time for achieving the project in the minimum deterministic or estimate time, or determining the minimum duration time for achieving the project in its minimum total cost. In these techniques we have the following:

[1]: The Sequence of the set of activities for the project means that we have to determine the starting and the ending event in the network graph.

[2]: The duration time for achieve the activities of the project is the main factor to draw the network diagram.

(1): <u>Critical Path Method or Technique (CPM)</u>:

The Critical Path defines a chain of critical activities which connect the starting and the ending nodes or events of the arrow in the network graph in a logical and organized sequence of activities for the project. The Critical path is a group of a sequence activities that must take a large importance and careful from the decision makers because any delay in the start for this critical activities will delay in the time for achieving the project.

In order to determine the critical path, we have to drawing the network graph for the set of activities for the project. The methods of determining the critical path is illustrated by either enumeration the set of paths in the network diagram or calculating the different types of floats in the set of activities for the project.

<u>Critical Path Calculations</u>:

(1): <u>Enumeration the set of paths in the network diagram</u>:

We can determine the critical path from enumerating all the set of paths by which connect the start and the end events in the network graph for the project. Then, the critical path is the path which have the long time for achieving the project.

(2): <u>Determining the types of times and floats for achieving the activities</u>:

The critical path calculations include two types of calculations:

<u>Firstly</u>: which called the forward computations, where compute two types of times named the Earliest Start and Earliest Completion time. In this computations we begin from the "start" event or node and moving to the "end" event, and at each event the time is computed representing the earliest occurrence time of the corresponding event. The Earliest start time (E_{si}) for the network graph for the project start at zero time, and the Earliest completion time (E_{cj}) which is computed by determining the maximum time for the sum of the earliest start plus the duration time(dij) for each activity decants or incoming in the event, i.e.,

$E_{sj} = \max (E_{si} + d_{ij})$, for all the defined (i , j) activities, where $E_{s1} = 0$.

<u>Secondly</u> : which called the backward computations, where compute two types of times named the Latest Start(L_{si}) and Latest completion time (L_{cj}) which is computed by determining the minimum time for the difference between the latest completion time minus the duration time(dij) for each activity coming to the event (i), i.e.,

$L_{si} = \min (L_{cj} - d_{ij})$, for all the defined (i , j) activities, where $L_{cn} = E_{sn}$.

The backward computation pass start from the end event. The objective of this computation is to compute L_{ci} , the latest completion time for all the activities coming into the event (i). Thus, if (i=n) is the end event , then, $L_{cn} = E_{sn}$ initiates the backward pass.

The following example illustrate these computations.

<u>Determination of the floats</u>:

The floats of time for the critical activities is naturally equal to zero, which is the fact and the main reason for critical. But, for the noncritical activities, there are two important types of float time: the total float(TF) of time and the free float(FF)of time. where:

* **<u>The Total Float (</u> TF_{ij}**) for the activity (i,j) is equal to the difference between the maximum time available to perform the activity ($= L_{cj} - E_{si}$) and the activity duration time (d_{ij}), i.e.,

$TF_{ij} = L_{cj} - E_{si} - d_{ij} = L_{cj} - (E_{si} + d_{ij}) = L_{cj} - E_{cij} = L_{sij} - E_{si}$The total float for the critical activities is always equal to zeros, but the total float for the noncritical activities is always positive values greater than zero, i.e.,

TF_{ij} = o for all the set of (ij) $^{\underline{th}}$ critical activities.

> 0 for all the set of (ij)$^{\underline{th}}$ noncritical activities.

*** <u>The Free Float (FF_{ij} </u>):**

The time free float is defined by assuming that all the activities start as early as possible. In this case, the time FF_{ij} for the activity (i , j)is the excess of available time(= E_{sj}– E_{si}) over its duration time(= D_{ij}), i.e.,

$$FF_{ij} = E_{sj} - E_{si} - D_{ij} = E_{sj} - (E_{si} + D_{ij}).$$

The time free float (FF_{ij}) for the critical activities is always equal to zeros, but the time free float for the noncritical activities is always at least zero or positive values, i.e.,

FF_{ij} = o for all the set of (ij)$^{\underline{th}}$ critical activities.

≥ 0 for all the set of (ij)$^{\underline{th}}$ noncritical activities.

The following example represents how can we draw the network graph and determining the critical path either by enumerating all the set of different paths in the network graph or by calculating the four types of times and floats.

Example (1) **:**

The following table {Table(1)} represents the time of achieving the set of activities for a specific project (week).

Table(1)

Activity	**Starting event**	**Ending event**	**Time**
A	1	2	4
B	1	3	6
C	1	4	5
D	2	3	4
E	3	5	5
F	3	7	7
G	4	7	5
H	2	6	2
I	5	6	5
J	6	8	4
K	7	8	2

Required**:**

(1) : Draw the network diagram (or graph) for this project.

(2) : From your previous result in (1) determine the critical path and then determine the total time for achieving this project.

(3) : Determine the different types of times for achieving the activities of this project.

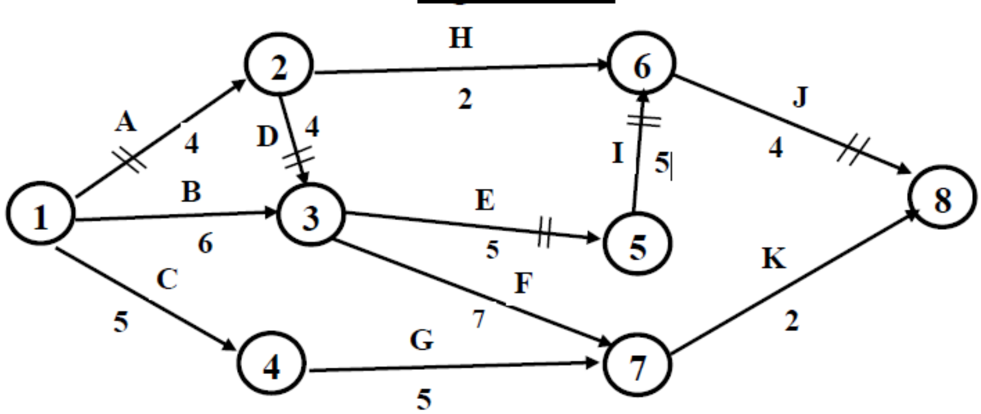
(4) : From your previous result in (3) determine the critical path for this project.

(5) : Calculate the time total float and the time free float for the activities for this project.

Solution:

(1): The following diagram represents the network graph for the set of activities for the project which shown the sequences of events and activities for the project as it be shown in Figure (1).

Figure No. 1



(2): From the preceding result in (1),i.e., from the network graph, we can determine the critical path and then determining the total time for achieving the project in two different ways:

<u>Firstly</u>: By enumeration all the set of paths in the network graph starting from the event (1) and ending with the latest event (8). Then the path which have the long duration time is considered the Critical Path.

The following table{ Table(2)} represents these calculations:

Table(2)

No.	Path	Total of duration time	Remark
1	A H J	4 + 2 + 4 = 10	
2	A D E I J	4 + 4 + 5 + 5 + 4 = 22	Critical Path
3	A D F K	4 + 4 + 7 + 2 = 17	
4	B E I J	6 + 5 + 5 + 4 = 20	
5	B F K	6 + 7 + 2 = 15	
6	C G K	5 + 5 + 2 = 12	

Therefore, the Critical Path for achieving the set of activities for the project is the path (A D E F K) which have the long duration time with total time for achievement is equal to 22 weeks.

Secondly: By calculating the different four types of times (E_s , E_c , L_s , L_c) as it be shown in the succeeding requirement.

(3): Determination the different types of times for achieving the activities for the project. From the previous diagram we can calculate these types as follows:

* Let E_{si} be the earliest start time of all the activities emanating from the event (i). Thus, E_{si} represents the earliest occurrence time of event (i). If i=1 is the "start" event, then conventionally, for the Critical Path calculations, Es1 = 0 . Let d_{ij} be the duration time for the activity (i , j). Then, the Forward pass calculations are obtained from the following formula:

$E_{sj} = \max \{ E_{si} + d_{ij} \}$, for all the defined (i , j) activities.

Then, the network graph starts with earliest starting time equal to zero, i.e., $E_{s1} = 0$, then the sequence of the earliest start and the earliest completion times are calculated by using the forward computation method as follows:

Then E_{s2} or $E_{c2} = E_{s1} + d_{12} = 0 + 4 = 4$, , since only the activity (A) is incoming in the event (2) ,

E_{s3} or $E_{c3} = \max\{(E_{s1} + d_{13}) , (E_{s2} + d_{23})\}$

$= \max\{ (0 + 6) , (4 + 4) \} = 8$, since each of the two activities (B) and (D) are incoming in the event(3) ,

E_{s4} or E_{c4} = E_{s1} + d_{14} = 0 + 5 = 5 , since only the activity (C) is incoming in the event (4) ,

E_{s5} or E_{c5} = E_{s3} + d_{35} = 8 + 5 = 13 , since only the activity (E) is incoming in the event (5) ,

E_{s6} or E_{c6} = max{(E_{s2} + d_{26}) , (E_{s5} + d_{56})}

= max{(4 + 2) , (13 + 5)} = 18 , since each of the two activities (H) and (I) are incoming in the event(6),

E_{s7} or E_{c7} = max{(E_{s3} + d_{37}) , (E_{s4} + d_{47})}

= max{(5 + 5) , (8 + 7)} = 15 , since each of the two activities (F) and (G) are incoming in the event(7), and finally:

E_{s8} or E_{c8} = max{(E_{s6} + d_{68}) , (E_{s7} + d_{78})}

= max{(18 + 4) , (15 + 2)} = 22 week , since each of the two activities (J) and (K) are incoming in the event(8).

On the other hand, the sequence of the Latest start and the Latest completion times are calculated by using the Backward computation method as follows:

The Backward pass starts from the "ending" node or event. And, the main objective of this phase is to compute the Latest completion time(L_{ci}) for all the activities coming into the event (i). Thus, if i=n is the "end" event, then, let $L_{cn} = E_{sn}$ initiates the Backward pass. And, in general, for any node(i),

$$L_{ci} = \min \{ L_{cj} - d_{ij} \} \text{, for all the defined (i , j) activities.}$$

Therefore, the values of L_c are determined as follows:

$L_{c8} = E_{s8} = 22$,

$L_{c6} = L_{c8} - d_{68} = 22 - 4 = 18$,

$L_{c7} = L_{c8} - d_{78} = 22 - 2 = 20$,

$L_{c5} = L_{c6} - d_{56} = 18 - 5 = 13$,

$L_{c4} = L_{c7} - d_{47} = 20 - 5 = 15$,

$L_{c3} = \min\{(L_{c7} - d_{37}),(L_{c5} - d_{35}) = \min\{(20 - 7),(13 - 5)\} = 8$,

$L_{c2} = \min\{(L_{c6} - d_{26}),(L_{c3} - d_{23}) = \min\{(18 - 2),(8 - 4)\} = 4$,

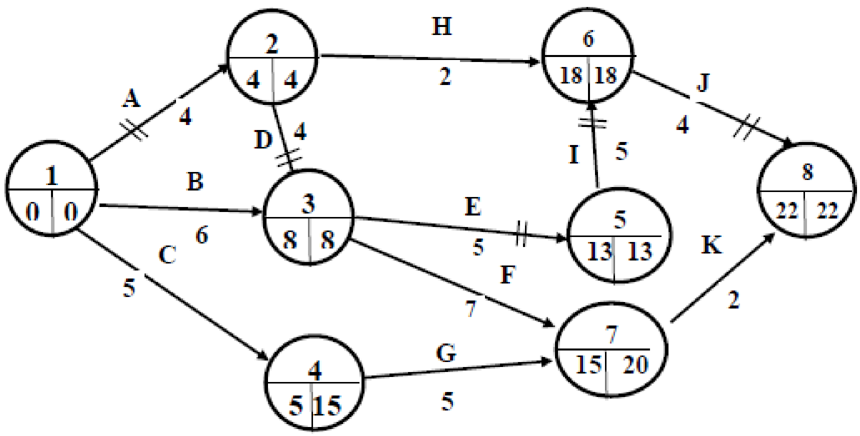
$L_{c1} = \min\{(L_{c4} - d_{14}),(L_{c3} - d_{13}),(L_{c2} - d_{12})\}$

$= \min\{(15 - 5), (8 - 6), (4 - 4)\} = 0$

This completes the Backward pass calculations.

Note that, all the results for either the Forward or the Backward pass calculations are stated in the following network graph illustrated in Figure (2).

Figure No. 2



(4): From the previous result in (3), we can determine the critical path for this project as follows :

Note that, the critical path is the minimum time for achieving the project, and it is not include any time float because the logical arrangements for achieving the sequence of the set of activities for achieving the project.

The critical path activities can now be identified by using the results of the Forward & Backward calculations. An activity (i,j) lies on the critical path if it satisfied the following three conditions:

(*) : $E_{si} = L_{ci}$,

(**) : $E_{sj} = L_{cj}$,

(***): $E_{sj} - E_{si} = L_{cj} - L_{ci} = d_{ij}$

All of these conditions actually indicate that there is no float or slack time between the earliest start(or completion) and the latest start(or completion) of the activity, i.e., this activity must be critical. In the preceding network graph, the critical activities are denoted by the sign (//) , i.e., the critical activities are (A , D , E , I , J) by which consist the critical path (A-D-E-I-J with 22 weeks as the minimum time for achieving all the set of activities for this project.

(5): Determination the time total float and the time free float for the activities for this project:

After the determination of either the critical activity and the critical path, it is desired to determine the floats for the noncritical activities, where the critical activities must have a zero float which is the main reason of criticality. In fact, there are two types of floats, Total float (TF) and Free Float (FF), where:

$$TF_{ij} = L_{cj} - (E_{si} + d_{ij}) \text{ , or:}$$

$$= L_{cj} - E_{cij} \text{ ,} \qquad \text{or:}$$

$$= L_{sij} - E_{si}$$

$$FF_{ij} = E_{sj} - (E_{si} + d_{ij}) .$$

The following table{ Table(3) } represents the calculation for determining the two types of floats from the network graph stated in Figure(2):

Table(3)

Activity	$TF_{ij} = L_{cj} - (E_{si} + d_{ij})$	$FF_{ij} = E_{sj} - (E_{si} + d_{ij})$
$A_{(1-2)}$	**$TF_{12}= L_{c2}-(E_{s1}+d_{12})=4-(0+4)=0$ ***	**$FF_{12}= E_{s2}-(E_{s1}+d_{12})=4-(0+4)=0$**
$B_{(1-3)}$	**$TF_{13}= L_{c3}-(E_{s1}+d_{13})=8-(0+6)=2$**	**$FF_{13}= E_{s3}-(E_{s1}+d_{13})=8-(0+6)=2$**
$C_{(1-4)}$	**$TF_{14}= L_{c4}-(E_{s1}+d_{14})=15-(0+5)=10$**	**$FF_{14}= E_{s4}-(E_{s1}+d_{14})=5-(0+5)=0$**
$D_{(2-3)}$	**$TF_{23}= L_{c3}-(E_{s2}+d_{23})=8-(4+4)=0$ ***	**$FF_{23}= E_{s3}-(E_{s2}+d_{23})=8-(4+4)=0$**
$E_{(3-5)}$	**$TF_{35}= L_{c5}-(E_{s3}+d_{35})=13-(8+5)=0$ ***	**$FF_{35}= E_{s5}-(E_{s3}+d_{35})=13-(8+5)=0$**
$F_{(3-7)}$	**$TF_{37}= L_{c7}-(E_{s3}+d_{37})=20-(8+7)=5$**	**$FF_{37}= E_{s7}-(E_{s3}+d_{37})=15-(8+7)=0$**
$G_{(4-7)}$	**$TF_{47}= L_{c7}-(E_{s4}+d_{47})=20-(5+5)=10$**	**$FF_{47}= E_{s7}-(E_{s4}+d_{47})=15-(5+5)=5$**
$H_{(2-6)}$	**$TF_{26}= L_{c6}-(E_{s2}+d_{26})=18-(4+2)=12$**	**$FF_{26}= E_{s6}-(E_{s2}+d_{26})=18-(4+2)=12$**
$I_{(5-6)}$	**$TF_{56}= L_{c6}-(E_{s5}+d_{56})=18-(13+5)=0$***	**$FF_{56}= E_{s6}-(E_{s5}+d_{56})=18-(13+5)=0$**
$J_{(6-8)}$	**$TF_{68}= L_{c8}-(E_{s6}+d_{68})=22-(18+4)=0$***	**$FF_{68}= E_{s8}-(E_{s6}+d_{68})=22-(18+4)=0$**
$K_{(7-8)}$	**$TF_{78}= L_{c8}-(E_{s7}+d_{78})=22-(15+2)=5$**	**$FF_{78}= E_{s8}-(E_{s7}+d_{78})=22-(15+2)=5$**

Note that, table (3) gives a summary about the critical path calculations, where, the critical activities and only the critical activity must have zero total float(TF), and the free float must also equal to zero when the total float is equal to zero. But, the converse is not true, however, in the sense that the noncritical activity may have zero free float. This is an accidental since all the events of the project happen to be on the critical path.

(2): **Project Evaluation and Review Technique (PERT):**

The critical path method based on that the duration times for achieving the set of activities for the project is deterministic. In this section, we will introduce a new technique when the duration times for achieving the set of activities for the project is probabilistic or uncertainty. In such cases, probability considerations are incorporated in project scheduling by assuming that the time estimate for each activity is based on three different values t_o , t_p , t_m , where:

t_o: is the optimistic time, which will be required if the execution for the set of the activities for the project goes extremely well.

t_p: is the pessimistic time, which will be required if the execution for the set of the activities for the project goes extremely badly and everything goes badly.

t_m: is the most likely time, which will be required if the execution for the set of the activities for the project goes normal.

In this case, because of these properties it is intuitively justified that the duration time for each activity follow the Beta distribution with mean time for each activity ($\bar{t}$) is equal to:

$\bar{t} = (t_o + 4t_m + t_p) / 6$, with variance: V^2 or $\sigma^2 = ((t_p - t_o)/ 6)^2$

Assuming that, all the set of activities in the network graph are statistically independent, then the normal distribution calculation can be used for determining the probabilities for achieving the project in at least (or at most) in a specific estimate duration time with a confidence level. One of the important advantage for PERT technique is that it is not stopped in the level of estimate the average or the expected time for achieving the project in at least (or at most) in a specific estimate duration time with a confidence level. In summary we can conclude the basic steps for PERT as follows:

1- Using the three times of estimates t_o, t_m, and t_p for estimate the expected duration time for achieving each activity in the project, and the standard deviation or the variance, where:

$\bar{t} = (t_o + 4t_m + t_p) / 6$, with variance:

V^2 or $\sigma^2 = ((t_p - t_o)/ 6)^2$

2- Drawing the network diagram by use the estimate times for the set of activities for achieving the project.

3- Determine the critical path by using any one of the previous methods.

4- If it is desired to know what is the probability (or the percentage of time)for achieving the project in at least(or at most) a specific estimate time, then we have to determine each of the two parameter for the time estimate(or the expected mean)time for achieving the project (the expected duration time for the critical path (μ_i) and Var(μ_i) which is the sum of the variances for the set of critical activities. Then, according to the Central Limit theorem the expected duration time for the critical path (μ_i) is approximately normally distributed with the mean E(μ_i) and variance Var(μ_i).

Example (2): The following table {Table(4)} represents the different types of time (optimistic (t_o) , most likely (t_m) and pessimistic (t_p)) for achieving a set of activities in a project by hour:

Table (4)

Activity	Starting event	Ending event	t_o	t_m	t_p
A	1	2	2	4	6
B	1	3	3	5	13
C	2	5	4	5	6
D	2	4	2	3	10
E	3	4	1	2	3
F	5	7	1	2	9
G	4	7	6	8	10
H	3	6	5	8	11
I	6	7	4	6	8

<u>Required</u>:

(1): By using the most likely time (t_m) graph the network diagram and determine the critical path.

(2): Calculate the four different types of times in the network diagram and determine the critical path.

(3): Determine the times free float(FF) and total float(TF) in the network graph.

(4): By using PERT technique, estimate the time for achieving the project.

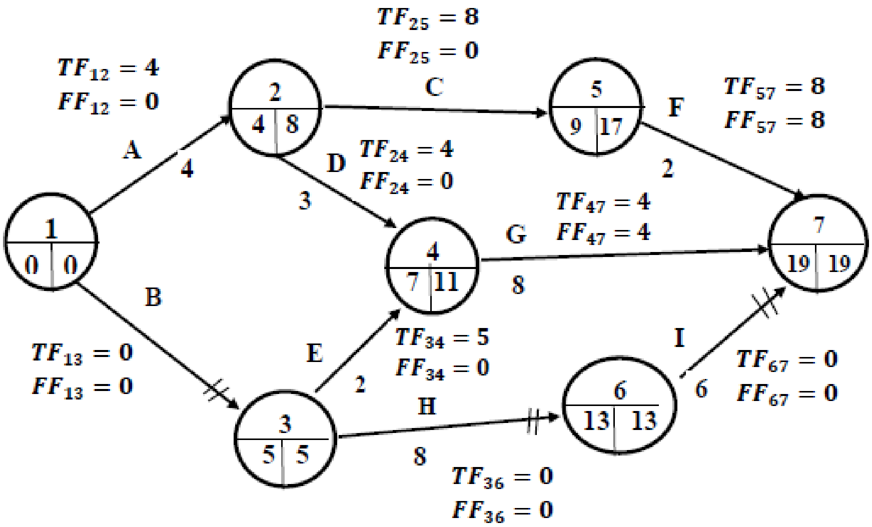
(5): Find the probability that the project is achieved at most in 23 hours.

(6): Determine the maximum level of time for achieving the project with 97.5% as a confidence level.

<u>Solution</u>:

(1): Graphing the network diagram using the most likely time t_m:

Figure No. 3



From Figure No.(3) we can determine the time estimate of the critical path by using the most likely time for achieving the set of activities through enumerating all the set of paths as it be shown in the following table:

Table (5)

A	Path	Achieving time	Remark
1	ACF	4 + 5 + 2 = 11 hour	
2	ADG	4 + 3 + 8 = 15	
3	BEG	5 + 2 + 8 = 15	
4	BHI	5 + 8 + 6 = 19 hour	Critical path

From the preceding table, we find that the path which have the longest most likely time from the different paths in the network graph is the path (B-H-I). And therefore, according to the critical path (B-H-I) the minimum time for achieving the project is 19 hour.

(2): The Earliest start and Earliest completion times are calculated and shown on the network graph by using the forward computation method, and the Latest start and Latest completion times are also calculated by using the backward computation method and shown in the network diagram. In the critical path, note that each of the three critical activities satisfied the following formulas:

(*) : $E_{si} = L_{ci}$,

(**) : $E_{sj} = L_{cj}$,

(***): $E_{sj} - E_{si} = L_{cj} - L_{ci} = d_{ij}$.

Therefore the path (B – H – I) is considered the critical path.

(3):Determining the time total float and the time free float on the network diagram as follows:

The two types of time floats (TF , FF) are illustrated in the preceding network graph (Figure No. (3)).

(4): By using PERT technique, the following table {Table (6)} represents the times estimate for achieving the set of activities for the project and its standard deviations by using the three different types of estimates (t_o , t_m , t_p), and its standard deviations:

Table (6)

Activity	T_o	T_m	T_p	$\bar{t} = (T_o+4T_m+T_p)/6$	$б_t = 1/6(T_p - T_o)$	$б_t^2$
A(1-2)	2	4	6	(2+(4x4)+6) / 6 = 4	(6 – 2) / 6 =2/3	4/9
B(1-3)	3	5	13	(3+(4x5)+13) / 6 = 6	(13 – 3) / 6 =5/3	25/9
C(2-5)	4	5	6	(4+(4x5)+6) / 6 =5	(6 – 4) / 6 =1/3	1/9
D(2-4)	2	3	10	(2+(4x3)+10) / 6 =4	(10 – 2) / 6 =4/3	16/9
E(3-4)	1	2	3	(1+(4x2)+3) / 6 =2	(3 – 1) / 6 =1/3	1/9
F(5-7)	1	2	9	(1+(4x2)+9) / 6 =3	(9 – 1) / 6 =4/3	16/9
G(4-7)	6	8	10	(6+(4x8)+10) / 6 =8	(10 – 6) / 6 =2/3	4/9
H(3-6)	5	8	11	(5+(4x8) +11) / 6 =8	(11 – 5) / 6 = 1	1
I(6-7)	4	6	8	(4+(4x6)+8) / 6 =6	(8–4) / 6 =2/3	4/9

After determining the time estimate (t) for each activity in the project, then, we can draw the network graph for this project by using these times of estimates as it be shown in the following graph (Figure No. (4)): in this graph, we represent the four different types of times(E_s ,E_c , L_s , L_c) for each activity on the network graph. And the critical path is B-H-I with duration time estimate equal to 20 hours for achieving the set of activities for this project.

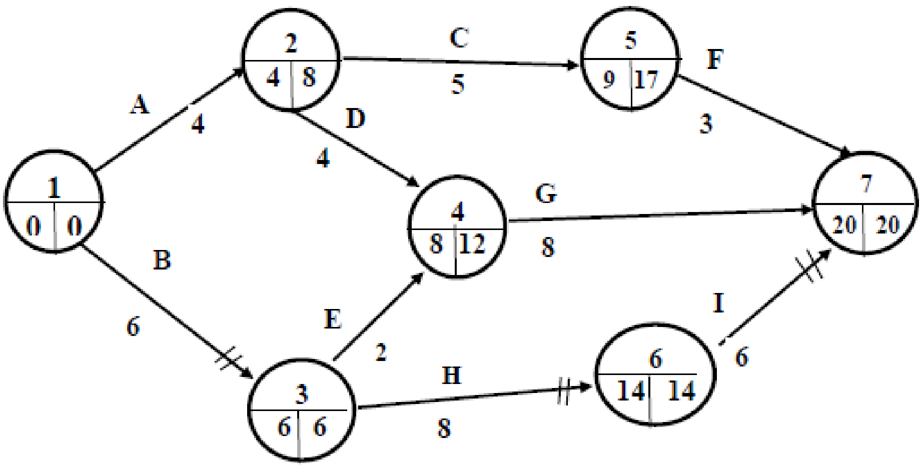
Note that, for the set of critical activities by which involved in the critical path, the following formulas are satisfied:

(*) : $E_{si} = L_{ci}$,

(**) : $E_{sj} = L_{cj}$,

(***): $E_{sj} - E_{si} = L_{cj} - L_{ci} = d_{ij}$.

<u>**Figure No. 4**</u>



(5): To find the probability that the project is achieved at most in 23 hours, suppose that the time estimate for achieving the project is a random variable(r.v.) say (x), then the r. v. (x) have a normal distribution with mean expected time equal to the time estimate for the critical path (i.e., with μ = 20 hour) and standard deviation б = square root to the sum of variances for the critical activities:

i.e., $б = (б^2_B + б^2_H + б^2_I)^{1/2} = ((25/9) +1 + (4/9))^{1/2}$ = 2.055 hour.

i.e., $x \sim N.D (\mu = 20 , б = 2.055$ hour). Then, we have to find:

$P (x \leq 23) = P \{((x - \mu) / б) \leq ((23 – 20) / 2.055))$

$= P (Z \leq 1.46) = \Phi (1.46)$,

Then, from the table for the standard normal distribution, we find that :

$P (x \leq 23) = \Phi (1.46) = 0.9279$

Therefore, the probability that the project is achieved at most in 23 hours = 0.9279. And by using the complement probability, then the probability that the project is achieved in at least 23 hour is equal to (= 1 – 0.9279) = 0.0721. In another word the percentage of time that the project is achieved in at least 23 hour is 7.21 % .

(6): In order to determine the maximum level of the time estimate

for achieving the project with 97.5% as a confidence level, we suppose that the required of this time estimate is equal to (a) hour, then we have to find the value of (a) by which satisfied the following formula:

$P (x \leq a) = 0.975$, i.e.,

$P \{ ((x - \mu) / б)) \leq ((a – 20) / 2.055)\} = 0.975$

$P\{ Z \leq ((a – 20) / 2.055)\} = 0.975$

Then , $\Phi ((a – 20) / 2.055) = 0.9279$,

Then, from the table for the standard normal distribution, we find that, the area 0.975 under the standard normal curve corresponding to the standard value 1.96 , then in order to determine the maximum level of the time estimate (a) for achieving the project with 97.5% as a confidence level, we have to solve the following equation:

$(a – 20) / 2.055 = 1.96$, i.e., $(a – 20) = 1.96 \times 2.055 = 4.0278$

Then: $a = 4.0278 + 20 = 24.0278$ hour.

Therefore, the maximum level of the time estimate for achieving the project with 97.5% as a confidence level is 24.0278 (Hour).

Formulation the Network Graph as a Linear Programming Model (LPM):

In order to formulate the network graph as a LPM, firstly we have to suppose a set of decision variables corresponding to the set of the project Events (x_1 , x_2 , x_3 ,...., x_n). Then, the main objective is to determine the values of these decision variables by which make the duration time for achieving the project (

which is equal to the difference between either the Earliest start(Es) or Earliest completion(Ec) time in the Latest event(x_n) and the start event(x_1)) in its minimum value, subjected by a set of constraints corresponding to the number of the project activities (each constraint means that the duration time for achieving that activity is at least equal to the deterministic duration time(d_{ij}) in the Critical Path Method(CPM) or the time estimate in the Project Evaluation and Review Technique (PERT), beside that there are a set of the non-negativity constraints over the set of decision variables.

For example, if we want to formulate the network graph for the project illustrated in example (1) as a LPM, then we have the following LPM:

Find the values of x_1 , x_2 , x_3 , ………. , x_8 by which make the duration time (T) for achieving the project :

$T = x_8 - x_1$ (Minimization)

Subject to :

$x_2 - x_1 \geq 4$ for the activity A ,

$x_3 - x_1 \geq 6$ for the activity B ,

$x_4 - x_1 \geq 5$, for the activity C ,

$x_3 - x_2 \geq 4$ for the activity D ,

$x_5 - x_3 \geq 5$ for the activity E ,

$x_6 - x_2 \geq 2$ for the activity H ,

$x_7 - x_3 \geq 7$ for the activity F ,

$x_7 - x_4 \geq 5$ for the activity G ,

$x_6 - x_5 \geq 5$ for the activity I ,

$x_8 - x_6 \geq 4$ for the activity J ,

$x_8 - x_7 \geq 2$ for the activity K ,

$X_i \geq 0$, for i = 1 : 8

We can rewrite the preceding LPM in an arrangement for the decision variables as follows:

Find the values of x_i : where i = 1,2, …8 by which make:

$$T = x_8 - x_1 \qquad \text{(min)}$$

Subject to:-

$$- x_1 + x_2 \geq 4$$

$$- x_1 + x_3 \geq 6$$

$$- x_1 + x_4 \geq 5$$

$$
\begin{array}{llllllll}
-x_2 + x_3 & & & & & & & \geq 4 \\
& -x_3 & & +x_5 & & & & \geq 5 \\
-x_2 & & & & +x_6 & & & \geq 2 \\
& -x_3 & & & & +x_7 & & \geq 7 \\
& & -x_4 & & & +x_7 & & \geq 5 \\
& & & -x_5 & +x_6 & & & \geq 5 \\
& & & & +x_6 & & +x_8 & \geq 4 \\
& & & & & -x_7 & +x_8 & \geq 2
\end{array}
$$

$x_i \geq$ zero where i = 1, 2, 3, …………… 8.

(3): **Cost Consideration in the Project Scheduling (PERT Cost Model):**

In the previous two scheduling techniques (CPM & PERT), we find that, the available data for each activity is the duration time for achieving the activity in the project only, and the main objective for these techniques is to find either the minimum deterministic duration time (CPM) or the minimum time estimate for achieving the project(PERT) , disregarding a very important component in the project scheduling operation which is the cost consideration in the project scheduling. That is to say, the main disadvantage of the previous two model is that they ignored an important side in the operation of decision making for the project performing which is the cost side for achieving the project activities. In this technique

(PERT/COST), two different data points of coordinates duration time and its corresponding cost for each activity in the project will be determined:

Normal Data: Normal Duration Time (T_n) for achieving the activity with a Normal Duration Cost (C_n) .

Crash Data: Crash Duration Time (T_c) for achieving the activity with a Crash Duration Cost (C_c) .

The point (T_n , C_n) represents the normal duration time(Tn) for achieving the activity in its associated cost(C_n) if the activity is executed under normal conditions. The normal duration time T_n can be compressed by increasing the allocated resources and hence increasing the cost directly. There is a limit for compressing the value of T_n, called the crash duration time, beyond which no further reduction in the duration time can be effected. At this point any increasing in resources will only increase the costs without reducing the duration time. The crash point is denoted by (T_c , C_c).

The straight line or the linear relationship between the two coordinates for the two Normal (T_n , C_n) and Crash (T_c , C_c) points are used mainly for convenience since it can be determined for each activity from the knowledge of the normal and crash points. Then, it becomes that if we want to reduce the normal duration time for a specific activity, we have to accept afford the additional costs (i.e., cost / time relationship, where there is a negative relationship), the value for this additional costs by which affords the project resulted from compressing the duration time is the cost slope (m) for that activity.

Therefore, we have the following two concepts for either the time or the cost for achieving each activity in the project:

1-Two types of duration times for each activity**:**

(a): Normal Time (T_n):

The Normal Time is the time for executed or achieved the activity in the level of normal cost.

(b): Crash time (C_c):

The Crash Time is the minimum level of duration time for achieving the activity.

2-Types of costs for achieving the activities**:**

(a): Normal costs:

It is the minimum cost to achieve the activity in its normal time.

(b): Crash cost:

It is the cost of achieving the activity in its crash time.

The main objective of PERT/COST technique is how can we use the two points coordinates {(T_n , C_n) , (T_c , C_c) for determining the minimum cost for achieving the project in its minimum time. This objective can be executed through the following steps:

1- Construct the network graph by using the normal duration time(T_n), and determine the time for the critical path with the total normal costs(C_n) which is the summation of the normal costs for all the set of activities in the project.

2- Construct the network graph by using the crash duration time(T_c), and determine the time for the critical path with the total crash costs (C_c) which is the summation of the crash costs for all the set of activities in the project.

3- Determine the difference between the duration times for the preceding two critical paths which represents the amount of time by which we can compressing the normal duration time for the normal critical path to reach the time for the crash critical path time.

4- Determine the Cost / Tim slope (m) for each activity in the network graph, where: $\mathbf{m = (C_c - C_n) / (T_n - T_c)}$, then determine the values for (m) for the critical activities.

5- Start in compressing or reduction in the normal network graph for the critical activity which have the minimum cost slope (m) without any deviation from criticality, i.e., the critical path still critical from the 1^{st} reduction to the latest reduction. In order to satisfy this condition of criticality, the reduction in the time for any specific critical activity must reduce by the value:

{Min (available reduction for the critical activity which have the minimum cost slope) , (minimum time of free float (FF) greater than zero on the normal network graph)}.

6- Continue in the preceding reductions until the time for the normal critical path reached to the time for the crash critical path.

(**) In all the operation for the steps of reductions in PERT, if there are at least two critical activities have the same minimum cost slope, select the critical activity which is exist in the largest number of paths on the network graph, specifically which have the most time limit of reductions.

Note that:

(I): If the network graph contains only one path, the succeeding operations of reductions in the activities is achieved accordingly the activity which have the minimum cost slope(m).

These procedures for this technique(PERT/COST) are illustrated in the following examples:

Example (3):

A project consisted of three activities A, B, C. The following table represents: the Normal Data {normal costs (in thousands L.E), the normal duration time (

in week) and the two different alternatives available with two crash times and costs as follows:

Table (7)

Activity	**Path**	**Normal data**		**Alternative (1)**		**Alternative (2)**	
		Time	**Cost**	**Crash time**	**Crash cost**	**Crash time**	**Crash cost**
A	1 – 2	10	20	9	25	8	28
B	2 – 3	12	14	11	21	6	26
C	3 – 4	18	35	14	41	10	49
∑		40	69	34	87	24	103

<u>Required</u>**:**

(1): Using PERT / Cost technique determine the minimum time for achieving this project in its minimum costs.

(2): what is the minimum cost for achieving the project in 23 weeks.

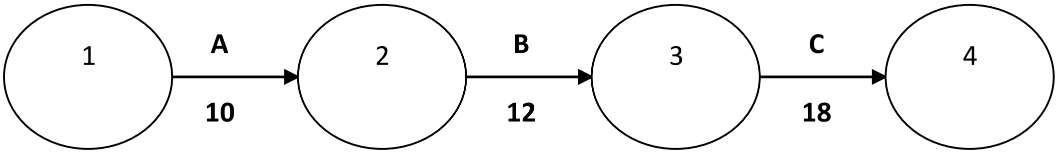
(3): Determine the following:

(a): The minimum cost for achieving the project in 20 weeks.

(b): The minimum time for achieving the project with a total cost of 83 thousand L.E.

Solution: -

(1): To determine the minimum cost for achieving the set of activities for this project in its minimum cost with minimum time using PERT/ COST technique, we have to draw the network graph as follows:



That is to say that, the network graph denoted that, there is only one path for the logical sequence of the set of activities with achieving normal time (10 + 12 + 18) = 40 weeks with the minimum normal cost 69,000 which are considered that the largest time to achieve the project with the minimum normal costs for achieving the project. Now, in order to determine the minimum cost for achieving the project in its minimum time by using Pert /Cost technique and the Crash Data for the two alternatives, we will compare between the Cost slope for each activity as it be shown in the following calculations:

(*): For the 1st alternative Crash Data**:**

By using the 1st alternative, we can achieve the project in:

9 + 11 + 14 = 34 weeks with total crash costs = 25 + 21 + 41

= 87 thousand (L.E.).

(*): <u>For the 2nd alternative Crash Data</u>:

By using the 2nd alternative, we can achieve the project in: 8 + 6 + 10 = 24 weeks with total crash costs:

= 28 + 26 + 49 = 103 thousand (L.E.).

Now, in order to determine the minimum cost for achieving the project in its minimum time, we have to calculate the cost slop for each activity in both the two different alternatives as follows:

Table(8)

Activity	**Normal data**		**Alternative (1)**				**Cost Slop**	**Alternative (2)**				
	Time	**Cost**	**Time**	**Cost**	**ΔC**	**Δ T**		**Time**	**Cost**	**Δ C**	**Δ T**	**slop**
A	10	20	9	25	5	1	5	8	28	8	2	4
B	12	14	11	21	7	1	7	6	26	12	6	2
C	18	35	14	41	6	4	1.5	10	49	14	8	1.75
	40	69	34	87				24	103			

From the two columns cost slops in the preceding table, we can arrange the two alternatives as a set of changes in each of the normal time and cost and calculate the modifications of the succeeding plans as follows:

(*): <u>The 1st modification</u>: According to the 1st alternative, the activity (C) can be reduced by (4) weeks with the unit of cost slope = 1.5 thousand L.E, which means that, reducing 4 weeks will increasing the normal total cost will increased by an additional costs = (4 × 1,5) = 6 thousand L.E, i.e., the minimum cost for achieving the project in(40 – 4 = 36 weeks) is:

(69 + 6 = 75) thousand L.E., and then we have the 1st modified planning.

(**): The 2nd modification**:**

According to the 2nd alternative, the activity (C) can be reduced by (8) weeks with the unit of cost slope = 1.75 thousand L.E, which means that, reducing 8 weeks will be increasing the normal total cost will have increased by an additional cost:

= (8 × 1.75) = 14 thousand L.E, i.e., the minimum cost for achieving the project in(36 – 8 = 28 weeks) is (75 + 14 = 89) thousand L.E., and then we have the 1st modified planning.

(***): The 3rd modification**:**

According to the 2nd alternative, the activity(B) can be reduced by (6) weeks with the unit of cost slope = 2 thousand L.E, which means that, reducing 6 weeks will increasing the normal total cost will increased by an additional costs = (2 × 6) = 12 thousand L.E, i.e., the minimum cost for achieving the project in(28 – 2 = 26 weeks) is (89 + 12 = 101) thousand L.E., and then we have the 3rd modified planning.

And so on for the set of successfully modified planning shown in the following table {Table (9)}:

Table (9)

Activities The no. of Plan:	(A) activity		(B) activity		(C) activity		Total	
	Time	Cost	Time	Cost	Time	Cost	Time	Cost
Normal data **1st Modification**	10	20	12	14	18 -4	35 +6	40 -4	69 + 6
1st modificat.Plan **2nd modification**	10	20	12	14	14 -8	41 +14	36 -8	75 +14
2nd modificat.Plan **3rd Modification**	10	20	12 -6	14 +12	6	55	28 -6	+89 +12
3rd modificat.Plan **4th Modification**	10 -2	20 +8	6	26	6	55	22 -2	+101 + 8
4th modificat.Plan **5th Modification**	8 -1	28 +5	6	26	6	55	20 -1	109 + 5
5th modificat.Plan **6th Modification**	7	33	6 - 1	26 + 7	6	55	19 -1	114 +7
6th modificat.Plan.	7	33	5	33	6	55	18	121

Therefore, the minimum time of achieving the project is 18 weeks with minimum costs 121 thousand L.E

(2): In order to determine the minimum cost for achieving the project in 23 weeks, we can do that by two different ways as follows:

Firstly: From the results of the preceding table specifically in the 3rd modification plan. where, we can have reduced the achieving time for the activity (B) by 5 weeks only instead of 6 weeks, then the minimum cost for achieving the project in 23 weeks becomes:

$= 89 + 5 \times 2 = 99$ thousand L.E.

Or, Secondly: since, it is required to determine the minimum cost for achieving the project in 23 weeks, and the 23th weeks lies between the 2nd and 3rd plans, as it be shown in the preceding table with corresponding total costs are 89 , 101 thousand L.E. respectively. Then, we can suppose that the minimum total cost for achieving the project in 23 weeks is (C), then the required cost can be determined by the relative law as follows:

Time	Total Cost
28 weeks $\longrightarrow$	89 thousand L.E.
23 weeks $\longrightarrow$	C thousand L.E.
22 weeks $\longrightarrow$	101 thousand L.E.

Then, in order to determine the value of (C) by the relative law, we have the following equation:

$$(22 - 28) / (22 - 23) = (101 - 89) / (101 - C)$$

, i.e., $(- 6) / (- 1) \quad = (12) / (101 - C)$

, i.e., $6 (101 - C) \quad = 12$

$606 - 6C \quad = 12$

$606 - 12 \quad = 594 \quad = 6\,C$

Therefore: $C = 594 / 6 \quad = 99$ thousand L.E.

i.e., the minimum total cost for achieving the project in 23 weeks is 99 thousand L.E., which is the same preceding results.

(3):

(a): To determine the minimum total cost for achieving the project in 20 weeks. It is directly from the preceding table, specifically for the 4th modification plan, we find that, the minimum total cost for achieving the project in 20 weeks is 109 thousand L.E.

(b): Also, to determine the minimum time for achieving the project with minimum total costs 83 thousand L.E., we can find that by using the two different preceding methods, for simplicity let us used the second method(Relative Law) as follows:

From the results of the preceding table specifically in the 2nd modification, i.e., between the 1st and the 2nd modified plans, i.e., the 83 thousand L.E. lies between 75 , 89 thousand L.E., then if we suppose that the minimum time for achieving the project in total cost 83 thousand L.E. is (y) weeks. Then, in order to determine the value of (y) we have the following data:

Time	Total Cost
36 $\longrightarrow$	75
y $\longrightarrow$	83
28 $\longrightarrow$	89

And then to find (y), we have the following equation:

$$\frac{(36-28)}{(36-y)} = \frac{(75-89)}{(75-83)}$$

i.e., $8 \times (-8) = -14\,(36 - y)$

$$-64 = -14 \times 36 + 14\,y$$

$504 - 64 = 14\,y$ i.e., $440 = 14\,y$

Then: $y = (440 / 14) = 31.4285714286 \approx 31.429$ weeks.

Therefore, the minimum time for achieving the project with minimum total costs 83 thousand L.E. is 31.429 weeks.

(II): <u>If the network graph contains a set of paths (at least two paths),</u> the succeeding operations of reductions in the critical activities is achieved according to the activity which have the minimum cost slope(m) by a number of time units equal to the {Min (activity available reduction) , (minimum time of free float (FF) greater than zero on the normal network graph)}. This is because we have to keep the criticality condition must be satisfied for all the steps of reductions, i.e., the critical path must be still critical for all the succeeding operations of reductions. These procedures for this technique (PERT/COST) are illustrated in the following examples if the network graph have at least two paths in the network graph:

Example (4):

The following table {Table (10)} represents the normal and crash data for achieving the set of activities for a specific project:

Table (10)

Activity	Path	Normal Data		Crash Data	
		Time (week)	Cost (thousand L. E.)	Time (week)	Cost (thousand L.E.)
A	(1 – 2)	10	7	9	7.4
B	(2 – 3)	9	18	6	20.4
C	(2 – 4)	6	18	3	24
D	(3 – 5)	7	3.5	4	6.5
E	(4 – 5)	9	5.4	7	7
F	(4 – 6)	5	5	4	6.5
G	(5 – 6)	4	8	4	8
H	(6 – 7)	7	8	5	12
Σ		57	72.9	42	91.8

Required:

(1): By Using PERT/ cost method, determine the minimum time for achieving the project in its minimum total costs.

(2): Determine the minimum cost for achieving the project in 31 weeks.

(3): Determine also the minimum time for achieving the project

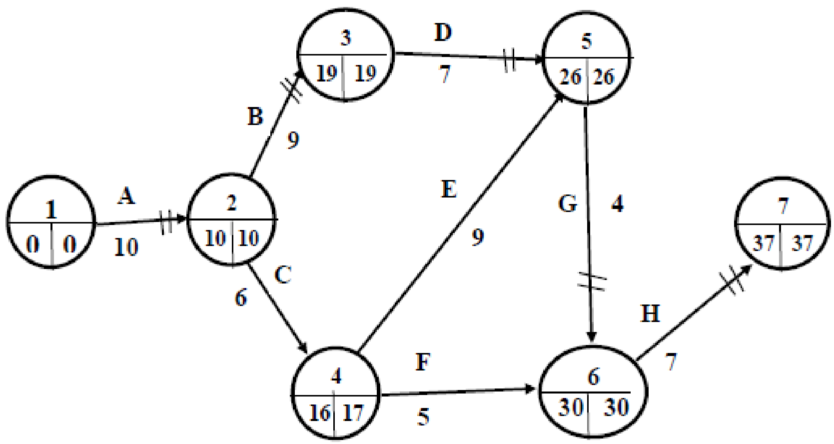
with 80 thousand total costs.

Solution:

(1): In order to determine the minimum time for achieving the project in its minimum total costs by using PERT/Cost technique, we have to follow the following steps:

(*) Draw the network graph for the project by using the normal time for achieving the activities. Then, calculate the four types of times, the earliest start and ending or completion times (E_s , E_c) and latest start and completion times(L_s , L_c) for the set of activities as it be shown in Figure No. (5) :

Figure No. 5



On that network graph, and by using the normal time for achieving the set of activities for the project, it shows that: The time of the normal critical path =

10 +9 + 7 + 4 + 7 = 37 weeks, with total costs 0f all the normal costs for all the set of achieving activities for the projects i.e.,

$T.C_n = 8 + 18 + 18 + 3.5 + 5.4 + 5 + 8 + 8 = 72.9$

(*): Draw the network graph for the project by using the crash times for activities and determine the crash critical path.

(Each student must graph this network).

According to the network graph by using the crash time, then, the minimum time for achieving the project (critical path) is 28 weeks with total costs are:

$T.C_c$= 7.4 + 20.4 + 24 + 6.5 + 7 + 6.5 + 8 + 12 = 91.8 thousand

The preceding calculation and results can be achieve from enumeration all the set of paths in the two network graphs as shown in the following table (table(11)):

Table(11)

No.	Path	Normal Data		Crash Data		Remarks
		Time	T.C	Time	T.C	
1	ABDGH	37 week	72.9 thousand L.E.	28 week	91.8 thousand L.E.	**
2	ACEGH	36 week		28 week		
3	ACFH	28 week		21 week		
∑						

Table (11), represents that the critical path which have the most or long duration time is the path (A-B-D-G-H) with 37 week as a normal duration time with minimum total costs 72.9 thousand L.E., And with 28 week as a duration crash time for either the two critical paths (ABDGH or ACEGH) with a duration crash time 28 week with the maximum total costs 91.8thousand L.E.

(*): By Comparing between the two critical paths: normal time critical path(37 weeks) by the crash time critical path (28weeks) , then, we can reduce the time for achieving the project from 37 week (normal time) to be 28 weeks (crash time) as follows: i.e.,

The limits of reduction = 37 - 28 = 9 weeks. Hence, in order to perform this reductions, we have to calculate the cost slope for all the activities in the project as it be shown in the following table {Table (12)}:

Table (12)

Data activity	Normal data		Crash data		Cost Increasing	Time Decreasing (Limits for Reduction)	Cost Slope	Successfully reductions 1	2	3	4	5
	Time	Cost	Time	Cost				A^1	B^1	B^2,E^2	C^4	C^5, D
A *	10	7	9	7.4	0.4 *	1	0.4	9	9	9	9	9
B *	9	18	6	20.4	2.4 *	3	0.8	9	8	6	6	6
C	6	18	3	24	6	3	2	6	6	6	6	3
D *	7	3.5	4	6.5	3 *	3	1	7	7	7	7	4
E	9	5.4	7	7	1.6	2	0.8	9	9	7	7	7
F	5	5	4	6.5	1.5	1	1.5	5	5	5	5	5
G *	4	8	4	8	Zero *	Zero	-	4	4	4	4	4
H *	7	8	5	12	4 *	2	2	7	7	7	5	5

Where, the Cost slope for each activity is the relative between the cost increasing and the time decreasing.

The following table (table(13)) represents the set of reductions operations instead of draw the network graph many times for this project and calculating the for types of times(E_s , E_c , L_s , L_c) and determining the time free floats for all the activities project, and reducing the critical activities by the value of the {Min (available reduction for the critical activity which have the minimum cost slope) , (minimum time of free float (FF) greater than zero on the normal network graph)}in a set of many different times. The following table { Table (13)} conclude these successfully reductions for the critical activities of the project .

Table (13)

The successfully reductions on the duration times for activities to determine the minimum total costs for achieving the project in its minimum time

Reduction / **Paths**	Normal time	(1) Reduce activity A by (1) week	(2) Reduce activity B by (2) weeks	(3) Reduce the activities (B), (E) by (2) weeks (parallel)	(4) Reduce activity H by (2) weeks	(5) Reduce both the activities D, C by (3) weeks
ABDGH	37	36	35	33	31	28
ACEGH	36	35	35	33	31	28
ACFH	28	27	27	27	25	22
Total Normal Costs	72.9	72.9	73.3	74.1	77.3	81.3
(+) Additional Reductions Cost	-	1 × 0.4 = (0.4)	1 × 0.8 = (0.8)	2 × 0.8 + 2 × 0.08 = (3.2)	2 × 2 = (4)	(3 × 2) + (3×1) = (9)
Total cost	72.9	73.3	74.1	77.3	81.3	90.3

In this table, we have the following:

(1): **1st reduction**: we search about the critical activities on the network graph(which is drown by the normal duration time),since the critical activity A have the smallest cost slope(0.4), then we will reduce the activity (A) with its reduction limit(1 week),{all the value for its limit for reduction, and hence there is no feasibility of any reduction in this activity A}, then the difference between the three paths still the same preceding difference before the 1st reduction since this activity exists in the three paths in the network graph. Then, the duration time for the three paths becomes 36 , 35 , 27 weeks respectively.

Then, the 1st reduction conclude that: " the minimum total costs for achieving the project at 36 weeks is = 72.9 + 0.4 = 73.3 thousand L.E."

(2): **2nd reduction**: we search on the critical activities (except A) on the network graph(which is drown by the normal duration time), what is the critical activity which have the smallest cost slope. Since the critical activity (B) have the smallest cost slope (0.8), then we will reduce the activity (B) by only 1 week from its reduction limit(3 weeks),{ a part from all the value for its limit for reduction, since the activity (B) is only exists in the critical path, that is because if it is reduced by a greater than one week, then, the critical path will changed and becomes noncritical path}. Therefore, according to the 2nd reduction we will reduce the activity (B) by only 1 week in the network graph. Note that, after this reduction, then we have two critical paths each one with a duration normal time equal to 35 weeks. Then, the duration time for the three paths becomes 35 , 35 , 27 weeks respectively and hence we have two critical paths.

Therefore, the 2nd reduction conclude that: " the minimum total costs for achieving the project at 35 weeks is = 73.3 + 0.8 = 74.1 thousand L.E."

(3): **<u>3rd reduction</u>**: firstly, if we search on the critical activities (on the network graph which is drown by the normal duration time) what is the critical activity which have the smallest cost slope. Since the critical activity (D) have the smallest cost slope (1.0), but the activity (D) doesn't exist or lies except for the critical path, then we can't reduce the activity (D) for no change on the critical path. Besides that, the activity (G)haven't any limits for reductions. Now, let we see for the activity (H), if we thought to reduced its normal duration time by its limit for reduction (2weeks) as a 1st alternative, it will costs an additional costs equal to 2 × 2 = 4 thousand L.E. , let us thought about other alternatives(2nd alternative) : if we reduced parallel each of the activities{(B) and(E)} by 2 week from its reduction limits(2 weeks),{ all the value for its limit for reduction}, since the activity (B) is only exists in the critical path and the activity (E) is only exists in the comparative critical path the (second path in table(13)) , therefore, according to the this reduction we will reduce the activity (B) and (E) parallel by 2 weeks in the network graph. This reduction will costs an additional cost = 2 × 0.8 + 2 × 0.8 = 3.2 thousand L.E.. Let us compare between the two alternatives. There is no doubt that is the 2nd alternative is the best since it costs less additional costs (3.2 $<$ 4).

Then the 3rd reduction conclude that : " the minimum total costs for achieving the project at 33 week is = 74.1 + 3.2 = 77.3 thousand L.E." . Then, each of the activities A, B, E and G haven't any limits of reductions. Then, the duration time for the three paths becomes 33 , 33 , 27 weeks respectively.

(4): **4th reduction**: reducing the critical activity (H) by 2weeks then, there is an additional cost = $2 \times 2 = 4$ thousand L.E. Note that the activity (H) exists in each of the three paths on the network graph. Therefore, the 4th reduction conclude that: " the minimum total costs for achieving the project at 31 week is = 77.3 + 4 = 81.3 thousand L.E." . Then, each of the activities A, B, E , G and H haven't any limits of reductions. Then, the duration time for the three paths becomes 31 , 31 , 25 weeks respectively

(5): **5th reduction**: reducing the two critical activities (C , D) by 3 weeks, then, there is an additional cost = $3 \times 2 + 3 \times 1 = 9$ thousand L.E. Note that the activity (C) exists in the second critical path and the activity(D) exists on the first critical path on the network graph. Therefore, the 5th reduction conclude that: " the minimum total costs for achieving the project at 31 week is = 81.3 + 9 = 90.3 thousand L.E." . Then, each of the critical activities A, B, E , G and H haven't any limits of reductions. Then, the duration time for the three paths becomes 28 , 28 , 22 weeks respectively.

<u>Note that</u>:

(*): After the 5th reduction, there is no feasibility of any reductions since all the normal duration time for the critical activities reached to its crash duration time.

(*): The minimum total costs for achieving the project in its minimum time resulted from PERT technique (91.3 thousand L.E. is already less than the total crash costs if the activities are achieved in its crash time which is equal to 91.8 thousand L.E.

(2): In order to determine the minimum total cost for achieving the project in 31 weeks is equal to 81.3 thousand L.E. as it be shown in the 4th reduction in table (13).

(3): In order to determine the minimum time for achieving the project with total costs equal to 80 thousand L.E. Let us suppose

That the minimum time is equal to (x) weeks. Then, in order to find the value of (x), note that the total cost 80 thousand L.E. lies between the total costs for the two successfully reductions 4th and 5th . Then, by using the relative equation from the following:

Time (week)		Total Cost (thousands L.E.)
33	$\longrightarrow$	77.3
X	$\longrightarrow$	80
31	$\longrightarrow$	81.3

Where, X is equal to the time of achieving project with 80 thousand L.E. , then we have the following relative equation:

$$\frac{31 - 33}{31 - X} = \frac{81.3 - 77.3}{81.3 - 80}$$

$$\frac{-2}{31 - X} = \frac{4}{1.3}$$

$$\frac{-1}{31 - X} = \frac{2}{1.3}$$

$- (1.3) = 2\ (31 - X)$

$- 1.3 = 62 - 2X$ i.e.,

$2X = 62 + 1.3$

$2X = 63.3$ $\rightarrow$ i.e., $X = 31.65$ weeks

Therefore, the minimum time for achieving the project with total costs = 80 thousand L.E. is 31.65 weeks.

Exercises:

(1): The following table represents the set of activities for a specific project with its achieving times by months

Activity	Starting event	Ending event	Time (month)
A	1	2	3
B	1	3	5
C	2	4	2
D	2	7	3
E	3	5	4
F	4	7	7
G	5	7	7
H	5	6	9
I	7	8	5

Required:

1- Draw the network graph for this project.

2- Determine the critical path by enumerating all the set of paths.

3- Determine the critical path from calculating the earliest start and earliest completion time for the starting and ending event for each activity and determine the total float time and free float time for all the network activities.

4- From your preceding result in (1) formulate the network graph as a LPM.

(2): The following table shows the different times for achieving the set of activities in a specific project and its time estimates

(optimistic, most likely and pessimistic) times for each activity:

Activity	Previous activities	Time		
		T_o	T_m	T_p
A	-	1	2	3
B	-	1	3	11
C	A	1	1	1
D	A	2	3	4
E	B	2	5	14
F	C	2	5	8
G	E	2	2	2
H	E	1/2	1	1.5
I	G, F, H	1	2	9

Required:

1- Draw the network diagram for the project

2- Calculate the expected time and standard deviation for the project activities.
3- Determine the critical path and the expected time for achieving the project.
4- Determine the earliest and latest for starting and ending activities and the total and free float times for each activity.
5- Find the probability of achieving the project in at least 17 weeks.
6- Determine the maximum value of time for achieving the project in a probably 95%.

(3): The following table shows the project activities A, B, C and their normal costs (thousands) and times (months) for achieving the activities and the crash costs & time for achieving these activities according to a two available alternative crash data.

Activity	Starting event	Ending event	Normal data		1st alternative		2nd alternative	
			Time	Cost	Time	cost	Time	cost
A	1	2	10	20	8	24	6	36
B	2	3	14	42	12	60	8	100
C	3	4	16	64	10	82	8	96

Required: Using PERT/Cost determine the minimum cost for achieving the project in its minimum time. And from your result what is the minimum cost for achieving the project in 33.5 months.

(4): The following table represents the normal data and crash data for achieving the set of activities for a specific project where the costs (in thousands $) and the times (in months):

Activity	Path	Normal data		Crash data	
		Time	Cost	Time	Cost
A	(1 – 2)	16	200	12	400
B	(1 – 3)	8	300	4	700
C	(2 – 4)	4	100	2	180
D	(2 – 5)	20	200	10	800
E	(3 – 4)	10	200	2	400
F	(4 – 5)	6	160	2	200

Required:

1- Draw the network diagram using the normal time of achieving the activities and determine the four types of times and the total and free float times.

2- Determine the same requirements in (1) by using the crash time for achieving the activities.

3- Determine the minimum cost for achieving the project in its minimum time.

(5): The following table shows the normal and crash data for achieving the activities for a specific project, where the costs is in thousands L.E., and times in weeks.

Activity	Path	Normal data		Crash data	
		Time	Cost	Time	Cost
A	(1 – 2)	10	7	9	7.4
B	(2 – 3)	9	18	6	20.4
C	(2 – 4)	6	18	3	24
D	(3 – 5)	7	3.5	4	6.5
E	(4 – 5)	9	5.4	7	7
F	(4 – 6)	5	5	4	6.5
G	(5 – 6)	4	8	4	8
H	(6 – 7)	7	8	5	12

Required: Determine the minimum possible total costs for achieving the project in its minimum time.

(6): The following table shows the normal time (months) and the normal cost (in thousands L.E.) for achieving the activities for a project and its crash data:

Activity	Path	Normal data		Crash data	
		Time	Cost	Time	Cost
A	(1 – 2)	12	42	9	2
B	(2 – 3)	7	30	5	4
C	(2 – 4)	10	26	7	2
D	(3 – 5)	4	10	3	1
E	(4 – 5)	11	34	6	3
F	(4 – 6)	8	14	6	1
G	(5 – 6)	7	10	5	1
H	(6 – 7)	3	13	2	4

Required: Determine the minimum possible cost for achieving the project in its minimum time by using PERT /Cost method.

South Valley University | May 2021

Faculty of Commerce | Final Exam

English Group | Time: 2Hours

Operations Research

4th year students

Answer the following Questions :

* 1st group of questions("Queuing Theory" from ques.(1) to ques.(3)).

At one-man barber shop, customers arrive according to a Poisson distribution with mean arrival rate of 9 customers per hour and the hair cutting time was exponentially distributed with an average hair cut taking 6 minutes. It is assumed that this man having a unique seat for haircut and because of his excellent reputation, customers were always willing to wait. Then:

1- The characteristics of this queuing system are:

(a): $(M_{(\lambda=9)}/M_{(\mu=6)}/1)$:(FCFS / N / N).

(b): $(M_{(\lambda=9)}/M_{(\mu=10)}/1)$:(FCFS / N / N).

"Continued..>>>>

(c): $(M_{(\lambda=9)}/M_{(\mu=15)}/1)$:(FCFS / N / N).

(d): Otherwise.

2- The 1^{st} three steady state probabilities are:

(a):(0.1,0.9,0.81). (b):(0.01,0.81,0.072).

(c):(0.1,0.09 ,0.081). (d): Otherwise.

3- The measures of effectiveness (L_s, L_q, W_s, W_q) respectively are:

(a):(8cars,9cars, 60minutes,54minutes).
(b):(9cars,8cars, 60minutes,54minutes).
(c):(9cars,8cars, 6 minutes,54minutes).
(d): otherwise.

* <u>2^{nd} group of questions:("Game theory": from ques. (4) to ques. (6)</u>)):

If you have the following pay-off matrix between two players (A) and (B):

$$\begin{array}{cc} & B \\ A & \begin{pmatrix} -2 & 3 \\ 5 & -1 \end{pmatrix} \end{array}$$

If x_1 , x_2 are the two probabilities that player (A) will use his 1^{st} and 2^{nd} strategies,

" Continued ...>>>>

also y_1 and y_2 are the two probabilities that player (B) will use his 1st and 2nd strategies and (V) is the value for the game. Then:

4- The percentage of time (x_1, x_2) that player (A) played his strategies are:

(a): (7/11, 4/11). (b): (4/11, 7/11).

(c): (6/11, 5/11). (d): Otherwise.

5- The percentage of time (y_1, y_2) that player (B) played his strategies are:

(a): (7/11, 4/11). (b): (4/11, 7/11).

(c): (6/11, 5/11). (d): Otherwise.

6- The value for the game (V) is:

(a): (13/11 to player (A)).

(b): (13/11 to player (B)).

(c): (7/11 to player (A)).

(d): Otherwise.

* <u>3rd group of questions: (transportation problem: from ques. (7) to ques. (11)</u>):

A company produced a specific identical or homogenous commodity through three sources A_1, A_2 and A_3 wanted to transport its production to four destinations B_1, B_2, B_3 and B_4. The following table represents the unit cost transported (L.E) from the

" Continued ...>>>>

set of sources to the set of destinations the number of produced (supply)units for the set of sources and the number of demand units to the set of destinations.

Des. Sou.	B_1	B_2	B_3	B_4	Supply units
A_1	3	4	6	11	200
A_2	5	7	2	8	250
A_3	10	8	3	9	300
Demand units	150	100	200	300	

Then:

7- The total transported cost for the north west corner technique is equal to:

(a):(3850L.E). (b):(3950 L.E).

(c):(4050 L.E). (d)Otherwise.

8- The total transported cost for the least cost technique is equal to:

(a):(3550 L.E). (b):(3650 L.E).

(c):(3875 L.E) (d):Otherwise.

*In the preceding transportation problem and from different point of view (i.e., from shipping company view),then the unit cost transported become unit profit transported . then:

" Continued ...>>>>

9- The total transported profit for the shipping company by using the north-west corner technique is equal to:

(a):(3550 L.E). (b):(3830 L.E).
(c):(7540 L.E). (d):otherwise.

10- The total transported profit for the shipping company by using the maximum profit technique is equal to:

(a):(5240 L.E). (b):(5700 L.E).
(c):(5780 L.E). (d):otherwise.

*<u>4th group of questions(Linear Programming Model(LPM): from ques.(11)to ques.(25</u>)):

If you have the following LPM:

$$F(x) = 5\ x_1 + 7\ x_2 \quad \text{(maximization)}$$

Subject to:

(1): $x_1 \leq 4$

(2): $x_2 \leq 6$

(3): $3\ x_1 + 2x_2 \leq 18$

(4): $x_1\ ,\ x_2 \geq 0$

<u>Required:(answer the following questions): From your graphical solution for the LPM , then</u>:

" Continued ...>>>>

11- The optimum solution for the LPM is verified when the values of x_1, x_2 respectively:

(a): (4,6). (b): (2,6).

(c): (4,3). (d): otherwise.

12- The number of basic solutions are:

(a): (8 points). (b): (7 points).

(c): (6 points). (d): otherwise.

13- The number of basic feasible solutions are:

(a): (5 points). (b): (4 points).

(c): (3 points). (d): otherwise.

14- The number of redundant constraint is:

(a): one constraint. (b): Two constraints.

(c): three constraints. (d): otherwise.

* By using the suitable simplex technique, then:

15- The pivot element and the corresponding values for the basic variables (x_2, x_3, x_5) in the 2^{nd} simplex tableau respectively are:

"Continued..>>>>

(a):(3 , 6 , 4 , 6). (b):(1 , 6 , 4 , 6).

(c):(2 , 4 , 6 , 6). (d): otherwise.

16- The corresponding coefficient for each of the basic and non-basic variables respectively are:

(a):(0,5 ,0,0,3). (b):(-3,0,0,5,0).

(c):(-5,0,0,7,0). (d): otherwise.

17- The corresponding values for each of the basic variables(x_1,x_2,x_3) and the value of the objective function in the optimum solution tableau respectively are:

(a):(6,2,6,24). (b):(2,6,2,52).

(c):(0,11/3,5,76). (d): otherwise.

18- From the optimum solution tableau and the concept of the shadow prices, the available capacity for the 1st resource is:

(a): Two units as unused capacity.

(b): Six units as unused capacity.

(c): full used capacity.

(d): otherwise. " Continued ...>>>>

19- From the optimum solution tableau and the concept of the shadow prices, the optimum value for $F(x)$ when the available capacity for the set of resources become 5 , 10 ,20 units respectively is:

(a):(60) (b):(70)

(c):(80) (d):otherwise.

20- From the optimum solution tableau and the concept of the shadow prices, the three resources respectively are:

(a): Full used capacity for the three resources.

(b): Unused capacity for the three resources.

(c): Unused capacity in the 1st resource and there is full used capacity in each of the 2nd and 3rd resources.

(d): otherwise.

21- The number of decision variables are:

(a):Two decision variables.

" Continued ...>>>>

(b): three decision variables.

(c): Four decision variables.

(d): otherwise.

22- From the optimum solution tableau, the optimum values for the decision variables to the dual problem respectively are:

(a): (0 , 6 , 2). (b): (0 , 2 , 6).

(c): (6 , 0 , 2). (d): otherwise.

23-From your preceding graphical solution,

suppose that the objective function is:

$F(x) = -5\ x_1 - 7\ x_2$, then: the optimum solution verified in the point :

(a): (4,0) (b): (0,0)

(c): (0,6) (d): otherwise.

24-From your preceding graphical solution, suppose that the objective function is $F(x)=5x_1-7x_2$, then, the optimum solution verified in the point :

(a): (6,0) (b): (0,0)

(c): (4,0) (d): otherwise.

"Continued ...>>>>

25-From your preceding graphical solution, suppose that the objective function is

$$F(x) = -5\ x_1 + 7\ x_2\ ,$$

Then, the optimum solution verified in the point:

(a):(4,0) (b):(0,0)

(c):(0,6) (d):otherwise.

"My Best Wishes & Good Luck ....."

References:

(1): **Arabic References**:

(1): د. محمد عبدالحميد الربع ، مقدمة فى بحوث العمليات ، كلية التجارة بقنا ، جامعة جنوب الوادى ، 2008 .

(2): د. أحمد على حسين ، مقدمة فى بحوث العمليات ، ترجمة مقدمة فى بحوث العمليات للدكتور حمدى طه : المكتبة الأكاديمية بالقاهرة ، 2010.

(3): د. نادية مكارى جرجس ، محاضرات فى بحوث العمليات ، كلية الإقتصاد والعلوم السياسية ، جامعة القاهرة ، 1980 .

(2): **English References**:

(1): Dantzig.G.B., Linear Programming and Extensions, Princeton University Press, Princeton N.J., 1963.

(2): Hadley, G., Linear Programming, Addison- Wesley, Reading, Mass. 1962.

(3): Hamdy A. Taha : " Operations Research- An Introduction", first edition , 1971.

	South Valley University Faculty of Commerce	Date: 1 / 6 / 2022 Time Allowed:3 hours	
	Operations Research For 4th year students		
	Final Exam	English Section	

You have 50 questions; please mark all your answers on the answer sheet provided to you. You have to submit both question papers and answer sheet.

Please: Make sure that the answer sheet form matches the question form.

-Choose the best answer for each of the following questions.

Hint: Two points are assigned for each MCQs questions from (Q_1) to (Q_{30}) and True & False questions from (Q_1) to (Q_{20}) of the following questions.

Firstly: The 1st group of questions is Multiple Choose Questions (MCQs) from Q(1) to Q(30).

Queuing Theory: The graduate student from the faculty of commerce in Qena arrival to the window of graduate affairs to get graduation certificates and their files according to a Poisson distribution with rate 70 graduate per hour. And the service rate for this window have a Poisson distribution with 80 graduate per hour.

Required: Answer the following question from $Q_{(1)}$ to $Q_{(4)}$:

$Q_{(1)}$: The characteristics for this queuing model concluded in the model:

(a): ($M_{(\lambda=7)}$ / $M_{(\mu=8)}$ /1) : (FCFS / N / N).

(b): ($M_{(\lambda=80)}$ / $M_{(\mu=70)}$ /1) : (FCFS / N / N).

(c): ($M_{(\lambda=70)}$ / $M_{(\mu=80)}$ /1) : (FCFS / N / N).

(d): Otherwise.

$Q_{(2)}$: The percentage of time that the service is idle (or available) and the probability that there are 3 graduates are in the system respectively:

(a):(12.5% , 0.084) (b):(84% , 0.125) (c):(33.3% , 0.333) (d):Otherwise.

$Q_{(3)}$: The expected number of graduates(grad.) in either the system or in the queue respectively are :

(a):(6 , 5 grad.)   (b): (7 , 6 grad.)   (c):(8 , 7 grad.)   (d):Otherwise.

$Q_{(4)}$: The expected spending time for each graduates in either the system or in the queue respectively are :

(a):(6 , 5.25 min.)   (b): (5.25 , 4.5 min.)   (c):(4 , 3 min.)   (d):Otherwise.

Game Theory :If you have the following Payoffs matrix between two players A and B :

$$\begin{array}{cc} & B \\ A & \begin{pmatrix} -7 & -2 & 3 \\ 6 & -3 & 4 \end{pmatrix} \end{array}$$

Then: Answer the following question from $Q_{(5)}$ to $Q_{(9)}$:

$Q_{(5)}$: From the concept of the elements for the Payoff matrix, if each of the two players are played with their 2^{nd} strategy, then:

(a): The player A will gain or win 6 points and B will lose 6 points.

(b): The player A will gain or win 4 points and B will lose 4 points.

(c): The player A will lose 3 points and B will gain or win 3 points.

(d): Otherwise.

$Q_{(6)}$: **As** a result for the absence of the saddle point and using the idea of the submatrices, then, the probabilities that the row player (A) will played with his 1^{st} strategy in the three sub matrices respectively are :

(a): (0.357 , 0 , 0.643).   (b): (0.357 , 0.643 , 0).

(c): (0.634 , 0.357 , 0).   (d): Otherwise.

$Q_{(7)}$: **As** a result for the absence of the saddle point and using the idea of the submatrices, then the probabilities that the player (B) will played with his 2^{nd} strategy in the three sub matrices respectively are:

(a): (0.357 , 0.929 , 1).   (b): (0.929 , 1 , 0).

(c): (1 , 0.929 , 0).   (d): Otherwise.

$Q_{(8)}$: The value for the game for each the three sub matrices respectively are:

(a): (-7 , 4 , 3 and each submatrix is zero sum game).
(b): (-3 , 4 , -2 and each submatrix is zero sum game).
(c): (3 , 4 , 6 and each submatrix is zero sum game).
(d): Otherwise.

$Q_{(9)}$: If we want to apply the dominance principal on the Payoffs matrix, then, the probabilities that the player (B) will played with his 1st and his 2nd strategy in the resulted payoffs matrix are respectively :
(a): (9/14 , 5/14). (b): (3/14 , 11/14). (c): (1/14 , 13/14). (d): Otherwise.

Transportation Problem: The following table represents the transported unit profit (L.E.) for one of a transshipment companies gained from transporting a homogeneous commodity produced in a specific company from four sources to three destinations, the available capacity for each source (thousand) from the commodity units and the demand units for each destination(thousand):

sou. \ Des.	B_1	B_2	B_3	Supply units
A_1	10	7	8	300
A_2	10	11	14	500
A_3	9	12	4	600
A_4	11	13	9	200
Demand units	500	600	500	

Required :(Answer the following questions from $Q_{(10)}$ to $Q_{(15)}$):

$Q_{(10)}$: If we formulate this transportation problem for this transshipment company, then the Objective Function (Obj.Fun.) is to …… in a LPM contains a set of ….. Decision Variables(Dec.Vra.) constrained with a set of ….. constraints (Con_S.) : (fill in the space with the correct answer):
(a): (Maximization the value of(Obj.Fun.) , 12 (Dec.Var.) , 20 (Con_S.).
(b): (Minimization the value of(Obj.Fun.) , 7 (Dec.Var.) , 10 (Con_S.).
(c): (Minimization the value of(Obj.Fun.) , 12 (Dec.Var.) , 20 (Con_S.).
(d): Otherwise.

$Q_{(11)}$: From the initial solution tableau for the North West Corner Technique, the total profits resulted from this technique is :

(a): (19400 L.E.). (b): (14900 L.E.). (c): (14500 L.E.). (d): Otherwise.

$Q_{(12)}$: From the initial solution tableau for the Descending Profits Technique, the total profits resulted from this technique is :

(a): (19200 L.E.). (b): (19300 L.E.). (c): (19450 L.E.). (d): Otherwise.

$Q_{(13)}$: From the initial solution tableau for the Vogel's Approximation Technique, the total profits resulted from this technique is :

(a): (19450 L.E.). (b): (19400 L.E.). (c): (19350 L.E.). (d): Otherwise.

$Q_{(14)}$: From your preceding result in the initial solution tableau for the North West Corner Technique, and by using one of the two techniques by which are used as a test of optimality, then the Evidence Improvement for the two cells A_1B_2 and A_1B_3 are respectively :

(a): (3 , 2). (b): (4 , 5). (c): (4 , -5). (d): Otherwise.

$Q_{(15)}$: From your preceding result in the initial solution tableau for the North West Corner Technique, and by using the closed path for the cell A_1B_3 For improve the initial solution tableau, then, from the Evidence Improvement for this cell the total profits after this improvement process becomes :

(a): (16400 L.E.). (b): (16800L.E.). (c):(16850 L.E.). (d): Otherwise.

<u>Network Analysis</u>: The following table represents the normal & crash costs(by million L.E.), and the normal & crash times (by month) for achieving the set of activities for a one of a specific government projects:

Activity	Path	Normal Data		Crash Data	
	Start→ End	Time	Cost	Time	Cost
A	(1 – 2)	16	200	12	400
B	(1 – 3)	8	300	4	700
C	(2 – 4)	4	100	2	180
D	(2 – 5)	20	200	15	800
E	(3 – 5)	10	200	5	400
F	(4 – 5)	6	160	4	200
$\sum$			1160		2680

Required :(Answer the following questions from $Q_{(16)}$ to $Q_{(20)}$):

$Q_{(16)}$ **:** From your computations about the four different types of times by using the Forward and Backward Methods, the time for the normal critical path is ……with minimum feasible costs …… respectively:
(a): (22 month , 400 million L.E.). (b): (24 month , 1160 million L.E.).
(c): (36 month , 1160 million L.E.). (d): Otherwise.

$Q_{(17)}$: From your preceding computations about the four different types of times by using the Forward and Backward Methods, the time total floats for each of three activities (B) , (C) and (E) respectively are …,…,…. :
(a): (1 , 0 , 1). (b): (2 , 0 , 2). (c): (3 , 0 , 2). (d): Otherwise.

$Q_{(18)}$: By using PERT/Cost technique, the minimum time for achieving this project is …… month with minimum feasible cost is ….. million L.E. are respectively:

(a): (26 month , 1160 million L.E.). (b): (27 month , 1960 million L.E.).
(c): (18 month , 1640 million L.E.). (d): Otherwise.

$Q_{(19)}$**:** According to PERT/Cost technique, the minimum total costs for achieving the set of activities for this project in 28 month is …… :

(a): (1810 million L.E.). (b): (1830 million L.E.).
(c): (1840 million L.E.). (d): Otherwise.

$Q_{(20)}$: According to the PERT/Cost technique, the minimum time for achieving the set of activities for this project with 1500 million L.E. total costs is …… :

(a): (27 month). (b): (30 month). (c): (31 month). (d): Otherwise.

Linear programming models: if you have the LPM:

$$F(x) = X_1 + 2 X_2 \qquad \text{(Maximization)}$$

Subject to :

$$- 3 X_1 + 3 X_2 \leq 9$$

$$X_1 - X_2 \leq 2$$

$$X_1 + X_2 \leq 6$$

$$X_1 + 3 X_2 \leq 6$$

$$X_1 , \quad X_2 \geq 0$$

Required: (Answer the following questions from $Q_{(21)}$ to $Q_{(30)}$):

$Q_{(21)}$: From the graphical solution for the LPM, the number of redundant constraints (R.C) is (are)….. :

(a): (One (R.C)). (b): (Two ($R.C_S$)). (c): (Three ($R.C_S$). (d): Otherwise.

$Q_{(22)}$: From the graphical solution for the LPM, the number of basic solutions, basic feasible solutions and optimal solution in case of maximizing the value of F(x) respectively are :

(a):(12 , 5,multiple optimum solutions). (b):(12 , 5,unique optimum solution).
(c):(10 , 4 , multiple optimum solutions.). (d): Otherwise.

$Q_{(23)}$**:** From the graphical solution for the LPM, the feasible solution for the LPM is ……. , and contains a number of ….. of feasible solutions respectively are **:**

(a): (Convex , infinite number). (b): (Non convex , finite number).
(c): (Convex , 12 points). (d): Otherwise.

$Q_{(24)}$**:** From your preceding results for the graphical solution to the LPM, and exclusion(deleting) the redundant constraint(s) if there is(are) exist and by using the suitable simplex method, the values for the pivot element from the initial solution tableau to the optimum solution are respectively:

(a): (1 , 4/3). (b): (3 , 4/3). (c): (1 , 1/3). (d): Otherwise.

$Q_{(25)}$**:** From your preceding results for the graphical solution to the LPM, and exclusion(deleting) the redundant constraint(s) if there is(are) exist and by determining the dual problem, the number of decision variable and the number of constraints for the dual problem are respectively:

(a): (4 , 2). (b): (3 , 2). (c): (3 , 3). (d): Otherwise.

$Q_{(26)}$**:** From your preceding results for the graphical solution to the LPM, and exclusion(deleting) the redundant constraint(s) if there is(are) exist and by using the suitable simplex method for the primal problem, we can conclude that the optimum values for the decision and slack variables and the value for the dual problem are respectively:

(a): (3 , 1 , 0 , 0 , 5). (b): (1/4 , 3/4 , 0 , 0 , 5).
(c): (0 , 0 , 1/4 , 3/4 , 5). (d): Otherwise.

Q**(27)**: From your preceding results for the graphical solution to the LPM, and exclusion(deleting) the redundant constraint(s) if there is(are) exist and by using the suitable simplex method, we can conclude that the elements for the optimality condition row in the optimum solution for the dual problem are respectively:

(a): (-2 , 0 , -3 , 0). (b): (0 , 0 , -3 , -1).
(c): (0 , 0 , -1 , -3). (d): Otherwise.

Q**(28)**: From your preceding results for the graphical solution to the LPM, and exclusion(deleting) the redundant constraint(s) if there is(are) exist and by using the suitable simplex method and the concept for the shadow prices for the primal problem, the value of the F(x) becomes ……. When the constants for the constraints is changed to become 10 , 5 , 10 , 10 respectively is equal to :

(a): (35/4). (b): (30/4). (c): (25/4). (d): Otherwise.

Q**(29)**: supposing that $F(x) = X_1 - 2X_2$, and from the preceding graphical solution, then, the optimum solution in either (Max) or (Min) for F(x) are respectively:

(a): (Unique solution in the two points (2 , 0) and (0 , 2)).
(b): (Unique solution in the two points (3 , 0) and (0 , 2)).

(c): (Multiple solutions in the two points (3 , 1) and (0 , 2)). (d): Otherwise.

$Q_{(30)}$: supposing that $F(x) = - X_1 + 2X_2$, and from the preceding graphical solution, then, the optimum solution in either (Max) or (Min)for F(x) are respectively:

(a): (Unique solution in the two points (2 , 0) and (0 , 2)).
(b): (Multiple solutions in the two points (2 , 0) and (0 , 2)).
(c): (Unique in the two points (3 , 1) and (0 , 2)). (d): Otherwise.

Secondly: The 2nd group of questions is True &False questions from $Q_{(1)}$ to $Q_{(20)}$: In the electronic paper sheet(Papal Sheet), select(T) for the correct answer and(F) for the false answer:

$Q_{(1)}$: The objective of the queuing theory is only minimizing the waiting and service time for the customer.

$Q_{(2)}$: If the service time in the system have an Exponential distribution with parameter five minutes mean time, then the service rate have a Poisson distribution with rate is five customers per minute.

$Q_{(3)}$: In the Game Theory, if there is a saddle point in the Payoffs game matrix , then, there are mixed strategies and then the optimum strategy for each player is that which have the most likely (or probability) strategy.

$Q_{(4)}$: In the Game Theory, particularly when we applied the dominance principal from the viewpoint for the column player, the strategy which have the most negative values must be dominates the other strategies.

$Q_{(5)}$: In the balance transportation problem table which have (n) sources and (m) destinations, the formulation process for the transportation problem as a LPM is to find the values for a set of (n + m) decision variables in the objective function constrained by a set of $(n \times m)$ constraints.

Q$_{(6)}$: In the unbalanced table for the transportation problem which have the total destinations demand greater than the total sources supply, then, we have to added a dummy column with zero cost (or profit) units to make the transportation problem in its equilibrium case or condition.

Q$_{(7)}$: If all the elements for the evidence improvements in the evaluation table for the non-basic(empty) cells for the transportation table are negative coefficients in case of minimizing the total transported costs(or positive coefficients in case of maximizing the total transportation profits), then the transportation table by which is evaluated is considered a multiple optimum solution.

Q$_{(8)}$: When we are improved a specific transportation table by finding the closed path for the non-basic cell(entering variable) ,then the total transportation costs (or total profits) will be decreased(or increased) by the value of the multiplication for the evidence improvement for this cell × the number of commodity units by which it will occupied in this cell which is determined according to its closed path .

Q$_{(9)}$: The initial solution tableau for an equilibrium transportation problem Which have a set of (n) sources and (m) destinations which contains a set of (n+m-1) occupied cells(basic variable)is considered a basic feasible solution.

Q$_{(10)}$: In the Project Network Analysis specifically in the Critical Path Technique, the critical activities only haven't times free floats.

Q$_{(11)}$: In the Project Network Analysis specifically in PERT/Cost Technique, before making any reduction in the normal time for the critical activities, the time for achieving the project according to the critical path which is considered the long time for achieving the project comparing by all the different paths, but that time is considered the minimum time for achieving the project in its minimum total cost.

$Q_{(12)}$: In the Project Network Analysis specifically in PERT/Cost Technique, When, we make the reduction in the normal time for the critical Activities, we can reduced the duration time for the critical activities parlay with non-critical activities according to that which have the minimum cost slope.

$Q_{(13)}$: When we put the LPM in its Canonical Form, we have to put all the constraints in the equation form except for the nonegativity constraints.

$Q_{(14)}$: When we put the LPM in either its Canonical or Standard Form, the absolute values constraints with the inequality greater than or equal to ($\geq$), we have to disassemble each one of these constraints into two sub constraints and we take the resulting two subconstraints together with the remainder constraints in the solution for this LPM.

$Q_{(15)}$: The feasible solution space resulted from the solution for the LPM is always a convex feasible solution space in either the feasible solution space is bounded or unbounded.

$Q_{(16)}$: In the optimum solution tableau, the zero coefficient under the nonbasic variable in the row of the optimality condition means that there are different and multiple optimum values for the variables in the LPM with a unique value for the objective function.

$Q_{(17)}$: We use the Charnize &Cooper rule for determining the pivot row in case of the multiple candidate leaving variables.

$Q_{(18)}$: If all the element in the pivot column during the improvement for the solution in the simplex techniques are only negative values and zeros, then, we have an unbounded feasible solution space for the set of constraints to the LPM and then the optimum solution is unbounded (infinite solution).

$Q_{(19)}$: Minimum value constraints for the decision variables considered redundant constraints.

$Q_{(20)}$: Existing an artificial variable in the optimum solution tableau as a basic variable with a value greater than zero in the feasibility condition means that the LPM haven't a feasible solution space.

My Best Wishes & Good Luck …………..