Calculus II
Integration Methods

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Contents

1. Indefinite Integrals

1.1 Introduction

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the following Table

Symbols/Terms/Phrases	**Meaning**
$\int f(x)\,dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\,dx$	Integrand
x in $\int f(x)\,dx$	Variable of integration
Integrate	Find the integral
An integral of f	A function F such that $\acute{F}(x) = f(x)$
Integration	The process of finding the integral
Constant of Integration	An arbitrary constant C, considered as constant function

1.2 Elementary Integrals

Applying the fundamental theorem of calculus, one can obtain the following integrals.

Derivatives		**Integrals (Anti derivatives)**
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$	$\rightleftarrows$	$\int x^n dx=\frac{x^{n+1}}{n+1}+\text{C},\ n\neq -1$
$\frac{d}{dx}(\sin x)=\cos x$	$\rightleftarrows$	$\int \cos x\,dx=\sin x+C$
$\frac{d}{dx}(-\cos x)=\sin x$	$\rightleftarrows$	$\int \sin x\,dx=-\cos x+C$
$\frac{d}{dx}(\tan x)=\sec^2 x$	$\rightleftarrows$	$\int \sec^2 x dx=\tan x+C$
$\frac{d}{dx}(-\cot x)=\text{cosec}^2 x$	$\rightleftarrows$	$\int \text{cosec}^2 x dx=-\cot x+C$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\rightleftarrows$	$\int \sec x\tan x dx=\sec x+\text{C}$
$\frac{d}{dx}(-\text{cosec}\, x)=\text{cosec}\, x\cot x$	$\rightleftarrows$	$\int \text{cosec}\, x\cot x dx=-\text{cosec}\, x+C$
$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$	$\rightleftarrows$	$\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+\text{C}$
$\frac{d}{dx}\left(-\cos^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$	$\rightleftarrows$	$\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1}x+\text{C}$
$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$	$\rightleftarrows$	$\int \frac{dx}{1+x^2}=\tan^{-1}x+\text{C}$
$\frac{d}{dx}\left(-\cot^{-1}x\right)=\frac{1}{1+x^2}$	$\rightleftarrows$	$\int \frac{dx}{1+x^2}=-\cot^{-1}x+\text{C}$
$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^2-1}}$	$\rightleftarrows$	$\int \frac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+\text{C}$
$\frac{d}{dx}\left(-\text{cosec}^{-1}x\right)=\frac{1}{x\sqrt{x^2-1}}$	$\rightleftarrows$	$\int \frac{dx}{x\sqrt{x^2-1}}=-\text{cosec}^{-1}x+\text{C}$
$\frac{d}{dx}(e^x)=e^x$	$\rightleftarrows$	$\int e^x dx=e^x+\text{C}$
$\frac{d}{dx}(\log\lvert x\rvert)=\frac{1}{x}$	$\rightleftarrows$	$\int \frac{1}{x}dx=\log\lvert x\rvert+\text{C}$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right)=a^x$	$\rightleftarrows$	$\int a^x dx=\frac{a^x}{\log a}+\text{C}$

Some properties of indefinite integral

1. $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$

2. $\int m\,f(x)\,dx = m\int f(x)\,dx,$ where m is a constant.

■ **Example 1.1** Write an anti derivative for each of the following functions using the method of inspection:

$$(a)\ \cos 2x \qquad (b)\ 3x^2+4x^3 \qquad (c)\ \frac{1}{x}\ , x \neq 0$$

Solution.

(a) We look for a function whose derivative is $\cos 2x$. Recall that

$$\frac{d}{dx}\sin 2x = 2\cos 2x$$

or

$$\cos 2x = \frac{1}{2}\frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

Therefore,

$$\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$$

(b)

$$\int (3x^2+4x^3)dx = 3\left(\frac{x^3}{3}\right)+4\left(\frac{x^4}{4}\right)+C = x^3+x^4+C$$

(c) We know that

$$\frac{d}{dx}\log x = \frac{1}{x}, \quad x>0,$$

so

$$\int \frac{1}{x}\,dx = \log x + C,$$

in general

$$\int \frac{\acute{f}(x)}{f(x)}\,dx = \log f(x) + C,$$

■

■ **Example 1.2** Find the following integrals:

$$(a)\int \frac{x^3-1}{x^2}\,dx \qquad (b)\int \left(x^{\frac{2}{3}}+1\right)dx \qquad (c)\int \left(x^{\frac{3}{2}}+2\,e^x-\frac{1}{x}\right)dx,$$

Solution.

(a)

$$\begin{aligned}\int \frac{x^3-1}{x^2}\,dx &= \int \frac{x^3}{x^2}\,dx - \int \frac{1}{x^2}\,dx \\ &= \int x\,dx - \int x^{-2}\,dx \\ &= \frac{x^2}{2} - \frac{x^{-2+1}}{-2+1} + C = \frac{x^2}{2} + \frac{1}{x} + C\end{aligned}$$

(b)

$$\int \left(x^{\frac{2}{3}}+1\right)dx = \int x^{\frac{2}{3}}\,dx + \int dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + x + C = \frac{5}{3}x^{\frac{5}{3}} + x + C$$

(c)

$$\begin{aligned}\int \left(x^{\frac{3}{2}}+2e^x-\frac{1}{x}\right)dx &= \int x^{\frac{3}{2}}\,dx + \int 2e^x\,dx - \int \frac{1}{x}\,dx \\ &= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2e^x - \log x + C \\ &= \frac{2}{5}x^{\frac{5}{2}} + 2e^x - \log x + C\end{aligned}$$

■

■ **Example 1.3** Find the following integrals:

(a) $\int (\sin x + \cos x)dx,$ (b) $\int \operatorname{cosec} x(\operatorname{cosec} x + \cot x)\,dx,$ (c) $\int \frac{1-\sin x}{\cos^2 x}dx.$

Solution.

(a)

$$\int (\sin x + \cos x)dx = -\cos x + \sin x + C$$

(b)

$$\begin{aligned}\int \operatorname{cosec} x(\operatorname{cosec} x + \cot x)\,dx &= \int \operatorname{cosec}^2 x\,dx + \int \operatorname{cosec} x \cot x\,dx \\ &= -\cot x - \operatorname{cosec} x + C\end{aligned}$$

(c)

$$\begin{aligned}\int \frac{1-\sin x}{\cos^2 x}dx &= \int \frac{1}{\cos^2 x}dx - \int \frac{\sin x}{\cos^2 x}dx \\ &= \int \sec^2 x\,dx - \int \tan x \sec x\,dx \\ &= \tan x - \sec x + C\end{aligned}$$

■

■ **Example 1.4** Find the anti derivative F of f defined by $f(x) = 4x^3 - 6$, where $F(0) = 3$.

Solution:

$$F(x) = \int (4x^3 - 6)dx = x^4 - 6x + C$$

where C is constant, given that $F(0) = 3$, which gives,

$$3 = 0 - 6 \times 0 + C, \quad \Longrightarrow \quad C = 3$$

Hence, the required anti derivative is the unique function F defined by

$$F(x) = x^4 - 6x + 3.$$

■

Exercise 1.1 Find an anti derivative (or integral) of the following functions by the method of inspection.

(1) $\sin 2x$.

(2) $\cos 3x$.

(3) e^{2x}.

(4) $(ax+b)^2$.

(5) $\sin 2x - 4e^{3x}$.

Find the following integrals in Exercises 6 to 20:

(6) $\int (4e^{3x}+1)dx.$

(7) $\int x^2\left(1-\frac{1}{x^2}\right)dx.$

(8) $\int (ax^2+bx+c)dx.$

(9) $\int \left(2x^2+e^x\right)dx.$

(10) $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx.$

(11) $\int \frac{x^3+5x^2-4}{x^2}dx.$

(12) $\int \frac{x^3+3x+4}{\sqrt{x}}dx.$

(13) $\int \frac{x^3-x^2+x-1}{x-1}dx.$

(14) $\int (1-x)\sqrt{x}dx.$

(15) $\int \sqrt{x}\left(3x^2+2x+3\right)dx.$

(16) $\int (2x-3\cos x+e^x)dx.$

(17) $\int \left(2x^2-3\sin x+5\sqrt{x}\right)dx.$

(18) $\int \sec x(\sec x+\tan x)dx.$

(19) $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}dx.$

(20) $\int \frac{2-3\sin x}{\cos^2 x}dx.$

2. Techniques of integration

In previous chapter, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function F whose derivative is f which led us to the integral of f. However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on:

1. Integration by Substitution
2. Integration using Partial Fractions
3. Integration by Parts

2.1 Integration by substitution

In this section, we consider the method of integration by substitution.
The given integral $\int f(x)dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.
Consider

$$I = \int f(x)dx$$

Put $x = g(t)$ so that

$$\frac{dx}{dt} = \acute{g}(t)$$

. We write

$$dx = \acute{g}(t)dt$$

Thus

$$I = \int f(x)dx = \int f(g(t))\acute{g}(t)dt$$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

■ **Example 2.1** Integrate the following functions w.r.t. x:

(a) $\sin mx$, (b) $2x\sin(x^2+1)$,

(c) $\dfrac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}$, (d) $\dfrac{\sin\tan^{-1}x}{1+x^2}$

Solution:

(a) We know that derivative of mx is m. Thus, we make the substitution $mx = t$ so that $mdx = dt$. Therefore,

$$\int \sin mx\,dx = \int \frac{1}{m}\sin t\,dt = -\frac{1}{m}\cos t + C = -\frac{1}{m}\cos mx + C$$

(b) Derivative of x^2+1 is $2x$. Thus, we use the substitution $x^2+1 = t$ so that $2x\,dx = dt$. Therefore,

$$\int 2x\sin(x^2+1)dx = \int \sin t\,dt = -\cos t + C = -\cos(x^2+1) + C$$

(c) Derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

Thus, we use the substitution $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}dx = dt$ giving $dx = 2tdt$.

Thus,

$$\int \frac{\tan^4\sqrt{x}\sec^2\sqrt{x}}{\sqrt{x}}dx = \int \frac{2t\tan^4 t\sec^2 t dt}{t} = 2\int \tan^4 t\sec^2 t dt$$

Again, we make another substitution $\tan t = u$ so that $\sec^2 t dt = du$ Therefore,

$$
\begin{aligned}
2\int \tan^4 t \sec^2 t dt = 2\int u^4 du = 2\frac{u^5}{5} + \mathrm{C} \\
&= \frac{2}{5}\tan^5 t + \mathrm{C}(\text{ since } u = \tan t) \\
&= \frac{2}{5}\tan^5 \sqrt{x} + \mathrm{C}(\text{ since } t = \sqrt{x})
\end{aligned}
$$

Hence,

$$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \frac{2}{5}\tan^5 \sqrt{x} + \mathrm{C}$$

Alternatively, make the substitution $\tan \sqrt{x} = t$

(d) Derivative of $\tan^{-1} x = \frac{1}{1+x^2}$. Thus, we use the substitution

$$\tan^{-1} x = t \text{ so that } \frac{dx}{1+x^2} = dt.$$

Therefore,

$$\int \frac{\sin\left(\tan^{-1} x\right)}{1+x^2} dx = \int \sin t dt = -\cos t + \mathrm{C} = -\cos\left(\tan^{-1} x\right) + \mathrm{C}$$

■

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\int \tan x dx = \log|\sec x| + C$

We have

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

Put $\cos x = t$ so that $\sin x dx = -dt$

Then

$$
\begin{aligned}
&\int \tan x dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C \\
&\int \tan x dx = \log|\sec x| + C
\end{aligned}
$$

(ii) $\int \cot x dx = \log|\sin x| + C$

We have

$$\int \cot x dx = \int \frac{\cos x}{\sin x} dx$$

Put $\sin x = t$ so that $\cos x dx = dt$

Then

$$\int \cot x dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$$

(iii) $\int \sec x dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$$

Put $\sec x + \tan x = t$ so that $\sec x(\tan x + \sec x) dx = dt$

Therefore,

$$\int \sec x dx = \int \frac{dt}{t} = \log|t| + \mathrm{C} = \log|\sec x + \tan x| + \mathrm{C}$$

(iv) $\int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$

We have

$$\int \operatorname{cosec} x dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{(\operatorname{cosec} x + \cot x)} dx$$

Put $\operatorname{cosec} x + \cot x = t$ so that $-\operatorname{cosec} x(\operatorname{cosec} x + \cot x) dx = dt$

So

$$\begin{aligned}
\int \operatorname{cosec} x dx &= -\int \frac{dt}{t} = -\log|t| = -\log|\operatorname{cosec} x + \cot x| + C \\
&= -\log\left|\frac{\operatorname{cosec}^2 x - \cot^2 x}{\operatorname{cosec} x - \cot x}\right| + C \\
&= \log|\operatorname{cosec} x - \cot x| + C
\end{aligned}$$

■ **Example 2.2** Find the following integrals:

(a) $\int \sin^3 x \cos^2 x dx.$

(b) $\int \frac{\sin x}{\sin(x+a)} dx.$

(c) $\int \frac{1}{1+\tan x} dx.$

■

Solution:

(a) We have

$$\begin{aligned}\int \sin^3 x \cos^2 x dx &= \int \sin^2 x \cos^2 x (\sin x) dx \\ &= \int \left(1-\cos^2 x\right) \cos^2 x (\sin x) dx\end{aligned}$$

Put $t = \cos x$ so that $dt = -\sin x dx$

Therefore,

$$\begin{aligned}\int \sin^2 x \cos^2 x (\sin x) dx &= -\int \left(1-t^2\right) t^2 dt \\ &= -\int \left(t^2 - t^4\right) dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + \mathrm{C} \\ &= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + \mathrm{C}\end{aligned}$$

(b) Put $x + a = t$. Then $dx = dt$.

Therefore

$$\begin{aligned}\int \frac{\sin x}{\sin(x+a)} dx &= \int \frac{\sin(t-a)}{\sin t} dt \\ &= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} dt \\ &= \cos a \int dt - \sin a \int \cot t dt \\ &= (\cos a)t - (\sin a)\left[\log|\sin t| + \mathrm{C}_1\right] \\ &= (\cos a)(x+a) - (\sin a)\left[\log|\sin(x+a)| + \mathrm{C}_1\right] \\ &= x\cos a + a\cos a - (\sin a)\log|\sin(x+a)| - \mathrm{C}_1 \sin a\end{aligned}$$

Hence,

$$\int \frac{\sin x}{\sin(x+a)} dx = x\cos a - \sin a \log|\sin(x+a)| + C$$

, where, $\mathrm{C} = -\mathrm{C}_1 \sin a + a\cos a$, is another arbitrary constant.

(c)

$$\begin{aligned}\int \frac{dx}{1+\tan x} &= \int \frac{\cos x dx}{\cos x + \sin x} \\ &= \frac{1}{2}\int \frac{(\cos x + \sin x + \cos x - \sin x) dx}{\cos x + \sin x} \\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x} dx \\ &= \frac{x}{2} + \frac{\mathrm{C}_1}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x} dx\end{aligned}$$

Now, consider

$$\mathrm{I} = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$$

Put $\cos x + \sin x = t$ so that $(\cos x - \sin x)dx = dt$

Therefore

$$\mathrm{I} = \int \frac{dt}{t} = \log|t| + \mathrm{C}_2 = \log|\cos x + \sin x| + \mathrm{C}_2$$

Putting it in (1), we get

$$\begin{aligned}
\int \frac{dx}{1+\tan x} &= \frac{x}{2} + \frac{\mathrm{C}_1}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{\mathrm{C}_2}{2} \\
&= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{\mathrm{C}_1}{2} + \frac{\mathrm{C}_2}{2} \\
&= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + \mathrm{C}, \left(\mathrm{C} = \frac{\mathrm{C}_1}{2} + \frac{\mathrm{C}_2}{2}\right)
\end{aligned}$$

Exercise 2.1 Integrate the following functions:

(1) $\dfrac{2x}{1+x^2}$.

(2) $\dfrac{(\log x)^2}{x}$.

(3) $\dfrac{1}{x+x\log x}$.

(4) $\sin x \sin(\cos x)$.

(5) $\sin(ax+b)\cos(ax+b)$.

(6) $\sqrt{ax+b}$.

(7) $x\sqrt{1+2x^2}$.

(8) $(4x+2)\sqrt{x^2+x+1}$.

(9) $\dfrac{1}{x-\sqrt{x}}$.

(10) $\left(x^3-1\right)^{\frac{1}{3}}x^5$.

(11) $\dfrac{x^2}{(2+3x^3)^3}$.

(12) $\dfrac{x}{9-4x^2}$.

(13) e^{2x+3}.

(14) $\dfrac{x}{e^{x^2}}$.

(15) $\dfrac{e^{tan^{-1}x}}{1+x^2}$.

Integration using trigonometric identities

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

Example 2.3 Find

(a) $\int \cos^2 x\,dx$.

(b) $\int \sin 2x \cos 3x\,dx$.

(c) $\int \sin^3 x\,dx$.

■

Solution:

(a) Recall the identity $\cos 2x = 2\cos^2 x - 1$, which gives

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Therefore,

$$\begin{aligned}\int \cos^2 x dx &= \frac{1}{2}\int (1+\cos 2x) dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx \\ &= \frac{x}{2} + \frac{1}{4}\sin 2x + C\end{aligned}$$

(b) Recall the identity

$$\sin x \cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$$

(Why?)

Then

$$\begin{aligned}\int \sin 2x \cos 3x dx &= \frac{1}{2}\left[\int \sin 5x dx \cdot \int \sin x dx\right] \\ &= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right] + C \\ &= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C\end{aligned}$$

(c) From the identity $\sin 3x = 3\sin x - 4\sin^3 x$, we find that

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Therefore,

$$\begin{aligned}\int \sin^3 x dx &= \frac{3}{4}\int \sin x dx - \frac{1}{4}\int \sin 3x dx \\ &= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C\end{aligned}$$

Alternatively,

$$\int \sin^3 x dx = \int \sin^2 x \sin x dx = \int \left(1-\cos^2 x\right) \sin x dx$$

Put $\cos x = t$ so that $-\sin x dx = dt$

Therefore,

$$\begin{aligned}\int \sin^3 x dx &= -\int \left(1-t^2\right) dt = -\int dt + \int t^2 dt = -t + \frac{t^3}{3} + \mathrm{C}\\ &= -\cos x + \frac{1}{3}\cos^3 x + \mathrm{C}\end{aligned}$$

R **Remark**: It can be shown using trigonometric identities that both answers are equivalent.

Exercise 2.2 Find the integrals of the following functions :

(1) $\sin^2(2x+5)$.

(2) $\sin 3x \cos 4x$.

(3) $\cos 2x \cos 4x \cos 6x$.

(4) $\sin^3(2x+1)$.

(5) $\sin^3 x \cos^3 x$.

(6) $\sin x \sin 2x \sin 3x$.

(7) $\sin 4x \sin 8x$.

(8) $\dfrac{1-\cos x}{1+\cos x}$.

(9) $\dfrac{\cos x}{1+\cos x}$.

(10) $\sin^4 x$.

(11) $\cos^4 2x$.

(12) $\dfrac{\sin^2 x}{1+\cos x}$.

(13) $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$.

(14) $\dfrac{\cos x - \sin x}{1+\sin 2x}$.

(15) $\tan^3 2x \sec 2x$.

(16) $\tan^4 x$.

(17) $\dfrac{\sin^3 x+\cos^3 x}{\sin^2 x\cos^2 x}$.

(18) $\dfrac{\cos 2x+2\sin^2 x}{\cos^2 x}$.

(19) $\dfrac{1}{\sin x\cos^3 x}$.

(20) $\dfrac{\cos 2x}{(\cos x+\sin x)^2}$.

(21) $\sin^{-1}(\cos x)$.

(22) $\dfrac{1}{\cos(x-a)\cos(x-b)}$.

Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1)
$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

(2)
$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

(3)
$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4)
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$$

(5)
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

(6)
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$$

We now prove the above results:

(1) We have

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)}$$
$$= \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right] = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

Therefore,

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$$
$$= \frac{1}{2a}[\log|(x-a)| - \log|(x+a)|] + \mathrm{C}$$
$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + \mathrm{C}$$

(2) In view of (1) above, we have

$$\frac{1}{a^2-x^2} = \frac{1}{2a}\left[\frac{(a+x)+(a-x)}{(a+x)(a-x)}\right] = \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$$

Therefore,

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\left[\int \frac{dx}{a-x} + \int \frac{dx}{a+x}\right]$$
$$= \frac{1}{2a}[-\log|a-x| + \log|a+x|] + \mathrm{C}$$
$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + \mathrm{C}$$

R **Note**: The technique used in (1) will be explained later.

(3) Put $x = a\tan\theta$. Then $dx = a\sec^2\theta d\theta$.

Therefore,

$$\int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta + a^2}$$
$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + \mathrm{C} = \frac{1}{a}\tan^{-1}\frac{x}{a} + \mathrm{C}$$

(4) Let $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta d\theta$.

Therefore,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2-a^2}} &= \int \frac{a\sec\theta\tan\theta d\theta}{\sqrt{a^2\sec^2\theta - a^2}} \\
&= \int \sec\theta d\theta = \log|\sec\theta+\tan\theta| + \mathrm{C}_1 \\
&= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right| + \mathrm{C}_1 \\
&= \log\left|x+\sqrt{x^2-a^2}\right| - \log|a| + \mathrm{C}_1 \\
&= \log\left|x+\sqrt{x^2-a^2}\right| + \mathrm{C},\ \text{where } \mathrm{C}_1 = \mathrm{C}_1 - \log|a|
\end{aligned}$$

(5) Let $x = a\sin\theta$. Then $dx = a\cos\theta \mathrm{d}\theta$.

Therefore,

$$\begin{aligned}
\int \frac{dx}{\sqrt{a^2-x^2}} &= \int \frac{a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}} \\
&= \int d\theta = \theta + \mathrm{C} = \sin^{-1}\frac{x}{a} + \mathrm{C}
\end{aligned}$$

(6) Let $x = a\tan\theta$. Then $dx = a\sec^2\theta \mathrm{d}\theta$.

Therefore,

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+a^2}} &= \int \frac{a\sec^2\theta d\theta}{\sqrt{a^2\tan^2\theta + a^2}} \\
&= \int \sec\theta d\theta = \log|(\sec\theta+\tan\theta)| + C_1 \\
&= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right| + \mathrm{C}_1 \\
&= \log\left|x+\sqrt{x^2+a^2}\right| - \log|a| + \mathrm{C}_1 \\
&= \log\left|x+\sqrt{x^2+a^2}\right| + \mathrm{C}, \text{ where } \mathrm{C} = \mathrm{C}_1 - \log|a|
\end{aligned}$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

(7) To find the integral

$$\int \frac{dx}{ax^2+bx+c},$$

we write

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

Now, put $x+\frac{b}{2a}=t$ so that $dx = dt$ and writing

$$\frac{c}{a}-\frac{b^2}{4a^2} = \pm k^2.$$

We find the integral reduced to the form

$$\frac{1}{a}\int\frac{dt}{t^2\pm k^2}$$

depending upon the sign of $\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)$ and hence can be evaluated.

(8) To find the integral of the type

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

, proceeding as in (7), we obtain the integral using the standard formulae.

(9) To find the integral of the type

$$\int\frac{px+q}{ax^2+bx+c}dx,$$

where p, q, a, b, c are constants, we are to find real numbers A, B such that

$$px+q = \mathrm{A}\frac{d}{dx}\left(ax^2+bx+c\right)+\mathrm{B} = \mathrm{A}(2ax+b)+\mathrm{B}$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type

$$\int\frac{(px+q)dx}{\sqrt{ax^2+bx+c}},$$

we proceed as in (9) and transform the integral into known standard forms. Let us illustrate the above methods by some examples.

■ **Example 2.4** Find the following integrals:

$$\text{(a)} \int \frac{dx}{x^2-16}, \qquad \text{(b)} \int \frac{dx}{\sqrt{2x-x^2}}.$$

■

Solution:

(a) We have

$$\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C.$$

(b)

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}.$$

Put $x-1=t$. Then $dx=dt$.

Therefore,

$$\begin{aligned}\int \frac{dx}{\sqrt{2x-x^2}} &= \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + \mathrm{C}\\ &= \sin^{-1}(x-1) + \mathrm{C}\end{aligned}$$

■ **Example 2.5** Find the following integrals:

$$\text{(a)} \int \frac{dx}{x^2-6x+13}, \qquad \text{(b)} \int \frac{dx}{3x^2+13x-10}, \qquad \text{(c)} \int \frac{dx}{\sqrt{5x^2-2x}}.$$

■

Solution:

(a)

$$\int \frac{dx}{x^2-6x+13},$$

We have

$$x^2-6x+13 = x^2-6x+3^2-3^2+13 = (x-3)^2+4$$

So,

$$\int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+2^2}dx$$

Let $x-3=t$. Then $dx=dt$

Therefore,

$$\begin{aligned}\int \frac{dx}{x^2-6x+13} &= \int \frac{dt}{t^2+2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + \mathrm{C}\\ &= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + \mathrm{C}\end{aligned}$$

(b) The given integral is of the form 2.1 (7).

$$\int \frac{dx}{3x^2+13x-10},$$

We write the denominator of the integrand,

$$\begin{aligned} 3x^2+13x-10 &= 3\left(x^2+\frac{13x}{3}-\frac{10}{3}\right) \\ &= 3\left[\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2\right] \text{ (completing the square)} \end{aligned}$$

Thus

$$\int \frac{dx}{3x^2+13x-10} = \frac{1}{3}\int \frac{dx}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2},$$

Put $x+\frac{13}{6}=t$. Then $dx=dt$.

Therefore,

$$\int \frac{dx}{3x^2+13x-10} = \frac{1}{3}\int \frac{dt}{t^2-\left(\frac{17}{6}\right)^2},$$

$$\begin{aligned} &= \frac{1}{3\times 2\times \frac{17}{6}}\log\left|\frac{t-\frac{17}{6}}{t+\frac{17}{6}}\right|+\mathrm{C}_1 \\ &= \frac{1}{17}\log\left|\frac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right|+\mathrm{C}_1 \\ &= \frac{1}{17}\log\left|\frac{6x-4}{6x+30}\right|+\mathrm{C}_1 \\ &= \frac{1}{17}\log\left|\frac{3x-2}{x+5}\right|+\mathrm{C}_1+\frac{1}{17}\log\frac{1}{3} \\ &= \frac{1}{17}\log\left|\frac{3x-2}{x+5}\right|+\mathrm{C}, \qquad \text{where, } \mathrm{C}=\mathrm{C}_1+\frac{1}{17}\log\frac{1}{3}. \end{aligned}$$

(c) We have

$$\begin{aligned} \int \frac{dx}{\sqrt{5x^2-2x}} &= \int \frac{dx}{\sqrt{5\left(x^2-\frac{2x}{5}\right)}} \\ &= \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2}} \text{ (completing the square)} \end{aligned}$$

Put $x - \frac{1}{5} = t$. Then $dx = dt$.

Therefore,

$$\begin{aligned}\int \frac{dx}{\sqrt{5x^2 - 2x}} &= \frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}} \\ &= \frac{1}{\sqrt{5}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C \\ &= \frac{1}{\sqrt{5}} \log \left| x - \frac{1}{5} + \sqrt{x^2 - \frac{2x}{5}} \right| + C\end{aligned}$$

Exercise 2.3 Integrate the following functions:

(1) $\dfrac{3x^2}{x^6+1}$.

(2) $\dfrac{1}{\sqrt{1+4x^2}}$.

(3) $\dfrac{1}{\sqrt{(2-x)^2+1}}$.

(4) $\dfrac{1}{\sqrt{9-25x^2}}$.

(5) $\dfrac{3x}{1+2x^4}$.

(6) $\dfrac{x^2}{1-x^6}$.

(7) $\dfrac{x-1}{\sqrt{x^2-1}}$.

(8) $\dfrac{x^2}{\sqrt{x^6+a^6}}$.

(9) $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$.

(10) $\dfrac{1}{\sqrt{x^2+2x+2}}$.

(11) $\dfrac{1}{9x^2+6x+5}$.

(12) $\dfrac{1}{\sqrt{7-6x-x^2}}$.

■

2.2 Integration by Partial Fractions

Recall that a rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If the degree of $P(x)$ is less than the degree of $Q(x)$, then the rational function is called proper, otherwise, it is called improper.The improper rational functions can be reduced to the proper rational functions by long division process. Thus, if $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is improper, then $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)} = \mathrm{T}(x) + \frac{\mathrm{P}_1(x)}{\mathrm{Q}(x)}$, where $\mathrm{T}(x)$ is a polynomial in x and $\frac{\mathrm{P}_1(x)}{\mathrm{Q}(x)}$ is a proper rational function. As we know how to integrate polynomials, the integration of any rational function is reduced to the integration of a proper rational function. The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors.

Assume that we want to evaluate $\int \frac{\mathrm{P}(x)}{\mathrm{Q}(x)} dx$, where $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods. The following Table indicates the types of simpler partial fractions that are to be associated with various kind of rational functions. In the above table,

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{x-b} + \frac{\mathrm{C}}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}}{(x-a)^2} + \frac{\mathrm{C}}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{\mathrm{A}}{x-a} + \frac{\mathrm{B}x+\mathrm{C}}{x^2+bx+c}$,
	where x^2+bx+c cannot be factorised further	

Table 2.1:

A, B and C are real numbers to be determined suitably.

Example 2.6 Find

$$\int \frac{dx}{(x+1)(x+2)}.$$

Solution: The integrand is a proper rational function. Therefore, by using the form of partial fraction, we write

$$\frac{1}{(x+1)(x+2)} = \frac{\mathrm{A}}{x+1} + \frac{\mathrm{B}}{x+2}, \tag{2.1}$$

where, real numbers A and B are to be determined suitably. This gives

$$1 = \mathbf{A}(x+2) + \mathbf{B}(x+1).$$

Equating the coefficients of x and the constant term, we get and

$$A + B = 0$$

$$2A + B = 1$$

Solving these equations, we get $\mathrm{A} = 1$ and $\mathrm{B} = -1$. Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

Therefore,

$$\begin{aligned}\int \frac{dx}{(x+1)(x+2)} &= \int \frac{dx}{x+1} - \int \frac{dx}{x+2} \\ &= \log|x+1| - \log|x+2| + C \\ &= \log\left|\frac{x+1}{x+2}\right| + C\end{aligned}$$

■

Remark The equation (2.1) above is an identity, i.e. a statement true for all (permissible) values of x. Some authors use the symbol ' $\equiv$ ' to indicate that the statement is an identity and use the symbol ' $=$ ' to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of x.

■ **Example 2.7** Find
$$\int \frac{x^2+1}{x^2-5x+6}dx.$$

Solution:

Here the integrand
$$\frac{x^2+1}{x^2-5x+6}$$
is not proper rational function, so we divide x^2+1 by x^2-5x+6 and find that
$$\frac{x^2+1}{x^2-5x+6} = 1+\frac{5x-5}{x^2-5x+6} = 1+\frac{5x-5}{(x-2)(x-3)}$$
Let
$$\frac{5x-5}{(x-2)(x-3)} = \frac{\mathrm{A}}{x-2}+\frac{\mathrm{B}}{x-3}$$
$$5x-5 = \mathrm{A}(x-3)+\mathrm{B}(x-2)$$
So that
$$5x-5 = \mathrm{A}(x-3)+\mathrm{B}(x-2)$$
Equating the coefficients of x and constant terms on both sides, we get $\mathrm{A}+\mathrm{B}=5$ and $3\ \mathrm{A}+2\ \mathrm{B}=5$. Solving these equations, we get $\mathrm{A}=-5$ and $\mathrm{B}=10$ Thus,
$$\frac{x^2+1}{x^2-5x+6} = 1-\frac{5}{x-2}+\frac{10}{x-3}$$
Therefore,
$$\int \frac{x^2+1}{x^2-5x+6}dx = \int dx - 5\int \frac{1}{x-2}dx + 10\int \frac{dx}{x-3}$$
$$= x-5\log|x-2|+10\log|x-3|+\ \mathrm{C}.$$
■

■ **Example 2.8** Find
$$\int \frac{3x-2}{(x+1)^2(x+3)}dx$$

Solution:

The integrand is of the type as given in Table 2.2 (4). We write
$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{\mathrm{A}}{x+1}+\frac{\mathrm{B}}{(x+1)^2}+\frac{\mathrm{C}}{x+3}$$
$$3x-2 = \mathrm{A}(x+1)(x+3)+\mathrm{B}(x+3)+\mathrm{C}(x+1)^2$$
$$= \mathrm{A}\left(x^2+4x+3\right)+\mathrm{B}(x+3)+\mathrm{C}\left(x^2+2x+1\right)$$

So that

$$3x-2 = A(x+1)(x+3)+B(x+3)+C(x+1)^2$$
$$= A\left(x^2+4x+3\right)+B(x+3)+C\left(x^2+2x+1\right)$$

Comparing coefficient of x^2, x and constant term on both sides, we get $A+C=0, 4\,A+B+2C=3$ and $3\,A+3\,B+C=-2$. Solving these equations, we get $A=\frac{11}{4}$, $B=\frac{-5}{2}$ and $C=\frac{-11}{4}$. Thus the integrand is given by

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)}$$

Therefore,

$$\begin{aligned}\int \frac{3x-2}{(x+1)^2(x+3)} &= \frac{11}{4}\int \frac{dx}{x+1} - \frac{5}{2}\int \frac{dx}{(x+1)^2} - \frac{11}{4}\int \frac{dx}{x+3}\\ &= \frac{11}{4}\log|x+1| + \frac{5}{2(x+1)} - \frac{11}{4}\log|x+3| + C\\ &= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C\end{aligned}$$

■

■ **Example 2.9** Find

$$\int \frac{x^2}{\left(x^2+1\right)\left(x^2+4\right)}dx$$

Solution:

put $x^2=y$. Then

$$\frac{x^2}{\left(x^2+1\right)\left(x^2+4\right)} = \frac{y}{(y+1)(y+4)}$$
$$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1}+\frac{B}{y+4}$$
$$y = A(y+4)+B(y+1)$$

So that

$$y = A(y+4)+B(y+1)$$

Comparing coefficients of y and constant terms on both sides, we get $A+B=1$ and $4\,A+B=0$, which give

$$A=-\frac{1}{3} \text{ and } B=\frac{4}{3}$$

Thus,

$$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

Therefore,

$$\begin{aligned}\int \frac{x^2 dx}{(x^2+1)(x^2+4)} &= -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4} \\ &= -\frac{1}{3}\tan^{-1}x + \frac{4}{3}\times\frac{1}{2}\tan^{-1}\frac{x}{2} + \mathrm{C} \\ &= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + \mathrm{C}\end{aligned}$$

■

In the above example, the substitution was made only for the partial fraction part and not for the integration part. Now, we consider an example, where the integration involves a combination of the substitution method and the partial fraction method.

■ **Example 2.10** Find

$$\int \frac{x^2+x+1dx}{(x+2)(x^2+1)}$$

Solution:

The integrand is a proper rational function. Decompose the rational function into partial fraction. Write

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{\mathrm{A}}{x+2} + \frac{\mathrm{B}x+\mathrm{C}}{(x^2+1)}$$

Therefore,

$$x^2+x+1 = \mathrm{A}\left(x^2+1\right) + (\mathrm{B}x+\mathrm{C})(x+2)$$

Equating the coefficients of x^2, x and of constant term of both sides, we get $\mathrm{A}+\mathrm{B} = 1, 2\,\mathrm{B}+\mathrm{C} = 1$ and $\mathrm{A}+2\mathrm{C} = 1$. Solving these equations, we get $\mathrm{A} = \frac{3}{5}$, $\mathrm{B} = \frac{2}{5}$ and $\mathrm{C} = \frac{1}{5}$

Thus, the integrand is given by

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{3}{5(x+2)} + \frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1} = \frac{3}{5(x+2)} + \frac{1}{5}\left(\frac{2x+1}{x^2+1}\right)$$

Therefore,

$$\int \frac{x^2+x+1}{(x^2+1)(x+2)}dx = \frac{3}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{2x}{x^2+1}dx + \frac{1}{5}\int \frac{1}{x^2+1}dx$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log\left|x^2+1\right| + \frac{1}{5}\tan^{-1}x + C.$$

■

Exercise 2.4 Integrate the following rational functions:

(1) $\dfrac{x}{(x+1)(x+2)}$.

(2) $\dfrac{1}{x^2-9}$.

(3) $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$.

(4) $\dfrac{x}{(x-1)(x-2)(x-3)}$.

(5) $\dfrac{2x}{x^2+3x+2}$.

(6) $\dfrac{1-x^2}{x(1-2x)}$.

(7) $\dfrac{x}{(x^2+1)(x-1)}$.

(8) $\dfrac{x}{(x-1)^2(x+2)}$.

(9) $\dfrac{3x+5}{x^3-x^2-x+1}$.

(10) $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

(11) $\dfrac{5x}{(x+1)(x^2-4)}$.

(12) $\dfrac{x^3+x+1}{x^2-1}$.

■

2.3 Integration by Parts

In this section, we describe one more method of integration, that is found quite useful in integrating products of functions.

If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have Integrating both sides, we get

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

or

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{dx}{dx}dx$$

Let

$$u = f(x) \text{ and } \frac{dv}{dx} = g(x). \text{ Then}$$

$$\frac{du}{dx} = f'(x) \text{ and } v = \int g(x)dx$$

Therefore, expression (1) can be rewritten as

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int \left[\int g(x)dx\right] f'(x)dx$$

$$\text{i.e., } \int f(x)g(x)dx = f(x)\int g(x)dx - \int \left[f'(x)\int g(x)dx\right] dx$$

If we take f as the first function and g as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (first function) × (integral of the second function) - Integral of the product of the derivative of the first function with integral of the second function".

■ **Example 2.11** Find

$$\int x\cos x dx$$

Solution:

Put $f(x) = x$ (first function) and $g(x) = \cos x$ (second function).

Then, integration by parts gives

$$\int x\cos x dx = x\int \cos x dx - \int \left[\frac{d}{dx}(x)\int \cos x dx\right] dx$$

$$= x\sin x - \int \sin x dx = x\sin x + \cos x + C$$

Suppose, we take

$$f(x) = \cos x \text{ and } g(x) = x. \text{ Then}$$

$$\int x\cos x dx = \cos x\int x dx - \int \left[\frac{d}{dx}(\cos x)\int x dx\right] dx$$

$$= (\cos x)\frac{x^2}{2} + \int \sin x\frac{x^2}{2}dx$$

Thus, it shows that the integral $\int x\cos x dx$ is reduced to the comparatively more complicated integral having more power of x. Therefore, the proper choice of the first function and the second function is significant. ■

Remarks:

- It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int \sqrt{x}\sin x dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}\sin x$.
- Observe that while finding the integral of the second function, we did not add any constant of integration. If we write the integral of the second function $\cos x$ as $\sin x + k$, where k is any constant, then

$$\begin{aligned}
\int x\cos x\, dx &= x(\sin x + k) - \int (\sin x + k) dx \\
&= x(\sin x + k) - \int \sin x dx - \int k dx \\
&= x(\sin x + k) - \cos x - kx + C \\
&= x\sin x + \cos x + C
\end{aligned}$$

  This shows that adding a constant to the integral of the second function is superfluous so far as the final result is concerned while applying the method of integration by parts.
- Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, in cases where other function is inverse trigonometric function or logarithmic function, then we take them as first function.

■ **Example 2.12** Find

$$\int \log x dx$$

Solution:

To start with, we are unable to guess a function whose derivative is $\log x$. We take $\log x$ as the first function and the constant function 1 as the second function. Then, the integral of the second function is x. Hence,

$$\int (\log x.1)dx = \log x \int 1dx - \int \left[\frac{d}{dx}(\log x)\int 1dx\right] dx$$
$$= (\log x)\cdot x - \int \frac{1}{x} x dx = x\log x - x + C.$$

■

■ **Example 2.13** Find

$$\int xe^x dx$$

Solution:

Take first function as x and second function as e. The integral of the second function is e^x. Therefore,

$$\int xe^x dx = xe^x - \int 1\cdot e^x dx = xe^x - e^x + \mathrm{C}$$

. ■

■ **Example 2.14** Find

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$$

Solution:

Let first function be $\sin^{-1}x$ and second function be $\frac{x}{\sqrt{1-x^2}}$.
First we find the integral of the second function, i.e.,

$$\int \frac{xdx}{\sqrt{1-x^2}}$$

. Put $t = 1-x^2$. Then $dt = -2xdx$ Therefore,

$$\int \frac{xdx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

Hence,

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = \left(\sin^{-1}x\right)\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)dx$$
$$= -\sqrt{1-x^2}\sin^{-1}x + x + \mathrm{C} = x - \sqrt{1-x^2}\sin^{-1}x + \mathrm{C}$$

Alternatively, this integral can also be worked out by making substitution $\sin^{-1}x = \theta$ and then integrating by parts. ■

■ **Example 2.15** Find

$$\int e^x \sin x dx$$

Solution:

Take e^x as the first function and sin x as second function. Then, integrating by parts, we have

$$I = \int e^x \sin x dx = e^x(-\cos x) + \int e^x \cos x dx$$
$$= -e^x \cos x + I_1(\text{say})$$

Taking e^x and $\cos x$ as the first and second functions, respectively, in I_1, we get

$$I_1 = e^x \sin x - \int e^x \sin x dx$$

Substituting the value of I_n in (1), we get

$$I = -e^x \cos x + e^x \sin x - I \quad \text{or} \quad 2I = e^x(\sin x - \cos x)$$

Hence,

$$\mathrm{I} = \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + \mathrm{C}$$

Alternatively, above integral can also be determined by taking sin x as the first function and e^x the second function. ■

Exercise 2.5 Integrate the functions:

(1) $x\sin x$.

(2) $x\sin 3x$.

(3) x^2e^x.

(4) $x\log 2x$.

(5) $x^2\log x$.

(6) $x\tan^{-1}x$.

(7) $\left(\sin^{-1}x\right)^2$.

(8) $\dfrac{x\cos^{-1}x}{\sqrt{1-x^2}}$.

(9) $x(\log x)^2$.

(10) $\left(x^2+1\right)\log x$.

(11) $\dfrac{xe^x}{(1+x)^2}$.

(12) $\dfrac{(x-3)e^x}{(x-1)^3}$.

■

3. Definite Integral

In the previous, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this Chapter, we shall study what is called definite integral of a function. The definite integral has a unique value.

A definite integral is denoted by $\int_a^b f(x)dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a,b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$.

3.1 Area function

Here, We have defined $\int_a^b f(x)dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and $x-$axis. Let x be a given point in $[a,b]$. Then $\int_a^b f(x)dx$ represents the area of the light shaded region in Fig 3.1 [Here it is assumed that $f(x) > 0$ for $x \in [a,b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x. In other words, the area of this shaded region is a function of x. In other words, the area of this shaded region is a function of x. We denote this function of x by $A(x)$.

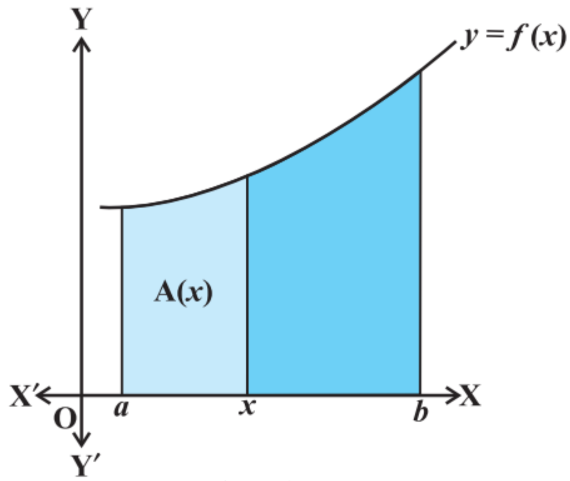


Figure 3.1:

We call the function $A(x)$ as Area function and is given by

$$A(x) = \int_a^b f(x)dx.$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

Theorem 3.1.1 Let f be a continuous function on the closed interval $[a,b]$ and let $A(x)$ be the area function. Then $\acute{A}(x) = f(x)$, for all $x \in [a,b]$

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 3.1.2 Let f be continuous function defined on the closed interval $[a,b]$ and F be an anti derivative of f. Then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

Steps for calculating $\int_a^b f(x)dx$.

(i) Find the indefinite integral $\int f(x)dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F$(x)+$C instead of F(x), we get $\int_a^b f(x)dx =$ $[\mathrm{F}(x)+\mathrm{C}]_a^b = [\mathrm{F}(b)+\mathrm{C}] - [\mathrm{F}(a)+\mathrm{C}] = \mathrm{F}(b) - \mathrm{F}(a)$. Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate $\mathrm{F}(b)-\mathrm{F}(a)=[\mathrm{F}(x)]_a^b$, which is the value of $\int_a^b f(x)dx$. We now consider some examples

Example 3.1 Evaluate the following integrals:

$$\text{(a)} \int_2^3 x^2 dx, \qquad \text{(b)} \int_4^9 \frac{\sqrt{x}}{\left(30-x^{\frac{3}{2}}\right)^2} dx$$

$$\text{(c)} \int_1^2 \frac{x dx}{(x+1)(x+2)}, \qquad \text{(d)} \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t dt$$

■

Solution:

(a) Let

$$\mathrm{I}=\int_2^3 x^2 dx.$$

Since

$$\int x^2 dx=\frac{x^3}{3}=\mathrm{F}(x),$$

Therefore, by the second fundamental theorem, we get

$$I=F(3)-F(2)=\frac{27}{3}-\frac{8}{3}=\frac{19}{3}$$

(b) Let

$$\int_4^9 \frac{\sqrt{x}}{\left(30-x^{\frac{3}{2}}\right)^2} dx.$$

We first find the anti derivative of the integrand.

Put $30-x^{\frac{3}{2}}=t$. Then $-\frac{3}{2}\sqrt{x}dx=dt$ or $\sqrt{x}dx=-\frac{2}{3}dt$ Thus,

$$\int \frac{\sqrt{x}}{\left(30-x^{\frac{3}{2}}\right)^2} dx=-\frac{2}{3}\int \frac{dt}{t^2}=\frac{2}{3}\left[\frac{1}{t}\right]=\frac{2}{3}\left[\frac{1}{\left(30-x^{\frac{3}{2}}\right)}\right]=\mathrm{F}(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$I=F(9)-F(4)=\frac{2}{3}\left[\frac{1}{\left(30-x^{\frac{3}{2}}\right)}\right]_4^9$$

$$=\frac{2}{3}\left[\frac{1}{(30-27)}-\frac{1}{30-8}\right]=\frac{2}{3}\left[\frac{1}{3}-\frac{1}{22}\right]=\frac{19}{99}$$

(c) Let

$$\mathrm{I} = \int_1^2 \frac{xdx}{(x+1)(x+2)}$$

Using partial fraction, we get

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

So

$$\int \frac{xdx}{(x+1)(x+2)} = -\log|x+1| + 2\log|x+2| = \mathrm{F}(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$\begin{aligned} I &= F(2) - F(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3] \\ &= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right) \end{aligned}$$

(d) Let

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t dt.$$

Consider

$$\int \sin^3 2t \cos 2t dt$$

Put $\sin 2t = u$ so that $2\cos 2t dt = du$ or $\cos 2t dt = \frac{1}{2}du$

So

$$\begin{aligned} \int \sin^3 2t \cos 2t dt &= \frac{1}{2}\int u^3 du \\ &= \frac{1}{8}\left[u^4\right] = \frac{1}{8}\sin^4 2t = \mathrm{F}(t) \text{ say} \end{aligned}$$

Therefore, by the second fundamental theorem of integral calculus

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{8}\left[\sin^4\frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

Exercise 3.1 Evaluate the following definite integrals:

(1) $\int_{-1}^{1}(x+1)dx$.

(2) $\int_{2}^{3}\frac{1}{x}dx$.

(3) $\int_{1}^{2}\left(4x^3-5x^2+6x+9\right)dx$.

(4) $\int_{0}^{\frac{\pi}{4}}\sin 2x dx$.

(5) $\int_{0}^{1}\frac{2x+3}{5x^2+1}dx$.

(6) $\int_{4}^{5}e^x dx$.

(7) $\int_{0}^{\frac{\pi}{4}}\tan x dx$.

(8) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\operatorname{cosec} x dx$.

(9) $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$.

(10) $\int_{0}^{1}\frac{dx}{1+x^2}$.

(11) $\int_{2}^{3}\frac{dx}{x^2-1}$.

(12) $\int_{0}^{\frac{\pi}{2}}\cos^2 x dx$.

■

3.2 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

$\mathbf{P}_0 : \int_a^b f(x)dx = \int_a^b f(t)dt$

$\mathbf{P}_1 : \int_a^b f(x)dx = -\int_b^a f(x)dx$. In particular, $\int_a^a f(x)dx = 0$

$\mathbf{P}_2 : \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$\mathbf{P}_3 : \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$\mathbf{P}_4 : \int_0^a f(x)dx = \int_0^a f(a-x)dx$

(Note that P_4 is a particular case of P_3)

$\mathbf{P}_5 : \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

$\mathbf{P}_6 : \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$ if $f(2a-x) = f(x)$ and 0 if $f(2a-x) = -f(x)$

$\mathbf{P}_7 :$ (i) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$, if f is an even function, i.e., if $f(-x) = f(x)$

(ii) $\int_{-a}^a f(x)dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.

We give the proofs of these properties one by one.

Proof of $\mathbf{P}_0$ It follows directly by making the substitution $x = t$.

Proof of $\mathbf{P}_1$: Let F be anti derivative of f. Then, by the second fundamental theorem of calculus, we have $\int_a^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a) = -[\mathrm{F}(a) - \mathrm{F}(b)] = -\int_b^a f(x)dx$ Here, we observe that, if $a = b$, then $\int_a^a f(x)dx = 0$.

Proof of $\mathbf{P}_2$: Let F be anti derivative of f. Then

$$\int_a^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a) \tag{3.1}$$

$$\int_a^c f(x)dx = \mathrm{F}(c) - \mathrm{F}(a) \tag{3.2}$$

$$\int_c^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(c) \tag{3.3}$$

Adding (3.2) and (3.2), we get

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a) = \int_a^b f(x)dx$$

This proves the property P_2.

Proof of $\mathbf{P}_3$: Let $t = a+b-x$. Then $dt = -dx$. When $x = a, t = b$ and when $x = b, t = a$. Therefore

$$\begin{aligned}\int_a^b f(x)dx &= -\int_b^a f(a+b-t)dt \\ &= \int_a^b f(a+b-t)dt \text{ (by } \mathrm{P}_1) \\ &= \int_a^b f(a+b-x)dx \text{ by } \mathrm{P}_0\end{aligned}$$

Proof of $\mathbf{P}_4$: Put $t = a-x$. Then $dt = -dx$. When $x = 0, t = a$ and when $x = a, t = 0$. Now proceed as in P_3.

Proof of $\mathbf{P}_5$: Using P_2, we have

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx.$$

Let $t = 2a - x$ in the second integral on the right hand side. Then $dt = -dx$. When $x = a, t = a$ and when $x = 2a, t = 0$. Also $x = 2a - t$.

Therefore, the second integral becomes

$$\int_a^{2a} f(x)dx = -\int_a^0 f(2a-t)dt = \int_0^a f(2a-t)dt = \int_0^a f(2a-x)dx$$

Hence

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

Proof of $\mathbf{P}_6$: Using P_5, we have

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx \tag{3.4}$$

Now, if $f(2a-x)=f(x)$, then Eq. (3.4) becomes

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx,$$

and if $f(2a-x)=-f(x)$, then Eq. (3.4) becomes

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

Proof of $\mathbf{P}_7$: Using P_2, we have

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx. \text{ Then}$$

Let $t=-x$ in the first integral on the right hand side. $dt=-dx$. When $x=-a, t=a$ and when $x=0, t=0$. Also $x=-t$.

Therefore

$$\begin{aligned}\int_{-a}^{a} f(x)dx &= -\int_a^0 f(-t)dt + \int_0^a f(x)dx \\ &= \int_0^a f(-x)dx + \int_0^a f(x)dx\end{aligned} \tag{3.5}$$

Eq. (3.5) Now, if f is an even function, then $f(-x)=f(x)$ and so Eq. (3.5) becomes

$$\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx = 2\int_0^a f(x)dx$$

(ii) If f is an odd function, then $f(-x)=-f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)dx = -\int_0^a f(x)dx + \int_0^a f(x)dx = 0$$

■ **Example 3.2** Evaluate

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx.$$

Solution: We observe that $\sin^2 x$ is an even function. Therefore, by P_7 (i), we get

$$\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx = 2\int_0^{\frac{\pi}{4}} \sin^2 x dx$$

$$\begin{aligned}&= 2\int_0^{\frac{\pi}{4}} \frac{(1-\cos 2x)}{2} dx = \int_0^{\frac{\pi}{4}} (1-\cos 2x)dx \\ &= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{\pi}{4} - \frac{1}{2}\end{aligned}$$

■

■ **Example 3.3** Evaluate

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

Solution:

Let

$$\mathrm{I} = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx.$$

Then, by P_4, we have

$$\begin{aligned}
\mathrm{I} &= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)dx}{1+\cos^2(\pi - x)} \\
&= \int_0^{\pi} \frac{(\pi - x)\sin x dx}{1+\cos^2 x} = \pi \int_0^{\pi} \frac{\sin x dx}{1+\cos^2 x} - \mathrm{I}
\end{aligned}$$

or

$$2\mathrm{I} = \pi \int_0^{\pi} \frac{\sin x dx}{1+\cos^2 x}$$

or

$$\mathrm{I} = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x dx}{1+\cos^2 x}$$

Put $\cos x = t$ so that $-\sin x dx = dt$. When $x = 0, t = 1$ and when $x = \pi, t = -1$. Therefore, (by P_1) we get

$$\begin{aligned}
\mathrm{I} &= \frac{-\pi}{2} \int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2} \int_{-1}^{1} \frac{dt}{1+t^2} \\
&= \pi \int_0^{1} \frac{dt}{1+t^2} \text{ (by } P_7, \text{ since } \frac{1}{1+t^2} \text{ is even function)} \\
&= \pi \left[\tan^{-1} t\right]_0^1 = \pi \left[\tan^{-1} 1 - \tan^{-1} 0\right] = \pi \left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}
\end{aligned}$$

■

Exercise 3.2 By using the properties of definite integrals, evaluate the integrals:

(1) $\int_0^{\frac{\pi}{2}} \cos^2 x dx$.

(2) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$.

(3) $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x dx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}$.

(4) $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x dx}{\sin^5 x+\cos^5 x}$.

(5) $\int_0^1 x(1-x)^n dx$.

(6) $\int_0^{\frac{\pi}{4}} \log(1+\tan x) dx$.

(7) $\int_0^2 x\sqrt{2-x} dx$.

(8) $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x) dx$.

(9) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x dx$.

(10) $\int_0^{\pi} \frac{x dx}{1+\sin x}$.

(11) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$.

(12) $\int_0^{2\pi} \cos^5 x dx$.

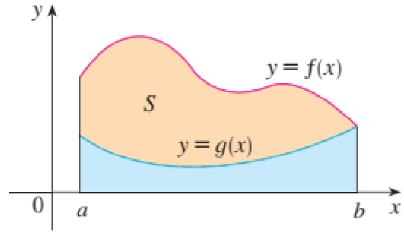
■

4. Applications of Integration

4.1 Areas between curves

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$



$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

■ **Example 4.1** Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Solution. The region is shown in Figure 2. The upper boundary curve is $y = e^x$ and the lower boundary curve is $y = x$. So we use the area formula with $f(x) = e^x, \mathrm{t}(x) = x$, $a = 0$, and $b=1$:

$$A=\int_0^1 (e^x-x)\,dx = e^x-\frac{1}{2}x^2\Big]_0^1$$
$$=e-\frac{1}{2}-1=e-1.5$$

■

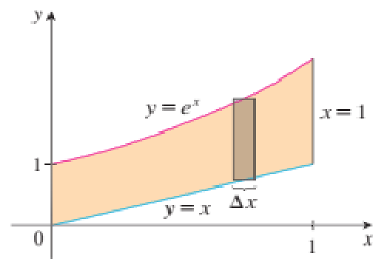


FIGURE 2

■ **Example 4.2** Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$.

Solution. First, we find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2=2x-x^2$, or $2x^2-2x=0$. Thus $2x(x-1)=0$, so $x=0$ or 1 . The points of intersection are $(0,0)$ and $(1,1)$. We see from Figure 3 that the top and bottom boundaries are

$$y_T=2x-x^2 \quad \text{and} \quad y_B=x^2$$

The area of a typical rectangle is

$$(y_T-y_B)\,\Delta x=\left(2x-x^2-x^2\right)\Delta x$$

and the region lies between $x=0$ and $x=1$. So the total area is

$$A=\int_0^1\left(2x-2x^2\right)dx=2\int_0^1\left(x-x^2\right)dx$$
$$=2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

■

■ **Example 4.3** Find the area of the region bounded by the curves $y=\sin x, y=\cos x$, $x=0$, and $x=\pi/2$.

Solution. The points of intersection occur when $\sin x=\cos x$, that is, when $x=\pi/4$

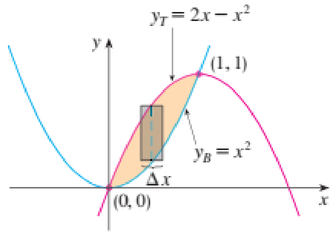


FIGURE 3

(since $0 \leqslant x \leqslant \pi/2$). The region is sketched in Figure 4. Observe that $\cos x \geqslant \sin x$ when $0 \leqslant x \leqslant \pi/4$ but $\sin x \geqslant \cos x$ when $\pi/4 \leqslant x \leqslant \pi/2$. Therefore the required area is

$$\begin{aligned}
A &= \int_0^{\pi/2} |\cos x - \sin x| dx = A_1 + A_2 \\
&= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\
&= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \\
&= 2\sqrt{2} - 2
\end{aligned}$$

In this particular example we could have saved some work by noticing that the region is symmetric about $x = \pi/4$ and so

$$A = 2A_1 = 2 \int_0^{\pi/4} (\cos x - \sin x) dx$$

■

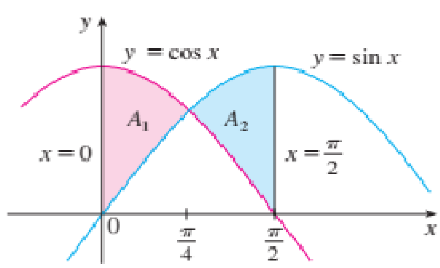


FIGURE 4

■ **Example 4.4** Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution. By solving the two equations we find that the points of intersection are $(-1, -2)$

and $(5,4)$. We solve the equation of the parabola for x and notice from Figure 5 that the left and right boundary curves are

$$x_L = \frac{1}{2}y^2 - 3 \quad x_R = y + 1$$

We must integrate between the appropriate y-values, $y = -2$ and $y = 4$. Thus

$$\begin{aligned}
A &= \int_{-2}^{4} (x_R - x_L)\,dy \\
&= \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3\right)\right] dy \\
&= \int_{-2}^{4} \left(-\frac{1}{2}y^2 + y + 4\right) dy \\
&= -\frac{1}{2}\left(\frac{y^3}{3}\right) + \frac{y^2}{2} + 4y\Big]_{-2}^{4} \\
&= -\frac{1}{6}(64) + 8 + 16 - \left(\frac{4}{3} + 2 - 8\right) = 18
\end{aligned}$$

■

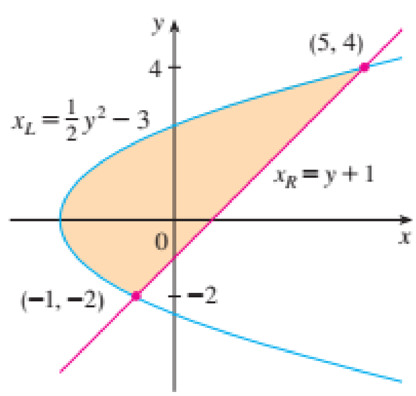
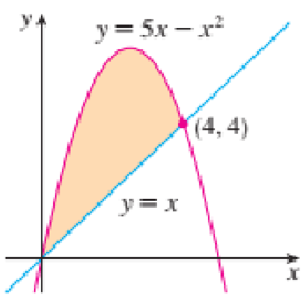


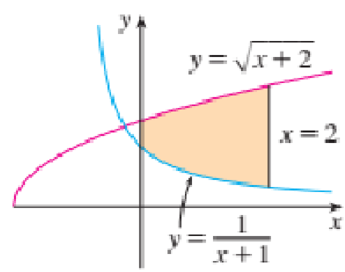
FIGURE 5

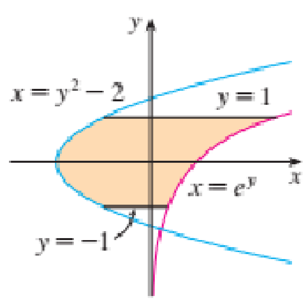
Exercise 4.1 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Then find the area of the region.

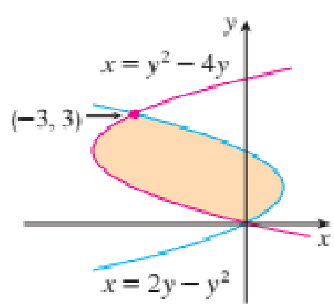
1. $y = x+1, \quad y = 9-x^2, \quad x = -1, \quad x = 2$
2. $y = \sin x, \quad y = e^x, \quad x = 0, \quad x = \pi/2$
3. $y = x, \quad y = x^2$
4. $y = x^2 - 2x, \quad y = x+4$
5. $y = 1/x, \quad y = 1/x^2, \quad x = 2$
6. $y = 1+\sqrt{x}, \quad y = (3+x)/3$
7. $y = x^2, \quad y^2 = x$
8. $y = x^2, \quad y = 4x - x^2$
9. $y = 12 - x^2, \quad y = x^2 - 6$
10. $y = \cos x, \quad y = 2 - \cos x, \quad 0 \leqslant x \leqslant 2\pi$
11. $y = \tan x, \quad y = 2\sin x, \quad -\pi/3 \leqslant x \leqslant \pi/3$
12. $y = x^3 - x, \quad y = 3x$
13. $y = \sqrt{x}, \quad y = \frac{1}{2}x, \quad x = 9$
14. $y = 8 - x^2, \quad y = x^2, \quad x = -3, \quad x = 3$
15. $x = 2y^2, \quad x = 4 + y^2$
16. $4x + y^2 = 12, x = y$

Exercise 4.2 Find the area of the shaded region.









4.2 Volumes

DEFINITION OF VOLUME Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \int_a^b A(x)dx$$

When we use the volume formula $V = \int_a^b A(x)dx$, it is important to remember that $A(x)$ is the area of a moving cross-section obtained by slicing through x perpendicular to the x-axis.

Notice that, for a cylinder, the cross-sectional area is constant: $A(x) = A$ for all x. So our definition of volume gives $V = \int_a^b A dx = A(b-a)$; this agrees with the formula $V = Ah$.

■ **Example 4.5** Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Solution> If we place the sphere so that its center is at the origin (see Figure 7), then the plane P_x intersects the sphere in a circle whose radius (from the Pythagorean Theorem) is $y = \sqrt{r^2 - x^2}$. So the cross-sectional area is

$$A(x) = \pi y^2 = \pi\left(r^2 - x^2\right)$$

Using the definition of volume with $a = -r$ and $b = r$, we have

$$\begin{aligned} V &= \int_{-r}^{r} A(x)dx = \int_{-r}^{r} \pi\left(r^2 - x^2\right)dx \\ &= 2\pi \int_0^r \left(r^2 - x^2\right)dx \qquad \text{(The integrand is even.)} \\ &= 2\pi\left[r^2 x - \frac{x^3}{3}\right]_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3 \end{aligned}$$

■

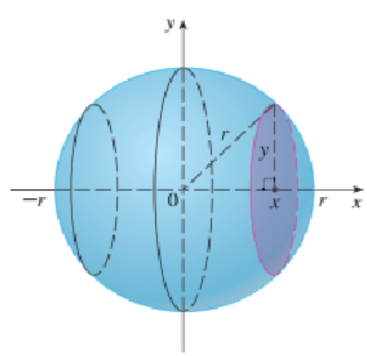


FIGURE7

■ **Example 4.6** Find the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 8$, and $x = 0$ about the y-axis.

Solution. The region is shown in Figure 8(a) and the resulting solid is shown in Figure 8(b). Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis and therefore to integrate with respect to y. If we slice at height y, we get a circular disk with radius x, where $x = \sqrt[3]{y}$. So the area of a cross section through y is

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

and the volume of the approximating cylinder pictured in Figure 8(b) is

$$A(y)\Delta y = \pi y^{2/3}\Delta y$$

Since the solid lies between $y = 0$ and $y = 8$, its volume is

$$V = \int_0^8 A(y)dy = \int_0^8 \pi y^{2/3} dy = \pi \left[\frac{3}{5} y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$

■

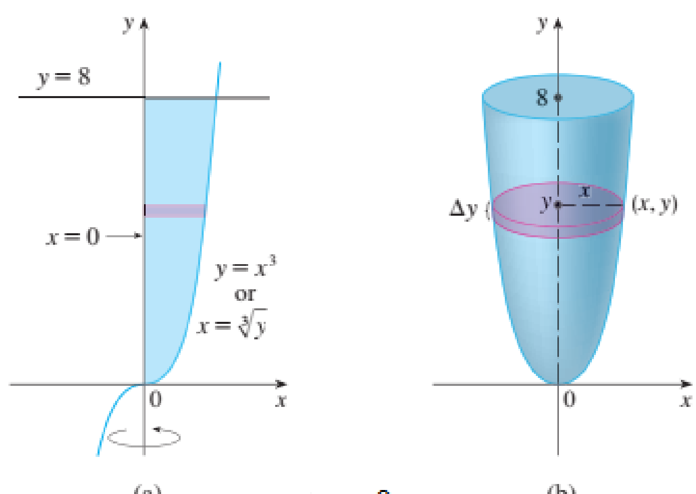


FIGURE 8

■ **Example 4.7** The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.

Solution. The curves $y = x$ and $y = x^2$ intersect at the points $(0,0)$ and $(1,1)$. The region between them, the solid of rotation, and a cross-section perpendicular to the x-axis are shown in Figure 9. A cross-section in the plane P_x has the shape of a washer (an annular ring) with inner radius x^2 and outer radius x, so we find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle:

$$A(x) = \pi x^2 - \pi \left(x^2\right)^2 = \pi \left(x^2 - x^4\right)$$

Therefore we have

$$V = \int_0^1 A(x)dx = \int_0^1 \pi \left(x^2 - x^4\right) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$

■

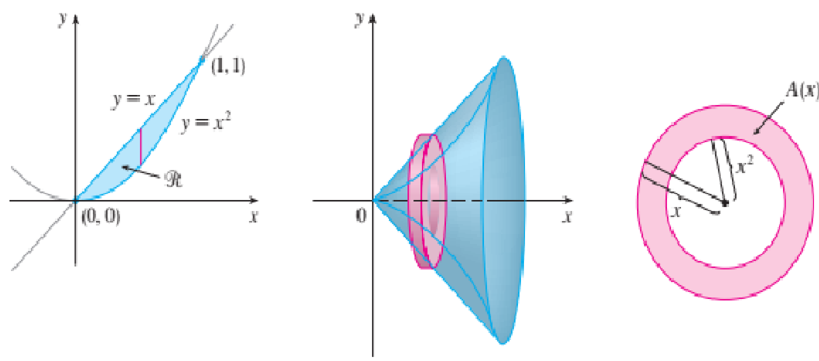


FIGURE 9 (a) (b) (c)

■ **Example 4.8** Find the volume of a pyramid whose base is a square with side L and whose height is h.

Solution. We place the origin O at the vertex of the pyramid and the x-axis along its central axis as in Figure 10. Any plane P_x that passes through x and is perpendicular to the x-axis intersects the pyramid in a square with side of length s, say. We can express s in terms of x by observing from the similar triangles in Figure 11 that

$$\frac{x}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$

and so $s = Lx/h$. [Another method is to observe that the line OP has slope $L/(2h)$ and so its equation is $y = Lx/(2h)$.] Thus the cross-sectional area is

$$A(x) = s^2 = \frac{L^2}{h^2}x^2$$

The pyramid lies between $x = 0$ and $x = h$, so its volume is

$$V = \int_0^h A(x)dx = \int_0^h \frac{L^2}{h^2} x^2 dx = \frac{L^2}{h^2} \frac{x^3}{3} \Big]_0^h = \frac{L^2 h}{3}$$

■

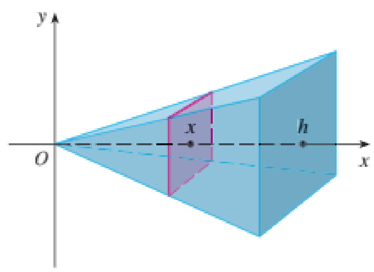


FIGURE 10

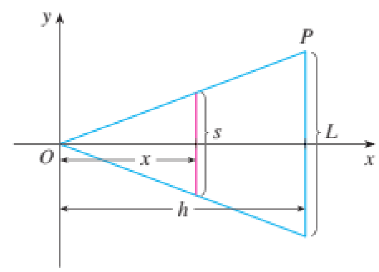


FIGURE 11

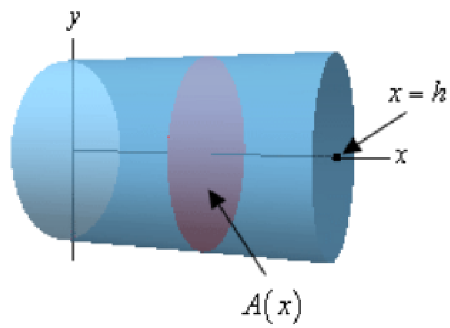
Exercise 4.3 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y = 2 - \frac{1}{2}x, y = 0, x = 1, x = 2$; about the x-axis
2. $y = 1 - x^2, y = 0$; about the x-axis
3. $y = 1/x, x = 1, x = 2, y = 0$; about the x-axis
4. $y = \sqrt{25 - x^2}, y = 0, x = 2, x = 4$; about the x-axis
5. $x = 2\sqrt{y}, x = 0, y = 9$; about the y-axis
6. $y = \ln x, y = 1, y = 2, x = 0$; about the y-axis
7. $y = x^3, y = x, x \geqslant 0$; about the x-axis
8. $y = \frac{1}{4}x^2, y = 5 - x^2$; about the x-axis
9. $y^2 = x, x = 2y$; about the y-axis
10. $y = \frac{1}{4}x^2, x = 2, y = 0$; about the y-axis
11. $y = x, y = \sqrt{x}$; about $y = 1$
12. $y = e^{-x}, y = 1, x = 2$; about $y = 2$
13. $y = 1 + \sec x, y = 3$; about $y = 1$
14. $y = 1/x, y = 0, x = 1, x = 3$; about $y = -1$ ■

4.3 More Volume Problems

■ **Example 4.9** Find the volume of a cylinder of radius r and height h. ■

Solution. We'll start off with the sketch of the cylinder below. What we need here is to get a



formula for the cross-sectional area at any x. In this case the cross-sectional area is constant and will be a disk of radius r. Therefore, for any x we'll have the following cross-sectional area,

$$A(x) = \pi r^2$$

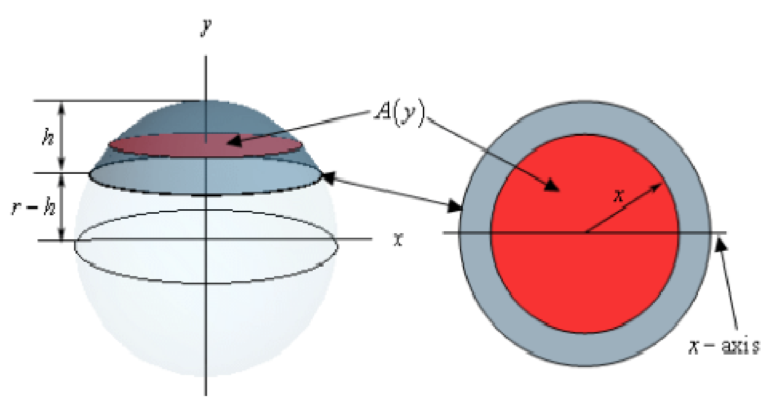
Next the limits for the integral will be $0 \le x \le h$ since that is the range of x in which the cylinder lives. Here is the integral for the volume,

$$V = \int_0^h \pi r^2 dx = \pi r^2 \int_0^h dx = \pi r^2 x\big|_0^h = \pi r^2 h$$

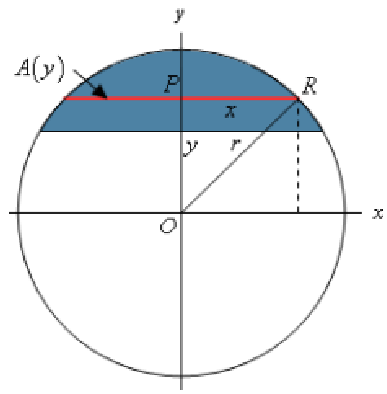
So, we get the expected formula.

■ **Example 4.10** For a sphere of radius r find the volume of the cap of height h. ■

Solution. A sketch is probably best to illustrate what we after have.



The area of the disk $A(y)$ is then, $A(y) = \pi x^2$. We need the cross-sectional area in terms of y. So, what we really need to determine what x will be for any given y at the cross-section. Let's look at the spherical cross-section.



We have

$$A(y) = \pi\left(r^2 - y^2\right),\ r-h \leq y \leq r$$

So the volume is,

$$\begin{aligned} V &= \int_{r-h}^{r} \pi\left(r^2 - y^2\right) dy \\ &= \pi\left(r^2 y - \frac{1}{3}y^3\right)\Big|_{r-h}^{r} \\ &= \pi\left(h^2 r - \frac{1}{3}h^3\right) = \pi h^2\left(r - \frac{1}{3}h\right) \end{aligned}$$

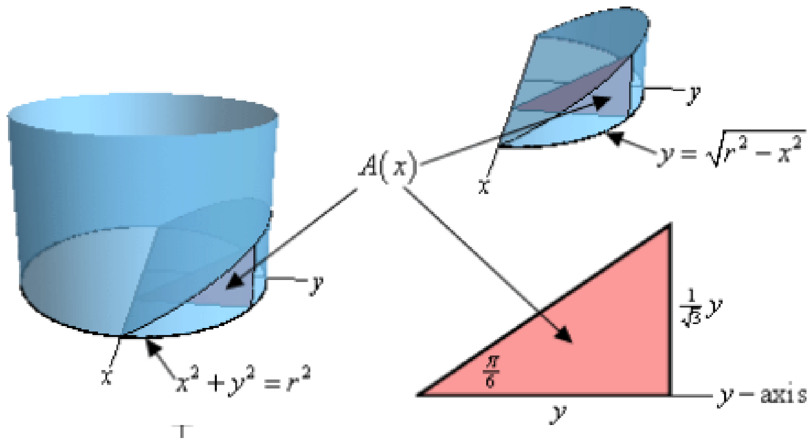
■ **Example 4.11** Find the volume of a wedge cut out of a cylinder of radius r if the angle between the top and bottom of the wedge is π ■

Solution. We should really start off with a sketch of just what we're looking for here. From the figure, we can compute the area of the cross-section $A(x)$ as,

$$A(x) = \frac{1}{2}(y)\left(\frac{1}{\sqrt{3}}y\right) = \frac{1}{2}\sqrt{r^2 - x^2}\left(\frac{1}{\sqrt{3}}\sqrt{r^2 - x^2}\right) = \frac{1}{2\sqrt{3}}\left(r^2 - x^2\right)$$

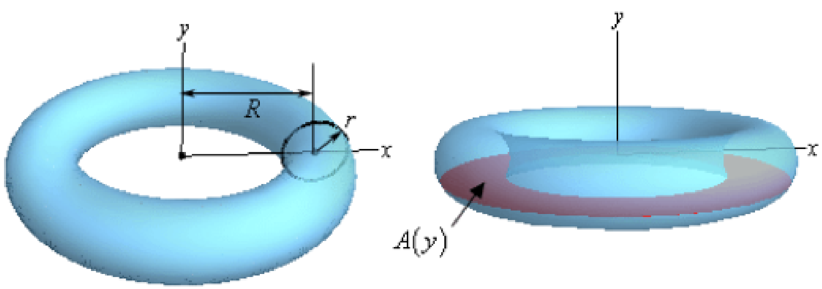
The limits on x are $-r \leq x \leq r$ and so the volume is then,

$$V = \int_{-r}^{r} \frac{1}{2\sqrt{3}}\left(r^2 - x^2\right) dx = \frac{1}{2\sqrt{3}}\left(r^2 x - \frac{1}{3}x^3\right)\Big|_{-r}^{r} = \frac{2r^3}{3\sqrt{3}}$$

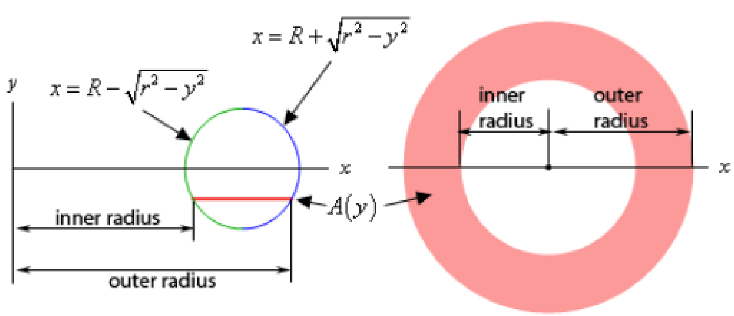


■ **Example 4.12** Find the volume of a torus with radii r and R. ■

Solution. First, just what is a torus? A torus is a donut shaped solid that is generated by rotating the circle of radius r and centered at $(R,0)$ about the y-axis. This is shown in the figire below.



Here, what we'll do is use a cross-section as shown in the sketch below. This cross-section is obtained by cuting the torus perpendicular to the y-axis we'll get a cross-section of a ring and finding the area.



The cross-sectional area is then,

$$
\begin{aligned}
A(y) &= \pi(\text{ outer radius })^2 - \pi(\text{ inner radius })^2 \\
&= \pi\left[\left(R+\sqrt{r^2-y^2}\right)^2 - \left(R-\sqrt{r^2-y^2}\right)^2\right] \\
&= \pi\left[R^2+2R\sqrt{r^2-y^2}+r^2-y^2-\left(R^2-2R\sqrt{r^2-y^2}+r^2-y^2\right)\right] \\
&= 4\pi R\sqrt{r^2-y^2}
\end{aligned}
$$

Next, the lowest cross-section will occur at $y=-r$ and the highest cross-section will occ at $y=r$ and so the limits for the integral will be $-r \le y \le r$. The integral giving the volume is then,

$$
\begin{aligned}
V &= \int_{-r}^{r} 4\pi R\sqrt{r^2-y^2}dy = 2\int_{0}^{r} 4\pi R\sqrt{r^2-y^2}dy \\
&= 8\pi R\int_{0}^{r}\sqrt{r^2-y^2}dy = 4\pi R\left(\frac{1}{4}\pi r^2\right) = 2\pi^2 r^2
\end{aligned}
$$

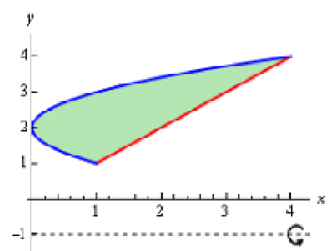
■ **Example 4.13** Determine the volume of the solid obtained by rotating the region bounded by $x=(y2)^2$ and $y=x$ about the line $y=1$. ■

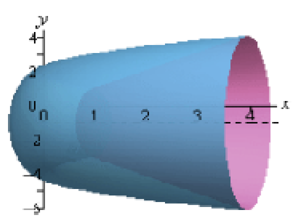
Solution. First, we should get the intersection points there. So, solving the equation

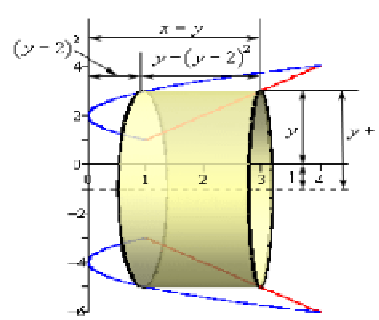
$$y=(y-2)^2 \;\; \textbf{or} \;\; y^2-5y+4=0$$

gives the intersection points $(1,1)$ and $(4,4)$.

Here is a sketch of the bounded region and the solid.







Here's the cross-sectional area for this cylinder.

$$\begin{aligned} A(y) &= 2\pi \text{ (radius) (width)} \\ &= 2\pi(y+1)\left(y-(y-2)^2\right) \\ &= 2\pi\left(-y^3+4y^2+y-4\right) \end{aligned}$$

The first cylinder will cut into the solid at $y=1$ and the final cylinder will cut in at $y=4$.

The volume is then,

$$\begin{aligned} V &= \int_c^d A(y)dy \\ &= 2\pi\int_1^4 -y^3+4y^2+y-4dy \\ &= 2\pi\left(-\frac{1}{4}y^4+\frac{4}{3}y^3+\frac{1}{2}y^2-4y\right)\bigg|_1^4 \\ &= \frac{63\pi}{2} \end{aligned}$$

4.4 Arc Length

THE ARC LENGTH FORMULA If f' is continuous on $[a,b]$, then the length of the curve $y=f(x), a\leqslant x\leqslant b$, is

$$L=\int_a^b\sqrt{1+[f'(x)]^2}dx$$

If we use Leibniz notation for derivatives, we can write the arc length formula as follows:

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

■ **Example 4.14** Find the length of the arc of the semi cubical parabola $y^2=x^3$ between the points $(1,1)$ and $(4,8)$.

Solution. For the top half of the curve we have

$$y=x^{3/2}, \qquad \frac{dy}{dx}=\frac{3}{2}x^{1/2}$$

and so the arc length formula gives

$$L=\int_1^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx=\int_1^4\sqrt{1+\frac{9}{4}x}dx$$

If we substitute $u = 1 + \frac{9}{4}x$, then $du = \frac{9}{4}dx$. When $x = 1, u = \frac{13}{4}$; when $x = 4, u = 10$. Therefore,

$$L = \frac{4}{9}\int_{13/4}^{10} \sqrt{u}du = \frac{4}{9} \cdot \frac{2}{3}u^{3/2}\Big]_{13/4}^{10}$$
$$= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$

■

If a curve has the equation $x = g(y), c \leqslant y \leqslant d$, and $g'(y)$ is continuous, then by interchanging the roles of x and y in Formula 2 or Equation 3, we obtain the following formula for its length:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy$$

■ **Example 4.15** Find the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.

Solution. Since $x = y^2$, we have $dx/dy = 2y$, gives

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy = \int_0^1 \sqrt{1 + 4y^2}dy$$

We make the trigonometric substitution $y = \frac{1}{2}\tan\theta$, which gives $dy = \frac{1}{2}\sec^2\theta d\theta$ and $\sqrt{1+4y^2} = \sqrt{1+\tan^2\theta} = \sec\theta$. When $y = 0, \tan\theta = 0$, so $\theta = 0$; when $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1} 2 = \alpha$, say. Thus

$$L = \int_0^\alpha \sec\theta \cdot \frac{1}{2}\sec^2\theta d\theta = \frac{1}{2}\int_0^\alpha \sec^3\theta d\theta$$
$$= \frac{1}{2} \cdot \frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^\alpha$$
$$= \frac{1}{4}(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|)$$

Since $\tan\alpha = 2$, we have $\sec^2\alpha = 1 + \tan^2\alpha = 5$, so $\sec\alpha = \sqrt{5}$ and

$$L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}$$

■

■ **Example 4.16** **(a)** Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1,1)$ to the point $\left(2, \frac{1}{2}\right)$.

Solution. **(a)** We have

$$y = \frac{1}{x} \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

and so the arc length is

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}} dx = \int_1^2 \frac{\sqrt{x^4+1}}{x^2} dx$$

■

The arc length function

We will find it useful to have a function that measures the arc length of a curve from a particular starting point to any other point on the curve. Thus if a smooth curve C has the equation $y = f(x), a \leqslant x \leqslant b$, let $s(x)$ be the distance along C from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$. Then s is a function, called **the arc length function**. It is defined by Formula

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt$$

(We have replaced the variable of integration by t so that x does not have two meanings.)

■ **Example 4.17** Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1,1)$ as the starting point.

Solution. If $f(x) = x^2 - \frac{1}{8}\ln x$, then

$$\begin{aligned}
f'(x) &= 2x - \frac{1}{8x} \\
1 + \left[f'(x)\right]^2 &= 1 + \left(2x - \frac{1}{8x}\right)^2 = 1 + 4x^2 - \frac{1}{2} + \frac{1}{64x^2} \\
&= 4x^2 + \frac{1}{2} + \frac{1}{64x^2} = \left(2x + \frac{1}{8x}\right)^2 \\
\sqrt{1 + [f'(x)]^2} &= 2x + \frac{1}{8x}
\end{aligned}$$

Thus the arc length function is given by

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1+[f'(t)]^2}\,dt \\ &= \int_1^x \left(2t - \frac{1}{8t}\right) dt = t^2 + \frac{1}{8}\ln t\Big]_1^x \\ &= x^2 + \frac{1}{8}\ln x - 1 \end{aligned}$$

For instance, the arc length along the curve from $(1,1)$ to $(3, f(3))$ is

$$s(3) = 3^2 + \frac{1}{8}\ln 3 - 1 = 8 + \frac{\ln 3}{8} \approx 8.1373$$

■

Exercise 4.4 Use the arc length formula to find the length of the curve $y = 2x-5, -1 \leqslant x \leqslant 3$. Check your answer by noting that the curve is a line segment and calculating its length by the distance formula. ■

Exercise 4.5 Use the arc length formula to find the length of the curve $y = \sqrt{2-x^2}, 0 \leqslant x \leqslant 1$. Check your answer by noting that the curve is part of a circle. ■

Exercise 4.6 Set up, but do not evaluate, an integral for the length of the curve.

1. $y = \cos x, \quad 0 \leqslant x \leqslant 2\pi$
2. $y = xe^{-x^2}, \quad 0 \leqslant x \leqslant 1$
3. $x = y + y^3, \quad 1 \leqslant y \leqslant 4$
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

■

Exercise 4.7 Find the length of the curve.

1. $y = 1 + 6x^{3/2}, \quad 0 \leqslant x \leqslant 1$
2. $y^2 = 4(x+4)^3, \quad 0 \leqslant x \leqslant 2, \quad y > 0$
3. $y = \frac{x^5}{6} + \frac{1}{10x^3}, \quad 1 \leqslant x \leqslant 2$
4. $x = \frac{y^4}{8} + \frac{1}{4y^2}, \quad 1 \leqslant y \leqslant 2$
5. $x = \frac{1}{3}\sqrt{y}(y-3), \quad 1 \leqslant y \leqslant 9$
6. $y = \ln(\cos x), \quad 0 \leqslant x \leqslant \pi/3$
7. $y = \ln(\sec x), \quad 0 \leqslant x \leqslant \pi/4$
8. $y = 3 + \frac{1}{2}\cosh 2x, \quad 0 \leqslant x \leqslant 1$

■

Final Exam with Model Answers

Question No.(3):

Find the following integrals

$$\int \frac{1+\sin 2x}{\cos^2 x}dx, \quad \int (x-7)^9 dx, \quad \int \frac{dx}{\sqrt{1-x^2}}, \quad \int x^2 e^{-3x}dx, \quad \int \frac{\sin\left(\tan^{-1}x\right)}{1+x^2}dx$$

Answer: Let c be an arbitrary constant.

$$\int \frac{1+\sin 2x}{\cos^2 x}dx = \int \left(\sec^2 x + \frac{2\sin x\cos x}{\cos^2 x}\right)dx = \tan x + \ln\cos^2 x + c.$$

$$\int (x-7)^9 dx = \frac{1}{10}(x-7)^{10} = c. \qquad \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c.$$

$$\int x^2 e^{-3x}dx = -\frac{1}{3}\left[x^2 e^{-3x} - 2\int xe^{-3x}dx\right] = \left(-\frac{1}{3}x^2 - \frac{2}{9}x - \frac{2}{27}\right)e^{-3x} + c$$

$$\int \frac{\sin\left(\tan^{-1}x\right)}{1+x^2}dx = -\cos\left(\tan^{-1}x\right) + c.$$

Question No.(4):

Compute the following integrals

$$\int_0^1 \cot^{-1}x dx, \quad \int_0^{\frac{\pi}{2}} \sin^3 x\cos^5 x dx, \quad \int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}}, \quad \int_1^2 \frac{dx}{(x+1)(x-3)}, \quad \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$$

Answer:

$$\int_0^1 \cot^{-1}x dx = \frac{\pi}{4} + \ln\sqrt{2}.$$

$$\int_0^{\frac{\pi}{2}} \sin^3 x\cos^5 x dx = \int_0^{\frac{\pi}{2}} (\cos^5 x - \cos^7 x)\sin x dx = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$$

Using $x = 2\sin\theta$

$$\int_1^{\sqrt{2}} \frac{dx}{x^2\sqrt{4-x^2}} = \frac{1}{4}\int_{\pi/6}^{\pi/4} \operatorname{cosec}^2\theta d\theta = \frac{1-\sqrt{3}}{4\sqrt{3}}.$$

$$\int_1^2 \frac{dx}{(x+1)(x-3)} = \frac{1}{4}\int_1^2 \left(\frac{1}{x-3} - \frac{1}{x+1}\right)dx = \frac{1}{4}\ln\left(\frac{1}{3}\right).$$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx.$$

So,

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}.$$

Question No.(5):

(1) Find the area of the region that is enclosed by the curves $y = x^2$ and $y = 2x - x^2$.

Answer: First, we find the points of intersection of the parabolas by solving their equations simultaneously. This gives $x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x-1) = 0$, so $x = 0$ or 1 . The points of intersection are $(0,0)$ and $(1,1)$. We see from Figure 3 that the top and bottom boundaries are

$$y_T = 2x - x^2 \quad \text{and} \quad y_B = x^2$$

The area of a typical rectangle is

$$(y_T - y_B)\,\Delta x = \left(2x - x^2 - x^2\right)\Delta x$$

and the region lies between $x = 0$ and $x = 1$. So the total area is

$$\begin{aligned} A &= \int_0^1 \left(2x - 2x^2\right) dx = 2\int_0^1 \left(x - x^2\right) dx \\ &= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 \end{aligned}$$

(2) Find the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 8$, and $x = 0$ about the y-axis.

Answer: Because the region is rotated about the y-axis, it makes sense to slice the solid perpendicular to the y-axis and therefore to integrate with respect to y. If we slice at height y, we get a circular disk with radius x, where $x = \sqrt[3]{y}$. So the area of a cross section through y is

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

and the volume of the approximating cylinder pictured in Figure 8(b) is

$$A(y)\Delta y = \pi y^{2/3}\Delta y$$

Since the solid lies between $y = 0$ and $y = 8$, its volume is

$$V = \int_0^8 A(y)dy = \int_0^8 \pi y^{2/3} dy = \pi\left[\frac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}$$