Chapter 30

Fundamental Integration Formulas

IF $F(x)$ IS A FUNCTION whose derivative $F'(x) = f(x)$ on a certain interval of the x axis, then $F(x)$ is called an *antiderivative* or *indefinite integral* of $f(x)$. The indefinite integral of a given function is not unique; for example, x^2, $x^2 + 5$, and $x^2 - 4$ are all indefinite integrals of $f(x) = 2x$, since $\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2 - 4) = 2x$. All indefinite integrals of $f(x) = 2x$ are then included in $F(x) = x^2 + C$, where C, called the *constant of integration*, is an arbitrary constant.

The symbol $\int f(x)\, dx$ is used to indicate the indefinite integral of $f(x)$. Thus we write $\int 2x\, dx = x^2 + C$. In the expression $\int f(x)\, dx$, the function $f(x)$ is called the *integrand*.

FUNDAMENTAL INTEGRATION FORMULAS. A number of the formulas below follow immediately from the standard differentiation formulas of earlier chapters, while others may be checked by differentiation. Formula 25, for example, may be checked by showing that

$$\frac{d}{dx}\left(\frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C\right) = \sqrt{a^2 - x^2}$$

Absolute value signs appear in certain of the formulas. For example, for formula 5 we write $\int \frac{dx}{x} = \ln|x| + C$ instead of

$$\int \frac{dx}{x} = \ln x + C \text{ for } x > 0 \qquad \text{and} \qquad \int \frac{dx}{x} = \ln(-x) + C \text{ for } x < 0$$

and for formula 10 we have $\int \tan x\, dx = \ln|\sec x| + C$ instead of

$$\int \tan x\, dx = \ln \sec x + C \qquad \text{for all } x \text{ such that } \sec x \geq 1$$

and

$$\int \tan x\, dx = \ln(-\sec x) + C \qquad \text{for all } x \text{ such that } \sec x \leq -1$$

1. $\int \frac{d}{dx}[f(x)]\, dx = f(x) + C$

2. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

3. $\int af(x)\, dx = a\int f(x)\, dx$, a any constant

4. $\int x^m\, dx = \frac{x^{m+1}}{m+1} + C$, $m \neq -1$

5. $\int \frac{dx}{x} = \ln|x| + C$

6. $\int a^x\, dx = \frac{a^x}{\ln a} + C$, $a > 0, a \neq 1$

7. $\int e^x\, dx = e^x + C$

8. $\int \sin x\, dx = -\cos x + C$

9. $\int \cos x \, dx = \sin x + C$

10. $\int \tan x \, dx = \ln |\sec x| + C$

11. $\int \cot x \, dx = \ln |\sin x| + C$

12. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

13. $\int \csc x \, dx = \ln |\csc x - \cot x| + C$

14. $\int \sec^2 x \, dx = \tan x + C$

15. $\int \csc^2 x \, dx = -\cot x + C$

16. $\int \sec x \tan x \, dx = \sec x + C$

17. $\int \csc x \cot x \, dx = -\csc x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C$

21. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$

22. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$

23. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln (x + \sqrt{x^2 + a^2}) + C$

24. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + C$

25. $\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C$

26. $\int \sqrt{x^2 + a^2} \, dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \ln (x + \sqrt{x^2 + a^2}) + C$

27. $\int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln |x + \sqrt{x^2 - a^2}| + C$

THE METHOD OF SUBSTITUTION. To evaluate an antiderivative $\int f(x) \, dx$, it is often useful to replace x with a new variable u by means of a *substitution* $x = g(u)$, $dx = g'(u) \, du$. The equation

$$\int f(x) \, dx = \int f(g(u))g'(u) \, du \tag{30.1}$$

is valid. After finding the right side of (*30.1*), we replace u with $g^{-1}(x)$; that is, we obtain the result in terms of x. To verify (*30.1*), observe that, if $F(x) = \int f(x) \, dx$, then $\frac{d}{du} F(x) = \frac{d}{dx} F(x) \frac{dx}{du} = f(x)g'(u) = f(g(u))g'(u)$. Hence, $F(x) = \int f(g(u))g'(u) \, du$, which is (*30.1*).

EXAMPLE 1: To evaluate $\int (x + 3)^{11} \, dx$, replace $x + 3$ with u; that is, let $x = u - 3$. Then $dx = du$, and we obtain

$$\int (x + 3)^{11} \, dx = \int u^{11} \, du = \tfrac{1}{12} u^{12} + C = \tfrac{1}{12}(x + 3)^{12} + C$$

QUICK INTEGRATION BY INSPECTION. Two simple formulas enable us to find antiderivatives almost immediately. The first is

$$\int g'(x)[g(x)]^r \, dx = \frac{1}{r + 1} [g(x)]^{r+1} + C \quad r \neq -1 \tag{30.2}$$

This formula is justified by noting that $\frac{d}{dx}\left\{\frac{1}{r+1}[g(x)]^{r+1}\right\} = g'(x)[g(x)]^r$.

EXAMPLE 2: (*a*) $\int \frac{(\ln x)^2}{x}\,dx = \int \frac{1}{x}(\ln x)^2\,dx = \frac{1}{3}(\ln x)^3 + C$

(*b*) $\int x\sqrt{x^2+3}\,dx = \frac{1}{2}\int (2x)(x^2+3)^{1/2}\,dx = \frac{1}{2}\left[\frac{1}{3/2}(x^2+3)^{3/2}\right] + C = \frac{1}{3}[\sqrt{x^2+3}]^3 + C$

The second quick integration formula is

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C \qquad (30.3)$$

This formula is justified by noting that $\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}$.

EXAMPLE 3: (*a*) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$

(*b*) $\int \frac{x^2}{x^3-5}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3-5}\,dx = \frac{1}{3}\ln|x^3-5| + C$

Solved Problems

In Problems 1 to 8, evaluate the indefinite integral at the left.

1. $\int x^5\,dx = \frac{x^6}{6} + C$

2. $\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

3. $\int \sqrt[3]{z}\,dz = \int z^{1/3}\,dz = \frac{z^{4/3}}{4/3} + C = \frac{3}{4}z^{4/3} + C$

4. $\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-2/3}\,dx = \frac{x^{1/3}}{1/3} + C = 3x^{1/3} + C$

5. $\int (2x^2 - 5x + 3)\,dx = 2\int x^2\,dx - 5\int x\,dx + 3\int dx = \frac{2x^3}{3} - \frac{5x^2}{2} + 3x + C$

6. $\int (1-x)\sqrt{x}\,dx = \int (x^{1/2} - x^{3/2})\,dx = \int x^{1/2}\,dx - \int x^{3/2}\,dx = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$

7. $\int (3s+4)^2\,ds = \int (9s^2 + 24s + 16)\,ds = 9(\frac{1}{3}s^3) + 24(\frac{1}{2}s^2) + 16s + C = 3s^3 + 12s^2 + 16s + C$

8. $\int \frac{x^3 + 5x^2 - 4}{x^2}\,dx = \int (x + 5 - 4x^{-2})\,dx = \frac{1}{2}x^2 + 5x - \frac{4x^{-1}}{-1} + C = \frac{1}{2}x^2 + 5x + \frac{4}{x} + C$

9. Evaluate (*a*) $\int (x^3+2)^2(3x^2)\,dx$, (*b*) $\int (x^3+2)^{1/2}x^2\,dx$, (*c*) $\int \frac{8x^2\,dx}{(x^3+2)^3}$, and (*d*) $\int \frac{x^2\,dx}{\sqrt[4]{(x^3+2)}}$ by means of (*30.2*).

(*a*) $\int (x^3+2)^2(3x^2)\,dx = \frac{1}{3}(x^3+2)^3 + C$

(*b*) $\int (x^3+2)^{1/2}x^2\,dx = \frac{1}{3}\int (x^3+2)^{1/2}(3x^2)\,dx = \frac{1}{3}\frac{2}{3}(x^3+2)^{3/2} + C = \frac{2}{9}(x^3+2)^{3/2} + C$

(*c*) $\int \frac{8x^2}{(x^3+2)^3}\,dx = \frac{8}{3}\int (x^3+2)^{-3}(3x^2)\,dx = \frac{8}{3}\left(-\frac{1}{2}\right)(x^3+2)^{-2} + C = -\frac{4}{3}\frac{1}{(x^3+2)^2} + C$

(*d*) $\int \frac{x^2}{\sqrt[4]{x^3+2}}\,dx = \frac{1}{3}\int (x^3+2)^{-1/4}(3x^2)\,dx = \frac{1}{3}\frac{4}{3}(x^3+2)^{3/4} + C = \frac{4}{9}(x^3+2)^{3/4} + C$

All four integrals can also be evaluated by making the substitution $u = x^3 + 2$, $du = 3x^2\,dx$.

10. Evaluate $\int 3x\sqrt{1-2x^2}\,dx$.

Formula (*30.2*) yields

$$\int 3x\sqrt{1-2x^2}\,dx = 3(-\tfrac{1}{4})\int (1-2x^2)^{1/2}(-4x)\,dx = -\tfrac{3}{4}\tfrac{2}{3}(1-2x^2)^{3/2} + C$$
$$= -\tfrac{1}{2}(1-2x^2)^{3/2} + C$$

We could also use the substitution $u = 1 - 2x^2$, $du = -4x\,dx$.

11. Evaluate $\int \frac{(x+3)\,dx}{(x^2+6x)^{1/3}}$.

Formula (*30.2*) yields

$$\int \frac{(x+3)\,dx}{(x^2+6x)^{1/3}} = \frac{1}{2}\int (x^2+6x)^{-1/3}(2x+6)\,dx = \frac{1}{2}\frac{3}{2}(x^2+6x)^{2/3} + C$$
$$= \frac{3}{4}(x^2+6x)^{2/3} + C$$

We could also use the substitution $u = x^2 + 6x$, $du = (2x+6)\,dx$.

In Problems 12 to 15, evaluate the indefinite integral on the left.

12. $\int \sqrt[3]{1-x^2}\,x\,dx = -\frac{1}{2}\int (1-x^2)^{1/3}(-2x\,dx) = -\frac{1}{2}\frac{3}{4}(1-x^2)^{4/3} + C = -\frac{3}{8}(1-x^2)^{4/3} + C$

13. $\int \sqrt{x^2-2x^4}\,dx = \int (1-2x^2)^{1/2}x\,dx = -\frac{1}{4}\int (1-2x^2)^{1/2}(-4x\,dx) - \frac{1}{4}\frac{2}{3}(1-2x^2)^{3/2} + C$

$= -\frac{1}{6}(1-2x^2)^{3/2} + C$

14. $\int \frac{(1+x)^2}{\sqrt{x}}\,dx = \int \frac{1+2x+x^2}{x^{1/2}}\,dx = \int (x^{-1/2} + 2x^{1/2} + x^{3/2})\,dx = 2x^{1/2} + \frac{4}{3}x^{3/2} + \frac{2}{5}x^{5/2} + C$

15. $\int \frac{x^2+2x}{(x+1)^2}\,dx = \int \left[1 - \frac{1}{(x+1)^2}\right]dx = x + \frac{1}{x+1} + C' = \frac{x^2}{x+1} + 1 + C' = \frac{x^2}{x+1} + C$

FORMULAS 5 TO 7

16. Evaluate $\int dx/x$.

Formula 5 gives $\int \frac{dx}{x} = \ln|x| + C$.

17. Evaluate $\int \frac{dx}{x+2}$, using *(30.3)*.

$\int \frac{dx}{x+2} = \ln|x+2| + C$. We also could use formula 5 and the substitution $u = x+2$, $du = dx$.

18. Evaluate $\int \frac{dx}{2x-3}$, using *(30.3)*.

$\int \frac{dx}{2x-3} = \frac{1}{2}\int \frac{2\,dx}{2x-3} = \frac{1}{2}\ln|2x-3| + C$. Another method is to make the substitution $u = 2x-3$, $du = 2\,dx$.

In Problems 19 to 27, evaluate the integral at the left.

19. $\int \frac{x\,dx}{x^2-1} = \frac{1}{2}\int \frac{2x\,dx}{x^2-1} = \frac{1}{2}\ln|x^2-1| + C = \frac{1}{2}\ln|x^2-1| + \ln c = \ln(c\sqrt{|x^2-1|}),\ c>0$

20. $\int \frac{x^2\,dx}{1-2x^3} = -\frac{1}{6}\int \frac{-6x^2\,dx}{1-2x^3} = -\frac{1}{6}\ln|1-2x^3| + C = \ln\frac{c}{\sqrt[6]{|1-2x^3|}},\ c>0$

21. $\int \frac{x+2}{x+1}\,dx = \int\left(1+\frac{1}{x+1}\right)dx = x + \ln|x+1| + C$

22. $\int e^{-x}\,dx = -\int e^{-x}(-dx) = -e^{-x} + C$

23. $\int a^{2x}\,dx = \frac{1}{2}\int a^{2x}(2\,dx) = \frac{1}{2}\frac{a^{2x}}{\ln a} + C$

24. $\int e^{3x}\,dx = \frac{1}{3}\int e^{3x}(3\,dx) = \frac{e^{3x}}{3} + C$

25. $\int \frac{e^{1/x}\,dx}{x^2} = -\int e^{1/x}\left(-\frac{dx}{x^2}\right) = -e^{1/x} + C$

26. $\int (e^x+1)^3 e^x\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(e^x+1)^4}{4} + C$, where $u = e^x+1$ and $du = e^x\,dx$, or

$\int (e^x+1)^3 e^x\,dx = \int (e^x+1)^3\,d(e^x+1) = \frac{(e^x+1)^4}{4} + C$

27. $\int \frac{dx}{e^x+1} = \int \frac{e^{-x}\,dx}{1+e^{-x}} = -\int \frac{-e^{-x}\,dx}{1+e^{-x}} = -\ln(1+e^{-x}) + C = \ln\frac{e^x}{1+e^x} + C$

$= x - \ln(1+e^x) + C$

The absolute-value sign is not needed here because $1+e^{-x}>0$ for all values of x.

FORMULAS 8 TO 17

In Problems 28 to 47, evaluate the integral at the left.

28. $\int \sin\frac{1}{2}x\,dx = 2\int(\sin\frac{1}{2}x)(\frac{1}{2}\,dx) = -2\cos\frac{1}{2}x + C$

29. $\int \cos 3x\,dx = \frac{1}{3}\int (\cos 3x)(3\,dx) = \frac{1}{3}\sin 3x + C$

30. $\int \sin^2 x \cos x\,dx = \int \sin^2 x(\cos x\,dx) = \dfrac{\sin^3 x}{3} + C$

31. $\int \tan x\,dx = \int \dfrac{\sin x}{\cos x}\,dx = -\int \dfrac{-\sin x\,dx}{\cos x} = -\ln|\cos x| + C = \ln|\sec x| + C$

32. $\int \tan 2x\,dx = \frac{1}{2}\int (\tan 2x)(2\,dx) = \frac{1}{2}\ln|\sec 2x| + C$

33. $\int x \cot x^2\,dx = \frac{1}{2}\int (\cot x^2)(2x\,dx) = \frac{1}{2}\ln|\sin x^2| + C$

34. $\int \sec x\,dx = \int \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + C$

35. $\int \sec\sqrt{x}\,\dfrac{dx}{\sqrt{x}} = 2\int (\sec x^{1/2})(\frac{1}{2}x^{-1/2}\,dx) = 2\ln|\sec\sqrt{x} + \tan\sqrt{x}| + C$

36. $\int \sec^2 2ax\,dx = \dfrac{1}{2a}\int (\sec^2 2ax)(2a\,dx) = \dfrac{\tan 2ax}{2a} + C$

37. $\int \dfrac{\sin x + \cos x}{\cos x}\,dx = \int (\tan x + 1)\,dx = \ln|\sec x| + x + C$

38. $\int \dfrac{\sin y\,dy}{\cos^2 y} = \int \tan y \sec y\,dy = \sec y + C$

39. $\int (1 + \tan x)^2\,dx = \int (1 + 2\tan x + \tan^2 x)\,dx = \int (\sec^2 x + 2\tan x)\,dx$

$= \tan x + 2\ln|\sec x| + C$

40. $\int e^x \cos e^x\,dx = \int (\cos e^x)(e^x\,dx) = \sin e^x + C$

41. $\int e^{3\cos 2x}\sin 2x\,dx = -\dfrac{1}{6}\int e^{3\cos 2x}(-6\sin 2x\,dx) = -\dfrac{e^{3\cos 2x}}{6} + C$

42. $\int \dfrac{dx}{1 + \cos x} = \int \dfrac{1 - \cos x}{1 - \cos^2 x}\,dx = \int \dfrac{1 - \cos x}{\sin^2 x}\,dx = \int (\csc^2 x - \cot x \csc x)\,dx$

$= -\cot x + \csc x + C$

43. $\int (\tan 2x + \sec 2x)^2\,dx = \int (\tan^2 2x + 2\tan 2x \sec 2x + \sec^2 2x)\,dx$

$= \int (2\sec^2 2x + 2\tan 2x \sec 2x - 1)\,dx = \tan 2x + \sec 2x - x + C$

44. $\int \csc u\,du = \int \dfrac{du}{\sin u} = \int \dfrac{du}{2\sin\frac{1}{2}u\cos\frac{1}{2}u} = \int \dfrac{(\sec^2\frac{1}{2}u)(\frac{1}{2}\,du)}{\tan\frac{1}{2}u} = \ln|\tan\frac{1}{2}u| + C$

45. $\int (\sec 4x - 1)^2\, dx = \int (\sec^2 4x - 2\sec 4x + 1)\, dx = \frac{1}{4}\tan 4x - \frac{1}{2}\ln|\sec 4x + \tan 4x| + x + C$

46. $\int \frac{\sec x \tan x\, dx}{a + b\sec x} = \frac{1}{b}\int \frac{(\sec x \tan x)(b\, dx)}{a + b\sec x} = \frac{1}{b}\ln|a + b\sec x| + C$

47. $\int \frac{dx}{\csc 2x - \cot 2x} = \int \frac{\sin 2x\, dx}{1 - \cos 2x} = \frac{1}{2}\int \frac{(\sin 2x)(2\, dx)}{1 - \cos 2x} = \frac{1}{2}\ln(1 - \cos 2x) + C'$

$$= \frac{1}{2}\ln(2\sin^2 x) + C' = \frac{1}{2}(\ln 2 + 2\ln|\sin x|) + C' = \ln|\sin x| + C$$

FORMULAS 18 TO 20

In Problems 48 to 72, evaluate the integral at the left.

48. $\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$

49. $\int \frac{dx}{1 + x^2} = \arctan x + C$

50. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \text{arcsec}\, x + C$

51. $\int \frac{dx}{\sqrt{4 - x^2}} = \arcsin \frac{x}{2} + C$

52. $\int \frac{dx}{9 + x^2} = \frac{1}{3}\arctan \frac{x}{3} + C$

53. $\int \frac{dx}{\sqrt{25 - 16x^2}} = \frac{1}{4}\int \frac{4\, dx}{\sqrt{5^2 - (4x)^2}} = \frac{1}{4}\arcsin \frac{4x}{5} + C$

54. $\int \frac{dx}{4x^2 + 9} = \frac{1}{2}\int \frac{2\, dx}{(2x)^2 + 3^2} = \frac{1}{6}\arctan \frac{2x}{3} + C$

55. $\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2\, dx}{2x\sqrt{(2x)^2 - 3^2}} = \frac{1}{3}\text{arcsec}\, \frac{2x}{3} + C$

56. $\int \frac{x^2\, dx}{\sqrt{1 - x^6}} = \frac{1}{3}\int \frac{3x^2\, dx}{\sqrt{1 - (x^3)^2}} = \frac{1}{3}\arcsin x^3 + C$

57. $\int \frac{x\, dx}{x^4 + 3} = \frac{1}{2}\int \frac{2x\, dx}{(x^2)^2 + 3} = \frac{1}{2}\frac{1}{\sqrt{3}}\arctan \frac{x^2}{\sqrt{3}} + C = \frac{\sqrt{3}}{6}\arctan \frac{x^2\sqrt{3}}{3} + C$

58. $\int \frac{dx}{x\sqrt{x^4 - 1}} = \frac{1}{2}\int \frac{2x\, dx}{x^2\sqrt{(x^2)^2 - 1}} = \frac{1}{2}\text{arcsec}\, x^2 + C = \frac{1}{2}\arccos \frac{1}{x^2} + C$

59. $\int \frac{dx}{\sqrt{4 - (x + 2)^2}} = \arcsin \frac{x + 2}{2} + C$

60. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\, dx}{e^{2x} + 1} = \arctan e^x + C$

61. $\int \frac{3x^3 - 4x^2 + 3x}{x^2 + 1}\, dx = \int \left(3x - 4 + \frac{4}{x^2 + 1}\right) dx = \frac{3x^2}{2} - 4x + 4\arctan x + C$

62. $$\int \frac{\sec x \tan x \, dx}{9 + 4\sec^2 x} = \frac{1}{2}\int \frac{2\sec x \tan x \, dx}{3^2 + (2\sec x)^2} = \frac{1}{6}\arctan\frac{2\sec x}{3} + C$$

63. $$\int \frac{(x+3)\,dx}{\sqrt{1-x^2}} = \int \frac{x\,dx}{\sqrt{1-x^2}} + 3\int \frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + 3\arcsin x + C$$

64. $$\int \frac{(2x-7)\,dx}{x^2+9} = \int \frac{2x\,dx}{x^2+9} - 7\int \frac{dx}{x^2+9} = \ln(x^2+9) - \frac{7}{3}\arctan\frac{x}{3} + C$$

65. $$\int \frac{dy}{y^2+10y+30} = \int \frac{dy}{(y^2+10y+25)+5} = \int \frac{dy}{(y+5)^2+5} = \frac{\sqrt{5}}{5}\arctan\frac{(y+5)\sqrt{5}}{5} + C$$

66. $$\int \frac{dx}{\sqrt{20+8x-x^2}} = \int \frac{dx}{\sqrt{36-(x^2-8x+16)}} = \int \frac{dx}{\sqrt{36-(x-4)^2}} = \arcsin\frac{x-4}{6} + C$$

67. $$\int \frac{dx}{2x^2+2x+5} = \int \frac{2\,dx}{4x^2+4x+10} = \int \frac{2\,dx}{(2x+1)^2+9} = \frac{1}{3}\arctan\frac{2x+1}{3} + C$$

68. $$\int \frac{x+1}{x^2-4x+8}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2-4x+8}\,dx = \frac{1}{2}\int \frac{(2x-4)+6}{x^2-4x+8}\,dx = \frac{1}{2}\int \frac{(2x-4)\,dx}{x^2-4x+8} + 3\int \frac{dx}{x^2-4x+8}$$
$$= \frac{1}{2}\int \frac{(2x-4)\,dx}{x^2-4x+8} + 3\int \frac{dx}{(x-2)^2+4} = \frac{1}{2}\ln(x^2-4x+8) + \frac{3}{2}\arctan\frac{x-2}{2} + C$$

The absolute-value sign is not needed here because $x^2 - 4x + 8 > 0$ for all values of x.

69. $$\int \frac{dx}{\sqrt{28-12x-x^2}} = \int \frac{dx}{\sqrt{64-(x^2+12x+36)}} = \int \frac{dx}{\sqrt{64-(x+6)^2}} = \arcsin\frac{x+6}{8} + C$$

70. $$\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x-6}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\int \frac{(-2x-4)-2}{\sqrt{5-4x-x^2}}\,dx$$
$$= -\frac{1}{2}\int \frac{-2x-4}{\sqrt{5-4x-x^2}}\,dx + \int \frac{dx}{\sqrt{5-4x-x^2}}$$
$$= -\frac{1}{2}\int \frac{-2x-4}{\sqrt{5-4x-x^2}}\,dx + \int \frac{dx}{\sqrt{9-(x+2)^2}}$$
$$= -\sqrt{5-4x-x^2} + \arcsin\frac{x+2}{3} + C$$

71. $$\int \frac{2x+3}{9x^2-12x+8}\,dx = \frac{1}{9}\int \frac{18x+27}{9x^2-12x+8}\,dx = \frac{1}{9}\int \frac{(18x-12)+39}{9x^2-12x+8}\,dx$$
$$= \frac{1}{9}\int \frac{18x-12}{9x^2-12x+8}\,dx + \frac{13}{3}\int \frac{dx}{(3x-2)^2+4}$$
$$= \frac{1}{9}\ln(9x^2-12x+8) + \frac{13}{18}\arctan\frac{3x-2}{2} + C$$

72. $$\int \frac{x+2}{\sqrt{4x-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}\,dx = -\frac{1}{2}\int \frac{(-2x+4)-8}{\sqrt{4x-x^2}}\,dx$$
$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx + 4\int \frac{dx}{\sqrt{4-(x-2)^2}} = -\sqrt{4x-x^2} + 4\arcsin\frac{x-2}{2} + C$$

FORMULAS 21 TO 24

In Problems 73 to 89, evaluate the integral at the left.

73. $\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

74. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

75. $\int \frac{dx}{x^2-4} = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

76. $\int \frac{dx}{9-x^2} = \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$

77. $\int \frac{dx}{\sqrt{x^2+1}} = \ln(x+\sqrt{x^2+1}) + C$

78. $\int \frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}| + C$

79. $\int \frac{dx}{\sqrt{4x^2+9}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln(2x+\sqrt{4x^2+9}) + C$

80. $\int \frac{dz}{\sqrt{9z^2-25}} = \frac{1}{3}\int \frac{3\,dz}{\sqrt{9z^2-25}} = \frac{1}{3}\ln|3z+\sqrt{9z^2-25}| + C$

81. $\int \frac{dx}{9x^2-16} = \frac{1}{3}\int \frac{3\,dx}{(3x)^2-16} = \frac{1}{24}\ln\left|\frac{3x-4}{3x+4}\right| + C$

82. $\int \frac{dy}{25-16y^2} = \frac{1}{4}\int \frac{4\,dy}{25-(4y)^2} = \frac{1}{40}\ln\left|\frac{5+4y}{5-4y}\right| + C$

83. $\int \frac{dx}{x^2+6x+8} = \int \frac{dx}{(x+3)^2-1} = \frac{1}{2}\ln\left|\frac{(x+3)-1}{(x+3)+1}\right| + C = \frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + C$

84. $\int \frac{dx}{4x-x^2} = \int \frac{dx}{4-(x-2)^2} = \frac{1}{4}\ln\left|\frac{2+(x-2)}{2-(x-2)}\right| + C = \frac{1}{4}\ln\left|\frac{x}{4-x}\right| + C$

85. $\int \frac{ds}{\sqrt{4s+s^2}} = \int \frac{ds}{\sqrt{(s+2)^2-4}} = \ln|s+2+\sqrt{4s+s^2}| + C$

86. $\int \frac{x+2}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2+9}} + 2\int \frac{dx}{\sqrt{x^2+9}}$

$$= \sqrt{x^2+9} + 2\ln(x+\sqrt{x^2+9}) + C$$

87. $\int \frac{2x-3}{4x^2-11}\,dx = \frac{1}{4}\int \frac{8x-12}{4x^2-11}\,dx = \frac{1}{4}\int \frac{8x\,dx}{4x^2-11} - \frac{3}{2}\int \frac{2\,dx}{4x^2-11}$

$$= \frac{1}{4}\ln|4x^2-11| - \frac{3\sqrt{11}}{44}\ln\left|\frac{2x-\sqrt{11}}{2x+\sqrt{11}}\right| + C$$

88. $\int \frac{x+2}{\sqrt{x^2+2x-3}}\,dx = \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x-3}}\,dx = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x-3}}\,dx + \int \frac{dx}{\sqrt{(x+1)^2-4}}$

$$= \sqrt{x^2+2x-3} + \ln|x+1+\sqrt{x^2+2x-3}| + C$$

89. $$\int \frac{2-x}{4x^2+4x-3}\,dx = -\frac{1}{8}\int \frac{8x-16}{4x^2+4x-3}\,dx = -\frac{1}{8}\int \frac{8x+4}{4x^2+4x-3}\,dx + \frac{5}{2}\int \frac{dx}{(2x+1)^2-4}$$
$$= -\frac{1}{8}\ln|4x^2+4x-3| + \frac{5}{16}\ln\left|\frac{2x-1}{2x+3}\right| + C$$

FORMULAS 25 TO 27

In Problems 90 to 95, evaluate the integral at the left.

90. $$\int \sqrt{25-x^2}\,dx = \frac{1}{2}x\sqrt{25-x^2} + \frac{25}{2}\arcsin\frac{x}{5} + C$$

91. $$\int \sqrt{3-4x^2}\,dx = \frac{1}{2}\int (\sqrt{3-4x^2})(2\,dx) = \frac{1}{2}\left(\frac{2x}{2}\sqrt{3-4x^2} + \frac{3}{2}\arcsin\frac{2x}{\sqrt{3}}\right) + C$$
$$= \frac{1}{2}x\sqrt{3-4x^2} + \frac{3}{4}\arcsin\frac{2x\sqrt{3}}{3} + C$$

92. $$\int \sqrt{x^2-36}\,dx = \tfrac{1}{2}x\sqrt{x^2-36} - 18\ln|x+\sqrt{x^2-36}| + C$$

93. $$\int \sqrt{3x^2+5}\,dx = \frac{1}{\sqrt{3}}\int \sqrt{3x^2+5}\sqrt{3}\,dx = \frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}x\sqrt{3x^2+5} + \frac{5}{2}\ln(\sqrt{3}x+\sqrt{3x^2+5})\right] + C$$
$$= \frac{1}{2}x\sqrt{3x^2+5} + \frac{5\sqrt{3}}{6}\ln(\sqrt{3}x+\sqrt{3x^2+5}) + C$$

94. $$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\arcsin\frac{x+1}{2} + C$$

95. $$\int \sqrt{4x^2-4x+5}\,dx = \frac{1}{2}\int (\sqrt{(2x-1)^2+4})(2\,dx)$$
$$= \frac{1}{2}\left[\frac{2x-1}{2}\sqrt{4x^2-4x+5} + 2\ln(2x-1+\sqrt{4x^2-4x+5})\right] + C$$
$$= \frac{2x-1}{4}\sqrt{4x^2-4x+5} + \ln(2x-1+\sqrt{4x^2-4x+5}) + C$$

Supplementary Problems

In Problems 96 to 200, evaluate the integral at the left.

96. $$\int (4x^3+3x^2+2x+5)\,dx = x^4+x^3+x^2+5x+C$$

97. $$\int (3-2x-x^4)\,dx = 3x - x^2 - \tfrac{1}{5}x^5 + C$$

98. $$\int (2-3x+x^3)\,dx = 2x - \tfrac{3}{2}x^2 + \tfrac{1}{4}x^4 + C$$

99. $$\int (x^2-1)^2\,dx = x^5/5 - 2x^3/3 + x + C$$

100. $\int (\sqrt{x} - \frac{1}{2}x + 2/\sqrt{x})\,dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4x^{1/2} + C$

101. $\int (a+x)^3\,dx = \frac{1}{4}(a+x)^4 + C$

102. $\int (x-2)^{3/2}\,dx = \frac{2}{5}(x-2)^{5/2} + C$

103. $\int \frac{dx}{x^3} = -\frac{1}{2x^2} + C$

104. $\int \frac{dx}{(x-1)^3} = -\frac{1}{2(x-1)^2} + C$

105. $\int \frac{dx}{\sqrt{x+3}} = 2\sqrt{x+3} + C$

106. $\int \sqrt{3x-1}\,dx = \frac{2}{9}(3x-1)^{3/2} + C$

107. $\int \sqrt{2-3x}\,dx = -\frac{2}{9}(2-3x)^{3/2} + C$

108. $\int (2x^2+3)^{1/3}x\,dx = \frac{3}{16}(2x^2+3)^{4/3} + C$

109. $\int (x-1)^2 x\,dx = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + C$

110. $\int (x^2-1)x\,dx = \frac{1}{4}(x^2-1)^2 + C$

111. $\int \sqrt{1+y^4}\,y^3\,dy = \frac{1}{6}(1+y^4)^{3/2} + C$

112. $\int (x^3+3)x^2\,dx = \frac{1}{6}(x^3+3)^2 + C$

113. $\int (4-x^2)^2 x^2\,dx = \frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7 + C$

114. $\int \frac{dy}{(2-y)^3} = \frac{1}{2(2-y)^2} + C$

115. $\int \frac{x\,dx}{(x^2+4)^3} = -\frac{1}{4(x^2+4)^2} + C$

116. $\int (1-x^3)^2\,dx = x - \frac{1}{2}x^4 + \frac{1}{7}x^7 + C$

117. $\int (1-x^3)^2 x\,dx = \frac{1}{2}x^2 - \frac{2}{5}x^5 + \frac{1}{8}x^8 + C$

118. $\int (1-x^3)^2 x^2\,dx = -\frac{1}{9}(1-x^3)^3 + C$

119. $\int (x^2-x)^4(2x-1)\,dx = \frac{1}{5}(x^2-x)^5 + C$

120. $\int \frac{3t\,dt}{\sqrt[3]{t^2+3}} = \frac{9}{4}(t^2+3)^{2/3} + C$

121. $\int \frac{(x+1)\,dx}{\sqrt{x^2+2x-4}} = \sqrt{x^2+2x-4} + C$

122. $\int \frac{dx}{(a+bx)^{1/3}} = \frac{3}{2b}(a+bx)^{2/3} + C$

123. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx = \frac{2}{3}(1+\sqrt{x})^3 + C$

124. $\int \sqrt{x}\,(3-5x)\,dx = 2x^{3/2}(1-x) + C$

125. $\int \frac{(x+1)(x-2)}{\sqrt{x}}\,dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} - 4x^{1/2} + C$

126. $\int \frac{dx}{x-1} = \ln|x-1| + C$

127. $\int \frac{dx}{3x+1} = \frac{1}{3}\ln|3x+1| + C$

128. $\int \frac{3x\,dx}{x^2+2} = \frac{3}{2}\ln(x^2+2) + C$

129. $\int \frac{x^2\,dx}{1-x^3} = -\frac{1}{3}\ln|1-x^3| + C$

130. $\int \frac{x-1}{x+1}\,dx = x - 2\ln|x+1| + C$

131. $\int \frac{x^2+2x+2}{x+2}\,dx = \frac{1}{2}x^2 + 2\ln|x+2| + C$

132. $\int \frac{x+1}{x^2+2x+2}\,dx = \frac{1}{2}\ln(x^2+2x+2) + C$

133. $\int \left(\frac{dx}{2x-1} - \frac{dx}{2x+1}\right) = \ln\sqrt{\left|\frac{2x-1}{2x+1}\right|} + C$

134. $\int a^{4x}\,dx = \frac{1}{4}\frac{a^{4x}}{\ln a} + C$

135. $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$

136. $\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}e^{1/x^2} + C$

137. $\int e^{-x^2+2}x\,dx = -\frac{1}{2}e^{-x^2+2} + C$

138. $\int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$

139. $\int (e^x+1)^2\,dx = \frac{1}{2}e^{2x} + 2e^x + x + C$

140. $\int (e^x - x^e)\,dx = e^x - \frac{x^{e+1}}{e+1} + C$

141. $\int (e^x+1)^2 e^x\,dx = \frac{1}{3}(e^x+1)^3 + C$

142. $\int \frac{e^{2x}}{e^{2x}+3}\,dx = \frac{1}{2}\ln(e^{2x}+3) + C$

143. $\int \left(e^x + \frac{1}{e^x}\right)^2 dx = \frac{1}{2}e^{2x} + 2x - \frac{1}{2e^{2x}} + C$

144. $\int \frac{e^x-1}{e^x+1}\,dx = \ln(e^x+1)^2 - x + C$

145. $\int \frac{e^{2x}-1}{e^{2x}+3}\,dx = \ln(e^{2x}+3)^{2/3} - \frac{1}{3}x + C$

146. $\int \frac{dx}{\sqrt{x}(1-\sqrt{x})} = \ln\frac{C}{(1-\sqrt{x})^2},\ C>0$

147. $\int \frac{dx}{x+x^{1/3}} = \frac{3}{2}\ln C(x^{2/3}+1),\ C>0$

148. $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

149. $\int \cos\frac{1}{2}x\,dx = 2\sin\frac{1}{2}x + C$

150. $\int \sec 3x\tan 3x\,dx = \frac{1}{3}\sec 3x + C$

151. $\int \csc^2 2x\,dx = -\frac{1}{2}\cot 2x + C$

152. $\int x\sec^2 x^2\,dx = \frac{1}{2}\tan x^2 + C$

153. $\int \tan^2 x\,dx = \tan x - x + C$

154. $\int \tan\frac{1}{2}x\,dx = 2\ln|\sec\frac{1}{2}x| + C$

155. $\int \csc 3x\,dx = \frac{1}{3}\ln|\csc 3x - \cot 3x| + C$

156. $\int b\sec ax\tan ax\,dx = \frac{b}{a}\sec ax + C$

157. $\int (\cos x - \sin x)^2\,dx = x + \frac{1}{2}\cos 2x + C$

158. $\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax + C$

$= -\frac{1}{2a}\cos^2 ax + C' = -\frac{1}{4a}\cos 2ax + C''$

159. $\int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

160. $\int \cos^4 x\sin x\,dx = -\frac{1}{5}\cos^5 x + C$

161. $\int \tan^5 x\sec^2 x\,dx = \frac{1}{6}\tan^6 x + C$

162. $\int \cot^4 3x\csc^2 3x\,dx = -\frac{1}{15}\cot^5 3x + C$

163. $\int \frac{dx}{1-\sin\frac{1}{2}x} = 2(\tan\frac{1}{2}x + \sec\frac{1}{2}x) + C$

164. $\int \frac{dx}{1+\cos 3x} = \frac{1-\cos 3x}{3\sin 3x} + C$

165. $\int \frac{dx}{1+\sec ax} = x + \frac{1}{a}(\cot ax - \csc ax) + C$

166. $\int \sec^2\frac{x}{a}\tan\frac{x}{a}\,dx = \frac{1}{2}a\tan^2\frac{x}{a} + C$

167. $\int \frac{\sec^2 3x}{\tan 3x}\,dx = \frac{1}{3}\ln|\tan 3x| + C$

168. $\int \frac{\sec^5 x}{\csc x}\,dx = \frac{1}{4}\sec^4 x + C$

169. $\int e^{\tan 2x}\sec^2 2x\,dx = \frac{1}{2}e^{\tan 2x} + C$

170. $\int e^{2\sin 3x}\cos 3x\,dx = \frac{1}{6}e^{2\sin 3x} + C$

171. $\int \frac{dx}{\sqrt{5-x^2}} = \arcsin\frac{x\sqrt{5}}{5} + C$

172. $\int \frac{dx}{5+x^2} = \frac{\sqrt{5}}{5}\arctan\frac{x\sqrt{5}}{5} + C$

173. $\int \frac{dx}{x\sqrt{x^2-5}} = \frac{\sqrt{5}}{5}\,\mathrm{arcsec}\,\frac{x\sqrt{5}}{5} + C$

174. $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}} = \arcsin e^x + C$

175. $\int \frac{e^{2x}\,dx}{1+e^{4x}} = \frac{1}{2}\arctan e^{2x} + C$

176. $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\arcsin\frac{3x}{2} + C$

177. $\int \frac{dx}{9x^2+4} = \frac{1}{6}\arctan\frac{3x}{2} + C$

178. $\int \frac{\sin 8x}{9+\sin^4 4x}\,dx = \frac{1}{12}\arctan\frac{\sin^2 4x}{3} + C$

179. $\int \frac{\sec^2 x\,dx}{\sqrt{1-4\tan^2 x}} = \frac{1}{2}\arcsin(2\tan x) + C$

180. $\int \frac{dx}{x\sqrt{4-9\ln^2 x}} = \frac{1}{3}\arcsin\ln x^{3/2} + C$

181. $\displaystyle\int \frac{2x^4 - x^2}{2x^2+1}\,dx = \frac{1}{3}x^3 - x + \frac{\sqrt{2}}{2}\arctan x\sqrt{2} + C$

182. $\displaystyle\int \frac{\cos 2x\,dx}{\sin^2 2x + 8} = \frac{\sqrt{2}}{8}\arctan\frac{\sin 2x}{2\sqrt{2}} + C$

183. $\displaystyle\int \frac{(2x-3)\,dx}{x^2+6x+13} = \int \frac{(2x+6)\,dx}{x^2+6x+13} - 9\int \frac{dx}{x^2+6x+13} = \ln(x^2+6x+13) - \frac{9}{2}\arctan\frac{x+3}{2} + C$

184. $\displaystyle\int \frac{(x-1)\,dx}{3x^2-4x+3} = \frac{1}{6}\int \frac{(6x-4)\,dx}{3x^2-4x+3} - \int \frac{dx}{9x^2-12x+9} = \frac{1}{6}\ln(3x^2-4x+3) - \frac{\sqrt{5}}{15}\arctan\frac{3x-2}{\sqrt{5}} + C$

185. $\displaystyle\int \frac{x\,dx}{\sqrt{27+6x-x^2}} = -\sqrt{27+6x-x^2} + 3\arcsin\frac{x-3}{6} + C$

186. $\displaystyle\int \frac{(5-4x)\,dx}{\sqrt{12x-4x^2-8}} = \sqrt{12x-4x^2-8} - \frac{1}{2}\arcsin(2x-3) + C$

187. $\displaystyle\int \frac{dx}{x^2-4} = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

188. $\displaystyle\int \frac{dx}{4x^2-9} = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$

189. $\displaystyle\int \frac{dx}{9-x^2} = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

190. $\displaystyle\int \frac{dx}{25-9x^2} = \frac{1}{30}\ln\left|\frac{3x+5}{3x-5}\right| + C$

191. $\displaystyle\int \frac{dx}{\sqrt{x^2+4}} = \ln(x+\sqrt{x^2+4}) + C$

192. $\displaystyle\int \frac{dx}{\sqrt{4x^2-25}} = \frac{1}{2}\ln|2x+\sqrt{4x^2-25}| + C$

193. $\displaystyle\int \sqrt{16-9x^2}\,dx = \frac{1}{2}x\sqrt{16-9x^2} + \frac{8}{3}\arcsin\frac{3x}{4} + C$

194. $\displaystyle\int \sqrt{x^2-16}\,dx = \tfrac{1}{2}x\sqrt{x^2-16} - 8\ln|x+\sqrt{x^2-16}| + C$

195. $\displaystyle\int \sqrt{4x^2+9}\,dx = \tfrac{1}{2}x\sqrt{4x^2+9} + \tfrac{9}{4}\ln(2x+\sqrt{4x^2+9}) + C$

196. $\displaystyle\int \sqrt{x^2-2x-3}\,dx = \tfrac{1}{2}(x-1)\sqrt{x^2-2x-3} - 2\ln|x-1+\sqrt{x^2-2x-3}| + C$

197. $\displaystyle\int \sqrt{12+4x-x^2}\,dx = \tfrac{1}{2}(x-2)\sqrt{12+4x-x^2} + 8\arcsin\tfrac{1}{4}(x-2) + C$

198. $\displaystyle\int \sqrt{x^2+4x}\,dx = \tfrac{1}{2}(x+2)\sqrt{x^2+4x} - 2\ln|x+2+\sqrt{x^2+4x}| + C$

199. $\displaystyle\int \sqrt{x^2-8x}\,dx = \tfrac{1}{2}(x-4)\sqrt{x^2-8x} - 8\ln|x-4+\sqrt{x^2-8x}| + C$

200. $\displaystyle\int \sqrt{6x-x^2}\,dx = \frac{1}{2}(x-3)\sqrt{6x-x^2} + \frac{9}{2}\arcsin\frac{x-3}{3} + C$

Chapter 31

Integration by Parts

INTEGRATION BY PARTS. When u and v are differentiable functions of x,

$$d(uv) = u\,dv + v\,du$$

or

$$u\,dv = d(uv) - v\,du$$

and

$$\int u\,dv = uv - \int v\,du \tag{31.1}$$

When (*31.1*) is to be used in a required integration, the given integral must be separated into two parts, one part being u and the other part, together with dx, being dv. (For this reason, integration by use of (*31.1*) is called *integration by parts*.) Two general rules can be stated:

1. The part selected as dv must be readily integrable.
2. $\int v\,du$ must not be more complex than $\int u\,dv$.

EXAMPLE 1: Find $\int x^3 e^{x^2}\,dx$.

Take $u = x^2$ and $dv = e^{x^2} x\,dx$; then $du = 2x\,dx$ and $v = \frac{1}{2}e^{x^2}$. Now by (*31.1*),

$$\int x^3 e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \int x\,e^{x^2}\,dx = \tfrac{1}{2}x^2\,e^{x^2} - \tfrac{1}{2}e^{x^2} + C$$

EXAMPLE 2: Find $\int \ln(x^2+2)\,dx$.

Take $u = \ln(x^2+2)$ and $dv = dx$; then $du = \dfrac{2x\,dx}{x^2+2}$ and $v = x$. By (*31.1*),

$$\int \ln(x^2+2)\,dx = x\ln(x^2+2) - \int \frac{2x^2\,dx}{x^2+2} = x\ln(x^2+2) - \int\left(2 - \frac{4}{x^2+2}\right)dx$$

$$= x\ln(x^2+2) - 2x + 2\sqrt{2}\arctan\frac{x}{\sqrt{2}} + C$$

(See Problems 1 to 10.)

REDUCTION FORMULAS. The labor involved in successive applications of integration by parts to evaluate an integral (see Problem 9) may be materially reduced by the use of *reduction formulas*. In general, a reduction formula yields a new integral of the same form as the original but with an exponent increased or reduced. A reduction formula succeeds if ultimately it produces an integral that can be evaluated. Among the reduction formulas are:

$$\int \frac{dx}{(a^2 \pm x^2)^m} = \frac{1}{a^2}\left[\frac{x}{(2m-2)(a^2 \pm x^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{dx}{(a^2 \pm x^2)^{m-1}}\right],\quad m \neq 1 \tag{31.2}$$

$$\int (a^2 \pm x^2)^m\,dx = \frac{x(a^2 \pm x^2)^m}{2m+1} + \frac{2ma^2}{2m+1}\int (a^2 \pm x^2)^{m-1}\,dx,\quad m \neq -1/2 \tag{31.3}$$

$$\int \frac{dx}{(x^2 - a^2)^m} = -\frac{1}{a^2}\left[\frac{x}{(2m-2)(x^2 - a^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{dx}{(x^2 - a^2)^{m-1}}\right],\quad m \neq 1 \tag{31.4}$$

$$\int (x^2 - a^2)^m\,dx = \frac{x(x^2 - a^2)^m}{2m+1} - \frac{2ma^2}{2m+1}\int (x^2 - a^2)^{m-1}\,dx,\quad m \neq -1/2 \tag{31.5}$$

$$\int x^m\,e^{ax}\,dx = \frac{1}{a}x^m\,e^{ax} - \frac{m}{a}\int x^{m-1}\,e^{ax}\,dx \tag{31.6}$$

$$\int \sin^m x\,dx = -\frac{\sin^{m-1}x\cos x}{m} + \frac{m-1}{m}\int \sin^{m-2}x\,dx \qquad (31.7)$$

$$\int \cos^m x\,dx = \frac{\cos^{m-1}x\sin x}{m} + \frac{m-1}{m}\int \cos^{m-2}x\,dx \qquad (31.8)$$

$$\int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx$$

$$= -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx\,,\quad m\neq -n \qquad (31.9)$$

$$\int x^m \sin bx\,dx = -\frac{x^m}{b}\cos bx + \frac{m}{b}\int x^{m-1}\cos bx\,dx \qquad (31.10)$$

$$\int x^m \cos bx\,dx = \frac{x^m}{b}\sin bx - \frac{m}{b}\int x^{m-1}\sin bx\,dx \qquad (31.11)$$

(See Problem 11.)

Solved Problems

1. Find $\int x\sin x\,dx$.

We have three choices: (*a*) $u = x\sin x$, $dv = dx$; (*b*) $u = \sin x$, $dv = x\,dx$; (*c*) $u = x$, $dv = \sin x\,dx$.

(*a*) Let $u = x\sin x$, $dv = dx$. Then $du = (\sin x + x\cos x)\,dx$, $v = x$, and

$$\int x\sin x\,dx = x\cdot x\sin x - \int x(\sin x + x\cos x)\,dx$$

The resulting integral is not as simple as the original, and this choice is discarded.

(*b*) Let $u = \sin x$, $dv = x\,dx$. Then $du = \cos x\,dx$, $v = \frac{1}{2}x^2$, and

$$\int x\sin x\,dx = \tfrac{1}{2}x^2\sin x - \int \tfrac{1}{2}x^2\cos x\,dx$$

The resulting integral is not as simple as the original, and this choice too is discarded.

(*c*) Let $u = x$, $dv = \sin x\,dx$. Then $du = dx$, $v = -\cos x$, and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

2. Find $\int xe^x\,dx$.

Let $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

3. Find $\int x^2\ln x\,dx$.

Let $u = \ln x$, $dv = x^2\,dx$. Then $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\,\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

4. Find $\int x\sqrt{1+x}\,dx$.

Let $u = x$, $dv = \sqrt{1+x}\,dx$. Then $du = dx$, $v = \frac{2}{3}(1+x)^{3/2}$, and

$$\int x\sqrt{1+x}\,dx = \tfrac{2}{3}x(1+x)^{3/2} - \tfrac{2}{3}\int (1+x)^{3/2}\,dx = \tfrac{2}{3}x(1+x)^{3/2} - \tfrac{4}{15}(1+x)^{5/2} + C$$

5. Find $\int \arcsin x\,dx$.

Let $u = \arcsin x$, $dv = dx$. Then $du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$, and

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C$$

6. Find $\int \sin^2 x\,dx$.

Let $u = \sin x$, $dv = \sin x\,dx$. Then $du = \cos x\,dx$, $v = -\cos x$, and

$$\int \sin^2 x\,dx = -\sin x\cos x + \int \cos^2 x\,dx = -\sin x\cos x + \int (1-\sin^2 x)\,dx$$
$$= -\tfrac{1}{2}\sin 2x + \int dx - \int \sin^2 x\,dx$$

Hence $\quad 2\int \sin^2 x\,dx = -\tfrac{1}{2}\sin 2x + x + C' \quad$ and $\quad \int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$

7. Find $\int \sec^3 x\,dx$.

Let $u = \sec x$, $dv = \sec^2 x\,dx$. Then $du = \sec x\tan x\,dx$, $v = \tan x$, and

$$\int \sec^3 x\,dx = \sec x\tan x - \int \sec x\tan^2 x\,dx = \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx$$
$$= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

Then $\quad 2\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx = \sec x\tan x + \ln|\sec x + \tan x| + C'$

and $\quad \int \sec^3 x\,dx = \tfrac{1}{2}\{\sec x\tan x + \ln|\sec x + \tan x|\} + C$

8. Find $\int x^2\sin x\,dx$.

Let $u = x^2$, $dv = \sin x\,dx$. Then $du = 2x\,dx$, $v = -\cos x$, and

$$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

For the resulting integral, let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$, $v = \sin x$, and

$$\int x^2\sin x\,dx = -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right) = -x^2\cos x + 2x\sin x + 2\cos x + C$$

9. Find $\int x^3 e^{2x}\,dx$.

Let $u = x^3$, $dv = e^{2x}\,dx$. Then $du = 3x^2\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{2}\int x^2e^{2x}\,dx$$

For the resulting integral, let $u = x^2$ and $dv = e^{2x}\,dx$. Then $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{2}\left(\tfrac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx\right) = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{2}\int xe^{2x}\,dx$$

For the resulting integral, let $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{2}\left(\tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,dx\right) = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + C$$

10. Find reduction formulas for (*a*) $\displaystyle\int \frac{x^2\,dx}{(a^2 \pm x^2)^m}$ and (*b*) $\displaystyle\int x^2(a^2 \pm x^2)^{m-1}\,dx$.

(*a*) Take $u = x$, $dv = \dfrac{x\,dx}{(a^2 \pm x^2)^m}$; then $du = dx$, $v = \dfrac{\mp 1}{(2m-2)(a^2 \pm x^2)^{m-1}}$, and

$$\int \frac{x^2\,dx}{(a^2 \pm x^2)^m} = \frac{\mp x}{(2m-2)(a^2 \pm x^2)^{m-1}} \pm \frac{1}{2m-2}\int \frac{dx}{(a^2 \pm x^2)^{m-1}}$$

(*b*) Take $u = x$, $dv = x(a^2 \pm x^2)^{m-1}\,dx$; then $du = dx$, $v = \dfrac{\pm 1}{2m}(a^2 \pm x^2)^m$, and

$$\int x^2(a^2 \pm x^2)^{m-1}\,dx = \frac{\pm x}{2m}(a^2 \pm x^2)^m \mp \frac{1}{2m}\int (a^2 \pm x^2)^m\,dx$$

11. Find: (*a*) $\displaystyle\int \frac{dx}{(1+x^2)^{5/2}}$ and (*b*) $\displaystyle\int (9+x^2)^{3/2}\,dx$.

(*a*) Since (*31.2*) reduces the exponent in the denominator by 1, we use this formula twice to obtain

$$\int \frac{dx}{(1+x^2)^{5/2}} = \frac{x}{3(1+x^2)^{3/2}} + \frac{2}{3}\int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{3(1+x^2)^{3/2}} + \frac{2}{3}\frac{x}{(1+x^2)^{1/2}} + C$$

(*b*) Using (*31.3*), we obtain

$$\int (9+x^2)^{3/2}\,dx = \tfrac{1}{4}x(9+x^2)^{3/2} + \tfrac{27}{4}\int (9+x^2)^{1/2}\,dx$$

$$= \tfrac{1}{4}x(9+x^2)^{3/2} + \tfrac{27}{8}[x(9+x^2)^{1/2} + 9\ln(x + \sqrt{9+x^2})] + C$$

12. Derive reduction formula (*31.7*): $\displaystyle\int \sin^m x\,dx = -\frac{\sin^{m-1}x\cos x}{m} + \frac{m-1}{m}\int \sin^{m-2}x\,dx$.

We use integration by parts: Let $u = \sin^{m-1}x$ and $dv = \sin x\,dx$; then $du = (m-1)\sin^{m-2}x \cos x\,dx$, $v = -\cos x$, and

$$\int \sin^m x\,dx = -\cos x\sin^{m-1}x + (m-1)\int \sin^{m-2}x\cos^2 x\,dx$$

$$= -\cos x\sin^{m-1}x + (m-1)\int (\sin^{m-2}x)(1-\sin^2 x)\,dx$$

$$= -\cos x\sin^{m-1}x + (m-1)\int \sin^{m-2}x\,dx - (m-1)\int \sin^m x\,dx$$

Hence,
$$m\int \sin^m x\,dx = -\cos x\sin^{m-1}x + (m-1)\int \sin^{m-2}x\,dx$$

and division by m yields (*31.7*).

Supplementary Problems

In Problems 13 to 29 and 32 to 40 evaluate the indefinite integral at left.

13. $\int x \cos x \, dx = x \sin x + \cos x + C$

14. $\int x \sec^2 3x \, dx = \frac{1}{3} x \tan 3x - \frac{1}{9} \ln|\sec 3x| + C$

15. $\int \arccos 2x \, dx = x \arccos 2x - \frac{1}{2}\sqrt{1-4x^2} + C$

16. $\int \arctan x \, dx = x \arctan x - \ln \sqrt{1+x^2} + C$

17. $\int x^2\sqrt{1-x} \, dx = -\frac{2}{105}(1-x)^{3/2}(15x^2 + 12x + 8) + C$

18. $\int \frac{xe^x \, dx}{(1+x)^2} = \frac{e^x}{1+x} + C$

19. $\int x \arctan x \, dx = \frac{1}{2}(x^2+1) \arctan x - \frac{1}{2}x + C$

20. $\int x^2 e^{-3x} \, dx = -\frac{1}{3}e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C$

21. $\int \sin^3 x \, dx = -\frac{2}{3}\cos^3 x - \sin^2 x \cos x + C$

22. $\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

23. $\int \frac{x \, dx}{\sqrt{a+bx}} = \frac{2(bx-2a)\sqrt{a+bx}}{3b^2} + C$

24. $\int \frac{x^2 \, dx}{\sqrt{1+x}} = \frac{2}{15}(3x^2 - 4x + 8)\sqrt{1+x} + C$

25. $\int x \arcsin x^2 \, dx = \frac{1}{2}x^2 \arcsin x^2 + \frac{1}{2}\sqrt{1-x^4} + C$

26. $\int \sin x \sin 3x \, dx = \frac{1}{8}\sin 3x \cos x - \frac{3}{8}\sin x \cos 3x + C$

27. $\int \sin(\ln x) \, dx = \frac{1}{2}x(\sin \ln x - \cos \ln x) + C$

28. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2+b^2} + C$

29. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C$

30. (*a*) Write $\int \frac{a^2 \, dx}{(a^2 \pm x^2)^m} = \int \frac{(a^2 \pm x^2) \mp x^2}{(a^2 \pm x^2)^m} dx = \int \frac{dx}{(a^2 \pm x^2)^{m-1}} \mp \int \frac{x^2 \, dx}{(a^2 \pm x^2)^m}$ and use the result of Problem 10(*a*) to obtain (*31.2*).

(*b*) Write $\int (a^2 \pm x^2)^m \, dx = a^2 \int (a^2 \pm x^2)^{m-1} \, dx \pm \int x^2(a^2 \pm x^2)^{m-1} \, dx$ and use the result of Problem 10(*b*) to obtain (*31.3*).

31. Derive reduction formulas (*31.4*) to (*31.11*).

32. $\int \frac{dx}{(1-x^2)^3} = \frac{x(5-3x^2)}{8(1-x^2)^2} + \frac{3}{16} \ln\left|\frac{1+x}{1-x}\right| + C$

33. $\int \frac{dx}{(4+x^2)^{3/2}} = \frac{x}{4(4+x^2)^{1/2}} + C$

34. $\int (4-x^2)^{3/2}\,dx = \frac{1}{4}x(10-x^2)\sqrt{4-x^2} + 6\arcsin\frac{1}{2}x + C$

35. $\int \frac{dx}{(x^2-16)^3} = \frac{1}{2048}\left[\frac{x(3x^2-80)}{(x^2-16)^2} + \frac{3}{8}\ln\left|\frac{x-4}{x+4}\right|\right] + C$

36. $\int (x^2-1)^{5/2}\,dx = \frac{1}{48}x(8x^4-26x^2+33)\sqrt{x^2-1} - \frac{5}{16}\ln|x+\sqrt{x^2-1}| + C$

37. $\int \sin^4 x\,dx = \frac{3}{8}x - \frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + C$

38. $\int \cos^5 x\,dx = \frac{1}{15}(3\cos^4 x + 4\cos^2 x + 8)\sin x + C$

39. $\int \sin^3 x\cos^2 x\,dx = -\frac{1}{5}\cos^3 x(\sin^2 x + \frac{2}{3}) + C$

40. $\int \sin^4 x\cos^5 x\,dx = \frac{1}{9}\sin^5 x(\cos^4 x + \frac{4}{7}\cos^2 x + \frac{8}{35}) + C$

An alternative procedure for some of the more tedious problems of this section can be found by noting (see Problem 9) that in

$$\int x^3e^{2x}\,dx = \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C \qquad (1)$$

the terms on the right, apart from the coefficients, are the different terms obtained by repeated differentiations of the integrand x^3e^{2x}. Thus, we may write at once

$$\int x^3e^{2x}\,dx = Ax^3e^{2x} + Bx^2e^{2x} + Dxe^{2x} + Ee^{2x} + C \qquad (2)$$

and from it obtain by differentiation

$$x^3e^{2x} = 2Ax^3e^{2x} + (3A+2B)x^2e^{2x} + (2B+2D)xe^{2x} + (D+2E)e^{2x}$$

Equating coefficients, we have

$$2A=1 \qquad 3A+2B=0 \qquad 2B+2D=0 \qquad D+2E=0$$

so that $A=\frac{1}{2}$, $B=-\frac{3}{2}A=-\frac{3}{4}$, $D=-B=\frac{3}{4}$, $E=-\frac{1}{2}D=-\frac{3}{8}$. Substituting for A, B, D, E in (*2*), we obtain (*1*).

This procedure may be used for finding $\int f(x)\,dx$ whenever repeated differentiation of $f(x)$ yields only a finite number of different terms.

41. Find $\int e^{2x}\cos 3x\,dx = \frac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C$, using

$$\int e^{2x}\cos 3x\,dx = Ae^{2x}\sin 3x + Be^{2x}\cos 3x + C$$

42. Find $\int e^{3x}(2\sin 4x - 5\cos 4x)\,dx = \frac{1}{25}e^{3x}(-14\sin 4x - 23\cos 4x) + C$, using

$$\int e^{3x}(2\sin 4x - 5\cos 4x)\,dx = Ae^{3x}\sin 4x + Be^{3x}\cos 4x + C$$

43. Find $\int \sin 3x\cos 2x\,dx = -\frac{1}{5}(2\sin 3x\sin 2x + 3\cos 3x\cos 2x) + C$, using

$$\int \sin 3x\cos 2x\,dx = A\sin 3x\sin 2x + B\cos 3x\cos 2x + D\cos 3x\sin 2x + E\sin 3x\cos 2x + C$$

44. Find $\int e^{3x}x^2\sin x\,dx = \frac{e^{3x}}{250}[25x^2(3\sin x - \cos x) - 10x(4\sin x - 3\cos x) + 9\sin x - 13\cos x] + C$.

Chapter 32

Trigonometric Integrals

THE FOLLOWING IDENTITIES are employed to find some of the trigonometric integrals of this chapter:

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin x \cos x = \frac{1}{2} \sin 2x$
7. $\sin x \cos y = \frac{1}{2}[\sin (x - y) + \sin (x + y)]$
8. $\sin x \sin y = \frac{1}{2}[\cos (x - y) - \cos (x + y)]$
9. $\cos x \cos y = \frac{1}{2}[\cos (x - y) + \cos (x + y)]$
10. $1 - \cos x = 2 \sin^2 \frac{1}{2}x$
11. $1 + \cos x = 2 \cos^2 \frac{1}{2}x$
12. $1 \pm \sin x = 1 \pm \cos (\frac{1}{2}\pi - x)$

TWO SPECIAL SUBSTITUTION RULES are useful in a few simple cases:

1. For $\int \sin^m x \cos^n x\, dx$: If m is odd, substitute $u = \cos x$. If n is odd, substitute $u = \sin x$.
2. For $\int \tan^m x \sec^n x\, dx$: If n is even, substitute $u = \tan x$. If m is odd, substitute $u = \sec x$.

Solved Problems

SINES AND COSINES

In Problems 1 to 17, evaluate the integral at the left.

1. $\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4} \sin 2x + C$

2. $\int \cos^2 3x\, dx = \int \frac{1}{2}(1 + \cos 6x)\, dx = \frac{1}{2}x + \frac{1}{12} \sin 6x + C$

3. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1 - \cos^2 x) \sin x\, dx = -\cos x + \frac{1}{3} \cos^3 x + C$

This solution is equivalent to using the substitution $u = \cos x$, $du = -\sin x\, dx$, as follows:

$$\int \sin^3 x\, dx = -\int (1 - u^2)\, du = -u + \frac{1}{3}u^3 + C = -\cos x + \frac{1}{3} \cos^3 x + C$$

4. $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx$

$$= \int \cos x\, dx - 2 \int \sin^2 x \cos x\, dx + \int \sin^4 x \cos x\, dx$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$$

This amounts to the use of the substitution $u = \sin x$. We have also used (*30.2*).

5. $\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx = \int \sin^2 x\,(1 - \sin^2 x) \cos x\, dx$

$= \int \sin^2 x \cos x\, dx - \int \sin^4 x \cos x\, dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$

6. $\int \cos^4 2x \sin^3 2x\, dx = \int \cos^4 2x \sin^2 2x \sin 2x\, dx = \int \cos^4 2x\,(1 - \cos^2 2x) \sin 2x\, dx$

$= \int \cos^4 2x \sin 2x\, dx - \int \cos^6 2x \sin 2x\, dx = -\frac{1}{10}\cos^5 2x + \frac{1}{14}\cos^7 2x + C$

7. $\int \sin^3 3x \cos^5 3x\, dx = \int (1 - \cos^2 3x) \cos^5 3x \sin 3x\, dx$

$= \int \cos^5 3x \sin 3x\, dx - \int \cos^7 3x \sin 3x\, dx = -\frac{1}{18}\cos^6 3x + \frac{1}{24}\cos^8 3x + C$

or $\int \sin^3 3x \cos^5 3x\, dx = \int \sin^3 3x\,(1 - \sin^2 3x)^2 \cos 3x\, dx$

$= \int \sin^3 3x \cos 3x\, dx - 2\int \sin^5 3x \cos 3x\, dx + \int \sin^7 3x \cos 3x\, dx$

$= \frac{1}{12}\sin^4 3x - \frac{1}{9}\sin^6 3x + \frac{1}{24}\sin^8 3x + C$

8. $\int \cos^3 \frac{x}{3}\, dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cos \frac{x}{3}\, dx = 3 \sin \frac{x}{3} - \sin^3 \frac{x}{3} + C$

9. $\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx = \frac{1}{4}\int (1 - \cos 2x)^2\, dx$

$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{4}\int \cos^2 2x\, dx$

$= \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{8}\int (1 + \cos 4x)\, dx$

$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

10. $\int \sin^2 x \cos^2 x\, dx = \frac{1}{4}\int \sin^2 2x\, dx = \frac{1}{8}\int (1 - \cos 4x)\, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C$

11. $\int \sin^4 3x \cos^2 3x\, dx = \int (\sin^2 3x \cos^2 3x) \sin^2 3x\, dx = \frac{1}{8}\int \sin^2 6x\,(1 - \cos 6x)\, dx$

$= \frac{1}{8}\int \sin^2 6x\, dx - \frac{1}{8}\int \sin^2 6x \cos 6x\, dx$

$= \frac{1}{16}\int (1 - \cos 12x)\, dx - \frac{1}{8}\int \sin^2 6x \cos 6x\, dx$

$= \frac{1}{16}x - \frac{1}{192}\sin 12x - \frac{1}{144}\sin^3 6x + C$

12. $\int \sin 3x \sin 2x\, dx = \int \frac{1}{2}[\cos(3x - 2x) - \cos(3x + 2x)]\, dx = \frac{1}{2}\int (\cos x - \cos 5x)\, dx$

$= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$

13. $\int \sin 3x \cos 5x\, dx = \int \frac{1}{2}[\sin(3x - 5x) + \sin(3x + 5x)]\, dx = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$

14. $\int \cos 4x \cos 2x\,dx = \frac{1}{2}\int (\cos 2x + \cos 6x)\,dx = \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C$

15. $\int \sqrt{1-\cos x}\,dx = \sqrt{2}\int \sin \frac{1}{2}x\,dx = -2\sqrt{2}\cos \frac{1}{2}x + C$

16. $\int (1+\cos 3x)^{3/2}\,dx = 2\sqrt{2}\int \cos^3 \frac{3}{2}x\,dx = 2\sqrt{2}\int (1-\sin^2 \frac{3}{2}x)\cos \frac{3}{2}x\,dx$

$$= 2\sqrt{2}\left(\tfrac{2}{3}\sin \tfrac{3}{2}x - \tfrac{2}{9}\sin^3 \tfrac{3}{2}x\right) + C$$

17. $\int \frac{dx}{\sqrt{1-\sin 2x}} = \int \frac{dx}{\sqrt{1-\cos(\frac{1}{2}\pi - 2x)}} = \frac{\sqrt{2}}{2}\int \frac{dx}{\sin(\frac{1}{4}\pi - x)} = \frac{\sqrt{2}}{2}\int \csc(\frac{1}{4}\pi - x)\,dx$

$$= -\frac{\sqrt{2}}{2}\ln\left|\csc(\tfrac{1}{4}\pi - x) - \cot(\tfrac{1}{4}\pi - x)\right| + C$$

TANGENTS, SECANTS, COTANGENTS, COSECANTS

Evaluate the integral at the left.

18. $\int \tan^4 x\,dx = \int \tan^2 x \tan^2 x\,dx = \int \tan^2 x(\sec^2 x - 1)\,dx = \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$

$$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx = \tfrac{1}{3}\tan^3 x - \tan x + x + C$$

19. $\int \tan^5 x\,dx = \int \tan^3 x \tan^2 x\,dx = \int \tan^3 x(\sec^2 x - 1)\,dx$

$$= \int \tan^3 x \sec^2 x\,dx - \int \tan^3 x\,dx = \int \tan^3 x \sec^2 x\,dx - \int \tan x(\sec^2 x - 1)\,dx$$

$$= \tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x + \ln|\sec x| + C$$

20. $\int \sec^4 2x\,dx = \int \sec^2 2x \sec^2 2x\,dx = \int \sec^2 2x(1+\tan^2 2x)\,dx$

$$= \int \sec^2 2x\,dx + \int \tan^2 2x \sec^2 2x\,dx = \tfrac{1}{2}\tan 2x + \tfrac{1}{6}\tan^3 2x + C$$

21. $\int \tan^3 3x \sec^4 3x\,dx = \int \tan^3 3x(1+\tan^2 3x)\sec^2 3x\,dx$

$$= \int \tan^3 3x \sec^2 3x\,dx + \int \tan^5 3x \sec^2 3x\,dx = \tfrac{1}{12}\tan^4 3x + \tfrac{1}{18}\tan^6 3x + C$$

22. $\int \tan^2 x \sec^3 x\,dx = \int (\sec^2 x - 1)\sec^3 x\,dx = \int \sec^5 x\,dx - \int \sec^3 x\,dx$

$$= \tfrac{1}{4}\sec^3 x \tan x - \tfrac{1}{8}\sec x \tan x - \tfrac{1}{8}\ln|\sec x + \tan x| + C \quad \text{(integrating by parts)}$$

23. $\int \tan^3 2x \sec^3 2x\,dx = \int (\tan^2 2x \sec^2 2x)(\sec 2x \tan 2x\,dx)$

$$= \int (\sec^2 2x - 1)(\sec^2 2x)(\sec 2x \tan 2x\,dx)$$

$$= \int (\sec^4 2x)(\sec 2x \tan 2x\,dx) - \int (\sec^2 2x)(\sec 2x \tan 2x\,dx)$$

$$= \tfrac{1}{10}\sec^5 2x - \tfrac{1}{6}\sec^3 2x + C$$

24. $\int \cot^3 2x\,dx = \int \cot 2x\,(\csc^2 2x - 1)\,dx = -\frac{1}{4}\cot^2 2x + \frac{1}{2}\ln|\csc 2x| + C$

25. $\int \cot^4 3x\,dx = \int \cot^2 3x\,(\csc^2 3x - 1)\,dx = \int \cot^2 3x\csc^2 3x\,dx - \int \cot^2 3x\,dx$

$= \int \cot^2 3x\csc^2 3x\,dx - \int (\csc^2 3x - 1)\,dx = -\frac{1}{9}\cot^3 3x + \frac{1}{3}\cot 3x + x + C$

26. $\int \csc^6 x\,dx = \int \csc^2 x(1+\cot^2 x)^2\,dx = \int \csc^2 x\,dx + 2\int \cot^2 x\csc^2 x\,dx + \int \cot^4 x\csc^2 x\,dx$

$= -\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C$

27. $\int \cot 3x\csc^4 3x\,dx = \int \cot 3x\,(1+\cot^2 3x)\csc^2 3x\,dx$

$= \int \cot 3x\csc^2 3x\,dx + \int \cot^3 3x\csc^2 3x\,dx = -\frac{1}{6}\cot^2 3x - \frac{1}{12}\cot^4 3x + C$

28. $\int \cot^3 x\csc^5 x\,dx = \int (\cot^2 x\csc^4 x)(\csc x\cot x\,dx) = \int (\csc^2 x - 1)(\csc^4 x)(\csc x\cot x\,dx)$

$= \int (\csc^6 x)(\csc x\cot x\,dx) - \int (\csc^4 x)(\csc x\cot x\,dx) = -\frac{1}{7}\csc^7 x + \frac{1}{5}\csc^5 x + C$

Supplementary Problems

In Problems 29 to 56, evaluate the integral at the left.

29. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

30. $\int \sin^3 2x\,dx = \frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + C$

31. $\int \sin^4 2x\,dx = \frac{3}{8}x - \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$

32. $\int \cos^4 \frac{1}{2}x\,dx = \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$

33. $\int \sin^7 x\,dx = \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + C$

34. $\int \cos^6 \frac{1}{2}x\,dx = \frac{5}{16}x + \frac{1}{2}\sin x + \frac{3}{32}\sin 2x - \frac{1}{24}\sin^3 x + C$

35. $\int \sin^2 x\cos^5 x\,dx = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

36. $\int \sin^3 x\cos^2 x\,dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

37. $\int \sin^3 x\cos^3 x\,dx = \frac{1}{48}\cos^3 2x - \frac{1}{16}\cos 2x + C$

38. $\int \sin^4 x \cos^4 x \, dx = \frac{1}{128}(3x - \sin 4x + \frac{1}{8}\sin 8x) + C$

39. $\int \sin 2x \cos 4x \, dx = \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

40. $\int \cos 3x \cos 2x \, dx = \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$

41. $\int \sin 5x \sin x \, dx = \frac{1}{8}\sin 4x - \frac{1}{12}\sin 6x + C$

42. $\int \frac{\cos^3 x \, dx}{1 - \sin x} = \sin x + \frac{1}{2}\sin^2 x + C$

43. $\int \frac{\cos^{2/3} x}{\sin^{8/3} x} dx = -\frac{3}{5}\cot^{5/3} x + C$

44. $\int \frac{\cos^3 x}{\sin^4 x} dx = \csc x - \frac{1}{3}\csc^3 x + C$

45. $\int x(\cos^3 x^2 - \sin^3 x^2) \, dx = \frac{1}{12}(\sin x^2 + \cos x^2)(4 + \sin 2x^2) + C$

46. $\int \tan^3 x \, dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

47. $\int \tan^3 3x \sec 3x \, dx = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

48. $\int \tan^{3/2} x \sec^4 x \, dx = \frac{2}{5}\tan^{5/2} x + \frac{2}{9}\tan^{9/2} x + C$

49. $\int \tan^4 x \sec^4 x \, dx = \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

50. $\int \csc^4 2x \, dx = -\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

51. $\int \cot^3 x \, dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

52. $\int \left(\frac{\sec x}{\tan x}\right)^4 dx = -\frac{1}{3\tan^3 x} - \frac{1}{\tan x} + C$

53. $\int \cot^3 x \csc^4 x \, dx = -\frac{1}{4}\cot^4 x - \frac{1}{6}\cot^6 x + C$

54. $\int \frac{\cot^3 x}{\csc x} dx = -\sin x - \csc x + C$

55. $\int \cot^3 x \csc^3 x \, dx = -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$

56. $\int \tan x \sqrt{\sec x} \, dx = 2\sqrt{\sec x} + C$

57. Use integration by parts to derive the reduction formulas

$$\int \sec^m u \, du = \frac{1}{m-1}\sec^{m-2} u \tan u + \frac{m-2}{m-1}\int \sec^{m-2} u \, du$$

and

$$\int \csc^m u \, du = -\frac{1}{m-1}\csc^{m-2} u \cot u + \frac{m-2}{m-1}\int \csc^{m-2} u \, du$$

Use the reduction formulas of Problem 57 to evaluate the left-hand integral in Problems 58 to 60.

58. $\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

59. $\int \csc^5 x \, dx = -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x + \frac{3}{8}\ln|\csc x - \cot x| + C$

60. $\int \sec^6 x \, dx = \frac{1}{5}\sec^4 x \tan x + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + C = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

Chapter 33

Trigonometric Substitutions

SOME INTEGRATIONS may be simplified with the following substitutions:

1. If an integrand contains $\sqrt{a^2 - x^2}$, substitute $x = a \sin z$.
2. If an integrand contains $\sqrt{a^2 + x^2}$, substitute $x = a \tan z$.
3. If an integrand contains $\sqrt{x^2 - a^2}$, substitute $x = a \sec z$.

More generally, an integrand that contains one of the forms $\sqrt{a^2 - b^2x^2}$, $\sqrt{a^2 + b^2x^2}$, or $\sqrt{b^2x^2 - a^2}$ but no other irrational factor may be transformed into another involving trigonometric functions of a new variable as follows:

For	Use	To obtain
$\sqrt{a^2 - b^2x^2}$	$x = \frac{a}{b} \sin z$	$a\sqrt{1 - \sin^2 z} = a \cos z$
$\sqrt{a^2 + b^2x^2}$	$x = \frac{a}{b} \tan z$	$a\sqrt{1 + \tan^2 z} = a \sec z$
$\sqrt{b^2x^2 - a^2}$	$x = \frac{a}{b} \sec z$	$a\sqrt{\sec^2 z - 1} = a \tan z$

In each case, integration yields an expression in the variable z. The corresponding expression in the original variable may be obtained by the use of a right triangle as shown in the solved problems that follow.

Solved Problems

1. Find $\int \frac{dx}{x^2\sqrt{4 + x^2}}$.

Let $x = 2 \tan z$, so that x and z are related as in Fig. 33-1. Then $dx = 2 \sec^2 z\, dz$ and $\sqrt{4 + x^2} = 2 \sec z$, and

$$\int \frac{dx}{x^2\sqrt{4 + x^2}} = \int \frac{2 \sec^2 z\, dz}{(4 \tan^2 z)(2 \sec z)} = \frac{1}{4} \int \frac{\sec z}{\tan^2 z}\, dz = \frac{1}{4} \int \sin^{-2} z \cos z\, dz$$

$$= -\frac{1}{4 \sin z} + C = -\frac{\sqrt{4 + x^2}}{4x} + C$$

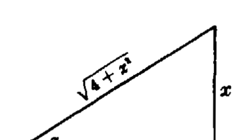

Fig. 33-1

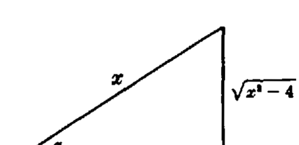

Fig. 33-2

2. Find $\int \frac{x^2}{\sqrt{x^2-4}}\,dx$.

Let $x = 2\sec z$, so that x and z are related as in Fig. 33-2. Then $dx = 2\sec z\tan z\,dz$ and $\sqrt{x^2-4} = 2\tan z$, and

$$\int \frac{x^2}{\sqrt{x^2-4}}\,dx = \int \frac{4\sec^2 z}{2\tan z}(2\sec z\tan z\,dz) = 4\int \sec^3 z\,dz$$

$$= 2\sec z\tan z + 2\ln|\sec z + \tan z| + C'$$

$$= \tfrac{1}{2}x\sqrt{x^2-4} + 2\ln|x + \sqrt{x^2-4}| + C$$

3. Find $\int \frac{\sqrt{9-4x^2}}{x}\,dx$.

Let $x = \frac{3}{2}\sin z$ (see Fig. 33-3); then $dx = \frac{3}{2}\cos z\,dz$ and $\sqrt{9-4x^2} = 3\cos z$, and

$$\int \frac{\sqrt{9-4x^2}}{x}\,dx = \int \frac{3\cos z}{\frac{3}{2}\sin z}\left(\frac{3}{2}\cos z\,dz\right) = 3\int \frac{\cos^2 z}{\sin z}\,dz = 3\int \frac{1-\sin^2 z}{\sin z}\,dx$$

$$= 3\int \csc z\,dz - 3\int \sin z\,dz = 3\ln|\csc z - \cot z| + 3\cos z + C'$$

$$= 3\ln\left|\frac{3-\sqrt{9-4x^2}}{x}\right| + \sqrt{9-4x^2} + C$$

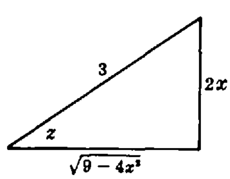

Fig. 33-3

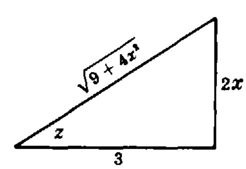

Fig. 33-4

4. Find $\int \frac{dx}{x\sqrt{9+4x^2}}$.

Let $x = \frac{3}{2}\tan z$ (see Fig. 33-4); then $dx = \frac{3}{2}\sec^2 z\,dz$ and $\sqrt{9+4x^2} = 3\sec z$, and

$$\int \frac{dx}{x\sqrt{9+4x^2}} = \int \frac{\frac{3}{2}\sec^2 z\,dz}{(\frac{3}{2}\tan z)(3\sec z)} = \frac{1}{3}\int \csc z\,dz = \frac{1}{3}\ln|\csc z - \cot z| + C'$$

$$= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{x}\right| + C$$

5. Find $\int \frac{(16-9x^2)^{3/2}}{x^6}\,dx$.

Let $x = \frac{4}{3}\sin z$ (see Fig. 33-5); then $dx = \frac{4}{3}\cos z\,dz$ and $\sqrt{16-9x^2} = 4\cos z$, and

$$\int \frac{(16-9x^2)^{3/2}}{x^6}\,dx = \int \frac{(64\cos^3 z)(\frac{4}{3}\cos z\,dz)}{\frac{4096}{729}\sin^6 z} = \frac{243}{16}\int \frac{\cos^4 z}{\sin^6 z}\,dz = \frac{243}{16}\int \cot^4 z\csc^2 z\,dz$$

$$= -\frac{243}{80}\cot^5 z + C = -\frac{243}{80}\,\frac{(16-9x^2)^{5/2}}{243x^5} + C = -\frac{1}{80}\,\frac{(16-9x^2)^{5/2}}{x^5} + C$$

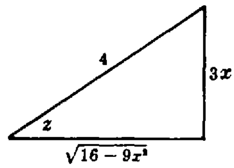

Fig. 33-5

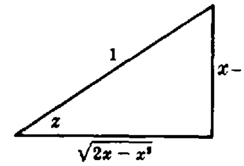

Fig. 33-6

6. Find $\int \frac{x^2\,dx}{\sqrt{2x-x^2}} = \int \frac{x^2\,dx}{\sqrt{1-(x-1)^2}}$.

Let $x-1=\sin z$ (see Fig. 33-6); then $dx=\cos z\,dz$ and $\sqrt{2x-x^2}=\cos z$, and

$$\int \frac{x^2\,dx}{\sqrt{2x-x^2}} = \int \frac{(1+\sin z)^2}{\cos z}\cos z\,dz = \int (1+\sin z)^2\,dz = \int \left(\frac{3}{2}+2\sin z-\frac{1}{2}\cos 2z\right)dz$$

$$= \frac{3}{2}z - 2\cos z - \frac{1}{4}\sin 2z + C = \frac{3}{2}\arcsin(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2} + C$$

$$= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}(x+3)\sqrt{2x-x^2} + C$$

7. Find $\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{dx}{[4(x-3)^2-9]^{3/2}}$.

Let $x-3=\frac{3}{2}\sec z$ (see Fig. 33-7); then $dx=\frac{3}{2}\sec z\tan z\,dz$ and $\sqrt{4x^2-24x+27}=3\tan z$, and

$$\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{\frac{3}{2}\sec z\tan z\,dz}{27\tan^3 z} = \frac{1}{18}\int \sin^{-2} z\cos z\,dz$$

$$= -\frac{1}{18}\csc z + C = -\frac{1}{9}\,\frac{x-3}{\sqrt{4x^2-24x+27}} + C$$

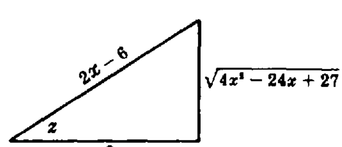

Fig. 33-7

Supplementary Problems

In Problems 8 to 22, integrate to obtain the given result.

8. $\int \frac{dx}{(4-x^2)^{3/2}} = \frac{x}{4\sqrt{4-x^2}} + C$

9. $\int \frac{\sqrt{25-x^2}}{x}\,dx = 5\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2} + C$

10. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2 x} + C$

11. $\int \sqrt{x^2 + 4}\, dx = \frac{1}{2}x\sqrt{x^2 + 4} + 2\ln(x + \sqrt{x^2 + 4}) + C$

12. $\int \frac{x^2\, dx}{(a^2 - x^2)^{3/2}} = \frac{x}{\sqrt{a^2 - x^2}} - \arcsin\frac{x}{a} + C$

13. $\int \sqrt{x^2 - 4}\, dx = \frac{1}{2}x\sqrt{x^2 - 4} - 2\ln|x + \sqrt{x^2 - 4}| + C$

14. $\int \frac{\sqrt{x^2 + a^2}}{x}\, dx = \sqrt{x^2 + a^2} + \frac{a}{2}\ln\frac{\sqrt{a^2 + x^2} - a}{\sqrt{a^2 + x^2} + a} + C$

15. $\int \frac{x^2\, dx}{(4 - x^2)^{5/2}} = \frac{x^3}{12(4 - x^2)^{3/2}} + C$

16. $\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 + x^2}} + C$

17. $\int \frac{dx}{x^2\sqrt{9 - x^2}} = -\frac{\sqrt{9 - x^2}}{9x} + C$

18. $\int \frac{x^2\, dx}{\sqrt{x^2 - 16}} = \frac{1}{2}x\sqrt{x^2 - 16} + 8\ln|x + \sqrt{x^2 - 16}| + C$

19. $\int x^3\sqrt{a^2 - x^2}\, dx = \frac{1}{5}(a^2 - x^2)^{5/2} - \frac{a^2}{3}(a^2 - x^2)^{3/2} + C$

20. $\int \frac{dx}{\sqrt{x^2 - 4x + 13}} = \ln(x - 2 + \sqrt{x^2 - 4x + 13}) + C$

21. $\int \frac{dx}{(4x - x^2)^{3/2}} = \frac{x - 2}{4\sqrt{4x - x^2}} + C$

22. $\int \frac{dx}{(9 + x^2)^2} = \frac{1}{54}\arctan\frac{x}{3} + \frac{x}{18(9 + x^2)} + C$

In Problems 23 and 24, integrate by parts and apply the method of this chapter.

23. $\int x\arcsin x\, dx = \frac{1}{4}(2x^2 - 1)\arcsin x + \frac{1}{4}x\sqrt{1 - x^2} + C$

24. $\int x\arccos x\, dx = \frac{1}{4}(2x^2 - 1)\arccos x - \frac{1}{4}x\sqrt{1 - x^2} + C$

Chapter 34

Integration by Partial Fractions

A POLYNOMIAL IN x is a function of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where the a's are constants, $a_0 \neq 0$, and n, called the *degree* of the polynomial, is a nonnegative integer.

If two polynomials of the same degree are equal for all values of the variable, then the coefficients of the like powers of the variable in the two polynomials are equal.

Every polynomial with real coefficients can be expressed (at least, theoretically) as a product of real linear factors of the form $ax + b$ and real irreducible quadratic factors of the form $ax^2 + bx + c$. (A polynomial of degree 1 or greater is said to be *irreducible* if it cannot be factored into polynomials of lower degree.) By the quadratic formula, $ax^2 + bx + c$ is irreducible if and only if $b^2 - 4ac < 0$. (In that case, the roots of $ax^2 + bx + c = 0$ are not real.)

EXAMPLE 1: (*a*) $x^2 - x + 1$ is irreducible, since $(-1)^2 - 4(1)(1) = -3 < 0$.
(*b*) $x^2 - x - 1$ is not irreducible, since $(-1)^2 - 4(1)(-1) = 5 > 0$. In fact, $x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$.

A FUNCTION $F(x) = f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials, is called a *rational fraction*.

If the degree of $f(x)$ is less than the degree of $g(x)$, $F(x)$ is called *proper*; otherwise, $F(x)$ is called *improper*.

An improper rational fraction can be expressed as the sum of a polynomial and a proper rational fraction. Thus, $\dfrac{x^3}{x^2+1} = x - \dfrac{x}{x^2+1}$.

Every proper rational fraction can be expressed (at least, theoretically) as a sum of simpler fractions (*partial fractions*) whose denominators are of the form $(ax + b)^n$ and $(ax^2 + bx + c)^n$, n being a positive integer. Four cases, depending upon the nature of the factors of the denominator, arise.

CASE I: DISTINCT LINEAR FACTORS. To each linear factor $ax + b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{A}{ax+b}$, where A is a constant to be determined. (See Problems 1 and 2.)

CASE II: REPEATED LINEAR FACTORS. To each linear factor $ax + b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

where the A's are constants to be determined. (See Problems 3 and 4.)

CASE III: DISTINCT QUADRATIC FACTORS. To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined. (See Problems 5 and 6.)

CASE IV: REPEATED QUADRATIC FACTORS. To each irreducible quadratic factor $ax^2 + bx + c$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where the A's and B's are constants to be determined. (See Problems 7 and 8.)

Solved Problems

1. Find $\int \frac{dx}{x^2 - 4}$.

We factor the denominator into $(x-2)(x+2)$ and write $\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$. Clearing of fractions yields

$$1 = A(x+2) + B(x-2) \qquad (1)$$

or

$$1 = (A+B)x + (2A - 2B) \qquad (2)$$

We can determine the constants by either of two methods.

General method: Equate coefficients of like powers of x in (2) and solve simultaneously for the constants. Thus, $A + B = 0$ and $2A - 2B = 1$; $A = \frac{1}{4}$ and $B = -\frac{1}{4}$.

Short method: Substitute in (1) the values $x = 2$ and $x = -2$ to obtain $1 = 4A$ and $1 = -4B$; then $A = \frac{1}{4}$ and $B = -\frac{1}{4}$, as before. (Note that the values of x used are those for which the denominators of the partial fractions become 0.)

By either method, we have $\frac{1}{x^2-4} = \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}$. Then

$$\int \frac{dx}{x^2-4} = \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2} = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

2. Find $\int \frac{(x+1)\,dx}{x^3 + x^2 - 6x}$.

Factoring yields $x^3 + x^2 - 6x = x(x-2)(x+3)$. Then $\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$ and

$$x + 1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2) \qquad (1)$$

$$x + 1 = (A + B + C)x^2 + (A + 3B - 2C)x - 6A \qquad (2)$$

General method: We solve simultaneously the system of equations

$$A + B + C = 0 \qquad A + 3B - 2C = 1 \qquad -6A = 1$$

to obtain $A = -\frac{1}{6}$, $B = \frac{3}{10}$, and $C = -\frac{2}{15}$.

Short method: We substitute in (1) the values $x = 0$, $x = 2$, and $x = -3$ to obtain $1 = -6A$ or $A = -1/6$, $3 = 10B$ or $B = 3/10$, and $-2 = 15C$ or $C = -2/15$.

By either method,

$$\int \frac{(x+1)\,dx}{x^3+x^2-6x} = -\frac{1}{6}\int \frac{dx}{x} + \frac{3}{10}\int \frac{dx}{x-2} - \frac{2}{15}\int \frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C = \ln\frac{|x-2|^{3/10}}{|x|^{1/6}|x+3|^{2/15}} + C$$

3. Find $\int \frac{(3x+5)\,dx}{x^3 - x^2 - x + 1}$.

$x^3 - x^2 - x + 1 = (x+1)(x-1)^2$. Hence, $\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ and

$$3x + 5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

For $x = -1$, $2 = 4A$ and $A = \frac{1}{2}$. For $x = 1$, $8 = 2C$ and $C = 4$. To determine the remaining constant, we use any other value of x, say $x = 0$; for $x = 0$, $5 = A - B + C$ and $B = -\frac{1}{2}$. Thus,

$$\int \frac{3x+5}{x^3 - x^2 - x + 1}\,dx = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| - \frac{4}{x-1} + C = -\frac{4}{x-1} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

4. Find $\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}\,dx$.

The integrand is an improper fraction. By division,

$$\frac{x^4 - x^3 - x - 1}{x^3 - x^2} = x - \frac{x+1}{x^3 - x^2} = x - \frac{x+1}{x^2(x-1)}$$

We write $\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ and obtain

$$x + 1 = Ax(x-1) + B(x-1) + Cx^2$$

For $x = 0$, $1 = -B$ and $B = -1$. For $x = 1$, $2 = C$. For $x = 2$, $3 = 2A + B + 4C$ and $A = -2$. Thus,

$$\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}\,dx = \int x\,dx + 2\int \frac{dx}{x} + \int \frac{dx}{x^2} - 2\int \frac{dx}{x-1}$$

$$= \frac{1}{2}x^2 + 2\ln|x| - \frac{1}{x} - 2\ln|x-1| + C = \frac{1}{2}x^2 - \frac{1}{x} + 2\ln\left|\frac{x}{x-1}\right| + C$$

5. Find $\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\,dx$.

$x^4 + 3x^2 + 2 = (x^2+1)(x^2+2)$. We write $\frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$ and obtain

$$x^3 + x^2 + x + 2 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$
$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$$

Hence $A + C = 1$, $B + D = 1$, $2A + C = 1$, and $2B + D = 2$. Solving simultaneously yields $A = 0$, $B = 1$, $C = 1$, $D = 0$. Thus,

$$\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\,dx = \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{x^2+2} = \arctan x + \frac{1}{2}\ln(x^2+2) + C$$

6. Solve the equation $\int \frac{x^2\,dx}{a^4 - x^4} = \int k\,dt$, which occurs in physical chemistry.

We write $\frac{x^2}{a^4 - x^4} = \frac{A}{a-x} + \frac{B}{a+x} + \frac{Cx+D}{a^2+x^2}$. Then

$$x^2 = A(a+x)(a^2+x^2) + B(a-x)(a^2+x^2) + (Cx+D)(a-x)(a+x)$$

For $x = a$, $a^2 = 4Aa^3$ and $A = 1/4a$. For $x = -a$, $a^2 = 4Ba^3$ and $B = 1/4a$. For $x = 0$, $0 = Aa^3 + Ba^3 + Da^2 = a^2/2 + Da^2$ and $D = -\frac{1}{2}$. For $x = 2a$, $4a^2 = 15Aa^3 - 5Ba^3 - 6Ca^3 - 3Da^2$ and $C = 0$. Thus,

$$\int \frac{x^2\,dx}{a^4 - x^4} = \frac{1}{4a}\int \frac{dx}{a-x} + \frac{1}{4a}\int \frac{dx}{a+x} - \frac{1}{2}\int \frac{dx}{a^2+x^2}$$
$$= -\frac{1}{4a}\ln|a-x| + \frac{1}{4a}\ln|a+x| - \frac{1}{2a}\arctan\frac{x}{a} + C$$

so that
$$\int k\,dt = kt = \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\arctan\frac{x}{a} + C$$

7. Find $\displaystyle\int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}\,dx$.

We write $\displaystyle\frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} + \frac{Ex+F}{(x^2+2)^3}$. Then

$$x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4 = (Ax+B)(x^2+2)^2 + (Cx+D)(x^2+2) + Ex + F$$
$$= Ax^5 + Bx^4 + (4A+C)x^3 + (4B+D)x^2 + (4A+2C+E)x + (4B+2D+F)$$

from which $A = 1$, $B = -1$, $C = 0$, $D = 0$, $E = 4$, $F = 0$. Thus the given integral is equal to

$$\int \frac{x-1}{x^2+2}\,dx + 4\int \frac{x\,dx}{(x^2+2)^3} = \frac{1}{2}\ln(x^2+2) - \frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}} - \frac{1}{(x^2+2)^2} + C$$

8. Find $\displaystyle\int \frac{2x^2+3}{(x^2+1)^2}\,dx$.

We write $\displaystyle\frac{2x^2+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$. Then

$$2x^2 + 3 = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

from which $A = 0$, $B = 2$, $A + C = 0$, $B + D = 3$. Thus $A = 0$, $B = 2$, $C = 0$, $D = 1$ and

$$\int \frac{2x^2+3}{(x^2+1)^2}\,dx = \int \frac{2\,dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$$

For the second integral on the right, let $x = \tan z$. Then

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2 z\,dz}{\sec^4 z} = \int \cos^2 z\,dz = \frac{1}{2}z + \frac{1}{4}\sin 2z + C$$

and $\displaystyle\int \frac{2x^2+3}{(x^2+1)^2}\,dx = 2\arctan x + \frac{1}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C = \frac{5}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C$

Supplementary Problems

In Problems 9 to 27, evaluate the integral at the left.

9. $\displaystyle\int \frac{dx}{x^2-9} = \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$

10. $\displaystyle\int \frac{dx}{x^2+7x+6} = \frac{1}{5}\ln\left|\frac{x+1}{x+6}\right| + C$

11. $\displaystyle\int \frac{x\,dx}{x^2-3x-4} = \frac{1}{5}\ln|(x+1)(x-4)^4| + C$

12. $\displaystyle\int \frac{x^2+3x-4}{x^2-2x-8}\,dx = x + \ln|(x+2)(x-4)^4| + C$

13. $\displaystyle\int \frac{x^2-3x-1}{x^3+x^2-2x}\,dx = \ln\left|\frac{x^{1/2}(x+2)^{3/2}}{x-1}\right| + C$

14. $\displaystyle\int \frac{x\,dx}{(x-2)^2} = \ln|x-2| - \frac{2}{x-2} + C$

15. $$\int \frac{x^4}{(1-x)^3}\,dx = -\frac{1}{2}x^2 - 3x - \ln(1-x)^6 - \frac{4}{1-x} + \frac{1}{2(1-x)^2} + C$$

16. $$\int \frac{dx}{x^3+x} = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$$

17. $$\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}\,dx = \ln\sqrt{x^2+3} + \arctan x + C$$

18. $$\int \frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}\,dx = \frac{1}{2}x^2 + \ln\left|\frac{x}{\sqrt{x^2-2x+3}}\right| + C$$

19. $$\int \frac{2x^3\,dx}{(x^2+1)^2} = \ln(x^2+1) + \frac{1}{x^2+1} + C$$

20. $$\int \frac{2x^3+x^2+4}{(x^2+4)^2}\,dx = \ln(x^2+4) + \frac{1}{2}\arctan\frac{1}{2}x + \frac{4}{x^2+4} + C$$

21. $$\int \frac{x^3+x-1}{(x^2+1)^2}\,dx = \ln\sqrt{x^2+1} - \frac{1}{2}\arctan x - \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C$$

22. $$\int \frac{x^4+8x^3-x^2+2x+1}{(x^2+x)(x^3+1)}\,dx = \ln\left|\frac{x^3-x^2+x}{(x+1)^2}\right| - \frac{3}{x+1} + \frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$$

23. $$\int \frac{x^3+x^2-5x+15}{(x^2+5)(x^2+2x+3)}\,dx = \ln\sqrt{x^2+2x+3} + \frac{5}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} - \sqrt{5}\arctan\frac{x}{\sqrt{5}} + C$$

24. $$\int \frac{x^6+7x^5+15x^4+32x^3+23x^2+25x-3}{(x^2+x+2)^2(x^2+1)^2}\,dx = \frac{1}{x^2+x+2} - \frac{3}{x^2+1} + \ln\frac{x^2+1}{x^2+x+2} + C$$

25. $$\int \frac{dx}{e^{2x}-3e^x} = \frac{1}{3e^x} + \frac{1}{9}\ln\left|\frac{e^x-3}{e^x}\right| + C$$ (*Hint*: Let $e^x = u$.)

26. $$\int \frac{\sin x\,dx}{\cos x\,(1+\cos^2 x)} = \ln\left|\frac{\sqrt{1+\cos^2 x}}{\cos x}\right| + C$$ (*Hint*: Let $\cos x = u$.)

27. $$\int \frac{(2+\tan^2\theta)\sec^2\theta\,d\theta}{1+\tan^3\theta} = \ln|1+\tan\theta| + \frac{2}{\sqrt{3}}\arctan\frac{2\tan\theta-1}{\sqrt{3}} + C$$

Chapter 35

Miscellaneous Substitutions

IF AN INTEGRAND IS RATIONAL except for a radical of the form

1. $\sqrt[n]{ax+b}$, then the substitution $ax + b = z^n$ will replace it with a rational integrand.
2. $\sqrt{q+px+x^2}$, then the substitution $q + px + x^2 = (z - x)^2$ will replace it with a rational integrand.
3. $\sqrt{q+px-x^2} = \sqrt{(\alpha+x)(\beta-x)}$, then the substitution $q + px - x^2 = (\alpha + x)^2z^2$ or $q + px - x^2 = (\beta - x)^2z^2$ will replace it with a rational integrand.

(See Problems 1 to 5.)

THE SUBSTITUTION $x = 2\arctan z$ will replace any rational function of $\sin x$ and $\cos x$ with a rational function of z, since

$$\sin x = \frac{2z}{1+z^2} \qquad \cos x = \frac{1-z^2}{1+z^2} \qquad \text{and} \qquad dx = \frac{2\,dz}{1+z^2}$$

(The first and second of these relations are obtained from Fig. 35-1, and the third by differentiating $x = 2\arctan z$.) After integrating, use $z = \tan \frac{1}{2}x$ to return to the original variable. (See Problems 6 to 10.)

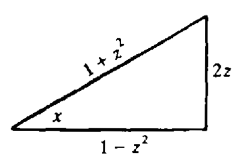

Fig. 35-1

EFFECTIVE SUBSTITUTIONS are often suggested by the form of the integrand. (See Problems 11 and 12.)

Solved Problems

1. Find $\displaystyle\int \frac{dx}{x\sqrt{1-x}}$.

Let $1 - x = z^2$. Then $x = 1 - z^2$, $dx = -2z\,dz$, and

$$\int \frac{dx}{x\sqrt{1-x}} = \int \frac{-2z\,dz}{(1-z^2)z} = -2\int \frac{dz}{1-z^2} = -\ln\left|\frac{1+z}{1-z}\right| + C = \ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right| + C$$

2. Find $\displaystyle\int \frac{dx}{(x-2)\sqrt{x+2}}$.

Let $x + 2 = z^2$. Then $x = z^2 - 2$, $dx = 2z\,dz$, and

$$\int \frac{dx}{(x-2)\sqrt{x+2}} = \int \frac{2z\,dz}{z(z^2-4)} = 2\int \frac{dz}{z^2-4} = \frac{1}{2}\ln\left|\frac{z-2}{z+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right| + C$$

3. Find $\int \frac{dx}{x^{1/2} - x^{1/4}}$.

Let $x = z^4$. Then $dx = 4z^3\,dz$ and

$$\int \frac{dx}{x^{1/2}-x^{1/4}} = \int \frac{4z^3\,dz}{z^2 - z} = 4\int \frac{z^2}{z-1}\,dz = 4\int\left(z + 1 + \frac{1}{z-1}\right)dz$$

$$= 4(\tfrac{1}{2}z^2 + z + \ln|z-1|) + C = 2\sqrt{x} + 4\sqrt[4]{x} + \ln(\sqrt[4]{x}-1)^4 + C$$

4. Find $\int \frac{dx}{x\sqrt{x^2+x+2}}$.

Let $x^2 + x + 2 = (z-x)^2$. Then

$$x = \frac{z^2-2}{1+2z} \qquad dx = \frac{2(z^2+z+2)\,dz}{(1+2z)^2} \qquad \sqrt{x^2+x+2} = \frac{z^2+z+2}{1+2z}$$

and

$$\int \frac{dx}{x\sqrt{x^2+x+2}} = \int \frac{\dfrac{2(z^2+z+2)}{(1+2z)^2}}{\dfrac{z^2-2}{1+2z}\,\dfrac{z^2+z+2}{1+2z}}\,dz = 2\int\frac{dz}{z^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{x^2+x+2}+x-\sqrt{2}}{\sqrt{x^2+x+2}+x+\sqrt{2}}\right| + C$$

5. Find $\int \frac{x\,dx}{(5-4x-x^2)^{3/2}}$.

Let $5 - 4x - x^2 = (5+x)(1-x) = (1-x)^2z^2$. Then

$$x = \frac{z^2-5}{1+z^2} \qquad dx = \frac{12z\,dz}{(1+z^2)^2} \qquad \sqrt{5-4x-x^2} = (1-x)z = \frac{6z}{1+z^2}$$

and

$$\int \frac{x\,dx}{(5-4x-x^2)^{3/2}} = \int \frac{\dfrac{z^2-5}{1+z^2}\,\dfrac{12z}{(1+z^2)^2}}{\dfrac{216z^3}{(1+z^2)^3}}\,dz = \frac{1}{18}\int\left(1 - \frac{5}{z^2}\right)dz$$

$$= \frac{1}{18}\left(z + \frac{5}{z}\right) + C = \frac{5-2x}{9\sqrt{5-4x-x^2}} + C$$

In Problems 6 to 10, evaluate the integral at the left.

6. $$\int \frac{dx}{1+\sin x - \cos x} = \int \frac{\dfrac{2\,dz}{1+z^2}}{1 + \dfrac{2z}{1+z^2} - \dfrac{1-z^2}{1+z^2}} = \int \frac{dz}{z(1+z)} = \ln|z| - \ln|1+z| + C$$

$$= \ln\left|\frac{z}{1+z}\right| + C = \ln\left|\frac{\tan\frac{1}{2}x}{1+\tan\frac{1}{2}x}\right| + C$$

7. $$\int \frac{dx}{3-2\cos x} = \int \frac{\dfrac{2\,dz}{1+z^2}}{3-2\dfrac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{1+5z^2} = \frac{2\sqrt{5}}{5}\arctan z\sqrt{5} + C$$
$$= \frac{2\sqrt{5}}{5}\arctan(\sqrt{5}\tan\tfrac{1}{2}x) + C$$

8. $$\int \sec x\,dx = \int \frac{1+z^2}{1-z^2}\frac{2\,dz}{1+z^2} = 2\int \frac{dz}{1-z^2} = \ln\left|\frac{1+z}{1-z}\right| + C = \ln\left|\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right| + C$$
$$= \ln\left|\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right| + C$$

9. $$\int \frac{dx}{2+\cos x} = \int \frac{\dfrac{2\,dz}{1+z^2}}{2+\dfrac{1-z^2}{1+z^2}} = 2\int \frac{dz}{3+z^2} = \frac{2}{\sqrt{3}}\arctan\frac{z}{\sqrt{3}} + C$$
$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\tan\frac{1}{2}x\right) + C$$

10. $$\int \frac{dx}{5+4\sin x} = \int \frac{\dfrac{2\,dz}{1+z^2}}{5+4\dfrac{2z}{1+z^2}} = \int \frac{2\,dz}{5+8z+5z^2} = \frac{2}{5}\int \frac{dz}{(z+\frac{4}{5})^2+\frac{9}{25}}$$
$$= \frac{2}{3}\arctan\frac{z+\frac{4}{5}}{\frac{3}{5}} + C = \frac{2}{3}\arctan\frac{5\tan\frac{1}{2}x+4}{3} + C$$

11. Use the substitution $1-x^3 = z^2$ to find $\int x^5\sqrt{1-x^3}\,dx$.

The substitution yields $x^3 = 1-z^2$, $3x^2\,dx = -2z\,dz$, and

$$\int x^5\sqrt{1-x^3}\,dx = \int x^3\sqrt{1-x^3}\,(x^2\,dx) = \int (1-z^2)z(-\tfrac{2}{3}z\,dz) = -\frac{2}{3}\int (1-z^2)z^2\,dz$$
$$= -\frac{2}{3}\left(\frac{z^3}{3}-\frac{z^5}{5}\right) + C = -\frac{2}{45}(1-x^3)^{3/2}(2+3x^3) + C$$

12. Use $x = \dfrac{1}{z}$ to find $\int \dfrac{\sqrt{x-x^2}}{x^4}\,dx$.

The substitution yields $dx = -dz/z^2$, $\sqrt{x-x^2} = \sqrt{z-1}/z$, and

$$\int \frac{\sqrt{x-x^2}}{x^4}\,dx = \int \frac{\dfrac{\sqrt{z-1}}{z}\left(-\dfrac{dz}{z^2}\right)}{1/z^4} = -\int z\sqrt{z-1}\,dz$$

Let $z-1 = s^2$. Then

$$-\int z\sqrt{z-1}\,dz = -\int (s^2+1)(s)(2s\,ds) = -2\left(\frac{s^5}{5}+\frac{s^3}{3}\right) + C$$
$$= -2\left[\frac{(z-1)^{5/2}}{5} + \frac{(z-1)^{3/2}}{3}\right] + C = -2\left[\frac{(1-x)^{5/2}}{5x^{5/2}} + \frac{(1-x)^{3/2}}{3x^{3/2}}\right] + C$$

13. Find $\int \dfrac{dx}{x^{1/2}+x^{1/3}}$.

Let $u = x^{1/6}$, so that $x = u^6$, $dx = 6u^5\,du$, $x^{1/2} = u^3$, and $x^{1/3} = u^2$. Then we obtain

$$\int \frac{6u^5\,du}{u^3 + u^2} = 6\int \frac{u^3}{u+1}\,du = 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right) du = 6\left(\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right) + C$$
$$= 2x^{1/2} - 3x^{1/3} + x^{1/6} - \ln|x^{1/6} + 1| + C$$

Supplementary Problems

In Problems 14 to 39, evaluate the integral at the left.

14. $\int \frac{\sqrt{x}}{1+x}\,dx = 2\sqrt{x} - 2\arctan\sqrt{x} + C$

15. $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\ln(1+\sqrt{x}) + C$

16. $\int \frac{dx}{3+\sqrt{x+2}} = 2\sqrt{x+2} - 6\ln(3+\sqrt{x+2}) + C$

17. $\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}\,dx = -x + \frac{4}{3}\left\{\sqrt{3x+2} - \ln(1+\sqrt{3x+2})\right\} + C$

18. $\int \frac{dx}{\sqrt{x^2-x+1}} = \ln|2\sqrt{x^2-x+1} + 2x - 1| + C$

19. $\int \frac{dx}{x\sqrt{x^2+x-1}} = 2\arctan(\sqrt{x^2+x-1} + x) + C$

20. $\int \frac{dx}{\sqrt{6+x-x^2}} = \arcsin\frac{2x-1}{5} + C$

21. $\int \frac{\sqrt{4x-x^2}}{x^3}\,dx = -\frac{(4x-x^2)^{3/2}}{6x^3} + C$

22. $\int \frac{dx}{(x+1)^{1/2} + (x+1)^{1/4}} = 2(x+1)^{1/2} - 4(x+1)^{1/4} + 4\ln(1+(x+1)^{1/4}) + C$

23. $\int \frac{dx}{2+\sin x} = \frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{1}{2}x + 1}{\sqrt{3}} + C$

24. $\int \frac{dx}{1-2\sin x} = \frac{\sqrt{3}}{3}\ln\left|\frac{\tan\frac{1}{2}x - 2 - \sqrt{3}}{\tan\frac{1}{2}x - 2 + \sqrt{3}}\right| + C$

25. $\int \frac{dx}{3+5\sin x} = \frac{1}{4}\ln\left|\frac{3\tan\frac{1}{2}x + 1}{\tan\frac{1}{2}x + 3}\right| + C$

26. $\int \frac{dx}{\sin x - \cos x - 1} = \ln|\tan\frac{1}{2}x - 1| + C$

27. $\int \frac{dx}{5+3\sin x} = \frac{1}{2}\arctan\frac{5\tan\frac{1}{2}x + 3}{4} + C$

28. $\int \frac{\sin x\,dx}{1+\sin^2 x} = \frac{\sqrt{2}}{4}\ln\left|\frac{\tan^2\frac{1}{2}x + 3 - 2\sqrt{2}}{\tan^2\frac{1}{2}x + 3 + 2\sqrt{2}}\right| + C$

29. $\int \frac{dx}{1+\sin x+\cos x} = \ln|1 + \tan\frac{1}{2}x| + C$

30. $\int \frac{dx}{2-\cos x} = \frac{2}{\sqrt{3}}\arctan(\sqrt{3}\tan\frac{1}{2}x) + C$

31. $\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

32. $\int \frac{dx}{x\sqrt{3x^2+2x-1}} = -\arcsin\frac{1-x}{2x} + C$ (*Hint*: Let $x = 1/z$.)

33. $\int \frac{(e^x-2)e^x}{e^x+1}\,dx = e^x - 3\ln(e^x+1) + C$ (*Hint*: Let $e^x + 1 = z$.)

34. $\int \frac{\sin x \cos x}{1 - \cos x}\, dx = \cos x + \ln(1 - \cos x) + C$ (*Hint*: Let $\cos x = z$.)

35. $\int \frac{dx}{x^2\sqrt{4 - x^2}} = -\frac{\sqrt{4 - x^2}}{4x} + C$ (*Hint*: Let $x = 2/z$.)

36. $\int \frac{dx}{x^2(4 + x^2)} = -\frac{1}{4x} + \frac{1}{8}\arctan\frac{2}{x} + C$

37. $\int \sqrt{1 + \sqrt{x}}\, dx = \frac{4}{5}(1 + \sqrt{x})^{5/2} - \frac{4}{3}(1 + \sqrt{x})^{3/2} + C$

38. $\int \frac{dx}{3(1 - x^2) - (5 + 4x)\sqrt{1 - x^2}} = \frac{2\sqrt{1 + x}}{3\sqrt{1 + x} - \sqrt{1 - x}} + C$

39. $\int \frac{x^{1/2}}{x^{1/5} + 1}\, dx = 10\left(\frac{1}{13}x^{13/10} - \frac{1}{11}x^{11/10} + \frac{1}{9}x^{9/10} - \frac{1}{7}x^{7/10} + \frac{1}{5}x^{5/10} - \frac{1}{3}x^{3/10} + x^{1/10} - \arctan x^{1/10}\right)$
$+ C$ (*Hint*: Let $u = x^{1/10}$.)

Chapter 36

Integration of Hyperbolic Functions

INTEGRATION FORMULAS. The following formulas are direct consequences of the differentiation formulas of Chapter 20.

28. $\int \sinh x\,dx = \cosh x + C$

29. $\int \cosh x\,dx = \sinh x + C$

30. $\int \tanh x\,dx = \ln \cosh x + C$

31. $\int \coth x\,dx = \ln |\sinh x| + C$

32. $\int \operatorname{sech}^2 x\,dx = \tanh x + C$

33. $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

34. $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$

35. $\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$

36. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C$

37. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C, \quad x > a > 0$

38. $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a} + C, \quad x^2 < a^2$

39. $\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \coth^{-1} \frac{x}{a} + C, \quad x^2 > a^2$

Solved Problems

In Problems 1 to 13, evaluate the integral at the left.

1. $\int \sinh \tfrac{1}{2}x\,dx = 2\int \sinh \tfrac{1}{2}x\,d(\tfrac{1}{2}x) = 2\cosh \tfrac{1}{2}x + C$

2. $\int \cosh 2x\,dx = \tfrac{1}{2}\int \cosh 2x\,d(2x) = \tfrac{1}{2}\sinh 2x + C$

3. $\int \operatorname{sech}^2 (2x-1)\,dx = \tfrac{1}{2}\int \operatorname{sech}^2 (2x-1)\,d(2x-1) = \tfrac{1}{2}\tanh (2x-1) + C$

4. $\int \operatorname{csch} 3x \coth 3x\,dx = \tfrac{1}{3}\int \operatorname{csch} 3x \coth 3x\,d(3x) = -\tfrac{1}{3}\operatorname{csch} 3x + C$

5. $\int \operatorname{sech} x\,dx = \int \frac{1}{\cosh x}\,dx = \int \frac{\cosh x}{\cosh^2 x} = \int \frac{\cosh x}{1 + \sinh^2 x}\,dx = \arctan (\sinh x) + C$

6. $\int \sinh^2 x\,dx = \tfrac{1}{2}\int (\cosh 2x - 1)\,dx = \tfrac{1}{4}\sinh 2x - \tfrac{1}{2}x + C$

7. $\int \tanh^2 2x\,dx = \int (1 - \operatorname{sech}^2 2x)\,dx = x - \tfrac{1}{2}\tanh 2x + C$

8. $\int \cosh^3 \frac{1}{2}x\,dx = \int (1 + \sinh^2 \frac{1}{2}x) \cosh \frac{1}{2}x\,dx = 2 \sinh \frac{1}{2}x + \frac{2}{3} \sinh^3 \frac{1}{2}x + C$

9. $\int \text{sech}^4 x\,dx = \int (1 - \tanh^2 x)\,\text{sech}^2 x\,dx = \tanh x - \frac{1}{3} \tanh^3 x + C$

10. $\int e^x \cosh x\,dx = \int e^x \frac{e^x + e^{-x}}{2}\,dx = \frac{1}{2} \int (e^{2x} + 1)\,dx = \frac{1}{4} e^{2x} + \frac{1}{2} x + C$

11. $\int x \sinh x\,dx = \int x \frac{e^x - e^{-x}}{2}\,dx = \frac{1}{2} \int xe^x\,dx - \frac{1}{2} \int xe^{-x}\,dx$

$$= \frac{1}{2}(xe^x - e^x) - \frac{1}{2}(-xe^{-x} - e^{-x}) + C = x \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} + C$$

$$= x \cosh x - \sinh x + C$$

12. $\int \frac{dx}{\sqrt{4x^2 - 9}} = \frac{1}{2} \cosh^{-1} \frac{2x}{3} + C$

13. $\int \frac{dx}{9x^2 - 25} = -\frac{1}{15} \coth^{-1} \frac{3x}{5} + C$

14. Find $\int \sqrt{x^2 + 4}\,dx$.

Let $x = 2 \sinh z$. Then $dx = 2 \cosh z\,dz$, $\sqrt{x^2 + 4} = 2 \cosh z$, and

$$\int \sqrt{x^2 + 4}\,dx = 4 \int \cosh^2 z\,dz = 2 \int (\cosh 2z + 1)\,dz = \sinh 2z + 2z + C$$

$$= 2 \sinh z \cosh z + 2z + C = \tfrac{1}{2}x\sqrt{x^2 + 4} + 2 \sinh^{-1} \tfrac{1}{2}x + C$$

15. Find $\int \frac{dx}{x\sqrt{1 - x^2}}$.

Let $x = \text{sech}\,z$. Then $dx = -\text{sech}\,z \tanh z\,dz$, $1 - x^2 = \tanh z$, and

$$\int \frac{dx}{x\sqrt{1 - x^2}} = -\int \frac{\text{sech}\,z \tanh z}{\text{sech}\,z \tanh z}\,dz = -\int dz = -z + C = -\text{sech}^{-1} x + C$$

Supplementary Problems

In Problems 16 to 39, evaluate the integral at the left.

16. $\int \sinh 3x\,dx = \frac{1}{3} \cosh 3x + C$

17. $\int \cosh \frac{1}{4}x\,dx = 4 \sinh \frac{1}{4}x + C$

18. $\int \coth \frac{3}{2}x\,dx = \frac{2}{3} \ln |\sinh \frac{3}{2}x| + C$

19. $\int \text{csch}^2 (1 + 3x)\,dx = -\frac{1}{3} \coth (1 + 3x) + C$

20. $\int \text{sech}\,2x \tanh 2x\,dx = -\frac{1}{2}\,\text{sech}\,2x + C$

21. $\int \text{csch}\,x\,dx = \ln \sqrt{\frac{\cosh x - 1}{\cosh x + 1}} + C$

22. $\int \cosh^2 \frac{1}{2}x\,dx = \frac{1}{2}(\sinh x + x) + C$

23. $\int \coth^2 3x\,dx = x - \frac{1}{3} \coth 3x + C$

24. $\int \sinh^3 x\,dx = \frac{1}{3} \cosh^3 x - \cosh x + C$

25. $\int e^x \sinh x\,dx = \frac{1}{4}e^{2x} - \frac{1}{2}x + C$

26. $\int e^{2x} \cosh x \, dx = \frac{1}{6}e^{3x} + \frac{1}{2}e^{x} + C$

27. $\int x \cosh x \, dx = x \sinh x - \cosh x + C$

28. $\int x^2 \sinh x \, dx = (x^2 + 2) \cosh x - 2x \sinh x + C$

29. $\int \sinh^3 x \cosh^2 x \, dx = \frac{1}{5} \cosh^5 x - \frac{1}{3} \cosh^3 x + C$

30. $\int \sinh x \ln \cosh^2 x \, dx = \cosh x \, (\ln \cosh^2 x - 2) + C$

31. $\int \frac{dx}{\sqrt{x^2 + 9}} = \sinh^{-1} \frac{x}{3} + C$

32. $\int \frac{dx}{\sqrt{x^2 - 25}} = \cosh^{-1} \frac{x}{5} + C$

33. $\int \frac{dx}{4 - 9x^2} = \frac{1}{6} \tanh^{-1} \frac{3}{2} x + C$

34. $\int \frac{dx}{16x^2 - 9} = -\frac{1}{12} \coth^{-1} \frac{4}{3} x + C$

35. $\int \sqrt{x^2 - 9} \, dx = \frac{x}{2} \sqrt{x^2 - 9} - \frac{9}{2} \cosh^{-1} \frac{x}{3} + C$

36. $\int \frac{dx}{\sqrt{x^2 - 2x + 17}} = \sinh^{-1} \frac{x - 1}{4} + C$

37. $\int \frac{dx}{4x^2 + 12x + 5} = -\frac{1}{4} \coth^{-1} \left(x + \frac{3}{2}\right) + C$

38. $\int \frac{x^2}{(x^2 + 4)^{3/2}} \, dx = \sinh^{-1} \frac{x}{2} - \frac{x}{\sqrt{x^2 + 4}} + C$

39. $\int \frac{\sqrt{x^2 + 1}}{x^2} \, dx = \sinh^{-1} x - \frac{\sqrt{1 + x^2}}{x} + C$

Chapter 37

Applications of Indefinite Integrals

WHEN THE EQUATION $y = f(x)$ of a curve is known, the slope m at any point $P(x, y)$ on it is given by $m = f'(x)$. Conversely, when the slope of a curve at a point $P(x, y)$ on it is given by $m = dy/dx = f'(x)$, a family of curves, $y = f(x) + C$, may be found by integration. To single out a particular curve of the family, it is necessary to assign or to determine a particular value of C. This may be done by prescribing that the curve pass through a given point. (See Problems 1 to 4.)

AN EQUATION $s = f(t)$, where s is the distance at time t of a body from a fixed point in its (straight-line) path, completely defines the motion of the body. The velocity and acceleration at time t are given by

$$v = \frac{ds}{dt} = f'(t) \qquad \text{and} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$$

Conversely, if the velocity (or acceleration) is known at time t, together with the position (or position and velocity) at some given instant, usually at $t = 0$, the equation of motion may be obtained. (See Problems 7 to 10.)

Solved Problems

1. Find the equation of the family of curves whose slope at any point is equal to the negative of twice the abscissa of the point. Find the curve of the family which passes through the point (1, 1).

We are given that $dy/dx = -2x$. Then $dy = -2x\,dx$, from which $\int dy = \int -2x\,dx$, and $y = -x^2 + C$. This is the equation of a family of parabolas.

Setting $x = 1$ and $y = 1$ in the equation of the family yields $1 = -1 + C$ or $C = 2$. The equation of the curve passing through the point (1, 1) is then $y = -x^2 + 2$.

2. Find the equation of the family of curves whose slope at any point $P(x, y)$ is $m = 3x^2y$. Find the equation of the curve of the family which passes through the point (0, 8).

Since $m = \frac{dy}{dx} = 3x^2y$, we have $\frac{dy}{y} = 3x^2\,dx$. Then $\ln y = x^3 + C = x^3 + \ln c$ and $y = ce^{x^3}$. When $x = 0$ and $y = 8$, then $8 = ce^0 = c$. The equation of the required curve is $y = 8e^{x^3}$.

3. At every point of a certain curve, $y'' = x^2 - 1$. Find the equation of the curve if it passes through the point (1, 1) and is there tangent to the line $x + 12y = 13$.

Here $\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = x^2 - 1$. Then $\int \frac{d}{dx}(y')\,dx = \int (x^2 - 1)\,dx$ and $y' = \frac{x^3}{3} - x + C_1$.

At (1, 1), the slope y' of the curve equals the slope $-\frac{1}{12}$ of the line. Then $-\frac{1}{12} = \frac{1}{3} - 1 + C_1$, from which $C_1 = \frac{7}{12}$. Hence $y' = dy/dx = \frac{1}{3}x^3 - x + \frac{7}{12}$, and integration yields

$$\int dy = \int \left(\tfrac{1}{3}x^3 - x + \tfrac{7}{12}\right) dx \qquad \text{or} \qquad y = \tfrac{1}{12}x^4 - \tfrac{1}{2}x^2 + \tfrac{7}{12}x + C_2$$

At $(1, 1)$, $1 = \frac{1}{12} - \frac{1}{2} + \frac{7}{12} + C_2$ and $C_2 = \frac{5}{6}$. The required equation is $y = \frac{1}{12}x^4 - \frac{1}{2}x^2 + \frac{7}{12}x + \frac{5}{6}$.

4. The family of *orthogonal trajectories* of a given system of curves is another system of curves, each of which cuts every curve of the given system at right angles. Find the equations of the orthogonal trajectories of the family of hyperbolas $x^2 - y^2 = c$.

At any point $P(x, y)$, the slope of the hyperbola through the point is given by $m_1 = x/y$, and the slope of the orthogonal trajectory through P is given by $m_2 = dy/dx = -y/x$. Then

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \qquad \text{so that} \qquad \ln|y| = -\ln|x| + \ln C' \qquad \text{or} \qquad |xy| = C'$$

The required equation is $xy = \pm C'$ or, simply, $xy = C$.

5. A certain quantity q increases at a rate proportional to itself. If $q = 25$ when $t = 0$ and $q = 75$ when $t = 2$, find q when $t = 6$.

Since $dq/dt = kq$, we have $dq/q = k\,dt$. Integration yields $\ln q = kt + \ln c$ or $q = ce^{kt}$.
When $t = 0$, $q = 25 = ce^0$; hence, $c = 25$ and $q = 25e^{kt}$.
When $t = 2$, $q = 25e^{2k} = 75$; then $e^{2k} = 3 = e^{1.10}$. So $k = 0.55$ and $q = 25e^{0.55t}$.
Finally, when $t = 6$, $q = 25e^{0.55t} = 25e^{3.3} = 25(e^{1.1})^3 = 25(27) = 675$.

6. A substance is being transformed into another at a rate proportional to the untransformed amount. If the original amount is 50 and is 25 when $t = 3$, when will $\frac{1}{10}$ of the substance remain untransformed?

Let q represent the amount transformed in time t. Then $dq/dt = k(50 - q)$, from which

$$\frac{dq}{50 - q} = k\,dt \qquad \text{so that} \qquad \ln(50 - q) = -kt + \ln c \qquad \text{or} \qquad 50 - q = ce^{-kt}$$

When $t = 0$, $q = 0$ and $c = 50$; thus $50 - q = 50e^{-kt}$.
When $t = 3$, $50 - q = 25 = 50e^{-3k}$; then $e^{-3k} = 0.5 = e^{-0.69}$, $k = 0.23$, and $50 - q = 50 - e^{-0.23t}$.
When the untransformed amount is 5, $50e^{-0.23t} = 5$; then $e^{-0.23t} = 0.1 = e^{-2.30}$ and $t = 10$.

7. A ball is rolled over a level lawn with initial velocity 25 ft/sec. Due to friction, the velocity decreases at the rate of 6 ft/sec^2. How far will the ball roll?

Here $dv/dt = -6$. So $v = -6t + C_1$. When $t = 0$, $v = 25$; hence $C_1 = 25$ and $v = -6t + 25$.
Since $v = ds/dt = -6t + 25$, integration yields $s = -3t^2 + 25t + C_2$. When $t = 0$, $s = 0$; hence $C_2 = 0$ and $s = -3t^2 + 25t$.
When $v = 0$, $t = \frac{25}{6}$; hence, the ball rolls for $\frac{25}{6}$ sec before coming to rest. In that time it rolls a distance $s = -3(\frac{25}{6})^2 + 25(\frac{25}{6}) = -\frac{625}{12} + \frac{625}{6} = \frac{625}{12}$ ft.

8. A stone is thrown straight down from a stationary balloon, 10,000 ft above the ground, with a speed of 48 ft/sec. Locate the stone and find its speed 20 sec later.

Take the upward direction as positive. When the stone leaves the balloon, it has acceleration $a = dv/dt = -32$ ft/sec^2 and velocity $v = -32t + C_1$.
When $t = 0$, $v = -48$; hence $C_1 = -48$. Then $v = ds/dt = -32t - 48$ and $s = -16t^2 - 48t + C_2$.
When $t = 0$, $s = 10{,}000$; hence $C_2 = 10{,}000$ and $s = -16t^2 - 48t + 10{,}000$.
When $t = 20$,

$$s = -16(20)^2 - 48(20) + 10{,}000 = 2640 \qquad \text{and} \qquad v = -32(20) - 48 = -688$$

After 20 sec, the stone is 2640 ft above the ground and its speed is 688 ft/sec.

9. A ball is dropped from a balloon that is 640 ft above the ground and rising at the rate of 48 ft/sec. Find (*a*) the greatest distance above the ground attained by the ball, (*b*) the time the ball is in the air, and (*c*) the speed of the ball when it strikes the ground.

Take the upward direction as positive. Then $a = dv/dt = -32$ ft/sec^2 and $v = -32t + C_1$.

When $t = 0$, $v = 48$; hence $C_1 = 48$. Then $v = ds/dt = -32t + 48$ and $s = -16t^2 + 48t + C_2$. When $t = 0$, $s = 640$; hence $C_2 = 640$ and $s = -16t^2 + 48t + 640$.

(*a*) When $v = 0$, $t = \frac{3}{2}$ and $s = -16(\frac{3}{2})^2 + 48(\frac{3}{2}) + 640 = 676$. The greatest height attained by the ball is 676 ft.

(*b*) When $s = 0$, $-16t^2 + 48t + 640 = 0$ and $t = -5, 8$. The ball is in the air for 8 sec.

(*c*) When $t = 8$, $v = -32(8) + 48 = -208$. The ball strikes the ground with speed 208 ft/sec.

10. The velocity with which water will flow from a small orifice in a tank, at a depth h ft below the surface, is $0.6\sqrt{2gh}$ ft/sec, where $g = 32$ ft/sec^2. Find the time required to empty an upright cylindrical tank of height 5 ft and radius 1 ft through a round 1-in hole in the bottom.

Let h be the depth of the water at time t. The water flowing out in time dt generates a cylinder of height $v\,dt$ ft, radius 1/24 ft, and volume $\pi(1/24)^2\,v\,dt = 0.6\pi(1/24)^2\sqrt{2gh}\,dt$ ft^3.

Let $-dh$ represent the corresponding drop in the surface level. The loss in volume is $-\pi(1)^2\,dh$ ft^3. Then $0.6\pi(1/24)^2(8\sqrt{h}\,dt) = -\pi\,dh$, or $dt = -(120\,dh)/\sqrt{h}$ and $t = -240\sqrt{h} + C$.

At $t = 0$, $h = 5$ and $C = 240\sqrt{5}$; thus $t = -240\sqrt{h} + 240\sqrt{5}$.

When the tank is empty, $h = 0$ and $t = 240\sqrt{5}$ sec = 9 min, approximately.

Supplementary Problems

11. Find the equation of the family of curves having the given slope, and the equation of the curve of the family which passes through the given point, in each of the following:

(*a*) $m = 4x$; (1, 5) (*b*) $m = \sqrt{x}$; (9, 18) (*c*) $m = (x-1)^3$; (3, 0)
(*d*) $m = 1/x^2$; (1, 2) (*e*) $m = x/y$; (4, 2) (*f*) $m = x^2/y^3$; (3, 2)
(*g*) $m = 2y/x$; (2, 8) (*h*) $m = xy/(1+x^2)$; (3, 5)

Ans. (*a*) $y = 2x^2 + C$, $y = 2x^2 + 3$; (*b*) $3y = 2x^{3/2} + C$, $3y = 2x^{3/2}$; (*c*) $4y = (x-1)^4 + C$, $4y = (x-1)^4 - 16$; (*d*) $xy = Cx - 1$, $xy = 3x - 1$; (*e*) $x^2 - y^2 = C$, $x^2 - y^2 = 12$; (*f*) $3y^4 = 4x^3 + C$, $3y^4 = 4x^3 - 60$; (*g*) $y = Cx^2$, $y = 2x^2$; (*h*) $y^2 = C(1+x^2)$, $2y^2 = 5(1+x^2)$

12. (*a*) For a certain curve, $y'' = 2$. Find its equation given that it passes through $P(2, 6)$ with slope 10. *Ans.* $y = x^2 + 6x - 10$

(*b*) For a certain curve, $y'' = 6x - 8$. Find its equation given that it passes through $P(1, 0)$ with slope 4. *Ans.* $y = x^3 - 4x^2 + 9x - 6$

13. A particle moves along a straight line from the origin (at $t = 0$) with the given velocity v. Find the distance the particle moves during the interval between the two given times t.

(*a*) $v = 4t + 1$; 0, 4 (*b*) $v = 6t + 3$; 1, 3 (*c*) $v = 3t^2 + 2t$; 2, 4
(*d*) $v = \sqrt{t} + 5$; 4, 9 (*e*) $v = 2t - 2$; 0, 5 (*f*) $v = t^2 - 3t + 2$; 0, 4

Ans. (*a*) 36; (*b*) 30; (*c*) 68; (*d*) $37\frac{2}{3}$; (*e*) 17; (*f*) $5\frac{2}{3}$

14. Find the equation of the family of orthogonal trajectories of the system of parabolas $y^2 = 2x + C$.
Ans. $y = Ce^{-x}$

15. A particle moves in a straight line from the origin (at $t = 0$) with given initial velocity v_0 and acceleration a. Find s at time t.
(*a*) $a = 32$, $v_0 = 2$ (*b*) $a = -32$; $v_0 = 96$ (*c*) $a = 12t^2 + 6t$; $v_0 = -3$ (*d*) $a = 1/\sqrt{t}$; $v_0 = 4$

Ans. (*a*) $s = 16t^2 + 2t$; (*b*) $s = -16t^2 + 96t$; (*c*) $s = t^4 + t^3 - 3t$; (*d*) $s = \frac{4}{3}(t^{3/2} + 3t)$

16. A car is slowing down at the rate 0.8 ft/sec^2. How far will the car move before it stops if its speed is initially 15 mi/hr? *Ans.* $302\frac{1}{2}$ ft

17. A particle is projected vertically upward from a point 112 ft above the ground with initial velocity 96 ft/sec. (*a*) How fast is it moving when it is 240 ft above the ground? (*b*) When will it reach the highest point in its path? (*c*) At what speed will it strike the ground?

Ans. (*a*) 32 ft/sec; (*b*) after 3 sec; (*c*) 128 ft/sec

18. A block of ice slides down a chute with acceleration 4 ft/sec^2. The chute is 60 ft long, and the ice reaches the bottom in 5 sec. What are the initial velocity of the ice and the velocity when it is 20 ft from the bottom of the chute? *Ans.* 2 ft/sec; 18 ft/sec

19. What constant acceleration is required (*a*) to move a particle 50 ft in 5 sec; (*b*) to slow a particle from a velocity of 45 ft/sec to a dead stop in 15 ft? *Ans.* (*a*) 4 ft/sec^2; (*b*) $-67\frac{1}{2}$ ft/sec^2

20. The bacteria in a certain culture increase according to $dN/dt = 0.25N$. If originally $N = 200$, find N when $t = 8$. *Ans.* 1478

Chapter 38

The Definite Integral

THE DEFINITE INTEGRAL. Let $a \le x \le b$ be an interval on which a given function $f(x)$ is continuous. Divide the interval into n subintervals $h_1, h_2, \ldots, h_n$ by the insertion of $n-1$ points $\xi_1, \xi_2, \ldots, \xi_{n-1}$, where $a < \xi_1 < \xi_2 < \cdots < \xi_{n-1} < b$, and relabel a as ξ_0 and b as ξ_n. Denote the length of the subinterval h_1 by $\Delta_1 x = \xi_1 - \xi_0$, of h_2 by $\Delta_2 x = \xi_2 - \xi_1, \ldots$, of h_n by $\Delta_n x = \xi_n - \xi_{n-1}$. (This is done in Fig. 38-1. The lengths are directed distances, each being positive in view of the above inequality.) On each subinterval select a point (x_1 on the subinterval h_1, x_2 on $h_2, \ldots, x_n$ on h_n) and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x = f(x_1)\,\Delta_1 x + f(x_2)\,\Delta_2 x + \cdots + f(x_n)\,\Delta_n x \qquad (38.1)$$

each term being the product of the length of a subinterval and the value of the function at the selected point on that subinterval. Denote by λ_n the length of the longest subinterval appearing in (*38.1*). Now let the number of subintervals increase indefinitely in such a manner that $\lambda_n \to 0$. (One way of doing this would be to bisect each of the original subintervals, then bisect each of these, and so on.) Then

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x \qquad (38.2)$$

exists and is the same for all methods of subdividing the interval $a \le x \le b$, so long as the condition $\lambda_n \to 0$ is met, and for all choices of the points x_k in the resulting subintervals.

Fig. 38-1

A proof of this theorem is beyond the scope of this book. In Problems 1 to 3 the limit is evaluated for selected functions $f(x)$. It must be understood, however, that for an arbitrary function this procedure is too difficult to attempt. Moreover, to succeed in the evaluations made here, it is necessary to prescribe some relation among the lengths of the subintervals (we take them all of equal length) and to follow some pattern in choosing a point on each subinterval (for example, choose the left-hand endpoint or the right-hand endpoint or the midpoint of each subinterval).

By agreement, we write

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x$$

The symbol $\int_a^b f(x)\,dx$ is read "the *definite integral* of $f(x)$, with respect to x, from $x = a$ to $x = b$." The function $f(x)$ is called the *integrand*; a and b are called, respectively, the *lower* and *upper limits* (boundaries) *of integration*. (See Problems 1 to 3.)

We have defined $\int_a^b f(x)\,dx$ when $a < b$. The other cases are taken care of by the following definitions:

$$\int_a^a f(x)\,dx = 0 \qquad (38.3)$$

$$\text{If } a < b, \text{ then } \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{38.4}$$

PROPERTIES OF DEFINITE INTEGRALS. If $f(x)$ and $g(x)$ are continuous on the interval of integration $a \le x \le b$, then

Property 38.1: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c

(For a proof, see Problem 4.)

Property 38.2: $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Property 38.3: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, for $a < c < b$

Property 38.4 (first mean-value theorem): $\int_a^b f(x)\,dx = (b-a)f(x_0)$ for at least one value $x = x_0$ between a and b.

(For a proof, see Problem 5.)

Property 38.5: If $F(u) = \int_a^u f(x)\,dx$, then $\frac{d}{du}F(u) = f(u)$

(For a proof, see Problem 6.)

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS. If $f(x)$ is continuous on the interval $a \le x \le b$, and if $F(x)$ is any indefinite integral of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

(For a proof, see Problem 7.)

EXAMPLE 1: (*a*) Take $f(x) = c$, a constant, and $F(x) = cx$; then $\int_a^b c\,dx = cx\Big|_a^b = c(b-a)$.
(*b*) Take $f(x) = x$ and $F(x) = \frac{1}{2}x^2$; then $\int_0^5 x\,dx = \frac{1}{2}x^2\Big|_0^5 = \frac{25}{2} - 0 = \frac{25}{2}$.
(*c*) Take $f(x) = x^3$ and $F(x) = \frac{1}{4}x^4$; then $\int_1^3 x^3\,dx = \frac{1}{4}x^4\Big|_1^3 = \frac{81}{4} - \frac{1}{4} = 20$.
These results should be compared with those of Problems 1 to 3. The reader can show that *any* indefinite integral of $f(x)$ may be used by redoing (*c*) with $F(x) = \frac{1}{4}x^4 + C$.

(See Problems 8 to 20.)

THE THEOREM OF BLISS. If $f(x)$ and $g(x)$ are continuous on the interval $a \le x \le b$, if the interval is divided into subintervals as before, and if two points are selected in each subinterval (that is, x_k and X_k in the kth subinterval), then

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k)g(X_k)\,\Delta_k x = \int_a^b f(x)g(x)\,dx$$

We note first that the theorem is true if the points x_k and X_k are identical. The force of the theorem is that when the points of each pair are distinct, the result is the same as if they were coincident. An intuitive feeling for the validity of the theorem follows from writing

$$\sum_{k=1}^{n} f(x_k)g(X_k)\,\Delta_k x = \sum_{k=1}^{n} f(x_k)g(x_k)\,\Delta_k x + \sum_{k=1}^{n} f(x_k)[g(X_k) - g(x_k)]\,\Delta_k x$$

and noting that as $n \to +\infty$ (that is, as $\Delta_k x \to 0$) x_k and X_k must become more nearly identical and, since $g(x)$ is continuous, $g(X_k) - g(x_k)$ must then go to zero.

In evaluating definite integrals directly from the definition, we sometimes make use of the following summation formulas:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{38.5}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{38.6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{38.7}$$

These formulas can be proved by mathematical induction.

Solved Problems

In Problems 1 to 3, evaluate the integral by setting up S_n and obtaining the limit as $n \to +\infty$.

1. $\displaystyle\int_a^b c\,dx = c(b-a)$, c constant

Let the interval $a \le x \le b$ be divided into n equal subintervals of length $\Delta x = (b-a)/n$. Since the integrand is $f(x) = c$, then $f(x_k) = c$ for any choice of the point x_k on the kth subinterval, and

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x = \sum_{k=1}^{n} c(\Delta x) = (c + c + \cdots + c)(\Delta x) = nc\,\Delta x = nc\,\frac{b-a}{n} = c(b-a)$$

Hence $$\int_a^b c\,dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} c(b-a) = c(b-a)$$

2. $\displaystyle\int_0^5 x\,dx = \tfrac{25}{2}$

Let the interval $0 \le x \le 5$ be divided into n equal subintervals of length $\Delta x = 5/n$. Take the points x_k as the right-hand endpoints of the subintervals; that is, $x_1 = \Delta x$, $x_2 = 2\,\Delta x, \ldots, x_n = n\,\Delta x$, as shown in Fig. 38-2. Then

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x = \sum_{k=1}^{n} (k\,\Delta x)\,\Delta x = (1 + 2 + \cdots + n)(\Delta x)^2 = \frac{n(n+1)}{2}\left(\frac{5}{n}\right)^2 = \frac{25}{2}\left(1 + \frac{1}{n}\right)$$

and $$\int_0^5 x\,dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{25}{2}$$

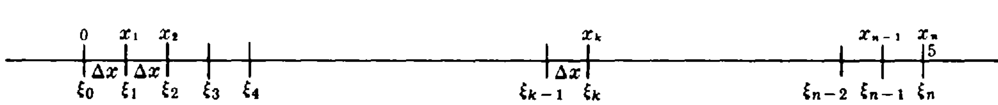

Fig. 38-2

3. $\int_1^3 x^3\,dx = 20$

Let the interval $1 \le x \le 3$ be divided into n subintervals of length $\Delta x = 2/n$.

First method: Take the points x_k as the left-hand endpoints of the subintervals, as in Fig. 38-3; that is, $x_1 = 1$, $x_2 = 1 + \Delta x, \ldots, x_n = 1 + (n-1)\,\Delta x$. Then

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} f(x_k)\,\Delta_k n = x_1^3\,\Delta x + x_2^3\,\Delta x + \cdots + x_n^3\,\Delta x \\
&= \{1 + (1+\Delta x)^3 + (1 + 2\,\Delta x)^3 + \cdots + [1 + (n-1)\,\Delta x]^3\}\,\Delta x \\
&= \{n + 3[1 + 2 + \cdots + (n-1)]\,\Delta x + 3[1^2 + 2^2 + \cdots + (n-1)^2](\Delta x)^2 \\
&\quad + [1^3 + 2^3 + \cdots + (n-1)^3](\Delta x)^3\}\,\Delta x \\
&= \left[n + 3\,\frac{(n-1)n}{1\cdot 2}\,\frac{2}{n} + 3\,\frac{(n-1)n(2n-1)}{1\cdot 2\cdot 3}\left(\frac{2}{n}\right)^2 + \frac{(n-1)^2n^2}{(1\cdot 2)^2}\left(\frac{2}{n}\right)^3\right]\frac{2}{n} \\
&= 2 + \left(6 - \frac{6}{n}\right) + \left(8 - \frac{12}{n} + \frac{4}{n^2}\right) + \left(4 - \frac{8}{n} + \frac{4}{n^2}\right) = 20 - \frac{26}{n} + \frac{8}{n^2}
\end{aligned}$$

and
$$\int_1^3 x^3\,dx = \lim_{n\to+\infty}\left(20 - \frac{26}{n} + \frac{8}{n^2}\right) = 20$$

Fig. 38-3

Second method: Take the points x_k as the midpoints of the subintervals, as in Fig. 38-4; that is, $x_1 = 1 + \frac{1}{2}\,\Delta x$, $x_2 = 1 + \frac{3}{2}\,\Delta x, \ldots, x_n = 1 + \dfrac{2n-1}{2}\,\Delta x$. Then

$$\begin{aligned}
S_n &= \left[\left(1 + \frac{1}{2}\,\Delta x\right)^3 + \left(1 + \frac{3}{2}\,\Delta x\right)^3 + \cdots + \left(1 + \frac{2n-1}{2}\,\Delta x\right)^3\right]\Delta x \\
&= \left\{\left[1 + 3\left(\frac{1}{2}\right)\Delta x + 3\left(\frac{1}{2}\right)^2(\Delta x)^2 + \left(\frac{1}{2}\right)^3(\Delta x)^3\right] + \left[1 + 3\left(\frac{3}{2}\right)(\Delta x) + 3\left(\frac{3}{2}\right)^2(\Delta x)^2 + \left(\frac{3}{2}\right)^3\left(\Delta x\right)^3\right] + \cdots\right. \\
&\quad \left. + \left[1 + 3\,\frac{2n-1}{2}\,\Delta x + 3\left(\frac{2n-1}{2}\right)^2(\Delta x)^2 + \left(\frac{2n-1}{2}\right)^3(\Delta x)^3\right]\right\}\Delta x \\
&= n\,\frac{2}{n} + \frac{3}{2}\,n^2\left(\frac{2}{n}\right)^2 + \frac{1}{4}\,(4n^3 - n)\left(\frac{2}{n}\right)^3 + \frac{1}{8}\,(2n^4 - n^2)\left(\frac{2}{n}\right)^4 \\
&= 2 + 6 + \left(8 - \frac{2}{n^2}\right) + \left(4 - \frac{2}{n^2}\right) = 20 - \frac{4}{n^2}
\end{aligned}$$

and
$$\int_1^3 x^3\,dx = \lim_{n\to+\infty}\left(20 - \frac{4}{n^2}\right) = 20$$

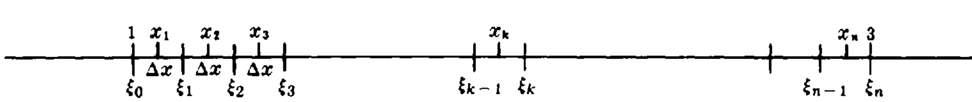

Fig. 38-4

4. Prove: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

For a proper subdivision of the interval $a \le x \le b$ and any choice of points on the subintervals,

$$S_n = \sum_{k=1}^{n} cf(x_k)\,\Delta_k x = c \sum_{k=1}^{n} f(x_k)\,\Delta_k x$$

Then $$\int_a^b cf(x)\,dx = c \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x = c\int_a^b f(x)\,dx$$

5. Prove the first mean-value theorem of the integral calculus: If $f(x)$ is continuous on the interval $a \le x \le b$, then $\int_a^b f(x)\,dx = (b-a)f(x_0)$ for at least one value $x = x_0$ between a and b.

The theorem is true, by Example 1(*a*), when $f(x) = c$, a constant. Otherwise, let m be the absolute minimum value, and M be the absolute maximum value, of $f(x)$ on the interval $a \le x \le b$. For any proper subdivision of the interval and any choice of the points x_k on the subintervals,

$$\sum_{k=1}^{n} m\,\Delta_k x < \sum_{k=1}^{n} f(x_k)\,\Delta_k x < \sum_{k=1}^{n} M\,\Delta_k x$$

Now when $n \to +\infty$, we have

$$\int_a^b m\,dx < \int_a^b f(x)\,dx < \int_a^b M\,dx$$

which, by Problem 1, becomes

$$m(b-a) < \int_a^b f(x)\,dx < M(b-a)$$

Then $$m < \frac{1}{b-a}\int_a^b f(x)\,dx < M$$

so that $\frac{1}{b-a}\int_a^b f(x)\,dx = N$, where N is some number between m and M. Now since $f(x)$ is continuous on the interval $a \le x \le b$, it must, by Property 8.1, take on at least once every value from m to M. Hence, there must be a value of x, say $x = x_0$, such that $f(x_0) = N$. Then

$$\frac{1}{b-a}\int_a^b f(x)\,dx = N = f(x_0) \qquad \text{and} \qquad \int_a^b f(x)\,dx = (b-a)f(x_0)$$

6. Prove: If $F(u) = \int_a^u f(x)\,dx$, then $\frac{d}{du}F(u) = f(u)$.

We have $$F(u+\Delta u) - F(u) = \int_a^{u+\Delta u} f(x)\,dx - \int_a^u f(x)\,dx$$

By Properties 38.3 and 38.4, this becomes

$$F(u+\Delta u) - F(u) = \int_u^a f(x)\,dx + \int_a^{u+\Delta u} f(x)\,dx = \int_u^{u+\Delta u} f(x)\,dx = f(u_0)\,\Delta u$$

where $u < u_0 < u + \Delta u$. Then

$$\frac{F(u+\Delta u) - F(u)}{\Delta u} = f(u_0) \qquad \text{and} \qquad \frac{dF}{du} = \lim_{\Delta u\to 0}\frac{F(u+\Delta u) - F(u)}{\Delta u} = \lim_{\Delta u\to 0} f(u_0) = f(u)$$

since $u_0 \to u$ as $\Delta u \to 0$.

This property is most frequently stated as:

$$\text{If } F(x) = \int_a^x f(x)\,dx, \text{ then } F'(x) = f(x). \qquad (1)$$

The use of the letter u above was merely an attempt to avoid the possibility of confusing the roles of the several x's. Note carefully in (*1*) that $F(x)$ is a function of the upper limit x of integration and not of the *dummy* letter x in $f(x)\,dx$. In other words, the property might also be stated as:

$$\text{If } F(x) = \int_a^x f(t)\, dt, \text{ then } F'(x) = f(x)\,.$$

It follows from (*1*) that $F(x)$ is simply an indefinite integral of $f(x)$.

7. Prove: If $f(x)$ is continuous on the interval $a \le x \le b$, and if $F(x)$ is any indefinite integral of $f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Use the last statement in Problem 6 to write $\int_a^x f(x)\, dx = F(x) + C$. When the upper limit of integration is $x = a$, we have

$$\int_a^a f(x)\, dx = 0 = F(a) + C \qquad \text{so} \qquad C = -F(a)$$

Then $\int_a^x f(x)\, dx = F(x) - F(a)$, and when the upper limit of integration is $x = b$, we have, as required, $\int_a^b f(x)\, dx = F(b) - F(a)$.

In Problems 8 to 17, use the fundamental theorem of integral calculus to evaluate the integral at the left.

8. $\int_{-1}^{1} (2x^2 - x^3)\, dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_{-1}^{1} = \left(\frac{2}{3} - \frac{1}{4}\right) - \left(-\frac{2}{3} - \frac{1}{4}\right) = \frac{4}{3}$

9. $\int_{-3}^{-1} \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_{-3}^{-1} = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{18}\right) = \frac{10}{9}$

10. $\int_1^4 \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_1^4 = 2(\sqrt{4} - \sqrt{1}) = 2$

11. $\int_{-2}^{3} e^{-x/2}\, dx = [-2e^{-x/2}]_{-2}^{3} = -2(e^{-3/2} - e) = 4.9904$

12. $\int_{-6}^{-10} \frac{dx}{x+2} = [\ln|x+2|]_{-6}^{-10} = \ln 8 - \ln 4 = \ln 2$

13. $\int_{\pi/2}^{3\pi/4} \sin x\, dx = [-\cos x]_{\pi/2}^{3\pi/4} = -(-\tfrac{1}{2}\sqrt{2} - 0) = \tfrac{1}{2}\sqrt{2}$

14. $\int_{-2}^{2} \frac{dx}{x^2+4} = \left[\frac{1}{2} \arctan \frac{1}{2} x\right]_{-2}^{2} = \frac{1}{2}\left[\frac{1}{4}\pi - \left(-\frac{1}{4}\pi\right)\right] = \frac{1}{4}\pi$

15. $\int_{-5}^{-3} \sqrt{x^2 - 4}\, dx = \left[\frac{1}{2} x\sqrt{x^2-4} - 2\ln|x + \sqrt{x^2-4}|\right]_{-5}^{-3} = \frac{5}{2}\sqrt{21} - \frac{3}{2}\sqrt{5} - 2\ln\frac{3-\sqrt{5}}{5-\sqrt{21}}$

16. $\int_{-1}^{2} \frac{dx}{x^2-9} = \left[\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|\right]_{-1}^{2} = \frac{1}{6}\left(\ln\frac{1}{5} - \ln 2\right) = \frac{1}{6}\ln 0.1$

17. $\int_1^e \ln x\, dx = [x\ln x - x]_1^e = (e\ln e - e) - (\ln 1 - 1) = 1$

18. Find $\int_3^6 xy\, dx$ when $x = 6\cos\theta$, $y = 2\sin\theta$.

We shall express x, y, and dx in the integral in terms of the parameter θ and $d\theta$, change the limits of integration to corresponding values of the parameter, and evaluate the resulting integral. We have, immediately, $dx = -6\sin\theta\, d\theta$. Also, when $x = 6\cos\theta = 6$, then $\theta = 0$; and when $x = 6\cos\theta = 3$, then $\theta = \pi/3$. Hence

$$\int_3^6 xy\,dx = \int_{\pi/3}^0 (6\cos\theta)(2\sin\theta)(-6\sin\theta)\,d\theta = -72\int_{\pi/3}^0 \sin^2\theta\cos\theta\,d\theta$$
$$= [-24\sin^3\theta]_{\pi/3}^0 = -24[0-(\sqrt{3}/2)^3] = 9\sqrt{3}$$

19. Find $\displaystyle\int_0^{2\pi/3} \frac{d\theta}{5+4\cos\theta}$.

The substitution $\theta = 2\arctan z$ (Fig. 38-5) yields $\displaystyle\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\frac{2\,dz}{1+z^2}}{5+4\frac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{9+z^2}$. To determine the z limits of integration, note that when $\theta = 0$, $z = 0$; when $\theta = 2\pi/3$, $\arctan z = \pi/3$ and $z = \sqrt{3}$. Then

$$\int_0^{2\pi/3} \frac{d\theta}{5+4\cos\theta} = 2\int_0^{\sqrt{3}} \frac{dz}{9+z^2} = \frac{2}{3}\left[\arctan\frac{z}{3}\right]_0^{\sqrt{3}} = \frac{\pi}{9}$$

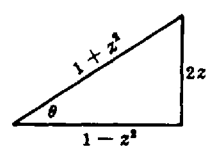

Fig. 38-5

20. Find $\displaystyle\int_0^{\pi/3} \frac{dx}{1-\sin x}$.

The substitution $x = 2\arctan z$ yields $\displaystyle\int \frac{dx}{1-\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{1-\frac{2z}{1+z^2}} = \int \frac{2\,dz}{(1-z)^2}$. When $x = 0$, $\arctan z = 0$ and $z = 0$; when $x = \pi/3$, $\arctan z = \pi/6$ and $z = \sqrt{3}/3$. Then

$$\int_0^{\pi/3} \frac{dx}{1-\sin x} = 2\int_0^{\sqrt{3}/3} \frac{dz}{(1-z)^2} = \left[\frac{2}{1-z}\right]_0^{\sqrt{3}/3} = \frac{2}{1-\sqrt{3}/3} - 2 = \sqrt{3}+1$$

Supplementary Problems

21. Evaluate $\displaystyle\int_a^b c\,dx$ of Problem 1 by dividing the interval $a \le x \le b$ into n subintervals of lengths $\Delta_1 x$, $\Delta_2 x, \ldots, \Delta_n x$. Note that $\displaystyle\sum_{k=1}^{n} \Delta_k x = b - a$.

22. Evaluate $\displaystyle\int_0^5 x\,dx$ of Problem 2 using subintervals of equal length and (*a*) choosing the points x_k as the left-hand endpoints of the subintervals; (*b*) choosing the points x_k as the midpoints of the subintervals; and (*c*) choosing the points x_k one-third of the way into each subinterval, that is, taking $x_1 = \frac{1}{3}\Delta x$, $x_2 = \frac{4}{3}\Delta x, \ldots$.

23. Evaluate $\int_1^4 x^2\,dx = 21$ using subintervals of equal length and choosing the points x_k as (*a*) the right-hand endpoints of the subintervals; (*b*) the left-hand endpoints of the subintervals; (*c*) the midpoints of the subintervals.

24. Using the same choice of subintervals and points as in Problem 23(*a*), evaluate $\int_1^4 x\,dx$ and $\int_1^4 (x^2 + x)\,dx$, and verify that $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

25. Evaluate $\int_1^2 x^2\,dx$ and $\int_2^4 x^2\,dx$. Compare the sum with the result of Problem 23 to verify that

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx \quad \text{for } a < c < b$$

26. Evaluate $\int_0^1 e^x\,dx = e - 1$. $\left(\textit{Hint}: \; S_n = \sum_{k=1}^{n} e^{k\,\Delta x}\,\Delta x = e^{\Delta x}(e-1)\dfrac{\Delta x}{e^{\Delta x} - 1}, \text{ and } \lim_{n\to+\infty} \dfrac{\Delta x}{e^{\Delta x} - 1} = \lim_{\Delta x\to 0} \dfrac{\Delta x}{e^{\Delta x} - 1} \text{ is indeterminate of the type } 0/0.\right)$

27. Prove Properties 38.2 and 38.3.

28. Use the fundamental theorem to evaluate each integral:

(*a*) $\int_0^2 (2 + x)\,dx = 6$

(*b*) $\int_0^2 (2 - x)^2\,dx = \frac{8}{3}$

(*c*) $\int_0^3 (3 - 2x + x^2)\,dx = 9$

(*d*) $\int_{-1}^2 (1 - t^2)t\,dt = -\frac{9}{4}$

(*e*) $\int_1^4 (1 - u)\sqrt{u}\,du = -\frac{116}{15}$

(*f*) $\int_1^8 \sqrt{1 + 3x}\,dx = 26$

(*g*) $\int_0^2 x^2(x^3 + 1)\,dx = \frac{40}{3}$

(*h*) $\int_0^3 \dfrac{dx}{\sqrt{1 + x}} = 2$

(*i*) $\int_0^1 x(1 - \sqrt{x})^2\,dx = \frac{1}{30}$

(*j*) $\int_4^8 \dfrac{x\,dx}{\sqrt{x^2 - 15}} = 6$

(*k*) $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{1}{4}a^2\pi$

(*l*) $\int_{-1}^1 x^2\sqrt{4 - x^2}\,dx = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$

(*m*) $\int_3^4 \dfrac{dx}{25 - x^2} = \dfrac{1}{5}\ln\dfrac{3}{2}$

(*n*) $\int_{-1/2}^0 \dfrac{x^3\,dx}{x^2 + x + 1} = \dfrac{\sqrt{3}\pi}{9} - \dfrac{5}{8}$

(*o*) $\int_2^4 \dfrac{\sqrt{16 - x^2}}{x}\,dx = 4\ln(2 + \sqrt{3}) - 2\sqrt{3}$

(*p*) $\int_8^{27} \dfrac{dx}{x - x^{1/3}} = \dfrac{3}{2}\ln\dfrac{8}{3}$

(*q*) $\int_0^1 \ln(x^2 + 1)\,dx = \ln 2 + \frac{1}{2}\pi - 2$

(*r*) $\int_0^{2\pi} \sin\frac{1}{2}t\,dt = 4$

(*s*) $\int_0^{\pi/3} x^2 \sin 3x\,dx = \frac{1}{27}(\pi^2 - 4)$

(*t*) $\int_0^{\pi/2} \dfrac{dx}{3 + \cos 2x} = \dfrac{\sqrt{2}\pi}{8}$

29. Show that $\int_3^5 \dfrac{dx}{\sqrt{x^2 + 16}} = \int_{-5}^{-3} \dfrac{dx}{\sqrt{x^2 + 16}}$.

30. Evaluate $\int_{\theta=0}^{\theta=2\pi} y\,dx = 3\pi$, given $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

31. Evaluate $\int_1^4 \sqrt{1 + (y')^2}\,dx = \frac{15}{2} + \frac{1}{4}\ln 2$, given $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$.

32. Evaluate $\int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{2}e^2(e-1)$, given $x = e^t \cos t$, $y = e^t \sin t$.

33. Use the appropriate reduction formulas (Chapter 31) to establish Wallis' formulas:

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \dfrac{1\cdot 3\cdots(n-3)(n-1)}{2\cdot 4\cdots(n-2)n}\,\dfrac{\pi}{2} & \text{if } n \text{ is even and } >0 \\[2ex] \dfrac{2\cdot 4\cdots(n-3)(n-1)}{1\cdot 3\cdots(n-2)n} & \text{if } n \text{ is odd and } >1 \end{cases}$$

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \begin{cases} \dfrac{1\cdot 3\cdots(m-1)\cdot 1\cdot 3\cdots(n-1)}{2\cdot 4\cdots(m+n-2)(m+n)}\,\dfrac{\pi}{2} & \text{if } m \text{ and } n \text{ are even and } >0 \\[2ex] \dfrac{2\cdot 4\cdots(m-3)(m-1)}{(n+1)(n+3)\cdots(n+m)} & \text{if } m \text{ is odd and } >1 \\[2ex] \dfrac{2\cdot 4\cdots(n-3)(n-1)}{(m+1)(m+3)\cdots(m+n)} & \text{if } n \text{ is odd and } >1 \end{cases}$$

34. Evaluate each integral:

(*a*) $\int_3^{11} \sqrt{2x+3}\, dx = \frac{98}{3}$ (*b*) $\int_0^{\pi/4} \frac{\cos 2x - 1}{\cos 2x + 1}\, dx = \frac{1}{4}\pi - 1$

(*c*) $\int_4^9 \frac{1-\sqrt{x}}{1+\sqrt{x}}\, dx = 4\ln\frac{3}{4} - 1$ (*d*) $\int_0^{\sqrt{2}} x^3 e^{x^2}\, dx = \frac{1}{2}(e^2+1)$

(*e*) $\int_{\pi/4}^{3\pi/4} \frac{\sin x\, dx}{\cos^2 x - 5\cos x + 4} = \frac{1}{3}\ln\frac{7+3\sqrt{2}}{7-3\sqrt{2}}$

(*f*) $\int_{-2}^{-1} \frac{x-1}{\sqrt{x^2-4x+3}}\, dx = \ln\frac{3-2\sqrt{2}}{4-\sqrt{15}} + 2\sqrt{2} - \sqrt{15}$

(*g*) $\int_{\pi/6}^{\pi/3} \frac{dx}{\sin 2x} = \ln\sqrt{3}$

(*h*) $\int_1^3 \ln(x+\sqrt{x^2-1})\, dx = 3\ln(3+2\sqrt{2}) - 2\sqrt{2}$

(*i*) $\int_{-1}^{-2} \frac{dx}{\sqrt{x^2+2x+2}} = \ln(\sqrt{2}-1)$ (*j*) $\int_{1/4}^{3/4} \frac{(x+1)\, dx}{x^2(x-1)} = 4\ln\frac{1}{3} - \frac{8}{3}$

(*k*) $\int_{-8}^{-3} \frac{(x+2)\, dx}{x(x-2)^2} = \frac{1}{2}\ln\frac{3}{4} + \frac{1}{5}$ (*l*) $\int_0^{\pi/4} \frac{dx}{2+\tan x} = \frac{1}{5}\ln\frac{3\sqrt{2}}{4} + \frac{\pi}{10}$

35. Prove (*38.5*) to (*38.7*).

36. Prove: $\frac{d}{dx}\int_x^b f(u)\, du = -f(x)$.

37. Prove: $\frac{d}{dx}\int_{h(x)}^{g(x)} f(u)\, du = f(g(x))g'(x) - f(h(x))h'(x)$.

38. Evaluate $\frac{d}{dx}\int_1^x \sin u\, du = \sin x$.

39. Evaluate $\frac{d}{dx}\int_x^0 u^2\, du = -x^2$.

40. Evaluate $\frac{d}{dx}\int_0^{\sin x} u^3\, du = \sin^3 x \cos x$.

41. Evaluate $\frac{d}{dx}\int_{x^2}^{4x} \cos u\, du = 4\cos 4x - 2x\cos x^2$.