South Valley University

Faculty of Science

Mathematics Department

Dynamic (2) : Lecture Notes

Prepared by

Department of Mathematics staff

Content

Chapter 1

Kinematics of Rigid Bodies

Deformable Body: Anybody that changes its shape and/or volume while being acted upon by any kind of external force.

Rigid body: A rigid body is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it

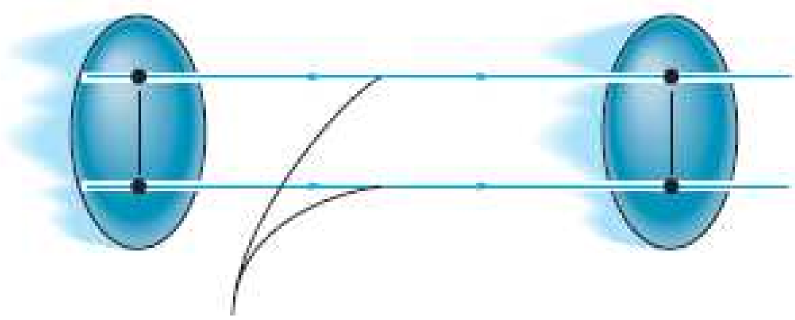
A deformable body is one that can distort. It would normally refer to a solid object so that as it deforms, it sort of deforms in a way that it could return to its starting shape if all the external forces were removed that caused it to deform.

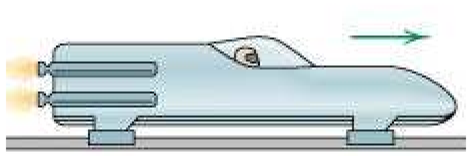
Types of Rigid Body Motion

Translation (Or Translation-al motion)

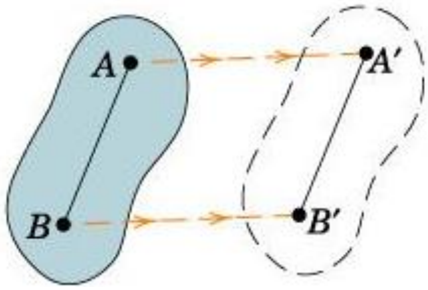
Translation. This type of motion occurs when a line in the body remains parallel to its original orientation throughout the motion.

Recti-linear translation: when the paths of motion for any two points on the body are parallel lines, the motion is called rectilinear translation



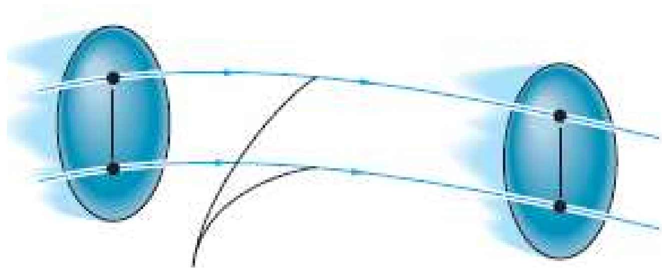


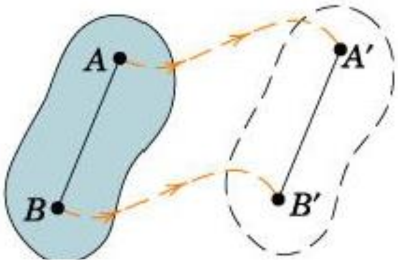
Rocket test sled

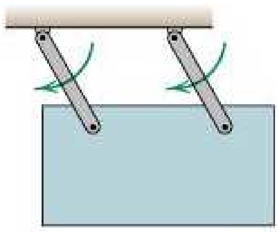


Curvi-linear Translation

If the paths of motion are along curved lines, the motion is called curvilinear translation



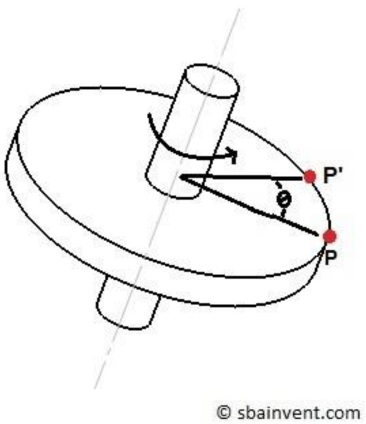


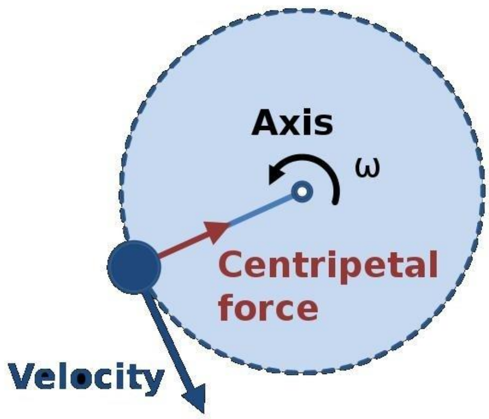


Parallel-link swinging plate

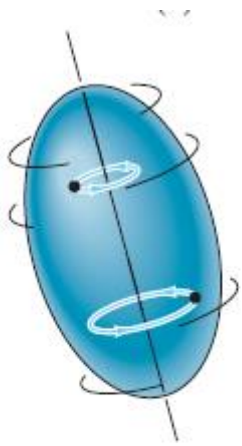
Rotation about a fixed axis

One straight line in the body is fixed. All other points in the body travel in circles around this line.



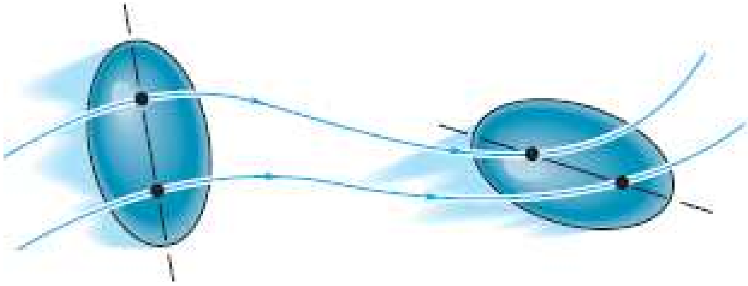


When a rigid body rotates about a fixed axis, all the particles of the body, except those which lies on the axis of rotation, move along circular paths except those which lies on the axis of rotation.



General plane motion

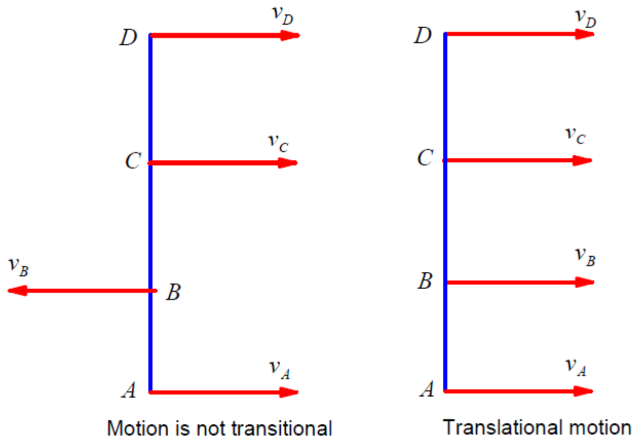
General plane motion. When a body is subjected to general plane motion, it undergoes a combination of translation and rotation, the translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane.



Velocity and Acceleration

In the Transitional motion , the velocity and acceleration of all points of the body at any moment are equal in magnitude and direction.

$\vec{V}_A = \vec{V}_B = \vec{V}_C = \vec{V}_D = \vec{V}$, $\vec{f}_A = \vec{f}_B = \vec{f}_C = \vec{f}_D = \vec{f}$



Rotational (Rotation) motion

Rotational motion is the motion of the body wrapping (Read: rapping) around its center

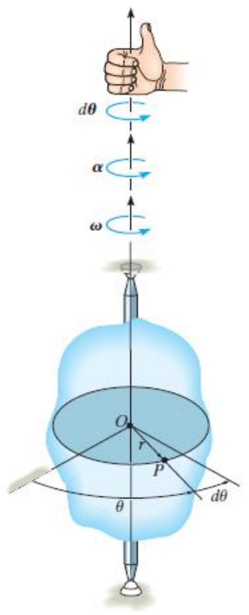


Fig. (a)

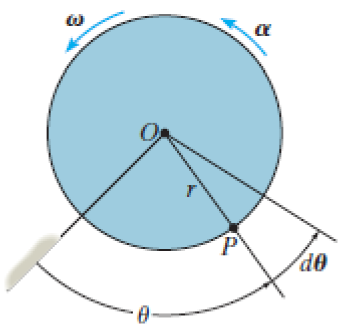


Fig. (b)

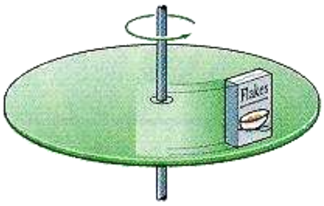


Fig. (c)

Note: One complete revolution is 360° = 2π radians.

Rotation about a Fixed Axis

When a body rotates about a fixed axis, any point P located in the body travels along a circular path. To study this motion it is first necessary to discuss the angular motion of the body about the axis.

Angular Motion. Since a point is without dimension, it cannot have angular motion. Only lines or bodies undergo angular motion. For example, consider the body shown in Figure and the angular motion of a radial line r located within the shaded plane.

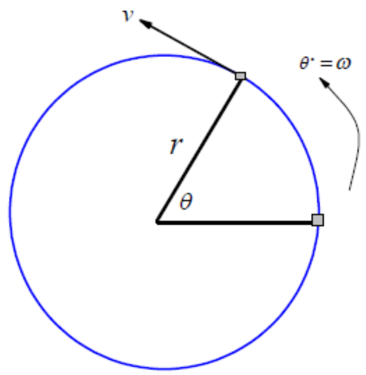
Angular Position. At the instant shown, the angular position of r is defined by the angle u, measured from a fixed reference line to r.

Angular Displacement. The change in the angular position, which can be measured as a differential dU, is called the angular displacement. This vector has a magnitude of dU, measured in degrees, radians, or revolutions, where 1 rev = 2p rad. Since motion is about a fixed axis, the direction of dU is always along this axis. Specifically, the direction is determined by the right-hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb, or dU, points upward, Fig. 16–4a. In two dimensions, as shown by the top view of the shaded plane, Fig. 16–4b, both u and du are counterclockwise, and so the thumb points outward from the page.

Angular Velocity. The time rate of change in the angular position is called the *angular velocity* V (omega). Since *d*U occurs during an,

Remember that

When a body moves in a circular path, we can write both the velocity and acceleration in the form



$\vec{v}=\dot{\vec{r}}\,(\,\dot{r}\ ,\ r\dot{\theta}),\quad \vec{f}\ =\ (\,\ddot{r}-r\dot{\theta}^2,\ \ r\ddot{\theta}+2\dot{r}\dot{\theta})$

If $r=\text{constant}$, then $r^{\bullet}=r^{\bullet\bullet}=0$. So, the velocity and acceleration becomes

$\vec{v}=\dot{\vec{r}}=(\,0\ ,\ r\dot{\theta}),\quad \vec{f}\ =\ (0-r\dot{\theta}^2,\ \ r\ddot{\theta}+0)$, or $\vec{v}=\dot{\vec{r}}=(\,0\ ,\ r\dot{\theta}),\quad \vec{f}\ =\ (\,-r\dot{\theta}^2,\ \ r\ddot{\theta}\,)$

The angular velocity in rotational (rotation) motion

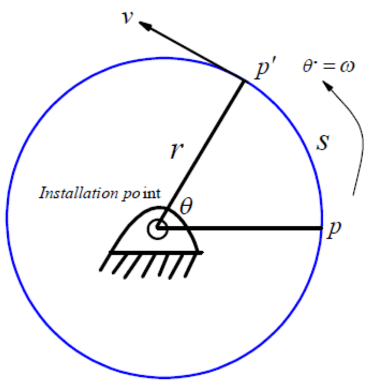
Angular velocity: The time rate of change in the angular position is called the angular velocity V (omega). Since dU occurs during an instant of time dt, then,

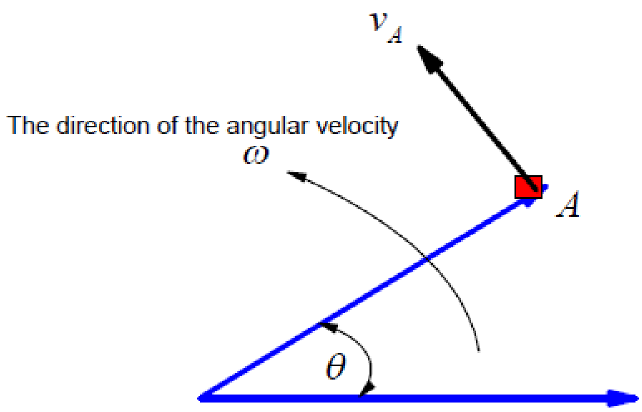
From the Figure $s=r\theta$, where θ is angler position and the angler velocity is $(\theta^{\bullet}=\omega)$.

It is clear that $\dfrac{d\theta}{dt}=\theta^{\bullet}=\omega$.

The relation between the angular velocity and transitional velocity is given from

$$\frac{ds}{dt}=s^{\bullet}=v=\frac{d(r\theta)}{dt}=r\frac{d\theta}{dt}=r\theta^{\bullet}=r\omega$$

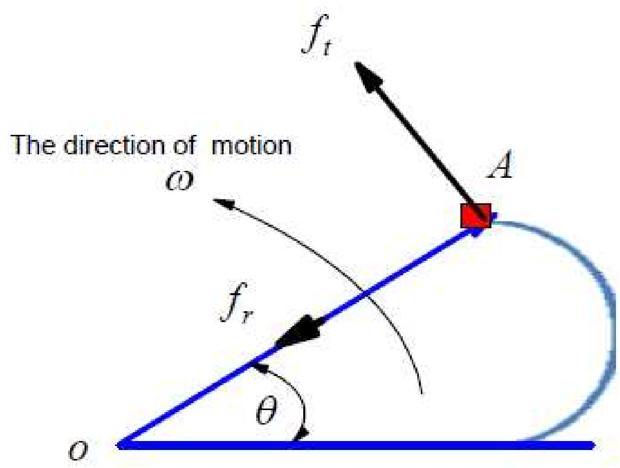




- The direction of translational velocity is in the same direction angular velocity

- The direction of the vector tells you the axis of the rotation, as well as whether the rotation is clockwise or counterclockwise.

- The relation between the angular acceleration and transitional acceleration is given from the transitional acceleration has two components, the first in Tangential direction (f_t) and the other in the normal direction (f_r)

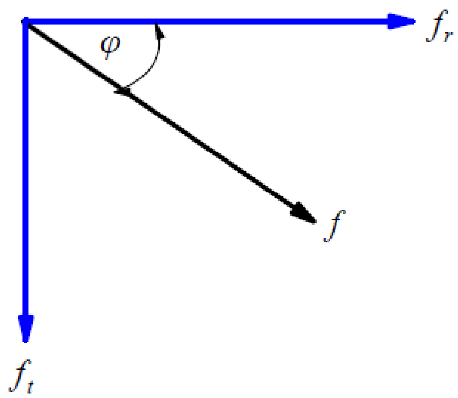


The components of the acceleration are given as

$f_t = r\omega^{\cdot}$, $f_r = r\omega^2$ Or $f_n = r\omega^2$

The Resultant of acceleration is given by $f = \sqrt{f_t^2 + f_r^2}$

While the direction is given by $\tan\varphi = \dfrac{f_t}{f_r}$



Special case of rotational motion

(i) In the case of constant angular velocity (ω = Constant) (i. e. the angular velocity does not change with time $\frac{d\omega}{dt}=0$) (ii) The pure rotational motion In the case of the angular acceleration is constant (i. e. $\omega^{\bullet}$ = Constant). Then $\frac{d\omega}{dt}=C$ and $\omega=Ct$

The relationship between the laws of motion in the case of linear motion with constant linear acceleration and rotational motion with constant angular acceleration

$\upsilon=\upsilon_o + a_c\, t,$ $\qquad\omega=\omega_o + \omega^{\bullet}\, t,$

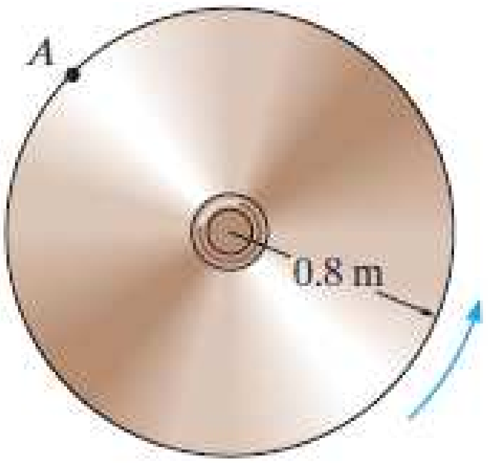
$x=\upsilon_o\, t + \frac{1}{2}a_c\, t^2,$ $\qquad\theta=\omega_o\, t + \frac{1}{2}\omega^{\bullet} t^2,$

$\upsilon^2=\upsilon_o^2 + 2\, a_c\, x,$ $\qquad\omega^2=\omega_o^2 + 2\,\omega^{\bullet}\theta$

General Plane Motion (Translation + Rotation)

If a rigid body moves with both translational and rotational motion, it is said to be in general plane motion.

Example 1: The angular velocity of the disk is defined by $\omega=\left(5t^2+2\right) rad/\sec$ where t is in seconds. Determine the magnitudes of the velocity and acceleration of point A on the disk when $t=0.5$ sec ?



Solution

$\omega = \left(5t^2 + 2\right) rad/\sec$

$\omega^{\bullet} = \frac{d\omega}{dt} = \left(10t\right) rad/\sec^2$

At $t = 0.5$ sec $\omega = \left(5(0.5)^2 + 2\right) = 3.25\ rad/\sec$, $\omega^{\bullet} = \left(10(0.5)\right) = 5\,rad/\sec^2$

But $v_A = (\omega)_{disk}(r)_{disk}$

Then $v_A = (3.25)(0.8) = 2.6\,m/\sec$,

$f_r = \omega^2 r,\quad f_t = \omega^{\bullet} r$

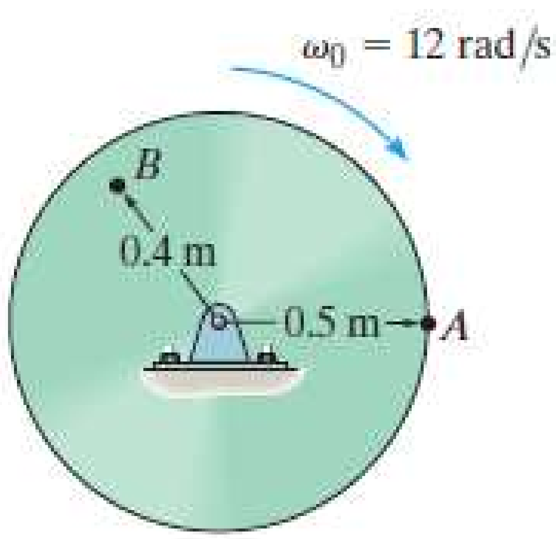
$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (3.25)^2(0.8\,m)) = (10.5625)(0.8) = 8.45\,m/\sec^2$

$f_{A_t} = (\omega^{\bullet})_{disk}(r\ )_{disk} = (5)(0.8\,m) = 4\,cm/\sec^2$ But $f_A = \sqrt{f_{A_t}^2 + f_{A_r}^2}$

$f_A = \sqrt{(8.45)^2 + (4)^2} = \sqrt{71.4025+16} = \sqrt{87.4025} = 9.349\,m/\sec^2 = 9.35\,m/\sec^2$

$\tan\phi = \frac{f_t}{f_r} \rightarrow \tan\phi = \frac{f_{A_t}}{f_{A_r}} = \frac{4}{8.45} = 0.47337 \rightarrow \phi = \tan^{-1}(0.47337) \rightarrow \phi = 25^{o}.33'$

Example 2: The angular acceleration of the disk is defined by $\omega^{\bullet} = \left(3t^2 + 12\right) rad/\sec^2$ where t is in seconds. If the disk is originally rotating at $\omega_0 = 12\ rad/\sec$. Determine the magnitude of the velocity and two components of acceleration of point A and B on the disk when $t = 2\sec$.



Solution

The angular acceleration of the disk is given by $\omega^{\bullet} = \left(3t^2 + 12\right) rad/\sec^2$.

While the angular velocity is given by $\omega = \int \omega^{\bullet}\, dt = \int \left(3t^2 + 12\right) dt$

$$\omega = \left(\frac{3}{3}t^3 + 12t\right) + c_1$$

At the start rotating point (i. e. $t = 0$) $\omega_0 = 12\ rad/\sec$. So $c_1 = 12$. Then

The angular velocity is given as $\omega = \left\{ t^3 + 12t + 12 \right\} rad/\sec$ At

$t = 2\sec$, we have

$$\omega = \left\{ (2)^3 + 12(2) + 12 \right\} = 44\, rad/\sec, \quad \omega^{\bullet} = \left(3(2)^2 + 12\right) = 24\, rad/\sec^2$$

From the two relation $f_r = \omega^2 r$, $f_t = \omega^{\bullet} r$,

<u>At the point A we have,</u> $(r)_{disk} = 0.5m$

$$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (44)^2(0.5m)) = (1936)(0.5) = 968\ m/\sec^2$$
$$f_{A_t} = (\omega^{\bullet})_{disk}(r)_{disk} = (24)(0.5m) = 12\ \ m/\sec^2$$

At the point A magnitude of the acceleration is given by $f_A = \sqrt{f_{A_t}^2 + f_{A_r}^2}$

$$f_A = \sqrt{(12)^2 + (968)^2} = \sqrt{144 + 937024} = \sqrt{937168} = 968.07\ m/\sec^2 = 968\, m/\sec^2$$

The direction of acceleration is given by $\tan\varphi_A = \frac{f_t}{f_r}$

$$\tan\varphi_A == \frac{f_{A_t}}{f_{A_r}} = \frac{12}{968} = 0.1239 \ \rightarrow\ \varphi_A = \tan^{-1}(0.1239) \ \rightarrow\ \varphi_A = 0.71024'$$

At the point B we have , $(r)_{disk} = 0.4m$

$$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (44)^2(0.4m)) = (1936)(0.4) = 774.4m/\sec^2$$

$$f_{A_t} = (\omega^{\bullet})_{disk}(r\ )_{disk} = (24)(0.4m) = 9.6m/\sec^2$$

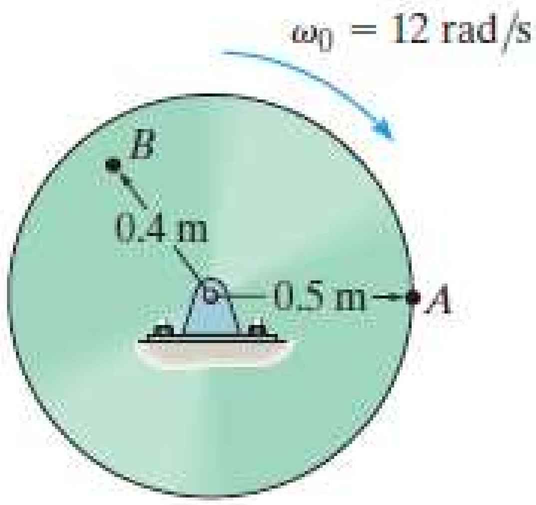
At the point B magnitude of the acceleration is given by $f_B = \sqrt{f_{B_t}^2 + f_{B_r}^2}$

$$f_B = \sqrt{(9.6)^2 + (774.4)^2} = \sqrt{92.16 + 599695} = \sqrt{599787} = 774.45\,m/\sec^2$$

The direction of the acceleration is given by $\tan\varphi_B = \dfrac{f_{B_t}}{f_{B_r}}$

$$\tan\varphi_B = \frac{f_{B_t}}{f_{B_r}} = \frac{9.6}{774.4} = 0.1239 \rightarrow \varphi_B = \tan^{-1}(0.1239) \rightarrow \varphi_B = 0.71024'$$

Example 3: The disk is originally rotating at $\omega_0 = 12\ rad/\sec$. If it is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$. Determine the magnitudes of the velocity and the two components of acceleration of point A at the instant $t = 2\sec$?



Solution

Where the disk is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$.

Then $\omega = \omega_o + \omega^{\bullet} t$, $\theta = \omega_o t + \frac{1}{2}\omega^{\bullet} t^2$, $\omega^2 = \omega_o^2 + 2\omega^{\bullet}\theta$

Where $\omega_0 = 12\ rad/\sec$, and after $t = 2\sec$ and form

Angular Motion: The angular velocity of the disk can be determined using from

$\omega = \omega_o + \omega^{\bullet} t$, we have $\omega = (12) + (20)(2) \rightarrow \omega = 52\ rad/\sec$

Motion of Point A. The magnitude of the velocity is given by

$$v_A = (\omega)_{disk}(r)_{disk} \rightarrow v_A = (52)_{disk}(0.5)_{disk} \rightarrow v_A = 26\ m/\sec$$

The tangential and normal component of acceleration are

$$f_{A_r} = (\omega^2)_{disk}(r)_{disk} = (52)^2(0.5m)) = (1936)(0.4) = 1352\ m/\sec^2$$

$$f_{A_t} = (\omega^{\bullet})_{disk}(r)_{disk} = (20)(0.5m) = 10\ m/\sec^2$$

At the point A magnitude of the acceleration is given by $f_A = \sqrt{f_{A_t}^2 + f_{A_r}^2}$

$$f_A = \sqrt{(10)^2 + (1352)^2} = 1352.04 m/\sec^2$$

The direction of the acceleration is given by $\tan\phi_A = \frac{f_t}{f_r}$

$$\tan\phi_A = \frac{f_{A_t}}{f_{A_r}} = \frac{10}{1352.04} \rightarrow \phi_A = \tan^{-1}(0.00739) \rightarrow \phi_A = 0.423778'$$, then from

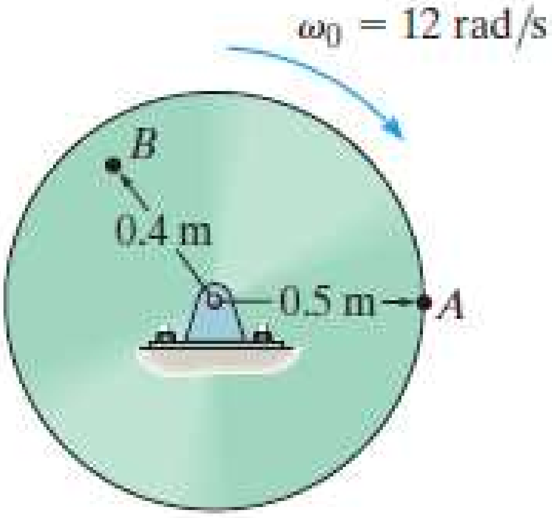
Eq. $\omega^2 = \omega_o^2 + 2\omega^{\bullet}\theta$, we have

$$(52)^2 = (12)^2 + 2(20)\theta \rightarrow \theta = \frac{2704 - 144}{40} = \frac{2560}{40} = \frac{256}{4} \rightarrow \theta = 64$$

The disk makes angle distance is given by ($\theta = 64$ rad)

The disk rotates laps $N = \frac{\theta}{2\pi} = \frac{64}{2\pi} = \frac{32}{\pi} \rightarrow \theta = 10.2\ rev$ (reflection-reversal)

Example 4: The disk is originally rotating at $\omega_0 = 12\ rad/\sec$. If it is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$. Determine the magnitudes of the velocity and the two components of acceleration of point B when the disk undergoes 2 revolutions?



Solution

Where the disk is subjected to a constant angular acceleration of $\omega^{\bullet} = 20\ rad/\sec^2$.Then

$$\omega = \omega_o + \omega^{\bullet} t, \qquad \theta = \omega_o t + \frac{1}{2}\omega^{\bullet} t^2, \qquad \omega^2 = \omega_o^2 + 2\,\omega^{\bullet}\theta$$

Where $\omega_0 = 12\ rad/\sec$.

When the disk undergoes 2 revolutions. Then $N = \dfrac{\theta}{2\pi} = 2 \rightarrow \quad \theta = 4\pi \ \ rev$

Angular Motion: The angular velocity of the disk can be determined using from

$\omega^2 = \omega_o^2 + 2\,\omega^{\bullet}\theta$, we have $\omega^2 = (12)^2 + 2\,(20)\,(4\pi) = 144 + 160\,\pi = 646.6548$,

$\omega = 25.43\ rad/\sec$

Motion of Point B. The magnitude of the velocity is given by

$$v_A = (\omega)_{disk}\,(r)_B \rightarrow v_A = (25.43)_{disk}\,(0.4)_B \rightarrow v_A = 10.1717\,m/\sec \rightarrow \quad v_A = 10.2\,m/\sec$$

The tangential and normal component of acceleration are

$$f_{B_r} = (\omega^2)_{disk}(r)_B = (25.43)^2(0.4m)) = 258.674\, m/\sec^2$$
$$f_{B_t} = (\omega^{\bullet})_{disk}(r\ )_B = (20)(0.4m) = 8\ m/\sec^2$$

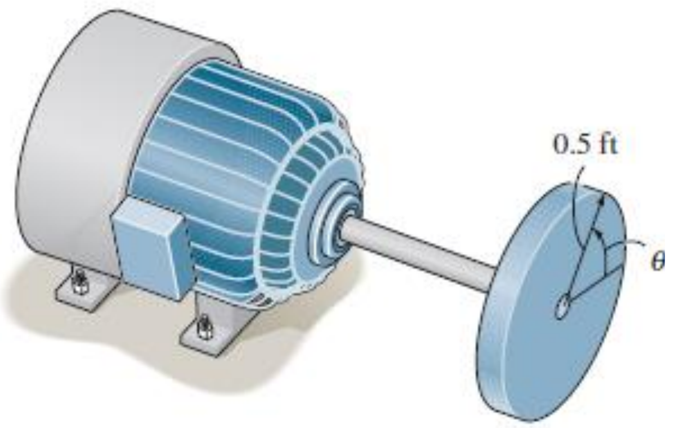
At the point A magnitude of the acceleration is given by $f_B = \sqrt{f_{B_t}^2 + f_{B_r}^2}$

$$f_B = \sqrt{(8)^2 + (\ 258.674)^2} = 258.798\, m/\sec^2$$

The direction of the acceleration is given by $\tan\varphi_B = \frac{f_t}{f_r}$

$$\tan\varphi_B == \frac{f_{B_t}}{f_{B_r}} = \frac{8}{258.674} \rightarrow \ \varphi_B = \tan^{-1}(0.03092) \ \rightarrow \ \varphi_B = 1^{o}.47714'$$

Example 5: The disk is driven by a motor such that the angular position of the disk is defined by $\theta = \left(20t + 4t^2\right) rad$ where t is in seconds. Determine the number of revolutions, the angular velocity, and angular acceleration of the disk when $t = 90\, s$?



Solution

At $t=90\sec$, we find that $\theta(t=90)=\left(20(90)+4(90)^2\right)rad=\left(1800+4(8100)\right)rad$

$\theta(t=90)=(1800+32400)\,rad=34200\ rad$

$\theta=34200\ rad\dfrac{(1)\,rev}{2\pi\ rad}=\dfrac{34200}{2\pi}rev \quad\rightarrow\quad \theta=5443\ \ rev$

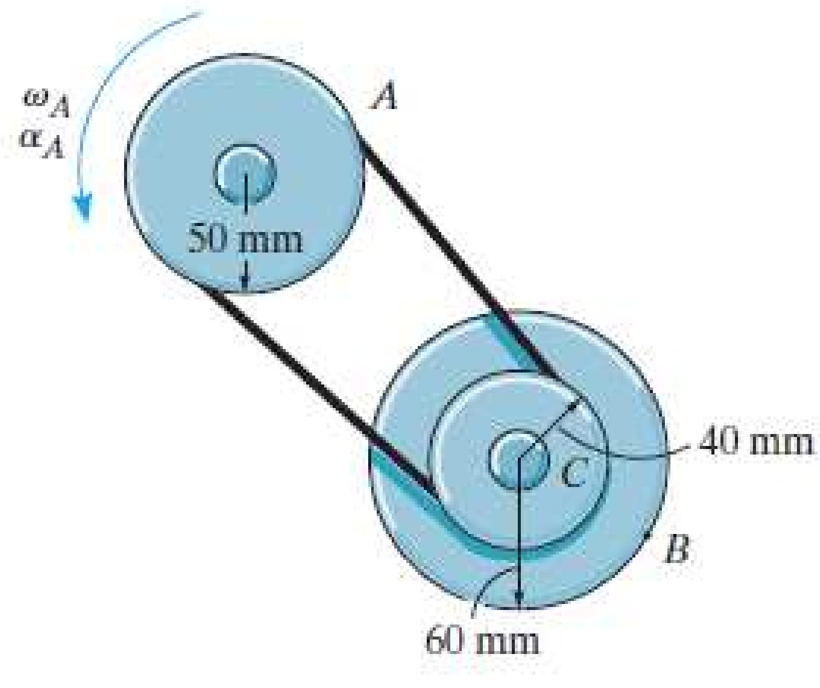
Angular Velocity: Applying Eq. $\omega=\dfrac{d\theta}{dt}$, we have

$\omega=\dfrac{d}{dt}\left(20t+4t^2\right)=20+8t$ and at $t=90\sec$, we have $\omega=20+8(90)=740\,rad/\sec$

Angular Acceleration: Applying Eq. $\dot{\omega}=\dfrac{d\omega}{dt}$, we have $\dot{\omega}=8$ A

$t=90\sec \quad \dot{\omega}=8\,rad/\sec^2$

Example 6: At the instant $\omega_A=5\ rad/\sec$ (it means initial the angular velocity), pulley A is given an angular acceleration $\dot{\omega}_A=6\,rad/\sec^2$. Determine the magnitude of acceleration of point B on pulley C when A rotates 2 revolutions. Pulley C has an inner hub which is fixed to its outer one and turns with it?



Solution

Given $\left(\omega_A\right)_0 = 5\ rad/\sec$, $\dot{\omega}_A = 6\, rad/\sec^2$,

Where the angular acceleration of pulley A is constant. So we have

$$\omega = \omega_o + \dot{\omega}\, t, \qquad \theta = \omega_o\, t + \frac{1}{2}\dot{\omega} t^2, \qquad \omega^2 = \omega_o^2 + 2\,\dot{\omega}\theta$$

When the pulley A rotates 2 revolutions. Then $N_A = \frac{\theta_A}{2\pi} = 2 \rightarrow \theta_A = 4\pi\ rev$

Angular Motion: The angular velocity of the pulley A can be determined from

$\omega^2 = \omega_o^2 + 2\,\dot{\omega}\theta$, we have $\omega^2 = (5)^2 + 2\,(6)\,(4\pi) = 25 + 48\,\pi = 175.79644$, $\omega = 13.2588\ rad/\sec$

Since pulleys A and C are connected by a non-slip belt. So, at any point on the pulleys A and C.

$v_A = v_C$, $\quad f_{A_t} = f_{C_t}$. Then

$$v_A = v_C \;\rightarrow\; \omega_A r_A = \omega_C r_C \;\rightarrow\; (13.2588)\,(50) = \omega_C (40) \;\rightarrow\; \omega_C = 16.57\ rad/\sec$$

Also $f_{A_t} = f_{C_t} \;\rightarrow\; r_A \dot{\omega}_A = r_C \dot{\omega}_C \rightarrow (50)\,(6) = (40)\,\dot{\omega}_C \rightarrow \quad \dot{\omega}_C = 7.5\ rad/\sec^2$

Motion of Point B. The tangential and normal component of acceleration of point B can be determined from,

$$f_{B_r} = (\omega^2)_C\,(r)_B = (16.57)^2\,(0.6\,m) = 164.739\,m/\sec^2$$

$$f_{B_t} = (\dot{\omega})_C\,(r)_B = (6)\,(0.6\,m) = 3.6\ m/\sec^2$$

$$f_B = \sqrt{(3.6)^2 + (164.77)^2} = 164.77\,m/\sec^2$$

Chapter 2

I. Mass Moment of Inertia

Definition of the Rigid body

In physics, a rigid body is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces exerted on it. A rigid body is usually considered as a continuous distribution of mass.

Definition of moment of inertia

Physical; A measure of the resistance of a body to angular acceleration about a given axis

For an object rotating about an axis, the resistance of a body to accelerate is called inertia of mass

Mathematic; The Moment of Inertia is equal to the sum of the products of each element of mass in the body and the square of the element's distance from the axis.

It is defined as the sum of second moment of area of individual section about an axis

(1) The basic shapes

(2) Systems of particles

(3) Composite bodies (shapes)

(4) Uninform shapes

The Moment of Inertia of mass (Second moment of mass)

The mass moment of inertia about a fixed axis is the property of a body that measures the body's resilience to rotational acceleration. The greater its value, the greater the moment required to provide a given acceleration about a fixed pivot. The moment of inertia must be specified with respect to a chosen axis of rotation.

(1)- For a single mass, the moment of inertia can is expressed as

For the element dm that is located a distance a from the L-axis, the Moment of inertia referenced to L-axis is given as

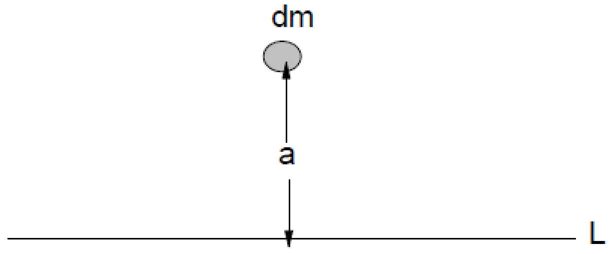


Fig. 1

$$I_{LL} = dm\,a^2$$

(2)- If a system consists of $n-$bodies, then the moment of inertia can be given as

For the $n-$elements, they have the mass $dm_1,\ dm_2,\ dm_3, \ldots\ldots, dm_n$ that is located a distance a from the L-axis, the moment of inertia referenced to L-axis is given as

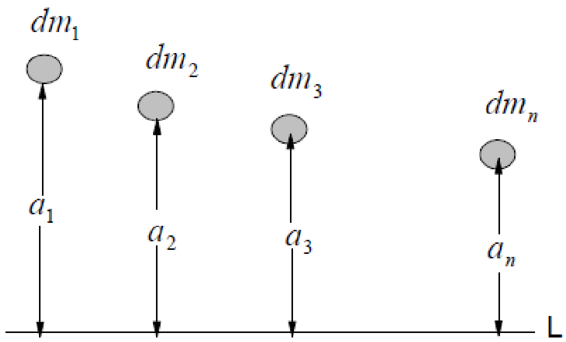


Fig. 2

$$I_{LL} = dm_1 a_1^2 + dm_2 a_2^2 + dm_3 a_3^2 + \dots\dots\dots + dm_n a_n^2 = \sum_{i=1}^{n} dm_i a_i^2$$

(3)- The Moment of Inertia in the plane

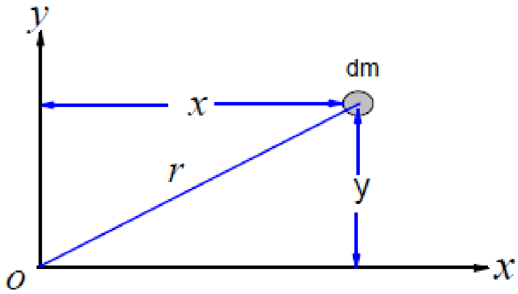


Fig. 3

Referenced to x-axis is given by $I_{xx} = dm\, y^2$,

Referenced to y-axis is given by $I_{yy} = dm\, x^2$,

Referenced to the original point (O) is given by

$$I_O = dm\, r^2 = m(x^2 + y^2) = I_{xx} + I_{yy}$$

I_O is called Polar moment inertial

(4)- The Moment of Inertia in the plane for number of elements

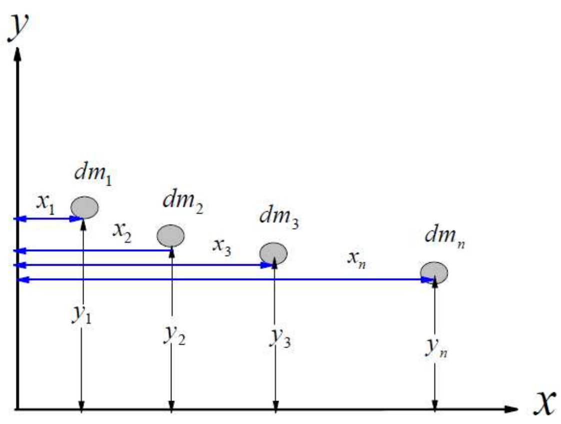


Fig. 4

Referenced to x-axis is given by $I_{xx} = \sum_{i=1}^{n} dm_i y_i^2$

Referenced to y-axis is given by $I_{yy} = \sum_{i=1}^{n} dm_i \, x_i^2$

(4)- The Moment of Inertia in space

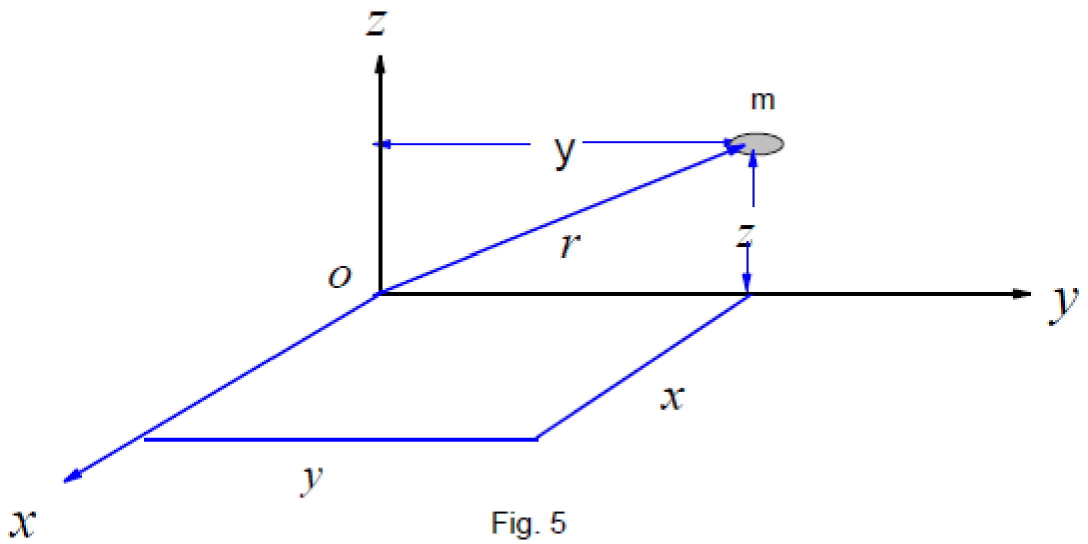


Fig. 5

Referenced to the original point (O) is given by

$$I_O = m r^2 = m(x^2 + y^2 + z^2) \qquad (1)$$

Referenced to x-axis is given by $I_{xx} = m(y^2 + z^2)$,

Referenced to y-axis is given by $I_{yy} = m(x^2 + z^2)$,

Referenced to z-axis is given by $I_x = m(x^2 + y^2)$,

Referenced to the plane $-x = 0$ is given by $I_{xx} = m(y^2 + z^2)$,

Referenced to the plane $-y=0$ is given by $I_y = m(x^2 + z^2)$,

Referenced to the plane $z=0$ is given by $I_z = m(x^2 + y^2)$,

From previous relation, we have

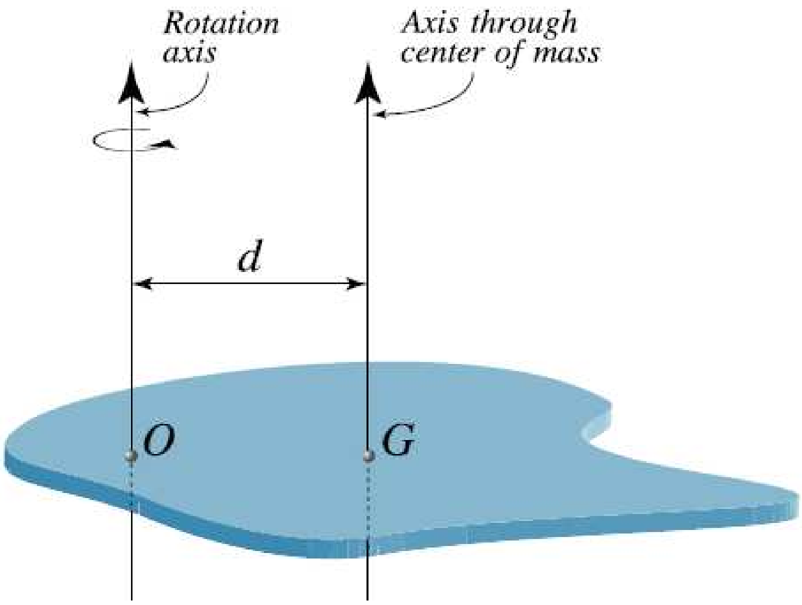
$$I_o = m r^2 = m(x^2 + y^2 + z^2) = I_{xoy} + I_{xoz} + I_{yoz}$$

$$I_o = mr^2 = m(x^2 + y^2 + z^2) = \frac{1}{2}(I_{xx} + I_{yy} + I_{zz}) \quad \text{or} \quad 2I_o = I_{xx} + I_{yy} + I_{zz}$$

$$I_{xx} = m(y^2 + z^2) = I_{xoy} + I_{xoz}$$
$$I_{yy} = m(x^2 + z^2) = I_{xoy} + I_{yoz}$$
$$I_{zz} = m(x^2 + y^2) = I_{xoz} + I_{yoz}$$

Parallel axis theorem

Parallel axis theorem is applicable to bodies of any shape. The theorem of parallel axis states that the moment of inertia of a body about an axis parallel to an axis passing through the centre of mass is equal to the sum of the moment of inertia of body about an axis passing through centre of mass and product of mass and square of the distance between the two axes. The parallel axis theorem is much easier to understand in equation form than in words. Here it is:



In physics, the parallel axis theorem can be used to determine the moment of inertia of a rigid object about any axis, given the moment of inertia of the object about the parallel axis through the object's center of mass and the perpendicular distance between the axes.
We consider an element (m) and its center is (x_{cm}, y_{cm}) (see below Figure)

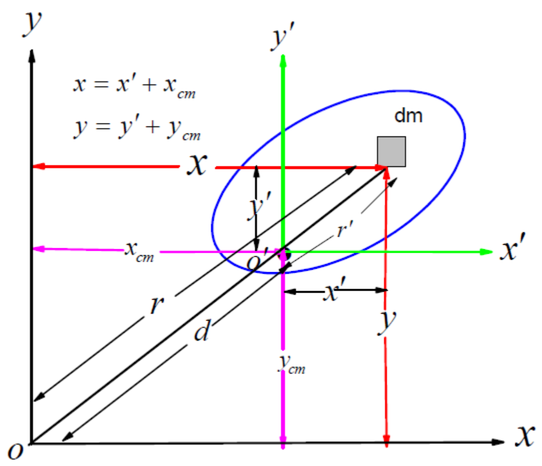


Fig. 7

$dI_{xx} = dm\, y^2$, the moment of inertial with respect to $x-$ axis

$dI_{yy} = dm\, x^2$, the moment of inertial with respect to $y-$ axis

$dI_O = dm\, r^2 = I_{xx} + I_{yy} = dm(x^2 + y^2)$, the moment of inertial with respect to the point(o)

$$I_O = \int r^2 dm = \int (x^2 + y^2) dm \qquad (1)$$

$$I_{cm} = \int r'^2 dm = \int (x'^2 + y'^2) dm \qquad (2)$$

$$x = x' + x_{cm}, \quad y = y' + y_{cm}$$

$$I_O = \int r^2 dm = \int \left\{ \left(x' + x_{cm} \right)^2 + \left(y' + y_{cm} \right)^2 \right\} dm$$

$$= \int \left\{ x'^2 + x_{cm}^2 + 2x' x_{cm} + y'^2 + y_{cm}^2 + 2y' y_{cm} \right\} dm$$

$$I_O = \underbrace{\int \left(x'^2 + y'^2 \right) dm}_{I_{cm}} + \int \underbrace{\left(x_{cm}^2 + y_{cm}^2 \right)}_{=d^2} dm + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm$$

$$I_O = I_{cm} + \int d^2\, dm + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm$$

$$I_O = I_{cm} + d^2 \int dm + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm$$

$$I_O = I_{cm} + d^2 m + 2x_{cm} \int x' dm + 2y_{cm} \int y' dm \qquad (3)$$

$$\bar{x} = \frac{\int x' dm}{\int dm} \rightarrow \int x' dm = \bar{x} \int dm, \qquad \bar{y} = \frac{\int y' dm}{\int dm} \rightarrow \int y' dm = \bar{y} \int dm \qquad (4)$$

$$I_O = I_{cm} + d^2 m + 2x_{cm} \left\{ \bar{x} \int dm \right\} + 2y_{cm} \left\{ \bar{y} \int dm \right\}$$

$$I_O = I_{cm} + d^2 m + 2x_{cm}\, \bar{x}\, m + 2y_{cm}\, \bar{y}\, m \qquad (5)$$

$I_O = I_{cm} + m d^2$ (6)

Question: Let I_A and I_B be moments of inertia of a body about two axes A and B respectively. The axis A passes through the centre of mass of the body but B does not, So.

(A) $I_A < I_B$ (B) $I_A > I_B$ (C) If the axes are parallel $I_A < I_B$

(D) If the axes are parallel $I_A > I_B$ (E) If the axes are not parallel $I_A > I_B$

The moment of inertia is always less for an axis passing through the center of mass than any other parallel axis. We cannot say anything of the moment of inertia about a non parallel axis. Thus C is correct.

Perpendicular Axis Theorem

This theorem is applicable only to the planar bodies. Bodies which are flat with very less or negligible thickness. This theorem states that the moment of inertia of a planar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with the perpendicular axis and lying in the plane of the body.

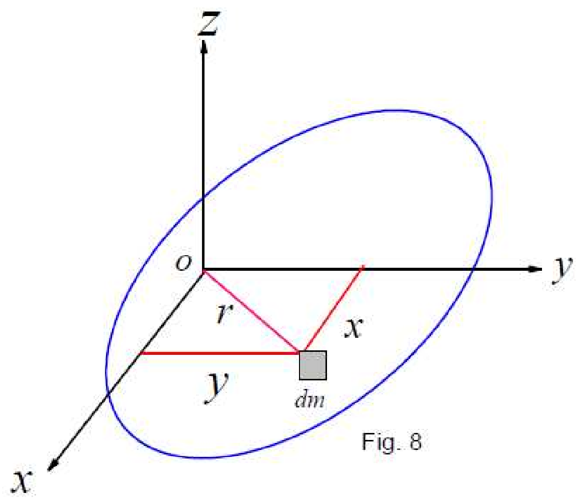


Fig. 8

$dI_{xx} = dm\, y^2$, the moment of inertial with respect to $x-$ axis

$dI_{yy} = dm\, x^2$, the moment of inertial with respect to $y-$ axis

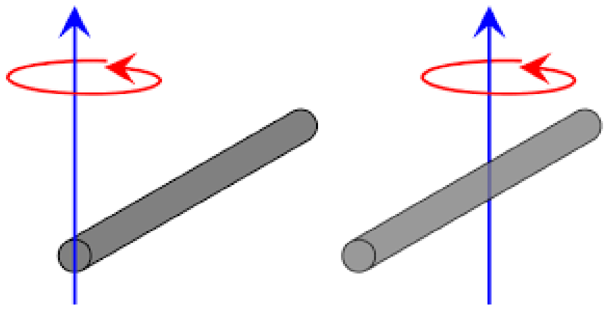
$dI_O = dm\, r^2 = I_{xx} + I_{yy} = dm(x^2 + y^2)$, , the moment of inertial with respect to the point (o)

$I_O = \int (x^2 + y^2)\, dm = \int r^2 dm = r^2 \int dm = r^2 m$ (1)

$I_{zz} = I_{xx} + I_{yy}$ (2)

Example:1 Find the Mass moment of inertia of a thin uniform rod?

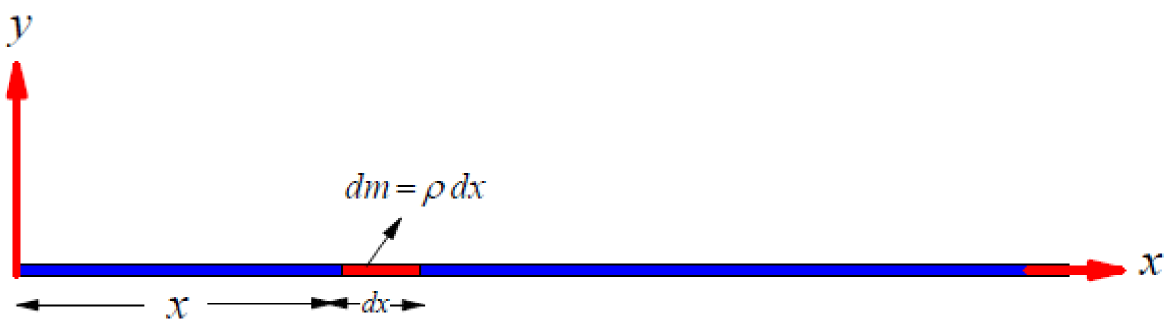
Solution

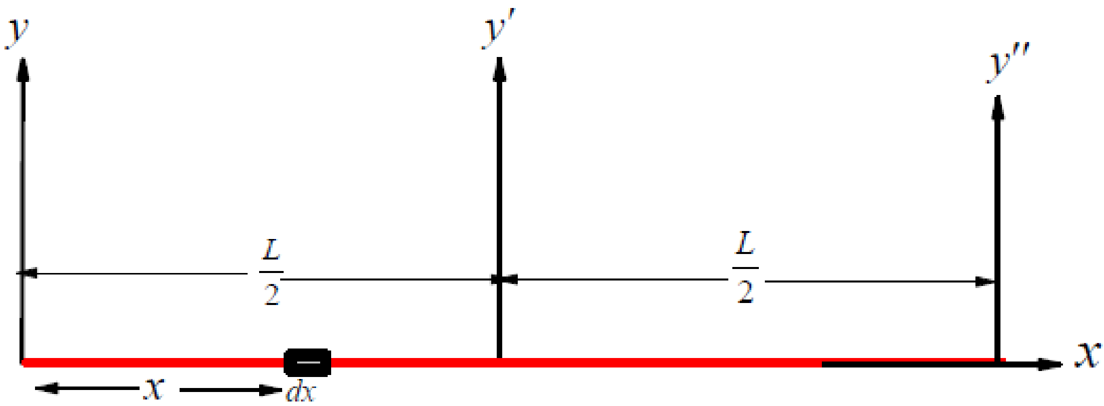


We consider L be the length of the Rod, M be the mass of the Rod and is the density ρ.

We divide the Rod into many small elements. We select one of them, that has length dx, mass dm and has the distance x from the left end of the Rod

For the small element $dm = \rho\, dx \rightarrow m = \int_0^L \rho\, dx = \rho \int_0^L dx = \rho\, x\Big|_0^L \rightarrow m = \rho\, L$





The moment of inertia about its end is given by

$$I_{yy} = \int x^2 dm = \int_0^L x^2(\rho\, dx) = \frac{1}{3}\rho L^3 = \frac{1}{3}\rho L^3 \frac{m}{\rho L^3} = \frac{1}{3} m L^2 \qquad \therefore\ I_{yy} = \frac{1}{3} m L^2$$

This the moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through one of its ends.

The moment of inertia of a thin uniform rod about an axis perpendicular to its length and passing through its center. From the Parallel axis theorem

$$I_{yy} = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \rightarrow \frac{1}{3} m L^2 = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \rightarrow\ I_{y'y'} = \frac{1}{3} m L^2 - \frac{1}{4} mL^2 = \left(\frac{4-3}{12}\right) mL^2 = \frac{1}{12} mL^2$$

$$\therefore\ I_{y'y'} = \frac{1}{12} mL^2$$

The moment of inertia about its other end is given as

$$I_{y''y''} = I_{y'y'} + m\left(\frac{1}{2}L\right)^2 \rightarrow I_{y''y''} = \frac{1}{12} m L^2 + m\left(\frac{1}{2}L\right)^2 = \frac{1}{12} m L^2 + \frac{1}{4} mL^2 = \left(\frac{1+3}{12}\right) mL^2 = \frac{4}{12} mL^2$$

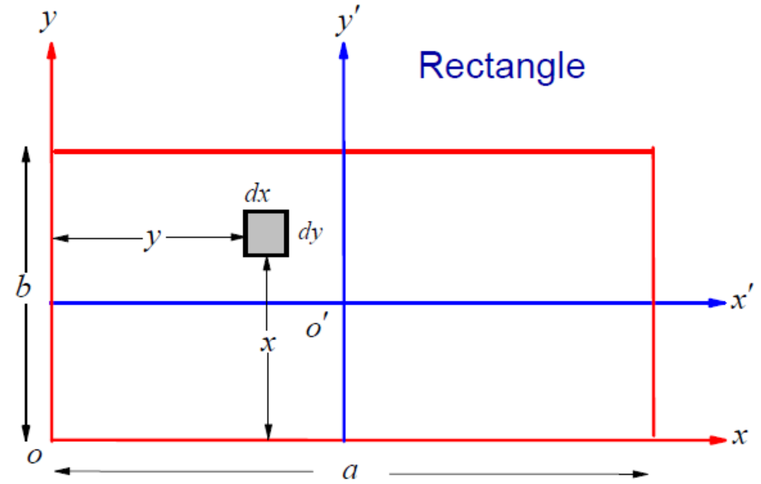
$$\therefore\ I_{y''y''} = \frac{1}{3} mL^2$$

Note: The moment of inertia for a thin uniform Rod that rotates about the axis perpendicular to the rod and passing through one end is $\frac{1}{3} mL^2$. If the axis of rotation passes through the center of the Rod, then the moment of inertia is $\frac{1}{12} mL^2$.

Example 2: Find the Mass moment of inertia of a thin uniform rectangular plate about its base and its one of edges axes?

Solution

We consider a uniform strip with the length (dx) and thickness (dy) as shown in below Figure, where the density is ρ.



$dm = \rho\, dx\, dy \rightarrow m = \rho \int_0^b \int_0^a dx\, dy \rightarrow m = \rho a b$. The moment of inertia about its corner is given by

$$dI_{yy} = x^2 dm = \rho x^2\, dx\, dy \quad \rightarrow I_{yy} = \rho \int_0^b \int_0^a x^2\, dx\, dy = \rho \left[\frac{x^3}{3} \right]_0^b [y]_0^b = \frac{b a^3}{3} \rho = \frac{b a^3}{3} \rho \frac{m}{\rho a b} \quad \therefore I_{yy} = \frac{1}{3} m a^2$$

If we select a vertical strip (sector, section), we have

$$dI_{yy} = x^2 dm = \rho x^2 (b\, dx) \quad \rightarrow I_{yy} = \rho b \int_0^a x^2\, dx = \rho b \left[\frac{x^3}{3} \right]_0^a = \frac{b a^3}{3} \rho = \frac{b a^3}{3} \rho \frac{m}{\rho a b}$$

$$\therefore I_{yy} = \frac{1}{3} m a^2 \qquad I_{yy} = I_{y'y'} + m\left(\frac{1}{2} a\right)^2 \rightarrow$$

$$\frac{1}{3} m a^2 = I_{y'y'} + m\left(\frac{1}{2} a\right)^2 \rightarrow I_{y'y'} = \frac{1}{3} m a^2 - \frac{1}{4} m a^2 = \left(\frac{4-3}{12}\right) m a^2 = \frac{1}{12} m a^2$$

$\therefore I_{y'y'} = \dfrac{1}{12} m a^2$ Similarly, if we select a horizontal strip, we can prove that:

$$I_{xx} = \frac{1}{3} m b^2, \qquad I_{x'x'} = \frac{1}{12} m b^2$$

For axis is perpendicular ox, oy $I_{zz} = I_{xx} + I_{yy} = \dfrac{1}{3} m b^2 + \dfrac{1}{3} m a^2 = \dfrac{1}{3} m (a^2 + b^2)$

For axis is perpendicular ox', oy': $I_{z'z'} = I_{x'x'} + I_{y'y'} = \dfrac{1}{12} m b^2 + \dfrac{1}{12} m a^2 = \dfrac{1}{12} m (a^2 + b^2)$

The moment of inertia about its corner is given by (Mass *moment of inertia)*

$I_{xx} = \frac{1}{3} m b^2 = \frac{1}{3}(ab) b^2 = \frac{1}{3} a b^3,$ $I_{yy} = \frac{1}{3} b a^3$ $I_o = I_{xx} + I_{yy} = \frac{1}{3} ab(a^2 + b^2)$

$I_{x'x'} = \frac{1}{12} a b^3,$ $I_{y'y'} = \frac{1}{12} b a^3$ $I_{o'} = I_{x'x'} + I_{y'y'} = \frac{1}{12} a b (a^2 + b^2)$

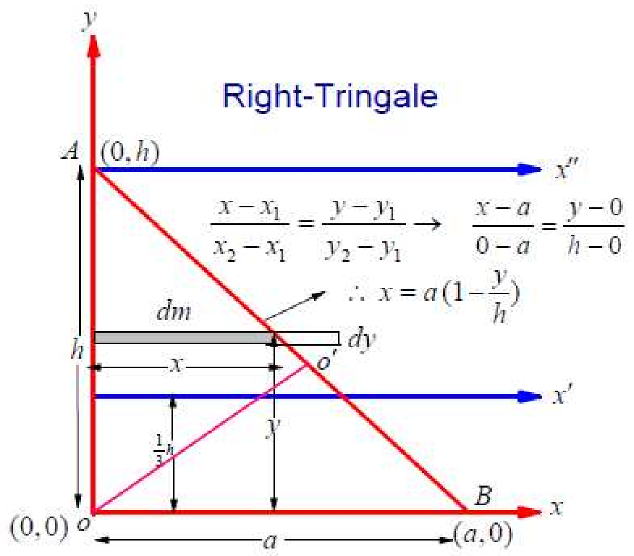
Uniform rectangular plate (a,b)	*Axis coincides with one of its sides*	*Axis passing through its centroid*	*Axis coincides to other side*
With respect to $I_{yy}-$ axis	$I_{yy} = \frac{1}{3} m a^2$	$I_{y'y'} = \frac{1}{12} m a^2$	$I_{y''y''} = \frac{1}{3} m a^2$
With respect to $I_{xx}-$ axis	$I_{xx} = \frac{1}{3} m b^2$	$I_{x'x'} = \frac{1}{12} m b^2$	$I_{x''x''} = \frac{1}{3} m b^2$
With respect to axis perpendicular to the plane oxy	$I_{zz} = \frac{1}{3} m(a^2 + b^2)$	$I_{z'z'} = \frac{1}{12} m(a^2 + b^2)$	$I_{z''z''} = \frac{1}{3} m(a^2 + b^2)$

Example 3: Determine the mass moment of inertia for right Triangular Plate (Right-angled triangle)?

Solution

We consider a uniform strip with the length (x) and thickness (dy), such that it is parallel to x- axis, as shown in below Figure. Then

$$dm = \rho x dy \rightarrow m = \rho \int_0^h x dy = \rho \int_0^h a(1 - \frac{y}{h}) dy = a\rho \left[y - \frac{y^2}{2h} \right]_0^h = a\rho \left[h - \frac{h^2}{2h} \right] \rightarrow m = \frac{1}{2} a h \rho$$



Then moment of Inertia with respect to $x-$ axis:

$$dI_{xx}=y^2 dm=\rho\, x\, y^2\, dy \quad \rightarrow I_{xx}=\rho\int_0^h x\, y^2\, dy \ \text{, but } \frac{x}{a}+\frac{y}{h}=1 \ \rightarrow \ x=a(1-\frac{y}{h})$$

$$I_{xx}=\rho\int_0^h a(1-\frac{y}{h})\, y^2\, dy=\rho\, a\int_0^h (y^2-\frac{y^3}{h})\, dy=\rho\, a\left[\frac{y^3}{3}-\frac{y^4}{4h}\right]_0^h$$

$$I_{xx}=\rho\, a\left[\frac{h^3}{3}-\frac{h^4}{4h}\right]=\frac{1}{12}\rho\, a\, h^3\,(4-3)=\frac{1}{12}\rho\, a\, h^3\,\frac{m}{\frac{1}{2}a\, h\rho}=\frac{1}{6}m\, h^2 \qquad \therefore\ I_{xx}=\frac{1}{6}m h^2$$

Then moment of Inertia with respect to $x'-$ axis:

$$I_{xx}=I_{x'x'}+m\left(\frac{1}{3}h\right)^2 \rightarrow \quad I_{x'x'}=\frac{1}{6}m h^2-\frac{1}{9}m h^2=\frac{1}{18}m h^2(3-2)=\frac{1}{18}m h^2 \qquad I_{x'x'}=\frac{1}{18}m h^2$$

Then moment of Inertia with respect to $x''-$ axis:

$$I_{x''x''}=I_{x'x'}+m\left(\frac{2}{3}h\right)^2=\frac{1}{18}m h^2+\frac{4}{9}m h^2=\frac{1}{18}m h^2(1+8)=\frac{9}{18}m h^2 \qquad I_{x''x''}=\frac{1}{2}m h^2$$

Also, $I_{yy}=\frac{1}{6}m a^2,\qquad I_{y'y'}=\frac{1}{18}m a^2,\qquad I_{y''y''}=\frac{1}{2}m a^2.$

$$I_{zz}=I_{xx}+I_{yy}=\frac{1}{6}m a^2+\frac{1}{6}m h^2=\frac{1}{6}m(a^2+h^2) \quad I_{z'z'}=I_{x'x'}+I_{y'y'}=\frac{1}{18}m a^2+\frac{1}{18}m h^2=\frac{1}{18}m(a^2+h^2)$$

Again, $I_{AB}=\frac{1}{6}m(oo')^2$

where $\frac{1}{2}(oo')\,AB$, $AB=\sqrt{(0-a)^2+(h-0)^2}=\sqrt{a^2+h^2}$

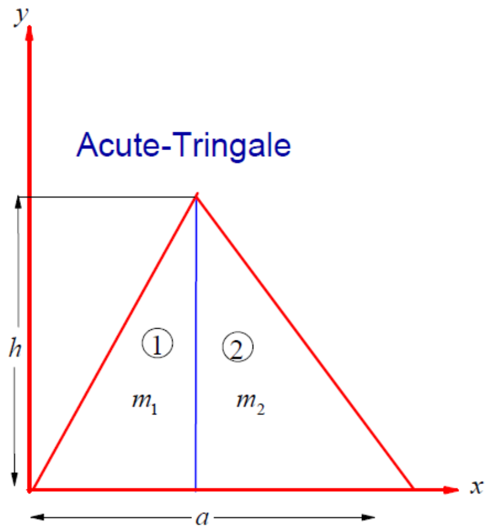
$$\frac{1}{2}a\,h=\frac{1}{2}(oo')\,AB=\frac{1}{2}(oo')\sqrt{a^2+h^2}\rightarrow oo'=\frac{a\,h}{\sqrt{a^2+h^2}}\text{ , Also } I_{AB}=\frac{1}{6}m(oo')^2=\frac{a^2h^2}{6(a^2+h^2)}m$$

Right Triangular Plate of height h and bass a	About its corner	*About its center of mass*	*About its* vertex
About its base	$I_{xx}=\frac{1}{6}mh^2$	$I_{x'x'}=\frac{1}{18}mh^2$	$I_{x''x''}=\frac{1}{2}mh^2$
About its height	$I_{yy}=\frac{1}{6}ma^2$	$I_{y'y'}=\frac{1}{18}ma^2$	$I_{y''y''}=\frac{1}{2}ma^2$
About vertical axis	$I_{zz}=\frac{1}{6}m(a^2+h^2)$	$I_{z'z'}=\frac{1}{18}m(a^2+h^2)$	$I_{z''z''}=\frac{1}{6}m(3a^2+h^2),\ I_{z''z''}=\frac{1}{6}m(a^2+3h^2)$

Example 4: The Mass Moment of inertia of acute triangular plate?

Solution

We divide the *acute triangular plate to two right triangular plate as is shown in Figure*



The Moment of inertia of about $x-$ axis for the two right triangular plate is given as

$$(I_{xx})_1=\frac{1}{6}m_1h^2,\qquad (I_{xx})_2=\frac{1}{6}m_2h^2,$$

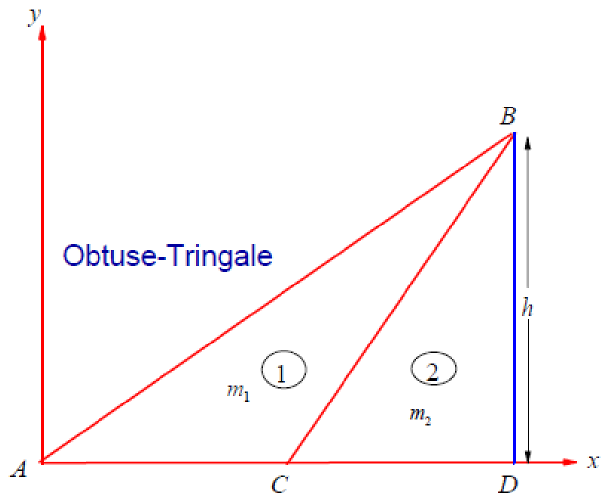
For the acute triangular plate

$$I_{xx} = \left(I_{xx}\right)_1 + \left(I_{xx}\right)_2 = \frac{1}{6}m_1 h^2 + \frac{1}{6}m_2 h^2 = \frac{1}{6}(m_1 + m_2)h^2 = \frac{1}{6}mh^2$$

Example 5: The Mass Moment of inertia of obtuse triangular plate?

Solution

We divide the *obtuse triangular plate to two right- triangular plate as is shown below Figure*



The Moment of inertia of about $x-$ axis for the two right triangular plate is given as

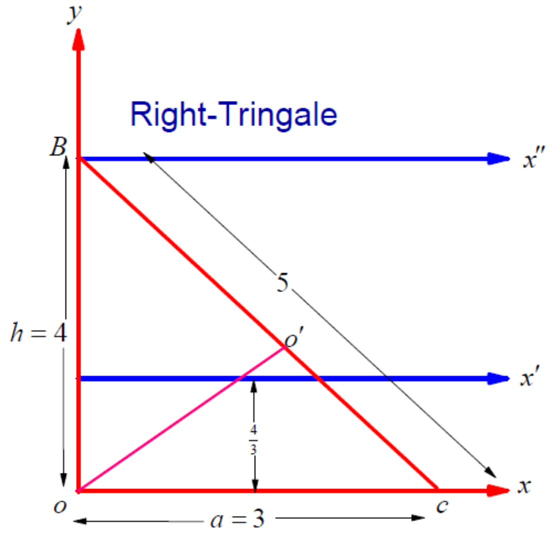
$$\left(I_{xx}\right)_{ABD} = \frac{1}{6}(m_1 + m_2)h^2, \quad \left(I_{xx}\right)_{CBD} = \frac{1}{6}m_2 h^2$$

For the acute triangular plate

$$(I_{xx})_{ABC} = \left(I_{xx}\right)_{ABD} + \left(I_{xx}\right)_{CBD} = \frac{1}{6}(m_1 + m_2)h^2 - \frac{1}{6}m_2 h^2 = \frac{1}{6}m_1 h^2$$

Example 6 : Find the Mass Moment of inertia of right- *triangular plate as is shown in figure about all different axes?*

Solution



From the Figure it is clear that $I_{xx} = \frac{1}{6} m h^2$, $I_{yy} = \frac{1}{6} m a^2$, $I_{BC} = \frac{a^2 h^2}{6(a^2 + h^2)} m$

$$I_{xx} = \frac{1}{6} m h^2 = \frac{1}{6} m (4)^2 = \frac{16}{6} m = \frac{8}{3} m, \qquad I_{yy} = \frac{1}{6} m a^2 = \frac{1}{6} m (3)^2 = \frac{9}{6} m = \frac{3}{2} m$$

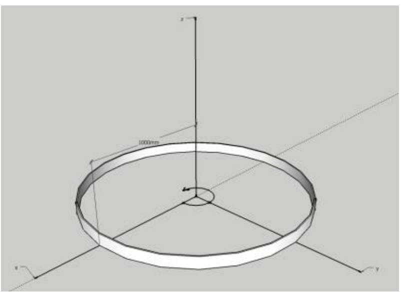
$$I_{BC} = \frac{a^2 h^2}{6(a^2 + h^2)} m = \frac{(3)^2 (4)^2}{6((3)^2 + (4)^2)} m = \frac{(9)(16)}{6(9+16)} m = \frac{(9)(16)}{6(25)} m = \frac{24}{25} m$$

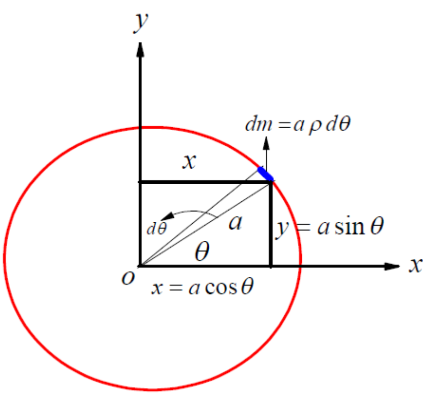
Note that $3 < 4 < 5, \quad I_{xx} = \frac{8}{3} m > I_{yy} = \frac{3}{2} m > I_{BC} = \frac{24}{25} m$

Example 7: The Mass Moment of inertia of Circular Ring?

Solution

We select a small element has the mass dm at any point located at distance (x, y) from the origin point





The Moment of inertia about $z-$axis (The axis is passing through the center (z-axis) and is perpendicular to the Ring) is given as

$$dI_{zz} = a^2 dm \,\ldots\ldots\ldots \quad I_{zz} = \int a^2\, dm = a^2 \int_0^m dm \rightarrow \quad I_{zz} = a^2 m$$

From the Perpendicular axis theorem (Here, the distance between the tangent and the diameter is a) $I_{zz} = I_{xx} + I_{yy}$. So $I_{xx} + I_{yy} = ma^2$

But I_{xx} and I_{yy} are symmetric, so $I_{xx} = I_{yy}$, Then

$I_{xx} = I_{yy} = \frac{1}{2} ma^2$ (The moment of inertia of a ring about of its diameter or the axis passes through the diameter)

From the parallel axis theorem $I_{y'y'} = I_{yy} + ma^2 \rightarrow I_{y'y'} = \frac{1}{2} ma^2 + ma^2 \rightarrow I_{y'y'} = \frac{1}{2} ma^2$

$$I_{x'x'} = I_{yy} + ma^2 \rightarrow I_{x'x'} = \frac{1}{2} ma^2 + ma^2 \rightarrow I_{x'x'} = \frac{1}{2} ma^2$$

Moment of inertia about an axis is passing through the edge of Ring and perpendicular to its plane and parallel an axis is passing through the center (z-axis) and is perpendicular to the Ring

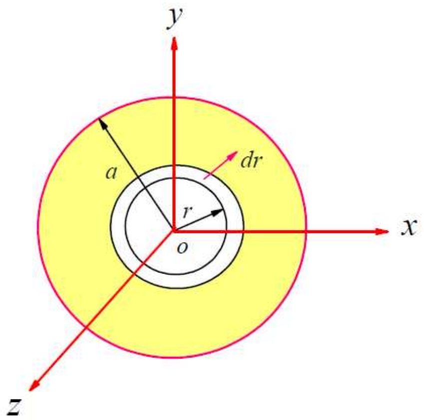
$$I_{z'z'} = I_{zz} + ma^2 \rightarrow I_{z'z'} = ma^2 + ma^2 \rightarrow I_{z'z'} = 2ma^2$$

Circular Ring	For Vertical axis	About axis in the plane of Circular Ring and passes in the its center *The moment of inertia of the ring about of its diameter*
Axis of rotation	$I_{zz} = ma^2$	$I_{xx} = I_{yy} = \frac{1}{2} ma^2$
Axis of rotation	$I_{z'z'} = 2ma^2$	$I_{x'x'} = I_{y'y'} = \frac{3}{2} ma^2$

Example 8: Find the Mass Moment of inertia of Circular area *?*

Solution

We divide the Circular area to the *small Circular Rings, we selected one of them has mass* (dm)*,thickness* (dr) *and raids* (r).



So, $dm = 2\pi r \rho dr \rightarrow m = 2\pi \rho \int_0^a r\, dr \rightarrow m = 2\pi \rho \left. \frac{r^2}{2} \right|_0^a = \pi a^2 \rho$

$$I_{zz} = \int r^2\, dm = \int r^2 (2\pi r \rho dr) = 2\pi\rho \int_0^a r^3\, dr = \left. \frac{2\pi\rho r^4}{4} \right|_0^a = \frac{\pi\rho a^4}{2} = \frac{\pi\rho a^4}{2} \frac{m}{m} = \frac{\pi\rho a^4}{2} \frac{m}{\pi a^2 \rho}$$

$$I_{zz} = \frac{1}{2} m a^2$$

From the Perpendicular axis theorem

$I_{zz} = I_{xx} + I_{yy}$. So $I_{xx} + I_{yy} = \frac{1}{2} m a^2$.

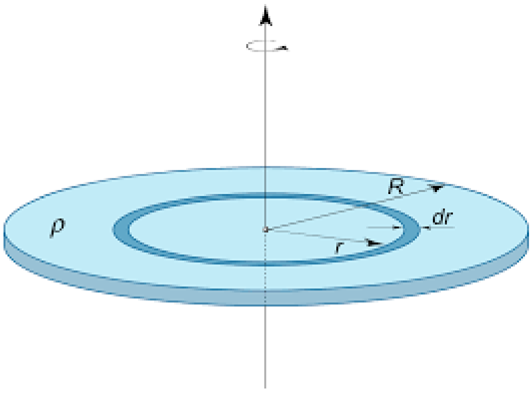
But I_{xx}, I_{yy} are symmetric, so $I_{xx} = I_{yy}$. Then $I_{xx} = I_{yy} = \frac{1}{4} m a^2$

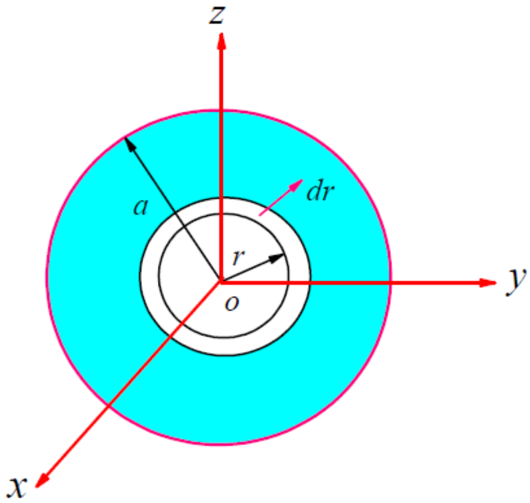
Circular area	For Vertical axis	About axis in the plane of Circular Ring and passes in the its center
Axis of rotation	$I_{zz} = \frac{1}{2} m a^2$	$I_{xx} = I_{yy} = \frac{1}{4} m a^2$
Axis of rotation	$I_{z'z'} = \frac{3}{2} m a^2$	$I_{x'x'} = I_{y'y'} = \frac{5}{4} m a^2$

Example 9: Find the Moment of inertia of Thin Disc?

Solution

We divide the solid Disc to the small Circular Rings, we selected one of them has mass (dm),thickness (dr),distraction thickness (Δz) and raids (r).





$$dm = 2\pi\, r\rho\,\Delta z\, dr \rightarrow m = 2\pi\,\rho\,\Delta z\int_0^a r\,dr \rightarrow m = 2\pi\,\rho\,\Delta z\left.\frac{r^2}{2}\right|_0^a = \pi\, a^2\rho\,\Delta z$$

So, the Moment of inertia of thin Disc is

$$I_{zz} = \int r^2\, dm = \int r^2\,(2\pi\, r\rho\,\Delta z\, dr) = 2\pi\rho\,\Delta z\int_0^a r^3\, dr = 2\pi\rho\,\Delta z\left.\frac{r^4}{4}\right|_0^a = \pi\rho\,\Delta z\,\frac{a^4}{2}$$

$$I_{zz} = \frac{\pi\rho\, a^4}{2}\frac{m}{m} = \frac{\pi\rho\Delta z\, a^4}{2}\frac{m}{\pi\, a^2\rho\,\Delta z} \rightarrow I_{zz} = \frac{1}{2}ma^2$$

From the Parallel axis theorem $I_{z'z'} = I_{zz} + ma^2 \rightarrow I_{z'z'} = \frac{3}{2}ma^2$

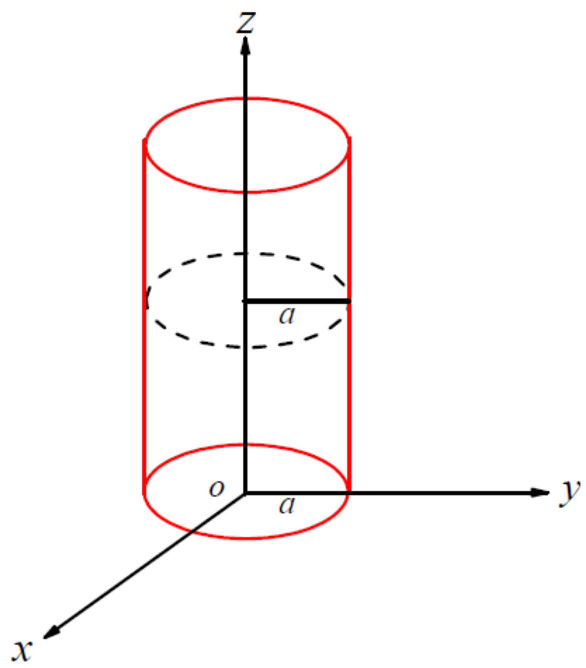
From the Perpendicular axis theorem $I_{zz} = I_{xx} + I_{yy}$. So $I_{xx} + I_{yy} = \frac{1}{2}ma^2$.

But I_{xx}, I_{yy} are symmetric, so $I_{xx} = I_{yy}$. Then $I_{xx} = I_{yy} = \frac{1}{4}ma^2$

Example: 10: Derive the *Mass* moment of inertia of Hollow Cylinder?

Solution

Take the hollow cylinder as the corresponding shape, divide it into an infinite number of regular circular rings and take one of these rings with the mass (dm) and the radius (a).



Then the moment of inertia of this ring is given as $dI_{zz} = a^2 dm$.

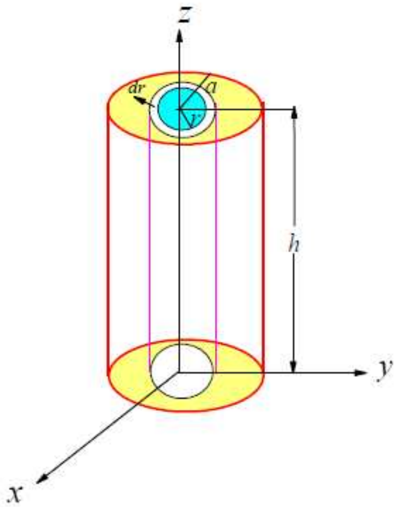
Then, the total moment of Hollow Cylinder

$$I_{zz} = \int_0^m a^2\, dm = m a^2 \quad \rightarrow \quad I_{zz} = m a^2$$

Example: 11: Derive the Mass moment of inertia of Solid Cylinder?

Solution

We divide the Solid Cylinder it into an infinite number of thin discs and take one of these discs with the mass (dm) and the radius (a).

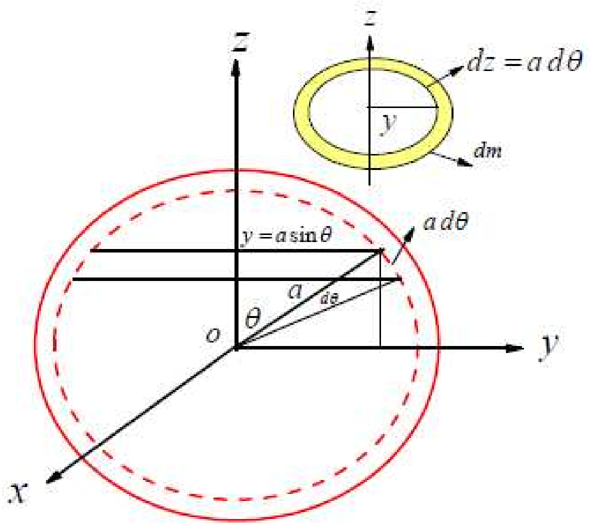


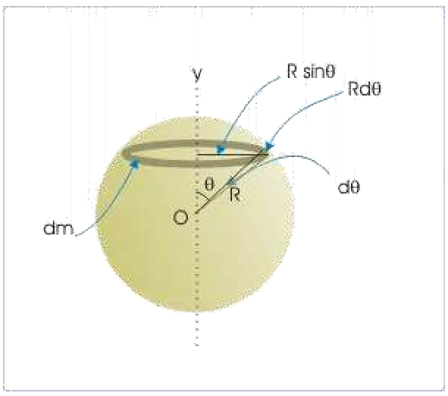
Then the moment of inertia of this disc is given as. $dI_{zz} = \frac{1}{2} a^2 \, dm$. Then the total moment of

Hollow Cylinder $I_{zz} = \int_0^m \frac{1}{2} a^2 \, dm = \frac{1}{2} m a^2 \quad \rightarrow \quad I_{zz} = \frac{1}{2} m a^2$

Example: 11: Derive the Mass moment of inertia of Hollow Sphere?

Solution





We divide the Hollow Sphere into a number of small circular rings and we consider one of them with the mass (dm), the radius (y) and thickness (dz).

$$dm = 2\pi \, y \, \rho \, dz = 2\pi (a \sin\theta) \rho \, a \, d\theta \rightarrow m = 2\pi \rho \, a^2 \int_0^{\pi} \sin\theta \, d\theta \rightarrow m = -2\pi \rho \, a^2 \cos\theta \Big|_0^{\pi} =$$
$$= -2\pi \rho \, a^2 \left(\cos(\pi) - \cos(0)\right) = -2\pi \rho \, a^2 \left(-1-1\right) = 2\pi \rho \, a^2 \left(1+1\right) == 4\pi \rho \, a^2$$

The moment of inertia of this circular ring is given as $dI_{zz} = y^2 dm$.

Then the total moment of Hollow Cylinder $I_{zz} = \int y^2 \, dm$, then

$$I_{zz} = \int y^2 \, dm = 2\pi \rho a^4 \int_0^{\pi} (\sin)^2 \sin\theta \, d\theta = 2\pi \rho a^4 \left[\int_0^{\pi} (1-\cos^2\theta)\sin\theta \, d\theta \right]$$

$$= 2\pi \rho a^4 \left[\int_0^{\pi} \sin\theta \, d\theta - \int_0^{\pi} (\cos\theta)^2 \, d(-\sin\theta) \right]$$

$$= 2\pi \rho a^4 \left[-\cos\theta + \frac{1}{3}(\cos\theta)^3 \right]_0^{\pi} = 2\pi \rho a^4 \left[-\cos(\pi) + \frac{1}{3}(\cos(\pi))^3 - \left\{ -\cos(0) + \frac{1}{3}(\cos(0))^3 \right\} \right]$$

$$= 2\pi \rho a^4 \left[1 + \frac{1}{3} - \left\{ -1 + \frac{1}{3} \right\} \right] = 2\pi \rho a^4 \left[1 + \frac{1}{3} + 1 - \frac{1}{3} \right] = 2\pi \rho a^4 \left[2 - \frac{2}{3} \right] = \frac{8}{3}\pi \rho a^4$$

$$I_{zz} = \frac{8}{3}\pi \rho a^4 \frac{m}{4\pi a^2 \rho} = \frac{2}{3} m a^2 \qquad \textit{Then} \quad I_{zz} = \frac{2}{3} m a^2$$

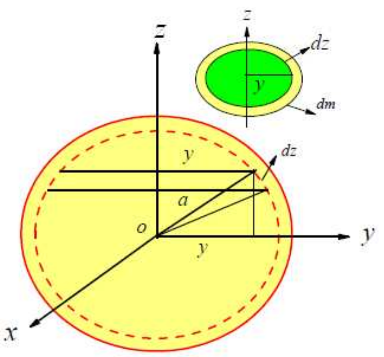
For the symmetric of axes $I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} m a^2$

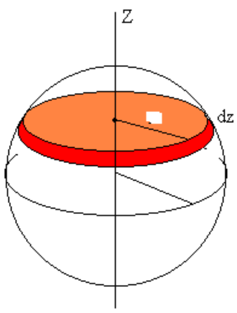
Also, we know $I_{xx} + I_{yy} + I_{zz} = 2I_o$, $\quad 2I_o = \frac{2}{3}ma^2 + \frac{2}{3}ma^2 + \frac{2}{3}ma^2 = \frac{6}{3}ma^2 = 2ma^2$

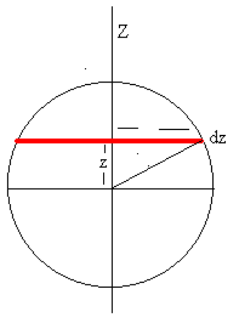
$$I_o = m a^2$$

Example: 12: Derive the Mass moment of inertia of Solid Sphere?

Solution







We divide the solid sphere into a number of hollow sphere and take one of these sphere with mass (dm), radius (r) and thickness (dr). Then the moment inertia of this sphere around oz

axis is $dI_{zz}=\frac{2}{3}(dm)r^2$, for whole sphere the moment inertia is given as $I_{zz}=\int\frac{2}{3}(dm)r^2$, where

$dm=4\pi r^2\rho\, dr\to\ m=4\pi\ \rho\int_0^a r^2\ dr=\frac{4}{3}\pi\ a^3\rho$. Then

$$I_{zz}=\int\frac{2}{3}(dm)r^2=\int_0^a\frac{2}{3}\left(4\pi r^2\rho\ dr\right)r^2=\frac{8}{3}\pi\ \rho\int_0^a r^4\,dr=\frac{8}{3}\pi\ \rho\frac{r^4}{5}\Bigg|_0^a=\frac{8}{15}\pi\ \rho a^5$$

$$I_{zz}=\frac{8}{15}\pi\ \rho a^5\frac{m}{m}=\frac{8}{15}\pi\ \rho a^5\ \frac{m}{\frac{4}{3}\rho a^3}=\frac{2}{5}ma^2\qquad \text{Then}\quad I_{zz}=\frac{2}{5}ma^2$$

Where the axes are Symmetrical $I_{xx}=I_{yy}=I_{zz}=\frac{2}{5}ma^2$

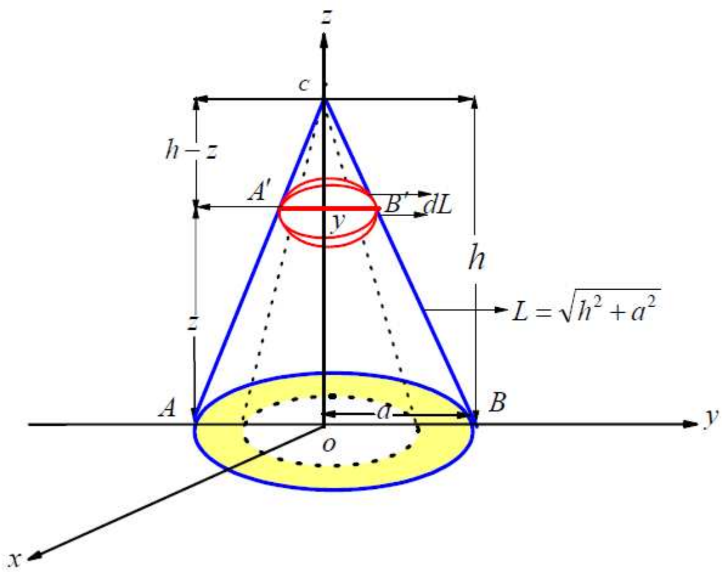
Also $I_{xx}+I_{yy}+I_{zz}=2I_o$, Then $2I_o=\frac{2}{5}ma^2+\frac{2}{5}ma^2+\frac{2}{5}ma^2=\frac{6}{5}ma^2$ $\quad I_o=\frac{3}{5}ma^2$

Example: 13: Find the Mass moment inertial for the Hollow Circular Cone ?

Solution

Divide the Hollow Circular Cone into a number of small circular rings and take one of these rings with mass (dm), radius (y) and thickness (dL), which is located higher (z) than the base of the cone with radius (a). Note that it is similar to triangles ABC and $A'B'C$, we have

$$\frac{h-z}{h}=\frac{y}{a}\ \to\ y=\frac{a}{h}(h-z)\to\ z=\frac{h}{a}(a-y)$$



The moment of inertia of this circular ring is given as $dI_{zz} = y^2 dm$.

Then the total moment of Hollow Circular Cone $I_{zz} = \int y^2\, dm$

Note that $dm = 2\pi\, y\, \rho\, dL \rightarrow\; m = 2\pi\, \rho \int_0^h y\, dL$, where

$$dL = \sqrt{1+\left(\frac{dz}{dy}\right)^2}\; dy = \sqrt{1+\left(\frac{h}{a}\right)^2}\; dy = \frac{1}{a}\sqrt{a^2+h^2}\; dy = \frac{L}{a}\, dy$$. Then

$$dm = 2\pi\, \rho \int_0 y\, dL = 2\pi\, \rho \int_0^a y \frac{L}{a} dy = \; 2\pi\, \rho \frac{L}{a}\frac{y^2}{2}\bigg|_0^a = 2\pi\, \rho \frac{L}{a}\frac{a^2}{2} \rightarrow m = \pi\, a\, L\, \rho$$. Then

$$I_{zz} = \int_0^a y^2\, dm = \int_0^a y^2 (2\pi\, y\, \rho\; dL) = 2\pi\;\; \rho \int_0^a y^3 \frac{L}{a} dy = 2\pi\;\; \rho \frac{L}{a}\frac{y^4}{4}\bigg|_0^a = 2\pi\;\; \rho \frac{L}{a}\frac{a^4}{4} = \pi\, L\rho\; \frac{a^3}{2}$$

$$= \pi\, L\rho\; \frac{a^3}{2}\; \frac{m}{m} = \pi\, L\rho\; \frac{a^3}{2}\;\; \frac{m}{\pi\, a\, L\, \rho} = \frac{1}{2} m a^2 \qquad\qquad I_{zz} = \frac{1}{2} m a^2$$

Again, $dL = \sqrt{1+\left(\frac{dy}{dz}\right)^2}\; dz = \sqrt{1+\left(\frac{a}{h}\right)^2}\; dz = \frac{1}{h}\sqrt{h^2+a^2}\; dz = \frac{L}{h}\, dz$

$$dm=2\pi\rho\int_0 y\,dL = 2\pi\rho\int_0^h \frac{a}{h}z\,\frac{L}{h}dz = 2\pi\rho a\frac{L}{h^2}\frac{z^2}{2}\Big|_0^h = 2\pi\rho a\frac{L}{h^2}\frac{h^2}{2} \to m=\pi a L\rho$$

$$I_{zz}=\int_0^a y^2\,dm=\int_0^a y^2(2\pi\,y\,\rho\,dL)=2\pi\;\rho\int_0^h\left(\frac{a}{h}z\right)^3\frac{L}{h}dz=2\pi\;\rho\frac{a^3L}{h^4}\int_0^h z^3\,dz==2\pi\;\rho\frac{a^3L}{h^4}\frac{z^4}{4}\Big|_0^a$$

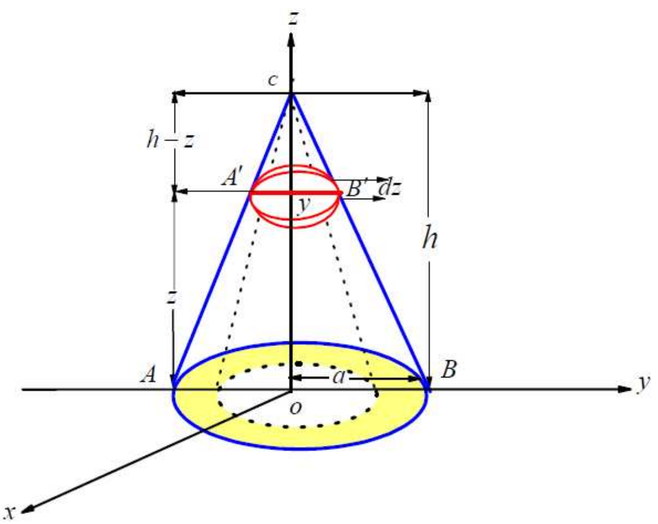
$$=2\pi\;\rho\frac{a^3L}{h^4}\frac{h^4}{4}=\pi L\rho\frac{a^3}{2}=\pi L\rho\frac{a^3}{2}\frac{m}{m}=\pi L\rho\frac{a^3}{2}\frac{m}{\pi a L\rho}=\frac{1}{2}ma^2 \qquad I_{zz}=\frac{1}{2}ma^2$$

Example 14: Find the Mass moment inertial for the Solid Circular Cone?

Solution

We divide the Solid Circular Cone into a number of small Disks and take one of them with mass (dm), radius (y) and thickness (dz), which is located higher (z) than the base of the cone with radius (a). Note that it is similar to triangles ABC and $A'B'C$, we have

$$\frac{h-z}{h}=\frac{y}{a} \;\to y=\frac{a}{h}(a-z)\to z=\frac{h}{a}(a-y)$$



$$dm=\pi\,y^2\,\rho\,dz\to\; m=\pi\,\rho\int_0^h\left(\frac{a}{h}(h-z)\right)^2 dz=\pi\,\rho\frac{a^2}{h^2}\int_0^h\left(h^2-2hz+z^2\right)dz$$

$$=\pi\,\rho\frac{a^2}{h^2}\left(h^2z-2h\frac{z^2}{2}+\frac{z^3}{3}\right)\Big|_0^h=\pi\,\rho\frac{a^2}{h^2}\left(h^3-h^3+\frac{h^3}{3}\right)\to m=\frac{1}{3}\pi\,a^2\,h\,\rho$$

The moment of inertia of this Disk is given as $dI_{zz}=y^2dm$.

Then the total moment of Solid Circular Cone $I_{zz}=\int y^2\,dm$, that is given

$$I_{zz}=\frac{1}{2}\int_0^a y^2\,dm=\frac{1}{2}\int_0^h y^2\left(\pi\,y^2\,\rho\,dz\right)=\frac{1}{2}\pi\,\rho\int_0^h y^4\,dz=\frac{1}{2}\pi\,\rho\int_0^h\left(\frac{a}{h}(h-z)\right)^4 dz=\frac{1}{2}\pi\,\rho\left(\frac{a}{h}\right)^4\int_0^h (h-z)^4\,dz$$

$$=\frac{1}{2}\pi\,\rho\left(\frac{a}{h}\right)^4\left.\frac{(h-z)^5}{-5}\right|_0^h=\frac{1}{2}\pi\,\rho\frac{a^4}{h^4}\frac{h^5}{5}=\frac{1}{10}\pi\,\rho a^4\,h=\frac{1}{10}\pi\,\rho a^4\,h\,\frac{m}{m}=\frac{1}{10}\pi\,\rho a^4 h\,\frac{m}{\frac{1}{3}\pi\,a^2\,h\,\rho}=\frac{3}{10}m\,a^2$$

$I_{zz}=\frac{3}{10}m\,a^2$, Also

$$I_o=\int_0^h z^2\,dm=\int_0^h z^2\left(\pi\,y^2\,\rho\,dz\right)=\pi\,\rho\int_0^h z^2y^2\,dz=\pi\,\rho\int_0^h z^2\left(\frac{a}{h}(h-z)\right)^2 dz$$

$$=\pi\,\rho\left(\frac{a}{h}\right)^2\int_0^h\left(h^2z^2-2h\,z^3+z^4\right)dz=\pi\,\rho\frac{a^2}{h^2}\left.\left(h^2\frac{z^3}{3}-h\frac{z^4}{2}+\frac{z^5}{5}\right)\right|_0^h=\pi\,\rho\frac{a^2}{h^2}h^5\left(\frac{10-15+6}{30}\right)$$

$$=\pi\,\rho\frac{a^2}{h^2}\frac{h^5}{30}=\frac{1}{30}\pi\,\rho a^2\,h^3=\frac{1}{30}\pi\,\rho a^2\,h^3\,\frac{m}{m}=\frac{1}{30}\pi\,\rho a^2\,h^3\,\frac{m}{\frac{1}{3}\pi\,a^2\,h\,\rho}=\frac{1}{10}m\,h^2.$$

Then $I_o=\frac{1}{10}mh^2$

II. Area Moment of Inertia

Area moment of inertia also known as second area moment or 2^{nd} moment of area is a property of a two-dimensional plane shape where it shows how its points are dispersed in an arbitrary axis in the cross-sectional plane. This property basically characterizes the deflection of the plane shape under some load.

Area moment of inertia is usually denoted by the letter I for an axis in a plane. The dimension unit of second area moment is Length to the power of four which is given as L^4. If we take the International System of Units, its unit of dimension is meter to the power of four or m^4. If we take the Imperial System of Units it can be inches to the fourth power, in^4.

We will come across this concept in the field of structural engineering often. Here the area moment of inertia is said to be the measure of the flexural stiffness of a beam. It is an important property that is used to measure the resistance offered by a beam to bending or in calculating a beam's deflection. Here we have to look at two cases.

First, a beam's resistance to bending can be easily described or defined by the planar second moment of area where the force lies perpendicular to the neutral axis.

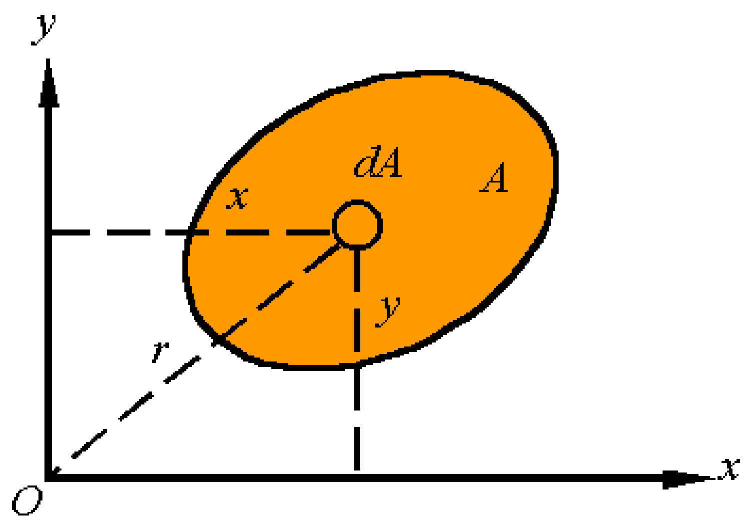
Secondly, the polar second moment of area can be used to determine the beam's resistance when the applied moment is parallel to its cross-section. It is basically the beams ability to resist torsion

Area Moment Of Inertia Formulas

The area moment of inertia for the area is given in below figure can be expressed mathematically as:

Referenced to x-axis is given by $I_{xx} = y^2\, dA,$

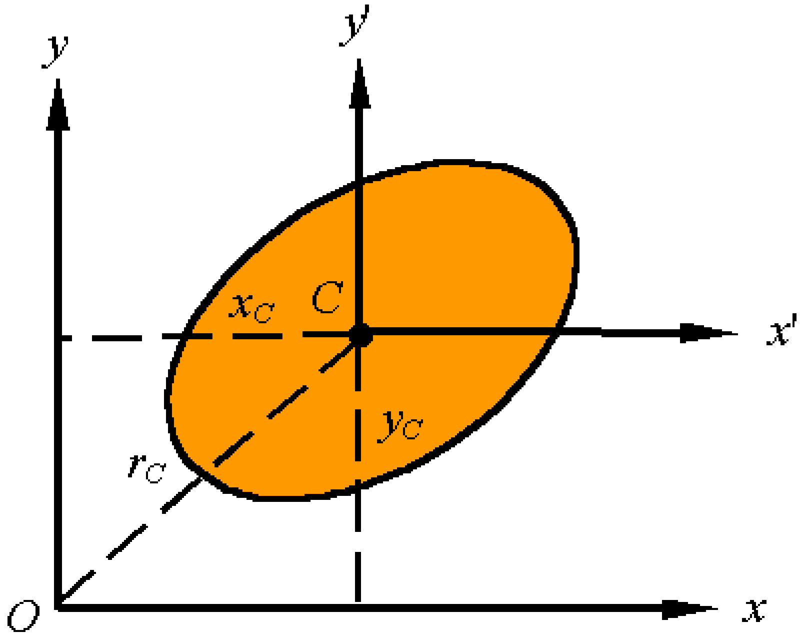
Referenced to y-axis is given by $I_{yy} = x^2\, dA,$



Referenced to o-point is given by $I_O = r^2 dA = (x^2 + y^2)\, dA = I_{xx} + I_{yy}$

The parallel axis theorem

The parallel axis theorem is a relation between the moment of inertia about an axis passing through the centroid and the moment of inertia about any parallel axis.



The parallel axis theorem states that

$$I_{xx} = I_{x'x'} + A\,\bar{y}^2, \quad I_{yy} = I_{y'y'} + A\,\bar{x}^2$$

A simple recap of the Basics:

- Moments of inertia are always positive.

- Minimum moments of inertia axes always pass through the center of mass.
- Moments of inertia are a measure of the mass distribution of a body about a set of axes. Think of a rotating ice skater. If the person stretches the arms out, she slows down and speeds up otherwise. Hence the smaller the inertia the more concentrated or closer the mass is about a particular axis.

- Area moments of inertia are for a particular section or a 2D surface.

- Products of inertia can be positive, negative or zero.

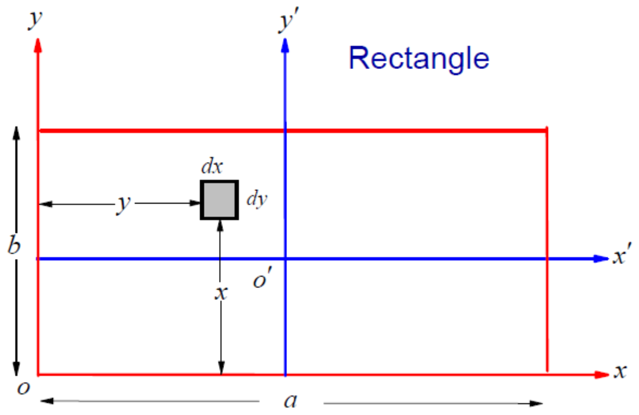
- Products of inertia are a measure of the symmetry of a body about a set of axes. They are zero about any axis normal to a plane of symmetry.

- For any given point on a section, for example the centroid or any other point, there exists a set of axes oriented in such a way that all products of inertia are zero.

Example 1: Find the Area moment of inertia of a rectangular section about a horizontal axis passing through base?

Solution

We consider a uniform element with the length (dx) and thickness (dy) as is shown in below Figure



The Area moment of inertia about its vertical corner is given by

$$dI_{yy} = x^2 dA = x^2\,dx\,dy \quad \rightarrow I_{yy} = \int_0^b \int_0^a x^2\,dx\,dy = \left[\frac{x^3}{3}\right]_0^b [y]_0^b = \frac{1}{3} b a^3 \quad \therefore\ I_{yy} = \frac{1}{3} m a^2$$

From the parallel Axis Theorem $\qquad I_{yy} = I_{y'y'} + m\left(\frac{1}{2}a\right)^2 \rightarrow$

$$I_{y'y'} = \frac{1}{3} b a^3 - \frac{1}{4}(ab)a^2 = \left(\frac{4-3}{12}\right) b a^3 = \frac{1}{12} b a^3 \qquad \therefore\ I_{y'y'} = \frac{1}{12} b a^3$$

Similarly, we can prove that: $I_{xx} = \frac{1}{3} a b^3$, $\qquad I_{x'x'} = \frac{1}{12} a b^3$

For axis is perpendicular ox, oy $\ I_{zz} = I_{xx} + I_{yy} = \frac{1}{3} a b^3 + \frac{1}{3} b a^3 = \frac{1}{3} a b(a^2 + b^2)$

For axis is perpendicular ox', oy': $\ I_{z'z'} = I_{x'x'} + I_{y'y'} = \frac{1}{12} a b^3 + \frac{1}{12} b a^3 = \frac{1}{12} a b(a^2 + b^2)$

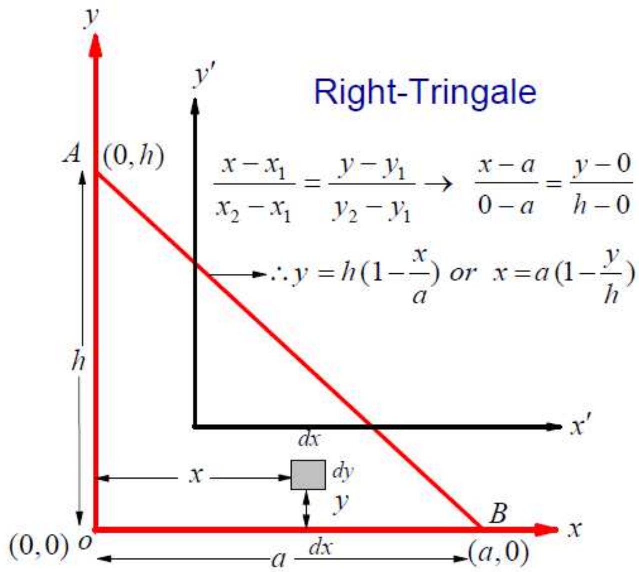
Uniform rectangular plate	Axis coincides with	Axis passing	Axis coincides to other

(a,b)	one of its sides	through its centroid	side
With respect to $I_{yy}-$ axis	$I_{yy}=\frac{1}{3}ba^3$	$I_{y'y'}=\frac{1}{12}ba^3$	$I_{y''y''}=\frac{1}{3}ba^3$
With respect to $I_{xx}-$ axis	$I_{xx}=\frac{1}{3}ab^3$	$I_{x'x'}=\frac{1}{12}ab^3$	$I_{x''x''}=\frac{1}{3}ab^3$
With respect to axis perpendicular to the plane oxy	$I_{zz}=\frac{1}{3}ab(a^2+b^2)$	$I_{z'z'}=\frac{1}{12}ab(a^2+b^2)$	$I_{z''z''}=\frac{1}{3}ab(a^2+b^2)$

Example 2: Find the Area moment of inertia of a triangular section about a horizontal axis passing through base?

Solution

We consider a uniform element with the length (dx) and thickness (dy) as is shown in below Figure



$$I_{yy} = \int \int x^2 \left(dx\,dy\right) \rightarrow I_{xy} = \int_0^h \int_0^{a(1-\frac{y}{h})} x^2 dx\,dy = \frac{a^3}{3} \int_0^h (1-\frac{y}{h})^3\, dy$$

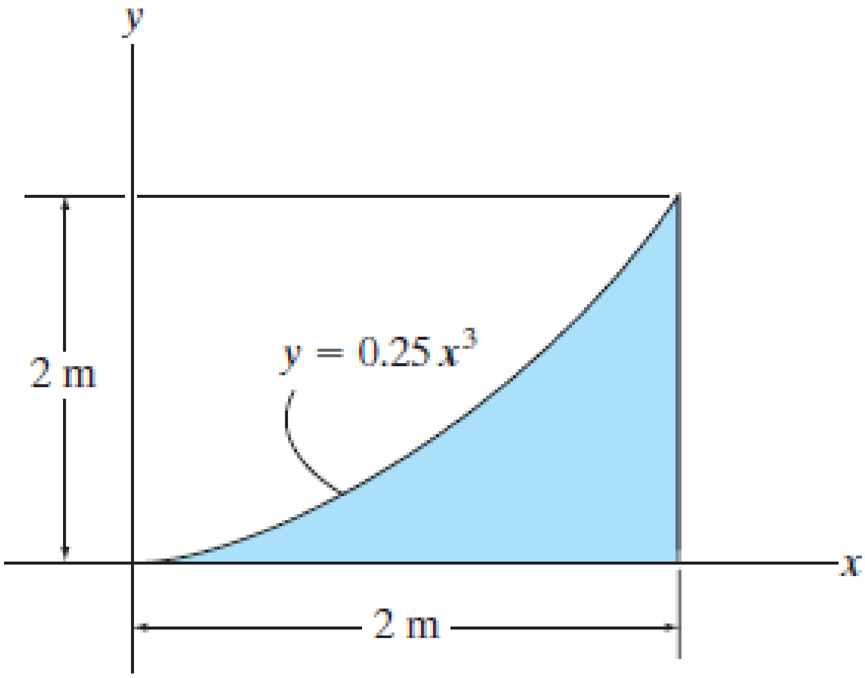
$$= \frac{a^3}{3h^3} \int_0^h (h-y)^3\, dy = \frac{a^3}{h^2} \int_0^h (h^3 - 3h^2 y + 3y^2 h - y^3)\, dy$$

$$= \frac{a^3}{h^3}\left[h^3 y - \frac{3y^2}{2} h^2 + y^3 h - \frac{y^4}{4} \right]_0^h = \frac{a^3}{3h^3}\left[h^4 - \frac{3}{2} h^4 + h^4 - \frac{h^4}{4} \right] = \frac{a^3 h^4}{12h^3}\left[8-6-1\right] = \frac{1}{12} a^3 h$$

$$\therefore\ I_{yy} = \frac{1}{12} h a^3$$

Right Triangular Plate of height h and bass a	About its corner	About its center of mass	About its vertex
About its base	$I_{xx} = \frac{1}{12}\, a h^3$	$I_{x'x'} = \frac{1}{36} a h^3$	$I_{x''x''} = \frac{1}{4}\, a h^3$
About its height	$I_{yy} = \frac{1}{12}\, h a^3$	$I_{y'y'} = \frac{1}{36}\, h a^3$	$I_{y''y''} = \frac{1}{4}\, h a^3$
About vertical axis	$I_{zz} = \frac{a h}{12}(a^2 + h^2)$	$I_{z'z'} = \frac{a h}{36}(a^2 + h^2)$	$I_{z''z''} = \frac{a h}{12}(a^2 + 3h^2)$, $I_{z''z''} = \frac{a h}{12}(3a^2 + h^2)$

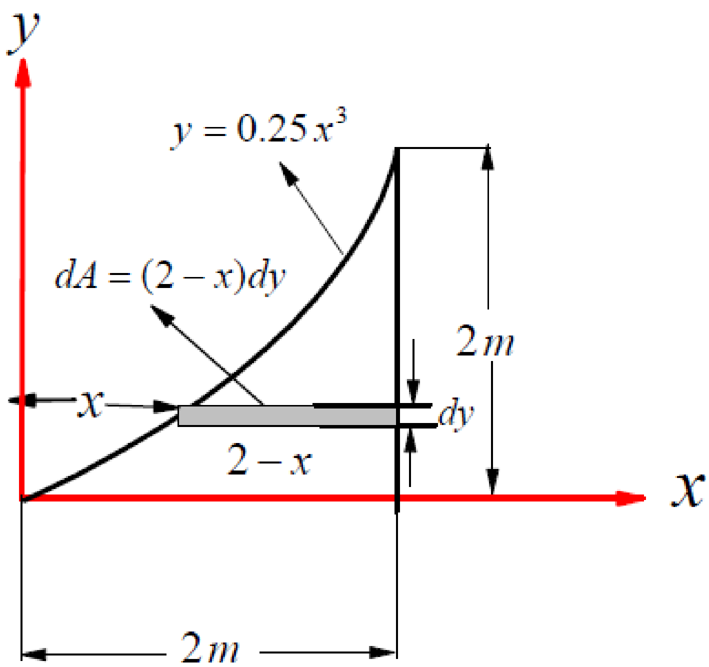
Example 3: Determine Area the moment of inertia of the shaded area with respect to $ox, oy-$axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $x-$axis with the length $(2-x)$ and thickness (dy) as is shown in below Figure. Then $dA = (2-x)\,dy$



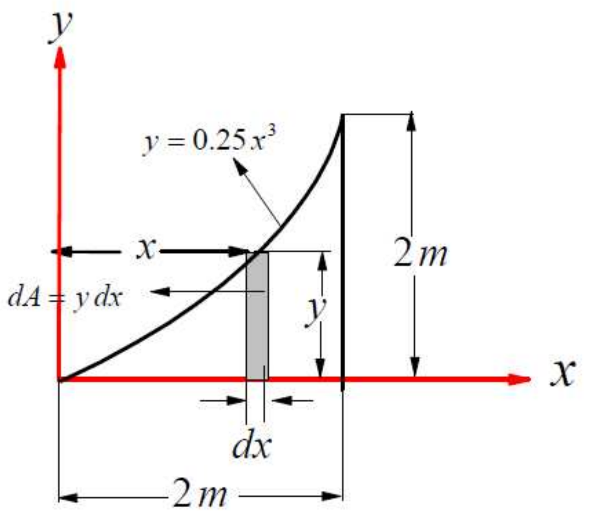
So, the area moment of inertia with respect to x-axis is given as

$$I_{xx}=\int y^2 dA=\int_0^2 y^2\left(2-x\right)dy=\int_0^2 y^2\left(2-\left(4y\right)^{\frac{1}{3}}\right)dy=\int_0^2\left(2y^2-\left(4\right)^{\frac{1}{3}}y^{\frac{7}{3}}\right)dy$$

$$I_{xx}=\left(\frac{2}{3}y^3-\frac{3}{10}\left(4\right)^{\frac{1}{3}}y^{\frac{10}{3}}\right)_0^2=\frac{2}{3}(2)^3-\frac{3}{10}\left(4\right)^{\frac{1}{3}}(2)^{\frac{10}{3}}=\frac{16}{3}-\frac{3}{10}\left(4\right)^{\frac{1}{3}}(2)^{\frac{10}{3}}=\frac{8}{15}=0.53334\,m^4$$

<u>The Area Moment of inertia with respect to y- axis</u>

While if we consider a uniform strip line parallels to the $y-$axis with the length (y) and thickness (dx) as is shown in below Figure. Then $dA=y\,dx$



So, the area moment of inertia with respect to y-axis is given as

$$I_{yy}=\int x^2 dA=\int_0^2 x^2 y\,dx=\int_0^2 x^2\left(0.25\right)x^3\,dx=\int_0^2\left(0.25\right)x^5\,dy=\frac{(2)^6}{24}=\frac{64}{24}=\frac{8}{3}=2.67\,m^4$$

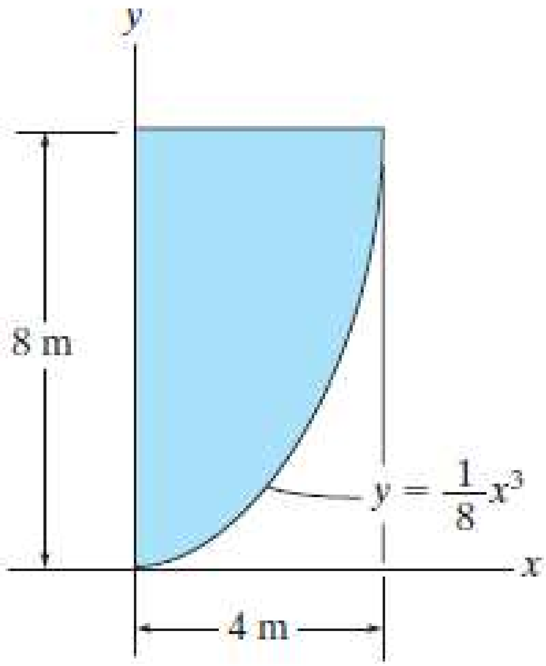
<u>Again, the Area Moment of inertia with respect to x- axis</u>

If we consider the previous Figure (second Figure) we can find the Area moment of inertia as

$$I_{xx}=\int\frac{1}{3}y^3dx=\int_0^2\frac{1}{3}y^3dx=\frac{1}{3}\int_0^2\left(0.25x^3\right)^3dx=\frac{1}{3}\left(0.25\right)^3\int_0^2 x^9\,dx=\frac{1}{3}\left(0.25\right)^3\frac{1}{10}\left(x^{10}\right)_0^2$$

$$I_{xx}=\frac{1}{3}\frac{15625}{1000000}\frac{1024}{10}=\frac{1}{3}\frac{16000000}{10000000}=\frac{1}{3}\frac{16}{10}=\frac{1}{3}\frac{8}{5}=\frac{8}{15}=0.53334\,m^4$$

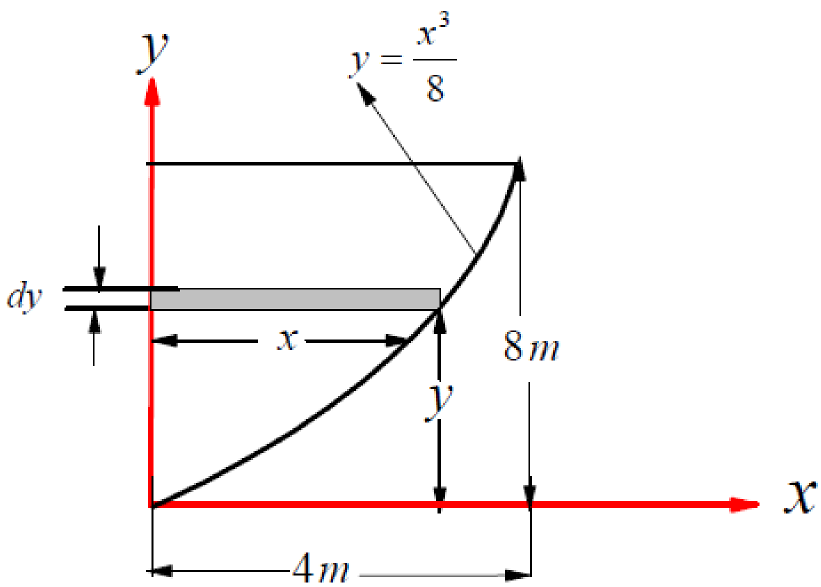
Example 4: Determine the Area moment of inertia of the shaded area with respect to $ox, oy-$axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $x-$axis with the length (x) and thickness (dy) as is shown in below Figure. Then $dA = x\,dy$



So, the area moment of inertia with respect to x-axis is given as

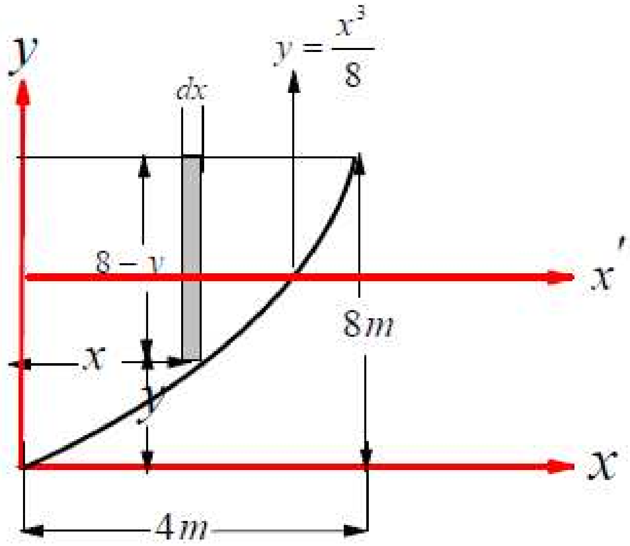
$$I_{xx} = \int y^2 dA = \int_0^8 y^2 x\, dy = \int_0^8 y^2 (2) y^{\frac{1}{3}}\, dy = 2\int_0^8 y^{\frac{7}{3}}\, dy = 2\frac{3(8)^{\frac{10}{3}}}{10} = \frac{3}{5}(1024) = 614.4\, m^4$$

From the above Figure, the area moment of inertia with respect to y-axis is given by

$$I_{yy} = \int \frac{1}{3}x^3 dy = \int_0^8 \frac{8}{3} y\, dy = \frac{8}{3}\frac{1}{2}\left(y^2\right)_0^8 = \frac{8}{3}\frac{1}{2}(64) = \frac{256}{3} = 85.3334 m^4$$

The Area Moment of inertia with respect to y- axis

While if we consider a uniform strip line parallels to the $y-$axis with the length $(8-y)$ and thickness (dx) as is shown in below Figure. Then $dA = (8-y)\, dx$.



So, the area moment of inertia with respect to y-axis is given as

$$I_{yy} = \int x^2 dA = \int_0^4 x^2 (8-y) dx = \int_0^4 x^2 \left(8 - \frac{x^3}{8}\right) dx = \int_0^4 \left(8x^2 - \frac{x^5}{8}\right) dx =$$

$$I_{yy} = \left(\frac{8}{3}x^3 - \frac{1}{48}y^6\right)_0^4 = \frac{8}{3}(4)^3 - \frac{1}{48}(4)^6 = (4)^4\left(\frac{2}{3} - \frac{16}{48}\right) = (4)^4\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{256}{3} = 85.3334 m^4$$

The Area Moment of inertia with respect to x- axis

If we consider the previous Figure (second Figure) we can find the Area moment of inertia with respect to $x-$ axis (from the parallel axis theorem) as $I_{xx} = I_{x'x'} + dm\,(\bar{y})^2$

Where $I_{x'x'} = \frac{1}{12}(8-y)^3 dx = \frac{1}{12}(512 - 192y + 24y^2 - y^3)\, dx$

$\bar{y} = \frac{1}{2}(8-y) + y = \frac{1}{2}(8+y) \;\rightarrow\; \bar{y}^2 = \left[\frac{1}{2}(8+y)\right]^2 = \frac{1}{4}(64 + 16y + y^2)$,

$dm = (8-y)\,dx$ Then $I_{xx} = I_{x'x'} + dm\,(\bar{y})^2$ becomes

$$I_{xx} = \frac{1}{12}(512 - 192y + 24y^2 - y^3)\,dx + (8-y)\;dx\left(\frac{1}{4}(64 + 16y + y^2)\right)$$

$$= \left\{\frac{1}{12}(512 - 192y + 24y^2 - y^3) + 128 + 32y + 2y^2 - \frac{1}{4}(64y + 16y^2 + y^3)\right\} dx$$

$$= \left\{\frac{128}{3} - \underset{2+6+8=0}{16y} + \underset{3+7+9=0}{2y^2} - \frac{1}{12}y^3 + 128 + \underset{2+6+8=0}{32y} + \underset{3+7+9=0}{2y^2} - \underset{2+6+8=0}{16y} - \underset{3+7+9=0}{4y^2} - \frac{1}{4}y^3\right\} dx$$

$$= \left\{\frac{512}{3} - \frac{1}{3}y^3\right\} dx = \frac{1}{3}\left\{512 - y^3\right\} dx = \frac{1}{3}\left\{512 - (\frac{x}{8})^3\right\} dx$$

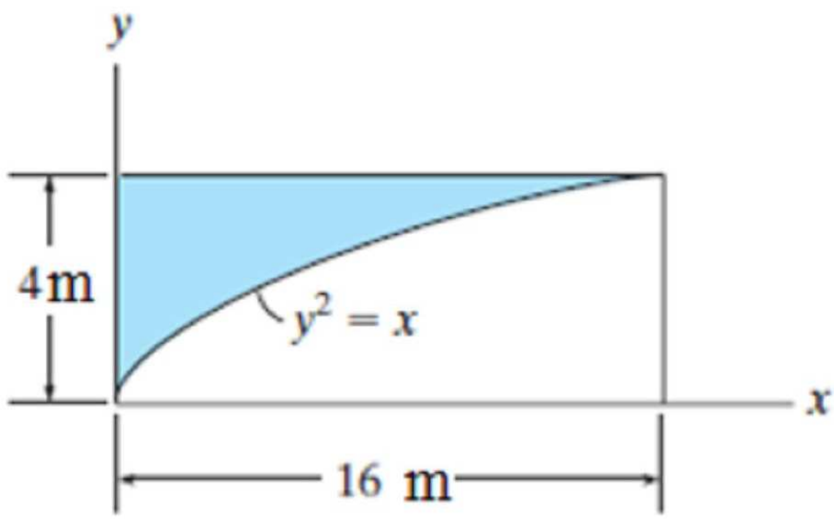
For all the Area, we have

$$I_{xx} = \int \frac{1}{3}\left\{512 - (\frac{x}{8})^3\right\} dx = \frac{1}{3}\int_0^4 \left\{512 - (\frac{1}{8})^3 x^3\right\} dx = \frac{1}{3}\left[512x - \frac{1}{4}(\frac{1}{8})^3 x^4\right]_0^4$$

$$= \frac{1}{3}\left[512(4) - \frac{1}{4}(\frac{1}{8})^3 (4)^4\right] = \frac{1}{3}\left[512(4) - (\frac{1}{8})^3 (4)^3\right] = \frac{1}{3}\left[512(4) - \frac{(4)(4)(4)}{(8)(8)(8)}\right]$$

$$= \frac{1}{3}\left[512(4) - \frac{1}{8}\right] = \frac{2}{7} m^4$$

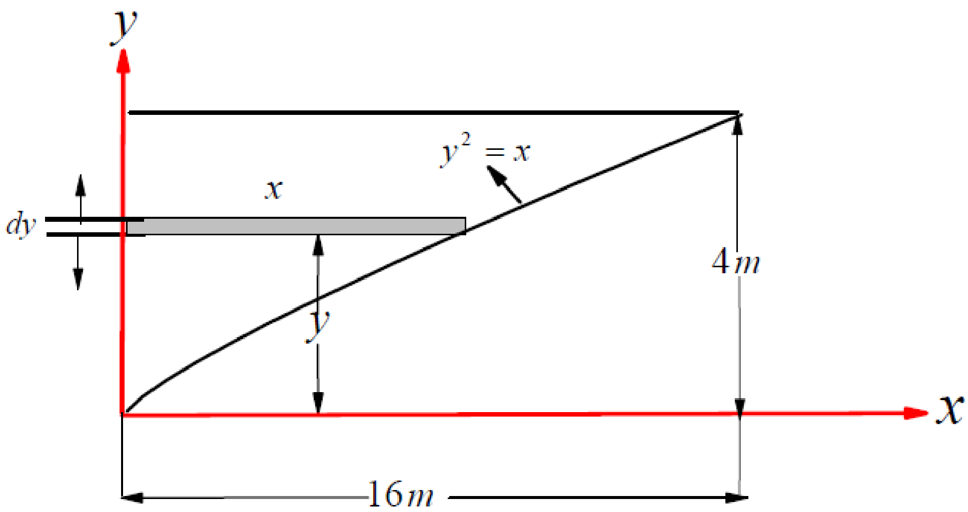
Example 5: Determine Area the moment of inertia of the shaded area with respect to $ox, oy-$axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $x-$axis with the length (x) and thickness (dy) as is shown in below Figure. Then $dA = x\,dy$

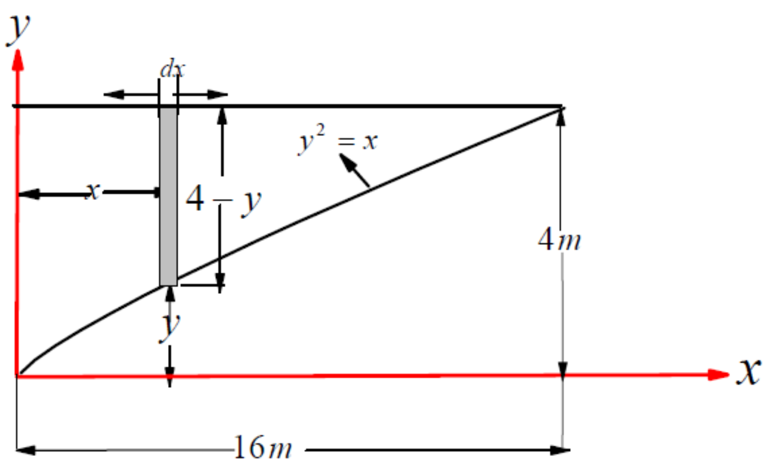


So, the area moment of inertia with respect to x-axis is given as

$$I_{xx} = \int y^2 dA = \int_0^4 y^2 x\,dy = \int_0^4 y^2 (y)^2\,dy = \int_0^4 y^4\,dy = \frac{(4)^5}{5} = \frac{1024}{5} = 204.8\,m^4$$

The Area Moment of inertia with respect to y- axis

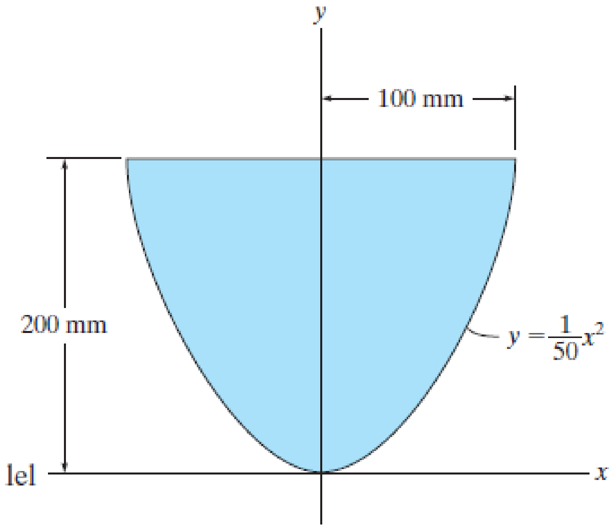
While if we consider a uniform strip line parallels to the $y-$axis with the length $(4-y)$ and thickness (dx) as is shown in below Figure. Then $dA = (4-y)\,dx$



$$I_{yy} = \int x^2 dA = \int_0^{16} x^2(4-y)dx = \int_0^{16} x^2\left(4-(x)^{\frac{1}{2}}\right)dx = \int_0^{16}\left(4x^2-(x)^{\frac{5}{2}}\right)dx = \frac{4(16)^3}{3} - \frac{2(16)^{\frac{7}{2}}}{7}$$

$$I_{yy} = 5461.333 - 4861.1428 = 780.2\,m^4$$

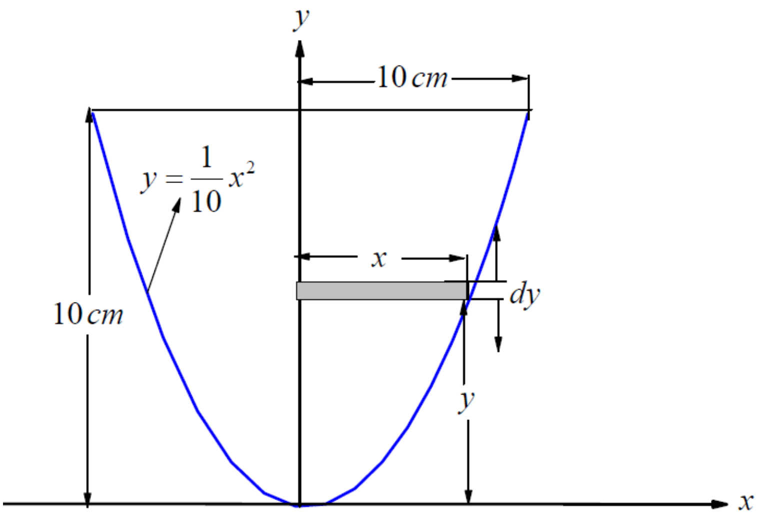
Example 6: Determine the Area moment of inertia of the shaded area with respect to ox, oy – axes?



Solution

<u>The Area Moment of inertia with respect to x- axis</u>

We consider a uniform strip line parallels to the x – axis with the length (x) and thickness (dy) as is shown in below Figure. Then $dA = x\,dy$



So, the area moment of inertia with respect to x-axis is given a

$$I_{xx} = 2\int y^2 dA = 2\int_0^{10} y^2 x\, dy = 2\int_0^{10} y^2 (10y)^{\frac{1}{2}} dy = 2(10)^{\frac{1}{2}} \int_0^{10} y^{\frac{5}{2}} dy = 2(10)^{\frac{1}{2}} \left(\frac{2}{7} y^{\frac{7}{2}} \right)_0^{10}$$

$$I_{xx} = 2(10)^{\frac{1}{2}} \left(\frac{2}{7} (10)^{\frac{7}{2}} \right) = \frac{4}{7}\, 10^4\, m^4$$

The Area Moment of inertia with respect to y- axis

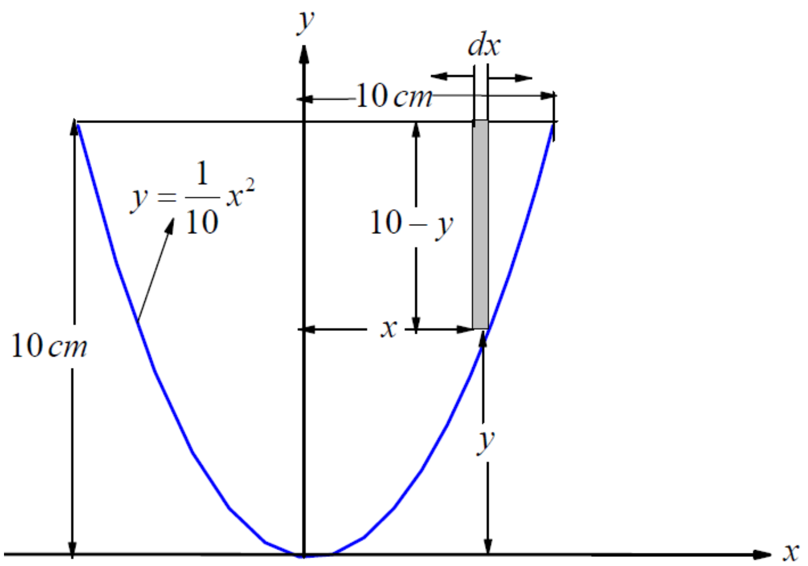
If we consider the previous Figure (first Figure) we can find the Area moment of inertia with respect to $y-$ axis as $I_{yy} = \int \frac{1}{3} x^3 dy$

$$I_{yy} = 2\int \frac{1}{3} x^3 dy = 2\int_0^{10} \frac{1}{3} (10y)^{\frac{3}{2}} dy = \frac{2(10)^{\frac{3}{2}}}{3} \int_0^{10} (y)^{\frac{3}{2}} dy = \frac{2\,(10)^{\frac{3}{2}}}{3} \frac{2\,(10)^{\frac{5}{2}}}{5}$$

$$I_{yy} = \frac{4}{15} (10)^{\frac{3}{2}+\frac{5}{2}} = \frac{4}{15} (10)^4 = \frac{4}{15}\, 10^4\, m^4$$

Again the Area Moment of inertia with respect to y- axis

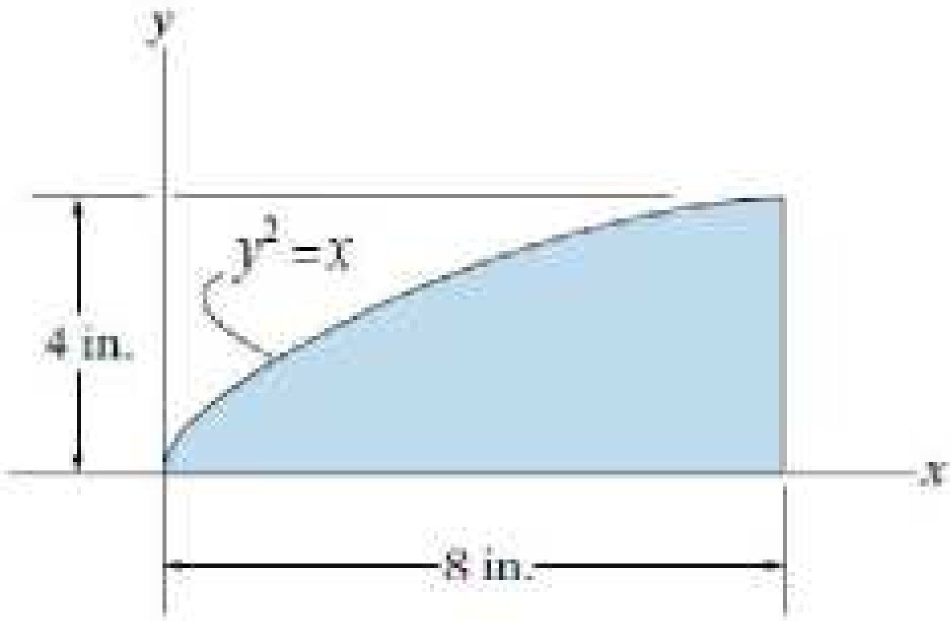
While if we consider a uniform strip line parallels to the $y-$axis with the length $(10-y)$ and thickness (dx) as is shown in below Figure. Then $dA = (10-y)\, dx$



$$I_{yy} = 2\int x^2 dA = 2\int_0^{10} x^2 (10-y) dx = 2\int_0^{10} x^2 \left(10 - \frac{1}{10} x^2 \right) dx = 2\int_0^{10} \left(10\, x^2 - \frac{1}{10} x^4 \right) dx$$

$$= 2\left(\frac{10}{3} x^3 - \frac{1}{50} x^5 \right)_0^{10} = 2\left(\frac{10^4}{3} - \frac{10^4}{50} \right) = \frac{2}{150} (50-30) 10^4 = \frac{40}{150} 10^4 = \frac{4}{15} 10^4$$

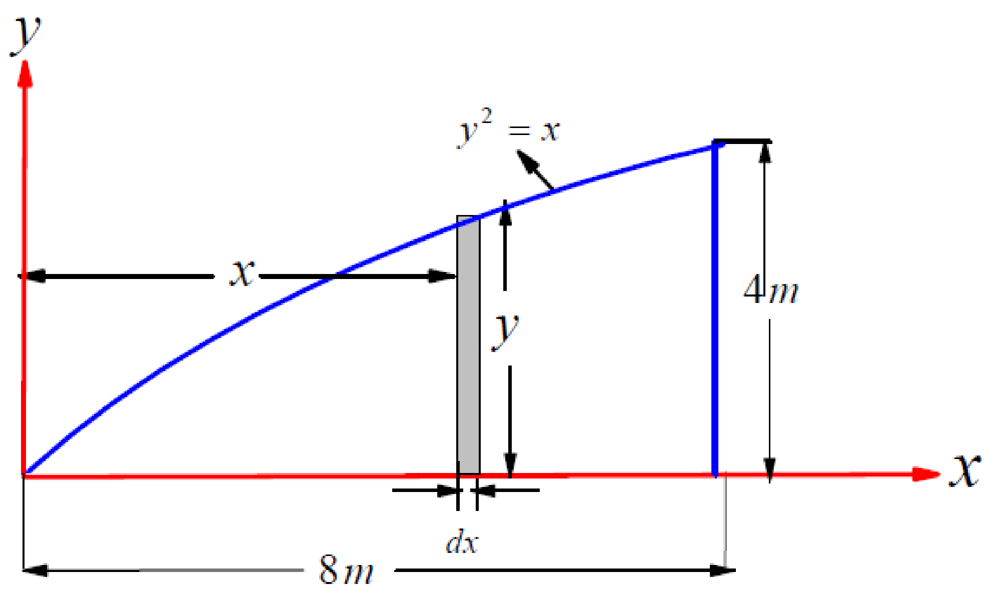
Example 7: Determine the Area moment of inertia of the shaded area with respect to ox, $oy-$axes?



Solution

The Area Moment of inertia with respect to x- axis

We consider a uniform strip line parallels to the $y-$axis with the length (y) and thickness (dy) as is shown in below Figure. Then $dA = y\,dx$



So, the area moment of inertia with respect to x-axis is given as

$$I_{xx} = \int \frac{1}{3} y^3 dx = \frac{1}{3}\int_0^4 y^3 (2y\,dy) = \frac{2}{3}\int_0^4 y^4 dy = \frac{2}{15}\left(y^5\right)_0^4 = \frac{2}{15}(4)^5 = \frac{2048}{15} m^4 = 136.533\, m^4$$

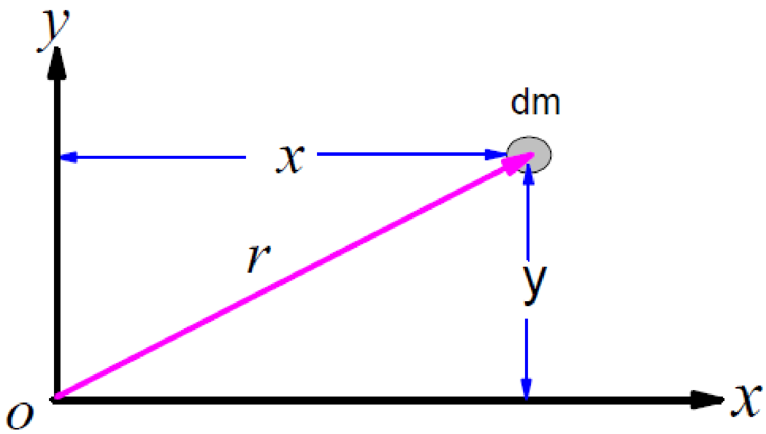
The Area Moment of inertia with respect to y- axis

$$I_{yy} = \int x^2 dA = \int_0^8 x^2 y\, dx = \int_0^8 x^2 (x)^{\frac{1}{2}}\, dx = \int_0^8 x^{\frac{5}{2}}\, dx = \frac{2}{7}\left[\, x^{\frac{7}{2}} \right]_0^8 = \frac{2048\sqrt{2}}{7} m^4$$

III. Products of Inertia of mass

Products of Inertia of mass

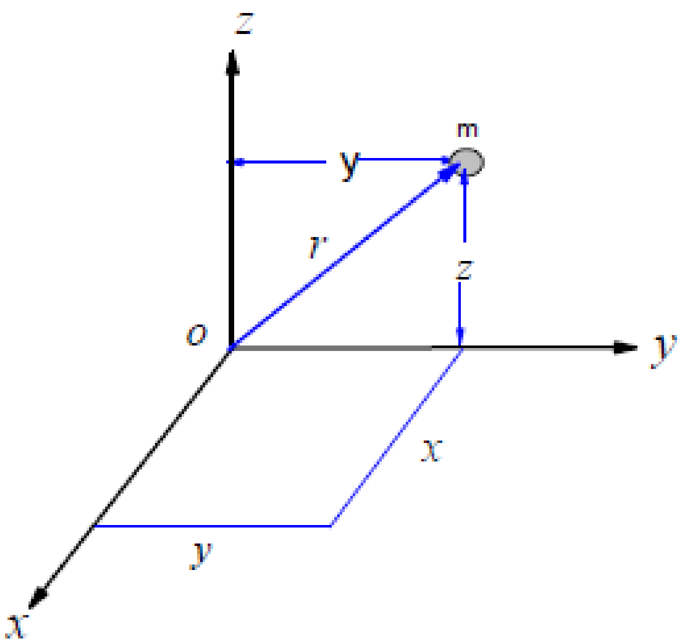
(1) If the body is located in a plane as shown below figure and has mass (dm). Then the product of inertia with respect to the axes ox, oy is given by



$I_{xy} = x\,y\,dm$ (1)

Note that $I_{xy} = x\,y\,dm = I_{yx} = y\,x\,dm$

(2) For the body in space



With respect to the axes ox, oy

$$I_{xy} = x\, y\, dm \qquad (2)$$

With respect to the axes ox, oz

$$I_{xz} = x\, z\, dm \qquad (3)$$

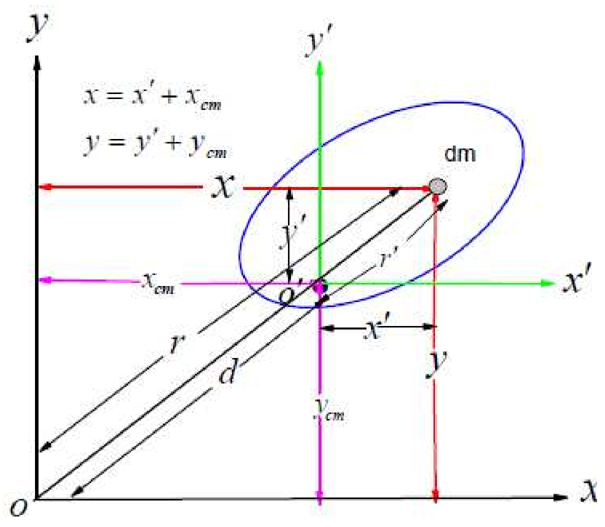
While, With respect to the axes oy, oz

$$I_{yz} = y\, z\, dm \qquad (4)$$

Product of inertia can be positive or negative value as oppose the moment of inertia. The calculation of the product of inertia isn't different much for the calculation of the moment of inertia. The units of the product of inertia are the same as for moment of inertia.

Parallel-axis theorem for products of inertia

For any rigid body has mass (m) and the center of mass (x_{cm} , y_{cm}) as shown below figure



Dividing the body into a number of small elements. Taking a small element whose mass (dm) and its coordinate with respect to the original axes is (x, y). With respect to axes parallel to the original axes and passing through the center of mass the element has the coordinate (x', y').

For the original axes (x, y), the inertial product of mass (dm) is given by

$$I_{xy} = xy\, dm \qquad (1)$$

For the total mass (m)

$$I_{xy} = \int x\,y\,dm \qquad (2)$$

From the above Figure $x = x' + x_{cm},\quad y = y' + y_{cm}$ and into Eq. (2), we have

$$I_{xy} = \int x\,ydm = \int \left\{ \left(x' + x_{cm} \right)\left(y' + y_{cm} \right) \right\} dm = \int \left\{ x'y' + x'y_{cm} + x_{cm}y' + x_{cm}\,y_{cm} \right\} dm$$

$$I_{xy} = \int x'y'dm + y_{cm}\int x'dm + x_{cm}\int y'dm + x_{cm}\,y_{cm}\int dm \qquad (3)$$

But, it is well-known that

$$\int x'y'dm = I_{x'y'},\quad x_{cm}\,y_{cm}\int dm = x_{cm}\,y_{cm}\,m,$$
$$\bar{x} = \frac{\int x'\,dm}{\int dm} \rightarrow \int x'\,dm = \bar{x}\int dm,\qquad \bar{y} = \frac{\int y'\,dm}{\int dm} \rightarrow \int y'\,dm = \bar{y}\int dm \qquad (4)$$

From Eq. (4) into Eq. (3), we have

$$I_{xy} = I_{x'y'} + y_{cm}\,\bar{x}\int dm + x_{cm}\,\bar{y}\int dm + x_{cm}\,y_{cm}m$$

$$I_{xy} = I_{x'y'} + y_{cm}\,\bar{x}\,m + x_{cm}\,\bar{y}\,m + x_{cm}\,y_{cm}m \qquad (5)$$

But the coordinate $(\bar{x},\,\bar{y})$ is the center of mass from the center of mass and it is equal to zero. Substituting in (5) we get

$$I_{xy} = I_{x'y'} + m\,x_{cm}\,y_{cm} \qquad (6)$$

Where I_{xy} is the product of inertia with respect to the two axes $ox,\ oy$, while $I_{x'y'}$ is the product of inertia with respect to the two axes $o'x'$, $o'y'$ and x_{cm}, y_{cm} are the distance of the center of gravity from the two axes $ox,\ oy$, respectively.

Notes

(1)-The product of inertia is a product of different coordinates, so it can be positive or negative quantity

(2)- For the product of inertia is $I_{xy} = I_{yx}$, $I_{yz} = I_{zy}$, $I_{zx} = I_{xz}$

(3)- If the Products of Inertia are zero with respect to any two planes. It is said that the axis of intersection of these two axes is a principal axis.

(4) If the inertia product is neglected with respect to any two principal coordinates ox, oy, it is said that the tow axis ox, oy are principal axes

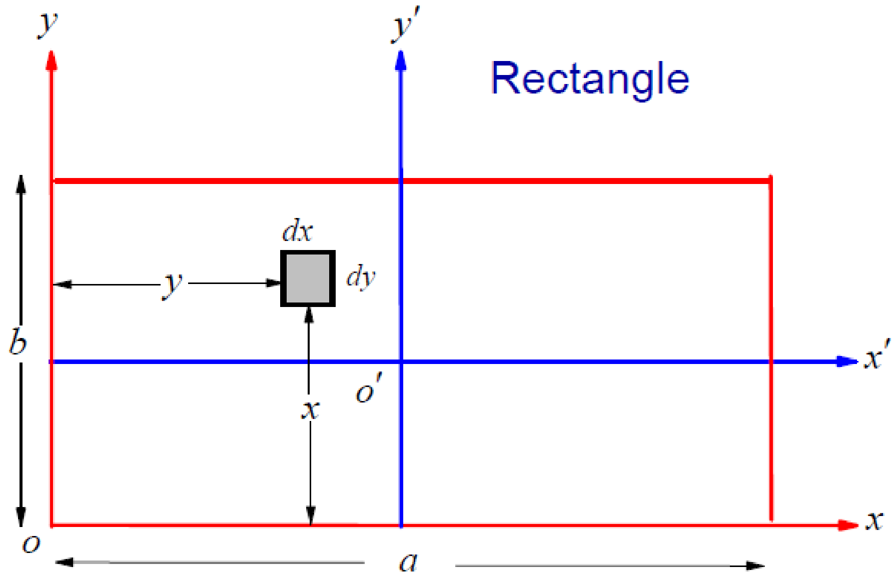
(5)- Any symmetry axis in a flat plate with any perpendicular axis , then these axes are called the principal axes

(6) - The product of inertia is finished for the two axes are perpendicular, if each other and one or both axes of symmetry.

Example 1: Find the Product of Inertia of a thin uniform rectangular plate?

Solution

We divide the plate to small uniform strip, we consider one of them with the length (dx) and thickness (dy) as in Figure, where the density is ρ.



$$dm=\rho\, dx dy \rightarrow m=\rho\int_0^b\int_0^a dx dy=\rho a\int_0^b dy=ab\rho \rightarrow\ m=ab\,\rho$$

With respect to $ox,\ oy$, we have $dI_{xy}=(dm)\,x\,y$.

For the total plate, we have

$$I_{xy}=\int\int x\,y\,(\rho\, dx dy)\rightarrow I_{xy}=\rho\int_0^b\int_0^a x\,y\,dx dy=\frac{a^2}{2}\frac{b^2}{2}\rho=\frac{a^2b^2}{4}\rho\frac{m}{m}=\frac{a^2b^2}{4}\rho\frac{m}{ab\rho}=\frac{1}{4}mab$$

$$I_{xy}=\frac{1}{4}mab$$

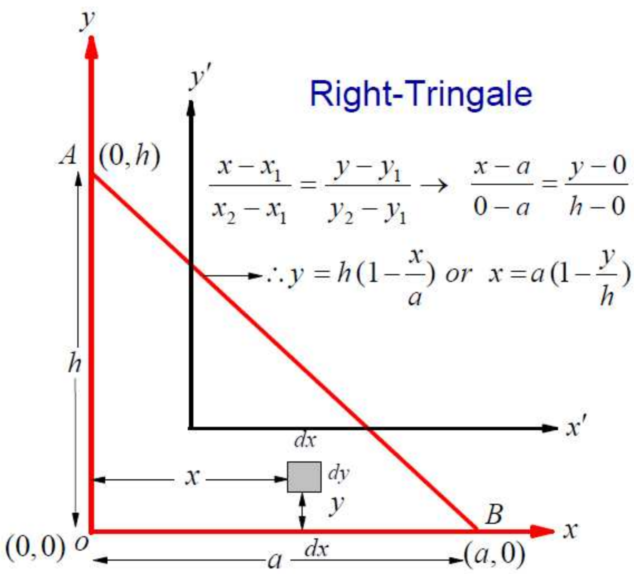
From the theory of parallel axes for the product of inertia, the product of inertia with respect to ox', oy' is given $I_{xy}=I_{x'y'}+m\,x_{cm}\,y_{cm}\ \rightarrow\ \frac{1}{4}mab=I_{x'y'}+m(\frac{1}{2}a)(\frac{1}{2}b)\rightarrow\ =I_{x'y'}=\frac{1}{4}mab-\frac{1}{4}mab=0$

$I_{x'y'}=0$. So, the axes ox', oy' are symmetric axes.

Example 2: Find the Product of Inertia of a thin uniform triangular plate?

Solution

$$dm=\rho\, dx dy \rightarrow m=\rho\int_0^h\int_0^{a(1-\frac{y}{h})} dx dy=\rho\int_0^h a(1-\frac{y}{h})dy=a\rho\left[y-\frac{y^2}{2h}\right]_0^h=a\rho\left[h-\frac{h^2}{2h}\right]=\frac{1}{2}ah\rho$$



With respect to $ox,\ oy$, we have $dI_{xy} = (dm)\,x\,y$. For the total plate, we have

$$I_{xy} = \int\int x\,y\,(\rho\,dx\,dy) \to I_{xy} = \rho\int_0^h\int_0^{a(1-\frac{y}{h})} x\,y\,dx\,dy = \frac{a^2}{2}\rho\int_0^h (1-\frac{y}{h})^2\,y\,dy$$

$$= \frac{a^2}{2h^2}\rho\int_0^h (h-y)^2\,y\,dy = \frac{a^2}{2h^2}\rho\int_0^h (h^2-2yh+y^2)\,y\,dy = \frac{a^2}{2h^2}\rho\int_0^h (yh^2-2y^2h+y^3)\,dy$$

$$= \frac{a^2}{2h^2}\rho\left[\frac{y^2h^2}{2}-\frac{2y^3}{3}h+\frac{y^4}{4}\right]_0^h = \frac{a^2}{2h^2}\rho\left[\frac{h^4}{2}-\frac{2h^4}{3}+\frac{h^4}{4}\right] = \frac{a^2h^4}{24h^2}\rho[6-8+3] = \frac{a^2h^4}{24h^2}\rho$$

$$= \frac{a^2h^2}{24}\rho\frac{m}{m} = \frac{a^2h^2}{24}\rho\frac{m}{\frac{1}{2}\rho ah} = \frac{1}{12}mah$$

$$I_{xy} = \frac{1}{12}mah$$

From the theory of parallel axes for the product of inertia, the product of inertia with respect to ox', oy' is given

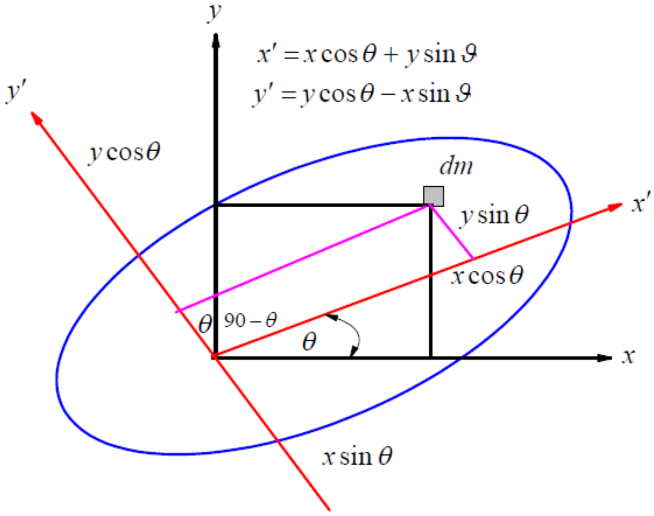
$$I_{xy} = I_{x'y'} + m\,x_{cm}\,y_{cm} \;\to\; \frac{1}{12}mah = I_{x'y'} + m(\frac{1}{3}a)(\frac{1}{3}h) \to\; = I_{x'y'} = \frac{1}{12}mah - \frac{1}{9}mah = \frac{1}{72}(6-8)$$

$$I_{x'y'} = -\frac{1}{36}mah$$

$$I_{x''y''} = I_{x'y'} + m\,x_{cm}\,y_{cm} \;\to\; I_{x''y''} = -\frac{1}{36}mah + m(\frac{2}{3}a)(\frac{-1}{3}h) \to\; = I_{x'y'} = -\frac{1}{36}mah - \frac{2}{9}mah = \frac{1}{36}(-1-8)mah$$

$$I_{x''y''} = \frac{-1}{4}mah$$

IV. Moments of Inertia about inclined axis



For ox' $\quad dI_{x'x'} = (dm)\, y'^2$

$$I_{x'x'} = \int y'^2\, dm \qquad (1)$$

$$y' = y\cos\theta - x\sin\theta$$

$$I_{x'x'} = \int (y\cos\theta - x\sin\theta)^2\, dm = \int y^2\cos^2\theta\, dm + \int x^2\sin^2\theta\, dm - 2\int xy\cos\theta\sin\theta\, dm$$

$$I_{x'x'} = \cos^2\theta \int y^2\, dm + \sin^2\theta \int x^2\, dm - 2\cos\theta\sin\theta \int xy\, dm$$

$$I_{x'x'} = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - I_{xy}\ \sin 2\theta \qquad (2)$$

For oy'

$$I_{y'y'} = \int x'^2\, dm \qquad (3)$$

$$x' = x\cos\theta + y\sin\theta$$

$$I_{y'y'} = \int (x\cos\theta + y\sin\theta)^2\, dm = \int x^2\cos^2\theta\, dm + \int y^2\sin^2\theta\, dm + 2\int xy\cos\theta\sin\theta\, dm$$

$$I_{y'y'} = \cos^2\theta \int x^2\, dm + \sin^2\theta \int y^2\, dm + 2\cos\theta\sin\theta \int xy\, dm$$

$$I_{y'y'} = I_{yy}\cos^2\theta + I_{xx}\sin^2\theta + I_{xy}\sin 2\theta \qquad (4)$$

For $I_{x'y'}$

$$I_{x'y'} = \int x'y'\,dm \qquad (5)$$

$$x' = x\cos\theta + y\sin\theta, \quad y' = y\cos\theta - x\sin\theta$$

$$I_{x'y'} = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dm$$

$$I_{x'y'} = \int x\,y\cos^2\theta\,dm - \int x^2\sin\theta\cos\theta\,dm + \int y^2\cos\theta\sin\theta\,dm - \int x\,y\sin^2\theta\,dm$$

$$I_{x'y'} = \cos^2\theta \int x\,y\,dm - \sin\theta\cos\theta \int x^2\,dm + \sin\theta\cos\theta \int y^2 dm - \sin^2\theta \int x\,y\,dm$$

$$I_{x'y'} = \cos^2\theta\, I_{xy} - \cos\theta\sin\theta\, I_{yy} + \cos\theta\sin\theta\, I_{xx} - \sin^2\theta\, I_{xy}$$

$$I_{x'y'} = (\cos^2\theta - \sin^2\theta) I_{xy} + \sin\theta\cos\theta(I_{xx} - I_{yy})$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta \qquad (6)$$

From Eq. (6), the maximum angle happens at $I_{x'y'} = 0$

$$\tan 2\theta = \frac{2I_{xy}}{I_{yy} - I_{xx}} \qquad (7)$$

Eq. (2)……. $I_{x'x'} = I_{xx}\cos^2\theta + I_{yy}\sin^2\theta - \sin 2\theta\, I_{xy}$

$$I_{x'x'} = I_{xx}\frac{1+\cos 2\theta}{2} + I_{yy}\frac{1-\cos 2\theta}{2} - \sin 2\theta\, I_{xy}$$

$$I_{x'x'} = \frac{I_{xx} + I_{yy}}{2} + \frac{I_{xx} - I_{yy}}{2}\cos 2\theta - I_{xy}\sin 2\theta \qquad (8)$$

Eq. (4)…….$I_{y'y'} = I_{yy}\cos^2\theta + I_{xx}\sin^2\theta + I_{xy}\sin 2\theta$

$$I_{y'y'} = I_{yy}\frac{1+\cos 2\theta}{2} + I_{xx}\frac{1-\cos 2\theta}{2} + I_{xy}\sin 2\theta$$

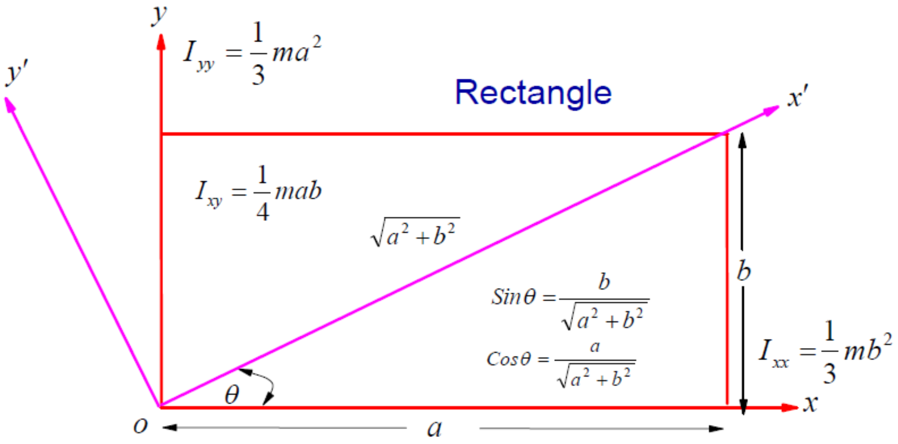
$$I_{y'y'} = \frac{I_{yy} + I_{xx}}{2} + \frac{I_{yy} - I_{xx}}{2}\cos 2\theta + I_{xy}\sin 2\theta \qquad (9)$$

Add 1+2 and 8+9, we have

$$I_{x'x'} + I_{y'y'} = I_{xx} + I_{yy} \qquad (10)$$

Example 3: Find the moment of inertia with respect to a diagonal of the rectangular plate?

Solution



It is well-known

$$I_{x'x'}=I_{xx}\cos^2\theta+I_{yy}\sin^2\theta-I_{xy}\sin 2\theta \quad (1)$$

$$I_{y'y'}=I_{yy}\cos^2\theta+I_{xx}\sin^2\theta+I_{xy}\sin 2\theta \quad (2)$$

$$I_{x'y'}=\frac{I_{xx}-I_{yy}}{2}\sin 2\theta+I_{xy}\cos 2\theta \quad (3)$$

Where

$$I_{xx}=\frac{1}{3}mb^2,\quad I_{yy}=\frac{1}{3}ma^2,\quad I_{xy}=\frac{1}{4}mab \quad (4)$$

Then From Eq. 1-4, we have

$$I_{x'x'}=I_{xx}\cos^2\theta+I_{yy}\sin^2\theta-I_{xy}\sin 2\theta \quad I_{x'x'}=\frac{1}{3}mb^2\cos^2\theta+\frac{1}{3}ma^2\sin^2\theta-\frac{1}{4}mab\,(2\sin\theta\cos\theta)$$

$$I_{x'x'}=\frac{1}{3}mb^2\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2+\frac{1}{3}ma^2\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2-\frac{2}{4}mab\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$I_{x'x'}=\frac{1}{3}m\frac{a^2b^2}{a^2+b^2}+\frac{1}{3}m\frac{a^2b^2}{a^2+b^2}-\frac{1}{2}m\frac{a^2b^2}{a^2+b^2}=m\frac{a^2b^2}{a^2+b^2}\frac{1}{6}(2+2-3)=\frac{1}{6}m\frac{a^2b^2}{a^2+b^2}$$

$$I_{x'x'}=\frac{1}{6}m\left(\frac{a^2b^2}{a^2+b^2}\right) \quad (5)$$

$$I_{y'y'}=I_{yy}\cos^2\theta+I_{xx}\sin^2\theta+I_{xy}\sin 2\theta \qquad I_{y'y'}=\frac{1}{3}ma^2\cos^2\theta+\frac{1}{3}mb^2\sin^2\theta+\frac{1}{4}mab\,(2\sin\theta\cos\theta)$$

$$I_{y'y'} = \frac{1}{3}ma^2\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \frac{1}{3}mb^2\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 + \frac{2}{4}mab\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

$$I_{y'y'} = \frac{1}{3}m\frac{a^4}{a^2+b^2} + \frac{1}{3}m\frac{b^4}{a^2+b^2} + \frac{1}{2}m\frac{a^2b^2}{a^2+b^2} = \frac{1}{3}m\frac{a^4+b^4}{a^2+b^2} + \frac{1}{2}m\frac{a^2b^2}{a^2+b^2}$$

$$I_{y'y'} = \frac{1}{6(a^2+b^2)}m\left(2a^4 + 3a^2b^2 + 2b^4\right) \qquad (6)$$

$$I_{x'y'} = \frac{I_{xx} - I_{yy}}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{x'y'} = (I_{xx} - I_{yy})\sin\theta\cos\theta + I_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$I_{x'y'} = \left(\frac{1}{3}mb^2 - \frac{1}{3}ma^2\right)\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right) + \frac{1}{4}mab\left\{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 - \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2\right\}$$

$$I_{x'y'} = \frac{1}{3}m\left(b^2 - a^2\right)\left(\frac{ab}{a^2+b^2}\right) + \frac{1}{4}mab\left\{\frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2}\right\} = \frac{1}{12}m\left(b^2 - a^2\right)\left(\frac{ab}{a^2+b^2}\right)(4-3)$$

$$I_{x'y'} = \frac{1}{12}m\left(b^2 - a^2\right)\left(\frac{ab}{a^2+b^2}\right) \qquad (7)$$

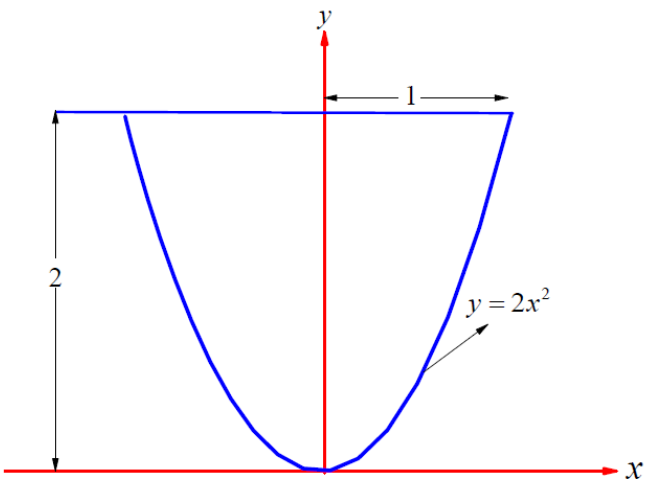
Note that at $\theta = 45^{o}$, we have $a = b$, Then

$$I_{x'x'} = \frac{1}{6}m\left(\frac{a^2b^2}{a^2+b^2}\right) = \frac{1}{12}m,$$

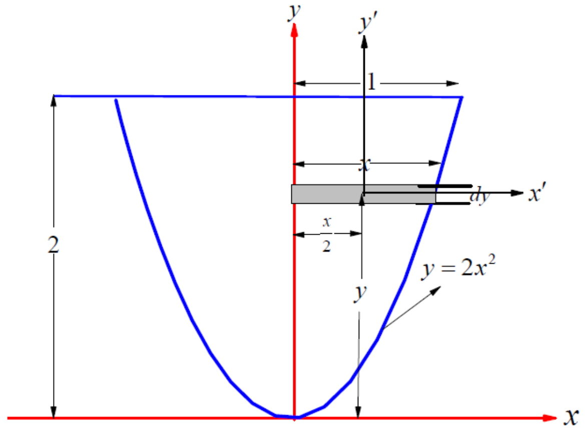
$$I_{y'y'} = \frac{1}{6(a^2+b^2)}m\left(2a^4 + 3a^2b^2 + 2b^4\right) = \frac{1}{6(2)}(7)m = \frac{7}{12}m, \qquad (8)$$

$$I_{x'y'} = \frac{1}{12}m\left(b^2 - a^2\right)\left(\frac{ab}{a^2+b^2}\right) = 0$$

Example 4: Determine the product of inertia I_{xy} of the right half of the parabolic area, bounded by the $y = 2m$ and $x = 0$?



Solution



$$dm = \rho x dy \rightarrow m = \rho\int_0^2 \left(\frac{y}{2}\right)^{\frac{1}{2}} dy = \frac{\rho}{\frac{3}{2}\sqrt{2}}[y]^{\frac{3}{2}}\Bigg|_0^2 = \frac{2\rho}{3\sqrt{2}}[2]^{\frac{3}{2}} = \frac{2\rho}{3\sqrt{2}}[8]^{\frac{1}{2}} = \frac{4\rho}{3\sqrt{2}}[2]^{\frac{1}{2}} \rightarrow m = \frac{4}{3}\rho$$

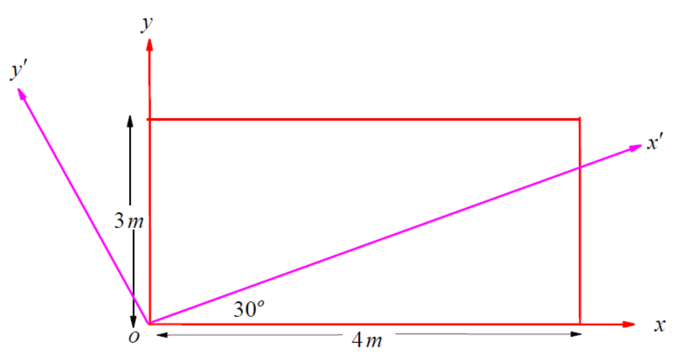
$$dI_{xy} = dI_{x'y'} + dm\, x_{cm}\, y_{cm}$$

$$I_{xy} = \int dm\, x_{cm}\, y_{cm} = \int(\rho x dy)(\frac{1}{2}x)\, y = \frac{1}{2}\rho\int_0^2 x^2\, y\, dy = \frac{1}{2}\rho\int_0^2 \frac{y}{2}\, y\, dy = \frac{1}{4}\rho\int_0^2 y^2\, dy$$

$$= \frac{1}{4}\rho\frac{y^3}{3}\Bigg|_0^2 = \frac{1}{4}\rho\frac{8}{3} = \frac{8}{12}\rho = \frac{2}{3}\rho = \frac{2}{3}\rho\frac{m}{m} = \frac{2}{3}\rho\frac{m}{\frac{4}{3}\rho} = \frac{1}{2}m \qquad I_{xy} = \frac{1}{2}m$$

Exercise

Find the moment of inertia about ox' and oy' axis also the product of inertia for rectangular plate as is shown Figure (3×4)?



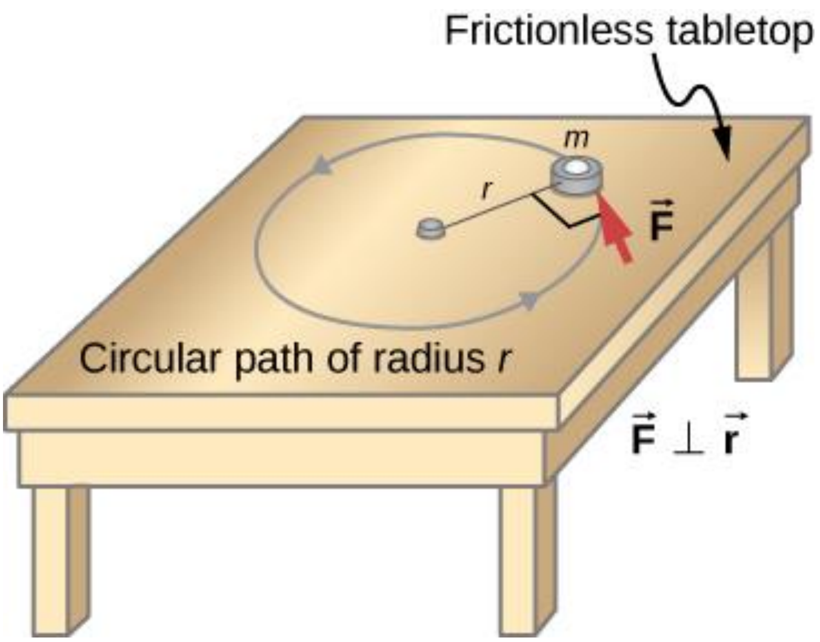
Chapter 3

Application

Newton's second law for rotation

We have thus far found many counterparts to the translational terms used throughout this text, most recently, torque, the rotational analog to force. This raises the question: Is there an analogous equation to Newton's second law $\sum \vec{F} = m\vec{a}$, which involves torque and rotational motion? To investigate this, we start with Newton's second law for a single particle rotating around an axis and executing circular motion. Let's exert a force $\vec{F}$ on a point mass m that is at a distance r from a pivot point (see below Figure). The particle is constrained to move in a circular path with fixed radius and the force is tangent to the circle. We apply Newton's second law to determine the magnitude of the acceleration $a = \frac{F}{m}$ in the direction of $\vec{F}$.

Recall that the magnitude of the tangential acceleration is proportional to the magnitude of the angular acceleration by $a = r\alpha$



Substituting this expression into Newton's second law, we obtain $F = m\,r\,\alpha$

Multiply both sides of this equation by r, we have $r\,F = m\,r^2\,\alpha$

Note that the left side of this equation is the torque about the axis of rotation, where r is the lever arm and F is the force, perpendicular to r. Recall that the moment of inertia for a point particle is $I = m\,r^2$. The torque applied perpendicularly to the point mass in above Figure is therefore $\tau = I\,\alpha$

The torque on the particle is equal to the moment of inertia about the rotation axis times the angular acceleration. We can generalize this equation to a rigid body rotating about a fixed axis.

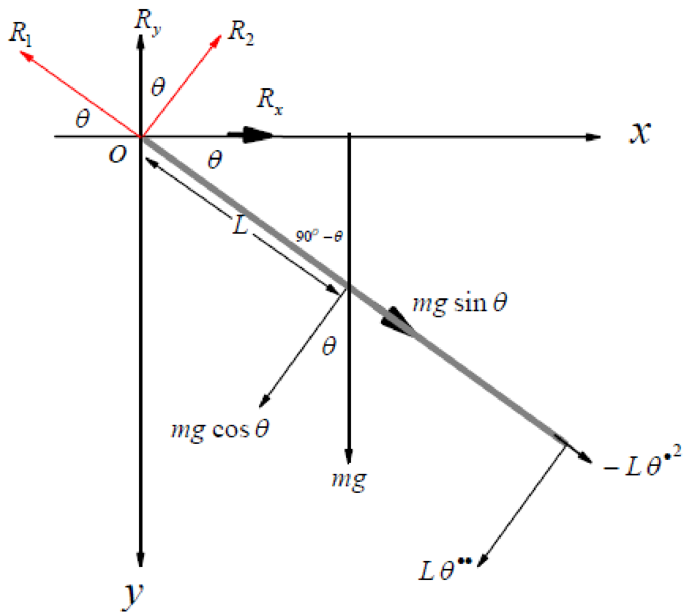
If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\sum_i \tau_i = I\,\alpha$$

The term $I\,\alpha$ is a scalar quantity and can be positive or negative (counterclockwise or clockwise) depending upon the sign of the net torque. Remember the convention that counterclockwise angular acceleration is positive. Thus, if a rigid body is rotating clockwise and experiences a positive torque (counterclockwise), the angular acceleration is positive.

Example-1: A uniform rod of length $2L$ and mass M is pivoted (is hinged) at one end and the other one is free to rotate in the vertical plane. If the rod is beginning the rotation when it was horizontally. Prove that the horizontal reaction will be maximum when the Rod tilts on the horizontal at an angle $\frac{\pi}{4}$ and in this case the vertical reaction is given as $\frac{11}{8}mg$?

Solution



The motion of center of Rod

$$m(-L\theta^{\bullet 2}) = mg\sin\theta - R_1 \;\rightarrow\; mL\theta^{\bullet 2} = R_1 - mg\sin\theta \qquad (1)$$

$$mL\theta^{\bullet\bullet} = mg\cos\theta - R_2 \qquad (2)$$

The rotation of motion (at then of Rod)

$$\frac{d}{dt}\left(I_O\,\theta^{\bullet}\right) = M_o \;\rightarrow\; I_O\,\theta^{\bullet\bullet} = M_o \qquad (3)$$

Eq. (3) maybe written as

$$\frac{1}{3}m(2L)^2\theta^{\bullet\bullet} = (mg\cos\theta)\;(L) \rightarrow \quad \theta^{\bullet\bullet} = \frac{3g}{4L}\cos\theta \qquad (4)$$

$$\theta^{\bullet}\frac{d\theta^{\bullet}}{d\theta} = \frac{3g}{4L}\cos\theta \;\rightarrow\; \int \theta^{\bullet}d\theta^{\bullet} = \frac{3g}{4L}\int \cos\theta\; d\theta \;\rightarrow\; \frac{\theta^{\bullet 2}}{2} = \frac{3g}{4L}\sin\theta + c_1$$

At the start point $\theta = 0$ and $\theta^{\bullet} = 0$, then $c_1 = 0$

$$\theta^{\bullet 2} = \frac{3g}{2L}\sin\theta \qquad (5)$$

From Eq, (5) into Eq. (1) $mL\left(\frac{3g}{2L}\sin\theta\right) = R_1 - mg\sin\theta \rightarrow R_1 = mL\left(\frac{3g}{2L}\sin\theta\right) + mg\sin\theta$

$$R_1 = \frac{5}{2}mg\;\sin\theta \qquad (6)$$

From Eq, (4) into Eq. (2)

$$mL\left(\frac{3g}{4L}\cos\theta\right)=mg\cos\theta-R_2 \quad\rightarrow\quad R_2=mg\cos\theta-\frac{3}{4}mg\cos\theta$$

$$R_2=\frac{1}{4}mg\cos\theta \qquad (7)$$

But

$$R_x=R_2\sin\theta-R_1\cos\theta \qquad (8)$$

$$R_y=R_1\sin\theta+R_2\cos\theta \qquad (9)$$

Then

$$R_x=(\frac{1}{4}mg\cos\theta)\sin\theta-(\frac{5}{2}mg\ \sin\theta)\cos\theta \rightarrow \quad R_x=-\frac{9}{4}mg\ \sin\theta\cos\theta$$

$$R_x=-\frac{9}{8}mg\ \sin 2\theta \qquad (10)$$

$$R_y=(\frac{5}{2}mg\ \sin\theta)\sin\theta+(\frac{1}{4}mg\cos\theta)\cos\theta \ \rightarrow\ R_y=\frac{5}{2}mg\ \sin^2\theta+\frac{1}{4}mg\cos^2\theta$$

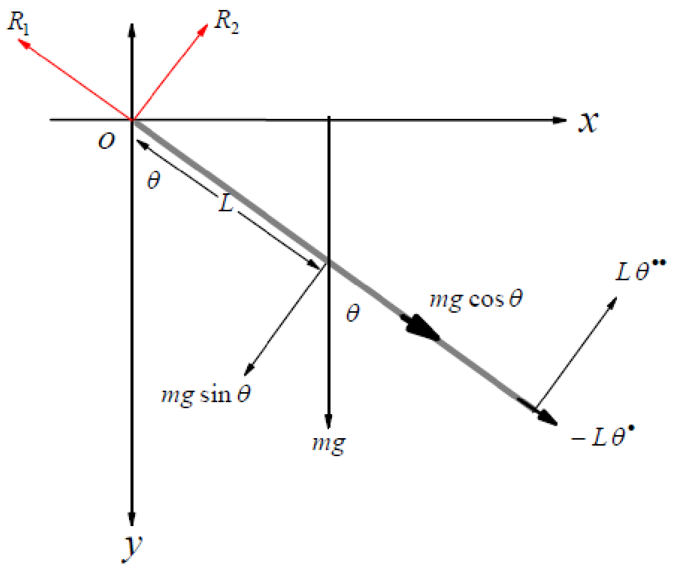
$$R_y=\left(\frac{5}{2}\sin^2\theta+\frac{1}{4}\cos^2\theta\right)mg \qquad (11)$$

From Eq. (9) $R_x=\frac{9}{8}mg\ \sin 2\theta$, R_x is maximum if $\sin 2\theta$ is maximum and $\sin 2\theta$ is maximum if $\sin 2\theta=1$, then $2\theta=\frac{\pi}{2}\rightarrow \quad \theta=\frac{\pi}{4}$

In this case $R_y(\theta=\frac{\pi}{4})=\left(\frac{5}{2}\left(\frac{1}{\sqrt{2}}\right)^2+\frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^2\right)mg \quad\rightarrow\quad R_y=\frac{11}{8}mg$

Example-2: A uniform rod of length L and mass M is pivoted (is hinged) at one end and the other one is free to rotate in the vertical plane. If the rod is beginning the rotation when it was vertically with angle velocity $\sqrt{\frac{3g}{L}}$. Find the reaction at the hinged point at $\theta=\frac{\pi}{3}$ and prove that the Rod move angle θ in time $t=2\sqrt{\frac{L}{3g}}\ln\left(\sec(\frac{\theta}{2})+\tan(\frac{\theta}{2})\right)$.

Solution



<u>The motion of center of Rod</u>

$$-mL\theta^{\bullet 2} = mg\cos\theta - R_1 \quad \rightarrow \quad mL\theta^{\bullet 2} = R_1 - mg\cos\theta \qquad (1)$$

$$mL\theta^{\bullet\bullet} = R_2 - mg\sin\theta \qquad (2)$$

<u>The rotation of motion (At then of Rod)</u>

$$\frac{d}{dt}\left(I_O\,\theta^{\bullet}\right) = M_o \quad \rightarrow \quad I_O\,\theta^{\bullet\bullet} = M_o \qquad (3)$$

Eq. (3) maybe written as

$$\frac{1}{3}m(2L)^2\theta^{\bullet\bullet} = (-mg\sin\theta)\ (L) \rightarrow \quad \theta^{\bullet\bullet} = -\frac{3g}{4L}\sin\theta \qquad (4)$$

$$\theta^{\bullet}\frac{d\theta^{\bullet}}{d\theta} = -\frac{3g}{4L}\sin\theta \quad \rightarrow \quad \int\theta^{\bullet}d\theta^{\bullet} = -\frac{3g}{4L}\int\sin\theta\, d\theta \quad \rightarrow \frac{\theta^{\bullet 2}}{2} = \frac{3g}{4L}\cos\theta + c_1$$

At the start point $\theta = 0$ and $\theta^{\bullet} = \sqrt{\frac{3g}{L}}$, then $c_1 = \frac{3g}{2L} - \frac{3g}{4L} = \frac{3g}{4L}$

$$\frac{\dot{\theta}^2}{2} = \frac{3g}{4L}\cos\theta + \frac{3g}{4L} \quad \rightarrow \quad \dot{\theta}^2 = \frac{3g}{2L}(1+\cos\theta) \qquad (5)$$

Note that

$$\cos(\theta) = \cos(\frac{\theta}{2}+\frac{\theta}{2}) = \cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2}) = \cos^2(\frac{\theta}{2}) - (1-\cos^2(\frac{\theta}{2})) = 2\cos^2(\frac{\theta}{2}) - 1$$

From Eq. (5), we have

$$\dot{\theta} = \frac{d\theta}{dt} = \sqrt{\frac{3g}{2L}\left(2\cos^2(\frac{\theta}{2})\right)} \quad \rightarrow \frac{d\theta}{dt} = \sqrt{\frac{3g}{L}}\cos(\frac{\theta}{2}) \quad \rightarrow \int \frac{1}{\cos(\frac{\theta}{2})} d\theta = \sqrt{\frac{3g}{L}} \int dt$$

$$2\int \sec(\frac{\theta}{2})\, d(\frac{\theta}{2}) = \sqrt{\frac{3g}{L}} \int dt \quad \rightarrow \quad = 2\ln\left(\sec(\frac{\theta}{2}) + \tan(\frac{\theta}{2})\right) = \sqrt{\frac{3g}{L}}\, t + c_2$$

At the start point $\theta = 0$ and $t = 0$

$$2\ln\left(\sec(0) + \tan(0)\right) = c_2 \quad \rightarrow \quad c_2 = 2\ln(1+0) = 0$$

$$t = 2\sqrt{\frac{L}{3g}}\ln\left(\sec(\frac{\theta}{2}) + \tan(\frac{\theta}{2})\right) \qquad (6)$$

$$mL\left(\frac{3g}{2L}(1+\cos\theta)\right) = R_1 - mg\cos\theta \rightarrow \quad R_1 = mL\left(\frac{3g}{2L}(1+\cos\theta)\right) + mg\cos\theta \;\; R_1 = \frac{1}{2}\left(3 + 5\cos\theta\right)mg$$

(7) $$mL\left(-\frac{3g}{4L}\sin\theta\right) = R_2 - mg\sin\theta \quad \rightarrow \quad R_2 = mg\sin\theta - \frac{3}{4}mg\sin\theta$$

$$R_2 = \frac{1}{4}mg\sin\theta \qquad (8)$$

At $\theta = \frac{\pi}{3}$

$$R_1 = \frac{1}{2}\left(3 + 5\cos(\frac{\pi}{3})\right)mg = \frac{1}{2}\left(3 + \frac{5}{2}\right)mg = \frac{11}{4}mg \quad \rightarrow \quad R_1 = \frac{11}{4}mg \qquad (9)$$

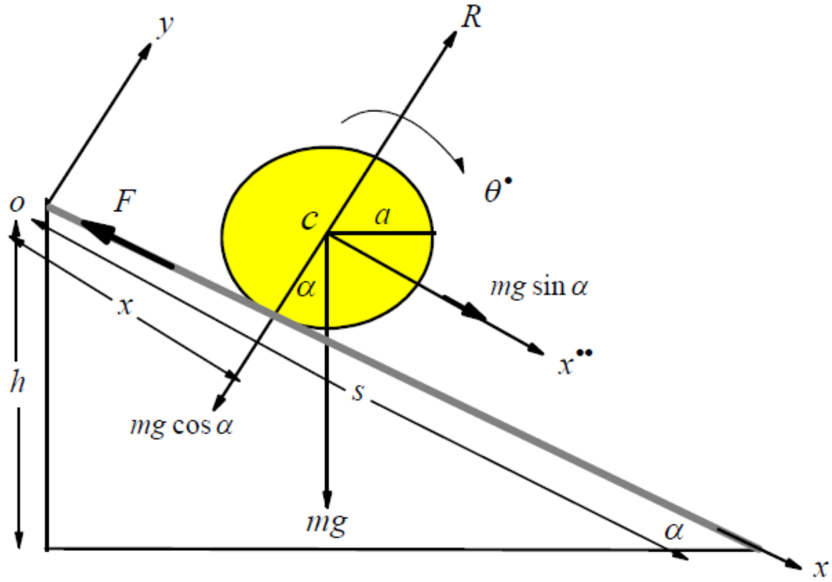
$$R_2 = \frac{1}{4}mg\sin\theta = \frac{1}{4}mg\sin(\frac{\pi}{3}) = \frac{1}{4}mg(\frac{\sqrt{3}}{2}) \quad \rightarrow \quad R_2 = \frac{\sqrt{3}}{8}mg \qquad (10)$$

Example 3: A body rolls down an inclined plane without slipping. Describe the motion of the body?

Solution

First draw a free body diagram of the body, which down the plane:

We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m\ddot{x} = mg\ \sin\alpha - F \qquad (1)$$

$$m(0) = R - mg\ \cos\alpha \qquad (2)$$

Equation of Rotational Motion of a Rigid Body

$$\frac{d}{dt}\left(I_c\,\dot{\theta}\right) = M_c \ \rightarrow\ I_c\,\ddot{\theta} = M_c \qquad (3)$$

$$I_c\,\ddot{\theta} = (F)\ (a) \rightarrow\ F = \frac{I_c}{a}\ddot{\theta} \qquad (4)$$

$$m\ddot{x} = mg\ \sin\alpha - \frac{I_c}{a}\ddot{\theta} \qquad (5)$$

Pure rolling

$\dot{x} = a\dot{\theta}$, then $\ddot{x} = a\ddot{\theta}$

$$m\ddot{x} = mg\sin\alpha - \frac{I_c}{a^2}\ddot{x} \rightarrow \ddot{x} + \frac{I_c}{ma^2}\ddot{x} = g\sin\alpha$$

$$\ddot{x} = \frac{g\sin\alpha}{1+\frac{I_c}{ma^2}} \rightarrow a\ddot{\theta} = \ddot{x} = \frac{g\sin\alpha}{1+\frac{I_c}{ma^2}} \qquad (6)$$

$$v^2 = v_0 + 2\ddot{x}x$$

$$v^2 = 0 + 2\left(\frac{g\sin\alpha}{1+\frac{I_c}{ma^2}}\right)s \rightarrow v^2 = 2\left(\frac{g\sin\alpha}{1+\frac{I_c}{ma^2}}\right)\frac{h}{\sin\alpha} \rightarrow v^2 = \frac{2gh}{1+\frac{I_c}{ma^2}}$$

$$v = \sqrt{\frac{2gh}{1+\frac{I_c}{ma^2}}} \qquad (7)$$

$$F = \frac{I_c}{a}\frac{1}{a}\left(\frac{g\sin\alpha}{1+\frac{I_c}{ma^2}}\right) \rightarrow F = \left(\frac{I_c}{ma^2 + I_c}\right)mg\sin\alpha \quad or \quad F = \frac{1}{\frac{ma^2}{I_c}+1}mg\sin\alpha \qquad (8)$$

$$F < \mu R$$

$$F < \mu R \rightarrow \mu > \frac{F}{R} \rightarrow \mu > \frac{\left(\frac{I_c}{ma^2 + I_c}\right)mg\sin\alpha}{mg\cos\alpha}$$

$$\mu > \left(\frac{I_c}{ma^2 + I_c}\right)\tan\alpha \qquad (9)$$

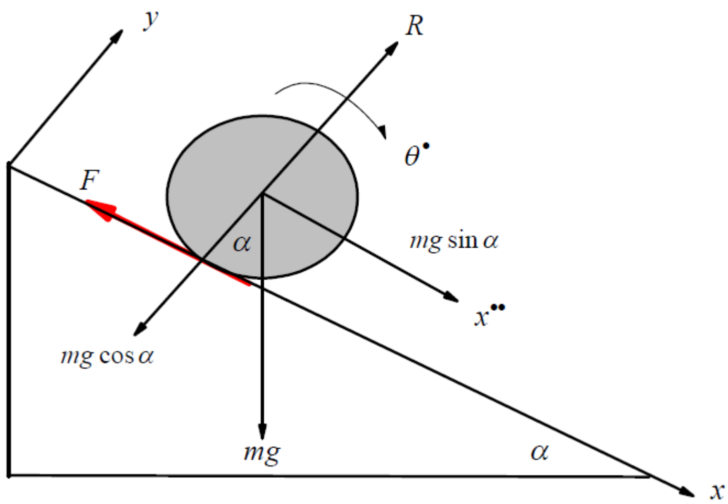
$$\mu = \left(\frac{I_c}{ma^2 + I_c}\right)\tan\alpha$$

Example 4: A Solid Cylinder of mass m and radius a rolls without slipping down an inclined plane whose incline angle with the horizontal is theta. Determine the acceleration of the cylinder's center of mass, and the minimum coefficient of friction that will allow the cylinder to roll without slipping on this incline?

Solution

First draw a free body diagram of the cylinder, which down the plane:

We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m\ddot{x} = mg\ \sin\alpha - F \qquad (1)$$

$$mg\ \cos\alpha = R \qquad (2)$$

Rotational Motion of a Cylinder

$$\frac{d}{dt}\left(I_c\dot{\theta}\right) = M_c \ \rightarrow \ I_c\ddot{\theta} = M_c \qquad (3)$$

that can be written as

$$\frac{1}{2}ma^2\ddot{\theta} = (F)\ (a) \rightarrow \quad F = \frac{1}{2}ma\ddot{\theta} \qquad (4)$$

The necessary condition for rolling without slipping is the contact point have zero velocity (the condition for no sliding is). i. e. $\dot{x} = a\dot{\theta} \rightarrow \ddot{x} = a\ddot{\theta}$. Substitute in Eq. (4), we have

$$F = \frac{1}{2} m \ddot{x} \qquad (5)$$

Again, Substituting from Eq. (5) into Eq. (1), we have

$$m\ddot{x} = mg \sin\alpha - \frac{1}{2} m \ddot{x} \rightarrow \ddot{x} + \frac{1}{2}\ddot{x} = g\sin\alpha \rightarrow \frac{3}{2}\ddot{x} = g\sin\alpha$$

$$\ddot{x} = \frac{2}{3} g\sin\alpha \qquad (6)$$

Substituting from Eq. (6) into Eq. (5), we have

$$F = \frac{1}{2} m \left(\frac{2}{3} g\sin\alpha\right) \rightarrow F = \frac{1}{3} m g \sin\alpha \qquad (7)$$

Again, the necessary condition for rolling without slipping is the static coefficient and is generally lower than the static coefficient of friction. i. e. $F < \mu R$

$$\mu > \frac{F}{R} \qquad (8)$$

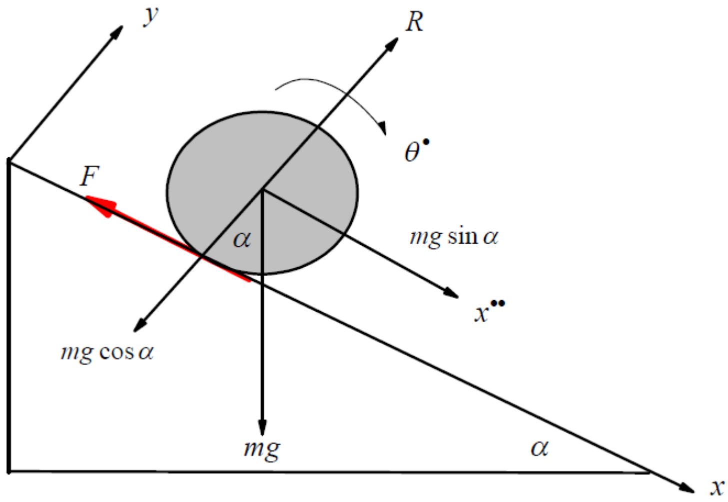
$$\mu > \frac{\frac{1}{3} m g \sin\alpha}{mg \cos\alpha} \rightarrow \mu > \frac{1}{3}\frac{\sin\alpha}{\cos\alpha} \rightarrow \mu > \frac{1}{3}\tan\alpha$$

Example 5: Calculate the minimum coefficient of friction necessary to keep a thin circular ring from sliding as it rolls down a plane inclined at an angle θ with respect to the horizontal plane.

Solution

First draw a free body diagram of the ring, which down the plane:

We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m x^{\bullet\bullet} = mg \sin\alpha - F \qquad (1)$$

$$mg \cos\alpha = R \qquad (2)$$

Rotational motion equations

$$\frac{d}{dt}\left(I_c \theta^{\bullet}\right) = M_c \rightarrow I_c \theta^{\bullet\bullet} = M_c \qquad (3)$$

that can be written as

$$m a^2 \theta^{\bullet\bullet} = (F)\ (a) \rightarrow F = m a \theta^{\bullet\bullet} \qquad (4)$$

The necessary condition for rolling without slipping is the contact point have zero velocity (the condition for no sliding is). i. e. $x^{\bullet} = a\theta^{\bullet} \rightarrow x^{\bullet\bullet} = a\theta^{\bullet\bullet}$. Substitute in Eq. (4), we have

$$F = m\ x^{\bullet\bullet} \qquad (5)$$

Again, Substituting from Eq.. (5) into Eq. (1), we have

$$m x^{\bullet\bullet} = mg \sin\alpha - m\ x^{\bullet\bullet} \rightarrow x^{\bullet\bullet} + x^{\bullet\bullet} = g \sin\alpha \rightarrow 2\ x^{\bullet\bullet} = g \sin\alpha$$

$$x^{\bullet\bullet} = \frac{1}{2}\ g \sin\alpha \qquad (6)$$

Substituting from Eq.. (6) into Eq. (5), we have

$$F = m\,(\frac{1}{2}\,g\sin\alpha) \quad \rightarrow \quad F = \frac{1}{2}m\,g\sin\alpha \qquad (7)$$

Again, the necessary condition for rolling without slipping is the static coefficient and is generally lower than the static coefficient of friction. i. e. $F < \mu R$

$$\mu > \frac{F}{R} \qquad (8)$$

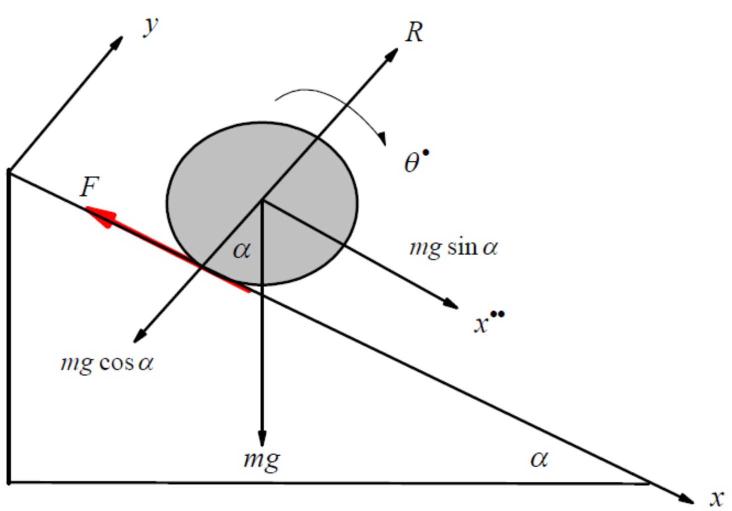
$$\mu > \frac{\frac{1}{2}m\,g\sin\alpha}{mg\;\cos\alpha} \quad \rightarrow \quad \mu > \frac{1}{2}\frac{\sin\alpha}{\cos\alpha} \quad \rightarrow \quad \mu > \frac{1}{2}\tan\alpha$$

Example 6: A uniform solid sphere of mass m and radius a rolls without slipping down an inclined plane whose incline angle with the horizontal is theta. Determine the acceleration of the ball's center of mass, and the minimum coefficient of friction that will allow the ball to roll without slipping on this incline?

Solution

First draw a free body diagram of the sphere, which down the plane:

We can write both of the Linear and rotation equations of motion



Linear equations of motion (Equations of motion of center of gravity)

$$m\ddot{x} = mg\sin\alpha - F \qquad (1)$$

$$mg\cos\alpha = R \qquad (2)$$

Rotational motion equations

$$\frac{d}{dt}\left(I_c\dot{\theta}\right) = M_c \rightarrow I_c\ddot{\theta} = M_c \qquad (3)$$

that can be written as

$$\frac{2}{5}ma^2\ddot{\theta} = (F)\,(a) \rightarrow F = \frac{2}{5}ma\ddot{\theta} \qquad (4)$$

The necessary condition for rolling without slipping is the contact point have zero velocity. i. e. $\dot{x} = a\dot{\theta} \rightarrow \ddot{x} = a\ddot{\theta}$. Substitute in Eq. (4), we have

$$F = \frac{2}{5}m\ddot{x} \qquad (5)$$

Again, Substituting from Eq. (5) into Eq. (1), we have

$$m\ddot{x} = mg\sin\alpha - \frac{2}{5}m\ddot{x} \rightarrow \ddot{x} + \frac{2}{5}\ddot{x} = g\sin\alpha \rightarrow \frac{7}{5}\ddot{x} = g\sin\alpha$$

$$\ddot{x} = \frac{5}{7}g\sin\alpha \qquad (6)$$

Substituting from Eq. (6) into Eq. (5), we have

$$F = \frac{2}{5}m\left(\frac{5}{7}g\sin\alpha\right) \rightarrow F = \frac{2}{7}mg\sin\alpha \qquad (7)$$

Again, the necessary condition for rolling without slipping is the static coefficient and is generally lower than the static coefficient of friction. i. e. $F < \mu R$

$$\mu > \frac{F}{R} \qquad (8)$$

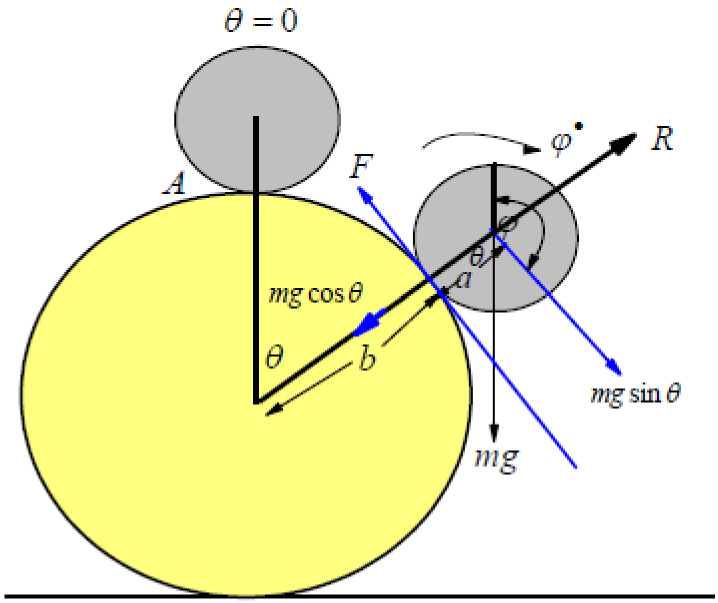
$$\mu > \frac{\frac{2}{7}mg\sin\alpha}{mg\cos\alpha} \rightarrow \mu > \frac{2}{7}\frac{\sin\alpha}{\cos\alpha} \rightarrow \mu > \frac{2}{7}\tan\alpha$$

Example 7: A uniform sphere of radius a initially at rest rolls without slipping down from the top of a rough sphere of radius b. Find the angular velocity of the ball at the instant it breaks off the sphere and show that the angle $\cos^{-1}\left(\frac{10}{17}\right)$ with the vertical?

Solution

First draw a free body diagram of the sphere, which down the plane:

We can write both of the Linear and rotation equations of motion



Note that

$$\vec{v} = (v_r, v_\theta) = (\dot{r}, r\dot{\theta})$$

$$\vec{a} = (a_r, a_\theta) = (\ddot{r} - r\dot{\theta}^2, r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

<u>Equations of motion of Center of Gravity</u>

$$m(a+b)\ddot{\theta} = mg \sin\theta - F \qquad (1)$$

$$-m(a+b)\theta^{\bullet^2} = mg\ \cos\theta - R \qquad (2)$$

Rotational motion equation

$$\frac{d}{dt}\left(I_c\,\varphi^{\bullet}\right) = M_c \ \rightarrow \ I_c\,\varphi^{\bullet\bullet} = M_c \qquad (3)$$

That can be written as

$$\frac{2}{5}ma^2\,\varphi^{\bullet\bullet} = (F)\ (a) \rightarrow \quad F = \frac{2}{5}ma\,\varphi^{\bullet\bullet} \qquad (4)$$

The condition for pure rolling is $(a+b)\theta = a\varphi \rightarrow (a+b)\theta^{\bullet} = a\varphi^{\bullet}$, then

$$(a+b)\theta^{\bullet\bullet} = a\varphi^{\bullet\bullet} \quad \rightarrow \quad \varphi^{\bullet\bullet} = \frac{a+b}{a}\theta^{\bullet\bullet} \qquad (5)$$

Substituting from Eq. (5) into Eq. (4), we have

$$F = \frac{2}{5}m(a+b)\ \theta^{\bullet\bullet} \qquad (6)$$

Again, substituting from Eq. (6) into Eq. (1), we have

$$m(a+b)\ \theta^{\bullet\bullet} = mg\ \sin\theta - \frac{2}{5}m(a+b)\ \theta^{\bullet\bullet} \ \rightarrow \ \frac{7}{5}m(a+b)\ \theta^{\bullet\bullet} = mg\ \sin\theta$$

$$\theta^{\bullet\bullet} = \frac{5}{7(a+b)}g\ \sin\theta \qquad (7)$$

$$\theta^{\bullet}\frac{d\theta^{\bullet}}{d\theta} = \frac{5}{7(a+b)}g\ \sin\theta \ \rightarrow \ \int\theta^{\bullet}d\theta^{\bullet} = \frac{5}{7(a+b)}g\int\sin\theta\,d\theta$$

$$\frac{\theta^{\bullet^2}}{2} = -\frac{5}{7(a+b)}g\ \cos\theta + c_1 \qquad (8)$$

At the initial motion $\theta = 0$, then $\theta^{\bullet} = 0$

Then in Eq. (8), we have $c_1 = \frac{5}{7(a+b)}g$ and again in Eq. (8), we have

$$\frac{\dot{\theta}^2}{2} = -\frac{5}{7(a+b)} g \cos\theta + \frac{5}{7(a+b)} g = \frac{5}{7(a+b)} g (1-\cos\theta)$$

$$\dot{\theta}^2 = \frac{10\,g}{7(a+b)}(1-\cos\theta) \quad \rightarrow \quad \dot{\theta}^2 (a+b) = \frac{10\,g}{7}(1-\cos\theta) \qquad (9)$$

Substituting from Eq. (9) into Eq. (2), we have

$$-m\left(\frac{10}{7} g (1-\cos\theta)\right) = mg \cos\theta - R$$

$$R = mg \cos\theta - \frac{10}{7} mg (1-\cos\theta) \quad \rightarrow \quad R = \frac{17}{7} mg \cos\theta - \frac{10}{7} mg \qquad (10)$$

When the ball instant breaks off the sphere

At the instant, that the ball breaks off the sphere, the reaction equals zero, so from Eq. (10), we have

$$\frac{17}{7} mg \cos\theta - \frac{10}{7} mg = 0 \quad \rightarrow \quad \frac{17}{7} mg \cos\theta = \frac{10}{7} mg \quad \rightarrow \quad 17 \cos\theta = 10$$

$$\cos\theta = \frac{10}{17} \quad \rightarrow \quad \theta = \cos^{-1}\left(\frac{10}{17}\right) \quad \rightarrow \quad \theta = 53.968^{o} \qquad (11)$$

In this case the angle will be maximum ($\theta = \theta_{max}$)

Where the velocity is given by $\vec{v} = (v_r, v_\theta) = (\dot{r}, r\dot{\theta})$ حيث $\vec{v} = (0, r\dot{\theta})$

$$v = r\dot{\theta} = (a+b)\sqrt{\frac{10\,g}{7(a+b)}(1-\cos\theta)} = \sqrt{\frac{10\,g}{7}(a+b)(1-\cos\theta)}$$

At the moment ($\theta = \theta_{max}$)

$$v = \sqrt{\frac{10\,g}{7}(a+b)(1-\frac{10}{17})} = \sqrt{\frac{10\,g}{7}(a+b)(\frac{17-10}{17})} = \sqrt{\frac{10\,g}{7}(a+b)(\frac{7}{17})}$$

$$v = \sqrt{\frac{10\,g}{17}(a+b)} \qquad (12)$$