محاضرات في
الهندسة التحليلية (2)

لطلاب الفرقة الثانية بكلية التربية

"برنامج اللغة"

العام الجامعي: 2024–2025

SECTION 9.1 The Ellipse

Objectives

1. Graph ellipses centered at the origin.
2. Write equations of ellipses in standard form.
3. Graph ellipses not centered at the origin.
4. Solve applied problems involving ellipses.

You took on a summer job driving a truck, delivering books that were ordered online. You're an avid reader, so just being around books sounded appealing. However, now you're feeling a bit shaky driving the truck for the first time. It's 10 feet wide and 9 feet high; compared to your compact car, it feels like you're behind the wheel of a tank. Up ahead you see a sign at the semielliptical entrance to a tunnel: Caution! Tunnel is 10 Feet High at Center Peak. Then you see another sign: Caution! Tunnel Is 40 Feet Wide. Will your truck clear the opening of the tunnel's archway?

Mathematics is present in the movements of planets, bridge and tunnel construction, navigational systems used to keep track of a ship's location, manufacture of lenses for telescopes, and even in a procedure for disintegrating kidney stones. The mathematics behind these applications involves conic sections. **Conic sections** are curves that result from the intersection of a right circular cone and a plane. Figure 9.1 illustrates the four conic sections: the circle, the ellipse, the parabola, and the hyperbola.

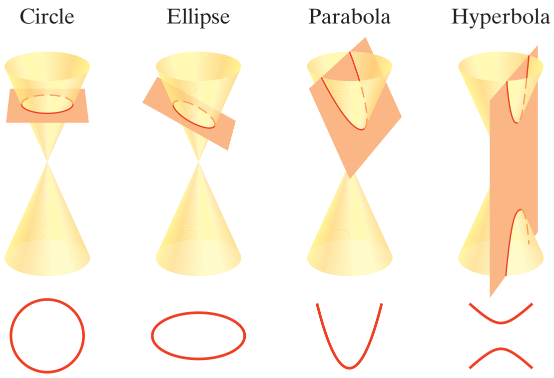


Figure 9.1 Obtaining the conic sections by intersecting a plane and a cone

In this section, we study the symmetric oval-shaped curve known as the ellipse. We will use a geometric definition for an ellipse to derive its equation. With this equation, we will determine if your delivery truck will clear the tunnel's entrance.

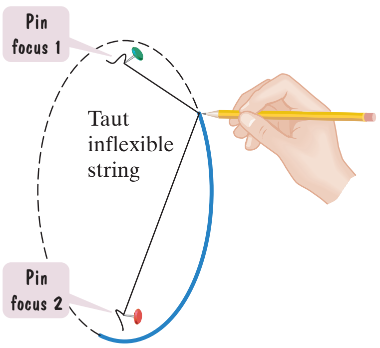


Figure 9.2 Drawing an ellipse

Definition of an Ellipse

Figure 9.2 illustrates how to draw an ellipse. Place pins at two fixed points, each of which is called a focus (plural: foci). If the ends of a fixed length of string are fastened to the pins and we draw the string taut with a pencil, the path traced by the pencil will be an ellipse. Notice that the sum of the distances of the pencil point from the foci remains constant because the length of the string is fixed. This procedure for drawing an ellipse illustrates its geometric definition.

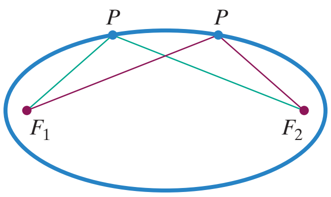


Figure 9.3

Definition of an Ellipse

An **ellipse** is the set of all points, P, in a plane the sum of whose distances from two fixed points, F_1 and F_2, is constant (see Figure 9.3). These two fixed points are called the **foci** (plural of **focus**). The midpoint of the segment connecting the foci is the **center** of the ellipse.

Figure 9.4 illustrates that an ellipse can be elongated in any direction. In this section, we will limit our discussion to ellipses that are elongated horizontally or vertically. The line through the foci intersects the ellipse at two points, called the **vertices** (singular: **vertex**). The line segment that joins the vertices is the **major axis**. Notice that the midpoint of the major axis is the center of the ellipse. The line segment whose endpoints are on the ellipse and that is perpendicular to the major axis at the center is called the **minor axis** of the ellipse.

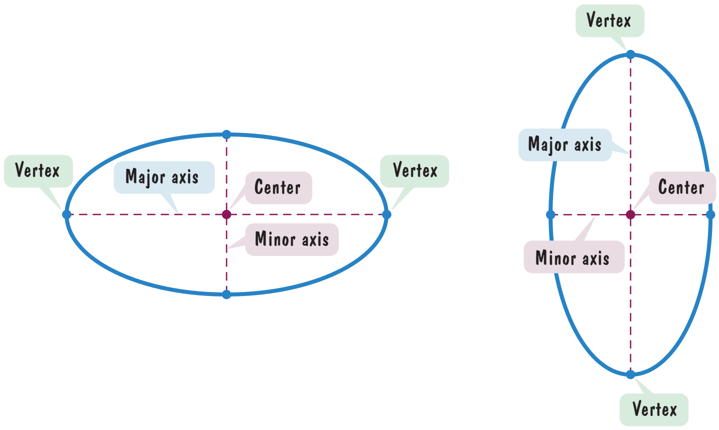


Figure 9.4 Horizontal and vertical elongations of an ellipse

Standard Form of the Equation of an Ellipse

The rectangular coordinate system gives us a unique way of describing an ellipse. It enables us to translate an ellipse's geometric definition into an algebraic equation.

We start with Figure 9.5 to obtain an ellipse's equation. We've placed an ellipse that is elongated horizontally into a rectangular coordinate system. The foci are on the x-axis at $(-c, 0)$ and $(c, 0)$, as in Figure 9.5. In this way, the center of the ellipse is at the origin. We let (x, y) represent the coordinates of any point on the ellipse.

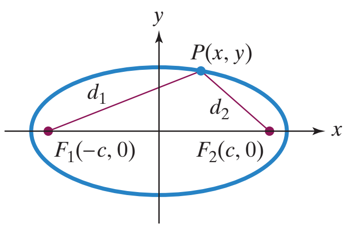


Figure 9.5

What does the definition of an ellipse tell us about the point (x, y) in Figure 9.5? For any point (x, y) on the ellipse, the sum of the distances to the two foci, $d_1 + d_2$, must be constant. As we shall see, it is convenient to denote this constant by $2a$. Thus, the point (x, y) is on the ellipse if and only if

$$d_1 + d_2 = 2a.$$

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a \qquad \text{Use the distance formula.}$$

After eliminating radicals and simplifying, we obtain

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

Look at the triangle in Figure 9.5. Notice that the distance from F_1 to F_2 is $2c$. Because the length of any side of a triangle is less than the sum of the lengths of the other two sides, $2c < d_1 + d_2$. Equivalently, $2c < 2a$ and $c < a$. Consequently, $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Substituting b^2 for $a^2 - c^2$ in the preceding equation, we obtain

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \qquad \text{Divide both sides by } a^2b^2.$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{Simplify.}$$

This last equation is the **standard form of the equation of an ellipse centered at the origin**. There are two such equations, one for a horizontal major axis and one for a vertical major axis.

Study Tip

The algebraic details behind eliminating the radicals and obtaining the equation shown can be found in the appendix. There you will find a step-by-step derivation of the ellipse's equation.

Standard Forms of the Equations of an Ellipse

The **standard form of the equation of an ellipse** with center at the origin, and major and minor axes of lengths $2a$ and $2b$ (where a and b are positive, and $a^2 > b^2$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Figure 9.6 illustrates that the vertices are on the major axis, a units from the center. The foci are on the major axis, c units from the center. For both equations, $b^2 = a^2 - c^2$. Equivalently, $c^2 = a^2 - b^2$.

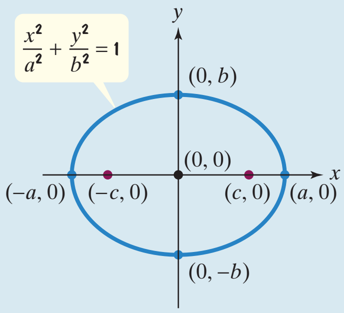


Figure 9.6(a) Major axis is horizontal with length $2a$.

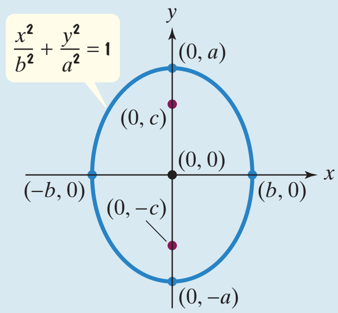


Figure 9.6(b) Major axis is vertical with length $2a$.

Study Tip

The form $c^2 = a^2 - b^2$ is the one you should remember. When finding the foci, this form is easy to manipulate.

The intercepts shown in Figure 9.6 can be obtained algebraically. Let's do this for

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

x-intercepts: Set $y = 0$.

$$\frac{x^2}{a^2} = 1$$
$$x^2 = a^2$$
$$x = \pm a$$

x-intercepts are $-a$ and a. The graph passes through $(-a, 0)$ and $(a, 0)$, which are the vertices.

y-intercepts: Set $x = 0$.

$$\frac{y^2}{b^2} = 1$$
$$y^2 = b^2$$
$$y = \pm b$$

y-intercepts are $-b$ and b. The graph passes through $(0, -b)$ and $(0, b)$.

Using the Standard Form of the Equation of an Ellipse

We can use the standard form of an ellipse's equation to graph the ellipse. Although the definition of the ellipse is given in terms of its foci, the foci are not part of the graph. A complete graph of an ellipse can be obtained without graphing the foci.

1 Graph ellipses centered at the origin.

EXAMPLE 1 Graphing an Ellipse Centered at the Origin

Graph and locate the foci: $\frac{x^2}{9} + \frac{y^2}{4} = 1.$

Solution The given equation is the standard form of an ellipse's equation with $a^2 = 9$ and $b^2 = 4$.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$a^2 = 9$. This is the larger of the two denominators.

$b^2 = 4$. This is the smaller of the two denominators.

Technology

We graph $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with a graphing utility by solving for y.

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}$$

$$y^2 = 4\left(1 - \frac{x^2}{9}\right)$$

$$y = \pm 2\sqrt{1 - \frac{x^2}{9}}$$

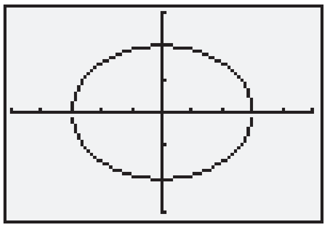
Notice that the square root property requires us to define two functions. Enter

$y_1 = 2$ [√] (1 [−] x [∧] 2 [÷] 9)

and

$y_2 = -y_1$.

To see the true shape of the ellipse, use the [ZOOM SQUARE] feature so that one unit on the y-axis is the same length as one unit on the x-axis.



[−5, 5, 1] by [−3, 3, 1]

Because the denominator of the x^2-term is greater than the denominator of the y^2-term, the major axis is horizontal. Based on the standard form of the equation, we know the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 9$, $a = 3$. Thus, the vertices are $(-3, 0)$ and $(3, 0)$, shown in Figure 9.7.

Now let us find the endpoints of the vertical minor axis. According to the standard form of the equation, these endpoints are $(0, -b)$ and $(0, b)$. Because $b^2 = 4$, $b = 2$. Thus, the endpoints of the minor axis are $(0, -2)$ and $(0, 2)$. They are shown in Figure 9.7.

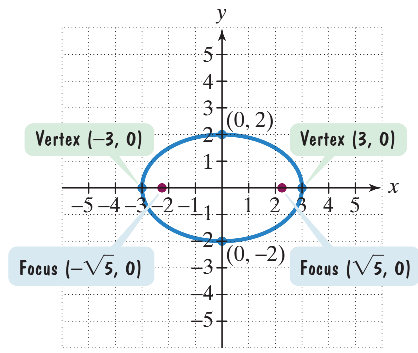


Figure 9.7 The graph of $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Finally, we find the foci, which are located at $(-c, 0)$ and $(c, 0)$. We can use the formula $c^2 = a^2 - b^2$ to do so. We know that $a^2 = 9$ and $b^2 = 4$. Thus,

$$c^2 = a^2 - b^2 = 9 - 4 = 5.$$

Because $c^2 = 5$, $c = \sqrt{5}$. The foci, $(-c, 0)$ and $(c, 0)$, are located at $\left(-\sqrt{5}, 0\right)$ and $\left(\sqrt{5}, 0\right)$. They are shown in Figure 9.7.

You can sketch the ellipse in Figure 9.7 by locating endpoints on the major and minor axes.

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

Endpoints of the major axis are 3 units to the right and left of the center.

Endpoints of the minor axis are 2 units up and down from the center.

Check Point 1 Graph and locate the foci: $\frac{x^2}{36} + \frac{y^2}{9} = 1$.

EXAMPLE 2 Graphing an Ellipse Centered at the Origin

Graph and locate the foci: $25x^2 + 16y^2 = 400$.

Solution We begin by expressing the equation in standard form. Because we want 1 on the right side, we divide both sides by 400.

$$\frac{25x^2}{400} + \frac{16y^2}{400} = \frac{400}{400}$$

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

$b^2 = 16$. This is the smaller of the two denominators.

$a^2 = 25$. This is the larger of the two denominators.

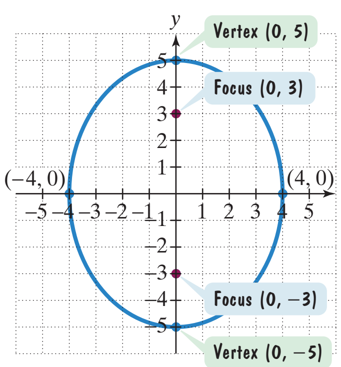


Figure 9.8 The graph of $25x^2 + 16y^2 = 400$, or $\frac{x^2}{16} + \frac{y^2}{25} = 1$

The equation is the standard form of an ellipse's equation with $a^2 = 25$ and $b^2 = 16$. Because the denominator of the y^2-term is greater than the denominator of the x^2-term, the major axis is vertical. Based on the standard form of the equation, we know the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 25$, $a = 5$. Thus, the vertices are $(0, -5)$ and $(0, 5)$, shown in Figure 9.8.

Now let us find the endpoints of the horizontal minor axis. According to the standard form of the equation, these endpoints are $(-b, 0)$ and $(b, 0)$. Because $b^2 = 16$, $b = 4$. Thus, the endpoints of the minor axis are $(-4, 0)$ and $(4, 0)$. They are shown in Figure 9.8.

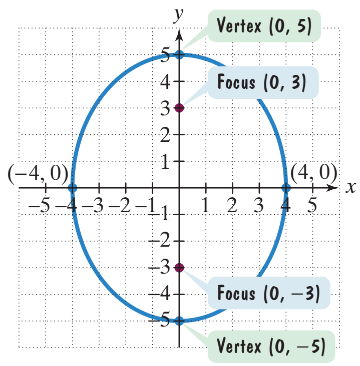


Figure 9.8 (repeated) The graph of $\frac{x^2}{16} + \frac{y^2}{25} = 1$

Finally, we find the foci, which are located at $(0, -c)$ and $(0, c)$. We can use the formula $c^2 = a^2 - b^2$ to do so. We know that $a^2 = 25$ and $b^2 = 16$. Thus,

$$c^2 = a^2 - b^2 = 25 - 16 = 9.$$

Because $c^2 = 9$, $c = 3$. The foci, $(0, -c)$ and $(0, c)$, are located at $(0, -3)$ and $(0, 3)$. They are shown in Figure 9.8.

You can sketch the ellipse in Figure 9.8 by locating endpoints on the major and minor axes.

$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$$

Endpoints of the minor axis are 4 units to the right and left of the center.

Endpoints of the major axis are 5 units up and down from the center.

Check Point 2 Graph and locate the foci: $16x^2 + 9y^2 = 144$.

In Examples 1 and 2, we used the equation of an ellipse to find its foci and vertices. In the next example, we reverse this procedure.

2 Write equations of ellipses in standard form.

EXAMPLE 3 Finding the Equation of an Ellipse from Its Foci and Vertices

Find the standard form of the equation of an ellipse with foci at $(-1, 0)$ and $(1, 0)$ and vertices $(-2, 0)$ and $(2, 0)$.

Solution Because the foci are located at $(-1, 0)$ and $(1, 0)$, on the x-axis, the major axis is horizontal. The center of the ellipse is midway between the foci, located at $(0, 0)$. Thus, the form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0, 0)$, to either vertex, $(-2, 0)$ or $(2, 0)$, is 2. Thus, $a = 2$.

$$\frac{x^2}{2^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{4} + \frac{y^2}{b^2} = 1$$

We must still find b^2. The distance from the center, $(0, 0)$, to either focus, $(-1, 0)$ or $(1, 0)$, is 1, so $c = 1$. Using $c^2 = a^2 - b^2$, we have

$$1^2 = 2^2 - b^2$$

and

$$b^2 = 2^2 - 1^2 = 4 - 1 = 3.$$

Substituting 3 for b^2 in $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$ gives us the standard form of the ellipse's equation. The equation is

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

Check Point 3 Find the standard form of the equation of an ellipse with foci at $(-2, 0)$ and $(2, 0)$ and vertices $(-3, 0)$ and $(3, 0)$.

3 Graph ellipses not centered at the origin.

Translations of Ellipses

Horizontal and vertical translations can be used to graph ellipses that are not centered at the origin. Figure 9.9 illustrates that the graphs of

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

have the same size and shape. However, the graph of the first equation is centered at (h, k) rather than at the origin.

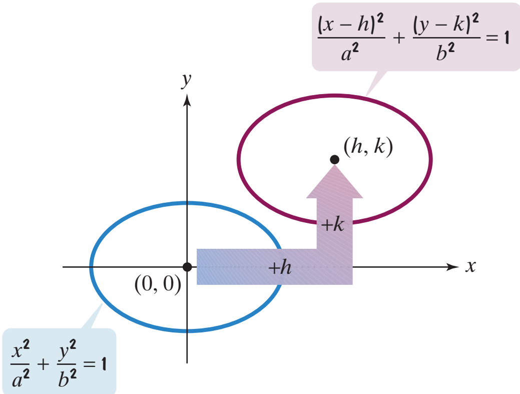
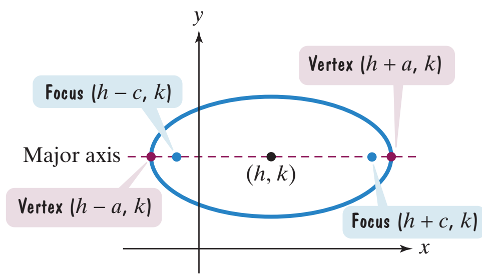
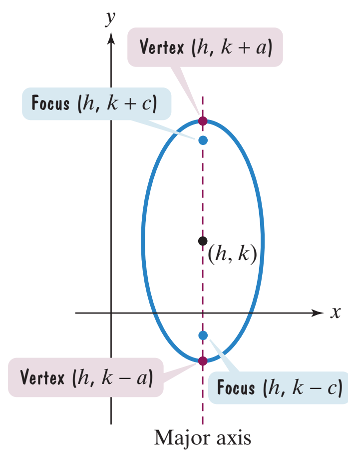


Figure 9.9 Translating an ellipse's graph

Table 9.1 gives the standard forms of equations of ellipses centered at (h, k) and shows their graphs.

Table 9.1 Standard Forms of Equations of Ellipses Centered at (h, k)

Equation	Center	Major Axis	Vertices	Graph
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ Endpoints of major axis are a units right and a units left of center. $a^2 > b^2$ Foci are c units right and c units left of center, where $c^2 = a^2 - b^2$.	(h, k)	Parallel to the x-axis, horizontal	$(h - a, k)$ $(h + a, k)$	
$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ $a^2 > b^2$ Endpoints of the major axis are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2 = a^2 - b^2$.	(h, k)	Parallel to the y-axis, vertical	$(h, k - a)$ $(h, k + a)$	

EXAMPLE 4 Graphing an Ellipse Centered at (h, k)

Graph: $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$. Where are the foci located?

Solution To graph the ellipse, we need to know its center, (h, k). In the standard forms of equations centered at (h, k), h is the number subtracted from x and k is the number subtracted from y.

This is $(x - h)^2$, with $h = 1$.

This is $(y - k)^2$, with $k = -2$.

$$\frac{(x-1)^2}{4} + \frac{(y-(-2))^2}{9} = 1$$

We see that $h = 1$ and $k = -2$. Thus, the center of the ellipse, (h, k), is $(1, -2)$. We can graph the ellipse by locating endpoints on the major and minor axes. To do this, we must identify a^2 and b^2.

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$

$b^2 = 4$. This is the smaller of the two denominators.

$a^2 = 9$. This is the larger of the two denominators.

The larger number is under the expression involving y. This means that the major axis is vertical and parallel to the y-axis.

We can sketch the ellipse by locating endpoints on the major and minor axes.

$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} = 1$$

Endpoints of the minor axis are 2 units to the right and left of the center.

Endpoints of the major axis (the vertices) are 3 units up and down from the center.

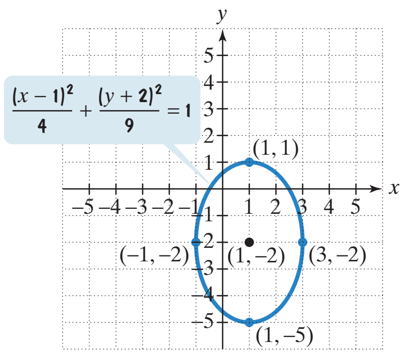


Figure 9.10 The graph of an ellipse centered at $(1, -2)$

We categorize the observations in the voice balloons as follows:

For a Vertical Major Axis with Center $(1, -2)$	
Vertices	**Endpoints of Minor Axis**
$(1, -2 + 3) = (1, 1)$	$(1 + 2, -2) = (3, -2)$
$(1, -2 - 3) = (1, -5)$	$(1 - 2, -2) = (-1, -2)$

3 units above and below center

2 units right and left of center

Using the center and these four points, we can sketch the ellipse shown in Figure 9.10.

With $c^2 = a^2 - b^2$, we have $c^2 = 9 - 4 = 5$. So the foci are located $\sqrt{5}$ units above and below the center, at $(1, -2 + \sqrt{5})$ and $(1, -2 - \sqrt{5})$.

Check Point 4 Graph: $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$. Where are the foci located?

In some cases, it is necessary to convert the equation of an ellipse to standard form by completing the square on x and y. For example, suppose that we wish to graph the ellipse whose equation is

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0.$$

Because we plan to complete the square on both x and y, we need to rearrange terms so that

- x-terms are arranged in descending order.
- y-terms are arranged in descending order.
- the constant term appears on the right.

$$9x^2 + 4y^2 - 18x + 16y - 11 = 0$$
This is the given equation.

$$(9x^2 - 18x) + (4y^2 + 16y) = 11$$
Group terms and add 11 to both sides.

$$9(x^2 - 2x + \square) + 4(y^2 + 4y + \square) = 11$$
To complete the square, coefficients of x^2 and y^2 must be 1. Factor out 9 and 4, respectively.

We added 9 · 1, or 9, to the left side.

We also added 4 · 4, or 16, to the left side.

$$9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = 11 + 9 + 16$$
Complete each square by adding the square of half the coefficient of x and y, respectively.

9 and 16, added on the left side, must also be added on the right side.

$$9(x-1)^2 + 4(y+2)^2 = 36$$
Factor.

$$\frac{9(x-1)^2}{36} + \frac{4(y+2)^2}{36} = \frac{36}{36}$$
Divide both sides by 36.

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$$
Simplify.

Study Tip

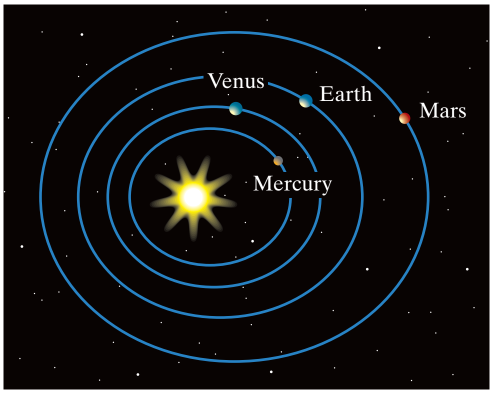
When completing the square, remember that changes made on the left side of the equation must also be made on the right side of the equation.

The equation is now in standard form. This is precisely the form of the equation that we graphed in Example 4.

4 Solve applied problems involving ellipses.

Applications

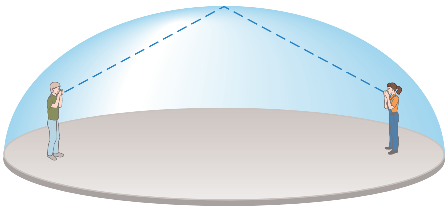
Ellipses have many applications. German scientist Johannes Kepler (1571–1630) showed that the planets in our solar system move in elliptical orbits, with the sun at a focus. Earth satellites also travel in elliptical orbits, with Earth at a focus.



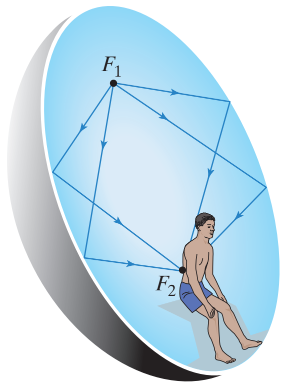
Planets move in elliptical orbits.

One intriguing aspect of the ellipse is that a ray of light or a sound wave emanating from one focus will be reflected from the ellipse to exactly the other focus. A whispering gallery is an elliptical room with an elliptical, dome-shaped ceiling. People standing at the foci can whisper and hear each other quite clearly, while persons in other locations in the room cannot hear them. Statuary Hall in the U.S. Capitol Building is elliptical. President John Quincy Adams, while a member of the House of Representatives, was aware of this acoustical phenomenon. He situated his desk at a focal point of the elliptical ceiling, easily eavesdropping on the private conversations of other House members located near the other focus.

The elliptical reflection principle is used in a procedure for disintegrating kidney stones. The patient is placed within a device that is elliptical in shape. The patient is placed so the kidney is centered at one focus, while ultrasound waves from the other focus hit the walls and are reflected to the kidney stone. The convergence of the ultrasound waves at the kidney stone causes vibrations that shatter it into fragments. The small pieces can then be passed painlessly through the patient's system. The patient recovers in days, as opposed to up to six weeks if surgery is used instead.



Whispering in an elliptical dome



Disintegrating kidney stones

Ellipses are often used for supporting arches of bridges and in tunnel construction. This application forms the basis of our next example.

EXAMPLE 5 An Application Involving an Ellipse

A semielliptical archway over a one-way road has a height of 10 feet and a width of 40 feet (see Figure 9.11). Your truck has a width of 10 feet and a height of 9 feet. Will your truck clear the opening of the archway?

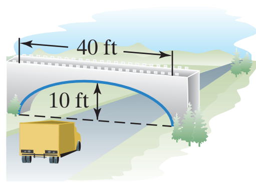


Figure 9.11 A semielliptical archway

Solution Because your truck's width is 10 feet, to determine the clearance, we must find the height of the archway 5 feet from the center. If that height is 9 feet or less, the truck will not clear the opening.

In Figure 9.12, we've constructed a coordinate system with the x-axis on the ground and the origin at the center of the archway. Also shown is the truck, whose height is 9 feet.

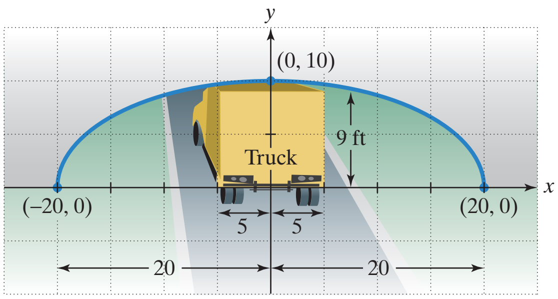


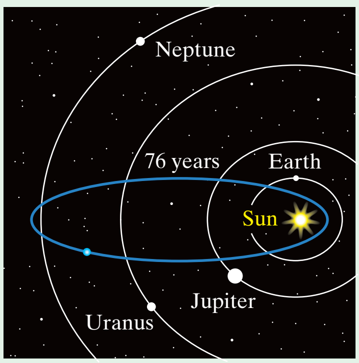
Figure 9.12

Using the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we can express the equation of the blue archway in Figure 9.12 as $\frac{x^2}{20^2} + \frac{y^2}{10^2} = 1$, or $\frac{x^2}{400} + \frac{y^2}{100} = 1$.

As shown in Figure 9.12, the edge of the 10-foot-wide truck corresponds to $x = 5$. We find the height of the archway 5 feet from the center by substituting 5 for x and solving for y.

Halley's Comet

Halley's Comet has an elliptical orbit with the sun at one focus. The comet returns every 76.3 years. The first recorded sighting was in 239 B.C. It was last seen in 1986. At that time, spacecraft went close to the comet, measuring its nucleus to be 7 miles long and 4 miles wide. By 2024, Halley's Comet will have reached the farthest point in its elliptical orbit before returning to be next visible from Earth in 2062.



The elliptical orbit of Halley's Comet

$$\frac{5^2}{400} + \frac{y^2}{100} = 1 \qquad \text{Substitute 5 for } x \text{ in } \frac{x^2}{400} + \frac{y^2}{100} = 1.$$

$$\frac{25}{400} + \frac{y^2}{100} = 1 \qquad \text{Square 5.}$$

$$400\left(\frac{25}{400} + \frac{y^2}{100}\right) = 400(1) \qquad \text{Clear fractions by multiplying both sides by 400.}$$

$$25 + 4y^2 = 400 \qquad \text{Use the distributive property and simplify.}$$

$$4y^2 = 375 \qquad \text{Subtract 25 from both sides.}$$

$$y^2 = \frac{375}{4} \qquad \text{Divide both sides by 4.}$$

$$y = \sqrt{\frac{375}{4}} \qquad \text{Take only the positive square root. The archway is above the x-axis, so y is nonnegative.}$$

$$\approx 9.68 \qquad \text{Use a calculator.}$$

Thus, the height of the archway 5 feet from the center is approximately 9.68 feet. Because your truck's height is 9 feet, there is enough room for the truck to clear the archway.

Check Point 5 Will a truck that is 12 feet wide and has a height of 9 feet clear the opening of the archway described in Example 5?

EXERCISE SET 9.1

Practice Exercises

In Exercises 1–18, graph each ellipse and locate the foci.

1. $\frac{x^2}{16} + \frac{y^2}{4} = 1$

2. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

3. $\frac{x^2}{9} + \frac{y^2}{36} = 1$

4. $\frac{x^2}{16} + \frac{y^2}{49} = 1$

5. $\frac{x^2}{25} + \frac{y^2}{64} = 1$

6. $\frac{x^2}{49} + \frac{y^2}{36} = 1$

7. $\frac{x^2}{49} + \frac{y^2}{81} = 1$

8. $\frac{x^2}{64} + \frac{y^2}{100} = 1$

9. $\frac{x^2}{\frac{9}{4}} + \frac{y^2}{\frac{25}{4}} = 1$

10. $\frac{x^2}{\frac{81}{4}} + \frac{y^2}{\frac{25}{16}} = 1$

11. $x^2 = 1 - 4y^2$

12. $y^2 = 1 - 4x^2$

13. $25x^2 + 4y^2 = 100$

14. $9x^2 + 4y^2 = 36$

15. $4x^2 + 16y^2 = 64$

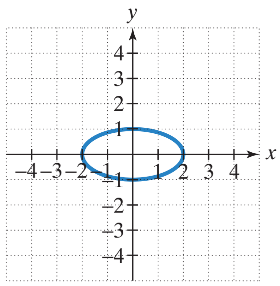
16. $4x^2 + 25y^2 = 100$

17. $7x^2 = 35 - 5y^2$

18. $6x^2 = 30 - 5y^2$

In Exercises 19–24, find the standard form of the equation of each ellipse and give the location of its foci.

19.



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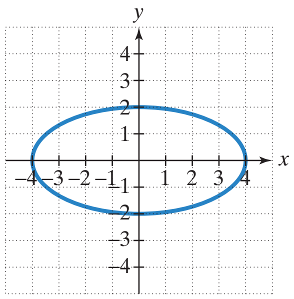

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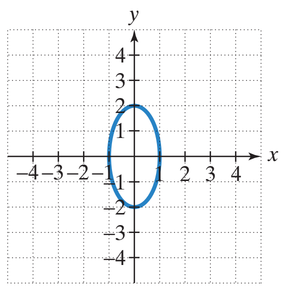

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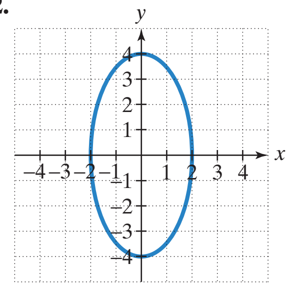

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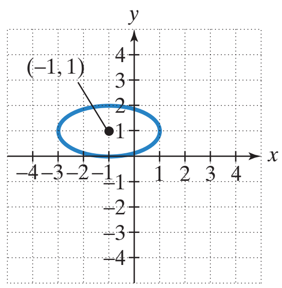

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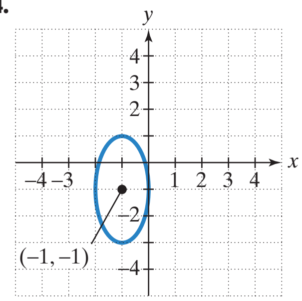


In Exercises 25–36, find the standard form of the equation of each ellipse satisfying the given conditions.

25. Foci: $(-5, 0)$, $(5, 0)$; vertices: $(-8, 0)$, $(8, 0)$

26. Foci: $(-2, 0)$, $(2, 0)$; vertices: $(-6, 0)$, $(6, 0)$

27. Foci: $(0, -4)$, $(0, 4)$; vertices: $(0, -7)$, $(0, 7)$

28. Foci: $(0, -3)$, $(0, 3)$; vertices: $(0, -4)$, $(0, 4)$

29. Foci: $(-2, 0)$, $(2, 0)$; y-intercepts: -3 and 3

30. Foci: $(0, -2)$, $(0, 2)$; x-intercepts: -2 and 2

31. Major axis horizontal with length 8; length of minor axis $= 4$; center: $(0, 0)$

32. Major axis horizontal with length 12; length of minor axis $= 6$; center: $(0, 0)$

33. Major axis vertical with length 10; length of minor axis $= 4$; center: $(-2, 3)$

34. Major axis vertical with length 20; length of minor axis $= 10$; center: $(2, -3)$

35. Endpoints of major axis: $(7, 9)$ and $(7, 3)$
Endpoints of minor axis: $(5, 6)$ and $(9, 6)$

36. Endpoints of major axis: $(2, 2)$ and $(8, 2)$
Endpoints of minor axis: $(5, 3)$ and $(5, 1)$

In Exercises 37–50, graph each ellipse and give the location of its foci.

37. $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

38. $\dfrac{(x-1)^2}{16} + \dfrac{(y+2)^2}{9} = 1$

39. $(x+3)^2 + 4(y-2)^2 = 16$

40. $(x-3)^2 + 9(y+2)^2 = 18$

41. $\dfrac{(x-4)^2}{9} + \dfrac{(y+2)^2}{25} = 1$

42. $\dfrac{(x-3)^2}{9} + \dfrac{(y+1)^2}{16} = 1$

43. $\dfrac{x^2}{25} + \dfrac{(y-2)^2}{36} = 1$

44. $\dfrac{(x-4)^2}{4} + \dfrac{y^2}{25} = 1$

45. $\dfrac{(x+3)^2}{9} + (y-2)^2 = 1$

46. $\dfrac{(x+2)^2}{16} + (y-3)^2 = 1$

47. $\dfrac{(x-1)^2}{2} + \dfrac{(y+3)^2}{5} = 1$

48. $\dfrac{(x+1)^2}{2} + \dfrac{(y-3)^2}{5} = 1$

49. $9(x-1)^2 + 4(y+3)^2 = 36$

50. $36(x+4)^2 + (y+3)^2 = 36$

In Exercises 51–56, convert each equation to standard form by completing the square on x and y. Then graph the ellipse and give the location of its foci.

51. $9x^2 + 25y^2 - 36x + 50y - 164 = 0$

52. $4x^2 + 9y^2 - 32x + 36y + 64 = 0$

53. $9x^2 + 16y^2 - 18x + 64y - 71 = 0$

54. $x^2 + 4y^2 + 10x - 8y + 13 = 0$

55. $4x^2 + y^2 + 16x - 6y - 39 = 0$

56. $4x^2 + 25y^2 - 24x + 100y + 36 = 0$

Practice Plus

In Exercises 57–62, find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

57. $x^2 + y^2 = 1$
$x^2 + 9y^2 = 9$

58. $x^2 + y^2 = 25$
$25x^2 + y^2 = 25$

59. $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$
$y = 3$

60. $\dfrac{x^2}{4} + \dfrac{y^2}{36} = 1$
$x = -2$

61. $4x^2 + y^2 = 4$
$2x - y = 2$

62. $4x^2 + y^2 = 4$
$x + y = 3$

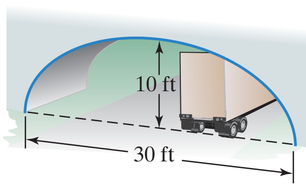
In Exercises 63–64, graph each semiellipse.

63. $y = -\sqrt{16 - 4x^2}$

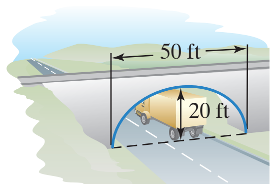
64. $y = -\sqrt{4 - 4x^2}$

Application Exercises

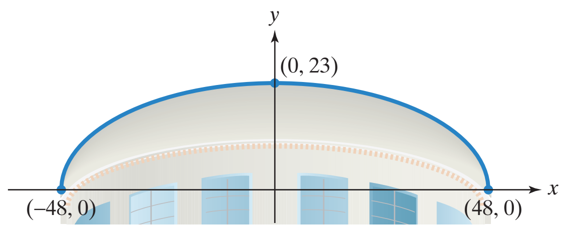
65. Will a truck that is 8 feet wide carrying a load that reaches 7 feet above the ground clear the semielliptical arch on the one-way road that passes under the bridge shown in the figure?



66. A semielliptic archway has a height of 20 feet and a width of 50 feet, as shown in the figure. Can a truck 14 feet high and 10 feet wide drive under the archway without going into the other lane?



67. The elliptical ceiling in Statuary Hall in the U.S. Capitol Building is 96 feet long and 23 feet tall.



a. Using the rectangular coordinate system in the figure shown, write the standard form of the equation of the elliptical ceiling.

b. John Quincy Adams discovered that he could overhear the conversations of opposing party leaders near the left side of the chamber if he situated his desk at the focus at the right side of the chamber. How far from the center of the ellipse along the major axis did Adams situate his desk? (Round to the nearest foot.)

68. If an elliptical whispering room has a height of 30 feet and a width of 100 feet, where should two people stand if they would like to whisper back and forth and be heard?

Writing in Mathematics

69. What is an ellipse?

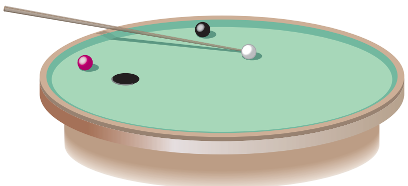
70. Describe how to graph $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

71. Describe how to locate the foci for $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

72. Describe one similarity and one difference between the graphs of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $\frac{x^2}{16} + \frac{y^2}{25} = 1$.

73. Describe one similarity and one difference between the graphs of $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $\frac{(x-1)^2}{25} + \frac{(y-1)^2}{16} = 1$.

74. An elliptipool is an elliptical pool table with only one pocket. A pool shark places a ball on the table, hits it in what appears to be a random direction, and yet it bounces off the edge, falling directly into the pocket. Explain why this happens.



Technology Exercises

75. Use a graphing utility to graph any five of the ellipses that you graphed by hand in Exercises 1–18.

76. Use a graphing utility to graph any three of the ellipses that you graphed by hand in Exercises 37–50. First solve the given equation for y by using the square root property. Enter each of the two resulting equations to produce each half of the ellipse.

77. Use a graphing utility to graph any one of the ellipses that you graphed by hand in Exercises 51–56. Write the equation as a quadratic equation in y and use the quadratic formula to solve for y. Enter each of the two resulting equations to produce each half of the ellipse.

78. Write an equation for the path of each of the following elliptical orbits. Then use a graphing utility to graph the two ellipses in the same viewing rectangle. Can you see why early astronomers had difficulty detecting that these orbits are ellipses rather than circles?

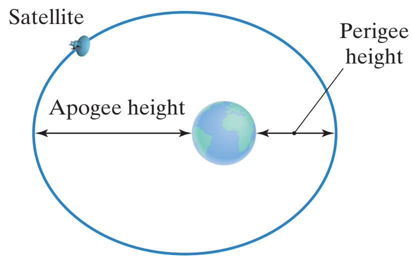
Earth's orbit:	Length of major axis: 186 million miles
	Length of minor axis: 185.8 million miles
Mars's orbit:	Length of major axis: 283.5 million miles
	Length of minor axis: 278.5 million miles

Critical Thinking Exercises

79. Find the standard form of the equation of an ellipse with vertices at $(0, -6)$ and $(0, 6)$, passing through $(2, -4)$.

80. An Earth satellite has an elliptical orbit described by

$$\frac{x^2}{(5000)^2} + \frac{y^2}{(4750)^2} = 1.$$



(All units are in miles.) The coordinates of the center of Earth are $(16, 0)$.

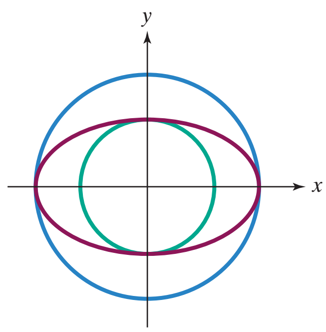
a. The perigee of the satellite's orbit is the point that is nearest Earth's center. If the radius of Earth is approximately 4000 miles, find the distance of the perigee above Earth's surface.

b. The apogee of the satellite's orbit is the point that is the greatest distance from Earth's center. Find the distance of the apogee above Earth's surface.

81. The equation of the red ellipse in the figure shown is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

Write the equation for each circle shown in the figure.



82. What happens to the shape of the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as $\frac{c}{a} \to 0$?

SECTION 9.2 The Hyperbola

Objectives

1. Locate a hyperbola's vertices and foci.
2. Write equations of hyperbolas in standard form.
3. Graph hyperbolas centered at the origin.
4. Graph hyperbolas not centered at the origin.
5. Solve applied problems involving hyperbolas.

St. Mary's Cathedral

Figure 9.13 Casting hyperbolic shadows

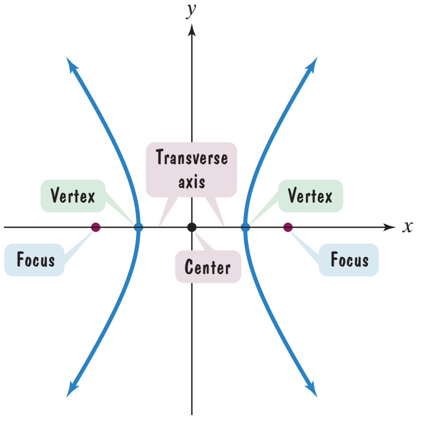


Figure 9.14 The two branches of a hyperbola

Conic sections are often used to create unusual architectural designs. The top of St. Mary's Cathedral in San Francisco is a 2135-cubic-foot dome with walls rising 200 feet above the floor and supported by four massive concrete pylons that extend 94 feet into the ground. Cross sections of the roof are parabolas and hyperbolas. In this section, we study the curve with two parts known as the hyperbola.

Definition of a Hyperbola

Figure 9.13 shows a cylindrical lampshade casting two shadows on a wall. These shadows indicate the distinguishing feature of hyperbolas: Their graphs contain two disjoint parts, called **branches**. Although each branch might look like a parabola, its shape is actually quite different.

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the *sum* of the distances to the foci is a constant. By contrast, for a hyperbola the *difference* of the distances to the foci is a constant.

Definition of a Hyperbola

A **hyperbola** is the set of points in a plane the difference of whose distances from two fixed points, called foci, is constant.

Figure 9.14 illustrates the two branches of a hyperbola. The line through the foci intersects the hyperbola at two points, called the **vertices**. The line segment that joins the vertices is the **transverse axis**. The midpoint of the transverse axis is the **center** of the hyperbola. Notice that the center lies midway between the vertices, as well as midway between the foci.

Standard Form of the Equation of a Hyperbola

The rectangular coordinate system enables us to translate a hyperbola's geometric definition into an algebraic equation. Figure 9.15 is our starting point for obtaining an equation. We place the foci, F_1 and F_2, on the x-axis at the points $(-c, 0)$ and $(c, 0)$. Note that the center of this hyperbola is at the origin. We let (x, y) represent the coordinates of any point, P, on the hyperbola.

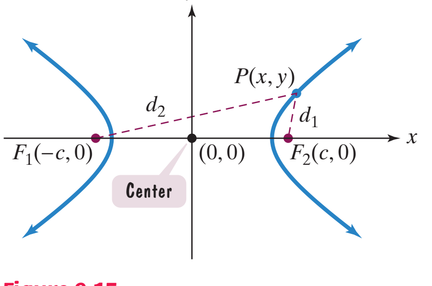


Figure 9.15

What does the definition of a hyperbola tell us about the point (x, y) in Figure 9.15? For any point (x, y) on the hyperbola, the absolute value of the difference of the distances from the two foci, $|d_2 - d_1|$, must be constant. We denote this constant by $2a$, just as we did for the ellipse. Thus, the point (x, y) is on the hyperbola if and only if

$$|d_2 - d_1| = 2a.$$

$$\left|\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2}\right| = 2a \quad \text{Use the distance formula.}$$

After eliminating radicals and simplifying, we obtain

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2).$$

For convenience, let $b^2 = c^2 - a^2$. Substituting b^2 for $c^2 - a^2$ in the preceding equation, we obtain

$$b^2x^2 - a^2y^2 = a^2b^2.$$

$$\frac{b^2x^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \quad \text{Divide both sides by } a^2b^2.$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Simplify.}$$

This last equation is called the **standard form of the equation of a hyperbola centered at the origin**. There are two such equations. The first is for a hyperbola in which the transverse axis lies on the x-axis. The second is for a hyperbola in which the transverse axis lies on the y-axis.

Study Tip

The form $c^2 = a^2 + b^2$ is the one you should remember. When finding the foci, this form is easy to manipulate.

Standard Forms of the Equations of a Hyperbola

The **standard form of the equation of a hyperbola** with center at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Figure 9.16(a) illustrates that for the equation on the left, the transverse axis lies on the x-axis. Figure 9.16(b) illustrates that for the equation on the right, the transverse axis lies on the y-axis. The vertices are a units from the center and the foci are c units from the center. For both equations, $b^2 = c^2 - a^2$. Equivalently, $c^2 = a^2 + b^2$.

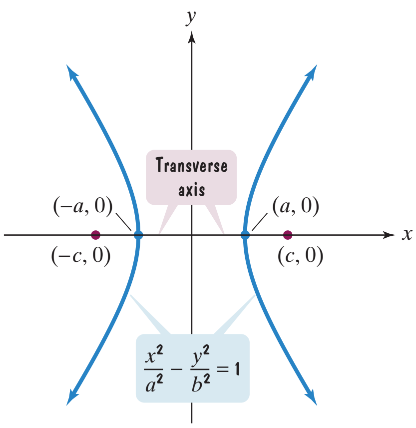


Figure 9.16(a) Transverse axis lies on the x-axis.

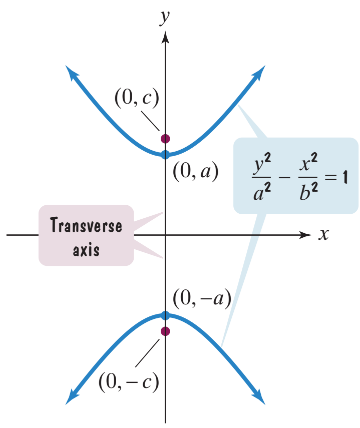


Figure 9.16(b) Transverse axis lies on the y-axis.

Study Tip

When the x^2-term is preceded by a plus sign, the transverse axis is horizontal. When the y^2-term is preceded by a plus sign, the transverse axis is vertical.

1 Locate a hyperbola's vertices and foci.

Study Tip

Notice the sign difference between the following equations:
Finding an ellipse's foci:
$$c^2 = a^2 - b^2$$
Finding a hyperbola's foci:
$$c^2 = a^2 + b^2.$$

Using the Standard Form of the Equation of a Hyperbola

We can use the standard form of the equation of a hyperbola to find its vertices and locate its foci. Because the vertices are a units from the center, begin by identifying a^2 in the equation. In the standard form of a hyperbola's equation, **a^2 is the number under the variable whose term is preceded by a plus sign** $(+)$. If the x^2-term is preceded by a plus sign, the transverse axis lies along the x-axis. Thus, the vertices are a units to the left and right of the origin. If the y^2-term is preceded by a plus sign, the transverse axis lies along the y-axis. Thus, the vertices are a units above and below the origin.

We know that the foci are c units from the center. The substitution that is used to derive the hyperbola's equation, $c^2 = a^2 + b^2$, is needed to locate the foci when a^2 and b^2 are known.

EXAMPLE 1 Finding Vertices and Foci from a Hyperbola's Equation

Find the vertices and locate the foci for each of the following hyperbolas with the given equation:

a. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ **b.** $\frac{y^2}{9} - \frac{x^2}{16} = 1.$

Solution Both equations are in standard form. We begin by identifying a^2 and b^2 in each equation.

a. The first equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$a^2 = 16$. This is the denominator of the term preceded by a plus sign.

$b^2 = 9$. This is the denominator of the term preceded by a minus sign.

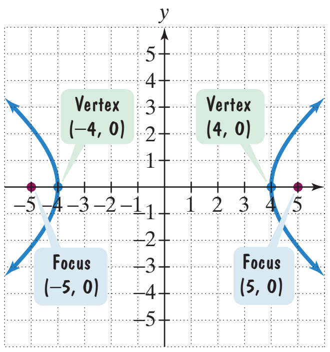


Figure 9.17 The graph of $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Because the x^2-term is preceded by a plus sign, the transverse axis lies along the x-axis. Thus, the vertices are a units to the *left* and *right* of the origin. Based on the standard form of the equation, we know the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 16$, $a = 4$. Thus, the vertices are $(-4, 0)$ and $(4, 0)$, shown in Figure 9.17.

We use $c^2 = a^2 + b^2$ to find the foci, which are located at $(-c, 0)$ and $(c, 0)$. We know that $a^2 = 16$ and $b^2 = 9$; we need to find c^2 in order to find c.

$$c^2 = a^2 + b^2 = 16 + 9 = 25$$

Because $c^2 = 25$, $c = 5$. The foci are located at $(-5, 0)$ and $(5, 0)$. They are shown in Figure 9.17.

b. The second given equation is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

$a^2 = 9$. This is the denominator of the term preceded by a plus sign.

$b^2 = 16$. This is the denominator of the term preceded by a minus sign.

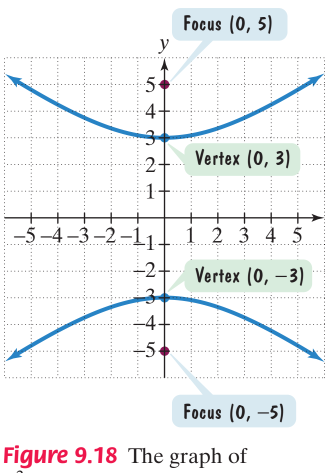


Figure 9.18 The graph of $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Because the y^2-term is preceded by a plus sign, the transverse axis lies along the y-axis. Thus, the vertices are a units *above* and *below* the origin. Based on the standard form of the equation, we know the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 9$, $a = 3$. Thus, the vertices are $(0, -3)$ and $(0, 3)$, shown in Figure 9.18.

We use $c^2 = a^2 + b^2$ to find the foci, which are located at $(0, -c)$ and $(0, c)$.

$$c^2 = a^2 + b^2 = 9 + 16 = 25$$

Because $c^2 = 25$, $c = 5$. The foci are located at $(0, -5)$ and $(0, 5)$. They are shown in Figure 9.18.

Check Point 1 Find the vertices and locate the foci for each of the following hyperbolas with the given equation:

a. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ **b.** $\frac{y^2}{25} - \frac{x^2}{16} = 1$.

In Example 1, we used equations of hyperbolas to find their foci and vertices. In the next example, we reverse this procedure.

2 Write equations of hyperbolas in standard form.

EXAMPLE 2 Finding the Equation of a Hyperbola from Its Foci and Vertices

Find the standard form of the equation of a hyperbola with foci at $(0, -3)$ and $(0, 3)$ and vertices $(0, -2)$ and $(0, 2)$, shown in Figure 9.19.

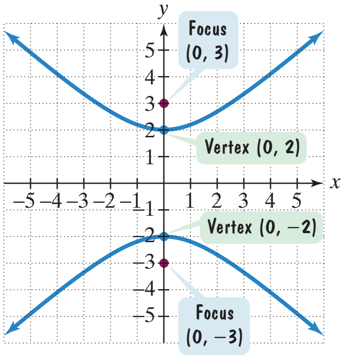


Figure 9.19

Solution Because the foci are located at $(0, -3)$ and $(0, 3)$, on the y-axis, the transverse axis lies on the y-axis. The center of the hyperbola is midway between the foci, located at $(0, 0)$. Thus, the form of the equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

We need to determine the values for a^2 and b^2. The distance from the center, $(0, 0)$, to either vertex, $(0, -2)$ or $(0, 2)$, is 2, so $a = 2$.

$$\frac{y^2}{2^2} - \frac{x^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{4} - \frac{x^2}{b^2} = 1$$

We must still find b^2. The distance from the center, $(0, 0)$, to either focus, $(0, -3)$ or $(0, 3)$, is 3. Thus, $c = 3$. Using $c^2 = a^2 + b^2$, we have

$$3^2 = 2^2 + b^2$$

and

$$b^2 = 3^2 - 2^2 = 9 - 4 = 5.$$

Substituting 5 for b^2 in $\frac{y^2}{4} - \frac{x^2}{b^2} = 1$ gives us the standard form of the hyperbola's equation. The equation is

$$\frac{y^2}{4} - \frac{x^2}{5} = 1.$$

Check Point 2 Find the standard form of the equation of a hyperbola with foci at $(0, -5)$ and $(0, 5)$ and vertices $(0, -3)$ and $(0, 3)$.

The Asymptotes of a Hyperbola

As x and y get larger, the two branches of the graph of a hyperbola approach a pair of intersecting straight lines, called **asymptotes**. The asymptotes pass through the center of the hyperbola and are helpful in graphing hyperbolas.

Figure 9.20 shows the asymptotes for the graphs of hyperbolas centered at the origin. The asymptotes pass through the corners of a rectangle. Note that the dimensions of this rectangle are $2a$ by $2b$. The line segment of length $2b$ is the **conjugate axis** of the hyperbola and is perpendicular to the transverse axis through the center of the hyperbola.

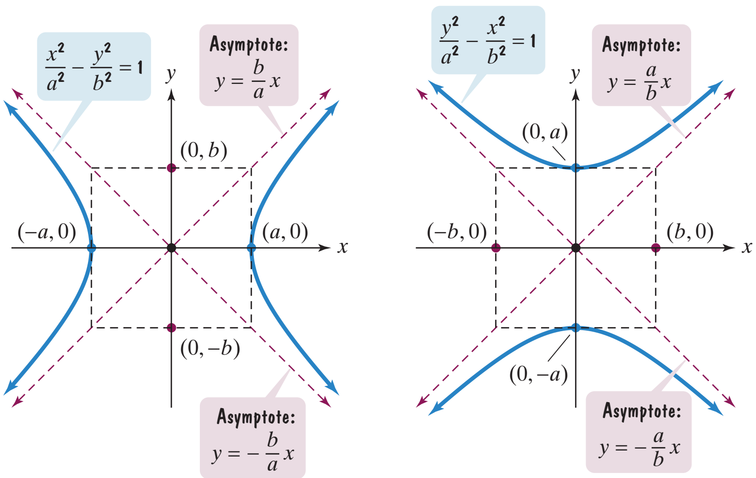


Figure 9.20 Asymptotes of a hyperbola

The Asymptotes of a Hyperbola Centered at the Origin

The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has a horizontal transverse axis and two asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

The hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has a vertical transverse axis and two asymptotes

$$y = \frac{a}{b}x \quad \text{and} \quad y = -\frac{a}{b}x.$$

Why are $y = \pm\frac{b}{a}x$ the asymptotes for a hyperbola whose transverse axis is horizontal? The proof can be found in the appendix.

3 Graph hyperbolas centered at the origin.

Graphing Hyperbolas Centered at the Origin

Hyperbolas are graphed using vertices and asymptotes.

Graphing Hyperbolas

1. Locate the vertices.
2. Use dashed lines to draw the rectangle centered at the origin with sides parallel to the axes, crossing one axis at $\pm a$ and the other at $\pm b$.
3. Use dashed lines to draw the diagonals of this rectangle and extend them to obtain the asymptotes.
4. Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes.

EXAMPLE 3 Graphing a Hyperbola

Graph and locate the foci: $\frac{x^2}{25} - \frac{y^2}{16} = 1$. What are the equations of the asymptotes?

Solution

Step 1 Locate the vertices. The given equation is in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, with $a^2 = 25$ and $b^2 = 16$.

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

$a^2 = 25$ $b^2 = 16$

Based on the standard form of the equation with the transverse axis on the x-axis, we know that the vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 25$, $a = 5$. Thus, the vertices are $(-5, 0)$ and $(5, 0)$, shown in Figure 9.21.

Step 2 Draw a rectangle. Because $a^2 = 25$ and $b^2 = 16$, $a = 5$ and $b = 4$. We construct a rectangle to find the asymptotes, using -5 and 5 on the x-axis (the vertices are located here) and -4 and 4 on the y-axis. The rectangle passes through these four points, shown using dashed lines in Figure 9.21.

Step 3 Draw extended diagonals for the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in Figure 9.21, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations for these asymptotes are

$$y = \pm\frac{b}{a}x \quad \text{or} \quad y = \pm\frac{4}{5}x.$$

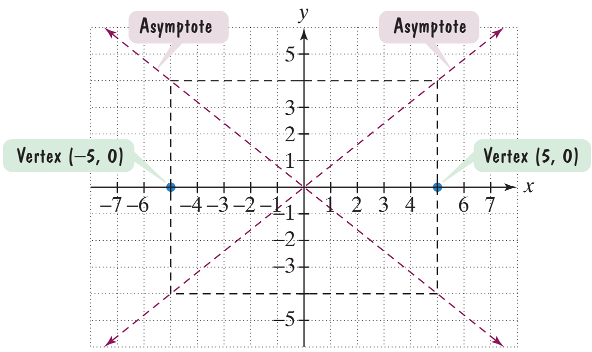


Figure 9.21 Preparing to graph $\frac{x^2}{25} - \frac{y^2}{16} = 1$

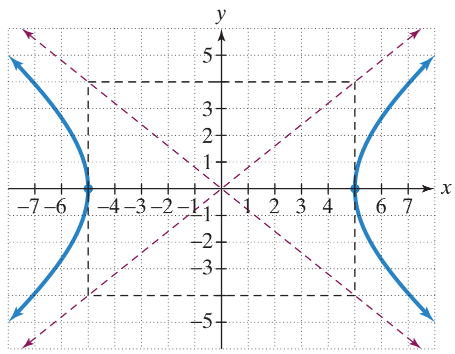


Figure 9.22 The graph of $\frac{x^2}{25} - \frac{y^2}{16} = 1$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in Figure 9.22.

We now consider the foci, located at $(-c, 0)$ and $(c, 0)$. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 25 + 16 = 41$$

Because $c^2 = 41$, $c = \sqrt{41}$. The foci are located at $\left(-\sqrt{41}, 0\right)$ and $\left(\sqrt{41}, 0\right)$, approximately $(-6.4, 0)$ and $(6.4, 0)$.

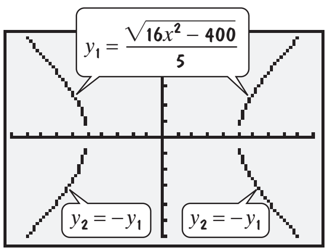
Check Point 3 Graph and locate the foci: $\frac{x^2}{36} - \frac{y^2}{9} = 1$. What are the equations of the asymptotes?

Technology

Graph $\frac{x^2}{25} - \frac{y^2}{16} = 1$ by solving for y:

$$y_1 = \frac{\sqrt{16x^2 - 400}}{5}$$

$$y_2 = -\frac{\sqrt{16x^2 - 400}}{5} = -y_1.$$



$[-10, 10, 1]$ by $[-6, 6, 1]$

EXAMPLE 4 Graphing a Hyperbola

Graph and locate the foci: $9y^2 - 4x^2 = 36$. What are the equations of the asymptotes?

Solution We begin by writing the equation in standard form. The right side should be 1, so we divide both sides by 36.

$$\frac{9y^2}{36} - \frac{4x^2}{36} = \frac{36}{36}$$

$$\frac{y^2}{4} - \frac{x^2}{9} = 1 \qquad \text{Simplify. The right side is now 1.}$$

Now we are ready to use our four-step procedure for graphing hyperbolas.

Step 1 Locate the vertices. The equation that we obtained is in the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, with $a^2 = 4$ and $b^2 = 9$.

$$\frac{y^2}{4} - \frac{x^2}{9} = 1$$

$a^2 = 4$ $\quad$ $b^2 = 9$

Based on the standard form of the equation with the transverse axis on the y-axis, we know that the vertices are $(0, -a)$ and $(0, a)$. Because $a^2 = 4$, $a = 2$. Thus, the vertices are $(0, -2)$ and $(0, 2)$, shown in Figure 9.23.

Step 2 Draw a rectangle. Because $a^2 = 4$ and $b^2 = 9$, $a = 2$ and $b = 3$. We construct a rectangle to find the asymptotes, using -2 and 2 on the y-axis (the vertices are located here) and -3 and 3 on the x-axis. The rectangle passes through these four points, shown using dashed lines in Figure 9.23.

Step 3 Draw extended diagonals of the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in Figure 9.23, to obtain the graph of the asymptotes. Based on the standard form of the hyperbola's equation, the equations of these asymptotes are

$$y = \pm\frac{a}{b}x \quad \text{or} \quad y = \pm\frac{2}{3}x.$$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in Figure 9.24.

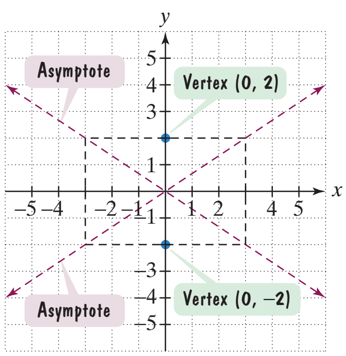


Figure 9.23 Preparing to graph $\frac{y^2}{4} - \frac{x^2}{9} = 1$

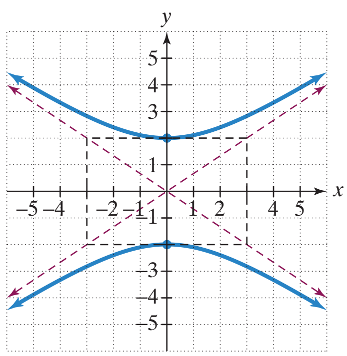


Figure 9.24 The graph of $\frac{y^2}{4} - \frac{x^2}{9} = 1$

We now consider the foci, located at $(0, -c)$ and $(0, c)$. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 4 + 9 = 13$$

Because $c^2 = 13$, $c = \sqrt{13}$. The foci are located at $(0, -\sqrt{13})$ and $(0, \sqrt{13})$, approximately $(0, -3.6)$ and $(0, 3.6)$.

Check Point 4 Graph and locate the foci: $y^2 - 4x^2 = 4$. What are the equations of the asymptotes?

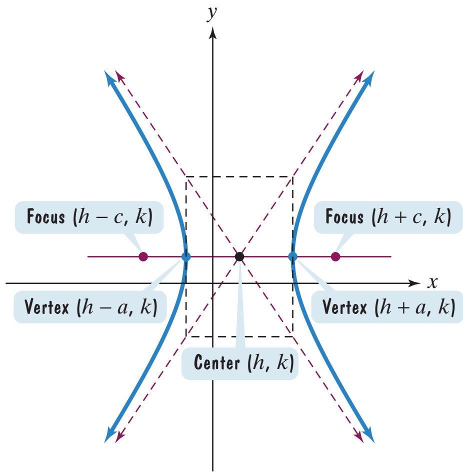
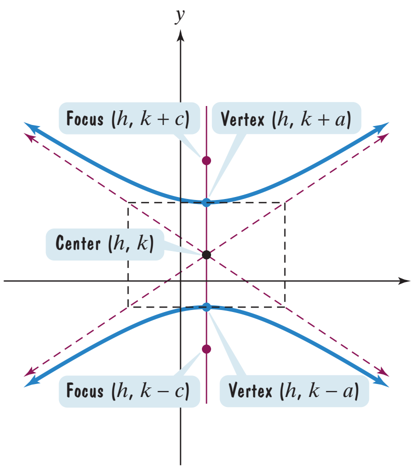
4 Graph hyperbolas not centered at the origin.

Translations of Hyperbolas

The graph of a hyperbola can be centered at (h, k), rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the hyperbola's equation.

Table 9.2 gives the standard forms of equations of hyperbolas centered at (h, k) and shows their graphs.

Table 9.2 Standard Forms of Equations of Hyperbolas Centered at (h, k)

Equation	Center	Transverse Axis	Vertices	Graph
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Vertices are a units right and a units left of center. Foci are c units right and c units left of center, where $c^2 = a^2 + b^2$.	(h, k)	Parallel to the x-axis, horizontal	$(h - a, k)$ $(h + a, k)$	
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ Vertices are a units above and a units below the center. Foci are c units above and c units below the center, where $c^2 = a^2 + b^2$.	(h, k)	Parallel to the y-axis, vertical	$(h, k - a)$ $(h, k + a)$	

EXAMPLE 5 Graphing a Hyperbola Centered at (h, k)

Graph: $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$. Where are the foci located? What are the equations of the asymptotes?

Solution In order to graph the hyperbola, we need to know its center, (h, k). In the standard forms of equations centered at (h, k), h is the number subtracted from x and k is the number subtracted from y.

This is $(x - h)^2$, with $h = 2$.

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

This is $(y - k)^2$, with $k = 3$.

We see that $h = 2$ and $k = 3$. Thus, the center of the hyperbola, (h, k), is $(2, 3)$. We can graph the hyperbola by using vertices, asymptotes, and our four-step graphing procedure.

Step 1 Locate the vertices. To do this, we must identify a^2.

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1 \qquad \text{The form of this equation is } \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

$a^2 = 16$ $\quad$ $b^2 = 9$

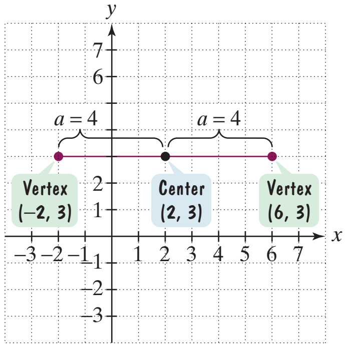


Figure 9.25 Locating a hyperbola's center and vertices

Based on the standard form of the equation with a horizontal transverse axis, the vertices are a units to the left and right of the center. Because $a^2 = 16$, $a = 4$. This means that the vertices are 4 units to the left and right of the center, $(2, 3)$. Four units to the left of $(2, 3)$ puts one vertex at $(2 - 4, 3)$, or $(-2, 3)$. Four units to the right of $(2, 3)$ puts the other vertex at $(2 + 4, 3)$, or $(6, 3)$. The vertices are shown in Figure 9.25.

Step 2 Draw a rectangle. Because $a^2 = 16$ and $b^2 = 9$, $a = 4$ and $b = 3$. The rectangle passes through points that are 4 units to the right and left of the center (the vertices are located here) and 3 units above and below the center. The rectangle is shown using dashed lines in Figure 9.26.

Step 3 Draw extended diagonals of the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in Figure 9.26, to obtain the graph of the asymptotes. The equations of the asymptotes of the unshifted hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ are $y = \pm\frac{b}{a}x$, or $y = \pm\frac{3}{4}x$. Thus, the asymptotes for the hyperbola that is shifted two units to the right and three units up, namely

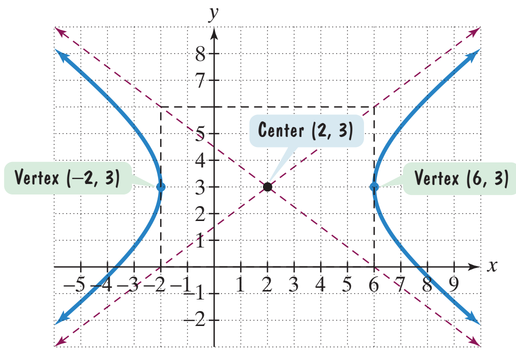


Figure 9.26 The graph of $\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

have equations that can be expressed as

$$y - 3 = \pm\frac{3}{4}(x - 2).$$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in Figure 9.26.

We now consider the foci, located c units to the right and left of the center. We find c using $c^2 = a^2 + b^2$.

$$c^2 = 16 + 9 = 25$$

Because $c^2 = 25$, $c = 5$. This means that the foci are 5 units to the left and right of the center, $(2, 3)$. Five units to the left of $(2, 3)$ puts one focus at $(2 - 5, 3)$, or $(-3, 3)$. Five units to the right of $(2, 3)$ puts the other focus at $(2 + 5, 3)$, or $(7, 3)$.

Study Tip

You can also use the point-slope form of a line's equation

$$y - y_1 = m(x - x_1)$$

to find the equations of the asymptotes. The center of the hyperbola, (h, k), is a point on each asymptote, so $x_1 = h$ and $y_1 = k$. The slopes, m, are $\pm\frac{b}{a}$ for a horizontal transverse axis and $\pm\frac{a}{b}$ for a vertical transverse axis.

Check Point 5 Graph: $\frac{(x-3)^2}{4} - \frac{(y-1)^2}{1} = 1$. Where are the foci located? What are the equations of the asymptotes?

In our next example, it is necessary to convert the equation of a hyperbola to standard form by completing the square on x and y.

EXAMPLE 6 Graphing a Hyperbola Centered at (h, k)

Graph: $4x^2 - 24x - 25y^2 + 250y - 489 = 0$. Where are the foci located? What are the equations of the asymptotes?

Solution We begin by completing the square on x and y.

$$4x^2 - 24x - 25y^2 + 250y - 489 = 0$$ This is the given equation.

$$(4x^2 - 24x) + (-25y^2 + 250y) = 489$$ Group terms and add 489 to both sides.

$$4(x^2 - 6x + \square) - 25(y^2 - 10y + \square) = 489$$ Factor out 4 and −25, respectively, so coefficients of x^2 and y^2 are 1.

$$4(x^2 - 6x + 9) - 25(y^2 - 10y + 25) = 489 + 36 + (-625)$$ Complete each square by adding the square of half the coefficient of x and y, respectively.

We added $4 \cdot 9$, or 36, to the left side. We added $-25 \cdot 25$, or -625, to the left side. Add $36 + (-625)$ to the right side.

$$4(x - 3)^2 - 25(y - 5)^2 = -100$$ Factor.

$$\frac{4(x - 3)^2}{-100} - \frac{25(y - 5)^2}{-100} = \frac{-100}{-100}$$ Divide both sides by −100.

$$\frac{(x - 3)^2}{-25} + \frac{(y - 5)^2}{4} = 1$$ Simplify.

$$\frac{(y - 5)^2}{4} - \frac{(x - 3)^2}{25} = 1$$ Write the equation in standard form, $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$.

This is $(y - k)^2$, with $k = 5$. This is $(x - h)^2$, with $h = 3$.

Study Tip

The hyperbola's center is $(3, 5)$ because the last equation shows that 3 is subtracted from x and 5 is subtracted from y. Many students tend to read the equation from left to right and get the center backward. The hyperbola's center is *not* $(5, 3)$.

We see that $h = 3$ and $k = 5$. Thus, the center of the hyperbola, (h, k), is $(3, 5)$. Because the x^2-term is being subtracted, the transverse axis is vertical and the hyperbola opens upward and downward.

We use our four-step procedure to obtain the graph of

$$\frac{(y - 5)^2}{4} - \frac{(x - 3)^2}{25} = 1.$$

$a^2 = 4$ $\quad$ $b^2 = 25$

Step 1 Locate the vertices. Based on the standard form of the equation with a vertical transverse axis, the vertices are a units above and below the center. Because $a^2 = 4$, $a = 2$. This means that the vertices are 2 units above and below the center, $(3, 5)$. This puts the vertices at $(3, 7)$ and $(3, 3)$, shown in Figure 9.27.

Step 2 Draw a rectangle. Because $a^2 = 4$ and $b^2 = 25$, $a = 2$ and $b = 5$. The rectangle passes through points that are 2 units above and below the center (the vertices are located here) and 5 units to the right and left of the center. The rectangle is shown using dashed lines in Figure 9.27.

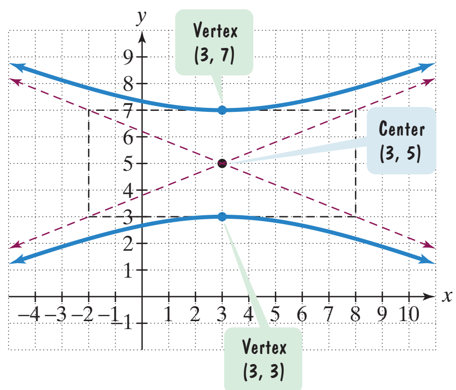


Figure 9.27 The graph of $\frac{(y - 5)^2}{4} - \frac{(x - 3)^2}{25} = 1$

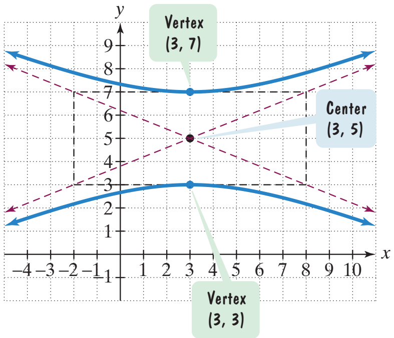


Figure 9.27 (repeated) The graph of $\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$

Step 3 Draw extended diagonals of the rectangle to obtain the asymptotes. We draw dashed lines through the opposite corners of the rectangle, shown in Figure 9.27, to obtain the graph of the asymptotes. The equations of the asymptotes of the unshifted hyperbola $\frac{y^2}{4} - \frac{x^2}{25} = 1$ are $y = \pm\frac{a}{b}x$, or $y = \pm\frac{2}{5}x$. Thus, the asymptotes for the hyperbola that is shifted three units to the right and five units up, namely

$$\frac{(y-5)^2}{4} - \frac{(x-3)^2}{25} = 1$$

have equations that can be expressed as

$$y - 5 = \pm\frac{2}{5}(x - 3).$$

Step 4 Draw the two branches of the hyperbola by starting at each vertex and approaching the asymptotes. The hyperbola is shown in Figure 9.27.

We now consider the foci, located c units above and below the center, (3, 5). We find c using $c^2 = a^2 + b^2$.

$$c^2 = 4 + 25 = 29.$$

Because $c^2 = 29$, $c = \sqrt{29}$. The foci are located at $(3, 5 + \sqrt{29})$ and $(3, 5 - \sqrt{29})$.

Check Point 6 Graph: $4x^2 - 24x - 9y^2 - 90y - 153 = 0$. Where are the foci located? What are the equations of the asymptotes?

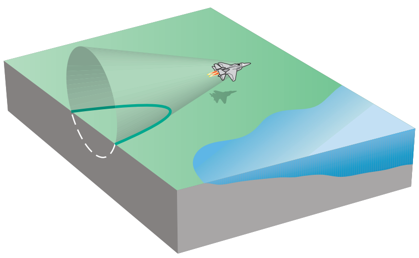
5 Solve applied problems involving hyperbolas.

Applications

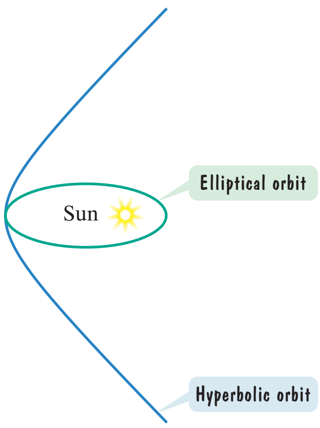
Hyperbolas have many applications. When a jet flies at a speed greater than the speed of sound, the shock wave that is created is heard as a sonic boom. The wave has the shape of a cone. The shape formed as the cone hits the ground is one branch of a hyperbola.

Halley's Comet, a permanent part of our solar system, travels around the sun in an elliptical orbit. Other comets pass through the solar system only once, following a hyperbolic path with the sun as a focus.

Hyperbolas are of practical importance in fields ranging from architecture to navigation. Cooling towers used in the design for nuclear power plants have cross sections that are both ellipses and hyperbolas. Three-dimensional solids whose cross sections are hyperbolas are used in some rather unique architectural creations, including the TWA building at Kennedy Airport in New York City and the St. Louis Science Center Planetarium.



The hyperbolic shape of a sonic boom



Orbits of comets

EXAMPLE 7 An Application Involving Hyperbolas

An explosion is recorded by two microphones that are 2 miles apart. Microphone M_1 received the sound 4 seconds before microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.

Solution We begin by putting the microphones in a coordinate system. Because 1 mile = 5280 feet, we place M_1 5280 feet on a horizontal axis to the right of the origin and M_2 5280 feet on a horizontal axis to the left of the origin. Figure 9.28 illustrates that the two microphones are 2 miles apart.

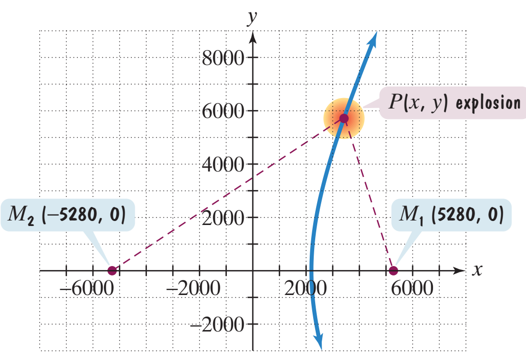


Figure 9.28 Locating an explosion on the branch of a hyperbola

We know that M_2 received the sound 4 seconds after M_1. Because sound travels at 1100 feet per second, the difference between the distance from P to M_1 and the distance from P to M_2 is 4400 feet. The set of all points P (or locations of the explosion) satisfying these conditions fits the definition of a hyperbola, with microphones M_1 and M_2 at the foci.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ Use the standard form of the hyperbola's equation. $P(x, y)$, the explosion point, lies on this hyperbola. We must find a^2 and b^2.

The difference between the distances, represented by $2a$ in the derivation of the hyperbola's equation, is 4400 feet. Thus, $2a = 4400$ and $a = 2200$.

$$\frac{x^2}{(2200)^2} - \frac{y^2}{b^2} = 1$$ Substitute 2200 for a.

We must still find b^2. We know that $a = 2200$. The distance from the center, (0, 0), to either focus, $(-5280, 0)$ or $(5280, 0)$, is 5280. Thus, $c = 5280$. Using $c^2 = a^2 + b^2$, we have

$$5280^2 = 2200^2 + b^2$$

and

$$b^2 = 5280^2 - 2200^2 = 23{,}038{,}400.$$

The equation of the hyperbola with a microphone at each focus is

$$\frac{x^2}{4{,}840{,}000} - \frac{y^2}{23{,}038{,}400} = 1$$ Substitute 23,038,400 for b^2.

We can conclude that the explosion occurred somewhere on the right branch (the branch closer to M_1) of the hyperbola given by this equation.

In Example 7, we determined that the explosion occurred somewhere along one branch of a hyperbola, but not exactly where on the hyperbola. If, however, we had received the sound from another pair of microphones, we could locate the sound along a branch of another hyperbola. The exact location of the explosion would be the point where the two hyperbolas intersect.

Check Point 7 Rework Example 7 assuming microphone M_1 receives the sound 3 seconds before microphone M_2.

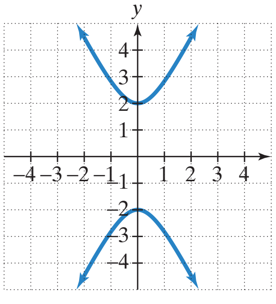
EXERCISE SET 9.2

Practice Exercises

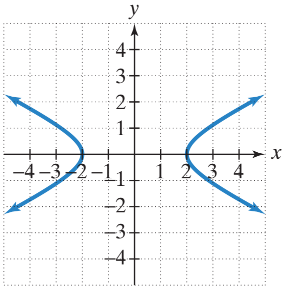
In Exercises 1–4, find the vertices and locate the foci of each hyperbola with the given equation. Then match each equation to one of the graphs that are shown and labeled (a)–(d).

1. $\dfrac{x^2}{4} - \dfrac{y^2}{1} = 1$
2. $\dfrac{x^2}{1} - \dfrac{y^2}{4} = 1$
3. $\dfrac{y^2}{4} - \dfrac{x^2}{1} = 1$
4. $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$

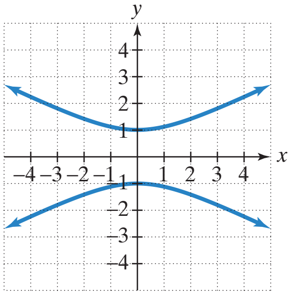
a.



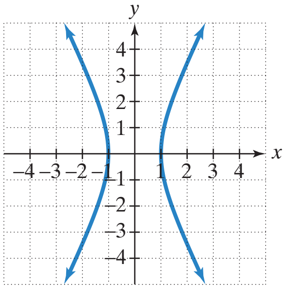
b.



c.



d.



In Exercises 5–12, find the standard form of the equation of each hyperbola satisfying the given conditions.

5. Foci: $(0, -3), (0, 3)$; vertices: $(0, -1), (0, 1)$
6. Foci: $(0, -6), (0, 6)$; vertices: $(0, -2), (0, 2)$
7. Foci: $(-4, 0), (4, 0)$; vertices: $(-3, 0), (3, 0)$
8. Foci: $(-7, 0), (7, 0)$; vertices: $(-5, 0), (5, 0)$
9. Endpoints of transverse axis: $(0, -6), (0, 6)$; asymptote: $y = 2x$
10. Endpoints of transverse axis: $(-4, 0), (4, 0)$; asymptote: $y = 2x$

11. Center: $(4, -2)$; Focus: $(7, -2)$; vertex: $(6, -2)$

12. Center: $(-2, 1)$; Focus: $(-2, 6)$; vertex: $(-2, 4)$

In Exercises 13–26, use vertices and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

13. $\frac{x^2}{9} - \frac{y^2}{25} = 1$

14. $\frac{x^2}{16} - \frac{y^2}{25} = 1$

15. $\frac{x^2}{100} - \frac{y^2}{64} = 1$

16. $\frac{x^2}{144} - \frac{y^2}{81} = 1$

17. $\frac{y^2}{16} - \frac{x^2}{36} = 1$

18. $\frac{y^2}{25} - \frac{x^2}{64} = 1$

19. $4y^2 - x^2 = 1$

20. $9y^2 - x^2 = 1$

21. $9x^2 - 4y^2 = 36$

22. $4x^2 - 25y^2 = 100$

23. $9y^2 - 25x^2 = 225$

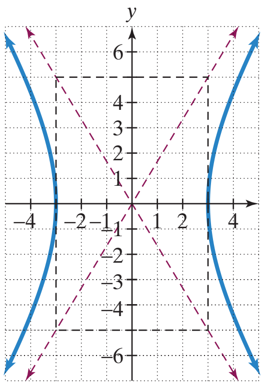
24. $16y^2 - 9x^2 = 144$

25. $y = \pm\sqrt{x^2 - 2}$

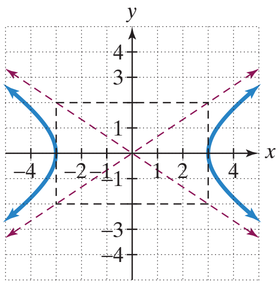
26. $y = \pm\sqrt{x^2 - 3}$

In Exercises 27–32, find the standard form of the equation of each hyperbola.

27.



28.



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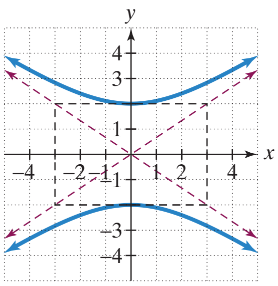

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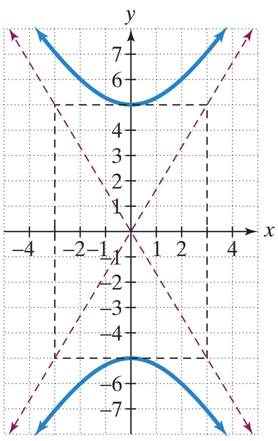
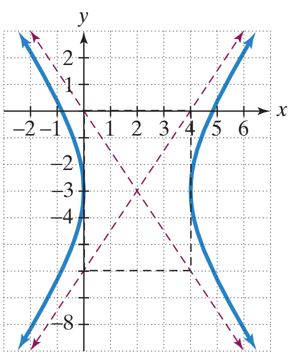
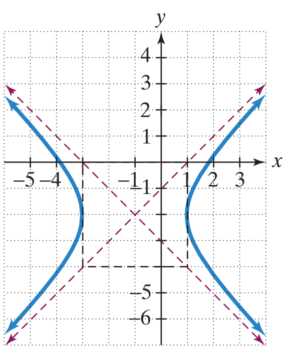


31.



32.



In Exercises 33–42, use the center, vertices, and asymptotes to graph each hyperbola. Locate the foci and find the equations of the asymptotes.

33. $\frac{(x + 4)^2}{9} - \frac{(y + 3)^2}{16} = 1$

34. $\frac{(x + 2)^2}{9} - \frac{(y - 1)^2}{25} = 1$

35. $\frac{(x + 3)^2}{25} - \frac{y^2}{16} = 1$

36. $\frac{(x + 2)^2}{9} - \frac{y^2}{25} = 1$

37. $\frac{(y + 2)^2}{4} - \frac{(x - 1)^2}{16} = 1$

38. $\frac{(y - 2)^2}{36} - \frac{(x + 1)^2}{49} = 1$

39. $(x - 3)^2 - 4(y + 3)^2 = 4$

40. $(x + 3)^2 - 9(y - 4)^2 = 9$

41. $(x - 1)^2 - (y - 2)^2 = 3$

42. $(y - 2)^2 - (x + 3)^2 = 5$

In Exercises 43–50, convert each equation to standard form by completing the square on x and y. Then graph the hyperbola. Locate the foci and find the equations of the asymptotes.

43. $x^2 - y^2 - 2x - 4y - 4 = 0$

44. $4x^2 - y^2 + 32x + 6y + 39 = 0$

45. $16x^2 - y^2 + 64x - 2y + 67 = 0$

46. $9y^2 - 4x^2 - 18y + 24x - 63 = 0$

47. $4x^2 - 9y^2 - 16x + 54y - 101 = 0$

48. $4x^2 - 9y^2 + 8x - 18y - 6 = 0$

49. $4x^2 - 25y^2 - 32x + 164 = 0$

50. $9x^2 - 16y^2 - 36x - 64y + 116 = 0$

Practice Plus

In Exercises 51–56, graph each relation. Use the relation's graph to determine its domain and range.

51. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

52. $\frac{x^2}{25} - \frac{y^2}{4} = 1$

53. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

54. $\frac{x^2}{25} + \frac{y^2}{4} = 1$

55. $\frac{y^2}{16} - \frac{x^2}{9} = 1$

56. $\frac{y^2}{4} - \frac{x^2}{25} = 1$

In Exercises 57–60, find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

57. $x^2 - y^2 = 4$
$x^2 + y^2 = 4$

58. $x^2 - y^2 = 9$
$x^2 + y^2 = 9$

59. $9x^2 + y^2 = 9$
$y^2 - 9x^2 = 9$

60. $4x^2 + y^2 = 4$
$y^2 - 4x^2 = 4$

Application Exercises

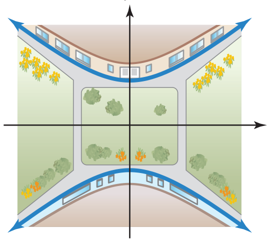
61. An explosion is recorded by two microphones that are 1 mile apart. Microphone M_1 received the sound 2 seconds before microphone M_2. Assuming sound travels at 1100 feet per second, determine the possible locations of the explosion relative to the location of the microphones.

62. Radio towers A and B, 200 kilometers apart, are situated along the coast, with A located due west of B. Simultaneous radio signals are sent from each tower to a ship, with the signal from B received 500 microseconds before the signal from A.

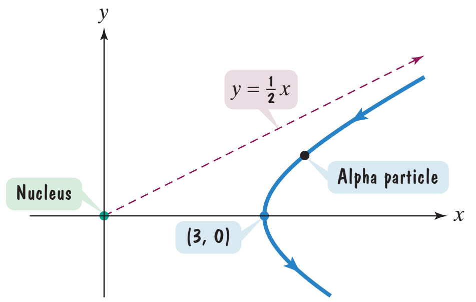
a. Assuming that the radio signals travel 300 meters per microsecond, determine the equation of the hyperbola on which the ship is located.

b. If the ship lies due north of tower B, how far out at sea is it?

63. An architect designs two houses that are shaped and positioned like a part of the branches of the hyperbola whose equation is $625y^2 - 400x^2 = 250{,}000$, where x and y are in yards. How far apart are the houses at their closest point?



64. Scattering experiments, in which moving particles are deflected by various forces, led to the concept of the nucleus of an atom. In 1911, the physicist Ernest Rutherford (1871– 1937) discovered that when alpha particles are directed toward the nuclei of gold atoms, they are eventually deflected along hyperbolic paths, illustrated in the figure. If a particle gets as close as 3 units to the nucleus along a hyperbolic path with an asymptote given by $y = \frac{1}{2}x$, what is the equation of its path?



Writing in Mathematics

65. What is a hyperbola?

66. Describe how to graph $\frac{x^2}{9} - \frac{y^2}{1} = 1$.

67. Describe how to locate the foci of the graph of $\frac{x^2}{9} - \frac{y^2}{1} = 1$.

68. Describe one similarity and one difference between the graphs of $\frac{x^2}{9} - \frac{y^2}{1} = 1$ and $\frac{y^2}{9} - \frac{x^2}{1} = 1$.

69. Describe one similarity and one difference between the graphs of $\frac{x^2}{9} - \frac{y^2}{1} = 1$ and $\frac{(x-3)^2}{9} - \frac{(y+3)^2}{1} = 1$.

70. How can you distinguish an ellipse from a hyperbola by looking at their equations?

71. In 1992, a NASA team began a project called Spaceguard Survey, calling for an international watch for comets that might collide with Earth. Why is it more difficult to detect a possible "doomsday comet" with a hyperbolic orbit than one with an elliptical orbit?

Technology Exercises

72. Use a graphing utility to graph any five of the hyperbolas that you graphed by hand in Exercises 13–26.

73. Use a graphing utility to graph any three of the hyperbolas that you graphed by hand in Exercises 33–42. First solve the given equation for y by using the square root property. Enter each of the two resulting equations to produce each branch of the hyperbola.

74. Use a graphing utility to graph any one of the hyperbolas that you graphed by hand in Exercises 43–50. Write the equation as a quadratic equation in y and use the quadratic formula to solve for y. Enter each of the two resulting equations to produce each branch of the hyperbola.

75. Use a graphing utility to graph $\frac{x^2}{4} - \frac{y^2}{9} = 0$. Is the graph a hyperbola? In general, what is the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$?

76. Graph $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ in the same viewing rectangle for values of a^2 and b^2 of your choice. Describe the relationship between the two graphs.

77. Write $4x^2 - 6xy + 2y^2 - 3x + 10y - 6 = 0$ as a quadratic equation in y and then use the quadratic formula to express y in terms of x. Graph the resulting two equations using a graphing utility in a $[-50, 70, 10]$ by $[-30, 50, 10]$ viewing rectangle. What effect does the xy-term have on the graph of the resulting hyperbola? What problems would you encounter if you attempted to write the given equation in standard form by completing the square?

78. Graph $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{x|x|}{16} - \frac{y|y|}{9} = 1$ in the same viewing rectangle. Explain why the graphs are not the same.

Critical Thinking Exercises

79. Which one of the following is true?

a. If one branch of a hyperbola is removed from a graph, then the branch that remains must define y as a function of x.

b. All points on the asymptotes of a hyperbola also satisfy the hyperbola's equation.

c. The graph of $\frac{x^2}{9} - \frac{y^2}{4} = 1$ does not intersect the line $y = -\frac{2}{3}x$.

d. Two different hyperbolas can never share the same asymptotes.

80. What happens to the shape of the graph of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as $\frac{c}{a} \to \infty$?

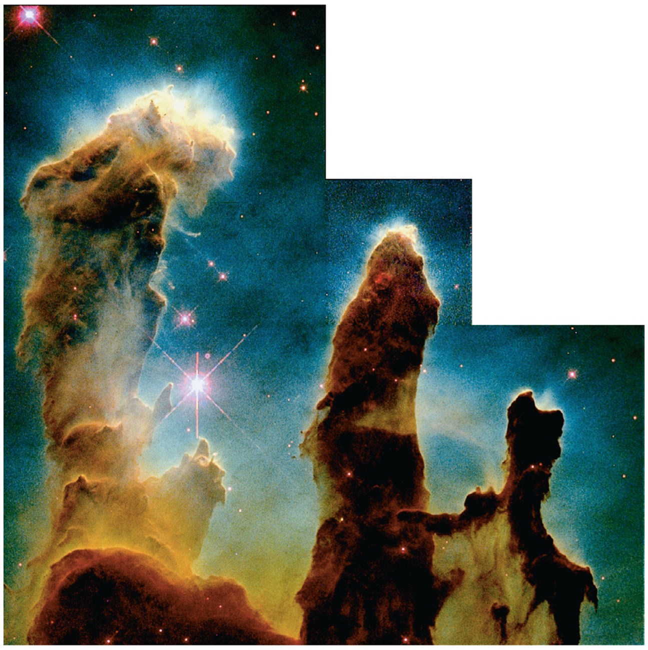
81. Find the standard form of the equation of the hyperbola with vertices $(5, -6)$ and $(5, 6)$, passing through $(0, 9)$.

82. Find the equation of a hyperbola whose asymptotes are perpendicular.

SECTION 9.3 The Parabola

Objectives

1. Graph parabolas with vertices at the origin.
2. Write equations of parabolas in standard form.
3. Graph parabolas with vertices not at the origin.
4. Solve applied problems involving parabolas.



At first glance, this image looks like columns of smoke rising from a fire into a starry sky. Those are, indeed, stars in the background, but you are not looking at ordinary smoke columns. These stand almost 6 trillion miles high and are 7000 light-years from Earth—more than 400 million times as far away as the sun.

This NASA photograph is one of a series of stunning images captured from the ends of the universe by the Hubble Space Telescope. The image shows infant star systems the size of our solar system emerging from the gas and dust that shrouded their creation. Using a parabolic mirror that is 94.5 inches in diameter, the Hubble has provided answers to many of the profound mysteries of the cosmos: How big and how old is the universe? How did the galaxies come to exist? Do other Earth-like planets orbit other sun-like stars? In this section, we study parabolas and their applications, including parabolic shapes that gather distant rays of light and focus them into spectacular images.

Definition of a Parabola

In Chapter 2, we studied parabolas, viewing them as graphs of quadratic functions in the form

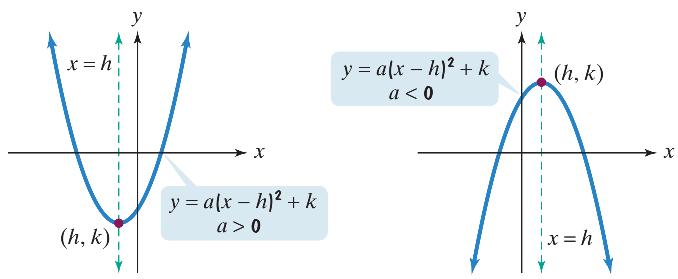
$$y = a(x - h)^2 + k \quad \text{or} \quad y = ax^2 + bx + c.$$

Study Tip

Here is a summary of what you should already know about graphing parabolas.

Graphing $y = a(x - h)^2 + k$ and $y = ax^2 + bx + c$

1. If $a > 0$, the graph opens upward. If $a < 0$, the graph opens downward.
2. The vertex of $y = a(x - h)^2 + k$ is (h, k).
3. The x-coordinate of the vertex of $y = ax^2 + bx + c$ is $x = -\frac{b}{2a}$.



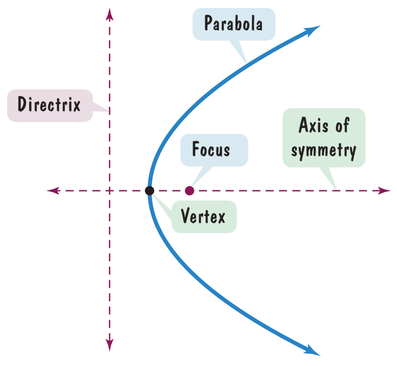


Figure 9.29

Parabolas can be given a geometric definition that enables us to include graphs that open to the left or to the right, as well as those that open obliquely. The definitions of ellipses and hyperbolas involved two fixed points, the foci. By contrast, the definition of a parabola is based on one point and a line.

Definition of a Parabola

A **parabola** is the set of all points in a plane that are equidistant from a fixed line, the **directrix**, and a fixed point, the **focus**, that is not on the line (see Figure 9.29).

In Figure 9.29, find the line passing through the focus and perpendicular to the directrix. This is the **axis of symmetry** of the parabola. The point of intersection of the parabola with its axis of symmetry is called the **vertex**. Notice that the vertex is midway between the focus and the directrix.

Standard Form of the Equation of a Parabola

The rectangular coordinate system enables us to translate a parabola's geometric definition into an algebraic equation. Figure 9.30 is our starting point for obtaining an equation. We place the focus on the x-axis at the point $(p, 0)$. The directrix has an equation given by $x = -p$. The vertex, located midway between the focus and the directrix, is at the origin.

What does the definition of a parabola tell us about the point (x, y) in Figure 9.30? For any point (x, y) on the parabola, the distance d_1 to the directrix is equal to the distance d_2 to the focus. Thus, the point (x, y) is on the parabola if and only if

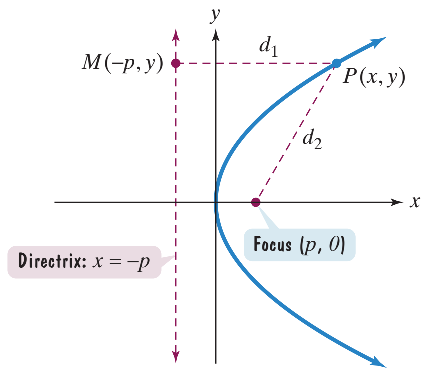


Figure 9.30

$$d_1 = d_2.$$

$$\sqrt{(x+p)^2 + (y-y)^2} = \sqrt{(x-p)^2 + (y-0)^2} \quad \text{Use the distance formula.}$$

$$(x+p)^2 = (x-p)^2 + y^2 \quad \text{Square both sides of the equation.}$$

$$x^2 + 2px + p^2 = x^2 - 2px + p^2 + y^2 \quad \text{Square } x + p \text{ and } x - p.$$

$$2px = -2px + y^2 \quad \text{Subtract } x^2 + p^2 \text{ from both sides of the equation.}$$

$$y^2 = 4px \quad \text{Solve for } y^2.$$

This last equation is called the **standard form of the equation of a parabola with its vertex at the origin**. There are two such equations, one for a focus on the x-axis and one for a focus on the y-axis.

Standard Forms of the Equations of a Parabola

The **standard form of the equation of a parabola** with vertex at the origin is

$$y^2 = 4px \quad \text{or} \quad x^2 = 4py.$$

Figure 9.31(a) on the next page illustrates that for the equation on the left, the focus is on the x-axis, which is the axis of symmetry. Figure 9.31(b) on the next page illustrates that for the equation on the right, the focus is on the y-axis, which is the axis of symmetry.

Study Tip

It is helpful to think of p as the *directed distance* from the vertex to the focus. If $p > 0$, the focus lies p units to the right of the vertex or p units above the vertex. If $p < 0$, the focus lies $|p|$ units to the left of the vertex or $|p|$ units below the vertex.

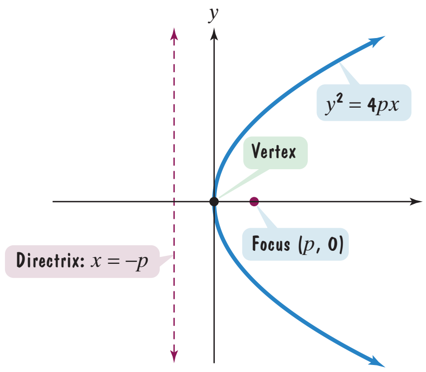


Figure 9.31(a) Parabola with the x-axis as the axis of symmetry. If $p > 0$, the graph opens to the right. If $p < 0$, the graph opens to the left.

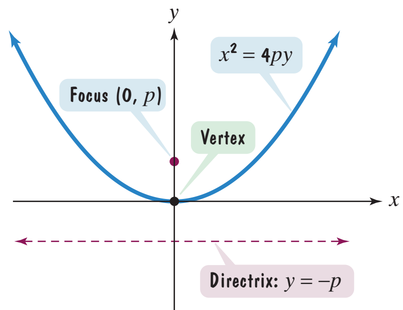


Figure 9.31(b) Parabola with the y-axis as the axis of symmetry. If $p > 0$, the graph opens upward. If $p < 0$, the graph opens downward.

1 Graph parabolas with vertices at the origin.

Using the Standard Form of the Equation of a Parabola

We can use the standard form of the equation of a parabola to find its focus and directrix. Observing the graph's symmetry from its equation is helpful in locating the focus.

$$y^2 = 4px$$

The equation does not change if y is replaced with $-y$. There is x-axis symmetry and the focus is on the x-axis at $(p, 0)$.

$$x^2 = 4py$$

The equation does not change if x is replaced with $-x$. There is y-axis symmetry and the focus is on the y-axis at $(0, p)$.

Although the definition of a parabola is given in terms of its focus and its directrix, the focus and directrix are not part of the graph. The vertex, located at the origin, is a point on the graph of $y^2 = 4px$ and $x^2 = 4py$. Example 1 illustrates how you can find two additional points on the parabola.

EXAMPLE 1 Finding the Focus and Directrix of a Parabola

Find the focus and directrix of the parabola given by $y^2 = 12x$. Then graph the parabola.

Solution The given equation is in the standard form $y^2 = 4px$, so $4p = 12$.

No change if y is replaced with $-y$. The parabola has x-axis symmetry.

$$y^2 = 12x$$

This is $4p$.

We can find both the focus and the directrix by finding p.

$$4p = 12$$
$$p = 3 \qquad \text{Divide both sides by 4.}$$

Because p is positive, the parabola, with its x-axis symmetry, opens to the right. The focus is 3 units to the right of the vertex, $(0, 0)$.

Focus: $(p, 0) = (3, 0)$

Directrix: $x = -p$; $x = -3$

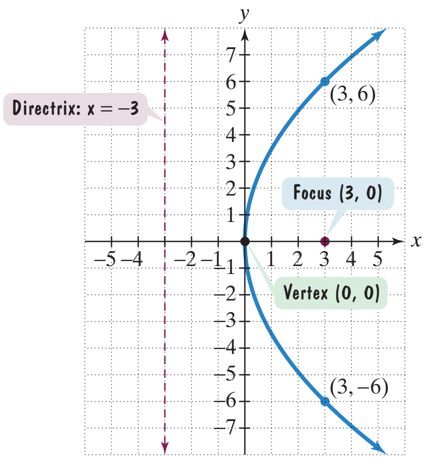


Figure 9.32 The graph of $y^2 = 12x$

The focus, $(3, 0)$, and directrix, $x = -3$, are shown in Figure 9.32.

To graph the parabola, we will use two points on the graph that lie directly above and below the focus. Because the focus is at $(3, 0)$, substitute 3 for x in the parabola's equation, $y^2 = 12x$.

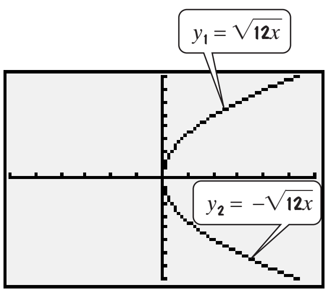
$$y^2 = 12 \cdot 3 \quad \text{Replace } x \text{ with 3 in } y^2 = 12x.$$
$$y^2 = 36 \quad \text{Simplify.}$$
$$y = \pm\sqrt{36} = \pm 6 \quad \text{Apply the square root property.}$$

The points on the parabola above and below the focus are $(3, 6)$ and $(3, -6)$. The graph is sketched in Figure 9.32.

Check Point 1 Find the focus and directrix of the parabola given by $y^2 = 8x$. Then graph the parabola.

Technology

We graph $y^2 = 12x$ with a graphing utility by first solving for y. The screen shows the graphs of $y = \sqrt{12x}$ and $y = -\sqrt{12x}$. The graph fails the vertical line test. Because $y^2 = 12x$ is not a function, you were not familiar with this form of the parabola's equation in Chapter 2.



[−6, 6, 1] by [−8, 8, 1]

In general, the points on a parabola $y^2 = 4px$ that lie above and below the focus, $(p, 0)$, are each at a distance $|2p|$ from the focus. This is because if $x = p$, then $y^2 = 4px = 4p^2$, so $y = \pm 2p$. The line segment joining these two points is called the *latus rectum*; its length is $|4p|$.

The Latus Rectum and Graphing Parabolas

The **latus rectum** of a parabola is a line segment that passes through its focus, is parallel to its directrix, and has its endpoints on the parabola. Figure 9.33 shows that the length of the latus rectum for the graphs of $y^2 = 4px$ and $x^2 = 4py$ is $|4p|$.

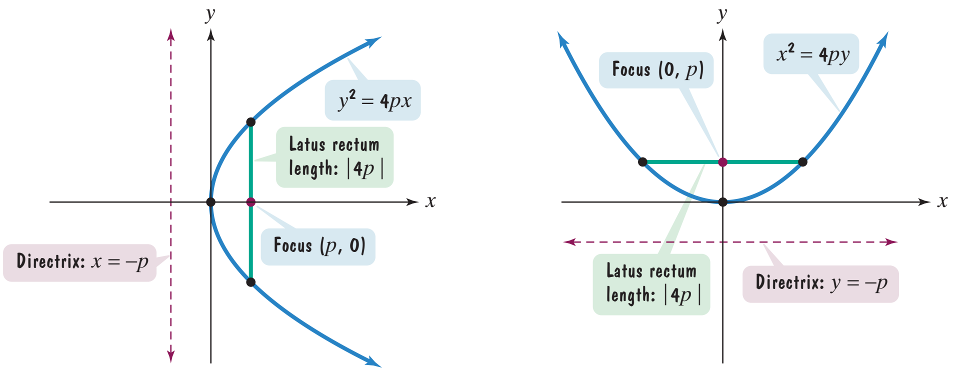


Figure 9.33 Endpoints of the latus rectum are helpful in determining a parabola's "width," or how it opens.

EXAMPLE 2 Finding the Focus and Directrix of a Parabola

Find the focus and directrix of the parabola given by $x^2 = -8y$. Then graph the parabola.

Solution The given equation is in the standard form $x^2 = 4py$, so $4p = -8$.

$$x^2 = -8y$$

No change if x is replaced with $-x$. The parabola has y-axis symmetry.

This is $4p$.

We can find both the focus and the directrix by finding p.

$$4p = -8$$
$$p = -2 \quad \text{Divide both sides by 4.}$$

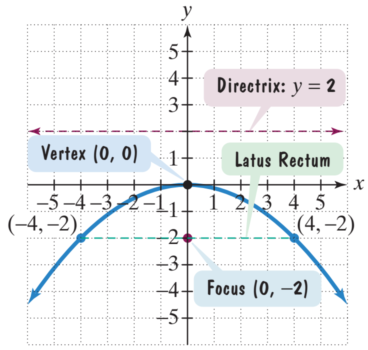


Figure 9.34 The graph of $x^2 = -8y$

Because p is negative, the parabola, with its y-axis symmetry, opens downward. The focus is 2 units below the vertex, (0, 0).

$$\text{Focus:} \quad (0, p) = (0, -2)$$
$$\text{Directrix:} \quad y = -p;\ y = 2$$

The focus and directrix are shown in Figure 9.34.

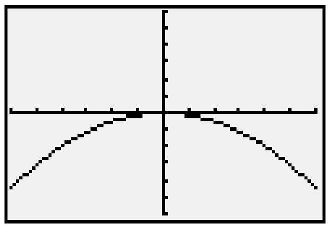
To graph the parabola, we will use the vertex, (0, 0), and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4(-2)| = |-8| = 8.$$

Because the graph has y-axis symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(0, -2)$. The endpoints of the latus rectum are $(-4, -2)$ and $(4, -2)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola, shown in Figure 9.34.

Technology

Graph $x^2 = -8y$ by first solving for y: $y = -\frac{x^2}{8}$. The graph passes the vertical line test. Because $x^2 = -8y$ is a function, you were familiar with the parabola's alternate algebraic form, $y = -\frac{1}{8}x^2$, in Chapter 2. The form is $y = ax^2 + bx + c$, with $a = -\frac{1}{8}$, $b = 0$, and $c = 0$.



[−6, 6, 1] by [−6, 6, 1]

Check Point 2 Find the focus and directrix of the parabola given by $x^2 = -12y$. Then graph the parabola.

In Examples 1 and 2, we used the equation of a parabola to find its focus and directrix. In the next example, we reverse this procedure.

2 Write equations of parabolas in standard form.

EXAMPLE 3 Finding the Equation of a Parabola from Its Focus and Directrix

Find the standard form of the equation of a parabola with focus $(5, 0)$ and directrix $x = -5$, shown in Figure 9.35.

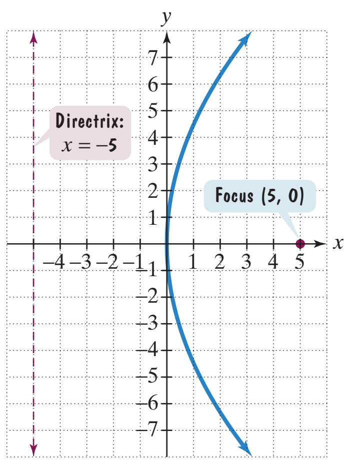


Figure 9.35

Solution The focus is $(5, 0)$. Thus, the focus is on the x-axis. We use the standard form of the equation in which there is x-axis symmetry, namely $y^2 = 4px$.

We need to determine the value of p. Figure 9.35 shows that the focus is 5 units to the right of the vertex, $(0, 0)$. Thus, p is positive and $p = 5$. We substitute 5 for p in $y^2 = 4px$ to obtain the standard form of the equation of the parabola. The equation is

$$y^2 = 4 \cdot 5x \quad \text{or} \quad y^2 = 20x.$$

Check Point 3 Find the standard form of the equation of a parabola with focus $(8, 0)$ and directrix $x = -8$.

3 Graph parabolas with vertices not at the origin.

Translations of Parabolas

The graph of a parabola can have its vertex at (h, k), rather than at the origin. Horizontal and vertical translations are accomplished by replacing x with $x - h$ and y with $y - k$ in the standard form of the parabola's equation.

Table 9.3 gives the standard forms of equations of parabolas with vertex at (h, k). Figure 9.36 shows their graphs.

Table 9.3 Standard Forms of Equations of Parabolas with Vertex at (*h*, *k*)

Equation	Vertex	Axis of Symmetry	Focus	Directrix	Description
$(y-k)^2 = 4p(x-h)$	(h, k)	Horizontal	$(h+p, k)$	$x = h - p$	If $p > 0$, opens to the right. If $p < 0$, opens to the left.
$(x-h)^2 = 4p(y-k)$	(h, k)	Vertical	$(h, k+p)$	$y = k - p$	If $p > 0$, opens upward. If $p < 0$, opens downward.

Study Tip

If y is the squared term, there is horizontal symmetry and the parabola's equation is not a function. If x is the squared term, there is vertical symmetry and the parabola's equation is a function. Continue to think of p as the directed distance from the vertex, (h, k), to the focus.

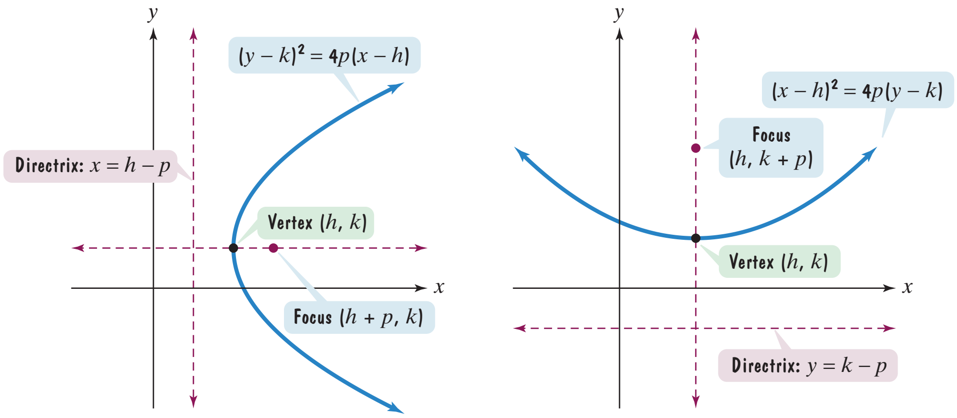


Figure 9.36 Graphs of parabolas with vertex at (h, k) and $p > 0$

The two parabolas shown in Figure 9.36 illustrate standard forms of equations for $p > 0$. If $p < 0$, a parabola with a horizontal axis of symmetry will open to the left and the focus will lie to the left of the directrix. If $p < 0$, a parabola with a vertical axis of symmetry will open downward and the focus will lie below the directrix.

EXAMPLE 4 Graphing a Parabola with Vertex at (*h*, *k*)

Find the vertex, focus, and directrix of the parabola given by

$$(x - 3)^2 = 8(y + 1).$$

Then graph the parabola.

Solution In order to find the focus and directrix, we need to know the vertex. In the standard forms of equations with vertex at (h, k), h is the number subtracted from x and k is the number subtracted from y.

$$(x - 3)^2 = 8(y - (-1))$$

This is $(x - h)^2$, with $h = 3$. This is $y - k$, with $k = -1$.

We see that $h = 3$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (3, -1)$.

Now that we have the vertex, we can find both the focus and directrix by finding p.

$$(x - 3)^2 = 8(y + 1)$$

This is $4p$.

The equation is in the standard form $(x - h)^2 = 4p(y - k)$. Because x is the squared term, there is vertical symmetry and the parabola's equation is a function.

Because $4p = 8$, $p = 2$. Based on the standard form of the equation, the axis of symmetry is vertical. With a positive value for p and a vertical axis of symmetry, the parabola opens upward. Because $p = 2$, the focus is located 2 units above the vertex, $(3, -1)$. Likewise, the directrix is located 2 units below the vertex.

Focus: $(h, k + p) = (3, -1 + 2) = (3, 1)$

The vertex, (h, k), is $(3, -1)$.

The focus is 2 units above the vertex, $(3, -1)$.

Directrix: $y = k - p$

$y = -1 - 2 = -3$

The directrix is 2 units below the vertex, $(3, -1)$.

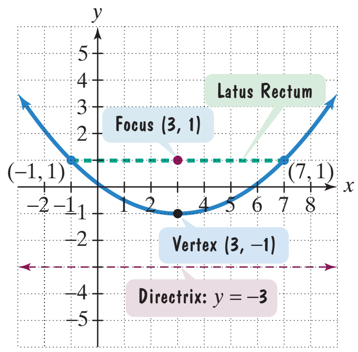


Figure 9.37 The graph of $(x - 3)^2 = 8(y + 1)$

Thus, the focus is $(3, 1)$ and the directrix is $y = -3$. They are shown in Figure 9.37. To graph the parabola, we will use the vertex, $(3, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4 \cdot 2| = |8| = 8.$$

Because the graph has vertical symmetry, the latus rectum extends 4 units to the left and 4 units to the right of the focus, $(3, 1)$. The endpoints of the latus rectum are $(3 - 4, 1)$, or $(-1, 1)$, and $(3 + 4, 1)$, or $(7, 1)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola, shown in Figure 9.37.

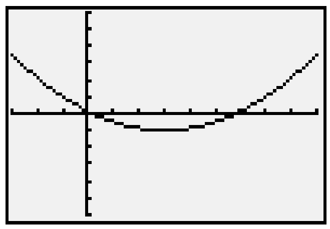
Technology

Graph $(x - 3)^2 = 8(y + 1)$ by first solving for y:

$$\frac{1}{8}(x - 3)^2 = y + 1$$

$$y = \frac{1}{8}(x - 3)^2 - 1.$$

The graph passes the vertical line test. Because $(x - 3)^2 = 8(y + 1)$ is a function, you were familiar with the parabola's alternate algebraic form, $y = \frac{1}{8}(x - 3)^2 - 1$, in Chapter 2. The form is $y = a(x - h)^2 + k$ with $a = \frac{1}{8}$, $h = 3$, and $k = -1$.



[−3, 9, 1] by [−6, 6, 1]

Check Point 4 Find the vertex, focus, and directrix of the parabola given by $(x - 2)^2 = 4(y + 1)$. Then graph the parabola.

In some cases, we need to convert the equation of a parabola to standard form by completing the square on x or y, whichever variable is squared. Let's see how this is done.

EXAMPLE 5 Graphing a Parabola with Vertex at (h, k)

Find the vertex, focus, and directrix of the parabola given by

$$y^2 + 2y + 12x - 23 = 0.$$

Then graph the parabola.

Solution We convert the given equation to standard form by completing the square on the variable y. We isolate the terms involving y on the left side.

$y^2 + 2y + 12x - 23 = 0$ — This is the given equation.

$y^2 + 2y = -12x + 23$ — Isolate the terms involving y.

$y^2 + 2y + 1 = -12x + 23 + 1$ — Complete the square by adding the square of half the coefficient of y.

$(y + 1)^2 = -12x + 24$ — Factor.

Technology

Graph $y^2 + 2y + 12x - 23 = 0$ by solving the equation for y.

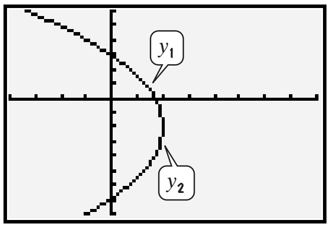
$$y^2 + 2y + (12x - 23) = 0$$

$a = 1$ $b = 2$ $c = 12x - 23$

Use the quadratic formula to solve for y and enter the resulting equations.

$$y_1 = \frac{-2 + \sqrt{4 - 4(12x - 23)}}{2}$$

$$y_2 = \frac{-2 - \sqrt{4 - 4(12x - 23)}}{2}$$



[−4, 8, 1] by [−8, 6, 1]

To express the equation $(y + 1)^2 = -12x + 24$ in the standard form $(y - k)^2 = 4p(x - h)$, we factor -12 on the right. The standard form of the parabola's equation is

$$(y + 1)^2 = -12(x - 2).$$

We use this form to identify the vertex, (h, k), and the value for p needed to locate the focus and the directrix.

$$[(y - (-1)]^2 = -12(x - 2)$$

This is $(y - k)^2$, with $k = -1$. This is $4p$. This is $x - h$, with $h = 2$.

The equation is in the standard form $(y - k)^2 = 4p(x - h)$. Because y is the squared term, there is horizontal symmetry and the parabola's equation is not a function.

We see that $h = 2$ and $k = -1$. Thus, the vertex of the parabola is $(h, k) = (2, -1)$. Because $4p = -12$, $p = -3$. Based on the standard form of the equation, the axis of symmetry is horizontal. With a negative value for p and a horizontal axis of symmetry, the parabola opens to the left. Because $p = -3$, the focus is located 3 units to the left of the vertex, $(2, -1)$. Likewise, the directrix is located 3 units to the right of the vertex.

Focus: $(h + p, k) = (2 + (-3), -1) = (-1, -1)$

The vertex, (h, k), is $(2, -1)$. The focus is 3 units to the left of the vertex, $(2, -1)$.

Directrix:

$$x = h - p$$
$$x = 2 - (-3) = 5$$

The directrix is 3 units to the right of the vertex, $(2, -1)$.

Thus, the focus is $(-1, -1)$ and the directrix is $x = 5$. They are shown in Figure 9.38.

To graph the parabola, we will use the vertex, $(2, -1)$, and the two endpoints of the latus rectum. The length of the latus rectum is

$$|4p| = |4(-3)| = |-12| = 12.$$

Because the graph has horizontal symmetry, the latus rectum extends 6 units above and 6 units below the focus, $(-1, -1)$. The endpoints of the latus rectum are $(-1, -1 + 6)$, or $(-1, 5)$, and $(-1, -1 - 6)$, or $(-1, -7)$. Passing a smooth curve through the vertex and these two points, we sketch the parabola shown in Figure 9.38.

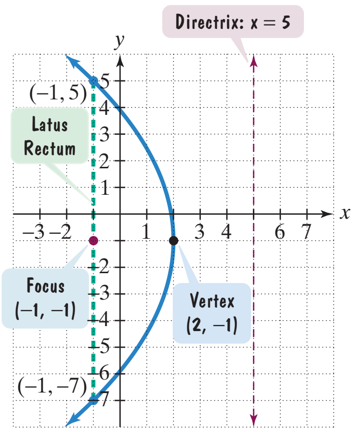


Figure 9.38 The graph of $y^2 + 2y + 12x - 23 = 0$, or $(y + 1)^2 = -12(x - 2)$

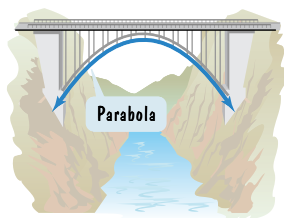
Check Point 5 Find the vertex, focus, and directrix of the parabola given by $y^2 + 2y + 4x - 7 = 0$. Then graph the parabola.

4 Solve applied problems involving parabolas.

Applications

Parabolas have many applications. Cables hung between structures to form suspension bridges form parabolas. Arches constructed of steel and concrete, whose main purpose is strength, are usually parabolic in shape.

Suspension bridge



Arch bridge

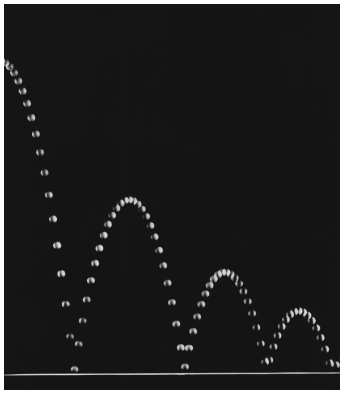


Figure 9.39 Multiflash photo showing the parabolic path of a ball thrown into the air

We have seen that comets in our solar system travel in orbits that are ellipses and hyperbolas. Some comets follow parabolic paths. Only comets with elliptical orbits, such as Halley's Comet, return to our part of the galaxy.

You throw a ball directly upward. As illustrated in Figure 9.39, the height of such a projectile as a function of time is parabolic.

If a parabola is rotated about its axis of symmetry, a parabolic surface is formed. Figure 9.40(a) shows how a parabolic surface can be used to reflect light. Light originates at the focus. Note how the light is reflected by the parabolic surface, so that the outgoing light is parallel to the axis of symmetry. The reflective properties of parabolic surfaces are used in the design of searchlights [see Figure 9.40(b)], automobile headlights, and parabolic microphones.

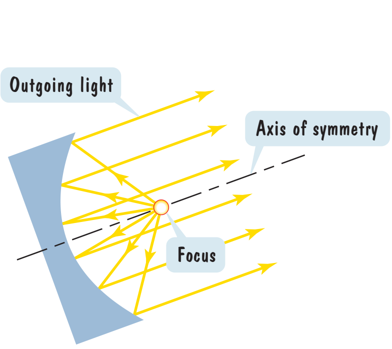


Figure 9.40(a) Parabolic surface reflecting light

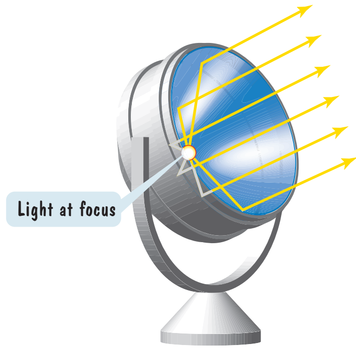


Figure 9.40(b) Light from the focus is reflected parallel to the axis of symmetry.

Figure 9.41(a) shows how a parabolic surface can be used to reflect *incoming* light. Note that light rays strike the surface and are reflected *to the focus*. This principle is used in the design of reflecting telescopes, radar, and television satellite dishes. Reflecting telescopes magnify the light from distant stars by reflecting the light from these bodies to the focus of a parabolic mirror [see Figure 9.41(b)].

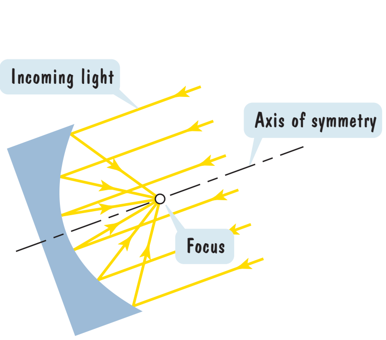


Figure 9.41(a) Parabolic surface reflecting incoming light

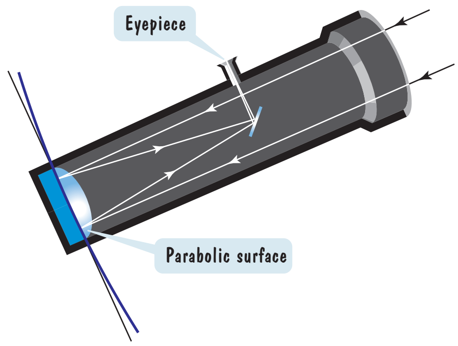


Figure 9.41(b) Incoming light rays are reflected to the focus.

The Hubble Space Telescope

The Hubble Space Telescope

For decades, astronomers hoped to create an observatory above the atmosphere that would provide an unobscured view of the universe. This vision was realized with the 1990 launching of the Hubble Space Telescope. The telescope initially had blurred vision due to problems with its parabolic mirror. The mirror had been ground two millionths of a meter smaller than design specifications. In 1993, astronauts from the Space Shuttle *Endeavor* equipped the telescope with optics to correct the blurred vision. "A small change for a mirror, a giant leap for astronomy," Christopher J. Burrows of the Space Telescope Science Institute said when clear images from the ends of the universe were presented to the public after the repair mission. Although these images have helped unravel some of the universe's deepest mysteries, by mid-2005, the uncertain fate of the Hubble Space Telescope sparked debate in the U.S. Congress and the scientific community due to the growing costs of keeping it among the stars.

EXAMPLE 6 Using the Reflection Property of Parabolas

An engineer is designing a flashlight using a parabolic reflecting mirror and a light source, shown in Figure 9.42. The casting has a diameter of 4 inches and a depth of 2 inches. What is the equation of the parabola used to shape the mirror? At what point should the light source be placed relative to the mirror's vertex?

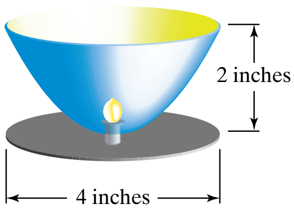


Figure 9.42 Designing a flashlight

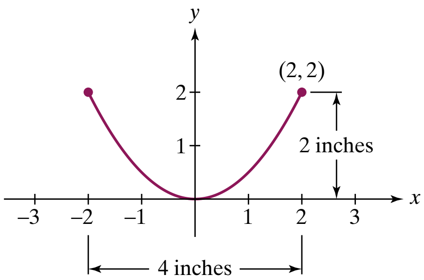


Figure 9.43

Solution We position the parabola with its vertex at the origin and opening upward (Figure 9.43). Thus, the focus is on the y-axis, located at $(0, p)$. We use the standard form of the equation in which there is y-axis symmetry, namely $x^2 = 4py$. We need to find p. Because $(2, 2)$ lies on the parabola, we let $x = 2$ and $y = 2$ in $x^2 = 4py$.

$$2^2 = 4p \cdot 2 \quad \text{Substitute 2 for } x \text{ and 2 for } y \text{ in } x^2 = 4py.$$

$$4 = 8p \quad \text{Simplify.}$$

$$p = \tfrac{1}{2} \quad \text{Divide both sides of the equation by 8 and reduce the resulting fraction.}$$

We substitute $\frac{1}{2}$ for p in $x^2 = 4py$ to obtain the standard form of the equation of the parabola. The equation of the parabola used to shape the mirror is

$$x^2 = 4 \cdot \tfrac{1}{2}y \quad \text{or} \quad x^2 = 2y.$$

The light source should be placed at the focus, $(0, p)$. Because $p = \frac{1}{2}$, the light should be placed at $\left(0, \frac{1}{2}\right)$, or $\frac{1}{2}$ inch above the vertex.

Check Point 6 In Example 6, suppose that the casting has a diameter of 6 inches and a depth of 4 inches. What is the equation of the parabola used to shape the mirror? At what point should the light source be placed relative to the mirror's vertex?

Degenerate Conic Sections

We opened the chapter by noting that conic sections are curves that result from the intersection of a cone and a plane. However, these intersections might not result in a conic section. Three degenerate cases occur when the cutting plane passes through the vertex. These **degenerate conic sections** are a point, a line, and a pair of intersecting lines, illustrated in Figure 9.44.

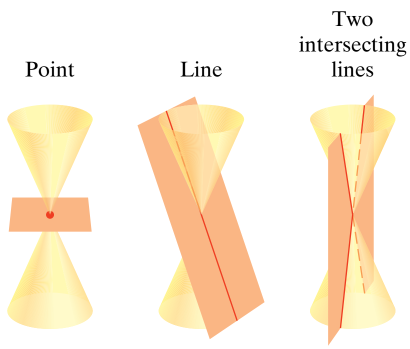


Figure 9.44 Degenerate conics

EXERCISE SET 9.3

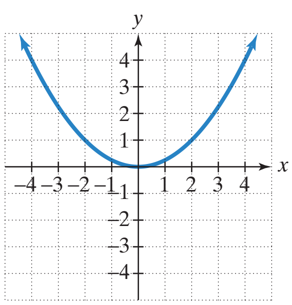
Practice Exercises

In Exercises 1–4, find the focus and directrix of each parabola with the given equation. Then match each equation to one of the graphs that are shown and labeled (a)–(d).

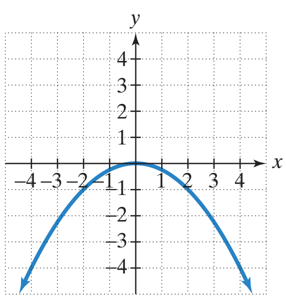
1. $y^2 = 4x$ **2.** $x^2 = 4y$

3. $x^2 = -4y$ **4.** $y^2 = -4x$

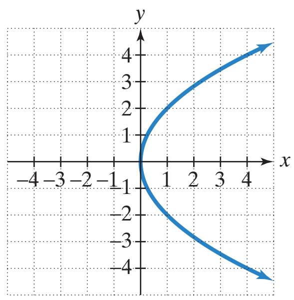
a.



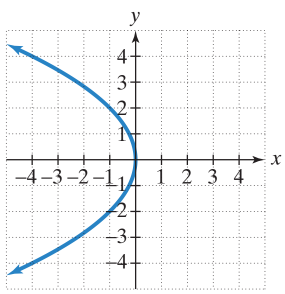
b.



c.



d.



In Exercises 5–16, find the focus and directrix of the parabola with the given equation. Then graph the parabola.

5. $y^2 = 16x$ **6.** $y^2 = 4x$

7. $y^2 = -8x$ **8.** $y^2 = -12x$

9. $x^2 = 12y$ **10.** $x^2 = 8y$

11. $x^2 = -16y$ **12.** $x^2 = -20y$

13. $y^2 - 6x = 0$ **14.** $x^2 - 6y = 0$

15. $8x^2 + 4y = 0$ **16.** $8y^2 + 4x = 0$

In Exercises 17–30, find the standard form of the equation of each parabola satisfying the given conditions.

17. Focus: $(7, 0)$; Directrix: $x = -7$

18. Focus: $(9, 0)$; Directrix: $x = -9$

19. Focus: $(-5, 0)$; Directrix: $x = 5$

20. Focus: $(-10, 0)$; Directrix: $x = 10$

21. Focus: $(0, 15)$; Directrix: $y = -15$

22. Focus: $(0, 20)$; Directrix: $y = -20$

23. Focus: $(0, -25)$; Directrix: $y = 25$

24. Focus: $(0, -15)$; Directrix: $y = 15$

25. Vertex: $(2, -3)$; Focus: $(2, -5)$

26. Vertex: $(5, -2)$; Focus: $(7, -2)$

27. Focus: $(3, 2)$; Directrix: $x = -1$

28. Focus: $(2, 4)$; Directrix: $x = -4$

29. Focus: $(-3, 4)$; Directrix: $y = 2$

30. Focus: $(7, -1)$; Directrix: $y = -9$

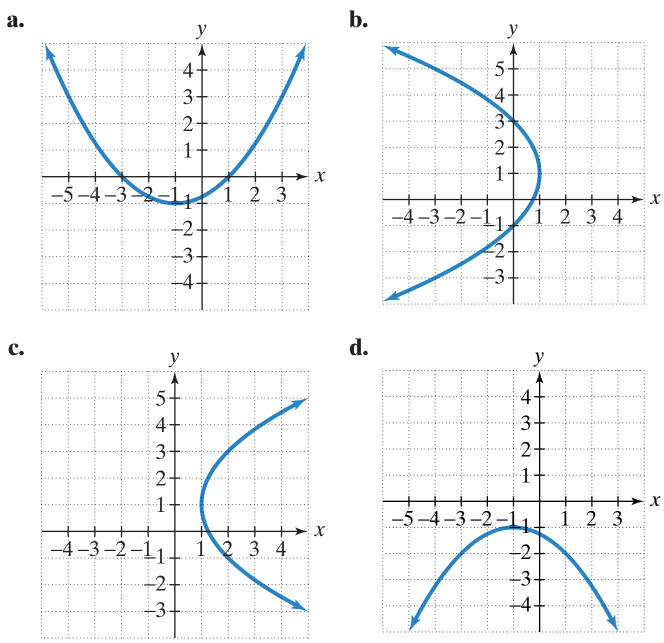
In Exercises 31–34, find the vertex, focus, and directrix of each parabola with the given equation. Then match each equation to one of the graphs that are shown and labeled (a)–(d).

31. $(y - 1)^2 = 4(x - 1)$

32. $(x + 1)^2 = 4(y + 1)$

33. $(x + 1)^2 = -4(y + 1)$

34. $(y - 1)^2 = -4(x - 1)$



In Exercises 35–42, find the vertex, focus, and directrix of each parabola with the given equation. Then graph the parabola.

35. $(x - 2)^2 = 8(y - 1)$ **36.** $(x + 2)^2 = 4(y + 1)$

37. $(x + 1)^2 = -8(y + 1)$ **38.** $(x + 2)^2 = -8(y + 2)$

39. $(y + 3)^2 = 12(x + 1)$ **40.** $(y + 4)^2 = 12(x + 2)$

41. $(y + 1)^2 = -8x$ **42.** $(y - 1)^2 = -8x$

In Exercises 43–48, convert each equation to standard form by completing the square on x or y. Then find the vertex, focus, and directrix of the parabola. Finally, graph the parabola.

43. $x^2 - 2x - 4y + 9 = 0$ **44.** $x^2 + 6x + 8y + 1 = 0$

45. $y^2 - 2y + 12x - 35 = 0$ **46.** $y^2 - 2y - 8x + 1 = 0$

47. $x^2 + 6x - 4y + 1 = 0$ **48.** $x^2 + 8x - 4y + 8 = 0$

Practice Plus

In Exercises 49–54, use the vertex and the direction in which the parabola opens to determine the relation's domain and range. Is the relation a function?

49. $y^2 + 6y - x + 5 = 0$ **50.** $y^2 - 2y - x - 5 = 0$

51. $y = -x^2 + 4x - 3$ **52.** $y = -x^2 - 4x + 4$

53. $x = -4(y - 1)^2 + 3$ **54.** $x = -3(y - 1)^2 - 2$

In Exercises 55–60, find the solution set for each system by graphing both of the system's equations in the same rectangular coordinate system and finding points of intersection. Check all solutions in both equations.

55. $(y - 2)^2 = x + 4$, $y = -\frac{1}{2}x$ **56.** $(y - 3)^2 = x - 2$, $x + y = 5$

57. $x = y^2 - 3$, $x = y^2 - 3y$ **58.** $x = y^2 - 5$, $x^2 + y^2 = 25$

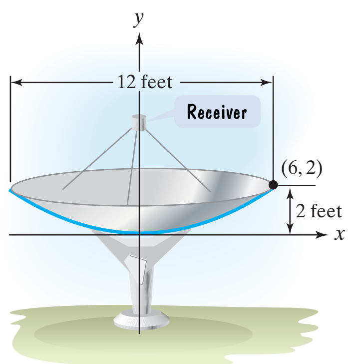
59. $x = (y + 2)^2 - 1$, $(x - 2)^2 + (y + 2)^2 = 1$ **60.** $x = 2y^2 + 4y + 5$, $(x + 1)^2 + (y - 2)^2 = 1$

Application Exercises

61. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 4 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?

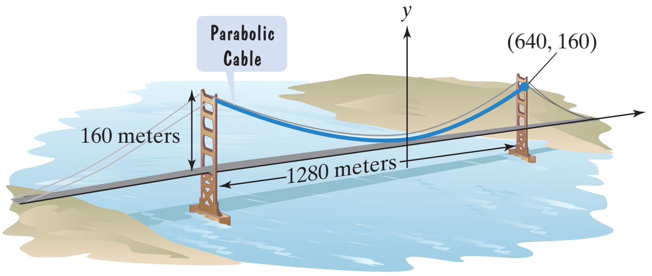
62. The reflector of a flashlight is in the shape of a parabolic surface. The casting has a diameter of 8 inches and a depth of 1 inch. How far from the vertex should the light bulb be placed?

63. A satellite dish, like the one shown below, is in the shape of a parabolic surface. Signals coming from a satellite strike the surface of the dish and are reflected to the focus, where the receiver is located. The satellite dish shown has a diameter of 12 feet and a depth of 2 feet. How far from the base of the dish should the receiver be placed?

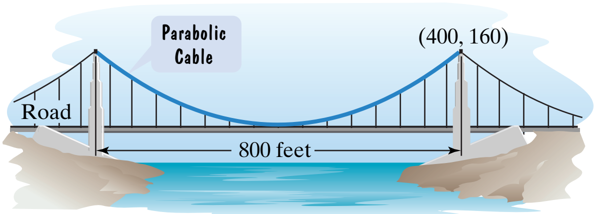


64. In Exercise 63, if the diameter of the dish is halved and the depth stays the same, how far from the base of the smaller dish should the receiver be placed?

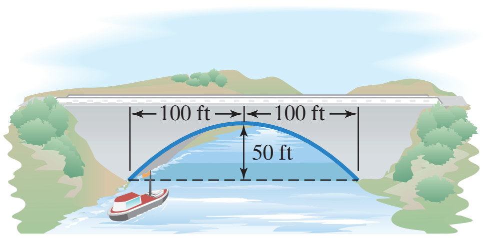
65. The towers of the Golden Gate Bridge connecting San Francisco to Marin County are 1280 meters apart and rise 160 meters above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 200 meters from a tower? Round to the nearest meter.



66. The towers of a suspension bridge are 800 feet apart and rise 160 feet above the road. The cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 100 feet from a tower?



67. The parabolic arch shown in the figure is 50 feet above the water at the center and 200 feet wide at the base. Will a boat that is 30 feet tall clear the arch 30 feet from the center?



68. A satellite dish in the shape of a parabolic surface has a diameter of 20 feet. If the receiver is to be placed 6 feet from the base, how deep should the dish be?

Writing in Mathematics

69. What is a parabola?

70. Explain how to use $y^2 = 8x$ to find the parabola's focus and directrix.

71. If you are given the standard form of the equation of a parabola with vertex at the origin, explain how to determine if the parabola opens to the right, left, upward, or downward.

72. Describe one similarity and one difference between the graphs of $y^2 = 4x$ and $(y-1)^2 = 4(x-1)$.

73. How can you distinguish parabolas from other conic sections by looking at their equations?

74. Look at the satellite dish shown in Exercise 63. Why must the receiver for a shallow dish be farther from the base of the dish than for a deeper dish of the same diameter?

Technology Exercises

75. Use a graphing utility to graph any five of the parabolas that you graphed by hand in Exercises 5–16.

76. Use a graphing utility to graph any three of the parabolas that you graphed by hand in Exercises 35–42. First solve the given equation for y, possibly using the square root property. Enter each of the two resulting equations to produce the complete graph.

Use a graphing utility to graph the parabolas in Exercises 77–78. Write the given equation as a quadratic equation in y and use the quadratic formula to solve for y. Enter each of the equations to produce the complete graph.

77. $y^2 + 2y - 6x + 13 = 0$

78. $y^2 + 10y - x + 25 = 0$

In Exercises 79–80, write each equation as a quadratic equation in y and then use the quadratic formula to express y in terms of x. Graph the resulting two equations using a graphing utility. What effect does the xy-term have on the graph of the resulting parabola?

79. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

80. $x^2 + 2\sqrt{3}xy + 3y^2 + 8\sqrt{3}x - 8y + 32 = 0$

Critical Thinking Exercises

81. Which one of the following is true?

a. The parabola whose equation is $x = 2y - y^2 + 5$ opens to the right.

b. If the parabola whose equation is $x = ay^2 + by + c$ has its vertex at $(3, 2)$ and $a > 0$, then it has no y-intercepts.

c. Some parabolas that open to the right have equations that define y as a function of x.

d. The graph of $x = a(y-k) + h$ is a parabola with vertex at (h, k).

82. Find the focus and directrix of a parabola whose equation is of the form $Ax^2 + Ey = 0$, $A \neq 0$, $E \neq 0$.

83. Write the standard form of the equation of a parabola whose points are equidistant from $y = 4$ and $(-1, 0)$.

Group Exercise

84. Consult the research department of your library or the Internet to find an example of architecture that incorporates one or more conic sections in its design. Share this example with other group members. Explain precisely how conic sections are used. Do conic sections enhance the appeal of the architecture? In what ways?

CHAPTER 9
MID-CHAPTER CHECK POINT

What You Know: We learned that the four conic sections are the circle, the ellipse, the hyperbola, and the parabola. Prior to this chapter, we graphed circles with center (h, k) and radius r:

$$(x-h)^2+(y-k)^2=r^2.$$

In this chapter, you learned to graph ellipses centered at the origin and ellipses centered at (h, k):

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad\text{or}\quad \frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a^2>b^2.$$

We saw that the larger denominator (a^2) determines whether the major axis is horizontal or vertical. We used vertices and asymptotes to graph hyperbolas centered at the origin and hyperbolas centered at (h, k):

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \quad\text{or}\quad \frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1.$$

We used $c^2=a^2-b^2$ to locate the foci of an ellipse. We used $c^2=a^2+b^2$ to locate the foci of a hyperbola. Finally, we used the vertex and the latus rectum to graph parabolas with vertices at the origin and parabolas with vertices at (h, k):

$$(y-k)^2=4p(x-h) \quad\text{or}\quad (x-h)^2=4p(y-k).$$

In Exercises 1–5, graph each ellipse. Give the location of the foci.

1. $\dfrac{x^2}{25}+\dfrac{y^2}{4}=1$

2. $9x^2+4y^2=36$

3. $\dfrac{(x-2)^2}{16}+\dfrac{(y+1)^2}{25}=1$

4. $\dfrac{(x+2)^2}{25}+\dfrac{(y-1)^2}{16}=1$

5. $x^2+9y^2-4x+54y+49=0$

In Exercises 6–11, graph each hyperbola. Give the location of the foci and the equations of the asymptotes.

6. $\dfrac{x^2}{9}-y^2=1$

7. $\dfrac{y^2}{9}-x^2=1$

8. $y^2-4x^2=16$

9. $4x^2-49y^2=196$

10. $\dfrac{(x-2)^2}{9}-\dfrac{(y+2)^2}{16}=1$

11. $4x^2-y^2+8x+6y+11=0$

In Exercises 12–13, graph each parabola. Give the location of the focus and the directrix.

12. $(x-2)^2=-12(y+1)$ **13.** $y^2-2x-2y-5=0$

In Exercises 14–21, graph each equation.

14. $x^2+y^2=4$

15. $x+y=4$

16. $x^2-y^2=4$

17. $x^2+4y^2=4$

18. $(x+1)^2+(y-1)^2=4$

19. $x^2+4(y-1)^2=4$

20. $(x-1)^2-(y-1)^2=4$

21. $(y+1)^2=4(x-1)$

In Exercises 22–27, find the standard form of the equation of the conic section satisfying the given conditions.

22. Ellipse; Foci: $(-4, 0)$, $(4, 0)$; Vertices: $(-5, 0)$, $(5, 0)$

23. Ellipse; Endpoints of major axis: $(-8, 2)$, $(10, 2)$ Foci: $(-4, 2)$, $(6, 2)$

24. Hyperbola; Foci: $(0, -3)$, $(0, 3)$; Vertices: $(0, -2)$, $(0, 2)$

25. Hyperbola; Foci: $(-4, 5)$, $(2, 5)$; Vertices: $(-3, 5)$, $(1, 5)$

26. Parabola; Focus: $(4, 5)$; Directrix: $y=-1$

27. Parabola; Focus: $(-2, 6)$; Directrix: $x=8$

28. A semielliptical archway over a one-way road has a height of 10 feet and a width of 30 feet. A truck has a width of 10 feet and a height of 9.5 feet. Will this truck clear the opening of the archway?

29. A lithotriper is used to disentegrate kidney stones. The patient is placed within an elliptical device with the kidney centered at one focus, while ultrasound waves from the other focus hit the walls and are reflected to the kidney stone, shattering the stone. Suppose that the length of the major axis of the ellipse is 40 centimeters and the length of the minor axis is 20 centimeters. How far from the kidney stone should the electrode that sends the ultrasound waves be placed in order to shatter the stone?

30. An explosion is recorded by two forest rangers, one at a primary station and the other at an outpost 6 kilometers away. The ranger at the primary station hears the explosion 6 seconds before the ranger at the outpost.

a. Assuming sound travels at 0.35 kilometer per second, write an equation in standard form that gives all the possible locations of the explosion. Use a coordinate system with the two ranger stations on the x-axis and the midpoint between the stations at the origin.

b. Graph the equation that gives the possible locations of the explosion. Show the locations of the ranger stations in your drawing.

31. A domed ceiling is a parabolic surface. Ten meters down from the top of the dome, the ceiling is 15 meters wide. For the best lighting on the floor, a light source should be placed at the focus of the parabolic surface. How far from the top of the dome, to the nearest tenth of a meter, should the light source be placed?

SECTION 9.4 Rotation of Axes

Objectives

1. Identify conics without completing the square.
2. Use rotation of axes formulas.
3. Write equations of rotated conics in standard form.
4. Identify conics without rotating axes.

Richard E. Prince "The Cone of Apollonius" (detail), fiberglass, steel, paint, graphite, 51 × 18 × 14 in. Private collection, Vancouver. Photo courtesy of Equinox Gallery, Vancouver, Canada.

To recognize a conic section, you often need to pay close attention to its graph. Graphs powerfully enhance our understanding of algebra and trigonometry. However, it is not possible for people who are blind—or sometimes, visually impaired—to see a graph. Creating informative materials for the blind and visually impaired is a challenge for instructors and mathematicians. Many people who are visually impaired "see" a graph by touching a three-dimensional representation of that graph, perhaps while it is described verbally.

Is it possible to identify conic sections in nonvisual ways? The answer is yes, and the methods for doing so are related to the coefficients in their equations. As we present these methods, think about how you learn them. How would your approach to studying mathematics change if we removed all graphs and replaced them with verbal descriptions?

1 Identify conics without completing the square.

Identifying Conic Sections without Completing the Square

Conic sections can be represented both geometrically (as intersecting planes and cones) and algebraically. The equations of the conic sections we have considered in the first three sections of this chapter can be expressed in the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

in which A and C are not both zero. You can use A and C, the coefficients of x^2 and y^2, respectively, to identify a conic section without completing the square.

Identifying a Conic Section without Completing the Square

A nondegenerate conic section of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

in which A and C are not both zero, is

- a circle if $A = C$,
- a parabola if $AC = 0$,
- an ellipse if $A \neq C$ and $AC > 0$, and
- a hyperbola if $AC < 0$.

EXAMPLE 1 Identifying a Conic Section without Completing the Square

Identify the graph of each of the following nondegenerate conic sections:

a. $4x^2 - 25y^2 - 24x + 250y - 489 = 0$

b. $x^2 + y^2 + 6x - 2y + 6 = 0$

c. $y^2 + 12x + 2y - 23 = 0$

d. $9x^2 + 25y^2 - 54x + 50y - 119 = 0$.

Solution We use A, the coefficient of x^2, and C, the coefficient of y^2, to identify each conic section.

a. $4x^2 - 25y^2 - 24x + 250y - 489 = 0$

$A = 4$ $C = -25$

$AC = 4(-25) = -100 < 0$

Because $AC < 0$, the graph of the equation is a hyperbola.

b. $x^2 + y^2 + 6x - 2y + 6 = 0$

$A = 1$ $C = 1$

Because $A = C$, the graph of the equation is a circle.

c. We can write $y^2 + 12x + 2y - 23 = 0$ as

$$0x^2 + y^2 + 12x + 2y - 23 = 0.$$

$A = 0$ $C = 1$

$AC = 0(1) = 0$

Because $AC = 0$, the graph of the equation is a parabola.

d. $9x^2 + 25y^2 - 54x + 50y - 119 = 0$

$A = 9$ $C = 25$

$AC = 9(25) = 225 > 0.$

Because $AC > 0$ and $A \neq C$, the graph of the equation is an ellipse.

Check Point 1 Identify the graph of each of the following nondegenerate conic sections:

a. $3x^2 + 2y^2 + 12x - 4y + 2 = 0$

b. $x^2 + y^2 - 6x + y + 3 = 0$

c. $y^2 - 12x - 4y + 52 = 0$

d. $9x^2 - 16y^2 - 90x + 64y + 17 = 0.$

2 Use rotation of axes formulas.

Rotation of Axes

Figure 9.45 shows the graph of

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0.$$

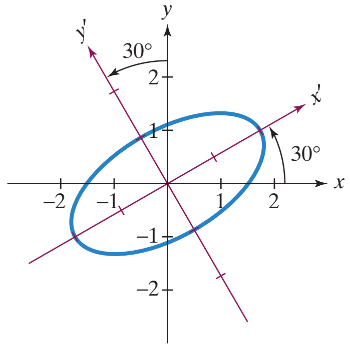


Figure 9.45 The graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$, a rotated ellipse

The graph looks like an ellipse, although its major axis neither lies along the x-axis nor is parallel to the x-axis. Do you notice anything unusual about the equation? It contains an xy-term. However, look at what happens if we rotate the x- and y-axes through an angle of 30°. In the rotated $x'y'$-system, the major axis of the ellipse lies along the x'-axis. We can write the equation of the ellipse in this rotated $x'y'$-system as

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1.$$

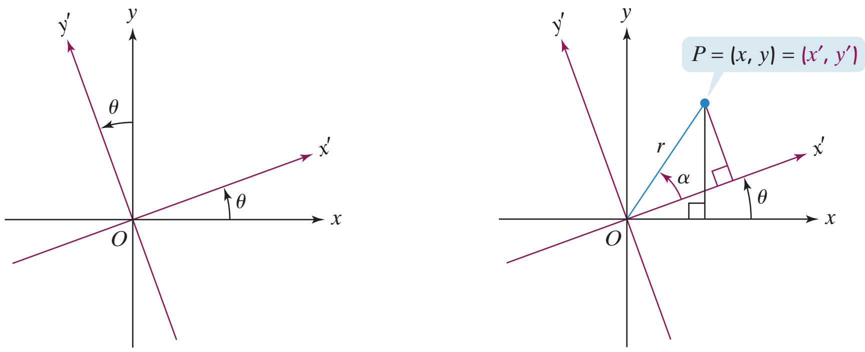
Observe that there is no $x'y'$-term in the equation.

Except for degenerate cases, the **general second-degree equation**

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

represents one of the conic sections. However, due to the xy-term in the equation, these conic sections are rotated in such a way that their axes are no longer parallel to the x- and y-axes. To reduce these equations to forms of the conic sections with which you are already familiar, we use a procedure called **rotation of axes**.

Suppose that the x- and y-axes are rotated through a positive angle θ, resulting in a new $x'y'$ coordinate system. This system is shown in Figure 9.46(a). The origin in the $x'y'$-system is the same as the origin in the xy-system. Point P in Figure 9.46(b) has coordinates (x, y) relative to the xy-system and coordinates (x', y') relative to the $x'y'$-system. Our goal is to obtain formulas relating the old and new coordinates. Thus, we need to express x and y in terms of x', y', and θ.



(a) Rotating the x- and y-axes through a positive angle θ

(b) Describing point P relative to the xy-system and the rotated $x'y'$-system

Figure 9.46 Rotating axes

Look at Figure 9.46(b). Notice that

r = the distance from the origin O to point P.

α = the angle from the positive x'-axis to the ray from O through P.

Using the definitions of sine and cosine, we obtain

$$\cos\alpha = \frac{x'}{r}: x' = r\cos\alpha$$

$$\sin\alpha = \frac{y'}{r}: y' = r\sin\alpha$$

This is from the right triangle with a leg along the x'-axis.

$$\cos(\theta + \alpha) = \frac{x}{r}: x = r\cos(\theta + \alpha)$$

$$\sin(\theta + \alpha) = \frac{y}{r}: y = r\sin(\theta + \alpha).$$

This is from the taller right triangle with a leg along the x-axis.

Thus,

$x = r\cos(\theta + \alpha)$ — This is the third of the preceding equations.

$= r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$ — Use the formula for the cosine of the sum of two angles.

$= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$ — Apply the distributive property and rearrange factors.

$= x'\cos\theta - y'\sin\theta.$ — Use the first and second of the preceding equations: $x' = r\cos\alpha$ and $y' = r\sin\alpha$.

Similarly,

$$y = r\sin(\theta + \alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = x'\sin\theta + y'\cos\theta.$$

Rotation of Axes Formulas

Suppose an xy-coordinate system and an $x'y'$-coordinate system have the same origin and θ is the angle from the positive x-axis to the positive x'-axis. If the coordinates of point P are (x, y) in the xy-system and (x', y') in the rotated $x'y'$-system, then

$$x = x' \cos\theta - y' \sin\theta$$
$$y = x' \sin\theta + y' \cos\theta.$$

EXAMPLE 2 Rotating Axes

Write the equation $xy = 1$ in terms of a rotated $x'y'$-system if the angle of rotation from the x-axis to the x'-axis is 45°. Express the equation in standard form. Use the rotated system to graph $xy = 1$.

Solution With $\theta = 45°$, the rotation formulas for x and y are

$$x = x' \cos\theta - y' \sin\theta = x' \cos 45° - y' \sin 45°$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) - y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' - y')$$
$$y = x' \sin\theta + y' \cos\theta = x' \sin 45° + y' \cos 45°$$
$$= x'\left(\frac{\sqrt{2}}{2}\right) + y'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}(x' + y').$$

Now substitute these expressions for x and y in the given equation, $xy = 1$.

$$xy = 1$$ This is the given equation.

$$\left[\frac{\sqrt{2}}{2}(x' - y')\right]\left[\frac{\sqrt{2}}{2}(x' + y')\right] = 1$$ Substitute the expressions for x and y from the rotation formulas.

$$\frac{2}{4}(x' - y')(x' + y') = 1$$ Multiply: $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{2}{4}$.

$$\frac{1}{2}(x'^2 - y'^2) = 1$$ Reduce $\frac{2}{4}$ and multiply the binomials.

$$\frac{x'^2}{2} - \frac{y'^2}{2} = 1$$ Write the equation in standard form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$a^2 = 2$ $b^2 = 2$

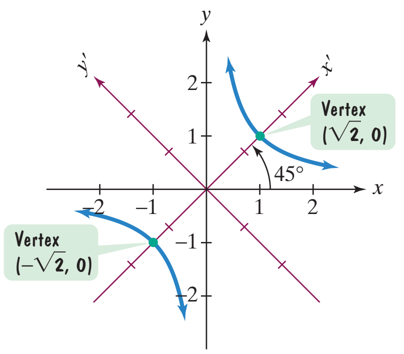


Figure 9.47 The graph of $xy = 1$ or $\frac{x'^2}{2} - \frac{y'^2}{2} = 1$

This equation expresses $xy = 1$ in terms of the rotated $x'y'$-system. Can you see that this is the standard form of the equation of a hyperbola? The hyperbola's center is at $(0, 0)$, with the transverse axis on the x'-axis. The vertices are $(-a, 0)$ and $(a, 0)$. Because $a^2 = 2$, the vertices are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$, located on the x'-axis. Based on the standard form of the hyperbola's equation, the equations for the asymptotes are

$$y' = \pm\frac{b}{a}x' \quad \text{or} \quad y' = \pm\frac{\sqrt{2}}{\sqrt{2}}x'.$$

The equations of the asymptotes can be simplified to $y' = x'$ and $y' = -x'$, which correspond to the original x- and y-axes. The graph of the hyperbola is shown in Figure 9.47.

Check Point 2 Write the equation $xy = 2$ in terms of a rotated $x'y'$-system if the angle of rotation from the x-axis to the x'-axis is 45°. Express the equation in standard form. Use the rotated system to graph $xy = 2$.

3 Write equations of rotated conics in standard form.

Using Rotations to Transform Equations with *xy*-Terms to Standard Equations of Conic Sections

We have noted that the appearance of the term Bxy $(B \neq 0)$ in the general second-degree equation indicates that the graph of the conic section has been rotated. A rotation of axes through an appropriate angle can transform the equation to one of the standard forms of the conic sections in x' and y' in which no $x'y'$-term appears.

Amount of Rotation Formula

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0$$

can be rewritten as an equation in x' and y' without an $x'y'$-term by rotating the axes through angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

Before we learn to apply this formula, let's see how it can be derived. We begin with the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0.$$

Then we rotate the axes through an angle θ. In terms of the rotated $x'y'$-system, the general second-degree equation can be written as

$$\begin{aligned} A(x'\cos\theta - y'\sin\theta)^2 &+ B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) \\ &+ C(x'\sin\theta + y'\cos\theta)^2 + D(x'\cos\theta - y'\sin\theta) \\ &+ E(x'\sin\theta + y'\cos\theta) + F = 0. \end{aligned}$$

After a lot of simplifying that involves expanding and collecting like terms, you will obtain the following equation:

We want a rotation that results in no $x'y'$-term.

$$\begin{aligned} (A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta)x'^2 &+ [B(\cos^2\theta - \sin^2\theta) + 2(C - A)(\sin\theta\cos\theta)]x'y' \\ &+ (A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta)y'^2 \\ &+ (D\cos\theta + E\sin\theta)x' \\ &+ (-D\sin\theta + E\cos\theta)y' + F = 0. \end{aligned}$$

If this looks somewhat ghastly, take a deep breath and focus only on the $x'y'$-term. We want to choose θ so that the coefficient of this term is zero. This will give the required rotation that results in no $x'y'$-term.

$B(\cos^2\theta - \sin^2\theta) + 2(C - A)\sin\theta\cos\theta = 0$ — Set the coefficient of the $x'y'$-term equal to 0.

$B\cos 2\theta + (C - A)\sin 2\theta = 0$ — Use the double-angle formulas: $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$B\cos 2\theta = -(C - A)\sin 2\theta$ — Subtract $(C - A)\sin 2\theta$ from both sides.

$B\cos 2\theta = (A - C)\sin 2\theta$ — Simplify

$\dfrac{B\cos 2\theta}{B\sin 2\theta} = \dfrac{(A - C)\sin 2\theta}{B\sin 2\theta}$ — Divide both sides by $B\sin 2\theta$.

$\dfrac{\cos 2\theta}{\sin 2\theta} = \dfrac{A - C}{B}$ — Simplify.

$\cot 2\theta = \dfrac{A - C}{B}$ — Apply a quotient identity: $\cot 2\theta = \dfrac{\cos 2\theta}{\sin 2\theta}$.

If $\cot 2\theta$ is positive, we will select θ so that $0° < \theta < 45°$. If $\cot 2\theta$ is negative, we will select θ so that $45° < \theta < 90°$. Thus θ, the angle of rotation, is always an acute angle.

Here is a step-by-step procedure for writing the equation of a rotated conic section in standard form:

Study Tip

What do you do after substituting the expressions for x and y from the rotation formulas into the given equation? You must simplify the resulting equation by expanding and collecting like terms. Work through this process slowly and carefully, allowing lots of room on your paper.

If your rotation equations are correct but you obtain an equation that has an $x'y'$-term, you have made an error in the algebraic simplification.

Writing the Equation of a Rotated Conic in Standard Form

1. Use the given equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0$$

to find $\cot 2\theta$.

$$\cot 2\theta = \frac{A - C}{B}$$

2. Use the expression for $\cot 2\theta$ to determine θ, the angle of rotation.
3. Substitute θ in the rotation formulas

$$x = x' \cos\theta - y' \sin\theta \quad \text{and} \quad y = x' \sin\theta + y' \cos\theta$$

and simplify.
4. Substitute the expressions for x and y from the rotation formulas in the given equation and simplify. The resulting equation should have no $x'y'$-term.
5. Write the equation involving x' and y' in standard form.

Using the equation in step 5, you can graph the conic section in the rotated $x'y'$-system.

EXAMPLE 3 Writing the Equation of a Rotated Conic Section in Standard Form

Rewrite the equation

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$

in a rotated $x'y'$-system without an $x'y'$-term. Express the equation in the standard form of a conic section. Graph the conic section in the rotated system.

Solution

Step 1 Use the given equation to find $\cot 2\theta$. We need to identify the constants A, B, and C in the given equation.

$$7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$$

A is the coefficient of the x^2-term: $A = 7$.

B is the coefficient of the xy-term: $B = -6\sqrt{3}$.

C is the coefficient of the y^2-term: $C = 13$.

The appropriate angle θ through which to rotate the axes satisfies the equation

$$\cot 2\theta = \frac{A - C}{B} = \frac{7 - 13}{-6\sqrt{3}} = \frac{-6}{-6\sqrt{3}} = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}.$$

Step 2 Use the expression for $\cot 2\theta$ to determine the angle of rotation. We have $\cot 2\theta = \frac{\sqrt{3}}{3}$. Based on our knowledge of exact values for trigonometric functions, we conclude that $2\theta = 60°$. Thus, $\theta = 30°$.

Step 3 Substitute θ in the rotation formulas $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$ and simplify. Substituting 30° for θ,

$$x = x'\cos 30^\circ - y'\sin 30^\circ = x'\left(\frac{\sqrt{3}}{2}\right) - y'\left(\frac{1}{2}\right) = \frac{\sqrt{3}x' - y'}{2}$$

$$y = x'\sin 30^\circ + y'\cos 30^\circ = x'\left(\frac{1}{2}\right) + y'\left(\frac{\sqrt{3}}{2}\right) = \frac{x' + \sqrt{3}y'}{2}.$$

Step 4 Substitute the expressions for x and y from the rotation formulas in the given equation and simplify.

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$
This is the given equation.

$$7\left(\frac{\sqrt{3}x' - y'}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}x' - y'}{2}\right)\left(\frac{x' + \sqrt{3}y'}{2}\right) + 13\left(\frac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$$
Substitute the expressions for x and y from the rotation formulas.

$$7\left(\frac{3x'^2 - 2\sqrt{3}x'y' + y'^2}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}x'^2 + 3x'y' - x'y' - \sqrt{3}y'^2}{4}\right) + 13\left(\frac{x'^2 + 2\sqrt{3}x'y' + 3y'^2}{4}\right) - 16 = 0$$
Square and multiply.

$$7\left(3x'^2 - 2\sqrt{3}x'y' + y'^2\right) - 6\sqrt{3}\left(\sqrt{3}x'^2 + 2x'y' - \sqrt{3}y'^2\right) + 13\left(x'^2 + 2\sqrt{3}x'y' + 3y'^2\right) - 64 = 0$$
Multiply both sides by 4.

$$21x'^2 - 14\sqrt{3}x'y' + 7y'^2 - 18x'^2 - 12\sqrt{3}x'y' + 18y'^2 + 13x'^2 + 26\sqrt{3}x'y' + 39y'^2 - 64 = 0$$
Distribute throughout parentheses.

$$21x'^2 - 18x'^2 + 13x'^2 - 14\sqrt{3}x'y' - 12\sqrt{3}x'y' + 26\sqrt{3}x'y' + 7y'^2 + 18y'^2 + 39y'^2 - 64 = 0$$
Rearrange terms.

$$16x'^2 + 64y'^2 - 64 = 0$$
Combine like terms.

Do you see how we "lost" the $x'y'$-term in the last equation?

$$-14\sqrt{3}x'y' - 12\sqrt{3}x'y' + 26\sqrt{3}x'y' = -26\sqrt{3}x'y' + 26\sqrt{3}x'y' = 0x'y' = 0$$

Step 5 Write the equation involving x' and y' in standard form. We can express $16x'^2 + 64y'^2 - 64 = 0$, an equation of an ellipse, in the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$16x'^2 + 64y'^2 - 64 = 0$$
This equation describes the ellipse relative to a system rotated through 30°.

$$16x'^2 + 64y'^2 = 64$$
Add 64 to both sides.

$$\frac{16x'^2}{64} + \frac{64y'^2}{64} = \frac{64}{64}$$
Divide both sides by 64.

$$\frac{x'^2}{4} + \frac{y'^2}{1} = 1$$
Simplify.

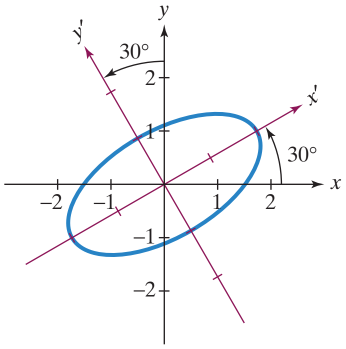


Figure 9.45 (repeated) The graph of $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$ or $\frac{x'^2}{4} - \frac{y'^2}{1} = 1$, a rotated ellipse

The last equation is the standard form of the equation of an ellipse. The major axis is on the x'-axis and the vertices are $(-2, 0)$ and $(2, 0)$. The minor axis is on the y'-axis with endpoints $(0, -1)$ and $(0, 1)$. The graph of the ellipse is shown in Figure 9.45. Does this graph look familiar? It should—you saw it earlier in this section on page 890.

Check Point 3 Rewrite the equation

$$2x^2 + \sqrt{3}xy + y^2 - 2 = 0$$

in a rotated $x'y'$-system without an $x'y'$-term. Express the equation in the standard form of a conic section. Graph the conic section in the rotated system.

Technology

In order to graph a general second-degree equation in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

using a graphing utility, it is necessary to solve for y. Rewrite the equation as a quadratic equation in y.

$$Cy^2 + (Bx + E)y + (Ax^2 + Dx + F) = 0$$

By applying the quadratic formula, the graph of this equation can be obtained by entering

$$y_1 = \frac{-(Bx + E) + \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}$$

and

$$y_2 = \frac{-(Bx + E) - \sqrt{(Bx + E)^2 - 4C(Ax^2 + Dx + F)}}{2C}.$$

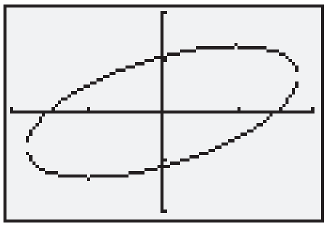
The graph of

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$$

is shown on the right in a $[-2, 2, 1]$ by $[-2, 2, 1]$ viewing rectangle. The graph was obtained by entering the equations for y_1 and y_2 shown above with

$$A = 7, B = -6\sqrt{3}, C = 13, D = 0, E = 0,$$

$$\text{and } F = -16.$$



In Example 3 and Check Point 3, we found θ, the angle of rotation, directly because we recognized $\frac{\sqrt{3}}{3}$ as the value of cot 60°. What do we do if $\cot 2\theta$ is not the cotangent of one of the familiar angles? We use $\cot 2\theta$ to find $\sin\theta$ and $\cos\theta$ as follows:

- Use a sketch of $\cot 2\theta$ to find $\cos 2\theta$.
- Find $\sin\theta$ and $\cos\theta$ using the identities

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}.$$

Because θ is an acute angle, the positive square roots are appropriate.

The resulting values for $\sin\theta$ and $\cos\theta$ are used to write the rotation formulas that give an equation with no $x'y'$-term.

EXAMPLE 4 Graphing the Equation of a Rotated Conic

Graph relative to a rotated $x'y'$-system in which the equation has no $x'y'$-term:

$$16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0.$$

Solution

Step 1 Use the given equation to find cot 2θ. With $A = 16$, $B = -24$, and $C = 9$, we have

$$\cot 2\theta = \frac{A - C}{B} = \frac{16 - 9}{-24} = -\frac{7}{24}.$$

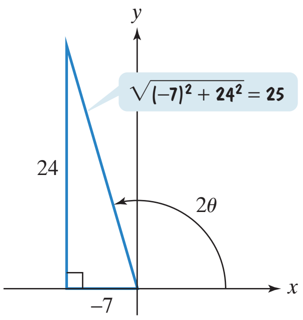


Figure 9.48 Using $\cot 2\theta$ to find $\cos 2\theta$

Step 2 Use the expression for $\cot 2\theta$ to determine $\sin\theta$ and $\cos\theta$. A rough sketch showing $\cot 2\theta$ is given in Figure 9.48. Because θ is always acute and $\cot 2\theta$ is negative, 2θ is in quadrant II. The third side of the triangle is found using $r = \sqrt{x^2 + y^2}$. Thus, $r = \sqrt{(-7)^2 + 24^2} = \sqrt{625} = 25$. By the definition of the cosine function,

$$\cos 2\theta = \frac{x}{r} = \frac{-7}{25} = -\frac{7}{25}.$$

Now we use identities to find values for $\sin\theta$ and $\cos\theta$.

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\frac{1-\left(-\frac{7}{25}\right)}{2}}$$

$$= \sqrt{\frac{\frac{25}{25}+\frac{7}{25}}{2}} = \sqrt{\frac{\frac{32}{25}}{2}} = \sqrt{\frac{32}{50}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\frac{1+\left(-\frac{7}{25}\right)}{2}}$$

$$= \sqrt{\frac{\frac{25}{25}-\frac{7}{25}}{2}} = \sqrt{\frac{\frac{18}{25}}{2}} = \sqrt{\frac{18}{50}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

Step 3 Substitute $\sin\theta$ and $\cos\theta$ in the rotation formulas

$$\boldsymbol{x = x'\cos\theta - y'\sin\theta \quad \text{and} \quad y = x'\sin\theta + y'\cos\theta}$$

and simplify. Substituting $\frac{4}{5}$ for $\sin\theta$ and $\frac{3}{5}$ for $\cos\theta$,

$$x = x'\left(\frac{3}{5}\right) - y'\left(\frac{4}{5}\right) = \frac{3x' - 4y'}{5}$$

$$y = x'\left(\frac{4}{5}\right) + y'\left(\frac{3}{5}\right) = \frac{4x' + 3y'}{5}.$$

Step 4 Substitute the expressions for x and y from the rotation formulas in the given equation and simplify.

$$16x^2 - 24xy + 9y^2 + 110x - 20y + 100 = 0$$

This is the given equation.

$$16\left(\frac{3x'-4y'}{5}\right)^2 - 24\left(\frac{3x'-4y'}{5}\right)\left(\frac{4x'+3y'}{5}\right) + 9\left(\frac{4x'+3y'}{5}\right)^2$$
$$+ 110\left(\frac{3x'-4y'}{5}\right) - 20\left(\frac{4x'+3y'}{5}\right) + 100 = 0$$

Substitute the expressions for x and y from the rotation formulas.

Take a few minutes to expand, multiply both sides of the equation by 25, and combine like terms. The resulting equation

$$y'^2 + 2x' - 4y' + 4 = 0$$

has no $x'y'$-term.

Step 5 Write the equation involving x' and y' in standard form. With only one variable that is squared, we have the equation of a parabola. We need to write the equation in the standard form $(y - k)^2 = 4p(x - h)$.

$y'^2 + 2x' - 4y' + 4 = 0$ This is the equation without an $x'y'$-term.

$y'^2 - 4y' = -2x' - 4$ Isolate the terms involving y'.

$y'^2 - 4y' + 4 = -2x' - 4 + 4$ Complete the square by adding the square of half the coefficient of y'.

$(y' - 2)^2 = -2x'$ Factor.

The standard form of the parabola's equation in the rotated $x'y'$-system is

$$(y' - 2)^2 = -2x'.$$

This is $(y' - k)^2$, with $k = 2$. This is $4p$. This is $x' - h$, with $h = 0$.

We see that $h = 0$ and $k = 2$. Thus, the vertex of the parabola in the $x'y'$-system is $(h, k) = (0, 2)$.

We can use the $x'y'$-system to graph the parabola. Using a calculator to solve $\sin\theta = \frac{4}{5}$, we find that $\theta = \sin^{-1}\frac{4}{5} \approx 53°$. Rotate the axes through approximately 53°. With $4p = -2$ and $p = -\frac{1}{2}$, the parabola's focus is $\frac{1}{2}$ unit to the left of the vertex, $(0, 2)$. Thus, the focus in the $x'y'$-system is $\left(-\frac{1}{2}, 2\right)$.

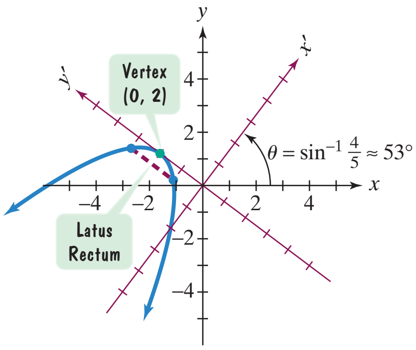


Figure 9.49 The graph of $(y' - 2)^2 = -2x'$ in a rotated $x'y'$-system

To graph the parabola, we use the vertex, $(0, 2)$, and the two endpoints of the latus rectum.

$$\text{length of latus rectum} = |4p| = |-2| = 2$$

The latus rectum extends 1 unit above and 1 unit below the focus, $\left(-\frac{1}{2}, 2\right)$. Thus, the endpoints of the latus rectum in the $x'y'$-system are $\left(-\frac{1}{2}, 3\right)$ and $\left(-\frac{1}{2}, 1\right)$. Using the rotated system, pass a smooth curve through the vertex and the two endpoints of the latus rectum. The graph of the parabola is shown in Figure 9.49.

Check Point 4 Graph relative to a rotated $x'y'$-system in which the equation has no $x'y'$-term:

$$4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0.$$

4 Identify conics without rotating axes.

Identifying Conic Sections without Rotating Axes

We now know that the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, B \neq 0$$

can be rewritten as

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

in a rotated $x'y'$-system. A relationship between the coefficients of the two equations is given by

$$B^2 - 4AC = -4A'C'.$$

We also know that A' and C' can be used to identify the graph of the rotated equation. Thus, $B^2 - 4AC$ can also be used to identify the graph of the general second-degree equation.

Identifying a Conic Section without a Rotation of Axes

A nondegenerate conic section of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is

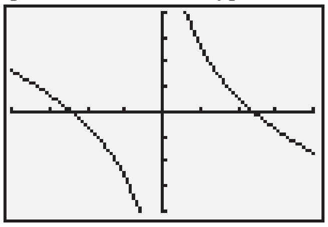
- a parabola if $B^2 - 4AC = 0$,
- an ellipse or a circle if $B^2 - 4AC < 0$, and
- a hyperbola if $B^2 - 4AC > 0$.

Technology

The graph of

$11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$

is shown in a $\left[-1, 1, \frac{1}{4}\right]$ by $\left[-1, 1, \frac{1}{4}\right]$ viewing rectangle. The graph verifies that the equation represents a rotated hyperbola.



EXAMPLE 5 Identifying a Conic Section without Rotating Axes

Identify the graph of

$$11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0.$$

Solution We use A, B, and C to identify the conic section.

$$11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$$

$A = 11$ $B = 10\sqrt{3}$ $C = 1$

$$B^2 - 4AC = \left(10\sqrt{3}\right)^2 - 4(11)(1) = 100 \cdot 3 - 44 = 256 > 0$$

Because $B^2 - 4AC > 0$, the graph of the equation is a hyperbola.

Check Point 5 Identify the graph of $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$.

EXERCISE SET 9.4

Practice Exercises

In Exercises 1–8, identify each equation without completing the square.

1. $y^2 - 4x + 2y + 21 = 0$

2. $y^2 - 4x - 4y = 0$

3. $4x^2 - 9y^2 - 8x - 36y - 68 = 0$

4. $9x^2 + 25y^2 - 54x - 200y + 256 = 0$

5. $4x^2 + 4y^2 + 12x + 4y + 1 = 0$

6. $9x^2 + 4y^2 - 36x + 8y + 31 = 0$

7. $100x^2 - 7y^2 + 90y - 368 = 0$

8. $y^2 + 8x + 6y + 25 = 0$

In Exercises 9–14, write each equation in terms of a rotated $x'y'$-system using θ, the angle of rotation. Write the equation involving x' and y' in standard form.

9. $xy = -1; \theta = 45°$

10. $xy = -4; \theta = 45°$

11. $x^2 - 4xy + y^2 - 3 = 0; \theta = 45°$

12. $13x^2 - 10xy + 13y^2 - 72 = 0; \theta = 45°$

13. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0; \theta = 30°$

14. $13x^2 - 6\sqrt{3}xy + 7y^2 - 16 = 0; \theta = 60°$

In Exercises 15–26, write the appropriate rotation formulas so that in a rotated system the equation has no $x'y'$-term.

15. $x^2 + xy + y^2 - 10 = 0$

16. $x^2 + 4xy + y^2 - 3 = 0$

17. $3x^2 - 10xy + 3y^2 - 32 = 0$

18. $5x^2 - 8xy + 5y^2 - 9 = 0$

19. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$

20. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$

21. $10x^2 + 24xy + 17y^2 - 9 = 0$

22. $32x^2 - 48xy + 18y^2 - 15x - 20y = 0$

23. $x^2 + 4xy - 2y^2 - 1 = 0$

24. $3xy - 4y^2 + 18 = 0$

25. $34x^2 - 24xy + 41y^2 - 25 = 0$

26. $6x^2 - 6xy + 14y^2 - 45 = 0$

In Exercises 27–38,

a. *Rewrite the equation in a rotated $x'y'$-system without an $x'y'$ term. Use the appropriate rotation formulas from Exercises 15–26.*

b. *Express the equation involving x' and y' in the standard form of a conic section.*

c. *Use the rotated system to graph the equation.*

27. $x^2 + xy + y^2 - 10 = 0$

28. $x^2 + 4xy + y^2 - 3 = 0$

29. $3x^2 - 10xy + 3y^2 - 32 = 0$

30. $5x^2 - 8xy + 5y^2 - 9 = 0$

31. $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$

32. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$

33. $10x^2 + 24xy + 17y^2 - 9 = 0$

34. $32x^2 - 48xy + 18y^2 - 15x - 20y = 0$

35. $x^2 + 4xy - 2y^2 - 1 = 0$

36. $3xy - 4y^2 + 18 = 0$

37. $34x^2 - 24xy + 41y^2 - 25 = 0$

38. $6x^2 - 6xy + 14y^2 - 45 = 0$

In Exercises 39–44, identify each equation without applying a rotation of axes.

39. $5x^2 - 2xy + 5y^2 - 12 = 0$

40. $10x^2 + 24xy + 17y^2 - 9 = 0$

41. $24x^2 + 16\sqrt{3}xy + 8y^2 - x + \sqrt{3}y - 8 = 0$

42. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

43. $23x^2 + 26\sqrt{3}xy - 3y^2 - 144 = 0$

44. $4xy + 3y^2 + 4x + 6y - 1 = 0$

Practice Plus

In Exercises 45–48,

- *If the graph of the equation is an ellipse, find the coordinates of the vertices on the minor axis.*
- *If the graph of the equation is a hyperbola, find the equations of the asymptotes.*
- *If the graph of the equation is a parabola, find the coordinates of the vertex.*

Express answers relative to an $x'y'$-system in which the given equation has no $x'y'$-term. Assume that the $x'y'$-system has the same origin as the xy-system.

45. $5x^2 - 6xy + 5y^2 - 8 = 0$

46. $2x^2 - 4xy + 5y^2 - 36 = 0$

47. $x^2 - 4xy + 4y^2 + 5\sqrt{5}y - 10 = 0$

48. $x^2 + 4xy - 2y^2 - 6 = 0$

Writing in Mathematics

49. Explain how to identify the graph of

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

50. If there is a 60° angle from the positive x-axis to the positive x'-axis, explain how to obtain the rotation formulas for x and y.

51. How do you obtain the angle of rotation so that a general second-degree equation has no $x'y'$-term in a rotated $x'y'$-system?

52. What is the most time-consuming part in using a graphing utility to graph a general second-degree equation with an xy-term?

53. Explain how to identify the graph of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Technology Exercises

In Exercises 54–60, use a graphing utility to graph each equation.

54. $x^2 + 4xy + y^2 - 3 = 0$

55. $7x^2 + 8xy + y^2 - 1 = 0$

56. $3x^2 + 4xy + 6y^2 - 7 = 0$

57. $3x^2 - 6xy + 3y^2 + 10x - 8y - 2 = 0$

58. $9x^2 + 24xy + 16y^2 + 90x - 130y = 0$

59. $x^2 + 4xy + 4y^2 + 10\sqrt{5}x - 9 = 0$

60. $7x^2 + 6xy + 2.5y^2 - 14x + 4y + 9 = 0$

Critical Thinking Exercises

61. Explain the relationship between the graph of $3x^2 - 2xy + 3y^2 + 2 = 0$ and the sound made by one hand clapping. Begin by following the directions for Exercises 27–38. (You will first need to write rotation formulas that eliminate the $x'y'$-term.)

62. What happens to the equation $x^2 + y^2 = r^2$ in a rotated $x'y'$-system?

In Exercises 63–64, let

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

be an equation of a conic section in an xy-coordinate system. Let $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$ be the equation of the conic section in the rotated $x'y'$-coordinate system. Use the coefficients A', B', and C', shown in the equation with the voice balloon pointing to B' on page 907, to prove the following relationships.

63. $A' + C' = A + C$

64. $B'^2 - 4A'C' = B^2 - 4AC$

Group Exercise

65. Many public and private organizations and schools provide educational materials and information for the blind and visually impaired. Using your library, resources on the World Wide Web, or local organizations, investigate how your group or college could make a contribution to enhance the study of mathematics for the blind and visually impaired. In relation to conic sections, group members should discuss how to create graphs in tactile, or touchable, form that show blind students the visual structure of the conics, including asymptotes, intercepts, end behavior, and rotations.

SECTION 9.5 Parametric Equations

Objectives

1. Use point plotting to graph plane curves described by parametric equations.
2. Eliminate the parameter.
3. Find parametric equations for functions.
4. Understand the advantages of parametric representations.

What a baseball game! You got to see the great Derek Jeter of the New York Yankees blast a powerful homer. In less than eight seconds, the parabolic path of his home run took the ball a horizontal distance of over 1000 feet. Is there a way to model this path that gives both the ball's location and the time that it is in each of its positions? In this section, we look at ways of describing curves that reveal the where and the when of motion.

Plane Curves and Parametric Equations

You throw a ball from a height of 6 feet, with an initial velocity of 90 feet per second and at an angle of 40° with the horizontal. After t seconds, the location of the ball can be described by

$$x = (90\cos 40°)t \quad \text{and} \quad y = 6 + (90\sin 40°)t - 16t^2.$$

This is the ball's horizontal distance, in feet.

This is the ball's vertical height, in feet.

Because we can use these equations to calculate the location of the ball at any time t, we can describe the path of the ball. For example, to determine the location when $t = 1$ second, substitute 1 for t in each equation:

$$x = (90\cos 40°)t = (90\cos 40°)(1) \approx 68.9 \text{ feet}$$

$$y = 6 + (90\sin 40°)t - 16t^2 = 6 + (90\sin 40°)(1) - 16(1)^2 \approx 47.9 \text{ feet.}$$

This tells us that after one second, the ball has traveled a horizontal distance of approximately 68.9 feet, and the height of the ball is approximately 47.9 feet. Figure 9.50 displays this information and the results for calculations corresponding to $t = 2$ seconds and $t = 3$ seconds.

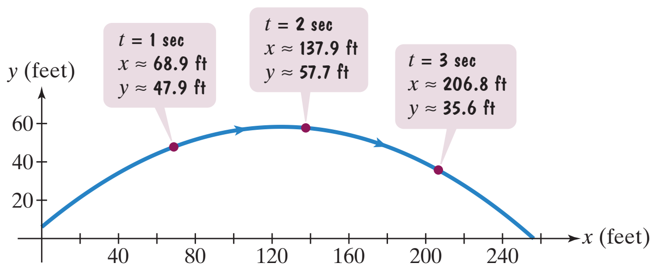


Figure 9.50 The location of a thrown ball after 1, 2, and 3 seconds

The voice balloons in Figure 9.50 tell where the ball is located and when the ball is at a given point (x, y) on its path. The variable t, called a **parameter**, gives the various times for the ball's location. The equations that describe where the ball is located express both x and y as functions of t, and are called **parametric equations**.

$$x = (90 \cos 40°)t \qquad y = 6 + (90 \sin 40°)t - 16t^2$$

This is the parametric equation for x.

This is the parametric equation for y.

The collection of points (x, y) in Figure 9.50 on the previous page is called a **plane curve**.

Plane Curves and Parametric Equations

Suppose that t is a number in an interval I. A **plane curve** is the set of ordered pairs (x, y), where

$$x = f(t), \quad y = g(t) \quad \text{for } t \text{ in interval } I.$$

The variable t is called a **parameter**, and the equations $x = f(t)$ and $y = g(t)$ are called **parametric equations** for the curve.

1 Use point plotting to graph plane curves described by parametric equations.

Graphing Plane Curves

Graphing a plane curve represented by parametric equations involves plotting points in the rectangular coordinate system and connecting them with a smooth curve.

Graphing a Plane Curve Described by Parametric Equations

1. Select some values of t on the given interval.
2. For each value of t, use the given parametric equations to compute x and y.
3. Plot the points (x, y) in the order of increasing t and connect them with a smooth curve.

Turn back a page and take a second look at Figure 9.50. Do you notice arrows along the curve? These arrows show the direction, or **orientation**, along the curve as t increases. After graphing a plane curve described by parametric equations, use arrows between the points to show the orientation of the curve corresponding to increasing values of t.

EXAMPLE 1 Graphing a Curve Defined by Parametric Equations

Graph the plane curve defined by the parametric equations:

$$x = t^2 - 1, \qquad y = 2t, \qquad -2 \le t \le 2.$$

Solution

Step 1 Select some values of t on the given interval. We will select integral values of t on the interval $-2 \le t \le 2$. Let $t = -2, -1, 0, 1$, and 2.

Step 2 For each value of t, use the given parametric equations to compute x and y. We organize our work in a table. The first column lists the choices for the parameter t. The next two columns show the corresponding values for x and y. The last column lists the ordered pair (x, y).

t	$x = t^2 - 1$	$y = 2t$	(x, y)
-2	$(-2)^2 - 1 = 4 - 1 = 3$	$2(-2) = -4$	$(3, -4)$
-1	$(-1)^2 - 1 = 1 - 1 = 0$	$2(-1) = -2$	$(0, -2)$
0	$0^2 - 1 = -1$	$2(0) = 0$	$(-1, 0)$
1	$1^2 - 1 = 0$	$2(1) = 2$	$(0, 2)$
2	$2^2 - 1 = 4 - 1 = 3$	$2(2) = 4$	$(3, 4)$

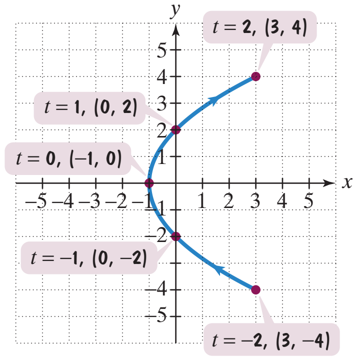


Figure 9.51 The plane curve defined by $x = t^2 - 1$, $y = 2t$, $-2 \le t \le 2$

2 Eliminate the parameter.

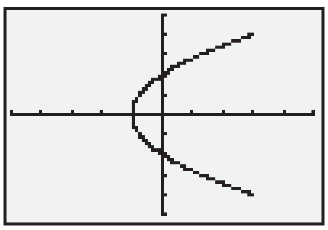
Technology

A graphing utility can be used to obtain a plane curve represented by parametric equations. Set the mode to parametric and enter the equations. You must enter the minimum and maximum values for t, and an increment setting for t ($t$step). The setting $t$step determines the number of points the graphing utility will plot.

Shown below is the plane curve for

$$x = t^2 - 1$$
$$y = 2t$$

in a $[-5, 5, 1]$ by $[-5, 5, 1]$ viewing rectangle with $t\text{min} = -2$, $t\text{max} = 2$, and $t\text{step} = 0.01$.



Step 3 Plot the points (x, y) in the order of increasing t and connect them with a smooth curve. The plane curve defined by the parametric equations on the given interval is shown in Figure 9.51. The arrows show the direction, or orientation, along the curve as t varies from -2 to 2.

Check Point 1 Graph the plane curve defined by the parametric equations:

$$x = t^2 + 1, \quad y = 3t, \quad -2 \le t \le 2.$$

Eliminating the Parameter

The graph in Figure 9.51 shows the plane curve for $x = t^2 - 1$, $y = 2t$, $-2 \le t \le 2$. Even if we examine the parametric equations carefully, we may not be able to tell that the corresponding plane curve is a parabola. By **eliminating the parameter**, we can write one equation in x and y that is equivalent to the two parametric equations. The voice balloons illustrate this process.

Begin with the parametric equations.	Solve for t in one of the equations.	Substitute the expression for t in the other parametric equation.
$x = t^2 - 1$ $y = 2t$	Using $y = 2t$, $t = \frac{y}{2}$.	Using $t = \frac{y}{2}$ and $x = t^2 - 1$, $x = \left(\frac{y}{2}\right)^2 - 1$.

The rectangular equation (the equation in x and y), $x = \frac{y^2}{4} - 1$, can be written as $y^2 = 4(x + 1)$. This is the standard form of the equation of a parabola with vertex at $(-1, 0)$ and axis of symmetry along the x-axis. Because the parameter t is restricted to the interval $[-2, 2]$, the plane curve in the technology box on the left shows only a part of the parabola.

Our discussion illustrates a second method for graphing a plane curve described by parametric equations. Eliminate the parameter t and graph the resulting rectangular equation in x and y. However, **you may need to change the domain of the rectangular equation to be consistent with the domain for the parametric equation in x.** This situation is illustrated in Example 2.

EXAMPLE 2 Finding and Graphing the Rectangular Equation of a Curve Defined Parametrically

Sketch the plane curve represented by the parametric equations

$$x = \sqrt{t} \quad \text{and} \quad y = \tfrac{1}{2}t + 1$$

by eliminating the parameter.

Solution We eliminate the parameter t and then graph the resulting rectangular equation.

Begin with the parametric equations.	Solve for t in one of the equations.	Substitute the expression for t in the other parametric equation.
$x = \sqrt{t}$ $y = \frac{1}{2}t + 1$	Using $x = \sqrt{t}$ and squaring both sides, $t = x^2$.	Using $t = x^2$ and $y = \frac{1}{2}t + 1$, $y = \frac{1}{2}x^2 + 1$.

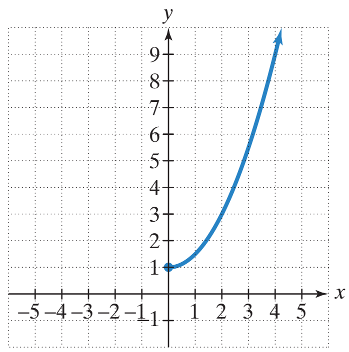


Figure 9.52 The plane curve for $x = \sqrt{t}$ and $y = \frac{1}{2}t + 1$, or $y = \frac{1}{2}x^2 + 1, x \ge 0$

Because t is not limited to a closed interval, you might be tempted to graph the entire U-shaped parabola whose equation is $y = \frac{1}{2}x^2 + 1$. However, take a second look at the parametric equation for x:

$$x = \sqrt{t}.$$

This equation is defined only when $t \ge 0$. Thus, x is nonnegative. The plane curve is the parabola given by $y = \frac{1}{2}x^2 + 1$ with the domain restricted to $x \ge 0$. The plane curve is shown in Figure 9.52.

Check Point 2 Sketch the plane curve represented by the parametric equations

$$x = \sqrt{t} \quad \text{and} \quad y = 2t - 1$$

by eliminating the parameter.

Eliminating the parameter is not always a simple matter. In some cases, it may not be possible. When this occurs, you can use point plotting to obtain a plane curve.

Trigonometric identities can be helpful in eliminating the parameter. For example, consider the plane curve defined by the parametric equations

$$x = \sin t, \quad y = \cos t, \quad 0 \le t < 2\pi.$$

We use the trigonometric identity $\sin^2 t + \cos^2 t = 1$ to eliminate the parameter. Square each side of each parametric equation and then add.

$$\begin{aligned} x^2 &= \sin^2 t \\ y^2 &= \cos^2 t \\ \hline x^2 + y^2 &= \sin^2 t + \cos^2 t \end{aligned}$$

This is the sum of the two equations above the horizontal lines.

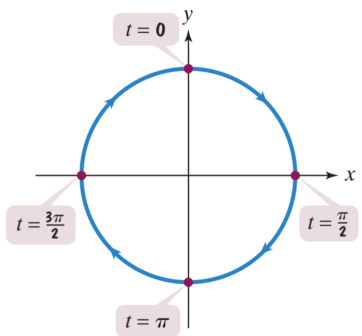


Figure 9.53 The plane curve defined by $x = \sin t, y = \cos t, 0 \le t < 2\pi$

Using a Pythagorean identity, we write this equation as $x^2 + y^2 = 1$. The plane curve is a circle with center $(0, 0)$ and radius 1. It is shown in Figure 9.53.

EXAMPLE 3 Finding and Graphing the Rectangular Equation of a Curve Defined Parametrically

Sketch the plane curve represented by the parametric equations

$$x = 5\cos t, \quad y = 2\sin t, \quad 0 \le t \le \pi$$

by eliminating the parameter.

Solution We eliminate the parameter using the identity $\cos^2 t + \sin^2 t = 1$. To apply the identity, divide the parametric equation for x by 5 and the parametric equation for y by 2.

$$\frac{x}{5} = \cos t \quad \text{and} \quad \frac{y}{2} = \sin t$$

Square and add these two equations.

$$\begin{aligned} \frac{x^2}{25} &= \cos^2 t \\ \frac{y^2}{4} &= \sin^2 t \\ \hline \frac{x^2}{25} + \frac{y^2}{4} &= \cos^2 t + \sin^2 t \end{aligned}$$

This is the sum of the two equations above the horizontal lines.

Using a Pythagorean identity, we write this equation as

$$\frac{x^2}{25} + \frac{y^2}{4} = 1.$$

This rectangular equation is the standard form of the equation for an ellipse centered at $(0, 0)$.

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

$a^2 = 25$: Endpoints of major axis are 5 units left and right of center.

$b^2 = 4$: Endpoints of minor axis are 2 units above and below center.

The ellipse is shown in Figure 9.54(a). However, this is not the plane curve. Because t is restricted to the interval $[0, \pi]$, the plane curve is only a portion of the ellipse. Use the starting and ending values for t, 0 and π, respectively, and a value of t in the interval $(0, \pi)$ to find which portion to include.

Begin at $t = 0$.

$$x = 5\cos t = 5\cos 0 = 5 \cdot 1 = 5$$
$$y = 2\sin t = 2\sin 0 = 2 \cdot 0 = 0$$

Increase to $t = \frac{\pi}{2}$.

$$x = 5\cos t = 5\cos\frac{\pi}{2} = 5 \cdot 0 = 0$$
$$y = 2\sin t = 2\sin\frac{\pi}{2} = 2 \cdot 1 = 2$$

End at $t = \pi$.

$$x = 5\cos t = 5\cos \pi = 5(-1) = -5$$
$$y = 2\sin t = 2\sin \pi = 2(0) = 0$$

Points on the plane curve include $(5, 0)$, which is the starting point, $(0, 2)$, and $(-5, 0)$, which is the ending point. The plane curve is the top half of the ellipse, shown in Figure 9.54(b).

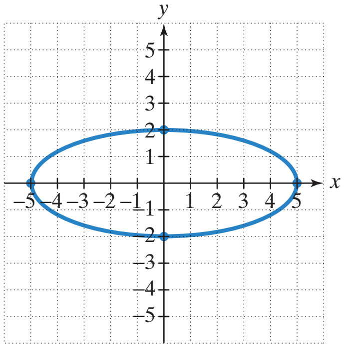


Figure 9.54(a) The graph of $\frac{x^2}{25} + \frac{y^2}{4} = 1$

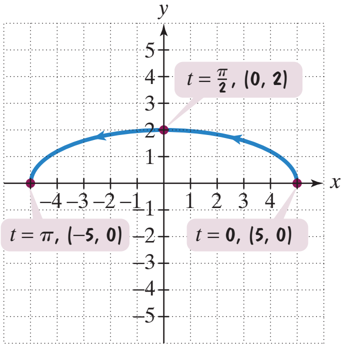


Figure 9.54(b) The plane curve for $x = 5\cos t$, $y = 2\sin t$, $0 \le t \le \pi$

Check Point 3 Sketch the plane curve represented by the parametric equations

$$x = 6\cos t, \; y = 4\sin t, \; \pi \le t \le 2\pi$$

by eliminating the parameter.

3 Find parametric equations for functions.

Finding Parametric Equations

Infinitely many pairs of parametric equations can represent the same plane curve. If the plane curve is defined by the function $y = f(x)$, here is a procedure for finding a set of parametric equations:

Parametric Equations for the Function $y = f(x)$

One set of parametric equations for the plane curve defined by $y = f(x)$ is

$$x = t \quad \text{and} \quad y = f(t),$$

in which t is in the domain of f.

EXAMPLE 4 Finding Parametric Equations

Find a set of parametric equations for the parabola whose equation is $y = 9 - x^2$.

Solution Let $x = t$. Parametric equations for $y = f(x)$ are $x = t$ and $y = f(t)$. Thus, parametric equations for $y = 9 - x^2$ are

$$x = t \quad \text{and} \quad y = 9 - t^2.$$

Check Point 4 Find a set of parametric equations for the parabola whose equation is $y = x^2 - 25$.

You can write other sets of parametric equations for $y = 9 - x^2$ by starting with a different parametric equation for x. Here are three more sets of parametric equations for

$$y = 9 - x^2:$$

- If $x = t^3$, $y = 9 - (t^3)^2 = 9 - t^6$.
  Parametric equations are $x = t^3$ and $y = 9 - t^6$.
- If $x = t + 1$, $y = 9 - (t + 1)^2 = 9 - (t^2 + 2t + 1) = 8 - t^2 - 2t$.
  Parametric equations are $x = t + 1$ and $y = 8 - t^2 - 2t$.
- If $x = \frac{t}{2}$, $y = 9 - \left(\frac{t}{2}\right)^2 = 9 - \frac{t^2}{4}$.
  Parametric equations are $x = \frac{t}{2}$ and $y = 9 - \frac{t^2}{4}$.

Can you start with any choice for the parametric equation for x? The answer is no. **The substitution for x must be a function that allows x to take on all the values in the domain of the given rectangular equation.** For example, the domain of the function $y = 9 - x^2$ is the set of all real numbers. If you incorrectly let $x = t^2$, these values of x exclude negative numbers that are included in $y = 9 - x^2$. The parametric equations

$$x = t^2 \quad \text{and} \quad y = 9 - (t^2)^2 = 9 - t^4$$

do not represent $y = 9 - x^2$ because only points for which $x \geq 0$ are obtained.

4 Understand the advantages of parametric representations.

Advantages of Parametric Equations over Rectangular Equations

We opened this section with parametric equations that described the horizontal distance and the vertical height of your thrown baseball after t seconds. Parametric equations are frequently used to represent the path of a moving object. If t represents time, parametric equations give the location of a moving object and tell when the object is located at each of its positions. Rectangular equations tell where the moving object is located but do not reveal when the object is in a particular position.

When using technology to obtain graphs, parametric equations that represent relations that are not functions are often easier to use than their corresponding rectangular equations. It is far easier to enter the equation of an ellipse given by the parametric equations

$$x = 2 + 3\cos t \quad \text{and} \quad y = 3 + 2\sin t$$

than to use the rectangular equivalent

$$\frac{(x - 2)^2}{9} + \frac{(y - 3)^2}{4} = 1.$$

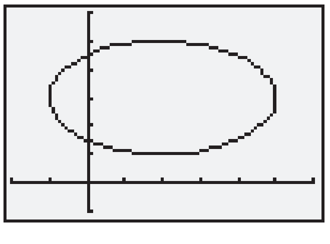
The rectangular equation must first be solved for y and then entered as two separate equations before a graphing utility reveals the ellipse.

Technology

The ellipse shown was obtained using the parametric mode and the radian mode of a graphing utility.

$$x(t) = 2 + 3\cos t$$
$$y(t) = 3 + 2\sin t$$

We used a $[-2, 6, 1]$ by $[-1, 6, 1]$ viewing rectangle with $t\text{min} = 0$, $t\text{max} = 6.2$, and $t\text{step} = 0.1$.



A curve that is used in physics for much of the theory of light is called a **cycloid**. The path of a fixed point on the circumference of a circle as it rolls along a line is a cycloid. A point on the rim of a bicycle wheel traces out a cycloid curve, shown in Figure 9.55. If the radius of the circle is a, the parametric equations of the cycloid are

$$x = a(t - \sin t) \quad \text{and} \quad y = a(1 - \cos t).$$

It is an extremely complicated task to represent the cycloid in rectangular form.

Cycloids are used to solve problems that involve the "shortest time." For example, Figure 9.56 shows a bead sliding down a wire. The shape of the wire a bead could slide down so that the distance between two points is traveled in the shortest time is an inverted cycloid.

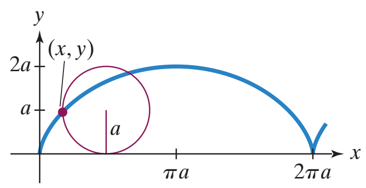


Figure 9.55 The curve traced by a fixed point on the circumference of a circle rolling along a straight line is a cycloid.

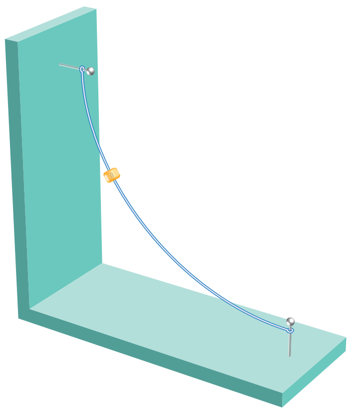
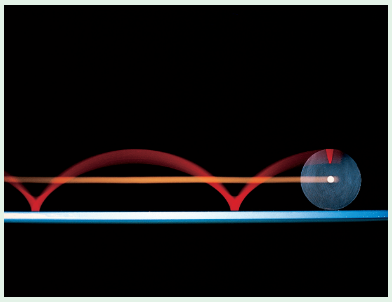


Figure 9.56

Rolling



Linear functions and cycloids are used to describe rolling motion. The light at the rolling circle's center shows that it moves linearly. By contrast, the light at the circle's edge has rotational motion and traces out a cycloid. A number of sites on the Internet illustrate rotational motion and show how the cycloid is created.

EXERCISE SET 9.5

Practice Exercises

In Exercises 1–8, parametric equations and a value for the parameter t are given. Find the coordinates of the point on the plane curve described by the parametric equations corresponding to the given value of t.

1. $x = 3 - 5t,\ y = 4 + 2t;\ t = 1$

2. $x = 7 - 4t,\ y = 5 + 6t;\ t = 1$

3. $x = t^2 + 1,\ y = 5 - t^3;\ t = 2$

4. $x = t^2 + 3,\ y = 6 - t^3;\ t = 2$

5. $x = 4 + 2\cos t,\ y = 3 + 5\sin t;\ t = \frac{\pi}{2}$

6. $x = 2 + 3\cos t,\ y = 4 + 2\sin t;\ t = \pi$

7. $x = (60\cos 30°)t,\ y = 5 + (60\sin 30°)t - 16t^2;\ t = 2$

8. $x = (80\cos 45°)t,\ y = 6 + (80\sin 45°)t - 16t^2;\ t = 2$

In Exercises 9–20, use point plotting to graph the plane curve described by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of t.

9. $x = t + 2,\ y = t^2;\ -2 \le t \le 2$

10. $x = t - 1,\ y = t^2;\ -2 \le t \le 2$

11. $x = t - 2,\ y = 2t + 1;\ -2 \le t \le 3$

12. $x = t - 3,\ y = 2t + 2;\ -2 \le t \le 3$

13. $x = t + 1,\ y = \sqrt{t};\ t \ge 0$ **14.** $x = \sqrt{t},\ y = t - 1;\ t \ge 0$

15. $x = \cos t,\ y = \sin t;\ 0 \le t < 2\pi$

16. $x = -\sin t,\ y = -\cos t;\ 0 \le t < 2\pi$

17. $x = t^2,\ y = t^3;\ -\infty < t < \infty$

18. $x = t^2 + 1,\ y = t^3 - 1;\ -\infty < t < \infty$

19. $x = 2t,\ y = |t - 1|;\ -\infty < t < \infty$

20. $x = |t + 1|,\ y = t - 2;\ -\infty < t < \infty$

In Exercises 21–40, eliminate the parameter t. Then use the rectangular equation to sketch the plane curve represented by the given parametric equations. Use arrows to show the orientation of the curve corresponding to increasing values of t. (If an interval for t is not specified, assume that $-\infty < t < \infty$.)

21. $x = t,\ y = 2t$

22. $x = t,\ y = -2t$

23. $x = 2t - 4,\ y = 4t^2$

24. $x = t - 2,\ y = t^2$

25. $x = \sqrt{t},\ y = t - 1$

26. $x = \sqrt{t},\ y = t + 1$

27. $x = 2\sin t,\ y = 2\cos t;\ 0 \le t < 2\pi$

28. $x = 3\sin t,\ y = 3\cos t;\ 0 \le t < 2\pi$

29. $x = 1 + 3\cos t,\ y = 2 + 3\sin t;\ 0 \le t < 2\pi$

30. $x = -1 + 2\cos t,\ y = 1 + 2\sin t;\ 0 \le t < 2\pi$

31. $x = 2\cos t,\ y = 3\sin t;\ 0 \le t < 2\pi$

32. $x = 3\cos t,\ y = 5\sin t;\ 0 \le t < 2\pi$

33. $x = 1 + 3\cos t,\ y = -1 + 2\sin t;\ 0 \le t \le \pi$

34. $x = 2 + 4\cos t,\ y = -1 + 3\sin t;\ 0 \le t \le \pi$

35. $x = \sec t,\ y = \tan t$

36. $x = 5\sec t,\ y = 3\tan t$

37. $x = t^2 + 2,\ y = t^2 - 2$

38. $x = \sqrt{t} + 2,\ y = \sqrt{t} - 2$

39. $x = 2^t,\ y = 2^{-t};\ t \ge 0$

40. $x = e^t,\ y = e^{-t};\ t \ge 0$

In Exercises 41–43, eliminate the parameter. Write the resulting equation in standard form.

41. A circle: $x = h + r\cos t,\ y = k + r\sin t$

42. An ellipse: $x = h + a\cos t,\ y = k + b\sin t$

43. A hyperbola: $x = h + a\sec t,\ y = k + b\tan t$

44. The parametric equations of the line through (x_1, y_1) and (x_2, y_2) are

$$x = x_1 + t(x_2 - x_1) \quad \text{and} \quad y = y_1 + t(y_2 - y_1).$$

Eliminate the parameter and write the resulting equation in point-slope form.

In Exercises 45–52, use your answers from Exercises 41–44 and the parametric equations given in Exercises 41–44 to find a set of parametric equations for the conic section or the line.

45. Circle: Center: (3, 5); Radius: 6

46. Circle: Center: (4, 6); Radius: 9

47. Ellipse: Center: (−2, 3); Vertices: 5 units to the left and right of the center; Endpoints of Minor Axis: 2 units above and below the center

48. Ellipse: Center: (4, −1); Vertices: 5 units above and below the center; Endpoints of Minor Axis: 3 units to the left and right of the center

49. Hyperbola: Vertices: (4, 0) and (−4, 0); Foci: (6, 0) and (−6, 0)

50. Hyperbola: Vertices: (0, 4) and (0, −4); Foci: (0, 5) and (0, −5)

51. Line: Passes through (−2, 4) and (1, 7)

52. Line: Passes through (3, −1) and (9, 12)

In Exercises 53–56, find two different sets of parametric equations for each rectangular equation.

53. $y = 4x - 3$

54. $y = 2x - 5$

55. $y = x^2 + 4$

56. $y = x^2 - 3$

In Exercises 57–58, the parametric equations of four plane curves are given. Graph each plane curve and determine how they differ from each other.

57. **a.** $x = t$ and $y = t^2 - 4$

b. $x = t^2$ and $y = t^4 - 4$

c. $x = \cos t$ and $y = \cos^2 t - 4$

d. $x = e^t$ and $y = e^{2t} - 4$

58. **a.** $x = t,\ y = \sqrt{4 - t^2};\ -2 \le t \le 2$

b. $x = \sqrt{4 - t^2},\ y = t;\ -2 \le t \le 2$

c. $x = 2\sin t,\ y = 2\cos t;\ 0 \le t < 2\pi$

d. $x = 2\cos t,\ y = 2\sin t;\ 0 \le t < 2\pi$

Practice Plus

In Exercises 59–62, sketch the plane curve represented by the given parametric equations. Then use interval notation to give each relation's domain and range.

59. $x = 4\cos t + 2,\ y = 4\cos t - 1$

60. $x = 2\sin t - 3,\ y = 2\sin t + 1$

61. $x = t^2 + t + 1,\ y = 2t$

62. $x = t^2 - t + 6,\ y = 3t$

In Exercises 63–68, sketch the function represented by the given parametric equations. Then use the graph to determine each of the following:

a. *intervals, if any, on which the function is increasing and intervals, if any, on which the function is decreasing.*

b. *the number, if any, at which the function has a maximum and this maximum value, or the number, if any, at which the function has a minimum and this minimum value.*

63. $x = 2^t,\ y = t$

64. $x = e^t,\ y = t$

65. $x = \dfrac{t}{2},\ y = 2t^2 - 8t + 3$

66. $x = \dfrac{t}{2},\ y = -2t^2 + 8t - 1$

67. $x = 2(t - \sin t),\ y = 2(1 - \cos t);\ 0 \le t \le 2\pi$

68. $x = 3(t - \sin t),\ y = 3(1 - \cos t);\ 0 \le t \le 2\pi$

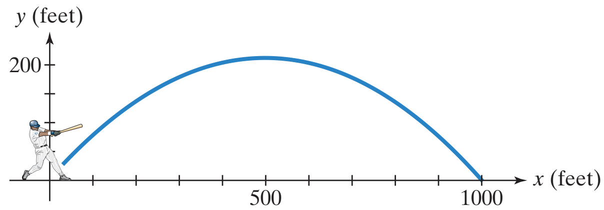
Application Exercises

The path of a projectile that is launched h feet above the ground with an initial velocity of v_0 feet per second and at an angle θ with the horizontal is given by the parametric equations

$$x = (v_0 \cos\theta)t \quad \text{and} \quad y = h + (v_0 \sin\theta)t - 16t^2,$$

where t is the time, in seconds, after the projectile was launched. The parametric equation for x gives the projectile's horizontal distance, in feet. The parametric equation for y gives the projectile's height, in feet. Use these parametric equations to solve Exercises 69–70.

69. The figure shows the path for a baseball hit by Derek Jeter. The ball was hit with an initial velocity of 180 feet per second at an angle of 40° to the horizontal. The ball was hit at a height 3 feet off the ground.



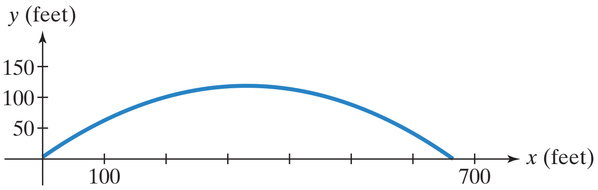
a. Find the parametric equations that describe the position of the ball as a function of time.

b. Describe the ball's position after 1, 2, and 3 seconds. Round to the nearest tenth of a foot. Locate your solutions on the plane curve.

c. How long, to the nearest tenth of a second, is the ball in flight? What is the total horizontal distance that it travels before it lands? Is your answer consistent with the figure shown?

d. You meet Derek Jeter and he asks you to tell him something interesting about the path of the baseball that he hit. Use the graph to respond to his request. Then verify your observation algebraically.

70. The figure shows the path for a baseball that was hit with an initial velocity of 150 feet per second at an angle of 35° to the horizontal. The ball was hit at a height of 3 feet off the ground.



a. Find the parametric equations that describe the position of the ball as a function of time.

b. Describe the ball's position after 1, 2, and 3 seconds. Round to the nearest tenth of a foot. Locate your solutions on the plane curve.

c. How long is the ball in flight? (Round to the nearest tenth of a second.) What is the total horizontal distance that it travels, to the nearest tenth of a foot, before it lands? Is your answer consistent with the figure shown?

d. Use the graph to describe something about the path of the baseball that might be of interest to the player who hit the ball. Then verify your observation algebraically.

Writing in Mathematics

71. What are plane curves and parametric equations?

72. How is point plotting used to graph a plane curve described by parametric equations? Give an example with your description.

73. What is the significance of arrows along a plane curve?

74. What does it mean to eliminate the parameter? What useful information can be obtained by doing this?

75. Explain how the rectangular equation $y = 5x$ can have infinitely many sets of parametric equations.

76. Discuss how the parametric equations for the path of a projectile (see Exercises 69–70) and the ability to obtain plane curves with a graphing utility can be used by a baseball coach to analyze performances of team players.

Technology Exercises

77. Use a graphing utility in a parametric mode to verify any five of your hand-drawn graphs in Exercises 9–40.

In Exercises 78–82, use a graphing utility to obtain the plane curve represented by the given parametric equations.

78. Cycloid: $x = 3(t - \sin t)$,
$y = 3(1 - \cos t)$; $[0, 60, 5] \times [0, 8, 1]$, $0 \le t < 6\pi$

79. Cycloid: $x = 2(t - \sin t)$,
$y = 2(1 - \cos t)$; $[0, 60, 5] \times [0, 8, 1]$, $0 \le t < 6\pi$

80. Witch of Agnesi: $x = 2 \cot t$, $y = 2 \sin^2 t$;
$[-6, 6, 1] \times [-4, 4, 1]$, $0 \le t < 2\pi$

81. Hypocycloid: $x = 4 \cos^3 t$, $y = 4 \sin^3 t$;
$[-5, 5, 1] \times [-5, 5, 1]$, $0 \le t < 2\pi$

82. Lissajous Curve: $x = 2 \cos t$, $y = \sin 2t$;
$[-3, 3, 1] \times [-2, 2, 1]$, $0 \le t < 2\pi$

Use the equations for the path of a projectile given prior to Exercises 69–70 to solve Exercises 83–85.

In Exercises 83–84, use a graphing utility to obtain the path of a projectile launched from the ground ($h = 0$) at the specified values of θ and v_0. In each exercise, use the graph to determine the maximum height and the time at which the projectile reaches its maximum height. Also use the graph to determine the range of the projectile and the time it hits the ground. Round all answers to the nearest tenth.

83. $\theta = 55°$, $v_0 = 200$ feet per second

84. $\theta = 35°$, $v_0 = 300$ feet per second

85. A baseball player throws a ball with an initial velocity of 140 feet per second at an angle of 22° to the horizontal. The ball leaves the player's hand at a height of 5 feet.

a. Write the parametric equations that describe the ball's position as a function of time.

b. Use a graphing utility to obtain the path of the baseball.

c. Find the ball's maximum height and the time at which it reaches this height. Round all answers to the nearest tenth.

d. How long is the ball in the air?

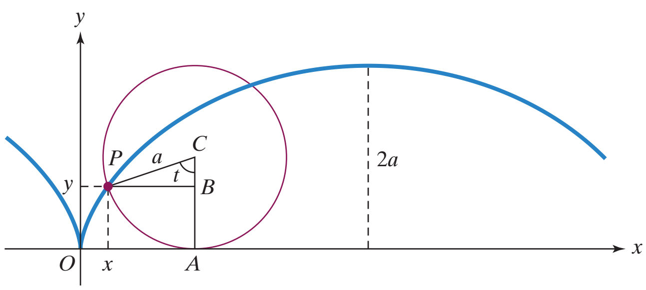
e. How far does the ball travel?

Critical Thinking Exercises

86. Eliminate the parameter: $x = \cos^3 t$ and $y = \sin^3 t$.

87. The plane curve described by the parametric equations $x = 3 \cos t$ and $y = 3 \sin t$, $0 \le t < 2\pi$, has a counterclockwise orientation. Alter one or both parametric equations so that you obtain the same plane curve with the opposite orientation.

88. The figure shows a circle of radius a rolling along a horizontal line. Point P traces out a cycloid. Angle t, in radians, is the angle through which the circle has rolled. C is the center of the circle.



Use the suggestions in parts (a) and (b) to prove that the parametric equations of the cycloid are $x = a(t - \sin t)$ and $y = a(1 - \cos t)$.

a. Derive the parametric equation for x using the figure and

$$x = OA - xA.$$

b. Derive the parametric equation for y using the figure and

$$y = AC - BC.$$

SECTION 9.6 Conic Sections in Polar Coordinates

Objectives

1. Define conics in terms of a focus and a directrix.
2. Graph the polar equations of conics.

John Glenn made the first U.S.-manned flight around Earth on *Friendship 7*.

On the morning of February 20, 1962, millions of Americans collectively held their breath as the world's newest pioneer swept across the threshold of one of our last frontiers. Roughly one hundred miles above Earth, astronaut John Glenn sat comfortably in the weightless environment of a $9\frac{1}{2}$-by-6-foot space capsule that offered the leg room of a Volkswagen "Beetle" and the aesthetics of a garbage can. Glenn became the first American to orbit Earth in a three-orbit mission that lasted slightly under 5 hours.

In this section, you will see how John Glenn's historic orbit can be described using conic sections in polar coordinates. To obtain this model, we begin with a definition that permits a unified approach to the conic sections.

The Focus-Directrix Definitions of the Conic Sections

1 Define conics in terms of a focus and a directrix.

The definition of a parabola is given in terms of a fixed point, the focus, and a fixed line, the directrix. By contrast, the definitions of an ellipse and a hyperbola are given in terms of two fixed points, the foci. It is possible to define each of these conic sections in terms of a point and a line. Figure 9.57 shows a conic section in the polar coordinate system. The fixed point, the focus, is at the pole. The fixed line, the directrix, is perpendicular to the polar axis.

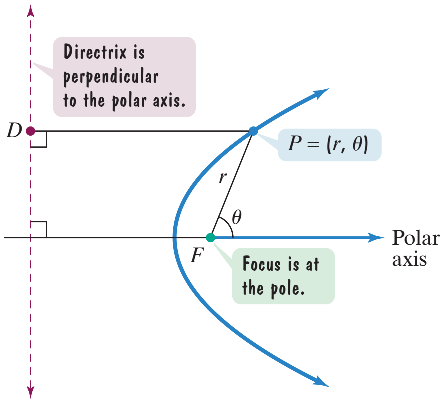


Figure 9.57 A conic in the polar coordinate system

Focus-Directrix Definitions of the Conic Sections

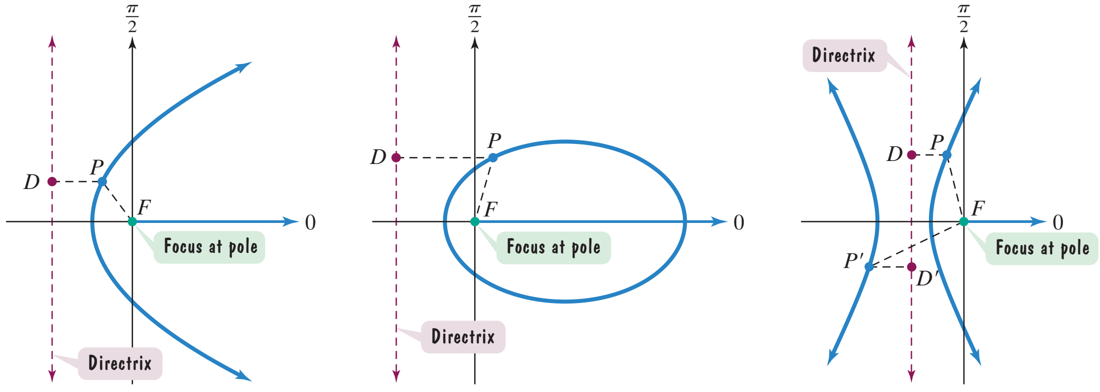
Let F be a fixed point, the focus, and let D be a fixed line, the directrix, in a plane (Figure 9.58). A **conic section**, or **conic**, is the set of all points P in the plane such that

$$\frac{PF}{PD} = e,$$

where e is a fixed positive number, called the **eccentricity**.

If $e = 1$, the conic is a parabola.
If $e < 1$, the conic is an ellipse.
If $e > 1$, the conic is a hyperbola.

Figure 9.58 illustrates the eccentricity for each type of conic. Notice that if $e = 1$, the definition of the parabola is the same as the focus-directrix definition with which you are familiar.



Parabola $\frac{PF}{PD} = e$, $e = 1$

Ellipse $\frac{PF}{PD} = e$, $e < 1$

Hyperbola $\frac{PF}{PD} = \frac{P'F}{P'D'} = e$, $e > 1$

Figure 9.58 The eccentricity for each conic

2 Graph the polar equations of conics.

Polar Equations of Conics

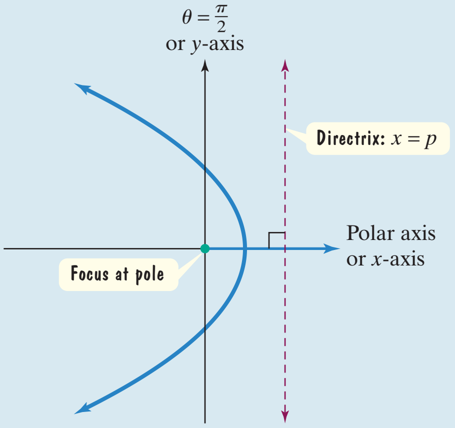
By locating a focus at the pole, all conics can be represented by similar equations in the polar coordinate system. In each of these equations,

- (r, θ) is a point on the graph of the conic.
- e is the eccentricity. (Remember that $e > 0$.)
- p is the distance between the focus (located at the pole) and the directrix.

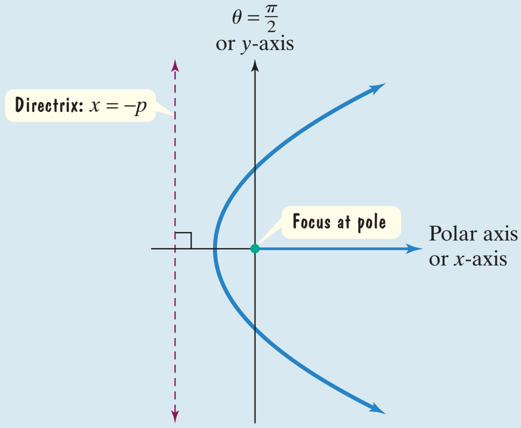
Standard Forms of the Polar Equations of Conics

Let the pole be a focus of a conic section of eccentricity e with the directrix p units from the focus. The equation of the conic is given by one of the four equations listed.

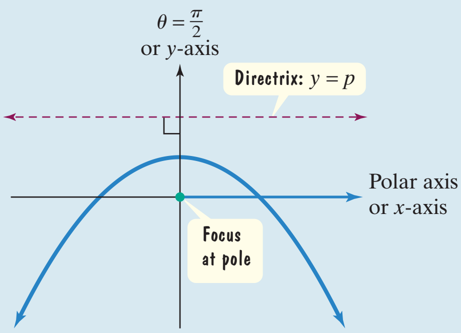
$$r = \frac{ep}{1 + e\cos\theta}$$



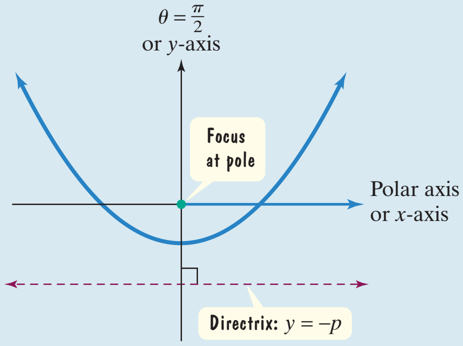
$$r = \frac{ep}{1 - e\cos\theta}$$



$$r = \frac{ep}{1 + e\sin\theta}$$



$$r = \frac{ep}{1 - e\sin\theta}$$



The graphs in the box on the previous page illustrate two kinds of symmetry—symmetry with respect to the polar axis and symmetry with respect to the y-axis. If the equation contains $\cos\theta$, the polar axis is an axis of symmetry. If the equation contains $\sin\theta$, the line $\theta = \frac{\pi}{2}$, or the y-axis, is an axis of symmetry. Take a moment to turn back a page and verify these observations.

We will derive one of the equations displayed in the previous box. The other three equations are obtained in a similar manner. In Figure 9.59, let $P = (r, \theta)$ be any point on a conic section.

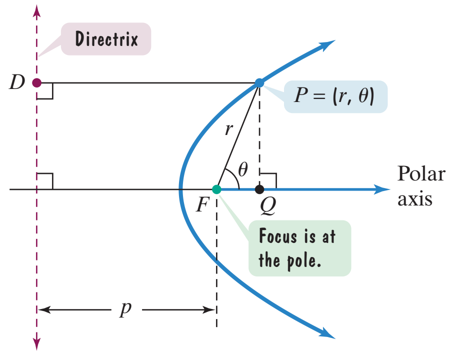


Figure 9.59

$$\frac{PF}{PD} = e$$ By definition, the ratio of the distance from P to the focus to the distance from P to the directrix equals the positive constant e.

$$\frac{r}{PD} = e$$ Figure 9.59 shows that the distance from P to the focus, located at the pole, is r: $PF = r$.

$$\frac{r}{p + FQ} = e$$ Figure 9.59 shows that the distance from P to the directrix is $p + FQ$: $PD = p + FQ$.

$$\frac{r}{p + r\cos\theta} = e$$ Using the triangle in the figure, $\cos\theta = \frac{FQ}{r}$ and $FQ = r\cos\theta$.

By solving this equation for r, we will obtain the desired equation. Clear fractions by multiplying both sides by the least common denominator.

$$r = e(p + r\cos\theta)$$ Multiply both sides by $p + r\cos\theta$.

$$r = ep + er\cos\theta$$ Apply the distributive property.

$$r - er\cos\theta = ep$$ Subtract $er\cos\theta$ from both sides to collect terms involving r on the same side.

$$r(1 - e\cos\theta) = ep$$ Factor out r from the two terms on the left.

$$r = \frac{ep}{1 - e\cos\theta}$$ Divide both sides by $1 - e\cos\theta$ and solve for r.

In summary, the standard forms of the polar equations of conics are

$$r = \frac{ep}{1 \pm e\cos\theta} \quad \text{and} \quad r = \frac{ep}{1 \pm e\sin\theta}.$$

In all forms, the constant term in the denominator is 1.

Graphing the Polar Equation of a Conic

1. If necessary, write the equation in one of the standard forms.
2. Use the standard form to determine values for e and p. Use the value of e to identify the conic.
3. Use the appropriate figure for the standard form of the equation shown in the box on page 911 to help guide the graphing process.

EXAMPLE 1 Graphing the Polar Equation of a Conic

Graph the polar equation:

$$r = \frac{4}{2 + \cos\theta}.$$

Solution

Step 1 Write the equation in one of the standard forms. The equation is not in standard form because the constant term in the denominator is not 1.

$$r = \frac{4}{2 + \cos\theta}$$

To obtain 1 in this position, divide the numerator and denominator by 2.

The equation in standard form is

$$r = \frac{2}{1 + \frac{1}{2}\cos\theta}.$$

$ep = 2$

$e = \frac{1}{2}$

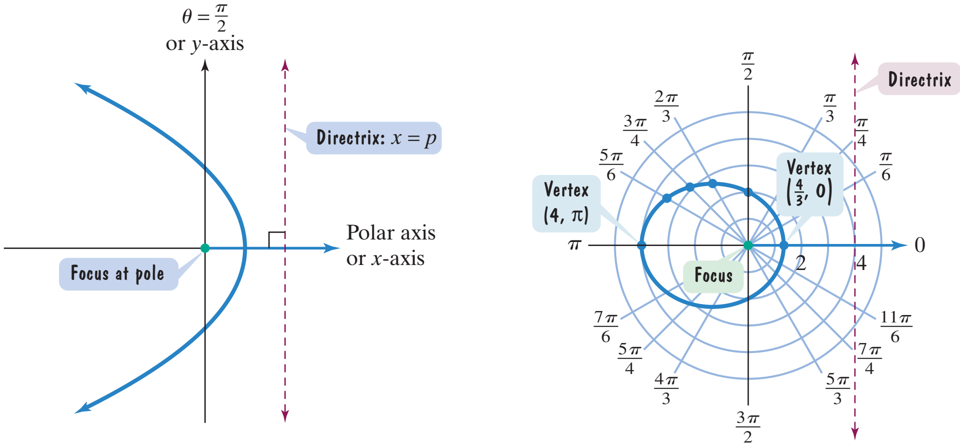
This equation is in the form $r = \dfrac{ep}{1 + e\cos\theta}$.

Step 2 Use the standard form to find e and p, and identify the conic. The voice balloons show that

$$e = \tfrac{1}{2} \quad \text{and} \quad ep = \tfrac{1}{2}p = 2.$$

Thus, $e = \frac{1}{2}$ and $p = 4$. Because $e = \frac{1}{2} < 1$, the conic is an ellipse.

Step 3 Use the figure for the equation's standard form to guide the graphing process. The figure for the conic's standard form is shown in Figure 9.60(a). We have symmetry with respect to the polar axis. One focus is at the pole and a directrix is $x = 4$, located four units to the right of the pole.



(a) Using $r = \dfrac{ep}{1 + e\cos\theta}$ to graph $r = \dfrac{2}{1 + \frac{1}{2}\cos\theta}$

(b) The graph of $r = \dfrac{4}{2 + \cos\theta}$ or $r = \dfrac{2}{1 + \frac{1}{2}\cos\theta}$

Figure 9.60

Technology

The graph of

$$r = \frac{4}{2 + \cos\theta}$$

is obtained using the polar mode with angle measure in radians. To verify the hand-drawn graph in Figure 9.60(b), we used $[-5, 5, 1]$ by $[-5, 5, 1]$ with $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \dfrac{\pi}{48}$.

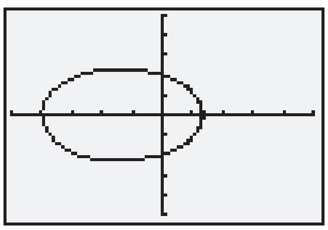


Figure 9.60(a) indicates that the major axis is on the polar axis. Thus, we find the vertices by selecting 0 and π for θ. The corresponding values for r are $\frac{4}{3}$ and 4, respectively. Figure 9.60(b) shows the vertices, $\left(\frac{4}{3}, 0\right)$ and $(4, \pi)$.

You can sketch the upper half of the ellipse by plotting some points from $\theta = 0$ to $\theta = \pi$.

$$r = \frac{4}{2 + \cos\theta}$$

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
r	2	2.7	3.1	3.5

Using symmetry with respect to the polar axis, you can sketch the lower half. The graph of the given equation is shown in Figure 9.60(b).

Check Point 1 Use the three steps shown in the box on page 912 to graph the polar equation:

$$r = \frac{4}{2 - \cos\theta}.$$

EXAMPLE 2 Graphing the Polar Equation of a Conic

Graph the polar equation:

$$r = \frac{12}{3 + 3\sin\theta}.$$

Solution

Step 1 Write the equation in one of the standard forms. The equation is not in standard form because the constant term in the denominator is not 1. Divide the numerator and denominator by 3 to write the standard form.

$$r = \frac{4}{1 + 1\sin\theta}$$

$ep = 4$

$e = 1$

This equation is in the form $r = \dfrac{ep}{1 + e\sin\theta}$.

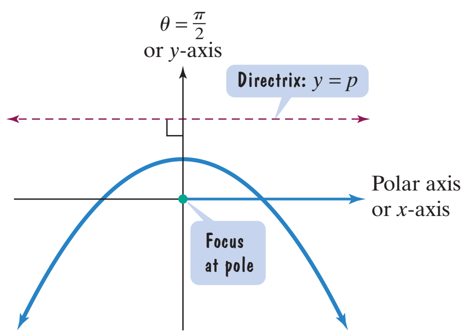
Step 2 Use the standard form to find *e* and *p*, and identify the conic. The voice balloons show that

$$e = 1 \quad \text{and} \quad ep = 1p = 4.$$

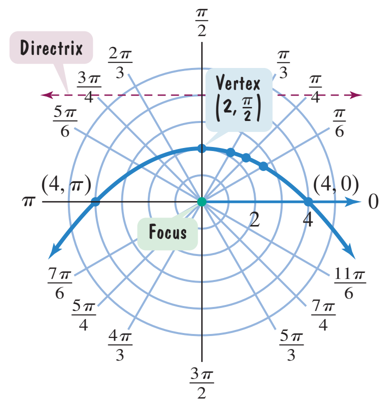
Thus, $e = 1$ and $p = 4$. Because $e = 1$, the conic is a parabola.

Step 3 Use the figure for the equation's standard form to guide the graphing process. Figure 9.61(a) indicates that we have symmetry with respect to $\theta = \frac{\pi}{2}$. The focus is at the pole and, with $p = 4$, the directrix is $y = 4$, located four units above the pole.

Figure 9.61(a) indicates that the vertex is on the line of $\theta = \frac{\pi}{2}$, or the y-axis. Thus, we find the vertex by selecting $\frac{\pi}{2}$ for θ. The corresponding value for r is 2. Figure 9.61(b) shows the vertex, $\left(2, \frac{\pi}{2}\right)$.



(a) Using $r = \dfrac{ep}{1 + e\sin\theta}$ to graph $r = \dfrac{4}{1 + \sin\theta}$



(b) The graph of $r = \dfrac{12}{3 + 3\sin\theta}$ or $r = \dfrac{4}{1 + \sin\theta}$

Figure 9.61

To find where the parabola crosses the polar axis, select $\theta = 0$ and $\theta = \pi$. The corresponding values for r are 4 and 4, respectively. Figure 9.61(b) shows the points $(4, 0)$ and $(4, \pi)$ on the polar axis.

Technology

The graph of

$$r = \frac{12}{3 + 3\sin\theta}$$

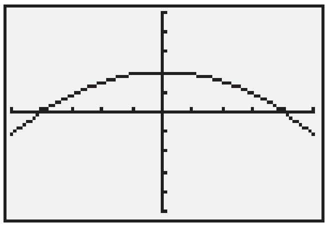
was obtained using

$[-5, 5, 1]$ by $[-5, 5, 1]$

with

$\theta\text{min} = 0, \quad \theta\text{max} = 2\pi,$

$\theta\text{step} = \frac{\pi}{48}.$



You can sketch the right half of the parabola by plotting some points from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

$$r = \frac{12}{3 + 3\sin\theta}$$

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
r	2.7	2.3	2.1

Using symmetry with respect to $\theta = \frac{\pi}{2}$, you can sketch the left half. The graph of the given equation is shown in Figure 9.61(b).

Check Point 2 Use the three steps shown in the box on page 912 to graph the polar equation:

$$r = \frac{8}{4 + 4\sin\theta}.$$

EXAMPLE 3 Graphing the Polar Equation of a Conic

Graph the polar equation:

$$r = \frac{9}{3 - 6\cos\theta}.$$

Solution

Step 1 Write the equation in one of the standard forms. We can obtain a constant term of 1 in the denominator by dividing each term by 3.

$$r = \frac{3}{1 - 2\cos\theta}$$

$ep = 3$

This equation is in the form $r = \frac{ep}{1 - e\cos\theta}$.

$e = 2$

Step 2 Use the standard form to find e and p, and identify the conic. The voice balloons show that

$$e = 2 \quad \text{and} \quad ep = 2p = 3.$$

Thus, $e = 2$ and $p = \frac{3}{2}$. Because $e = 2 > 1$, the conic is a hyperbola.

Step 3 Use the figure for the equation's standard form to guide the graphing process. Figure 9.62(a) indicates that we have symmetry with respect to the polar axis. One focus is at the pole and, with $p = \frac{3}{2}$, a directrix is $x = -\frac{3}{2}$, located 1.5 units to the left of the pole.

Figure 9.62(a) indicates that the transverse axis is horizontal and the vertices lie on the polar axis. Thus, we find the vertices by selecting 0 and π for θ. Figure 9.62(b) shows the vertices, $(-3, 0)$ and $(1, \pi)$.

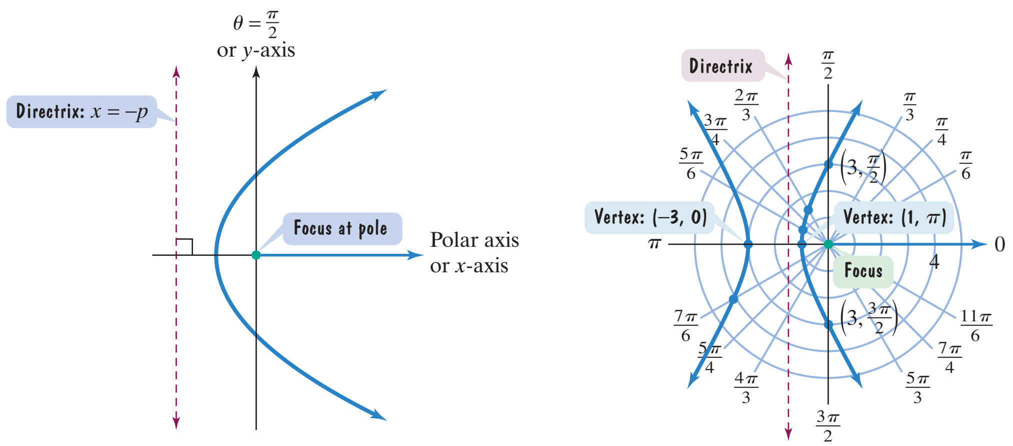


Figure 9.62 **(a)** Using $r = \frac{ep}{1 - e\cos\theta}$ to graph $r = \frac{3}{1 - 2\cos\theta}$ **(b)** The graph of $r = \frac{9}{3 - 6\cos\theta}$ or $r = \frac{3}{1 - 2\cos\theta}$

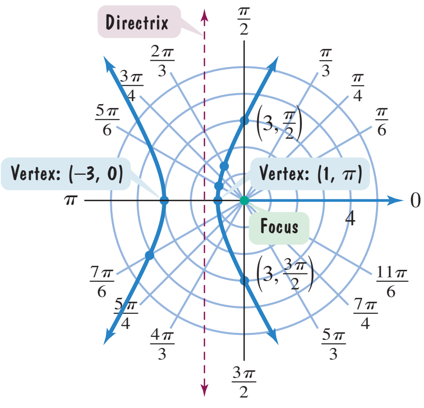


Figure 9.62(b) (repeated) The graph of $r = \frac{9}{3 - 6\cos\theta}$ or $r = \frac{3}{1 - 2\cos\theta}$

To find where the hyperbola crosses the line $\theta = \frac{\pi}{2}$, select $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ for θ. Figure 9.62(b) shows the points $\left(3, \frac{\pi}{2}\right)$ and $\left(3, \frac{3\pi}{2}\right)$ on the graph.

We sketch the upper half of the hyperbola by plotting some points from $\theta = 0$ to $\theta = \pi$.

$$r = \frac{3}{1 - 2\cos\theta}$$

θ	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
r	−4.1	1.5	1.1

Figure 9.62(b) shows the points $\left(\frac{\pi}{6}, -4.1\right)$, $\left(\frac{2\pi}{3}, 1.5\right)$, and $\left(\frac{5\pi}{6}, 1.1\right)$ on the graph. Observe that $\left(\frac{\pi}{6}, -4.1\right)$ is on the lower half of the hyperbola. Using symmetry with respect to the polar axis, we sketch the entire lower half. The graph of the given equation is shown in Figure 9.62(b).

Check Point 3 Use the three steps shown in the box on page 912 to graph the polar equation:

$$r = \frac{9}{3 - 9\cos\theta}.$$

Modeling Planetary Motion

Polish astronomer Nicolaus Copernicus (1473–1543) was correct in stating that planets in our solar system revolve around the sun and not Earth. However, he incorrectly believed that celestial orbits move in perfect circles, calling his system "the ballet of the planets."

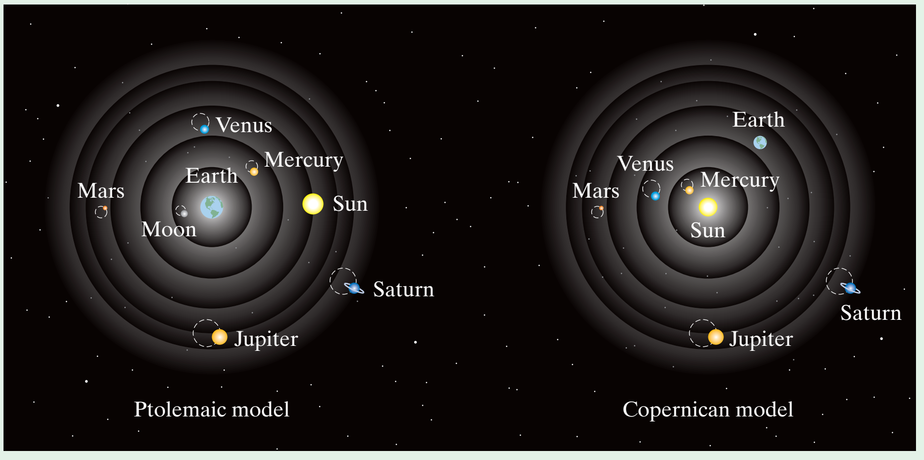


Table 9.4 indicates that the planets in our solar system have orbits with eccentricities that are much closer to 0 than to 1. Most of these orbits are almost circular, which made it difficult for early astronomers to detect that they are actually ellipses.

German scientist and mathematician Johannes Kepler (1571–1630) discovered that planets move in elliptical orbits with the sun at one focus. The polar equation for these orbits is

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta},$$

Table 9.4 Eccentricities of Planetary Orbits

Mercury	0.2056	Saturn	0.0543
Venus	0.0068	Uranus	0.0460
Earth	0.0167	Neptune	0.0082
Mars	0.0934	Pluto	0.2481
Jupiter	0.0484		

where the length of the orbit's major axis is $2a$. Describing planetary orbits, Kepler wrote, "The heavenly motions are nothing but a continuous song for several voices, to be perceived by the intellect, not by the ear."

EXERCISE SET 9.6

Practice Exercises

In Exercises 1–8,

a. *Identify the conic section that each polar equation represents.*

b. *Describe the location of a directrix from the focus located at the pole.*

1. $r = \dfrac{3}{1 + \sin\theta}$

2. $r = \dfrac{3}{1 + \cos\theta}$

3. $r = \dfrac{6}{3 - 2\cos\theta}$

4. $r = \dfrac{6}{3 + 2\cos\theta}$

5. $r = \dfrac{8}{2 + 2\sin\theta}$

6. $r = \dfrac{8}{2 - 2\sin\theta}$

7. $r = \dfrac{12}{2 - 4\cos\theta}$

8. $r = \dfrac{12}{2 + 4\cos\theta}$

In Exercises 9–20, use the three steps shown in the box on page 912 to graph each polar equation.

9. $r = \dfrac{1}{1 + \sin\theta}$

10. $r = \dfrac{1}{1 + \cos\theta}$

11. $r = \dfrac{2}{1 - \cos\theta}$

12. $r = \dfrac{2}{1 - \sin\theta}$

13. $r = \dfrac{12}{5 + 3\cos\theta}$

14. $r = \dfrac{12}{5 - 3\cos\theta}$

15. $r = \dfrac{6}{2 - 2\sin\theta}$

16. $r = \dfrac{6}{2 + 2\sin\theta}$

17. $r = \dfrac{8}{2 - 4\cos\theta}$

18. $r = \dfrac{8}{2 + 4\cos\theta}$

19. $r = \dfrac{12}{3 - 6\cos\theta}$

20. $r = \dfrac{12}{3 - 3\cos\theta}$

Practice Plus

In Exercises 21–28, describe a viewing rectangle, or window, such as $[-30, 30, 3]$ *by* $[-8, 4, 1]$*, that shows a complete graph of each polar equation and minimizes unused portions of the screen.*

21. $r = \dfrac{15}{3 - 2\cos\theta}$

22. $r = \dfrac{16}{5 - 3\cos\theta}$

23. $r = \dfrac{8}{1 - \cos\theta}$

24. $r = \dfrac{8}{1 + \cos\theta}$

25. $r = \dfrac{4}{1 + 3\cos\theta}$

26. $r = \dfrac{16}{3 + 5\cos\theta}$

27. $r = \dfrac{4}{5 + 5\sin\theta}$

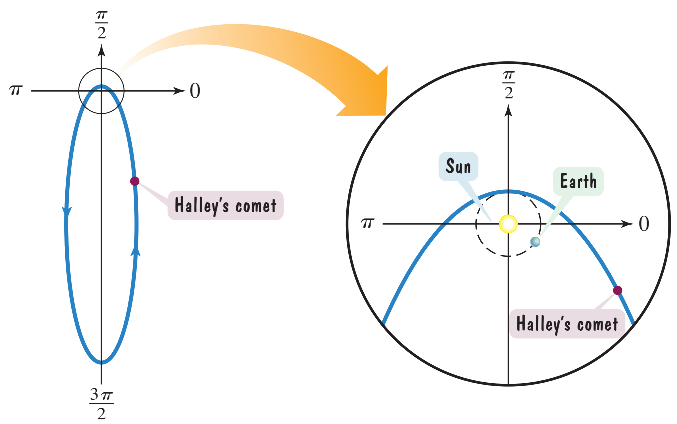
28. $r = \dfrac{2}{3 + 3\sin\theta}$

Application Exercises

Halley's comet has an elliptical orbit with the sun at one focus. Its orbit, shown in the figure at the top of the next column, is given approximately by

$$r = \frac{1.069}{1 + 0.967\sin\theta}.$$

In the formula, r is measured in astronomical units. (One astronomical unit is the average distance from Earth to the sun, approximately 93 million miles.) Use the given formula and the figure to solve Exercises 29–30. Round to the nearest hundredth of an astronomical unit and the nearest million miles.



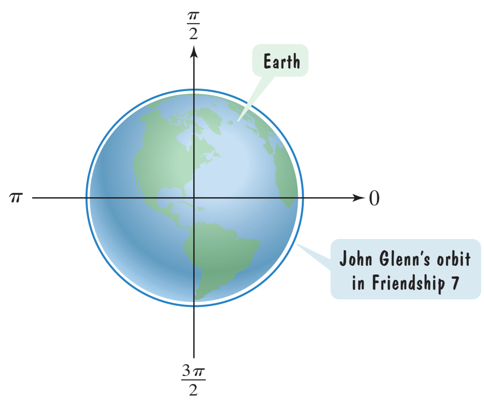
29. Find the distance from Halley's comet to the sun at its shortest distance from the sun.

30. Find the distance from Halley's comet to the sun at its greatest distance from the sun.

On February 20, 1962, John Glenn made the first U.S.-manned flight around the Earth for three orbits on Friendship 7. With Earth at one focus, the orbit of Friendship 7 is given approximately by

$$r = \frac{4090.76}{1 - 0.0076\cos\theta},$$

where r is measured in miles from Earth's center. Use the formula and the figure shown to solve Exercises 31–32.



31. How far from Earth's center was John Glenn at his greatest distance from the planet? Round to the nearest mile. If the radius of Earth is 3960 miles, how far was he from Earth's surface at this point on the flight?

32. How far from Earth's center was John Glenn at his closest distance from the planet? Round to the nearest mile. If the radius of Earth is 3960 miles, how far was he from Earth's surface at this point on the flight?

Writing in Mathematics

33. How are the conics described in terms of a fixed point and a fixed line?

34. If all conics are defined in terms of a fixed point and a fixed line, how can you tell one kind of conic from another?

35. If you are given the standard form of the polar equation of a conic, how do you determine its eccentricity?

36. If you are given the standard form of the polar equation of a conic, how do you determine the location of a directrix from the focus at the pole?

37. Describe a strategy for graphing $r = \dfrac{1}{1 + \sin\theta}$.

38. You meet John Glenn and he asks you to tell him something of interest about the elliptical orbit of his first space voyage in 1962. Describe how to use the polar equation for orbits in the essay on page 916, the equation for his 1962 journey in Exercises 31–32, and a graphing utility to provide him with an interesting visual analysis.

Technology Exercises

Use the polar mode of a graphing utility with angle measure in radians to solve Exercises 39–42. Unless otherwise indicated, use θmin $= 0$, θmax $= 2\pi$, *and* θ step $= \dfrac{\pi}{48}$. *If you are not satisfied with the quality of the graph, experiment with smaller values for* θ *step.*

39. Use a graphing utility to verify any five of your hand-drawn graphs in Exercises 9–20.

In Exercises 40–42, identify the conic that each polar equation represents. Then use a graphing utility to graph the equation.

40. $r = \dfrac{16}{4 - 3\cos\theta}$ **41.** $r = \dfrac{12}{4 + 5\sin\theta}$ **42.** $r = \dfrac{18}{6 - 6\cos\theta}$

In Exercises 43–44, use a graphing utility to graph the equation. Then answer the given question.

43. $r = \dfrac{4}{1 - \sin\left(\theta - \dfrac{\pi}{4}\right)}$; How does the graph differ from the graph of $r = \dfrac{4}{1 - \sin\theta}$?

44. $r = \dfrac{3}{2 + 6\cos\left(\theta + \dfrac{\pi}{3}\right)}$; How does the graph differ from the graph of $r = \dfrac{3}{2 + 6\cos\theta}$?

45. Use the polar equation for planetary orbits,

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta},$$

to find the polar equation of the orbit for Mercury and Earth.

Mercury: $e = 0.2056$ and $a = 36.0 \times 10^6$ miles

Earth: $e = 0.0167$ and $a = 92.96 \times 10^6$ miles

Use a graphing utility to graph both orbits in the same viewing rectangle. What do you see about the orbits from their graphs that is not obvious from their equations?

Critical Thinking Exercises

46. Identify the conic and graph the equation:

$$r = \frac{4\sec\theta}{2\sec\theta - 1}.$$

In Exercises 47–48, write a polar equation of the conic that is named and described.

47. Ellipse: a focus at the pole; vertex: $(4, 0)$; $e = \frac{1}{2}$

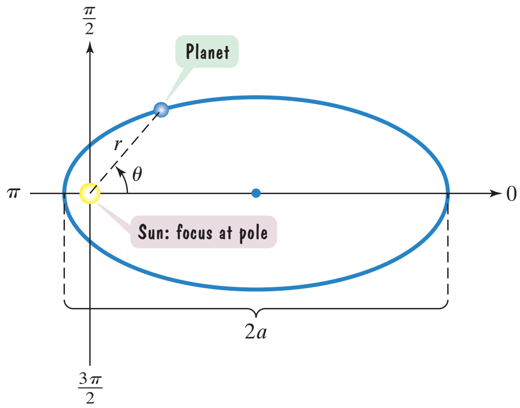
48. Hyperbola: a focus at the pole; directrix: $x = -1$; $e = \frac{3}{2}$

49. Identify the conic and write its equation in rectangular coordinates: $r = \dfrac{1}{2 - 2\cos\theta}$.

50. Prove that the polar equation of a planet's elliptical orbit is

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta},$$

where e is the eccentricity and $2a$ is the length of the major axis.



Chapter 9 Summary, Review, and Test

Summary

DEFINITIONS AND CONCEPTS	EXAMPLES
9.1 The Ellipse	
a. An ellipse is the set of all points in a plane the sum of whose distances from two fixed points, the foci, is constant.	
b. Standard forms of the equations of an ellipse with center at the origin are $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ [foci: $(-c, 0)$, $(c, 0)$] and $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ [foci: $(0, -c)$, $(0, c)$], where $c^2 = a^2 - b^2$ and $a^2 > b^2$. See the box on page 852 and Figure 9.6.	Ex. 1, p. 852; Ex. 2, p. 853; Ex. 3, p. 854

DEFINITIONS AND CONCEPTS	EXAMPLES
c. Standard forms of the equations of an ellipse centered at (h, k) are $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ and $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$, $a^2 > b^2$. See Table 9.1 on page 855.	Ex. 4, p. 856

9.2 The Hyperbola

a. A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points, the foci, is constant.	
b. Standard forms of the equations of a hyperbola with center at the origin are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ [foci: $(-c, 0)$, $(c, 0)$] and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ [foci: $(0, -c)$, $(0, c)$], where $c^2 = a^2 + b^2$. See the box on page 863 and Figure 9.16.	Ex. 1, p. 864; Ex. 2, p. 865
c. Asymptotes for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$. Asymptotes for $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are $y = \pm\frac{a}{b}x$.	
d. A procedure for graphing hyperbolas is given in the box on page 866.	Ex. 3, p. 867; Ex. 4, p. 868
e. Standard forms of the equations of a hyperbola centered at (h, k) are $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ and $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. See Table 9.2 on page 869.	Ex. 5, p. 869; Ex. 6, p. 871

9.3 The Parabola

a. A parabola is the set of all points in a plane that are equidistant from a fixed line, the directrix, and a fixed point, the focus.	
b. Standard forms of the equations of parabolas with vertex at the origin are $y^2 = 4px$ [focus: $(p, 0)$] and $x^2 = 4py$ [focus: $(0, p)$]. See the box on page 877 and Figure 9.31 on page 878.	Ex. 1, p. 878; Ex. 3, p. 880
c. A parabola's latus rectum is a line segment that passes through its focus, is parallel to its directrix, and has its endpoints on the parabola. The length of the latus rectum for $y^2 = 4px$ and $x^2 = 4py$ is $\lvert 4p \rvert$. A parabola can be graphed using the vertex and endpoints of the latus rectum.	Ex. 2, p. 879
d. Standard forms of the equations of a parabola with vertex at (h, k) are $(y-k)^2 = 4p(x-h)$ and $(x-h)^2 = 4p(y-k)$. See Table 9.3 on page 881 and Figure 9.36.	Ex. 4, p. 881; Ex. 5, p. 882

9.4 Rotation of Axes

a. A nondegenerate conic section with equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ in which A and C are not both zero is **1.** a circle if $A = C$; **2.** a parabola if $AC = 0$; **3.** an ellipse if $A \neq C$ and $AC > 0$; **4.** a hyperbola if $AC < 0$.	Ex. 1, p. 889
b. Rotation of Axes Formulas θ is the angle from the positive x-axis to the positive x'-axis. $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$	Ex. 2, p. 892
c. Amount of Rotation Formula The general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten in x' and y' without an $x'y'$-term by rotating the axes through angle θ, where $\cot 2\theta = \frac{A-C}{B}$ and θ is an acute angle.	
d. If 2θ in $\cot 2\theta$ is one of the familiar angles such as 30°, 45°, or 60°, write the equation of a rotated conic in standard form using the five-step procedure in the box on page 894.	Ex. 3, p. 894

DEFINITIONS AND CONCEPTS | **EXAMPLES**

e. If $\cot 2\theta$ is not the cotangent of one of the more familiar angles, use a sketch of $\cot 2\theta$ to find $\cos 2\theta$. Then use

$$\sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} \quad \text{and} \quad \cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$$

to find values for $\sin\theta$ and $\cos\theta$ in the rotation formulas. — Ex. 4, p. 896

f. A nondegenerate conic section of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is **1.** a parabola if $B^2 - 4AC = 0$; **2.** an ellipse or a circle if $B^2 - 4AC < 0$; **3.** a hyperbola if $B^2 - 4AC > 0$. — Ex. 5, p. 899

9.5 *Parametric Equations*

a. The relationship between the parametric equations $x = f(t)$ and $y = g(t)$ and plane curves is described in the first box on page 902.

b. Point plotting can be used to graph a plane curve described by parametric equations. See the second box on page 902. — Ex. 1, p. 902

c. Plane curves can be sketched by eliminating the parameter t and graphing the resulting rectangular equation. It is sometimes necessary to change the domain of the rectangular equation to be consistent with the domain for the parametric equation in x. — Ex. 2, p. 903; Ex. 3, p. 904

d. Infinitely many pairs of parametric equations can represent the same plane curve. One pair for $y = f(x)$ is $x = t$ and $y = f(t)$, in which t is in the domain of f. — Ex. 4, p. 906

9.6 *Conic Sections in Polar Coordinates*

a. The focus-directrix definitions of the conic sections are given in the box on page 910. For all points on a conic, the ratio of the distance from a fixed point (focus) and the distance from a fixed line (directrix) is constant and is called its eccentricity. If $e = 1$, the conic is a parabola. If $e < 1$, the conic is an ellipse. If $e > 1$, the conic is a hyperbola.

b. Standard forms of the polar equations of conics are

$$r = \frac{ep}{1 \pm e\cos\theta} \quad \text{and} \quad r = \frac{ep}{1 \pm e\sin\theta},$$

in which (r, θ) is a point on the conic's graph, e is the eccentricity, and p is the distance between the focus (located at the pole) and the directrix. Details are shown in the box on page 911. — Ex. 1, p. 912; Ex. 2, p. 914; Ex. 3, p. 915

c. A procedure for graphing the polar equation of a conic is given in the box on page 912.

Review Exercises

9.1

In Exercises 1–8, graph each ellipse and locate the foci.

1. $\frac{x^2}{36} + \frac{y^2}{25} = 1$

2. $\frac{y^2}{25} + \frac{x^2}{16} = 1$

3. $4x^2 + y^2 = 16$

4. $4x^2 + 9y^2 = 36$

5. $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$

6. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{16} = 1$

7. $4x^2 + 9y^2 + 24x - 36y + 36 = 0$

8. $9x^2 + 4y^2 - 18x + 8y - 23 = 0$

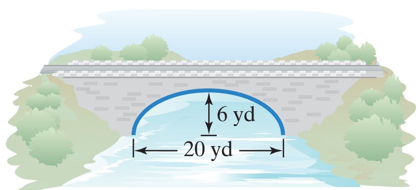
In Exercises 9–11, find the standard form of the equation of each ellipse satisfying the given conditions.

9. Foci: $(-4, 0)$, $(4, 0)$; Vertices: $(-5, 0)$, $(5, 0)$

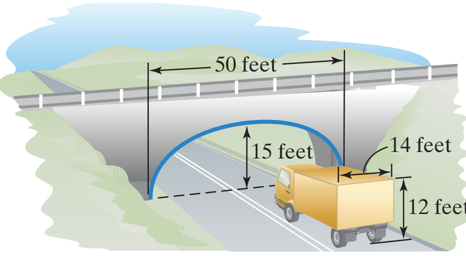
10. Foci: $(0, -3)$, $(0, 3)$; Vertices: $(0, -6)$, $(0, 6)$

11. Major axis horizontal with length 12; length of minor axis $= 4$; center: $(-3, 5)$.

12. A semielliptical arch supports a bridge that spans a river 20 yards wide. The center of the arch is 6 yards above the river's center. Write an equation for the ellipse so that the x-axis coincides with the water level and the y-axis passes through the center of the arch.



13. A semielliptic archway has a height of 15 feet at the center and a width of 50 feet, as shown in the figure. The 50-foot width consists of a two-lane road. Can a truck that is 12 feet high and 14 feet wide drive under the archway without going into the other lane?



14. An elliptical pool table has a ball placed at each focus. If one ball is hit toward the side of the table, explain what will occur.

9.2

In Exercises 15–22, graph each hyperbola. Locate the foci and find the equations of the asymptotes.

15. $\frac{x^2}{16} - y^2 = 1$

16. $\frac{y^2}{16} - x^2 = 1$

17. $9x^2 - 16y^2 = 144$

18. $4y^2 - x^2 = 16$

19. $\frac{(x-2)^2}{25} - \frac{(y+3)^2}{16} = 1$

20. $\frac{(y+2)^2}{25} - \frac{(x-3)^2}{16} = 1$

21. $y^2 - 4y - 4x^2 + 8x - 4 = 0$

22. $x^2 - y^2 - 2x - 2y - 1 = 0$

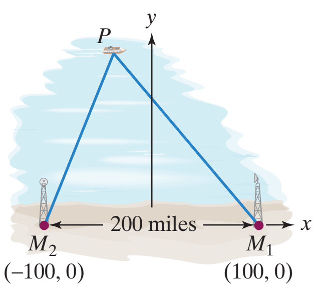
In Exercises 23–24, find the standard form of the equation of each hyperbola satisfying the given conditions.

23. Foci: $(0, -4)$, $(0, 4)$; Vertices: $(0, -2)$, $(0, 2)$

24. Foci: $(-8, 0)$, $(8, 0)$; Vertices: $(-3, 0)$, $(3, 0)$

25. Explain why it is not possible for a hyperbola to have foci at $(0, -2)$ and $(0, 2)$ and vertices at $(0, -3)$ and $(0, 3)$.

26. Radio tower M_2 is located 200 miles due west of radio tower M_1. The situation is illustrated in the figure shown, where a coordinate system has been superimposed. Simultaneous radio signals are sent from each tower to a ship, with the signal from M_2 received 500 microseconds before the signal from M_1. Assuming that radio signals travel at 0.186 mile per microsecond, determine the equation of the hyperbola on which the ship is located.



9.3

In Exercises 27–33, find the vertex, focus, and directrix of each parabola with the given equation. Then graph the parabola.

27. $y^2 = 8x$

28. $x^2 + 16y = 0$

29. $(y-2)^2 = -16x$

30. $(x-4)^2 = 4(y+1)$

31. $x^2 + 4y = 4$

32. $y^2 - 4x - 10y + 21 = 0$

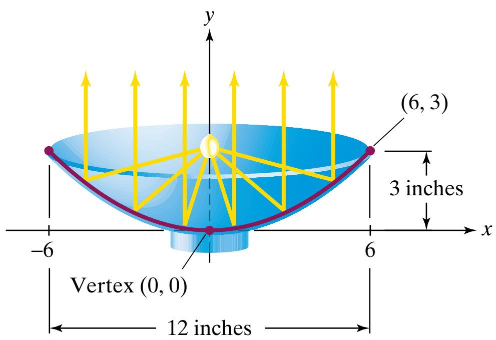
33. $x^2 - 4x - 2y = 0$

In Exercises 34–35, find the standard form of the equation of each parabola satisfying the given conditions.

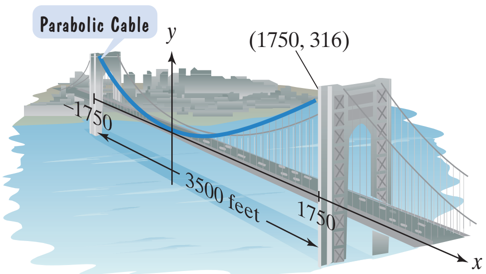
34. Focus: $(12, 0)$; Directrix: $x = -12$

35. Focus: $(0, -11)$; Directrix: $y = 11$

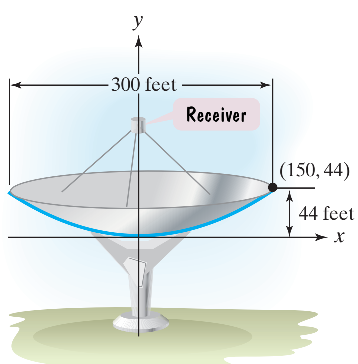
36. An engineer is designing headlight units for automobiles. The unit has a parabolic surface with a diameter of 12 inches and a depth of 3 inches. The situation is illustrated in the figure, where a coordinate system has been superimposed. What is the equation of the parabola in this system? Where should the light source be placed? Describe this placement relative to the vertex.



37. The George Washington Bridge spans the Hudson River from New York to New Jersey. Its two towers are 3500 feet apart and rise 316 feet above the road. As shown in the figure, the cable between the towers has the shape of a parabola and the cable just touches the sides of the road midway between the towers. What is the height of the cable 1000 feet from a tower?



38. The giant satellite dish in the figure shown is in the shape of a parabolic surface. Signals strike the surface and are reflected to the focus, where the receiver is located. The diameter of the dish is 300 feet and its depth is 44 feet. How far, to the nearest foot, from the base of the dish should the receiver be placed?



9.4

In Exercises 39–46, identify the conic represented by each equation without completing the square or using a rotation of axes.

39. $y^2 + 4x + 2y - 15 = 0$

40. $x^2 + 16y^2 - 160y + 384 = 0$

41. $16x^2 + 64x + 9y^2 - 54y + 1 = 0$

42. $4x^2 - 9y^2 - 8x + 12y - 144 = 0$

43. $5x^2 + 2\sqrt{3}xy + 3y^2 - 18 = 0$

44. $5x^2 - 8xy + 7y^2 - 9\sqrt{5}x - 9 = 0$

45. $x^2 + 6xy + 9y^2 - 2y = 0$

46. $x^2 - 2xy + 3y^2 + 2x + 4y - 1 = 0$

In Exercises 47–51,

a. *Rewrite the equation in a rotated $x'y'$-system without an $x'y'$-term.*

b. *Express the equation involving x' and y' in the standard form of a conic section.*

c. *Use the rotated system to graph the equation.*

47. $xy - 4 = 0$

48. $x^2 + xy + y^2 - 1 = 0$

49. $4x^2 + 10xy + 4y^2 - 9 = 0$

50. $6x^2 - 6xy + 14y^2 - 45 = 0$

51. $x^2 + 2\sqrt{3}xy + 3y^2 - 12\sqrt{3}x + 12y = 0$

9.5

In Exercises 52–57, eliminate the parameter and graph the plane curve represented by the parametric equations. Use arrows to show the orientation of each plane curve.

52. $x = 2t - 1, y = 1 - t; -\infty < t < \infty$

53. $x = t^2, y = t - 1; -1 \le t \le 3$

54. $x = 4t^2, y = t + 1; -\infty < t < \infty$

55. $x = 4 \sin t, y = 3 \cos t; 0 \le t < \pi$

56. $x = 3 + 2 \cos t, y = 1 + 2 \sin t; 0 \le t < 2\pi$

57. $x = 3 \sec t, y = 3 \tan t; 0 \le t \le \frac{\pi}{4}$

58. Find two different sets of parametric equations for $y = x^2 + 6$.

59. The path of a projectile that is launched h feet above the ground with an initial velocity of v_0 feet per second and at an angle θ with the horizontal is given by the parametric equations

$$x = (v_0 \cos \theta)t \quad \text{and} \quad y = h + (v_0 \sin \theta)t - 16t^2,$$

where t is the time, in seconds, after the projectile was launched. A football player throws a football with an initial velocity of 100 feet per second at an angle of 40° to the horizontal. The ball leaves the player's hand at a height of 6 feet.

a. Find the parametric equations that describe the position of the ball as a function of time.

b. Describe the ball's position after 1, 2, and 3 seconds. Round to the nearest tenth of a foot.

c. How long, to the nearest tenth of a second, is the ball in flight? What is the total horizontal distance that it travels before it lands?

d. Graph the parametric equations in part (a) using a graphing utility. Use the graph to determine when the ball is at its maximum height. What is its maximum height? Round all answers to the nearest tenth.

9.6

In Exercises 60–65,

a. *If necessary, write the equation in one of the standard forms for a conic in polar coordinates.*

b. *Determine values for e and p. Use the value of e to identify the conic section.*

c. *Graph the given polar equation.*

60. $r = \dfrac{4}{1 - \sin \theta}$

61. $r = \dfrac{6}{1 + \cos \theta}$

62. $r = \dfrac{6}{2 + \sin \theta}$

63. $r = \dfrac{2}{3 - 2 \cos \theta}$

64. $r = \dfrac{6}{3 + 6 \sin \theta}$

65. $r = \dfrac{8}{4 + 16 \cos \theta}$

Chapter 9 Test

In Exercises 1–5, graph the conic section with the given equation. For ellipses, find the foci. For hyperbolas, find the foci and give the equations of the asymptotes. For parabolas, find the vertex, focus, and directrix.

1. $9x^2 - 4y^2 = 36$

2. $x^2 = -8y$

3. $\dfrac{(x + 2)^2}{25} + \dfrac{(y - 5)^2}{9} = 1$

4. $4x^2 - y^2 + 8x + 2y + 7 = 0$

5. $(x + 5)^2 = 8(y - 1)$

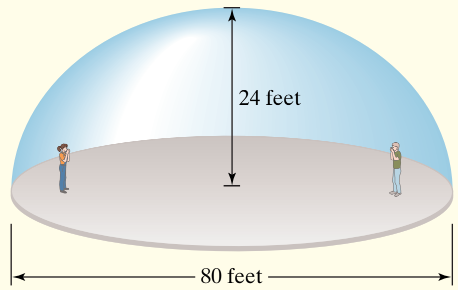
In Exercises 6–8, find the standard form of the equation of the conic section satisfying the given conditions.

6. Ellipse; Foci: $(-7, 0), (7, 0)$; Vertices: $(-10, 0), (10, 0)$

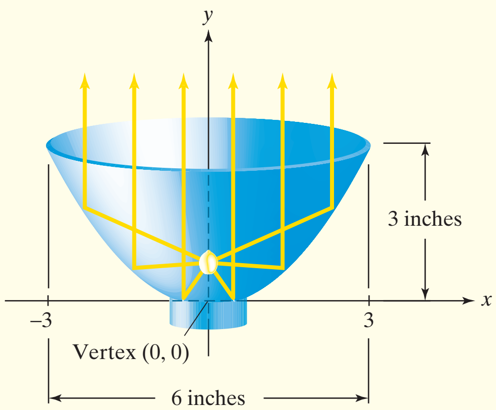
7. Hyperbola; Foci: $(0, -10), (0, 10)$; Vertices: $(0, -7), (0, 7)$

8. Parabola; Focus: $(50, 0)$; Directrix: $x = -50$

9. A sound whispered at one focus of a whispering gallery can be heard at the other focus. The figure at the top of the next page shows a whispering gallery whose cross section is a semielliptical arch with a height of 24 feet and a width of 80 feet. How far from the room's center should two people stand so that they can whisper back and forth and be heard?



10. An engineer is designing headlight units for cars. The unit shown in the figure below has a parabolic surface with a diameter of 6 inches and a depth of 3 inches.



a. Using the coordinate system that has been positioned on the unit, find the parabola's equation.

b. If the light source is located at the focus, describe its placement relative to the vertex.

In Exercises 11–12, identify each equation without completing the square or using a rotation of axes.

11. $x^2 + 9y^2 + 10x - 18y + 25 = 0$

12. $x^2 + y^2 + xy + 3x - y - 3 = 0$

13. For the equation

$$7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0,$$

determine what angle of rotation would eliminate the $x'y'$-term in a rotated $x'y'$-system.

In Exercises 14–15, eliminate the parameter and graph the plane curve represented by the parametric equations. Use arrows to show the orientation of each plane curve.

14. $x = t^2,\ y = t - 1;\ -\infty < t < \infty$

15. $x = 1 + 3\sin t,\ y = 2\cos t;\ 0 \le t < 2\pi$

In Exercises 16–17, identify the conic section and graph the polar equation.

16. $r = \dfrac{2}{1 - \cos\theta}$ **17.** $r = \dfrac{4}{2 + \sin\theta}$

Cumulative Review Exercises (Chapters P–9)

Solve each equation or inequality in Exercises 1–7.

1. $2(x - 3) + 5x = 8(x - 1)$

2. $-3(2x - 4) > 2(6x - 12)$

3. $x - 5 = \sqrt{x + 7}$ **4.** $(x - 2)^2 = 20$

5. $|2x - 1| \ge 7$ **6.** $3x^3 + 4x^2 - 7x + 2 = 0$

7. $\log_2(x + 1) + \log_2(x - 1) = 3$

Solve each system in Exercises 8–10.

8. $3x + 4y = 2$
$2x + 5y = -1$

9. $2x^2 - y^2 = -8$
$x - y = 6$

10. (Use matrices.)

$$\begin{aligned} x - y + z &= 17 \\ -4x + y + 5z &= -2 \\ 2x + 3y + z &= 8 \end{aligned}$$

In Exercises 11–13, graph each equation, function, or system in a rectangular coordinate system.

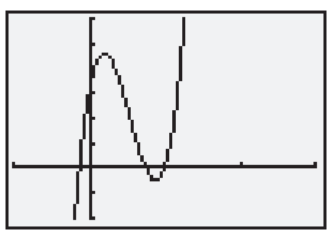
11. $f(x) = (x - 1)^2 - 4$ **12.** $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

13. $5x + y \le 10$
$y \ge \frac{1}{4}x + 2$

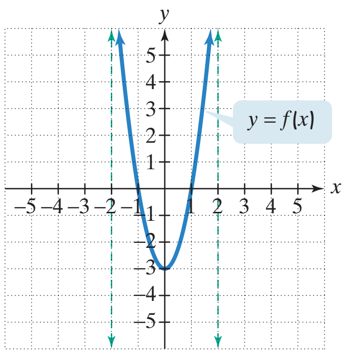
14. a. List all possible rational roots of

$$32x^3 - 52x^2 + 17x + 3 = 0.$$

b. The graph of $f(x) = 32x^3 - 52x^2 + 17x + 3$ is shown in a $[-1, 3, 1]$ by $[-2, 6, 1]$ viewing rectangle. Use the graph of f and synthetic division to solve the equation in part (a).



15. The figure shows the graph of $y = f(x)$ and its two vertical asymptotes.



a. Find the domain and the range of f.

b. What is the relative minimum and where does it occur?

c. Find the interval on which f is increasing.

d. Find $f(-1) - f(0)$.

e. Find $(f \circ f)(1)$.

f. Use arrow notation to complete this statement:

$$f(x) \to \infty \text{ as } \_\_\_\_\_\_ \text{ or as } \_\_\_\_\_\_.$$

g. Graph $g(x) = f(x - 2) + 1$.

h. Graph $h(x) = -f(2x)$.

16. If $f(x) = x^2 - 4$ and $g(x) = x + 2$, find $(g \circ f)(x)$.

17. Expand using logarithmic properties. Where possible, evaluate logarithmic expressions.

$$\log_5\left(\frac{x^3\sqrt{y}}{125}\right)$$

18. Write the slope-intercept form of the equation of the line passing through $(1, -4)$ and $(-5, 8)$.

19. Rent-a-Truck charges a daily rental rate for a truck of \$39 plus \$0.16 a mile. A competing agency, Ace Truck Rentals, charges \$25 a day plus \$0.24 a mile for the same truck. How many miles must be driven in a day to make the daily cost of both agencies the same? What will be the cost?

20. The local cable television company offers two deals. Basic cable service with one movie channel costs \$35 per month. Basic service with two movie channels cost \$45 per month. Find the charge for the basic cable service and the charge for each movie channel.

21. Verify the identity: $\dfrac{\csc\theta - \sin\theta}{\sin\theta} = \cot^2\theta$.

22. Graph one complete cycle of $y = 2\cos(2x + \pi)$.

23. If $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j}$, find $(\mathbf{v} \cdot \mathbf{w})\mathbf{w}$.

24. Solve for θ: $\sin 2\theta = \sin\theta$, $0 \le \theta < 2\pi$.

25. In oblique triangle ABC, $A = 64°$, $B = 72°$, and $a = 13.6$. Solve the triangle. Round lengths to the nearest tenth.