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Calculus I

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Chapter 0

Before Calculus

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0

BEFORE CALCULUS

The development of calculus in the seventeenth and eighteenth centuries was motivated by the need to understand physical phenomena such as the tides, the phases of the moon, the nature of light, and gravity.

One of the important themes in calculus is the analysis of relationships between physical or mathematical quantities. Such relationships can be described in terms of graphs, formulas, numerical data, or words. In this chapter we will develop the concept of a "function," which is the basic idea that underlies almost all mathematical and physical relationships, regardless of the form in which they are expressed. We will study properties of some of the most basic functions that occur in calculus.

0.1 FUNCTIONS

In this section we will define and develop the concept of a "function," which is the basic mathematical object that scientists and mathematicians use to describe relationships between variable quantities. Functions play a central role in calculus and its applications.

DEFINITION OF A FUNCTION

Many scientific laws and engineering principles describe how one quantity depends on another. This idea was formalized in 1673 by Gottfried Wilhelm Leibniz (see p. xx) who coined the term *function* to indicate the dependence of one quantity on another, as described in the following definition.

0.1.1 DEFINITION If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that ***y is a function of x***.

Four common methods for representing functions are:

- Numerically by tables
- Algebraically by formulas
- Geometrically by graphs
- Verbally

Table 0.1.1

INDIANAPOLIS 500
QUALIFYING SPEEDS

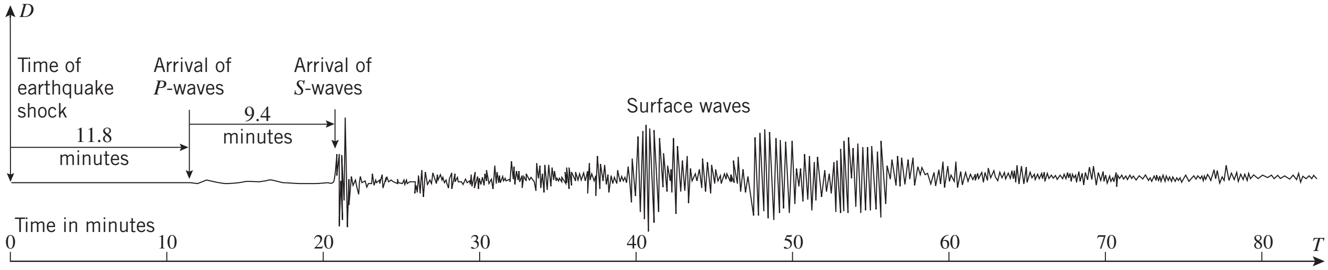
YEAR t	SPEED S (mi/h)
1994	228.011
1995	231.604
1996	233.100
1997	218.263
1998	223.503
1999	225.179
2000	223.471
2001	226.037
2002	231.342
2003	231.725
2004	222.024
2005	227.598
2006	228.985
2007	225.817
2008	226.366
2009	224.864
2010	227.970
2011	227.472

The method of representation often depends on how the function arises. For example:

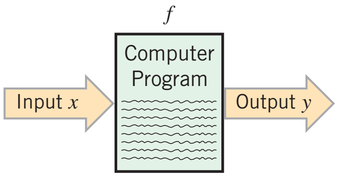
- Table 0.1.1 shows the top qualifying speed S for the Indianapolis 500 auto race as a function of the year t. There is exactly one value of S for each value of t.
- Figure 0.1.1 is a graphical record of an earthquake recorded on a seismograph. The graph describes the deflection D of the seismograph needle as a function of the time T elapsed since the wave left the earthquake's epicenter. There is exactly one value of D for each value of T.
- Some of the most familiar functions arise from formulas; for example, the formula $C = 2\pi r$ expresses the circumference C of a circle as a function of its radius r. There is exactly one value of C for each value of r.
- Sometimes functions are described in words. For example, Isaac Newton's Law of Universal Gravitation is often stated as follows: The gravitational force of attraction between two bodies in the Universe is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. This is the verbal description of the formula

$$F = G\frac{m_1 m_2}{r^2}$$

  in which F is the force of attraction, m_1 and m_2 are the masses, r is the distance between them, and G is a constant. If the masses are constant, then the verbal description defines F as a function of r. There is exactly one value of F for each value of r.



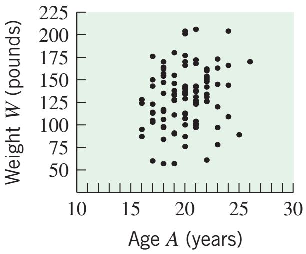
▲ **Figure 0.1.1**



▲ **Figure 0.1.2**

In the mid-eighteenth century the Swiss mathematician Leonhard Euler (pronounced "oiler") conceived the idea of denoting functions by letters of the alphabet, thereby making it possible to refer to functions without stating specific formulas, graphs, or tables. To understand Euler's idea, think of a function as a computer program that takes an *input* x, operates on it in some way, and produces exactly one *output* y. The computer program is an object in its own right, so we can give it a name, say f. Thus, the function f (the computer program) associates a unique output y with each input x (Figure 0.1.2). This suggests the following definition.

0.1.2 DEFINITION A ***function*** f is a rule that associates a unique output with each input. If the input is denoted by x, then the output is denoted by $f(x)$ (read "f of x").



▲ **Figure 0.1.3**

In this definition the term *unique* means "exactly one." Thus, a function cannot assign two different outputs to the same input. For example, Figure 0.1.3 shows a plot of weight versus age for a random sample of 100 college students. This plot does *not* describe W as a function of A because there are some values of A with more than one corresponding value of W. This is to be expected, since two people with the same age can have different weights.

INDEPENDENT AND DEPENDENT VARIABLES

For a given input x, the output of a function f is called the ***value*** of f at x or the ***image*** of x under f. Sometimes we will want to denote the output by a single letter, say y, and write

$$y = f(x)$$

This equation expresses y as a function of x; the variable x is called the ***independent variable*** (or ***argument***) of f, and the variable y is called the ***dependent variable*** of f. This terminology is intended to suggest that x is free to vary, but that once x has a specific value a corresponding value of y is determined. For now we will only consider functions in which the independent and dependent variables are real numbers, in which case we say that f is a ***real-valued function of a real variable***. Later, we will consider other kinds of functions.

Table 0.1.2

x	0	1	2	3
y	3	4	−1	6

Example 1 Table 0.1.2 describes a functional relationship $y = f(x)$ for which

$f(0) = 3$	f associates $y = 3$ with $x = 0$.
$f(1) = 4$	f associates $y = 4$ with $x = 1$.
$f(2) = -1$	f associates $y = -1$ with $x = 2$.
$f(3) = 6$	f associates $y = 6$ with $x = 3$. ◄

Example 2 The equation

$$y = 3x^2 - 4x + 2$$

has the form $y = f(x)$ in which the function f is given by the formula

$$f(x) = 3x^2 - 4x + 2$$

Leonhard Euler (1707–1783) Euler was probably the most prolific mathematician who ever lived. It has been said that "Euler wrote mathematics as effortlessly as most men breathe." He was born in Basel, Switzerland, and was the son of a Protestant minister who had himself studied mathematics. Euler's genius developed early. He attended the University of Basel, where by age 16 he obtained both a Bachelor of Arts degree and a Master's degree in philosophy. While at Basel, Euler had the good fortune to be tutored one day a week in mathematics by a distinguished mathematician, Johann Bernoulli. At the urging of his father, Euler then began to study theology. The lure of mathematics was too great, however, and by age 18 Euler had begun to do mathematical research. Nevertheless, the influence of his father and his theological studies remained, and throughout his life Euler was a deeply religious, unaffected person. At various times Euler taught at St. Petersburg Academy of Sciences (in Russia), the University of Basel, and the Berlin Academy of Sciences. Euler's energy and capacity for work were virtually boundless. His collected works form more than 100 quarto-sized volumes and it is believed that much of his work has been lost. What is particularly astonishing is that Euler was blind for the last 17 years of his life, and this was one of his most productive periods! Euler's flawless memory was phenomenal. Early in his life he memorized the entire *Aeneid* by Virgil, and at age 70 he could not only recite the entire work but could also state the first and last sentence on each page of the book from which he memorized the work. His ability to solve problems in his head was beyond belief. He worked out in his head major problems of lunar motion that baffled Isaac Newton and once did a complicated calculation in his head to settle an argument between two students whose computations differed in the fiftieth decimal place.

Following the development of calculus by Leibniz and Newton, results in mathematics developed rapidly in a disorganized way. Euler's genius gave coherence to the mathematical landscape. He was the first mathematician to bring the full power of calculus to bear on problems from physics. He made major contributions to virtually every branch of mathematics as well as to the theory of optics, planetary motion, electricity, magnetism, and general mechanics.

[*Image: http://commons.wikimedia.org/wiki/File:Leonhard_Euler_by_Handmann_.png*]

For each input x, the corresponding output y is obtained by substituting x in this formula. For example,

$$f(0) = 3(0)^2 - 4(0) + 2 = 2$$

f associates $y = 2$ with $x = 0$.

$$f(-1.7) = 3(-1.7)^2 - 4(-1.7) + 2 = 17.47$$

f associates $y = 17.47$ with $x = -1.7$.

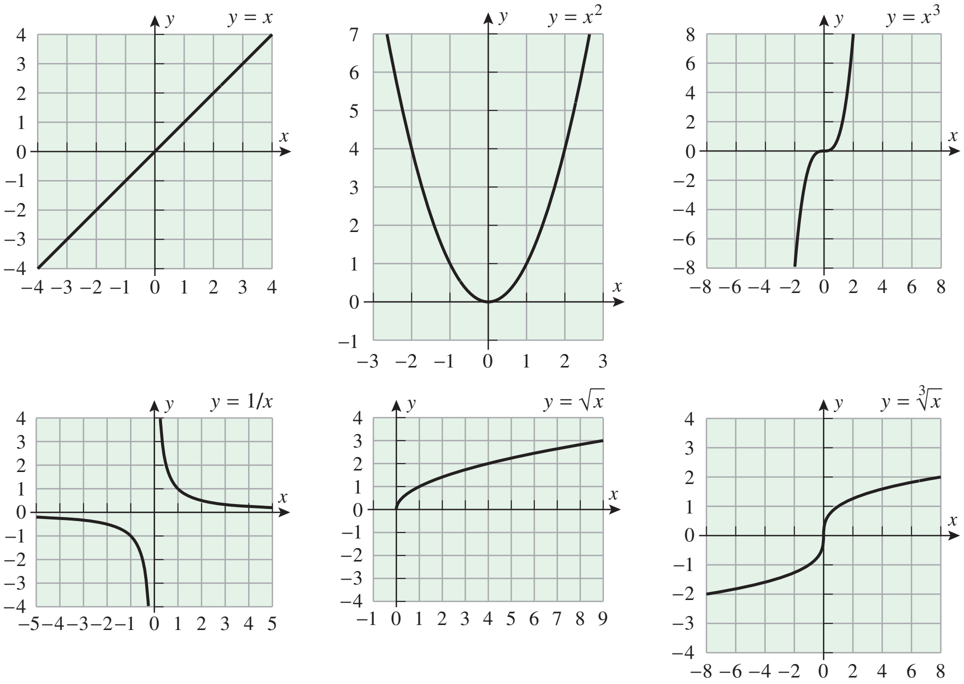
$$f(\sqrt{2}) = 3(\sqrt{2})^2 - 4\sqrt{2} + 2 = 8 - 4\sqrt{2}$$

f associates $y = 8 - 4\sqrt{2}$ with $x = \sqrt{2}$. ◄

GRAPHS OF FUNCTIONS

If f is a real-valued function of a real variable, then the ***graph*** of f in the xy-plane is defined to be the graph of the equation $y = f(x)$. For example, the graph of the function $f(x) = x$ is the graph of the equation $y = x$, shown in Figure 0.1.4. That figure also shows the graphs of some other basic functions that may already be familiar to you. In Appendix A we discuss techniques for graphing functions using graphing technology.

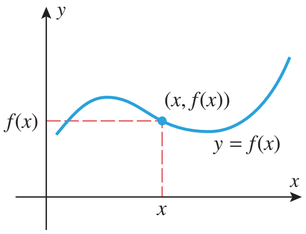
Figure 0.1.4 shows only portions of the graphs. Where appropriate, and unless indicated otherwise, it is understood that graphs shown in this text extend indefinitely beyond the boundaries of the displayed figure.



▲ **Figure 0.1.4**

Since $\sqrt{x}$ is imaginary for negative values of x, there are no points on the graph of $y = \sqrt{x}$ in the region where $x < 0$.

Graphs can provide valuable visual information about a function. For example, since the graph of a function f in the xy-plane is the graph of the equation $y = f(x)$, the points on the graph of f are of the form $(x, f(x))$; that is, *the y-coordinate of a point on the graph of f is the value of f at the corresponding x-coordinate* (Figure 0.1.5). The values of x for which $f(x) = 0$ are the x-coordinates of the points where the graph of f intersects the x-axis (Figure 0.1.6). These values are called the ***zeros*** of f, the ***roots*** of $f(x) = 0$, or the ***x-intercepts*** of the graph of $y = f(x)$.

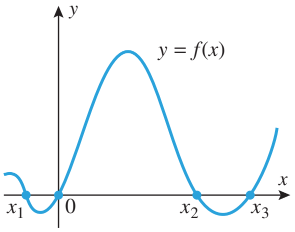


▲ **Figure 0.1.5** The y-coordinate of a point on the graph of $y = f(x)$ is the value of f at the corresponding x-coordinate.

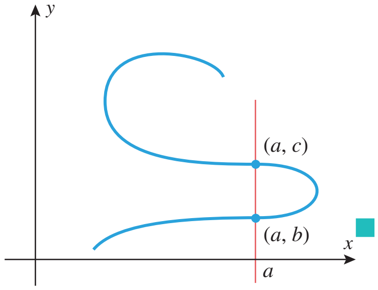
THE VERTICAL LINE TEST

Not every curve in the xy-plane is the graph of a function. For example, consider the curve in Figure 0.1.7, which is cut at two distinct points, (a, b) and (a, c), by a vertical line. This curve cannot be the graph of $y = f(x)$ for any function f; otherwise, we would have

$$f(a) = b \quad \text{and} \quad f(a) = c$$



▲ **Figure 0.1.6** f has zeros at x_1, 0, x_2, and x_3.



▲ **Figure 0.1.7** This curve cannot be the graph of a function.

which is impossible, since f cannot assign two different values to a. Thus, there is no function f whose graph is the given curve. This illustrates the following general result, which we will call the ***vertical line test***.

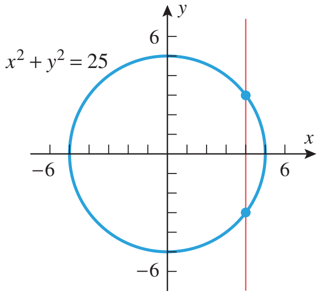
0.1.3 THE VERTICAL LINE TEST *A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.*

▶ **Example 3** The graph of the equation

$$x^2 + y^2 = 25$$

is a circle of radius 5 centered at the origin and hence there are vertical lines that cut the graph more than once (Figure 0.1.8). Thus this equation does not define y as a function of x. ◀

Symbols such as $+x$ and $-x$ are deceptive, since it is tempting to conclude that $+x$ is positive and $-x$ is negative. However, this need not be so, since x itself can be positive or negative. For example, if x is negative, say $x = -3$, then $-x = 3$ is positive and $+x = -3$ is negative.



▲ **Figure 0.1.8**

THE ABSOLUTE VALUE FUNCTION

Recall that the ***absolute value*** or ***magnitude*** of a real number x is defined by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative. Thus,

$$|5| = 5, \quad \left|-\tfrac{4}{7}\right| = \tfrac{4}{7}, \quad |0| = 0$$

A more detailed discussion of the properties of absolute value is given in Web Appendix F. However, for convenience we provide the following summary of its algebraic properties.

0.1.4 PROPERTIES OF ABSOLUTE VALUE *If a and b are real numbers, then*

(a)	$\lvert -a\rvert = \lvert a\rvert$	A number and its negative have the same absolute value.
(b)	$\lvert ab\rvert = \lvert a\rvert\,\lvert b\rvert$	The absolute value of a product is the product of the absolute values.
(c)	$\lvert a/b\rvert = \lvert a\rvert/\lvert b\rvert,\ b \neq 0$	The absolute value of a ratio is the ratio of the absolute values.
(d)	$\lvert a+b\rvert \leq \lvert a\rvert + \lvert b\rvert$	The ***triangle inequality***

The graph of the function $f(x) = |x|$ can be obtained by graphing the two parts of the equation

$$y = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

separately. Combining the two parts produces the V-shaped graph in Figure 0.1.9.

Absolute values have important relationships to square roots. To see why this is so, recall from algebra that every positive real number x has two square roots, one positive and one negative. By definition, the symbol $\sqrt{x}$ denotes the *positive* square root of x.

Care must be exercised in simplifying expressions of the form $\sqrt{x^2}$, since it is *not* always true that $\sqrt{x^2} = x$. This equation is correct if x is nonnegative, but it is false if x is negative. For example, if $x = -4$, then

$$\sqrt{x^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq x$$

WARNING

To denote the negative square root you must write $-\sqrt{x}$. For example, the positive square root of 9 is $\sqrt{9} = 3$, whereas the negative square root of 9 is $-\sqrt{9} = -3$. (Do not make the mistake of writing $\sqrt{9} = \pm 3$.)

TECHNOLOGY MASTERY

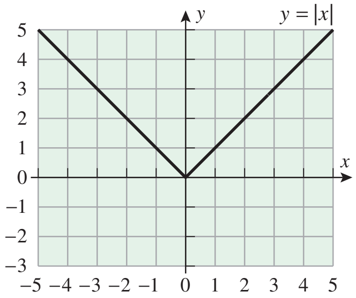
Verify (1) by using a graphing utility to show that the equations $y = \sqrt{x^2}$ and $y = |x|$ have the same graph.

A statement that is correct for all real values of x is

$$\sqrt{x^2} = |x| \tag{1}$$

PIECEWISE-DEFINED FUNCTIONS

The absolute value function $f(x) = |x|$ is an example of a function that is defined ***piecewise*** in the sense that the formula for f changes, depending on the value of x.

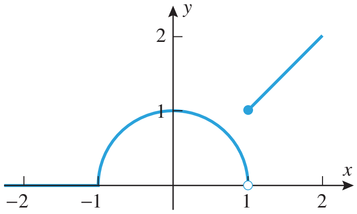


▲ **Figure 0.1.9**

▶ **Example 4** Sketch the graph of the function defined piecewise by the formula

$$f(x) = \begin{cases} 0, & x \le -1 \\ \sqrt{1 - x^2}, & -1 < x < 1 \\ x, & x \ge 1 \end{cases}$$

Solution. The formula for f changes at the points $x = -1$ and $x = 1$. (We call these the ***breakpoints*** for the formula.) A good procedure for graphing functions defined piecewise is to graph the function separately over the open intervals determined by the breakpoints, and then graph f at the breakpoints themselves. For the function f in this example the graph is the horizontal ray $y = 0$ on the interval $(-\infty, -1]$, it is the semicircle $y = \sqrt{1 - x^2}$ on the interval $(-1, 1)$, and it is the ray $y = x$ on the interval $[1, +\infty)$. The formula for f specifies that the equation $y = 0$ applies at the breakpoint -1 [so $y = f(-1) = 0$], and it specifies that the equation $y = x$ applies at the breakpoint 1 [so $y = f(1) = 1$]. The graph of f is shown in Figure 0.1.10. ◀



▲ **Figure 0.1.10**

REMARK In Figure 0.1.10 the solid dot and open circle at the breakpoint $x = 1$ serve to emphasize that the point on the graph lies on the ray and not the semicircle. There is no ambiguity at the breakpoint $x = -1$ because the two parts of the graph join together continuously there.

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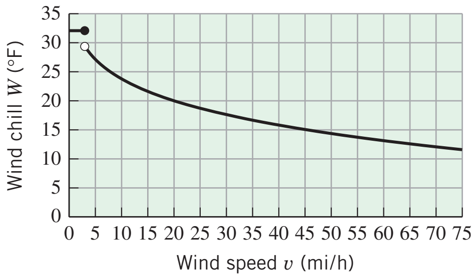
The wind chill index measures the sensation of coldness that we feel from the combined effect of temperature and wind speed.

▶ **Example 5** Increasing the speed at which air moves over a person's skin increases the rate of moisture evaporation and makes the person feel cooler. (This is why we fan ourselves in hot weather.) The ***wind chill index*** is the temperature at a wind speed of 4 mi/h that would produce the same sensation on exposed skin as the current temperature and wind speed combination. An empirical formula (i.e., a formula based on experimental data) for the wind chill index W at 32°F for a wind speed of v mi/h is

$$W = \begin{cases} 32, & 0 \le v \le 3 \\ 55.628 - 22.07v^{0.16}, & 3 < v \end{cases}$$

A computer-generated graph of $W(v)$ is shown in Figure 0.1.11. ◀



▶ **Figure 0.1.11** Wind chill versus wind speed at 32°F

DOMAIN AND RANGE

If x and y are related by the equation $y = f(x)$, then the set of all allowable inputs (x-values) is called the ***domain*** of f, and the set of outputs (y-values) that result when x varies over the domain is called the ***range*** of f. For example, if f is the function defined by the table in Example 1, then the domain is the set $\{0, 1, 2, 3\}$ and the range is the set $\{-1, 3, 4, 6\}$.

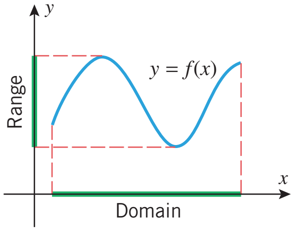
Sometimes physical or geometric considerations impose restrictions on the allowable inputs of a function. For example, if y denotes the area of a square of side x, then these variables are related by the equation $y = x^2$. Although this equation produces a unique value of y for every real number x, the fact that lengths must be nonnegative imposes the requirement that $x \geq 0$.

One might argue that a physical square cannot have a side of length zero. However, it is often convenient mathematically to allow zero lengths, and we will do so throughout this text where appropriate.

When a function is defined by a mathematical formula, the formula itself may impose restrictions on the allowable inputs. For example, if $y = 1/x$, then $x = 0$ is not an allowable input since division by zero is undefined, and if $y = \sqrt{x}$, then negative values of x are not allowable inputs because they produce imaginary values for y and we have agreed to consider only real-valued functions of a real variable. In general, we make the following definition.

0.1.5 DEFINITION If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain*** of the function.

The domain and range of a function f can be pictured by projecting the graph of $y = f(x)$ onto the coordinate axes as shown in Figure 0.1.12.



▲ **Figure 0.1.12** The projection of $y = f(x)$ on the x-axis is the set of allowable x-values for f, and the projection on the y-axis is the set of corresponding y-values.

▶ **Example 6** Find the natural domain of

(a) $f(x) = x^3$ (b) $f(x) = 1/[(x-1)(x-3)]$

(c) $f(x) = \tan x$ (d) $f(x) = \sqrt{x^2 - 5x + 6}$

Solution (a). The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

Solution (b). The function f has real values for all real x, except $x = 1$ and $x = 3$, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 \text{ and } x \neq 3\} = (-\infty, 1) \cup (1, 3) \cup (3, +\infty)$$

For a review of trigonometry see Appendix B.

Solution (c). Since $f(x) = \tan x = \sin x / \cos x$, the function f has real values except where $\cos x = 0$, and this occurs when x is an odd integer multiple of $\pi/2$. Thus, the natural domain consists of all real numbers except

$$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Solution (d). The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2 - 5x + 6 = (x-3)(x-2) \geq 0$$

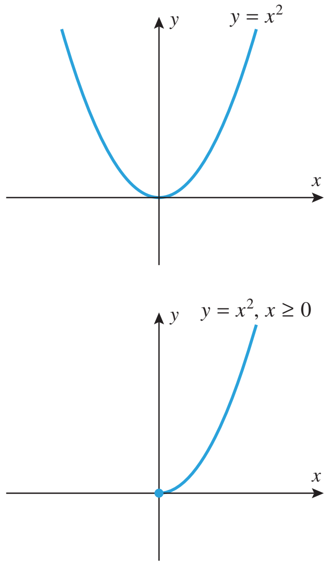
This inequality is satisfied if $x \leq 2$ or $x \geq 3$ (verify), so the natural domain of f is

$$(-\infty, 2] \cup [3, +\infty)$$ ◀

In some cases we will state the domain explicitly when defining a function. For example, if $f(x) = x^2$ is the area of a square of side x, then we can write

$$f(x) = x^2, \quad x \geq 0$$

to indicate that we take the domain of f to be the set of nonnegative real numbers (Figure 0.1.13).



▲ **Figure 0.1.13**

THE EFFECT OF ALGEBRAIC OPERATIONS ON THE DOMAIN

Algebraic expressions are frequently simplified by canceling common factors in the numerator and denominator. However, care must be exercised when simplifying formulas for functions in this way, since this process can alter the domain.

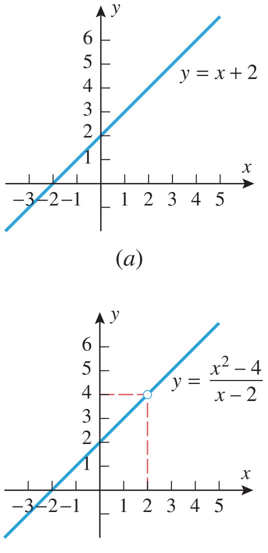
▶ **Example 7** The natural domain of the function

$$f(x) = \frac{x^2 - 4}{x - 2} \tag{2}$$

consists of all real x except $x = 2$. However, if we factor the numerator and then cancel the common factor in the numerator and denominator, we obtain

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2 \tag{3}$$

Since the right side of (3) has a value of $f(2) = 4$ and $f(2)$ was undefined in (2), the algebraic simplification has changed the function. Geometrically, the graph of (3) is the line in Figure 0.1.14*a*, whereas the graph of (2) is the same line but with a hole at $x = 2$, since the function is undefined there (Figure 0.1.14*b*). In short, the geometric effect of the algebraic cancellation is to eliminate the hole in the original graph. ◀



(*a*)

(*b*)

▲ **Figure 0.1.14**

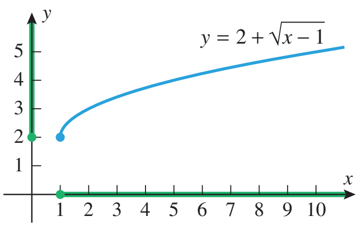
Sometimes alterations to the domain of a function that result from algebraic simplification are irrelevant to the problem at hand and can be ignored. However, if the domain must be preserved, then one must impose the restrictions on the simplified function explicitly. For example, if we wanted to preserve the domain of the function in Example 7, then we would have to express the simplified form of the function as

$$f(x) = x + 2, \quad x \neq 2$$

▶ **Example 8** Find the domain and range of

(a) $f(x) = 2 + \sqrt{x-1}$ (b) $f(x) = (x+1)/(x-1)$

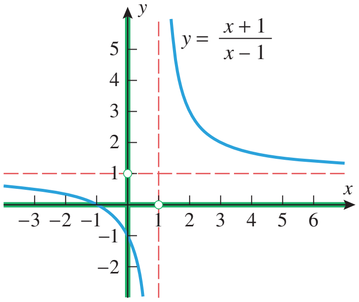
Solution (a). Since no domain is stated explicitly, the domain of f is its natural domain, $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $f(x) = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f. The domain and range are highlighted in green on the x- and y-axes in Figure 0.1.15.



▲ **Figure 0.1.15**

Solution (b). The given function f is defined for all real x, except $x = 1$, so the natural domain of f is

$$\{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty)$$



▲ **Figure 0.1.16**

To determine the range it will be convenient to introduce a dependent variable

$$y = \frac{x+1}{x-1} \tag{4}$$

Although the set of possible y-values is not immediately evident from this equation, the graph of (4), which is shown in Figure 0.1.16, suggests that the range of f consists of all y, except $y = 1$. To see that this is so, we solve (4) for x in terms of y:

$$\begin{aligned} (x-1)y &= x+1 \\ xy - y &= x+1 \\ xy - x &= y+1 \\ x(y-1) &= y+1 \\ x &= \frac{y+1}{y-1} \end{aligned}$$

It is now evident from the right side of this equation that $y = 1$ is not in the range; otherwise we would have a division by zero. No other values of y are excluded by this equation, so the range of the function f is $\{y : y \neq 1\} = (-\infty, 1) \cup (1, +\infty)$, which agrees with the result obtained graphically. ◀

DOMAIN AND RANGE IN APPLIED PROBLEMS

In applications, physical considerations often impose restrictions on the domain and range of a function.

▶ **Example 9** An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 0.1.17*a*).

(a) Let V be the volume of the box that results when the squares have sides of length x. Find a formula for V as a function of x.

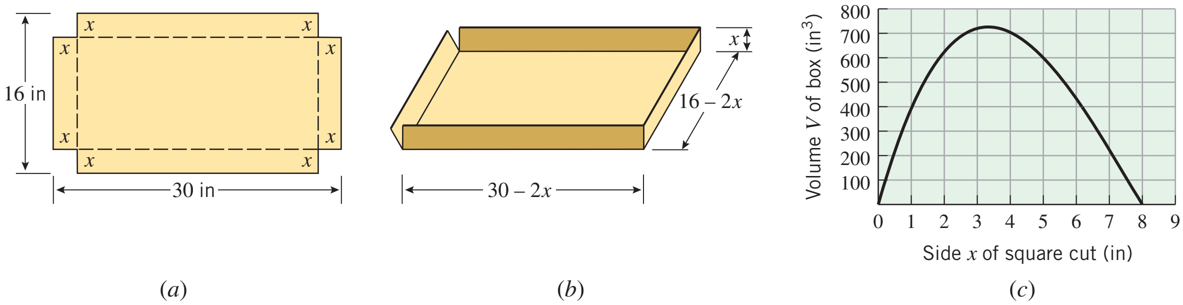
(b) Find the domain of V.

(c) Use the graph of V given in Figure 0.1.17*c* to estimate the range of V.

(d) Describe in words what the graph tells you about the volume.

Solution (a). As shown in Figure 0.1.17*b*, the resulting box has dimensions $16 - 2x$ by $30 - 2x$ by x, so the volume $V(x)$ is given by

$$V(x) = (16-2x)(30-2x)x = 480x - 92x^2 + 4x^3$$



▲ **Figure 0.1.17**

Solution (b). The domain is the set of x-values and the range is the set of V-values. Because x is a length, it must be nonnegative, and because we cannot cut out squares whose sides are more than 8 in long (why?), the x-values in the domain must satisfy

$$0 \le x \le 8$$

Solution (c). From the graph of V versus x in Figure 0.1.17c we estimate that the V-values in the range satisfy

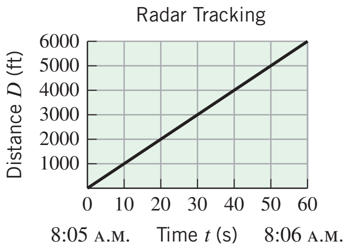
$$0 \le V \le 725$$

Note that this is an approximation. Later we will show how to find the range exactly.

Solution (d). The graph tells us that the box of maximum volume occurs for a value of x that is between 3 and 4 and that the maximum volume is approximately 725 in^3. The graph also shows that the volume decreases toward zero as x gets closer to 0 or 8, which should make sense to you intuitively. ◀

In applications involving time, formulas for functions are often expressed in terms of a variable t whose starting value is taken to be $t = 0$.

▶ **Example 10** At 8:05 A.M. a car is clocked at 100 ft/s by a radar detector that is positioned at the edge of a straight highway. Assuming that the car maintains a constant speed between 8:05 A.M. and 8:06 A.M., find a function $D(t)$ that expresses the distance traveled by the car during that time interval as a function of the time t.



▲ **Figure 0.1.18**

Solution. It would be clumsy to use the actual clock time for the variable t, so let us agree to use the *elapsed* time in seconds, starting with $t = 0$ at 8:05 A.M. and ending with $t = 60$ at 8:06 A.M. At each instant, the distance traveled (in ft) is equal to the speed of the car (in ft/s) multiplied by the elapsed time (in s). Thus,

$$D(t) = 100t, \quad 0 \le t \le 60$$

The graph of D versus t is shown in Figure 0.1.18. ◀

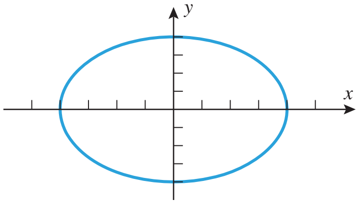
ISSUES OF SCALE AND UNITS

In geometric problems where you want to preserve the "true" shape of a graph, you must use units of equal length on both axes. For example, if you graph a circle in a coordinate system in which 1 unit in the y-direction is smaller than 1 unit in the x-direction, then the circle will be squashed vertically into an elliptical shape (Figure 0.1.19).

However, sometimes it is inconvenient or impossible to display a graph using units of equal length. For example, consider the equation

$$y = x^2$$

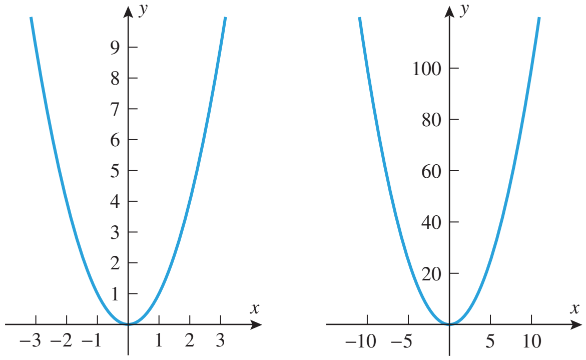
If we want to show the portion of the graph over the interval $-3 \le x \le 3$, then there is no problem using units of equal length, since y only varies from 0 to 9 over that interval. However, if we want to show the portion of the graph over the interval $-10 \le x \le 10$, then there is a problem keeping the units equal in length, since the value of y varies between 0 and 100. In this case the only reasonable way to show all of the graph that occurs over the interval $-10 \le x \le 10$ is to compress the unit of length along the y-axis, as illustrated in Figure 0.1.20.



The circle is squashed because 1 unit on the y-axis has a smaller length than 1 unit on the x-axis.

▲ **Figure 0.1.19**

In applications where the variables on the two axes have unrelated units (say, centimeters on the y-axis and seconds on the x-axis), then nothing is gained by requiring the units to have equal lengths; choose the lengths to make the graph as clear as possible.



▶ **Figure 0.1.20**

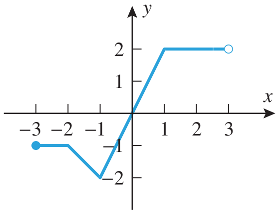
✔ QUICK CHECK EXERCISES 0.1 *(See page 15 for answers.)*

1. Let $f(x) = \sqrt{x+1} + 4$.
   (a) The natural domain of f is ________.
   (b) $f(3) =$ ________
   (c) $f(t^2 - 1) =$ ________
   (d) $f(x) = 7$ if $x =$ ________
   (e) The range of f is ________.

2. Line segments in an xy-plane form "letters" as depicted.

   (a) If the y-axis is parallel to the letter I, which of the letters represent the graph of $y = f(x)$ for some function f?
   (b) If the y-axis is perpendicular to the letter I, which of the letters represent the graph of $y = f(x)$ for some function f?

3. The accompanying figure shows the complete graph of $y = f(x)$.
   (a) The domain of f is ________.
   (b) The range of f is ________.
   (c) $f(-3) =$ ________
   (d) $f\left(\frac{1}{2}\right) =$ ________
   (e) The solutions to $f(x) = -\frac{3}{2}$ are $x =$ ________ and $x =$ ________.



◀ **Figure Ex-3**

4. The accompanying table gives a 5-day forecast of high and low temperatures in degrees Fahrenheit (°F).
   (a) Suppose that x and y denote the respective high and low temperature predictions for each of the 5 days. Is y a function of x? If so, give the domain and range of this function.
   (b) Suppose that x and y denote the respective low and high temperature predictions for each of the 5 days. Is y a function of x? If so, give the domain and range of this function.

	MON	TUE	WED	THURS	FRI
HIGH	75	71	65	70	73
LOW	52	56	48	50	52

▲ **Table Ex-4**

5. Let l, w, and A denote the length, width, and area of a rectangle, respectively, and suppose that the width of the rectangle is half the length.
   (a) If l is expressed as a function of w, then $l =$ ________.
   (b) If A is expressed as a function of l, then $A =$ ________.
   (c) If w is expressed as a function of A, then $w =$ ________.

EXERCISE SET 0.1

Graphing Utility

1. Use the accompanying graph to answer the following questions, making reasonable approximations where needed.

(a) For what values of x is $y = 1$?
(b) For what values of x is $y = 3$?
(c) For what values of y is $x = 3$?
(d) For what values of x is $y \le 0$?
(e) What are the maximum and minimum values of y and for what values of x do they occur?

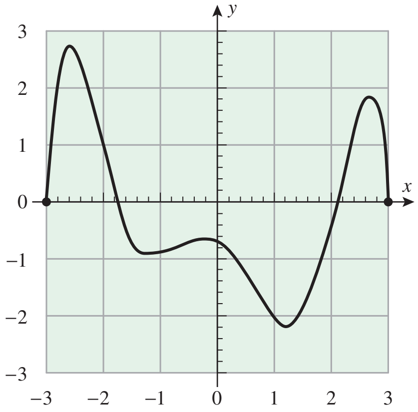


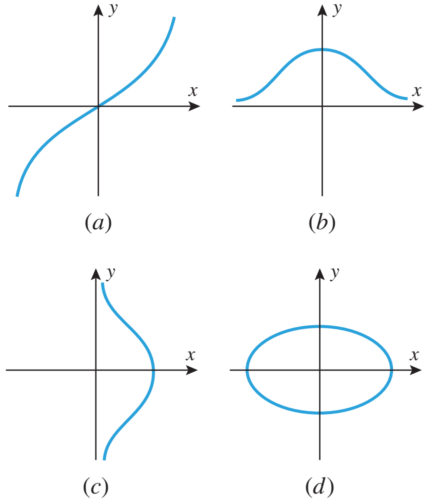
Figure Ex-1

2. Use the accompanying table to answer the questions posed in Exercise 1.

x	−2	−1	0	2	3	4	5	6
y	5	1	−2	7	−1	1	0	9

Table Ex-2

3. In each part of the accompanying figure, determine whether the graph defines y as a function of x.



(a) (b)

(c) (d)

Figure Ex-3

4. In each part, compare the natural domains of f and g.

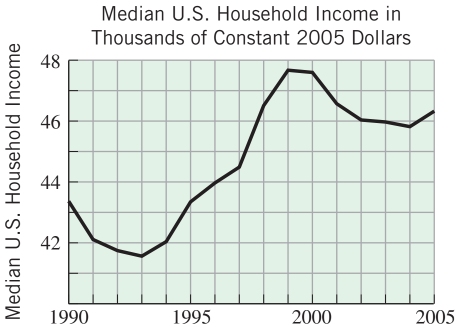
(a) $f(x) = \dfrac{x^2 + x}{x + 1}$; $g(x) = x$

(b) $f(x) = \dfrac{x\sqrt{x} + \sqrt{x}}{x + 1}$; $g(x) = \sqrt{x}$

FOCUS ON CONCEPTS

5. The accompanying graph shows the median income in U.S. households (adjusted for inflation) between 1990 and 2005. Use the graph to answer the following questions, making reasonable approximations where needed.

(a) When was the median income at its maximum value, and what was the median income when that occurred?
(b) When was the median income at its minimum value, and what was the median income when that occurred?
(c) The median income was declining during the 2-year period between 2000 and 2002. Was it declining more rapidly during the first year or the second year of that period? Explain your reasoning.



Source: U.S. Census Bureau, August 2006.

Figure Ex-5

6. Use the median income graph in Exercise 5 to answer the following questions, making reasonable approximations where needed.

(a) What was the average yearly growth of median income between 1993 and 1999?
(b) The median income was increasing during the 6-year period between 1993 and 1999. Was it increasing more rapidly during the first 3 years or the last 3 years of that period? Explain your reasoning.
(c) Consider the statement: "After years of decline, median income this year was finally higher than that of last year." In what years would this statement have been correct?

7. Find $f(0)$, $f(2)$, $f(-2)$, $f(3)$, $f(\sqrt{2})$, and $f(3t)$.

(a) $f(x) = 3x^2 - 2$ (b) $f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$

8. Find $g(3)$, $g(-1)$, $g(\pi)$, $g(-1.1)$, and $g(t^2 - 1)$.

(a) $g(x) = \dfrac{x+1}{x-1}$ (b) $g(x) = \begin{cases} \sqrt{x+1}, & x \ge 1 \\ 3, & x < 1 \end{cases}$

9–10 Find the natural domain and determine the range of each function. If you have a graphing utility, use it to confirm that your result is consistent with the graph produced by your graphing utility. [*Note:* Set your graphing utility in radian mode when graphing trigonometric functions.]

9. (a) $f(x) = \dfrac{1}{x-3}$ (b) $F(x) = \dfrac{x}{|x|}$

(c) $g(x) = \sqrt{x^2 - 3}$ (d) $G(x) = \sqrt{x^2 - 2x + 5}$

(e) $h(x) = \dfrac{1}{1 - \sin x}$ (f) $H(x) = \sqrt{\dfrac{x^2 - 4}{x - 2}}$

10. (a) $f(x) = \sqrt{3 - x}$ (b) $F(x) = \sqrt{4 - x^2}$
(c) $g(x) = 3 + \sqrt{x}$ (d) $G(x) = x^3 + 2$
(e) $h(x) = 3 \sin x$ (f) $H(x) = (\sin \sqrt{x})^{-2}$

FOCUS ON CONCEPTS

11. (a) If you had a device that could record the Earth's population continuously, would you expect the graph of population versus time to be a continuous (unbroken) curve? Explain what might cause breaks in the curve.

(b) Suppose that a hospital patient receives an injection of an antibiotic every 8 hours and that between injections the concentration C of the antibiotic in the bloodstream decreases as the antibiotic is absorbed by the tissues. What might the graph of C versus the elapsed time t look like?

12. (a) If you had a device that could record the temperature of a room continuously over a 24-hour period, would you expect the graph of temperature versus time to be a continuous (unbroken) curve? Explain your reasoning.

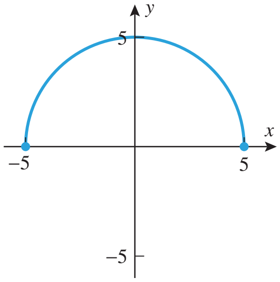
(b) If you had a computer that could track the number of boxes of cereal on the shelf of a market continuously over a 1-week period, would you expect the graph of the number of boxes on the shelf versus time to be a continuous (unbroken) curve? Explain your reasoning.

13. A boat is bobbing up and down on some gentle waves. Suddenly it gets hit by a large wave and sinks. Sketch a rough graph of the height of the boat above the ocean floor as a function of time.

14. A cup of hot coffee sits on a table. You pour in some cool milk and let it sit for an hour. Sketch a rough graph of the temperature of the coffee as a function of time.

15–18 As seen in Example 3, the equation $x^2 + y^2 = 25$ does not define y as a function of x. Each graph in these exercises is a portion of the circle $x^2 + y^2 = 25$. In each case, determine whether the graph defines y as a function of x, and if so, give a formula for y in terms of x.

15.



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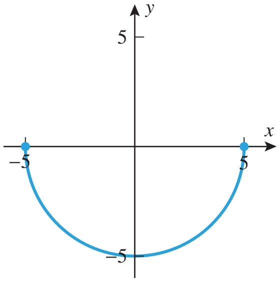
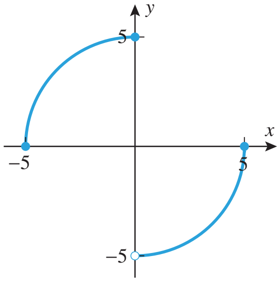
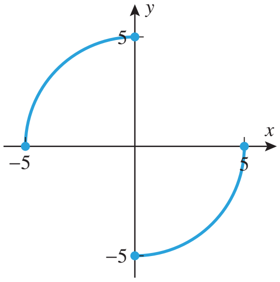


17.



18.



19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. A curve that crosses the x-axis at two different points cannot be the graph of a function.

20. The natural domain of a real-valued function defined by a formula consists of all those real numbers for which the formula yields a real value.

21. The range of the absolute value function is all positive real numbers.

22. If $g(x) = 1/\sqrt{f(x)}$, then the domain of g consists of all those real numbers x for which $f(x) \ne 0$.

23. Use the equation $y = x^2 - 6x + 8$ to answer the following questions.
(a) For what values of x is $y = 0$?
(b) For what values of x is $y = -10$?
(c) For what values of x is $y \ge 0$?
(d) Does y have a minimum value? A maximum value? If so, find them.

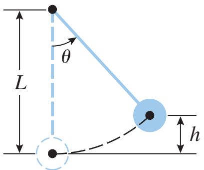
24. Use the equation $y = 1 + \sqrt{x}$ to answer the following questions.
(a) For what values of x is $y = 4$?
(b) For what values of x is $y = 0$?
(c) For what values of x is $y \ge 6$?

(*cont.*)

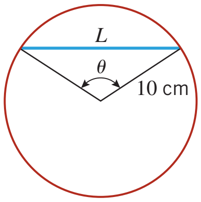
(d) Does y have a minimum value? A maximum value? If so, find them.

25. As shown in the accompanying figure, a pendulum of constant length L makes an angle θ with its vertical position. Express the height h as a function of the angle θ.

26. Express the length L of a chord of a circle with radius 10 cm as a function of the central angle θ (see the accompanying figure).



▲ **Figure Ex-25**



▲ **Figure Ex-26**

27–28 Express the function in piecewise form without using absolute values. [*Suggestion:* It may help to generate the graph of the function.] ■

27. (a) $f(x) = |x| + 3x + 1$ (b) $g(x) = |x| + |x - 1|$

28. (a) $f(x) = 3 + |2x - 5|$ (b) $g(x) = 3|x - 2| - |x + 1|$

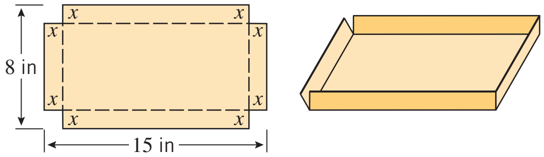
29. As shown in the accompanying figure, an open box is to be constructed from a rectangular sheet of metal, 8 in by 15 in, by cutting out squares with sides of length x from each corner and bending up the sides.

(a) Express the volume V as a function of x.

(b) Find the domain of V.

(c) Plot the graph of the function V obtained in part (a) and estimate the range of this function.

(d) In words, describe how the volume V varies with x, and discuss how one might construct boxes of maximum volume.



▲ **Figure Ex-29**

30. Repeat Exercise 29 assuming the box is constructed in the same fashion from a 6-inch-square sheet of metal.

31. A construction company has adjoined a 1000 ft^2 rectangular enclosure to its office building. Three sides of the enclosure are fenced in. The side of the building adjacent to the enclosure is 100 ft long and a portion of this side is used as the fourth side of the enclosure. Let x and y be the dimensions of the enclosure, where x is measured parallel to the building, and let L be the length of fencing required for those dimensions.

(a) Find a formula for L in terms of x and y.

(b) Find a formula that expresses L as a function of x alone.

(c) What is the domain of the function in part (b)?

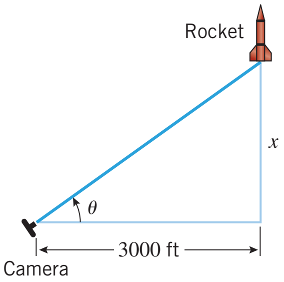
(d) Plot the function in part (b) and estimate the dimensions of the enclosure that minimize the amount of fencing required.

32. As shown in the accompanying figure, a camera is mounted at a point 3000 ft from the base of a rocket launching pad. The rocket rises vertically when launched, and the camera's elevation angle is continually adjusted to follow the bottom of the rocket.

(a) Express the height x as a function of the elevation angle θ.

(b) Find the domain of the function in part (a).

(c) Plot the graph of the function in part (a) and use it to estimate the height of the rocket when the elevation angle is $\pi/4 \approx 0.7854$ radian. Compare this estimate to the exact height.



◀ **Figure Ex-32**

33. A soup company wants to manufacture a can in the shape of a right circular cylinder that will hold 500 cm^3 of liquid. The material for the top and bottom costs 0.02 cent/cm^2, and the material for the sides costs 0.01 cent/cm^2.

(a) Estimate the radius r and the height h of the can that costs the least to manufacture. [*Suggestion:* Express the cost C in terms of r.]

(b) Suppose that the tops and bottoms of radius r are punched out from square sheets with sides of length $2r$ and the scraps are waste. If you allow for the cost of the waste, would you expect the can of least cost to be taller or shorter than the one in part (a)? Explain.

(c) Estimate the radius, height, and cost of the can in part (b), and determine whether your conjecture was correct.

34. The designer of a sports facility wants to put a quarter-mile (1320 ft) running track around a football field, oriented as in the accompanying figure on the next page. The football field is 360 ft long (including the end zones) and 160 ft wide. The track consists of two straightaways and two semicircles, with the straightaways extending at least the length of the football field.

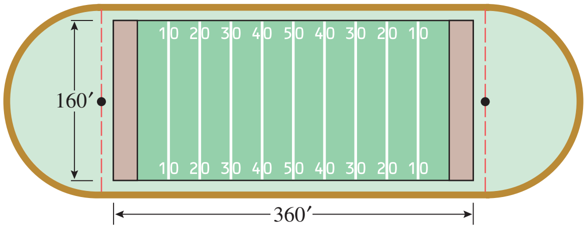
(a) Show that it is possible to construct a quarter-mile track around the football field. [*Suggestion:* Find the shortest track that can be constructed around the field.]

(b) Let L be the length of a straightaway (in feet), and let x be the distance (in feet) between a sideline of the football field and a straightaway. Make a graph of L versus x.

(*cont.*)

(c) Use the graph to estimate the value of x that produces the shortest straightaways, and then find this value of x exactly.

(d) Use the graph to estimate the length of the longest possible straightaways, and then find that length exactly.



▲ **Figure Ex-34**

35–36 (i) Explain why the function f has one or more holes in its graph, and state the x-values at which those holes occur. (ii) Find a function g whose graph is identical to that of f, but without the holes. ■

35. $f(x) = \dfrac{(x+2)(x^2-1)}{(x+2)(x-1)}$ **36.** $f(x) = \dfrac{x^2+|x|}{|x|}$

37. In 2001 the National Weather Service introduced a new wind chill temperature (WCT) index. For a given outside temperature T and wind speed v, the wind chill temperature index is the equivalent temperature that exposed skin would feel with a wind speed of v mi/h. Based on a more accurate model of cooling due to wind, the new formula is

$$\text{WCT} = \begin{cases} T, & 0 \le v \le 3 \\ 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}, & 3 < v \end{cases}$$

where T is the temperature in °F, v is the wind speed in mi/h, and WCT is the equivalent temperature in °F. Find the WCT to the nearest degree if $T = 25$°F and
(a) $v = 3$ mi/h (b) $v = 15$ mi/h (c) $v = 46$ mi/h.

Source: Adapted from UMAP Module 658, *Windchill*, W. Bosch and L. Cobb, COMAP, Arlington, MA.

38–40 Use the formula for the wind chill temperature index described in Exercise 37. ■

38. Find the air temperature to the nearest degree if the WCT is reported as −60°F with a wind speed of 48 mi/h.

39. Find the air temperature to the nearest degree if the WCT is reported as −10°F with a wind speed of 48 mi/h.

40. Find the wind speed to the nearest mile per hour if the WCT is reported as 5°F with an air temperature of 20°F.

✔ QUICK CHECK ANSWERS 0.1

1. (a) $[-1, +\infty)$ (b) 6 (c) $|t| + 4$ (d) 8 (e) $[4, +\infty)$ **2.** (a) M (b) I **3.** (a) $[-3, 3)$ (b) $[-2, 2]$ (c) -1 (d) 1 (e) $-\frac{3}{4}$; $-\frac{3}{2}$ **4.** (a) yes; domain: $\{65, 70, 71, 73, 75\}$; range: $\{48, 50, 52, 56\}$ (b) no **5.** (a) $l = 2w$ (b) $A = l^2/2$ (c) $w = \sqrt{A/2}$

0.2 NEW FUNCTIONS FROM OLD

Just as numbers can be added, subtracted, multiplied, and divided to produce other numbers, so functions can be added, subtracted, multiplied, and divided to produce other functions. In this section we will discuss these operations and some others that have no analogs in ordinary arithmetic.

■ ARITHMETIC OPERATIONS ON FUNCTIONS

Two functions, f and g, can be added, subtracted, multiplied, and divided in a natural way to form new functions $f + g$, $f - g$, fg, and f/g. For example, $f + g$ is defined by the formula

$$(f + g)(x) = f(x) + g(x) \tag{1}$$

which states that for each input the value of $f + g$ is obtained by adding the values of f and g. Equation (1) provides a formula for $f + g$ but does not say anything about the domain of $f + g$. However, for the right side of this equation to be defined, x must lie in the domains of both f and g, so we define the domain of $f + g$ to be the intersection of these two domains. More generally, we make the following definition.

If f is a constant function, that is, $f(x) = c$ for all x, then the product of f and g is cg, so multiplying a function by a constant is a special case of multiplying two functions.

0.2.1 DEFINITION Given functions f and g, we define

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$(f/g)(x) = f(x)/g(x)$$

For the functions $f+g$, $f-g$, and fg we define the domain to be the intersection of the domains of f and g, and for the function f/g we define the domain to be the intersection of the domains of f and g but with the points where $g(x) = 0$ excluded (to avoid division by zero).

▶ Example 1 Let

$$f(x) = 1 + \sqrt{x-2} \quad \text{and} \quad g(x) = x - 3$$

Find the domains and formulas for the functions $f+g$, $f-g$, fg, f/g, and $7f$.

Solution. First, we will find the formulas and then the domains. The formulas are

$$(f+g)(x) = f(x) + g(x) = (1 + \sqrt{x-2}\,) + (x-3) = x - 2 + \sqrt{x-2} \tag{2}$$
$$(f-g)(x) = f(x) - g(x) = (1 + \sqrt{x-2}\,) - (x-3) = 4 - x + \sqrt{x-2} \tag{3}$$
$$(fg)(x) = f(x)g(x) = (1 + \sqrt{x-2}\,)(x-3) \tag{4}$$
$$(f/g)(x) = f(x)/g(x) = \frac{1 + \sqrt{x-2}}{x-3} \tag{5}$$
$$(7f)(x) = 7f(x) = 7 + 7\sqrt{x-2} \tag{6}$$

The domains of f and g are $[2, +\infty)$ and $(-\infty, +\infty)$, respectively (their natural domains). Thus, it follows from Definition 0.2.1 that the domains of $f+g$, $f-g$, and fg are the intersection of these two domains, namely,

$$[2, +\infty) \cap (-\infty, +\infty) = [2, +\infty) \tag{7}$$

Moreover, since $g(x) = 0$ if $x = 3$, the domain of f/g is (7) with $x = 3$ removed, namely,

$$[2, 3) \cup (3, +\infty)$$

Finally, the domain of $7f$ is the same as the domain of f. ◀

We saw in the last example that the domains of the functions $f+g$, $f-g$, fg, and f/g were the natural domains resulting from the formulas obtained for these functions. The following example shows that this will not always be the case.

▶ Example 2 Show that if $f(x) = \sqrt{x}$, $g(x) = \sqrt{x}$, and $h(x) = x$, then the domain of fg is not the same as the natural domain of h.

Solution. The natural domain of $h(x) = x$ is $(-\infty, +\infty)$. Note that

$$(fg)(x) = \sqrt{x}\sqrt{x} = x = h(x)$$

on the domain of fg. The domains of both f and g are $[0, +\infty)$, so the domain of fg is

$$[0, +\infty) \cap [0, +\infty) = [0, +\infty)$$

by Definition 0.2.1. Since the domains of fg and h are different, it would be misleading to write $(fg)(x) = x$ without including the restriction that this formula holds only for $x \geq 0$. ◀

COMPOSITION OF FUNCTIONS

We now consider an operation on functions, called *composition*, which has no direct analog in ordinary arithmetic. Informally stated, the operation of composition is performed by substituting some function for the independent variable of another function. For example, suppose that

$$f(x) = x^2 \quad \text{and} \quad g(x) = x + 1$$

If we substitute $g(x)$ for x in the formula for f, we obtain a new function

$$f(g(x)) = (g(x))^2 = (x + 1)^2$$

which we denote by $f \circ g$. Thus,

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x + 1)^2$$

In general, we make the following definition.

Although the domain of $f \circ g$ may seem complicated at first glance, it makes sense intuitively: To compute $f(g(x))$ one needs x in the domain of g to compute $g(x)$, and one needs $g(x)$ in the domain of f to compute $f(g(x))$.

0.2.2 DEFINITION Given functions f and g, the ***composition*** of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g for which $g(x)$ is in the domain of f.

▶ **Example 3** Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

Solution (a). The formula for $f(g(x))$ is

$$f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x}\,)^2 + 3 = x + 3$$

Since the domain of g is $[0, +\infty)$ and the domain of f is $(-\infty, +\infty)$, the domain of $f \circ g$ consists of all x in $[0, +\infty)$ such that $g(x) = \sqrt{x}$ lies in $(-\infty, +\infty)$; thus, the domain of $f \circ g$ is $[0, +\infty)$. Therefore,

$$(f \circ g)(x) = x + 3, \quad x \geq 0$$

Solution (b). The formula for $g(f(x))$ is

$$g(f(x)) = \sqrt{f(x)} = \sqrt{x^2 + 3}$$

Since the domain of f is $(-\infty, +\infty)$ and the domain of g is $[0, +\infty)$, the domain of $g \circ f$ consists of all x in $(-\infty, +\infty)$ such that $f(x) = x^2 + 3$ lies in $[0, +\infty)$. Thus, the domain of $g \circ f$ is $(-\infty, +\infty)$. Therefore,

$$(g \circ f)(x) = \sqrt{x^2 + 3}$$

There is no need to indicate that the domain is $(-\infty, +\infty)$, since this is the natural domain of $\sqrt{x^2 + 3}$. ◀

Note that the functions $f \circ g$ and $g \circ f$ in Example 3 are not the same. Thus, the order in which functions are composed can (and usually will) make a difference in the end result.

Compositions can also be defined for three or more functions; for example, $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

In other words, first find $h(x)$, then find $g(h(x))$, and then find $f(g(h(x)))$.

▸ **Example 4** Find $(f \circ g \circ h)(x)$ if

$$f(x) = \sqrt{x}, \quad g(x) = 1/x, \quad h(x) = x^3$$

Solution.

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3)) = f(1/x^3) = \sqrt{1/x^3} = 1/x^{3/2} \blacktriangleleft$$

EXPRESSING A FUNCTION AS A COMPOSITION

Many problems in mathematics are solved by "decomposing" functions into compositions of simpler functions. For example, consider the function h given by

$$h(x) = (x + 1)^2$$

To evaluate $h(x)$ for a given value of x, we would first compute $x + 1$ and then square the result. These two operations are performed by the functions

$$g(x) = x + 1 \quad \text{and} \quad f(x) = x^2$$

We can express h in terms of f and g by writing

$$h(x) = (x + 1)^2 = [g(x)]^2 = f(g(x))$$

so we have succeeded in expressing h as the composition $h = f \circ g$.

The thought process in this example suggests a general procedure for decomposing a function h into a composition $h = f \circ g$:

- Think about how you would evaluate $h(x)$ for a specific value of x, trying to break the evaluation into two steps performed in succession.
- The first operation in the evaluation will determine a function g and the second a function f.
- The formula for h can then be written as $h(x) = f(g(x))$.

For descriptive purposes, we will refer to g as the "inside function" and f as the "outside function" in the expression $f(g(x))$. The inside function performs the first operation and the outside function performs the second.

▸ **Example 5** Express $\sin(x^3)$ as a composition of two functions.

Solution. To evaluate $\sin(x^3)$, we would first compute x^3 and then take the sine, so $g(x) = x^3$ is the inside function and $f(x) = \sin x$ the outside function. Therefore,

$$\sin(x^3) = f(g(x)) \qquad \boxed{g(x) = x^3 \text{ and } f(x) = \sin x} \quad \blacktriangleleft$$

Table 0.2.1 gives some more examples of decomposing functions into compositions.

Table 0.2.1

COMPOSING FUNCTIONS

FUNCTION	$g(x)$ INSIDE	$f(x)$ OUTSIDE	COMPOSITION
$(x^2+1)^{10}$	x^2+1	x^{10}	$(x^2+1)^{10} = f(g(x))$
$\sin^3 x$	$\sin x$	x^3	$\sin^3 x = f(g(x))$
$\tan (x^5)$	x^5	$\tan x$	$\tan (x^5) = f(g(x))$
$\sqrt{4-3x}$	$4-3x$	$\sqrt{x}$	$\sqrt{4-3x} = f(g(x))$
$8+\sqrt{x}$	$\sqrt{x}$	$8+x$	$8+\sqrt{x} = f(g(x))$
$\dfrac{1}{x+1}$	$x+1$	$\dfrac{1}{x}$	$\dfrac{1}{x+1} = f(g(x))$

REMARK There is always more than one way to express a function as a composition. For example, here are two ways to express $(x^2+1)^{10}$ as a composition that differ from that in Table 0.2.1:

$$(x^2+1)^{10} = [(x^2+1)^2]^5 = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^2 \text{ and } f(x) = x^5}$$

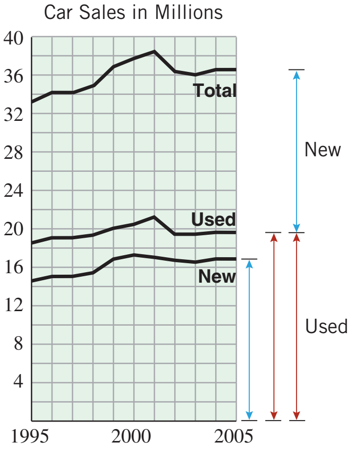
$$(x^2+1)^{10} = [(x^2+1)^3]^{10/3} = f(g(x)) \qquad \boxed{g(x) = (x^2+1)^3 \text{ and } f(x) = x^{10/3}}$$

NEW FUNCTIONS FROM OLD

The remainder of this section will be devoted to considering the geometric effect of performing basic operations on functions. This will enable us to use known graphs of functions to visualize or sketch graphs of related functions. For example, Figure 0.2.1 shows the graphs of yearly new car sales $N(t)$ and used car sales $U(t)$ over a certain time period. Those graphs can be used to construct the graph of the total car sales

$$T(t) = N(t) + U(t)$$

by adding the values of $N(t)$ and $U(t)$ for each value of t. In general, the graph of $y = f(x) + g(x)$ can be constructed from the graphs of $y = f(x)$ and $y = g(x)$ by adding corresponding y-values for each x.



Source: NADA.

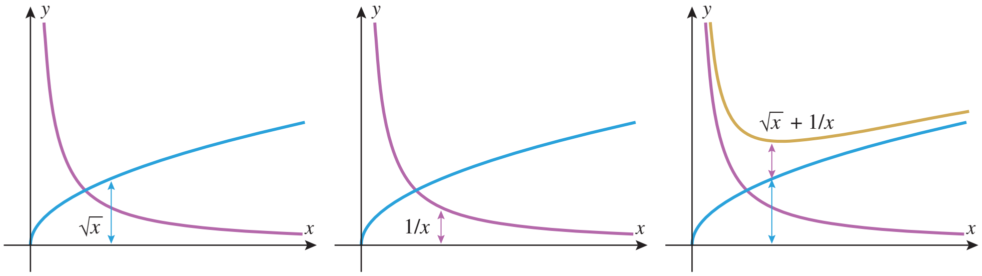
▲ **Figure 0.2.1**

▶ **Example 6** Referring to Figure 0.1.4 for the graphs of $y = \sqrt{x}$ and $y = 1/x$, make a sketch that shows the general shape of the graph of $y = \sqrt{x} + 1/x$ for $x \geq 0$.

Solution. To add the corresponding y-values of $y = \sqrt{x}$ and $y = 1/x$ graphically, just imagine them to be "stacked" on top of one another. This yields the sketch in Figure 0.2.2. ◀

Use the technique in Example 6 to sketch the graph of the function

$$\sqrt{x} - \frac{1}{x}$$



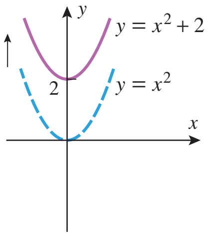
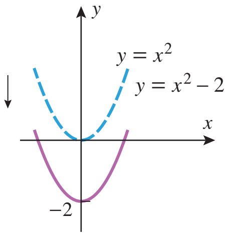
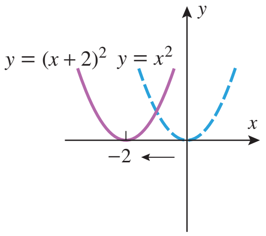
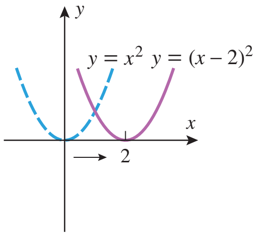
▶ **Figure 0.2.2**

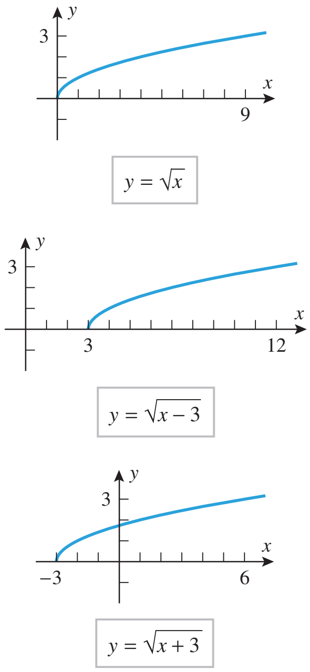
Add the y-coordinates of $\sqrt{x}$ and $1/x$ to obtain the y-coordinate of $\sqrt{x} + 1/x$.

TRANSLATIONS

Table 0.2.2 illustrates the geometric effect on the graph of $y = f(x)$ of adding or subtracting a *positive* constant c to f or to its independent variable x. For example, the first result in the table illustrates that adding a positive constant c to a function f adds c to each y-coordinate of its graph, thereby shifting the graph of f up by c units. Similarly, subtracting c from f shifts the graph down by c units. On the other hand, if a positive constant c is added to x, then the value of $y = f(x + c)$ at $x - c$ is $f(x)$; and since the point $x - c$ is c units to the left of x on the x-axis, the graph of $y = f(x + c)$ must be the graph of $y = f(x)$ shifted left by c units. Similarly, subtracting c from x shifts the graph of $y = f(x)$ right by c units.

Table 0.2.2
TRANSLATION PRINCIPLES

OPERATION ON $y = f(x)$	Add a positive constant c to $f(x)$	Subtract a positive constant c from $f(x)$	Add a positive constant c to x	Subtract a positive constant c from x
NEW EQUATION	$y = f(x) + c$	$y = f(x) - c$	$y = f(x + c)$	$y = f(x - c)$
GEOMETRIC EFFECT	Translates the graph of $y = f(x)$ up c units	Translates the graph of $y = f(x)$ down c units	Translates the graph of $y = f(x)$ left c units	Translates the graph of $y = f(x)$ right c units
EXAMPLE				



▲ **Figure 0.2.3**

Before proceeding to the next examples, it will be helpful to review the graphs in Figures 0.1.4 and 0.1.9.

▶ **Example 7** Sketch the graph of

$$\text{(a)}\ \ y = \sqrt{x - 3} \qquad \text{(b)}\ \ y = \sqrt{x + 3}$$

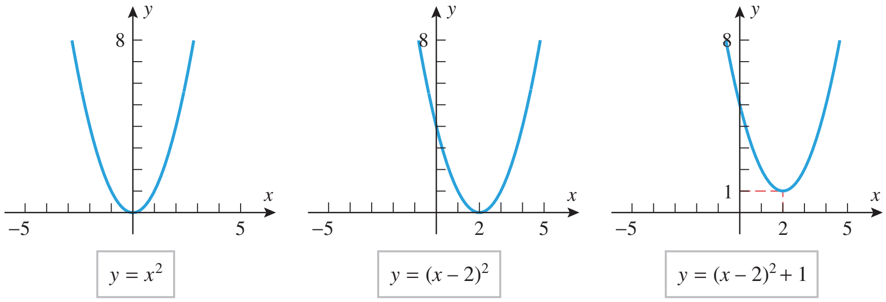
Solution. Using the translation principles given in Table 0.2.2, the graph of the equation $y = \sqrt{x - 3}$ can be obtained by translating the graph of $y = \sqrt{x}$ right 3 units. The graph of $y = \sqrt{x + 3}$ can be obtained by translating the graph of $y = \sqrt{x}$ left 3 units (Figure 0.2.3). ◀

▶ **Example 8** Sketch the graph of $y = x^2 - 4x + 5$.

Solution. Completing the square on the first two terms yields

$$y = (x^2 - 4x + 4) - 4 + 5 = (x - 2)^2 + 1$$

(see Web Appendix H for a review of this technique). In this form we see that the graph can be obtained by translating the graph of $y = x^2$ right 2 units because of the $x - 2$, and up 1 unit because of the $+1$ (Figure 0.2.4). ◀



$y = x^2$ $y = (x - 2)^2$ $y = (x - 2)^2 + 1$

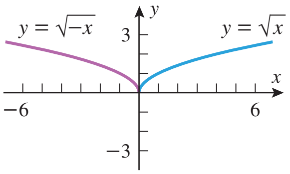
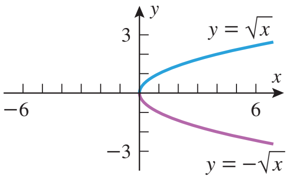
▶ **Figure 0.2.4**

REFLECTIONS

The graph of $y = f(-x)$ is the reflection of the graph of $y = f(x)$ about the y-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(-x, y)$. Similarly, the graph of $y = -f(x)$ is the reflection of the graph of $y = f(x)$ about the x-axis because the point (x, y) on the graph of $f(x)$ is replaced by $(x, -y)$ [the equation $y = -f(x)$ is equivalent to $-y = f(x)$]. This is summarized in Table 0.2.3.

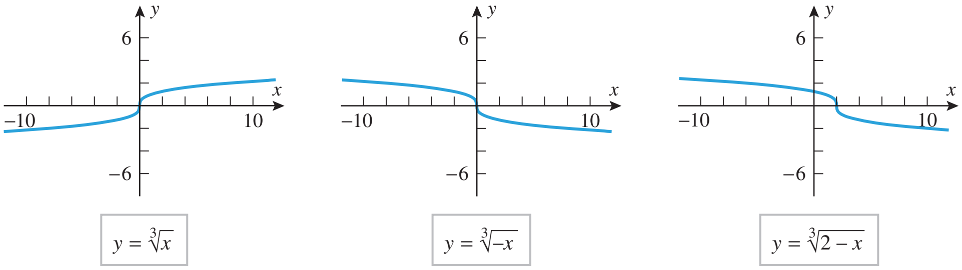
Table 0.2.3

REFLECTION PRINCIPLES

OPERATION ON $y = f(x)$	Replace x by $-x$	Multiply $f(x)$ by -1
NEW EQUATION	$y = f(-x)$	$y = -f(x)$
GEOMETRIC EFFECT	Reflects the graph of $y = f(x)$ about the y-axis	Reflects the graph of $y = f(x)$ about the x-axis
EXAMPLE		

▶ **Example 9** Sketch the graph of $y = \sqrt[3]{2 - x}$.

Solution. Using the translation and reflection principles in Tables 0.2.2 and 0.2.3, we can obtain the graph by a reflection followed by a translation as follows: First reflect the graph of $y = \sqrt[3]{x}$ about the y-axis to obtain the graph of $y = \sqrt[3]{-x}$, then translate this graph right 2 units to obtain the graph of the equation $y = \sqrt[3]{-(x - 2)} = \sqrt[3]{2 - x}$ (Figure 0.2.5). ◀

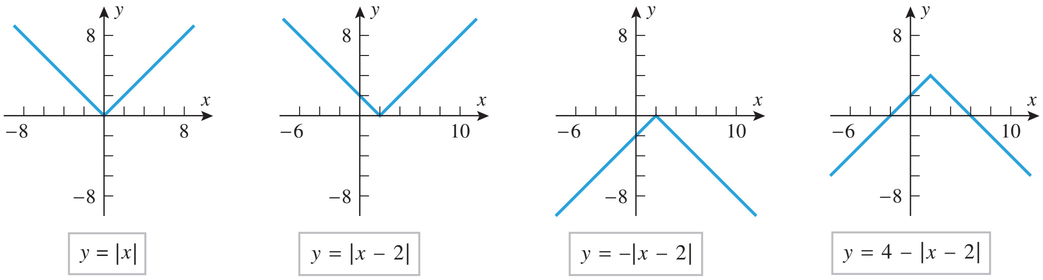


$y = \sqrt[3]{x}$ $y = \sqrt[3]{-x}$ $y = \sqrt[3]{2 - x}$

▶ **Figure 0.2.5**

Example 10 Sketch the graph of $y = 4 - |x - 2|$.

Solution. The graph can be obtained by a reflection and two translations: First translate the graph of $y = |x|$ right 2 units to obtain the graph of $y = |x - 2|$; then reflect this graph about the x-axis to obtain the graph of $y = -|x - 2|$; and then translate this graph up 4 units to obtain the graph of the equation $y = -|x - 2| + 4 = 4 - |x - 2|$ (Figure 0.2.6). ◀



$y = |x|$ $y = |x - 2|$ $y = -|x - 2|$ $y = 4 - |x - 2|$

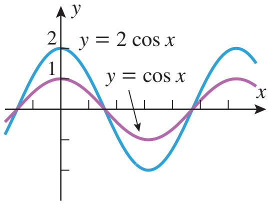
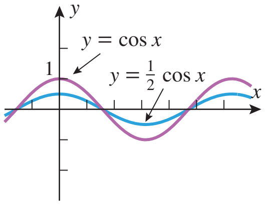
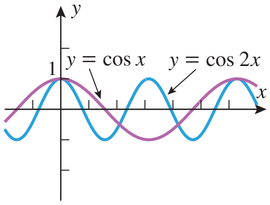
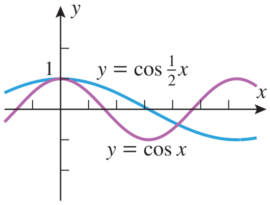
Figure 0.2.6

STRETCHES AND COMPRESSIONS

Multiplying $f(x)$ by a *positive* constant c has the geometric effect of stretching the graph of $y = f(x)$ in the y-direction by a factor of c if $c > 1$ and compressing it in the y-direction by a factor of $1/c$ if $0 < c < 1$. For example, multiplying $f(x)$ by 2 doubles each y-coordinate, thereby stretching the graph vertically by a factor of 2, and multiplying by $\frac{1}{2}$ cuts each y-coordinate in half, thereby compressing the graph vertically by a factor of 2. Similarly, multiplying x by a *positive* constant c has the geometric effect of compressing the graph of $y = f(x)$ by a factor of c in the x-direction if $c > 1$ and stretching it by a factor of $1/c$ if $0 < c < 1$. [If this seems backwards to you, then think of it this way: The value of $2x$ changes twice as fast as x, so a point moving along the x-axis from the origin will only have to move half as far for $y = f(2x)$ to have the same value as $y = f(x)$, thereby creating a horizontal compression of the graph.] All of this is summarized in Table 0.2.4.

Describe the geometric effect of multiplying a function f by a *negative* constant in terms of reflection and stretching or compressing. What is the geometric effect of multiplying the independent variable of a function f by a *negative* constant?

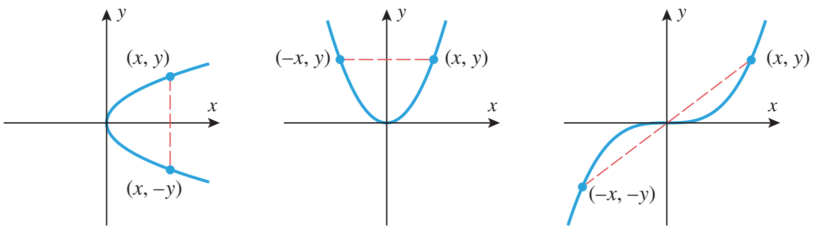
Table 0.2.4
STRETCHING AND COMPRESSING PRINCIPLES

OPERATION ON $y = f(x)$	Multiply $f(x)$ by c $(c > 1)$	Multiply $f(x)$ by c $(0 < c < 1)$	Multiply x by c $(c > 1)$	Multiply x by c $(0 < c < 1)$
NEW EQUATION	$y = cf(x)$	$y = cf(x)$	$y = f(cx)$	$y = f(cx)$
GEOMETRIC EFFECT	Stretches the graph of $y = f(x)$ vertically by a factor of c	Compresses the graph of $y = f(x)$ vertically by a factor of $1/c$	Compresses the graph of $y = f(x)$ horizontally by a factor of c	Stretches the graph of $y = f(x)$ horizontally by a factor of $1/c$
EXAMPLE				

SYMMETRY

Figure 0.2.7 illustrates three types of symmetries: ***symmetry about the x-axis***, ***symmetry about the y-axis***, and ***symmetry about the origin***. As illustrated in the figure, a curve is symmetric about the x-axis if for each point (x, y) on the graph the point $(x, -y)$ is also on the graph, and it is symmetric about the y-axis if for each point (x, y) on the graph the point $(-x, y)$ is also on the graph. A curve is symmetric about the origin if for each point (x, y) on the graph, the point $(-x, -y)$ is also on the graph. (Equivalently, a graph is symmetric about the origin if rotating the graph 180° about the origin leaves it unchanged.) This suggests the following symmetry tests.

Explain why the graph of a nonzero function cannot be symmetric about the x-axis.



Symmetric about the x-axis

Symmetric about the y-axis

Symmetric about the origin

▶ **Figure 0.2.7**

0.2.3 THEOREM (***Symmetry Tests***)

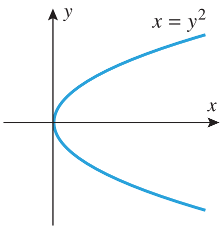
(*a*) *A plane curve is symmetric about the* y*-axis if and only if replacing* x *by* $-x$ *in its equation produces an equivalent equation.*

(*b*) *A plane curve is symmetric about the* x*-axis if and only if replacing* y *by* $-y$ *in its equation produces an equivalent equation.*

(*c*) *A plane curve is symmetric about the origin if and only if replacing both* x *by* $-x$ *and* y *by* $-y$ *in its equation produces an equivalent equation.*

▶ **Example 11** Use Theorem 0.2.3 to identify symmetries in the graph of $x = y^2$.

Solution. Replacing y by $-y$ yields $x = (-y)^2$, which simplifies to the original equation $x = y^2$. Thus, the graph is symmetric about the x-axis. The graph is not symmetric about the y-axis because replacing x by $-x$ yields $-x = y^2$, which is not equivalent to the original equation $x = y^2$. Similarly, the graph is not symmetric about the origin because replacing x by $-x$ and y by $-y$ yields $-x = (-y)^2$, which simplifies to $-x = y^2$, and this is again not equivalent to the original equation. These results are consistent with the graph of $x = y^2$ shown in Figure 0.2.8. ◀



▲ **Figure 0.2.8**

EVEN AND ODD FUNCTIONS

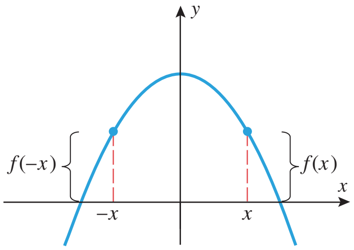
A function f is said to be an ***even function*** if

$$f(-x) = f(x) \tag{8}$$

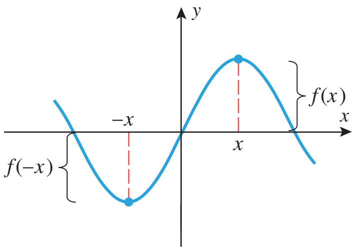
and is said to be an ***odd function*** if

$$f(-x) = -f(x) \tag{9}$$

Geometrically, the graphs of even functions are symmetric about the y-axis because replacing x by $-x$ in the equation $y = f(x)$ yields $y = f(-x)$, which is equivalent to the original equation $y = f(x)$ by (8) (see Figure 0.2.9). Similarly, it follows from (9) that graphs of odd functions are symmetric about the origin (see Figure 0.2.10). Some examples of even functions are x^2, x^4, x^6, and $\cos x$; and some examples of odd functions are x^3, x^5, x^7, and $\sin x$.



▲ **Figure 0.2.9** This is the graph of an even function since $f(-x) = f(x)$.



▲ **Figure 0.2.10** This is the graph of an odd function since $f(-x) = -f(x)$.

✔QUICK CHECK EXERCISES 0.2 *(See page 27 for answers.)*

1. Let $f(x) = 3\sqrt{x} - 2$ and $g(x) = |x|$. In each part, give the formula for the function and state the corresponding domain.
(a) $f + g$: ________ Domain: ________
(b) $f - g$: ________ Domain: ________
(c) fg: ________ Domain: ________
(d) f/g: ________ Domain: ________

2. Let $f(x) = 2 - x^2$ and $g(x) = \sqrt{x}$. In each part, give the formula for the composition and state the corresponding domain.
(a) $f \circ g$: ________ Domain: ________
(b) $g \circ f$: ________ Domain: ________

3. The graph of $y = 1 + (x - 2)^2$ may be obtained by shifting the graph of $y = x^2$ ________ (left/right) by ________ unit(s) and then shifting this new graph ________ (up/down) by ________ unit(s).

4. Let

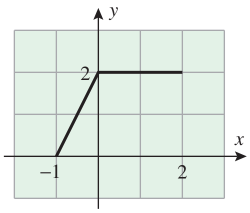
$$f(x) = \begin{cases} |x+1|, & -2 \le x \le 0 \\ |x-1|, & 0 < x \le 2 \end{cases}$$

(a) The letter of the alphabet that most resembles the graph of f is ________.
(b) Is f an even function?

EXERCISE SET 0.2 Graphing Utility

FOCUS ON CONCEPTS

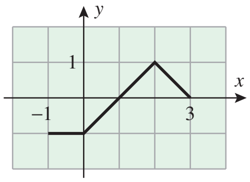
1. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
(a) $y = f(x) - 1$ (b) $y = f(x - 1)$
(c) $y = \frac{1}{2}f(x)$ (d) $y = f\left(-\frac{1}{2}x\right)$



◀ **Figure Ex-1**

2. Use the graph in Exercise 1 to sketch the graphs of the following equations.
(a) $y = -f(-x)$ (b) $y = f(2 - x)$
(c) $y = 1 - f(2 - x)$ (d) $y = \frac{1}{2}f(2x)$

3. The graph of a function f is shown in the accompanying figure. Sketch the graphs of the following equations.
(a) $y = f(x + 1)$ (b) $y = f(2x)$
(c) $y = |f(x)|$ (d) $y = 1 - |f(x)|$



◀ **Figure Ex-3**

4. Use the graph in Exercise 3 to sketch the graph of the equation $y = f(|x|)$.

5–24 Sketch the graph of the equation by translating, reflecting, compressing, and stretching the graph of $y = x^2$, $y = \sqrt{x}$, $y = 1/x$, $y = |x|$, or $y = \sqrt[3]{x}$ appropriately. Then use a graphing utility to confirm that your sketch is correct.

5. $y=-2(x+1)^2-3$

6. $y=\frac{1}{2}(x-3)^2+2$

7. $y=x^2+6x$

8. $y=\frac{1}{2}(x^2-2x+3)$

9. $y=3-\sqrt{x+1}$

10. $y=1+\sqrt{x-4}$

11. $y=\frac{1}{2}\sqrt{x}+1$

12. $y=-\sqrt{3x}$

13. $y=\dfrac{1}{x-3}$

14. $y=\dfrac{1}{1-x}$

15. $y=2-\dfrac{1}{x+1}$

16. $y=\dfrac{x-1}{x}$

17. $y=|x+2|-2$

18. $y=1-|x-3|$

19. $y=|2x-1|+1$

20. $y=\sqrt{x^2-4x+4}$

21. $y=1-2\sqrt[3]{x}$

22. $y=\sqrt[3]{x-2}-3$

23. $y=2+\sqrt[3]{x+1}$

24. $y+\sqrt[3]{x-2}=0$

25. (a) Sketch the graph of $y=x+|x|$ by adding the corresponding y-coordinates on the graphs of $y=x$ and $y=|x|$.
(b) Express the equation $y=x+|x|$ in piecewise form with no absolute values, and confirm that the graph you obtained in part (a) is consistent with this equation.

26. Sketch the graph of $y=x+(1/x)$ by adding corresponding y-coordinates on the graphs of $y=x$ and $y=1/x$. Use a graphing utility to confirm that your sketch is correct.

27–28 Find formulas for $f+g$, $f-g$, fg, and f/g, and state the domains of the functions.

27. $f(x)=2\sqrt{x-1},\ g(x)=\sqrt{x-1}$

28. $f(x)=\dfrac{x}{1+x^2},\ g(x)=\dfrac{1}{x}$

29. Let $f(x)=\sqrt{x}$ and $g(x)=x^3+1$. Find
(a) $f(g(2))$ (b) $g(f(4))$ (c) $f(f(16))$
(d) $g(g(0))$ (e) $f(2+h)$ (f) $g(3+h)$.

30. Let $g(x)=\sqrt{x}$. Find
(a) $g(5s+2)$ (b) $g(\sqrt{x}+2)$ (c) $3g(5x)$
(d) $\dfrac{1}{g(x)}$ (e) $g(g(x))$ (f) $(g(x))^2-g(x^2)$
(g) $g(1/\sqrt{x})$ (h) $g((x-1)^2)$ (i) $g(x+h)$.

31–34 Find formulas for $f\circ g$ and $g\circ f$, and state the domains of the compositions.

31. $f(x)=x^2,\ g(x)=\sqrt{1-x}$

32. $f(x)=\sqrt{x-3},\ g(x)=\sqrt{x^2+3}$

33. $f(x)=\dfrac{1+x}{1-x},\ g(x)=\dfrac{x}{1-x}$

34. $f(x)=\dfrac{x}{1+x^2},\ g(x)=\dfrac{1}{x}$

35–36 Find a formula for $f\circ g\circ h$.

35. $f(x)=x^2+1,\ g(x)=\dfrac{1}{x},\ h(x)=x^3$

36. $f(x)=\dfrac{1}{1+x},\ g(x)=\sqrt[3]{x},\ h(x)=\dfrac{1}{x^3}$

37–42 Express f as a composition of two functions; that is, find g and h such that $f=g\circ h$. [*Note:* Each exercise has more than one solution.]

37. (a) $f(x)=\sqrt{x+2}$ (b) $f(x)=|x^2-3x+5|$

38. (a) $f(x)=x^2+1$ (b) $f(x)=\dfrac{1}{x-3}$

39. (a) $f(x)=\sin^2 x$ (b) $f(x)=\dfrac{3}{5+\cos x}$

40. (a) $f(x)=3\sin(x^2)$ (b) $f(x)=3\sin^2 x+4\sin x$

41. (a) $f(x)=\left(1+\sin(x^2)\right)^3$ (b) $f(x)=\sqrt{1-\sqrt[3]{x}}$

42. (a) $f(x)=\dfrac{1}{1-x^2}$ (b) $f(x)=|5+2x|$

43–46 True–False Determine whether the statement is true or false. Explain your answer.

43. The domain of $f+g$ is the intersection of the domains of f and g.

44. The domain of $f\circ g$ consists of all values of x in the domain of g for which $g(x)\neq 0$.

45. The graph of an even function is symmetric about the y-axis.

46. The graph of $y=f(x+2)+3$ is obtained by translating the graph of $y=f(x)$ right 2 units and up 3 units.

FOCUS ON CONCEPTS

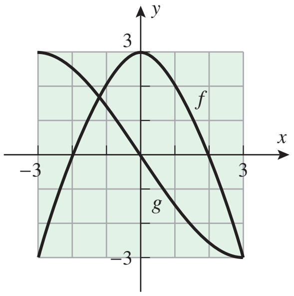
47. Use the data in the accompanying table to make a plot of $y=f(g(x))$.

x	−3	−2	−1	0	1	2	3
$f(x)$	−4	−3	−2	−1	0	1	2
$g(x)$	−1	0	1	2	3	−2	−3

▲ **Table Ex-47**

48. Find the domain of $g\circ f$ for the functions f and g in Exercise 47.

49. Sketch the graph of $y=f(g(x))$ for the functions graphed in the accompanying figure.



◀ **Figure Ex-49**

50. Sketch the graph of $y=g(f(x))$ for the functions graphed in Exercise 49.

51. Use the graphs of f and g in Exercise 49 to estimate the solutions of the equations $f(g(x))=0$ and $g(f(x))=0$.

52. Use the table given in Exercise 47 to solve the equations $f(g(x))=0$ and $g(f(x))=0$.

53–56 Find

$$\frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \frac{f(w)-f(x)}{w-x}$$

Simplify as much as possible.

53. $f(x) = 3x^2 - 5$

54. $f(x) = x^2 + 6x$

55. $f(x) = 1/x$

56. $f(x) = 1/x^2$

57. Classify the functions whose values are given in the accompanying table as even, odd, or neither.

x	−3	−2	−1	0	1	2	3
$f(x)$	5	3	2	3	1	−3	5
$g(x)$	4	1	−2	0	2	−1	−4
$h(x)$	2	−5	8	−2	8	−5	2

▲ **Table Ex-57**

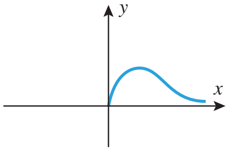
58. Complete the accompanying table so that the graph of $y = f(x)$ is symmetric about
(a) the y-axis (b) the origin.

x	−3	−2	−1	0	1	2	3
$f(x)$	1		−1	0		−5	

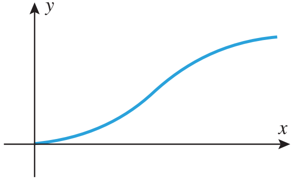
▲ **Table Ex-58**

59. The accompanying figure shows a portion of a graph. Complete the graph so that the entire graph is symmetric about
(a) the x-axis (b) the y-axis (c) the origin.

60. The accompanying figure shows a portion of the graph of a function f. Complete the graph assuming that
(a) f is an even function (b) f is an odd function.



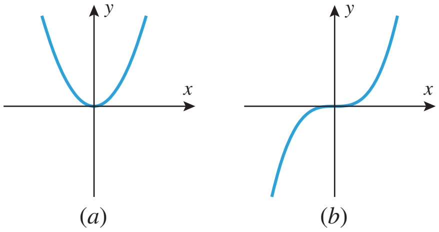
▲ **Figure Ex-59**



▲ **Figure Ex-60**

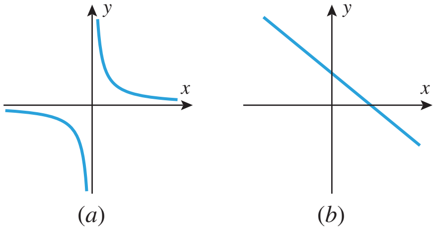
61–62 Classify the functions graphed in the accompanying figures as even, odd, or neither.

61.



▲ **Figure Ex-61**

62.



▲ **Figure Ex-62**

63. In each part, classify the function as even, odd, or neither.
(a) $f(x) = x^2$ (b) $f(x) = x^3$
(c) $f(x) = |x|$ (d) $f(x) = x + 1$
(e) $f(x) = \dfrac{x^5 - x}{1 + x^2}$ (f) $f(x) = 2$

64. Suppose that the function f has domain all real numbers. Determine whether each function can be classified as even or odd. Explain.
(a) $g(x) = \dfrac{f(x) + f(-x)}{2}$ (b) $h(x) = \dfrac{f(x) - f(-x)}{2}$

65. Suppose that the function f has domain all real numbers. Show that f can be written as the sum of an even function and an odd function. [*Hint:* See Exercise 64.]

66–67 Use Theorem 0.2.3 to determine whether the graph has symmetries about the x-axis, the y-axis, or the origin.

66. (a) $x = 5y^2 + 9$ (b) $x^2 - 2y^2 = 3$
(c) $xy = 5$

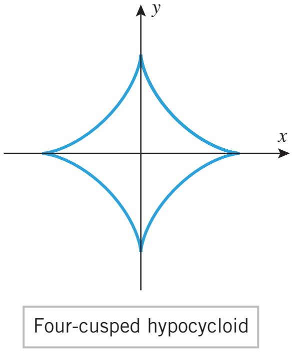
67. (a) $x^4 = 2y^3 + y$ (b) $y = \dfrac{x}{3 + x^2}$
(c) $y^2 = |x| - 5$

68–69 (i) Use a graphing utility to graph the equation in the first quadrant. [*Note:* To do this you will have to solve the equation for y in terms of x.] (ii) Use symmetry to make a hand-drawn sketch of the entire graph. (iii) Confirm your work by generating the graph of the equation in the remaining three quadrants.

68. $9x^2 + 4y^2 = 36$

69. $4x^2 + 16y^2 = 16$

70. The graph of the equation $x^{2/3} + y^{2/3} = 1$, which is shown in the accompanying figure, is called a ***four-cusped hypocycloid***.
(a) Use Theorem 0.2.3 to confirm that this graph is symmetric about the x-axis, the y-axis, and the origin.
(b) Find a function f whose graph in the first quadrant coincides with the four-cusped hypocycloid, and use a graphing utility to confirm your work.
(c) Repeat part (b) for the remaining three quadrants.



◀ **Figure Ex-70**

71. The equation $y = |f(x)|$ can be written as

$$y = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$$

which shows that the graph of $y = |f(x)|$ can be obtained from the graph of $y = f(x)$ by retaining the portion that lies

35–36 Find the amplitude and period, and sketch at least two periods of the graph by hand. If you have a graphing utility, use it to check your work.

35. (a) $y = 3\sin 4x$ (b) $y = -2\cos \pi x$
(c) $y = 2 + \cos\left(\dfrac{x}{2}\right)$

36. (a) $y = -1 - 4\sin 2x$ (b) $y = \frac{1}{2}\cos(3x - \pi)$
(c) $y = -4\sin\left(\dfrac{x}{3} + 2\pi\right)$

37. Equations of the form

$$x = A_1 \sin \omega t + A_2 \cos \omega t$$

arise in the study of vibrations and other periodic motion. Express the equation

$$x = \sqrt{2}\sin 2\pi t + \sqrt{6}\cos 2\pi t$$

in the form $x = A\sin(\omega t + \theta)$, and use a graphing utility to confirm that both equations have the same graph.

38. Determine the number of solutions of $x = 2\sin x$, and use a graphing or calculating utility to estimate them.

✔QUICK CHECK ANSWERS 0.3

1. even; odd; negative **2.** $(-\infty, +\infty)$ **3.** (a) $[0, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(0, +\infty)$ (d) $(-\infty, 0) \cup (0, +\infty)$ **4.** (a) algebraic (b) polynomial (c) not algebraic (d) rational (e) rational **5.** $|A|$; $2\pi/|B|$

0.4 INVERSE FUNCTIONS

In everyday language the term "inversion" conveys the idea of a reversal. For example, in meteorology a temperature inversion is a reversal in the usual temperature properties of air layers, and in music a melodic inversion reverses an ascending interval to the corresponding descending interval. In mathematics the term ***inverse*** *is used to describe functions that reverse one another in the sense that each undoes the effect of the other. In this section we discuss this fundamental mathematical idea.*

INVERSE FUNCTIONS

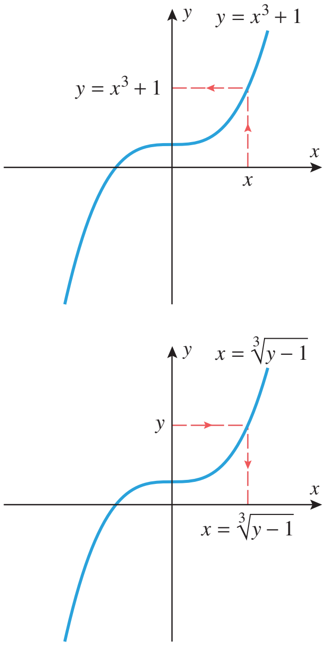
The idea of solving an equation $y = f(x)$ for x as a function of y, say $x = g(y)$, is one of the most important ideas in mathematics. Sometimes, solving an equation is a simple process; for example, using basic algebra the equation

$$y = x^3 + 1 \qquad \boxed{y = f(x)}$$

can be solved for x as a function of y:

$$x = \sqrt[3]{y - 1} \qquad \boxed{x = g(y)}$$

The first equation is better for computing y if x is known, and the second is better for computing x if y is known (Figure 0.4.1).



▲ **Figure 0.4.1**

Our primary interest in this section is to identify relationships that may exist between the functions f and g when an equation $y = f(x)$ is expressed as $x = g(y)$, or conversely. For example, consider the functions $f(x) = x^3 + 1$ and $g(y) = \sqrt[3]{y - 1}$ discussed above. When these functions are composed in either order, they cancel out the effect of one another in the sense that

$$\begin{aligned} g(f(x)) &= \sqrt[3]{f(x) - 1} = \sqrt[3]{(x^3 + 1) - 1} = x \\ f(g(y)) &= [g(y)]^3 + 1 = (\sqrt[3]{y - 1})^3 + 1 = y \end{aligned} \tag{1}$$

Pairs of functions with these two properties are so important that there is special terminology for them.

0.4.1 DEFINITION If the functions f and g satisfy the two conditions

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$
$$f(g(y)) = y \text{ for every } y \text{ in the domain of } g$$

then we say that ***f* is an inverse of *g*** and ***g* is an inverse of *f*** or that ***f* and *g* are inverse functions**.

WARNING

If f is a function, then the -1 in the symbol f^{-1} always denotes an inverse and *never* an exponent. That is,

$$f^{-1}(x) \quad \textit{never means} \quad \frac{1}{f(x)}$$

It can be shown (Exercise 35) that if a function f has an inverse, then that inverse is unique. Thus, if a function f has an inverse, then we are entitled to talk about "the" inverse of f, in which case we denote it by the symbol f^{-1}.

▶ **Example 1** The computations in (1) show that $g(y) = \sqrt[3]{y-1}$ is the inverse of $f(x) = x^3 + 1$. Thus, we can express g in inverse notation as

$$f^{-1}(y) = \sqrt[3]{y-1}$$

and we can express the equations in Definition 0.4.1 as

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(y)) &= y \quad \text{for every } y \text{ in the domain of } f^{-1} \end{aligned} \tag{2}$$

We will call these the ***cancellation equations*** for f and f^{-1}. ◀

CHANGING THE INDEPENDENT VARIABLE

The formulas in (2) use x as the independent variable for f and y as the independent variable for f^{-1}. Although it is often convenient to use different independent variables for f and f^{-1}, there will be occasions on which it is desirable to use the same independent variable for both. For example, if we want to graph the functions f and f^{-1} together in the same xy-coordinate system, then we would want to use x as the independent variable and y as the dependent variable for both functions. Thus, to graph the functions $f(x) = x^3 + 1$ and $f^{-1}(y) = \sqrt[3]{y-1}$ of Example 1 in the same xy-coordinate system, we would change the independent variable y to x, use y as the dependent variable for both functions, and graph the equations

$$y = x^3 + 1 \quad \text{and} \quad y = \sqrt[3]{x-1}$$

We will talk more about graphs of inverse functions later in this section, but for reference we give the following reformulation of the cancellation equations in (2) using x as the independent variable for both f and f^{-1}:

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in the domain of } f^{-1} \end{aligned} \tag{3}$$

▶ **Example 2** Confirm each of the following.

(a) The inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.

(b) The inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.

The results in Example 2 should make sense to you intuitively, since the operations of multiplying by 2 and multiplying by $\frac{1}{2}$ in either order cancel the effect of one another, as do the operations of cubing and taking a cube root.

Solution (a).

$$f^{-1}(f(x)) = f^{-1}(2x) = \tfrac{1}{2}(2x) = x$$
$$f(f^{-1}(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

Solution (b).

$$f^{-1}(f(x)) = f^{-1}(x^3) = \left(x^3\right)^{1/3} = x$$
$$f(f^{-1}(x)) = f(x^{1/3}) = \left(x^{1/3}\right)^3 = x \quad ◀$$

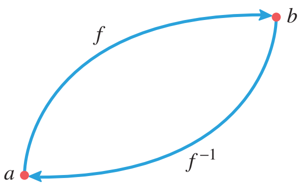
In general, if a function f has an inverse and $f(a) = b$, then the procedure in Example 3 shows that $a = f^{-1}(b)$; that is, f^{-1} maps each output of f back into the corresponding input (Figure 0.4.2).

▶ **Example 3** Given that the function f has an inverse and that $f(3) = 5$, find $f^{-1}(5)$.

Solution. Apply f^{-1} to both sides of the equation $f(3) = 5$ to obtain

$$f^{-1}(f(3)) = f^{-1}(5)$$

and now apply the first equation in (3) to conclude that $f^{-1}(5) = 3$. ◀



▲ **Figure 0.4.2** If f maps a to b, then f^{-1} maps b back to a.

DOMAIN AND RANGE OF INVERSE FUNCTIONS

The equations in (3) imply the following relationships between the domains and ranges of f and f^{-1}:

$$\begin{aligned} \text{domain of } f^{-1} &= \text{range of } f \\ \text{range of } f^{-1} &= \text{domain of } f \end{aligned} \tag{4}$$

One way to show that two sets are the same is to show that each is a subset of the other. Thus we can establish the first equality in (4) by showing that the domain of f^{-1} is a subset of the range of f and that the range of f is a subset of the domain of f^{-1}. We do this as follows: The first equation in (3) implies that f^{-1} is defined at $f(x)$ for all values of x in the domain of f, and this implies that the range of f is a subset of the domain of f^{-1}. Conversely, if x is in the domain of f^{-1}, then the second equation in (3) implies that x is in the range of f because it is the image of $f^{-1}(x)$. Thus, the domain of f^{-1} is a subset of the range of f. We leave the proof of the second equation in (4) as an exercise.

A METHOD FOR FINDING INVERSE FUNCTIONS

At the beginning of this section we observed that solving $y = f(x) = x^3 + 1$ for x as a function of y produces $x = f^{-1}(y) = \sqrt[3]{y-1}$. The following theorem shows that this is not accidental.

0.4.2 THEOREM *If an equation $y = f(x)$ can be solved for x as a function of y, say $x = g(y)$, then f has an inverse and that inverse is $g(y) = f^{-1}(y)$.*

PROOF Substituting $y = f(x)$ into $x = g(y)$ yields $x = g(f(x))$, which confirms the first equation in Definition 0.4.1, and substituting $x = g(y)$ into $y = f(x)$ yields $y = f(g(y))$, which confirms the second equation in Definition 0.4.1. ■

Theorem 0.4.2 provides us with the following procedure for finding the inverse of a function.

A Procedure for Finding the Inverse of a Function f

Step 1. Write down the equation $y = f(x)$.

Step 2. If possible, solve this equation for x as a function of y.

Step 3. The resulting equation will be $x = f^{-1}(y)$, which provides a formula for f^{-1} with y as the independent variable.

Step 4. If y is acceptable as the independent variable for the inverse function, then you are done, but if you want to have x as the independent variable, then you need to interchange x and y in the equation $x = f^{-1}(y)$ to obtain $y = f^{-1}(x)$.

An alternative way to obtain a formula for $f^{-1}(x)$ with x as the independent variable is to reverse the roles of x and y at the outset and solve the equation $x = f(y)$ for y as a function of x.

▸ **Example 4** Find a formula for the inverse of $f(x) = \sqrt{3x-2}$ with x as the independent variable, and state the domain of f^{-1}.

Solution. Following the procedure stated above, we first write

$$y = \sqrt{3x-2}$$

Then we solve this equation for x as a function of y:

$$\begin{aligned} y^2 &= 3x-2 \\ x &= \tfrac{1}{3}(y^2+2) \end{aligned}$$

which tells us that

$$f^{-1}(y) = \tfrac{1}{3}(y^2+2) \tag{5}$$

Since we want x to be the independent variable, we reverse x and y in (5) to produce the formula

$$f^{-1}(x) = \tfrac{1}{3}(x^2+2) \tag{6}$$

We know from (4) that the domain of f^{-1} is the range of f. In general, this need not be the same as the natural domain of the formula for f^{-1}. Indeed, in this example the natural domain of (6) is $(-\infty, +\infty)$, whereas the range of $f(x) = \sqrt{3x-2}$ is $[0, +\infty)$. Thus, if we want to make the domain of f^{-1} clear, we must express it explicitly by rewriting (6) as

$$f^{-1}(x) = \tfrac{1}{3}(x^2+2), \quad x \geq 0 \blacktriangleleft$$

EXISTENCE OF INVERSE FUNCTIONS

The procedure we gave above for finding the inverse of a function f was based on solving the equation $y = f(x)$ for x as a function of y. This procedure can fail for two reasons—the function f may not have an inverse, or it may have an inverse but the equation $y = f(x)$ cannot be solved explicitly for x as a function of y. Thus, it is important to establish conditions that ensure the existence of an inverse, even if it cannot be found explicitly.

If a function f has an inverse, then it must assign distinct outputs to distinct inputs. For example, the function $f(x) = x^2$ cannot have an inverse because it assigns the same value to $x = 2$ and $x = -2$, namely,

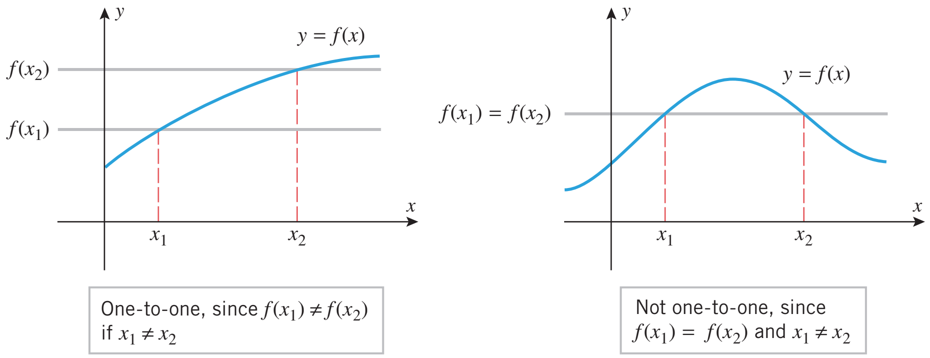
$$f(2) = f(-2) = 4$$

Thus, if $f(x) = x^2$ were to have an inverse, then the equation $f(2) = 4$ would imply that $f^{-1}(4) = 2$, and the equation $f(-2) = 4$ would imply that $f^{-1}(4) = -2$. But this is impossible because $f^{-1}(4)$ cannot have two different values. Another way to see that $f(x) = x^2$ has no inverse is to attempt to find the inverse by solving the equation $y = x^2$ for x as a function of y. We run into trouble immediately because the resulting equation $x = \pm\sqrt{y}$ does not express x as a *single* function of y.

A function that assigns distinct outputs to distinct inputs is said to be ***one-to-one*** or ***invertible***, so we know from the preceding discussion that if a function f has an inverse, then it must be one-to-one. The converse is also true, thereby establishing the following theorem.

0.4.3 THEOREM *A function has an inverse if and only if it is one-to-one.*

Stated algebraically, a function f is one-to-one if and only if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$; stated geometrically, a function f is one-to-one if and only if the graph of $y = f(x)$ is cut at most once by any horizontal line (Figure 0.4.3). The latter statement together with Theorem 0.4.3 provides the following geometric test for determining whether a function has an inverse.



One-to-one, since $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

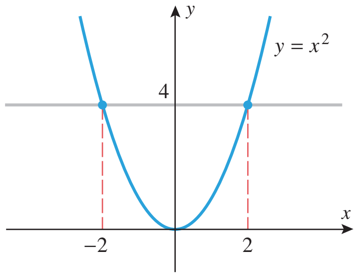
Not one-to-one, since $f(x_1) = f(x_2)$ and $x_1 \neq x_2$

▶ **Figure 0.4.3**

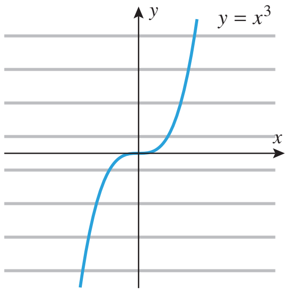
0.4.4 THEOREM (*The Horizontal Line Test*) *A function has an inverse function if and only if its graph is cut at most once by any horizontal line.*

▶ **Example 5** Use the horizontal line test to show that $f(x) = x^2$ has no inverse but that $f(x) = x^3$ does.

Solution. Figure 0.4.4 shows a horizontal line that cuts the graph of $y = x^2$ more than once, so $f(x) = x^2$ is not invertible. Figure 0.4.5 shows that the graph of $y = x^3$ is cut at most once by any horizontal line, so $f(x) = x^3$ is invertible. [Recall from Example 2 that the inverse of $f(x) = x^3$ is $f^{-1}(x) = x^{1/3}$.] ◀



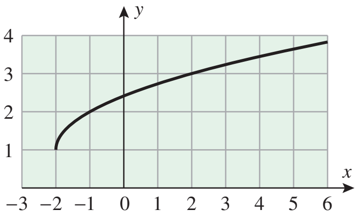
▲ **Figure 0.4.4**



▲ **Figure 0.4.5**

▶ **Example 6** Explain why the function f that is graphed in Figure 0.4.6 has an inverse, and find $f^{-1}(3)$.

Solution. The function f has an inverse since its graph passes the horizontal line test. To evaluate $f^{-1}(3)$, we view $f^{-1}(3)$ as that number x for which $f(x) = 3$. From the graph we see that $f(2) = 3$, so $f^{-1}(3) = 2$. ◀



▲ **Figure 0.4.6**

The function $f(x) = x^3$ in Figure 0.4.5 is an example of an increasing function. Give an example of a decreasing function and compute its inverse.

INCREASING OR DECREASING FUNCTIONS ARE INVERTIBLE

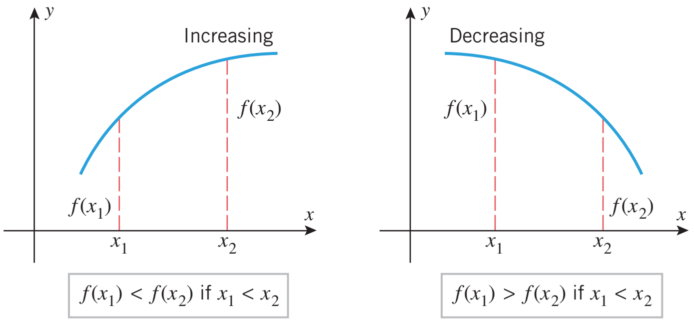
A function whose graph is always rising as it is traversed from left to right is said to be an ***increasing function***, and a function whose graph is always falling as it is traversed from left to right is said to be a ***decreasing function***. If x_1 and x_2 are points in the domain of a function f, then f is increasing if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2$$

and f is decreasing if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2$$

(Figure 0.4.7). It is evident geometrically that increasing and decreasing functions pass the horizontal line test and hence are invertible.



$f(x_1) < f(x_2)$ if $x_1 < x_2$

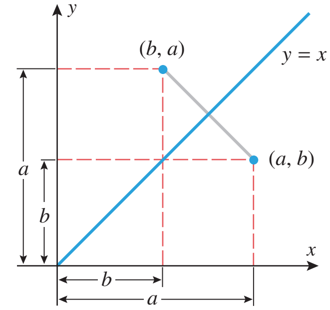
$f(x_1) > f(x_2)$ if $x_1 < x_2$

▶ **Figure 0.4.7**

GRAPHS OF INVERSE FUNCTIONS

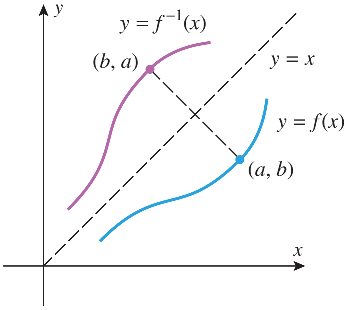
Our next objective is to explore the relationship between the graphs of f and f^{-1}. For this purpose, it will be desirable to use x as the independent variable for both functions so we can compare the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

If (a, b) is a point on the graph $y = f(x)$, then $b = f(a)$. This is equivalent to the statement that $a = f^{-1}(b)$, which means that (b, a) is a point on the graph of $y = f^{-1}(x)$. In short, reversing the coordinates of a point on the graph of f produces a point on the graph of f^{-1}. Similarly, reversing the coordinates of a point on the graph of f^{-1} produces a point on the graph of f (verify). However, the geometric effect of reversing the coordinates of a point is to reflect that point about the line $y = x$ (Figure 0.4.8), and hence the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about this line (Figure 0.4.9). In summary, we have the following result.



The points (a, b) and (b, a) are reflections about $y = x$.

▲ **Figure 0.4.8**

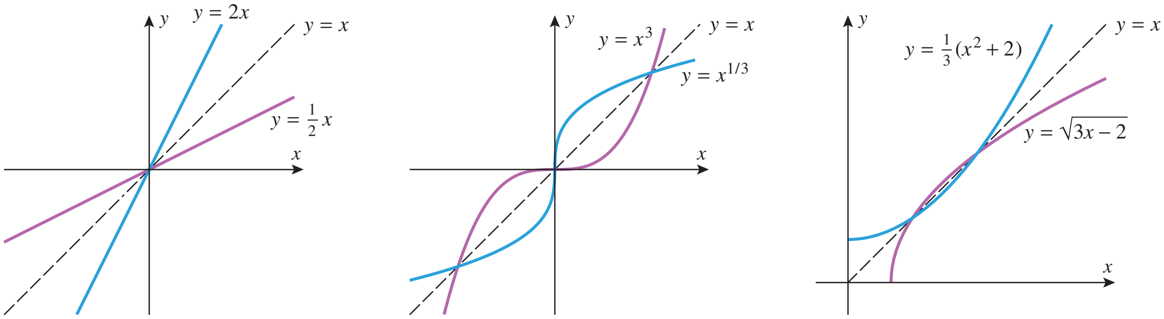


The graphs of f and f^{-1} are reflections about $y = x$.

▲ **Figure 0.4.9**

0.4.5 THEOREM *If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.*

▶ **Example 7** Figure 0.4.10 shows the graphs of the inverse functions discussed in Examples 2 and 4. ◀



▲ **Figure 0.4.10**

RESTRICTING DOMAINS FOR INVERTIBILITY

If a function g is obtained from a function f by placing restrictions on the domain of f, then g is called a ***restriction*** of f. Thus, for example, the function

$$g(x) = x^3, \quad x \geq 0$$

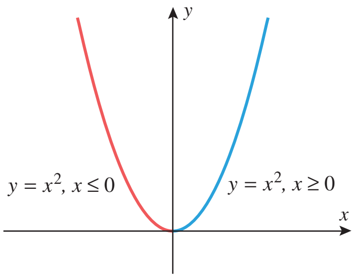
is a restriction of the function $f(x) = x^3$. More precisely, it is called the restriction of x^3 to the interval $[0, +\infty)$.

Sometimes it is possible to create an invertible function from a function that is not invertible by restricting the domain appropriately. For example, we showed earlier that $f(x) = x^2$ is not invertible. However, consider the restricted functions

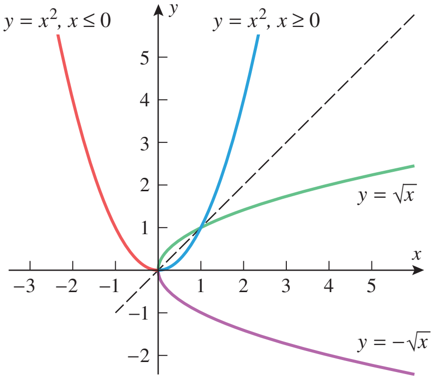
$$f_1(x) = x^2, \quad x \geq 0 \quad \text{and} \quad f_2(x) = x^2, \quad x \leq 0$$

the union of whose graphs is the complete graph of $f(x) = x^2$ (Figure 0.4.11). These restricted functions are each one-to-one (hence invertible), since their graphs pass the horizontal line test. As illustrated in Figure 0.4.12, their inverses are

$$f_1^{-1}(x) = \sqrt{x} \quad \text{and} \quad f_2^{-1}(x) = -\sqrt{x}$$



▲ **Figure 0.4.11**



▲ **Figure 0.4.12**

QUICK CHECK EXERCISES 0.4 *(See page 46 for answers.)*

1. In each part, determine whether the function f is one-to-one.
   (a) $f(t)$ is the number of people in line at a movie theater at time t.
   (b) $f(x)$ is the measured high temperature (rounded to the nearest °F) in a city on the xth day of the year.
   (c) $f(v)$ is the weight of v cubic inches of lead.

2. A student enters a number on a calculator, doubles it, adds 8 to the result, divides the sum by 2, subtracts 3 from the quotient, and then cubes the difference. If the resulting number is x, then ________ was the student's original number.

3. If $(3, -2)$ is a point on the graph of an odd invertible function f, then ________ and ________ are points on the graph of f^{-1}.

EXERCISE SET 0.4 Graphing Utility

1. In (a)–(d), determine whether f and g are inverse functions.
   (a) $f(x) = 4x, \; g(x) = \frac{1}{4}x$
   (b) $f(x) = 3x + 1, \; g(x) = 3x - 1$
   (c) $f(x) = \sqrt[3]{x - 2}, \; g(x) = x^3 + 2$
   (d) $f(x) = x^4, \; g(x) = \sqrt[4]{x}$

2. Check your answers to Exercise 1 with a graphing utility by determining whether the graphs of f and g are reflections of one another about the line $y = x$.

3. In each part, use the horizontal line test to determine whether the function f is one-to-one.
   (a) $f(x) = 3x + 2$
   (b) $f(x) = \sqrt{x - 1}$
   (c) $f(x) = |x|$
   (d) $f(x) = x^3$
   (e) $f(x) = x^2 - 2x + 2$
   (f) $f(x) = \sin x$

4. In each part, generate the graph of the function f with a graphing utility, and determine whether f is one-to-one.
(a) $f(x) = x^3 - 3x + 2$ (b) $f(x) = x^3 - 3x^2 + 3x - 1$

FOCUS ON CONCEPTS

5. In each part, determine whether the function f defined by the table is one-to-one.

(a)

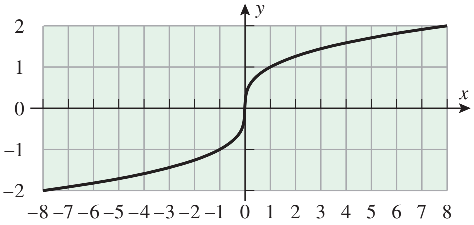
x	1	2	3	4	5	6
$f(x)$	–2	–1	0	1	2	3

(b)

x	1	2	3	4	5	6
$f(x)$	4	–7	6	–3	1	4

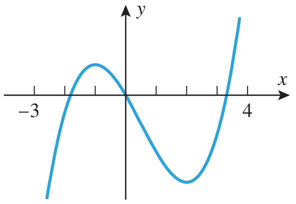
6. A face of a broken clock lies in the xy-plane with the center of the clock at the origin and 3:00 in the direction of the positive x-axis. When the clock broke, the tip of the hour hand stopped on the graph of $y = f(x)$, where f is a function that satisfies $f(0) = 0$.
(a) Are there any times of the day that cannot appear in such a configuration? Explain.
(b) How does your answer to part (a) change if f must be an invertible function?
(c) How do your answers to parts (a) and (b) change if it was the tip of the minute hand that stopped on the graph of f?

7. (a) The accompanying figure shows the graph of a function f over its domain $-8 \le x \le 8$. Explain why f has an inverse, and use the graph to find $f^{-1}(2)$, $f^{-1}(-1)$, and $f^{-1}(0)$.
(b) Find the domain and range of f^{-1}.
(c) Sketch the graph of f^{-1}.



▲ **Figure Ex-7**

8. (a) Explain why the function f graphed in the accompanying figure has no inverse function on its domain $-3 \le x \le 4$.
(b) Subdivide the domain into three adjacent intervals on each of which the function f has an inverse.



◀ **Figure Ex-8**

9–16 Find a formula for $f^{-1}(x)$. ■

9. $f(x) = 7x - 6$ **10.** $f(x) = \dfrac{x+1}{x-1}$

11. $f(x) = 3x^3 - 5$ **12.** $f(x) = \sqrt[5]{4x+2}$

13. $f(x) = 3/x^2, \quad x < 0$ **14.** $f(x) = 5/(x^2+1), \quad x \ge 0$

15. $f(x) = \begin{cases} 5/2 - x, & x < 2 \\ 1/x, & x \ge 2 \end{cases}$

16. $f(x) = \begin{cases} 2x, & x \le 0 \\ x^2, & x > 0 \end{cases}$

17–20 Find a formula for $f^{-1}(x)$, and state the domain of the function f^{-1}. ■

17. $f(x) = (x+2)^4, \quad x \ge 0$

18. $f(x) = \sqrt{x+3}$ **19.** $f(x) = -\sqrt{3-2x}$

20. $f(x) = x - 5x^2, \quad x \ge 1$

21–24 True–False Determine whether the statement is true or false. Explain your answer. ■

21. If f is an invertible function such that $f(2) = 2$, then $f^{-1}(2) = \frac{1}{2}$.

22. If f and g are inverse functions, then f and g have the same domain.

23. A one-to-one function is invertible.

24. If the graph of a function f is symmetric about the line $y = x$, then f is invertible.

25. Let $f(x) = ax^2 + bx + c$, $a > 0$. Find f^{-1} if the domain of f is restricted to
(a) $x \ge -b/(2a)$ (b) $x \le -b/(2a)$.

FOCUS ON CONCEPTS

26. The formula $F = \frac{9}{5}C + 32$, where $C \ge -273.15$ expresses the Fahrenheit temperature F as a function of the Celsius temperature C.
(a) Find a formula for the inverse function.
(b) In words, what does the inverse function tell you?
(c) Find the domain and range of the inverse function.

27. (a) One meter is about 6.214×10^{-4} miles. Find a formula $y = f(x)$ that expresses a length y in meters as a function of the same length x in miles.
(b) Find a formula for the inverse of f.
(c) Describe what the formula $x = f^{-1}(y)$ tells you in practical terms.

28. Let $f(x) = x^2$, $x > 1$, and $g(x) = \sqrt{x}$.
(a) Show that $f(g(x)) = x$, $x > 1$, and $g(f(x)) = x$, $x > 1$.
(b) Show that f and g are *not* inverses by showing that the graphs of $y = f(x)$ and $y = g(x)$ are not reflections of one another about $y = x$.
(c) Do parts (a) and (b) contradict one another? Explain.

29. (a) Show that $f(x) = (3 - x)/(1 - x)$ is its own inverse.
 (b) What does the result in part (a) tell you about the graph of f?

30. Sketch the graph of a function that is one-to-one on $(-\infty, +\infty)$, yet not increasing on $(-\infty, +\infty)$ and not decreasing on $(-\infty, +\infty)$.

31. Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

32. Let $f(x) = \dfrac{x^3}{x^2 + 1}$. Find x if $f^{-1}(x) = 2$.

33. Prove that if $a^2 + bc \neq 0$, then the graph of

$$f(x) = \frac{ax + b}{cx - a}$$

is symmetric about the line $y = x$.

34. (a) Prove: If f and g are one-to-one, then so is the composition $f \circ g$.
 (b) Prove: If f and g are one-to-one, then

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

35. Prove: A one-to-one function f cannot have two different inverses.

✔QUICK CHECK ANSWERS 0.4

1. (a) not one-to-one (b) not one-to-one (c) one-to-one **2.** $\sqrt[3]{x} - 1$ **3.** $(-2, 3)$; $(2, -3)$

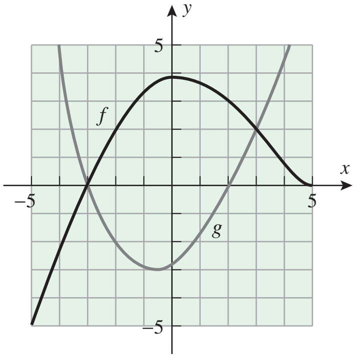
CHAPTER 0 REVIEW EXERCISES

Graphing Utility

1. Sketch the graph of the function

$$f(x) = \begin{cases} -1, & x \le -5 \\ \sqrt{25 - x^2}, & -5 < x < 5 \\ x - 5, & x \ge 5 \end{cases}$$

2. Use the graphs of the functions f and g in the accompanying figure to solve the following problems.
 (a) Find the values of $f(-2)$ and $g(3)$.
 (b) For what values of x is $f(x) = g(x)$?
 (c) For what values of x is $f(x) < 2$?
 (d) What are the domain and range of f?
 (e) What are the domain and range of g?
 (f) Find the zeros of f and g.



◀ **Figure Ex-2**

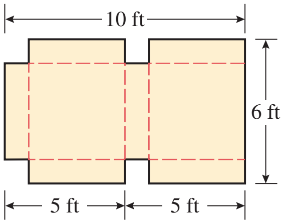
3. A glass filled with water that has a temperature of 40°F is placed in a room in which the temperature is a constant 70°F. Sketch a rough graph that reasonably describes the temperature of the water in the glass as a function of the elapsed time.

4. You want to paint the top of a circular table. Find a formula that expresses the amount of paint required as a function of the radius, and discuss all of the assumptions you have made in finding the formula.

5. A rectangular storage container with an open top and a square base has a volume of 8 cubic meters. Material for the base costs \$5 per square meter and material for the sides \$2 per square meter.
 (a) Find a formula that expresses the total cost of materials as a function of the length of a side of the base.
 (b) What is the domain of the cost function obtained in part (a)?

6. A ball of radius 3 inches is coated uniformly with plastic.
 (a) Express the volume of the plastic as a function of its thickness.
 (b) What is the domain of the volume function obtained in part (a)?

7. A box with a closed top is to be made from a 6 ft by 10 ft piece of cardboard by cutting out four squares of equal size (see the accompanying figure), folding along the dashed lines, and tucking the two extra flaps inside.
 (a) Find a formula that expresses the volume of the box as a function of the length of the sides of the cut-out squares.
 (b) Find an inequality that specifies the domain of the function in part (a).
 (c) Use the graph of the volume function to estimate the dimensions of the box of largest volume.



◀ **Figure Ex-7**

8. Let C denote the graph of $y = 1/x$, $x > 0$.

(a) Express the distance between the point $P(1, 0)$ and a point Q on C as a function of the x-coordinate of Q.

(b) What is the domain of the distance function obtained in part (a)?

(c) Use the graph of the distance function obtained in part (a) to estimate the point Q on C that is closest to the point P.

9. Sketch the graph of the equation $x^2 - 4y^2 = 0$.

10. Generate the graph of $f(x) = x^4 - 24x^3 - 25x^2$ in two different viewing windows, each of which illustrates a different property of f. Identify each viewing window and a characteristic of the graph of f that is illustrated well in the window.

11. Complete the following table.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	0	-1	2	1	3	-2	-3	4	-4
$g(x)$	3	2	1	-3	-1	-4	4	-2	0
$(f\circ g)(x)$									
$(g\circ f)(x)$									

▲ **Table Ex-11**

12. Let $f(x) = -x^2$ and $g(x) = 1/\sqrt{x}$. Find formulas for $f \circ g$ and $g \circ f$ and state the domain of each composition.

13. Given that $f(x) = x^2 + 1$ and $g(x) = 3x + 2$, find all values of x such that $f(g(x)) = g(f(x))$.

14. Let $f(x) = (2x - 1)/(x + 1)$ and $g(x) = 1/(x - 1)$.

(a) Find $f(g(x))$.

(b) Is the natural domain of the function $h(x) = (3 - x)/x$ the same as the domain of $f \circ g$? Explain.

15. Given that

$$f(x) = \frac{x}{x-1}, \quad g(x) = \frac{1}{x}, \quad h(x) = x^2 - 1$$

find a formula for $f \circ g \circ h$ and state the domain of this composition.

16. Given that $f(x) = 2x + 1$ and $h(x) = 2x^2 + 4x + 1$, find a function g such that $f(g(x)) = h(x)$.

17. In each part, classify the function as even, odd, or neither.

(a) $x^2 \sin x$ (b) $\sin^2 x$ (c) $x + x^2$ (d) $\sin x \tan x$

18. (a) Write an equation for the graph that is obtained by reflecting the graph of $y = |x - 1|$ about the y-axis, then stretching that graph vertically by a factor of 2, then translating that graph down 3 units, and then reflecting that graph about the x-axis.

(b) Sketch the original graph and the final graph.

19. In each part, describe the family of curves.

(a) $(x - a)^2 + (y - a^2)^2 = 1$

(b) $y = a + (x - 2a)^2$

20. Find an equation for a parabola that passes through the points $(2, 0)$, $(8, 18)$, and $(-8, 18)$.

21. Suppose that the expected low temperature in Anchorage, Alaska (in °F), is modeled by the equation

$$T = 50\sin\frac{2\pi}{365}(t - 101) + 25$$

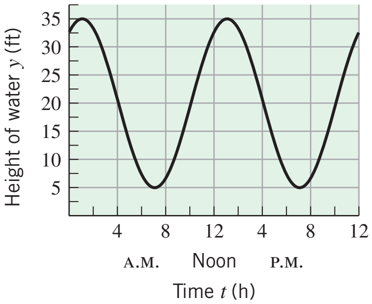
where t is in days and $t = 0$ corresponds to January 1.

(a) Sketch the graph of T versus t for $0 \le t \le 365$.

(b) Use the model to predict when the coldest day of the year will occur.

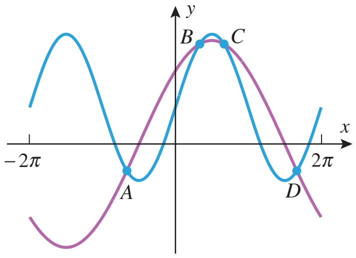
(c) Based on this model, how many days during the year would you expect the temperature to be below 0°F?

22. The accompanying figure shows a model for the tide variation in an inlet to San Francisco Bay during a 24-hour period. Find an equation of the form $y = y_0 + y_1 \sin(at + b)$ for the model, assuming that $t = 0$ corresponds to midnight.



◀ **Figure Ex-22**

23. The accompanying figure shows the graphs of the equations $y = 1 + 2\sin x$ and $y = 2\sin(x/2) + 2\cos(x/2)$ for $-2\pi \le x \le 2\pi$. Without the aid of a calculator, label each curve by its equation, and find the coordinates of the points A, B, C, and D. Explain your reasoning.



◀ **Figure Ex-23**

24. The electrical resistance R in ohms (Ω) for a pure metal wire is related to its temperature T in °C by the formula

$$R = R_0(1 + kT)$$

in which R_0 and k are positive constants.

(a) Make a hand-drawn sketch of the graph of R versus T, and explain the geometric significance of R_0 and k for your graph.

(b) In theory, the resistance R of a pure metal wire drops to zero when the temperature reaches absolute zero ($T = -273$°C). What information does this give you about k?

(cont.)

(c) A tungsten bulb filament has a resistance of 1.1 Ω at a temperature of 20°C. What information does this give you about R_0 for the filament?

(d) At what temperature will the tungsten filament have a resistance of 1.5 Ω?

25. (a) State conditions under which two functions, f and g, will be inverses, and give several examples of such functions.

(b) In words, what is the relationship between the graphs of $y = f(x)$ and $y = g(x)$ when f and g are inverse functions?

(c) What is the relationship between the domains and ranges of inverse functions f and g?

(d) What condition must be satisfied for a function f to have an inverse? Give some examples of functions that do not have inverses.

26. Let $f(x) = (ax + b)/(cx + d)$. What conditions on a, b, c, and d guarantee that f^{-1} exists? Find $f^{-1}(x)$.

27. In each part, find $f^{-1}(x)$ if the inverse exists.

(a) $f(x) = 8x^3 - 1$
(b) $f(x) = x^2 - 2x + 1$
(c) $f(x) = 3/(x + 1)$
(d) $f(x) = (x + 2)/(x - 1)$

Chapter I

Limits and Continuity

1

LIMITS AND CONTINUITY

Joe McBride/Stone/Getty Images

Air resistance prevents the velocity of a skydiver from increasing indefinitely. The velocity approaches a limit, called the "terminal velocity."

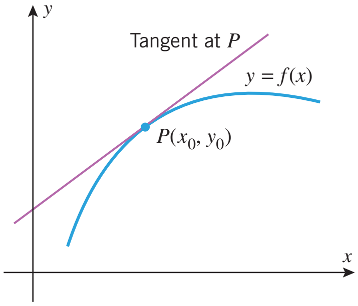
The development of calculus in the seventeenth century by Newton and Leibniz provided scientists with their first real understanding of what is meant by an "instantaneous rate of change" such as velocity and acceleration. Once the idea was understood conceptually, efficient computational methods followed, and science took a quantum leap forward. The fundamental building block on which rates of change rest is the concept of a "limit," an idea that is so important that all other calculus concepts are now based on it.

In this chapter we will develop the concept of a limit in stages, proceeding from an informal, intuitive notion to a precise mathematical definition. We will also develop theorems and procedures for calculating limits, and we will conclude the chapter by using the limits to study "continuous" curves.

1.1 LIMITS (AN INTUITIVE APPROACH)

The concept of a "limit" is the fundamental building block on which all calculus concepts are based. In this section we will study limits informally, with the goal of developing an intuitive feel for the basic ideas. In the next three sections we will focus on computational methods and precise definitions.

Many of the ideas of calculus originated with the following two geometric problems:



▲ **Figure 1.1.1**

THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at P (Figure 1.1.1).

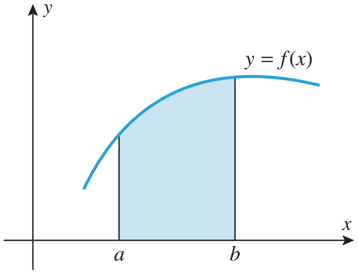
THE AREA PROBLEM Given a function f, find the area between the graph of f and an interval $[a, b]$ on the x-axis (Figure 1.1.2).

Traditionally, that portion of calculus arising from the tangent line problem is called ***differential calculus*** and that arising from the area problem is called ***integral calculus***. However, we will see later that the tangent line and area problems are so closely related that the distinction between differential and integral calculus is somewhat artificial.

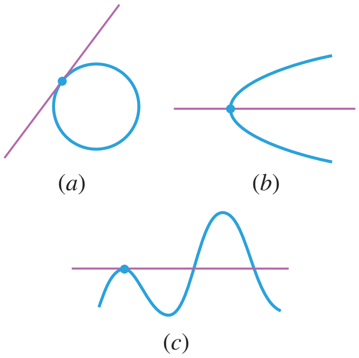
TANGENT LINES AND LIMITS

In plane geometry, a line is called *tangent* to a circle if it meets the circle at precisely one point (Figure 1.1.3*a*). Although this definition is adequate for circles, it is not appropriate for more general curves. For example, in Figure 1.1.3*b*, the line meets the curve exactly once but is obviously not what we would regard to be a tangent line; and in Figure 1.1.3*c*, the line appears to be tangent to the curve, yet it intersects the curve more than once.

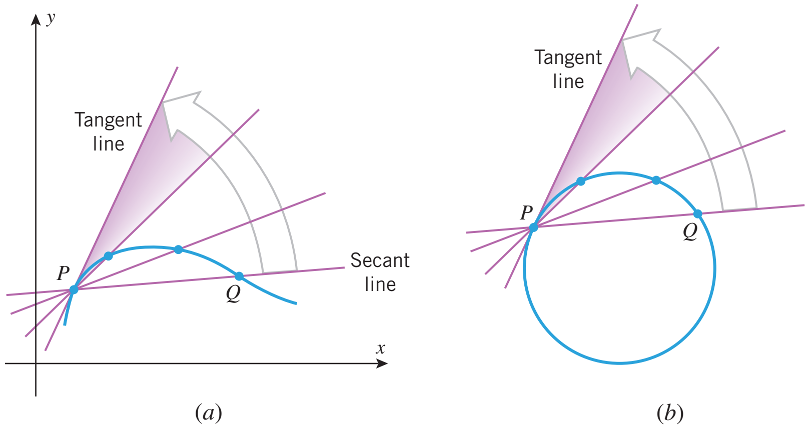
To obtain a definition of a tangent line that applies to curves other than circles, we must view tangent lines another way. For this purpose, suppose that we are interested in the tangent line at a point P on a curve in the xy-plane and that Q is any point that lies on the curve and is different from P. The line through P and Q is called a ***secant line*** for the curve at P. Intuition suggests that if we move the point Q along the curve toward P, then the secant line will rotate toward a *limiting position*. The line in this limiting position is what we will consider to be the ***tangent line*** at P (Figure 1.1.4*a*). As suggested by Figure 1.1.4*b*, this new concept of a tangent line coincides with the traditional concept when applied to circles.



▲ **Figure 1.1.2**



▲ **Figure 1.1.3**



(*a*) (*b*)

▶ **Figure 1.1.4**

▶ **Example 1** Find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution. If we can find the slope $m_{\tan}$ of the tangent line at P, then we can use the point P and the point-slope formula for a line (Web Appendix G) to write the equation of the tangent line as

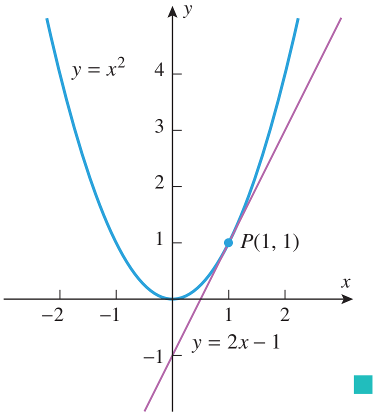
$$y - 1 = m_{\tan}(x - 1) \tag{1}$$

To find the slope $m_{\tan}$, consider the secant line through P and a point $Q(x, x^2)$ on the parabola that is distinct from P. The slope $m_{\sec}$ of this secant line is

Why are we requiring that P and Q be distinct?

$$m_{\sec} = \frac{x^2 - 1}{x - 1} \tag{2}$$

Figure 1.1.4*a* suggests that if we now let Q move along the parabola, getting closer and closer to P, then the limiting position of the secant line through P and Q will coincide with that of the tangent line at P. This in turn suggests that the value of $m_{\sec}$ will get closer and closer to the value of $m_{\tan}$ as P moves toward Q along the curve. However, to say that $Q(x, x^2)$ gets closer and closer to $P(1, 1)$ is algebraically equivalent to saying that x gets closer and closer to 1. Thus, the problem of finding $m_{\tan}$ reduces to finding the "limiting value" of $m_{\sec}$ in Formula (2) as x gets closer and closer to 1 (but with $x \neq 1$ to ensure that P and Q remain distinct).



▲ **Figure 1.1.5**

We can rewrite (2) as

$$m_{\text{sec}} = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)} = x + 1$$

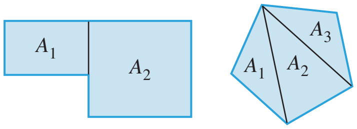
where the cancellation of the factor $(x - 1)$ is allowed because $x \neq 1$. It is now evident that m_{sec} gets closer and closer to 2 as x gets closer and closer to 1. Thus, $m_{\tan} = 2$ and (1) implies that the equation of the tangent line is

$$y - 1 = 2(x - 1) \quad \text{or equivalently} \quad y = 2x - 1$$

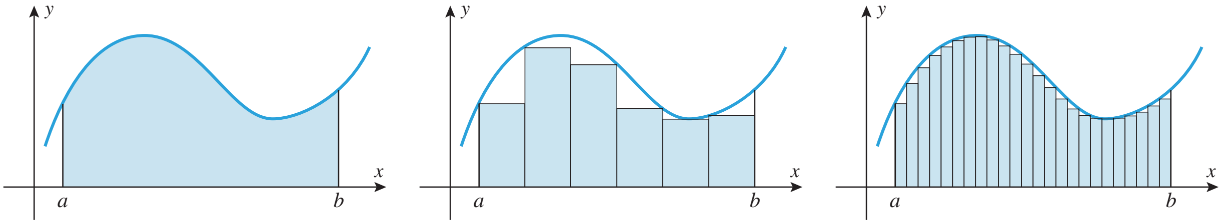
Figure 1.1.5 shows the graph of $y = x^2$ and this tangent line. ◄

AREAS AND LIMITS

Just as the general notion of a tangent line leads to the concept of *limit*, so does the general notion of area. For plane regions with straight-line boundaries, areas can often be calculated by subdividing the region into rectangles or triangles and adding the areas of the constituent parts (Figure 1.1.6). However, for regions with curved boundaries, such as that in Figure 1.1.7*a*, a more general approach is needed. One such approach is to begin by approximating the area of the region by inscribing a number of rectangles of equal width under the curve and adding the areas of these rectangles (Figure 1.1.7*b*). Intuition suggests that if we repeat that approximation process using more and more rectangles, then the rectangles will tend to fill in the gaps under the curve, and the approximations will get closer and closer to the exact area under the curve (Figure 1.1.7*c*). This suggests that we can define the area under the curve to be the limiting value of these approximations. This idea will be considered in detail later, but the point to note here is that once again the concept of a limit comes into play.



▲ **Figure 1.1.6**



(*a*) (*b*) (*c*)

▲ **Figure 1.1.7**

DECIMALS AND LIMITS

Limits also arise in the familiar context of decimals. For example, the decimal expansion of the fraction $\frac{1}{3}$ is

$$\frac{1}{3} = 0.33333\ldots \tag{3}$$

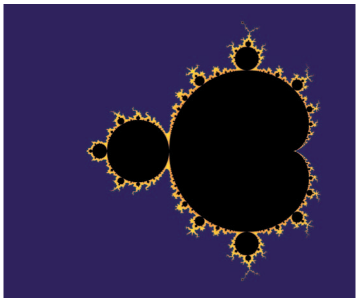
in which the dots indicate that the digit 3 repeats indefinitely. Although you may not have thought about decimals in this way, we can write (3) as

$$\frac{1}{3} = 0.33333\ldots = 0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 + \cdots \tag{4}$$

which is a sum with “infinitely many” terms. As we will discuss in more detail later, we interpret (4) to mean that the succession of finite sums

$$0.3, \quad 0.3 + 0.03, \quad 0.3 + 0.03 + 0.003, \quad 0.3 + 0.03 + 0.003 + 0.0003, \ldots$$

gets closer and closer to a limiting value of $\frac{1}{3}$ as more and more terms are included. Thus, limits even occur in the familiar context of decimal representations of real numbers.



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This figure shows a region called the ***Mandelbrot Set****. It illustrates how complicated a region in the plane can be and why the notion of area requires careful definition.*

LIMITS

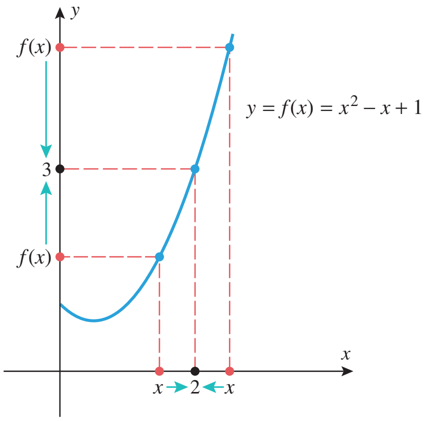
Now that we have seen how limits arise in various ways, let us focus on the limit concept itself.

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value. For example, let us examine the behavior of the function

$$f(x) = x^2 - x + 1$$

for x-values closer and closer to 2. It is evident from the graph and table in Figure 1.1.8 that the values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2. We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3 \tag{5}$$



x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

▲ **Figure 1.1.8**

This leads us to the following general idea.

1.1.1 LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L \tag{6}$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The expression in (6) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a \tag{7}$$

Since x is required to be different from a in (6), the value of f at a, or even whether f is defined at a, has no bearing on the limit L. The limit describes the behavior of f *close to* a but not *at* a.

Example 2 Use numerical evidence to make a conjecture about the value of

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} \tag{8}$$

Solution. Although the function

$$f(x) = \frac{x-1}{\sqrt{x}-1} \tag{9}$$

is undefined at $x = 1$, this has no bearing on the limit. Table 1.1.1 shows sample x-values approaching 1 from the left side and from the right side. In both cases the corresponding values of $f(x)$, calculated to six decimal places, appear to get closer and closer to 2, and hence we conjecture that

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} = 2$$

This is consistent with the graph of f shown in Figure 1.1.9. In the next section we will show how to obtain this result algebraically. ◀

TECHNOLOGY MASTERY

Use a graphing utility to generate the graph of the equation $y = f(x)$ for the function in (9). Find a window containing $x = 1$ in which all values of $f(x)$ are within 0.5 of $y = 2$ and one in which all values of $f(x)$ are within 0.1 of $y = 2$.

Table 1.1.1

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

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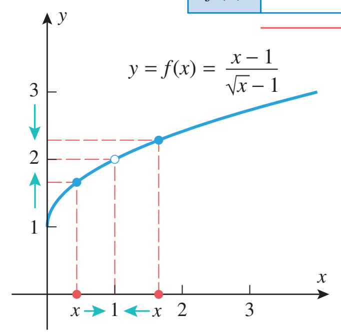


▲ **Figure 1.1.9**

Example 3 Use numerical evidence to make a conjecture about the value of

$$\lim_{x\to 0} \frac{\sin x}{x} \tag{10}$$

Solution. With the help of a calculating utility set in radian mode, we obtain Table 1.1.2. The data in the table suggest that

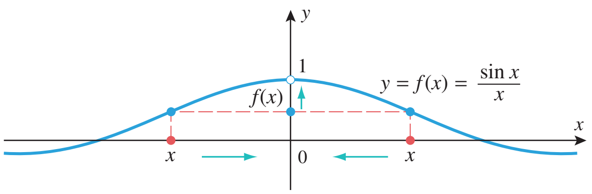
$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{11}$$

The result is consistent with the graph of $f(x) = (\sin x)/x$ shown in Figure 1.1.10. Later in this chapter we will give a geometric argument to prove that our conjecture is correct. ◀

Use numerical evidence to determine whether the limit in (11) changes if x is measured in degrees.

Table 1.1.2

x (RADIANS)	$y = \frac{\sin x}{x}$
±1.0	0.84147
±0.9	0.87036
±0.8	0.89670
±0.7	0.92031
±0.6	0.94107
±0.5	0.95885
±0.4	0.97355
±0.3	0.98507
±0.2	0.99335
±0.1	0.99833
±0.01	0.99998



As x approaches 0 from the left or right, $f(x)$ approaches 1.

▶ **Figure 1.1.10**

SAMPLING PITFALLS

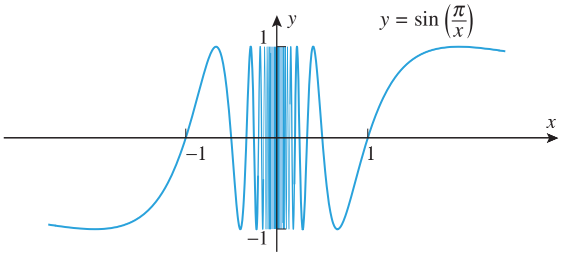
Numerical evidence can sometimes lead to incorrect conclusions about limits because of roundoff error or because the sample values chosen do not reveal the true limiting behavior. For example, one might *incorrectly* conclude from Table 1.1.3 that

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$$

The fact that this is not correct is evidenced by the graph of f in Figure 1.1.11. The graph reveals that the values of f oscillate between -1 and 1 with increasing rapidity as $x \to 0$ and hence do not approach a limit. The data in the table deceived us because the x-values selected all happened to be x-intercepts for $f(x)$. This points out the need for having alternative methods for corroborating limits conjectured from numerical evidence.

Table 1.1.3

x	$\frac{\pi}{x}$	$f(x) = \sin\left(\frac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
⋮	⋮	⋮

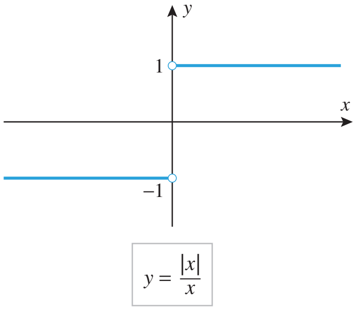


▲ **Figure 1.1.11**

ONE-SIDED LIMITS

The limit in (6) is called a ***two-sided limit*** because it requires the values of $f(x)$ to get closer and closer to L as values of x are taken from *either* side of $x = a$. However, some functions exhibit different behaviors on the two sides of an x-value a, in which case it is necessary to distinguish whether values of x near a are on the left side or on the right side of a for purposes of investigating limiting behavior. For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{12}$$



▲ **Figure 1.1.12**

which is graphed in Figure 1.1.12. As x approaches 0 from the *right*, the values of $f(x)$ approach a limit of 1 [in fact, the values of $f(x)$ are exactly 1 for all such x], and similarly, as x approaches 0 from the *left*, the values of $f(x)$ approach a limit of -1. We denote these limits by writing

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = -1 \tag{13}$$

With this notation, the superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

This leads to the general idea of a ***one-sided limit***.

As with two-sided limits, the one-sided limits in (14) and (15) can also be written as

$$f(x) \to L \quad \text{as} \quad x \to a^+$$

and

$$f(x) \to L \quad \text{as} \quad x \to a^-$$

respectively.

1.1.2 ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \tag{14}$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \tag{15}$$

Expression (14) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right." Similarly, expression (15) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

THE RELATIONSHIP BETWEEN ONE-SIDED LIMITS AND TWO-SIDED LIMITS

In general, there is no guarantee that a function f will have a two-sided limit at a given point a; that is, the values of $f(x)$ may not get closer and closer to any *single* real number L as $x \to a$. In this case we say that

$$\lim_{x \to a} f(x) \quad \textit{\textbf{does not exist}}$$

Similarly, the values of $f(x)$ may not get closer and closer to a single real number L as $x \to a^+$ or as $x \to a^-$. In these cases we say that

$$\lim_{x \to a^+} f(x) \quad \textit{\textbf{does not exist}}$$

or that

$$\lim_{x \to a^-} f(x) \quad \textit{\textbf{does not exist}}$$

In order for the two-sided limit of a function $f(x)$ to exist at a point a, the values of $f(x)$ must approach some real number L as x approaches a, and this number must be the same regardless of whether x approaches a from the left or the right. This suggests the following result, which we state without formal proof.

1.1.3 THE RELATIONSHIP BETWEEN ONE-SIDED AND TWO-SIDED LIMITS The two-sided limit of a function $f(x)$ exists at a if and only if both of the one-sided limits exist at a and have the same value; that is,

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

▶ **Example 4** Explain why

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist.

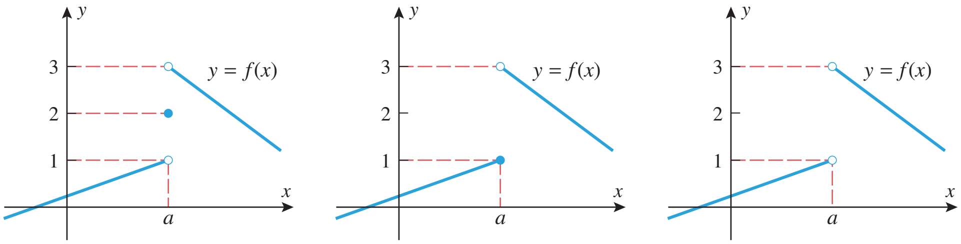
Solution. As x approaches 0, the values of $f(x) = |x|/x$ approach -1 from the left and approach 1 from the right [see (13)]. Thus, the one-sided limits at 0 are not the same. ◀

▶ **Example 5** For the functions in Figure 1.1.13, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are

$$\lim_{x \to a^+} f(x) = 3 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 1$$

In all three cases the two-sided limit does not exist as $x \to a$ because the one-sided limits are not equal. ◀



▶ **Figure 1.1.13**

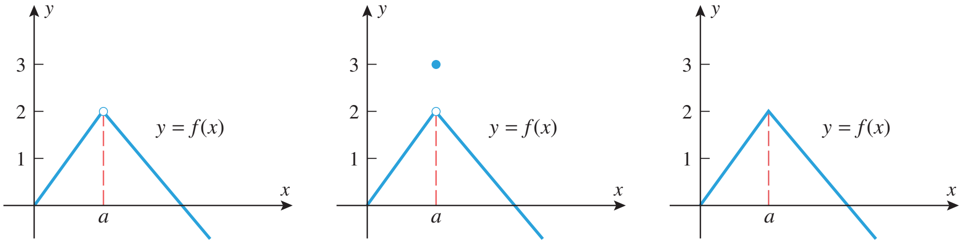
▶ Example 6 For the functions in Figure 1.1.14, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution. As in the preceding example, the value of f at $x = a$ has no bearing on the limits as $x \to a$, so in all three cases we have

$$\lim_{x \to a^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to a^-} f(x) = 2$$

Since the one-sided limits are equal, the two-sided limit exists and

$$\lim_{x \to a} f(x) = 2 \ ◀$$



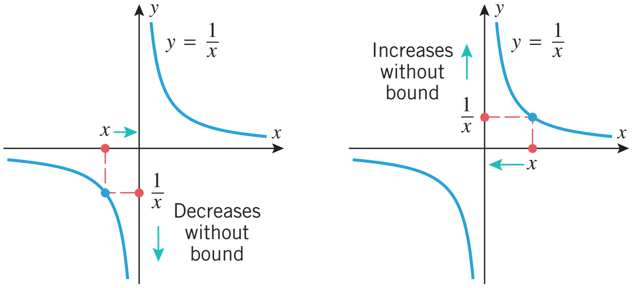
▲ **Figure 1.1.14**

INFINITE LIMITS

Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound. For example, consider the behavior of $f(x) = 1/x$ for values of x near 0. It is evident from the table and graph in Figure 1.1.15 that as x-values are taken closer and closer to 0 from the right, the values of $f(x) = 1/x$ are positive and increase without bound; and as x-values are taken closer and closer to 0 from the left, the values of $f(x) = 1/x$ are negative and decrease without bound. We describe these limiting behaviors by writing

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The symbols $+\infty$ and $-\infty$ here are *not* real numbers; they simply describe particular ways in which the limits fail to exist. Do not make the mistake of manipulating these symbols using rules of algebra. For example, it is *incorrect* to write $(+\infty) - (+\infty) = 0$.



x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

Left side → ← Right side

▲ **Figure 1.1.15**

1.1.4 INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x\to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = +\infty$$

Similarly, the expressions

$$\lim_{x\to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = -\infty$$

▶ **Example 7** For the functions in Figure 1.1.16, describe the limits at $x = a$ in appropriate limit notation.

Solution (a). In Figure 1.1.16*a*, the function increases without bound as x approaches a from the right and decreases without bound as x approaches a from the left. Thus,

$$\lim_{x\to a^+} \frac{1}{x-a} = +\infty \quad \text{and} \quad \lim_{x\to a^-} \frac{1}{x-a} = -\infty$$

Solution (b). In Figure 1.1.16*b*, the function increases without bound as x approaches a from both the left and right. Thus,

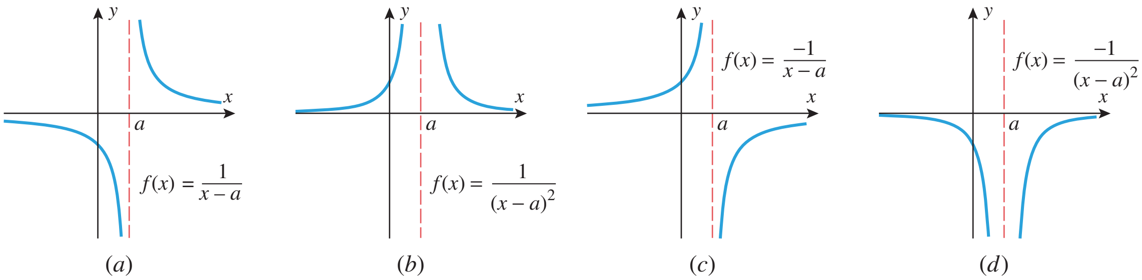
$$\lim_{x\to a} \frac{1}{(x-a)^2} = \lim_{x\to a^+} \frac{1}{(x-a)^2} = \lim_{x\to a^-} \frac{1}{(x-a)^2} = +\infty$$

Solution (c). In Figure 1.1.16*c*, the function decreases without bound as x approaches a from the right and increases without bound as x approaches a from the left. Thus,

$$\lim_{x\to a^+} \frac{-1}{x-a} = -\infty \quad \text{and} \quad \lim_{x\to a^-} \frac{-1}{x-a} = +\infty$$

Solution (d). In Figure 1.1.16*d*, the function decreases without bound as x approaches a from both the left and right. Thus,

$$\lim_{x\to a} \frac{-1}{(x-a)^2} = \lim_{x\to a^+} \frac{-1}{(x-a)^2} = \lim_{x\to a^-} \frac{-1}{(x-a)^2} = -\infty \blacktriangleleft$$



(*a*) (*b*) (*c*) (*d*)

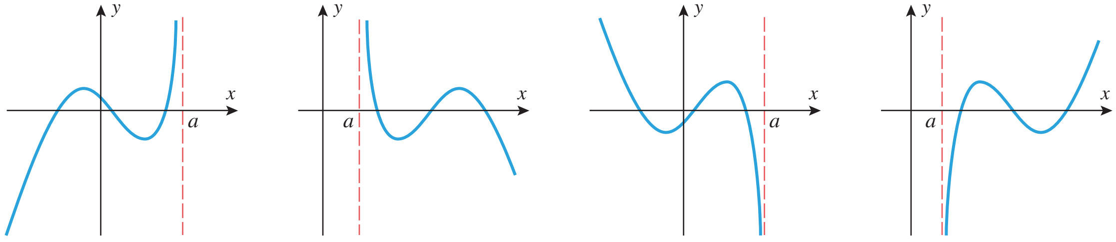
▲ **Figure 1.1.16**

VERTICAL ASYMPTOTES

Figure 1.1.17 illustrates geometrically what happens when any of the following situations occur:

$$\lim_{x\to a^-} f(x) = +\infty, \quad \lim_{x\to a^+} f(x) = +\infty, \quad \lim_{x\to a^-} f(x) = -\infty, \quad \lim_{x\to a^+} f(x) = -\infty$$

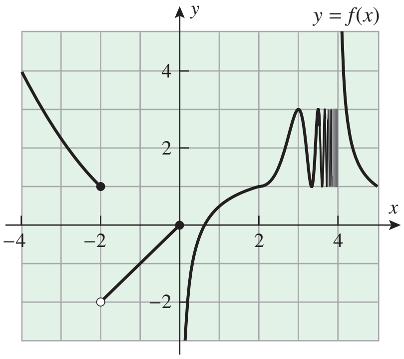
In each case the graph of $y = f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit. The line $x = a$ is called a ***vertical asymptote*** of the curve $y = f(x)$ (from the Greek word *asymptotos*, meaning "nonintersecting").



$\lim_{x\to a^-} f(x) = +\infty$ $\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$

▲ **Figure 1.1.17**

In general, the graph of a single function can display a wide variety of limits.



▲ **Figure 1.1.18**

▶ **Example 8** For the function f graphed in Figure 1.1.18, find

(a) $\lim_{x\to -2^-} f(x)$ (b) $\lim_{x\to -2^+} f(x)$ (c) $\lim_{x\to 0^-} f(x)$ (d) $\lim_{x\to 0^+} f(x)$
(e) $\lim_{x\to 4^-} f(x)$ (f) $\lim_{x\to 4^+} f(x)$ (g) the vertical asymptotes of the graph of f.

Solution (a) and (b).

$$\lim_{x\to -2^-} f(x) = 1 = f(-2) \quad\text{and}\quad \lim_{x\to -2^+} f(x) = -2$$

Solution (c) and (d).

$$\lim_{x\to 0^-} f(x) = 0 = f(0) \quad\text{and}\quad \lim_{x\to 0+} f(x) = -\infty$$

Solution (e) and (f).

$$\lim_{x\to 4^-} f(x) \text{ does not exist due to oscillation} \quad\text{and}\quad \lim_{x\to 4^+} f(x) = +\infty$$

Solution (g). The y-axis and the line $x = 4$ are vertical asymptotes for the graph of f. ◀

QUICK CHECK EXERCISES 1.1 *(See page 61 for answers.)*

1. We write $\lim_{x\to a} f(x) = L$ provided the values of ________ can be made as close to ________ as desired, by taking values of ________ sufficiently close to ________ but not ________.

2. We write $\lim_{x\to a^-} f(x) = +\infty$ provided ________ increases without bound, as ________ approaches ________ from the left.

3. State what must be true about

$$\lim_{x\to a^-} f(x) \quad\text{and}\quad \lim_{x\to a^+} f(x)$$

in order for it to be the case that

$$\lim_{x\to a} f(x) = L$$

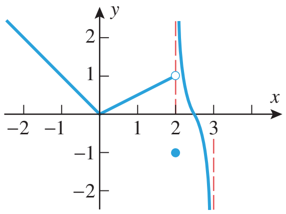
4. Use the accompanying graph of $y = f(x)$ $(-\infty < x < 3)$ to determine the limits.

(a) $\lim_{x\to 0} f(x) =$ ________

(b) $\lim_{x\to 2^-} f(x) =$ ________

(c) $\lim_{x\to 2^+} f(x) =$ ________

(d) $\lim_{x\to 3^-} f(x) =$ ________



◀ **Figure Ex-4**

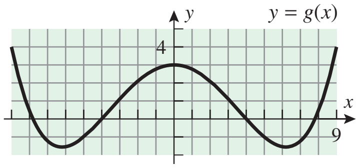
5. The slope of the secant line through $P(2, 4)$ and $Q(x, x^2)$ on the parabola $y = x^2$ is $m_{\text{sec}} = x + 2$. It follows that the slope of the tangent line to this parabola at the point P is ________.

EXERCISE SET 1.1

Graphing Utility C CAS

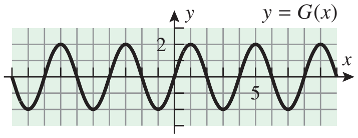
1–10 In these exercises, make reasonable assumptions about the graph of the indicated function outside of the region depicted. ■

1. For the function g graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} g(x)$ (b) $\lim_{x\to 0^+} g(x)$
(c) $\lim_{x\to 0} g(x)$ (d) $g(0)$.



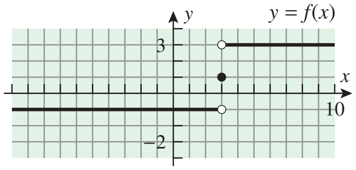
◀ **Figure Ex-1**

2. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
(c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.



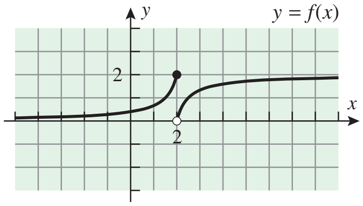
◀ **Figure Ex-2**

3. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.



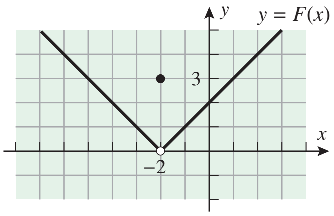
◀ **Figure Ex-3**

4. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$
(c) $\lim_{x\to 2} f(x)$ (d) $f(2)$.



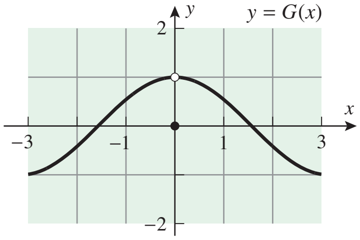
◀ **Figure Ex-4**

5. For the function F graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} F(x)$ (b) $\lim_{x\to -2^+} F(x)$
(c) $\lim_{x\to -2} F(x)$ (d) $F(-2)$.



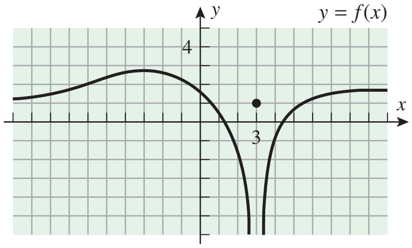
◀ **Figure Ex-5**

6. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
(c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.



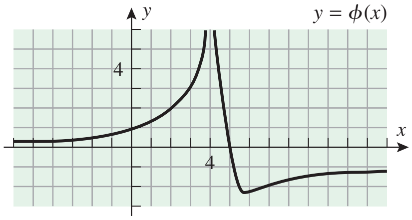
◀ **Figure Ex-6**

7. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.



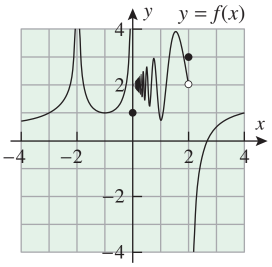
◀ **Figure Ex-7**

8. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to 4^-} \phi(x)$ (b) $\lim_{x\to 4^+} \phi(x)$
(c) $\lim_{x\to 4} \phi(x)$ (d) $\phi(4)$.



◀ **Figure Ex-8**

9. For the function f graphed in the accompanying figure on the next page, find
(a) $\lim_{x\to -2} f(x)$ (b) $\lim_{x\to 0^-} f(x)$
(c) $\lim_{x\to 0^+} f(x)$ (d) $\lim_{x\to 2^-} f(x)$
(e) $\lim_{x\to 2^+} f(x)$
(f) the vertical asymptotes of the graph of f.



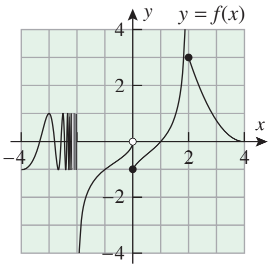
◀ **Figure Ex-9**

10. For the function f graphed in the accompanying figure, find

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $\lim_{x \to 0^-} f(x)$

(d) $\lim_{x \to 0^+} f(x)$ (e) $\lim_{x \to 2^-} f(x)$ (f) $\lim_{x \to 2^+} f(x)$

(g) the vertical asymptotes of the graph of f.



◀ **Figure Ex-10**

11–12 (i) Complete the table and make a guess about the limit indicated. (ii) Confirm your conclusions about the limit by graphing a function over an appropriate interval. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.] ■

11. $f(x) = \dfrac{\sin 2x}{x}$; $\lim_{x \to 0} f(x)$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

▲ **Table Ex-11**

12. $f(x) = \dfrac{\cos x - 1}{x^2}$; $\lim_{x \to 0} f(x)$

x	−0.5	−0.05	−0.005	0.005	0.05	0.5
$f(x)$						

▲ **Table Ex-12**

13–16 (i) Make a guess at the limit (if it exists) by evaluating the function at the specified x-values. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.] ■

13. (a) $\lim_{x \to 1} \dfrac{x-1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 0, 0.5, 0.9, 0.99, 0.999$

(b) $\lim_{x \to 1^+} \dfrac{x+1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 1.0001$

(c) $\lim_{x \to 1^-} \dfrac{x+1}{x^3-1}$; $x = 0, 0.5, 0.9, 0.99, 0.999, 0.9999$

14. (a) $\lim_{x \to 0} \dfrac{\sqrt{x+1}-1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\lim_{x \to 0^+} \dfrac{\sqrt{x+1}+1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$

(c) $\lim_{x \to 0^-} \dfrac{\sqrt{x+1}+1}{x}$; $x = -0.25, -0.1, -0.001, -0.0001$

15. (a) $\lim_{x \to 0} \dfrac{\sin 3x}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

(b) $\lim_{x \to -1} \dfrac{\cos x}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

16. (a) $\lim_{x \to -1} \dfrac{\tan(x+1)}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

(b) $\lim_{x \to 0} \dfrac{\sin(5x)}{\sin(2x)}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

17–20 True–False Determine whether the statement is true or false. Explain your answer. ■

17. If $f(a) = L$, then $\lim_{x \to a} f(x) = L$.

18. If $\lim_{x \to a} f(x)$ exists, then so do $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

19. If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist, then so does $\lim_{x \to a} f(x)$.

20. If $\lim_{x \to a^+} f(x) = +\infty$, then $f(a)$ is undefined.

21–26 Sketch a possible graph for a function f with the specified properties. (Many different solutions are possible.) ■

21. (i) the domain of f is $[-1, 1]$

(ii) $f(-1) = f(0) = f(1) = 0$

(iii) $\lim_{x \to -1^+} f(x) = \lim_{x \to 0} f(x) = \lim_{x \to 1^-} f(x) = 1$

22. (i) the domain of f is $[-2, 1]$

(ii) $f(-2) = f(0) = f(1) = 0$

(iii) $\lim_{x \to -2^+} f(x) = 2$, $\lim_{x \to 0} f(x) = 0$, and $\lim_{x \to 1^-} f(x) = 1$

23. (i) the domain of f is $(-\infty, 0]$

(ii) $f(-2) = f(0) = 1$

(iii) $\lim_{x \to -2} f(x) = +\infty$

24. (i) the domain of f is $(0, +\infty)$

(ii) $f(1) = 0$

(iii) the y-axis is a vertical asymptote for the graph of f

(iv) $f(x) < 0$ if $0 < x < 1$

25. (i) $f(-3) = f(0) = f(2) = 0$

(ii) $\lim_{x\to -2^-} f(x) = +\infty$ and $\lim_{x\to -2^+} f(x) = -\infty$

(iii) $\lim_{x\to 1} f(x) = +\infty$

26. (i) $f(-1) = 0,\ f(0) = 1,\ f(1) = 0$

(ii) $\lim_{x\to -1^-} f(x) = 0$ and $\lim_{x\to -1^+} f(x) = +\infty$

(iii) $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = +\infty$

27–30 Modify the argument of Example 1 to find the equation of the tangent line to the specified graph at the point given. ■

27. the graph of $y = x^2$ at $(-1, 1)$

28. the graph of $y = x^2$ at $(0, 0)$

29. the graph of $y = x^4$ at $(1, 1)$

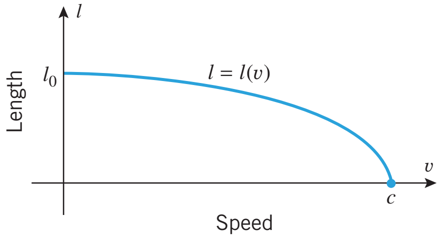
30. the graph of $y = x^4$ at $(-1, 1)$

FOCUS ON CONCEPTS

31. In the special theory of relativity the length l of a narrow rod moving longitudinally is a function $l = l(v)$ of the rod's speed v. The accompanying figure, in which c denotes the speed of light, displays some of the qualitative features of this function.

(a) What is the physical interpretation of l_0?

(b) What is $\lim_{v\to c^-} l(v)$? What is the physical significance of this limit?

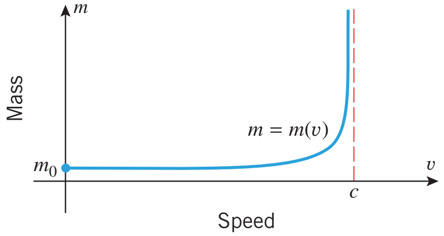


▲ **Figure Ex-31**

32. In the special theory of relativity the mass m of a moving object is a function $m = m(v)$ of the object's speed v. The accompanying figure, in which c denotes the speed of light, displays some of the qualitative features of this function.

(a) What is the physical interpretation of m_0?

(b) What is $\lim_{v\to c^-} m(v)$? What is the physical significance of this limit?



▲ **Figure Ex-32**

33. Let

$$f(x) = (1 + x^2)^{1.1/x^2}$$

(a) Graph f in the window

$$[-1, 1] \times [2.5, 3.5]$$

and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(b) Graph f in the window

$$[-0.001, 0.001] \times [2.5, 3.5]$$

and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(c) Graph f in the window

$$[-0.000001, 0.000001] \times [2.5, 3.5]$$

and use the calculator's trace feature to make a conjecture about the limit of $f(x)$ as $x \to 0$.

(d) Later we will be able to show that

$$\lim_{x\to 0} (1 + x^2)^{1.1/x^2} \approx 3.00416602$$

What flaw do your graphs reveal about using numerical evidence (as revealed by the graphs you obtained) to make conjectures about limits?

34. Writing Two students are discussing the limit of $\sqrt{x}$ as x approaches 0. One student maintains that the limit is 0, while the other claims that the limit does not exist. Write a short paragraph that discusses the pros and cons of each student's position.

35. Writing Given a function f and a real number a, explain informally why

$$\lim_{x\to 0} f(x + a) = \lim_{x\to a} f(x)$$

(Here "equality" means that either both limits exist and are equal or that both limits fail to exist.)

✔QUICK CHECK ANSWERS 1.1

1. $f(x)$; L; x; a; a **2.** $f(x)$; x; a **3.** Both one-sided limits must exist and equal L. **4.** (a) 0 (b) 1 (c) $+\infty$ (d) $-\infty$ **5.** 4

1.2 COMPUTING LIMITS

In this section we will discuss techniques for computing limits of many functions. We base these results on the informal development of the limit concept discussed in the preceding section. A more formal derivation of these results is possible after Section 1.4.

SOME BASIC LIMITS

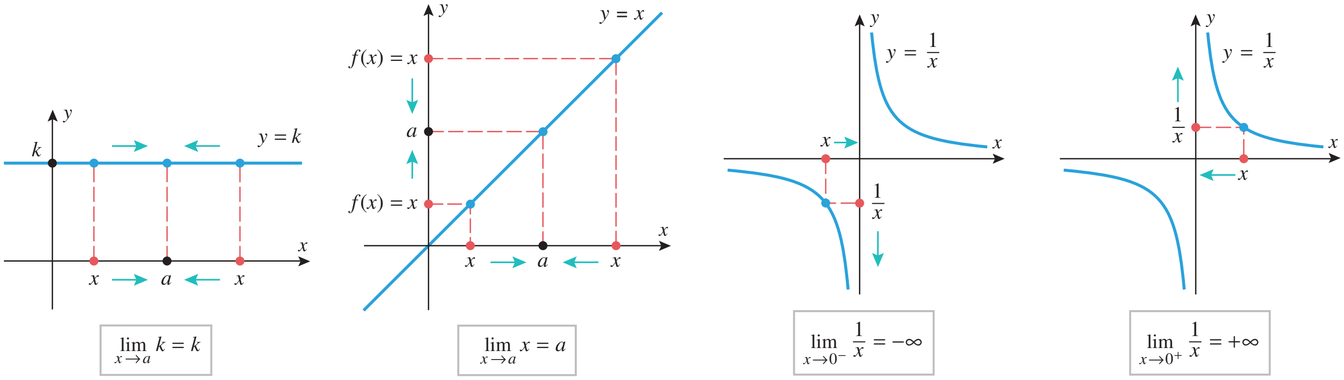
Our strategy for finding limits algebraically has two parts:

- First we will obtain the limits of some simple functions.
- Then we will develop a repertoire of theorems that will enable us to use the limits of those simple functions as building blocks for finding limits of more complicated functions.

We start with the following basic results, which are illustrated in Figure 1.2.1.

1.2.1 THEOREM *Let a and k be real numbers.*

(*a*) $\lim_{x\to a} k = k$ (*b*) $\lim_{x\to a} x = a$ (*c*) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (*d*) $\lim_{x\to 0^+} \frac{1}{x} = +\infty$



▲ **Figure 1.2.1**

The following examples explain these results further.

Do not confuse the algebraic size of a number with its closeness to zero. For positive numbers, the smaller the number the closer it is to zero, but for negative numbers, the larger the number the closer it is to zero. For example, -2 is larger than -4, but it is closer to zero.

▶ **Example 1** If $f(x) = k$ is a constant function, then the values of $f(x)$ remain fixed at k as x varies, which explains why $f(x)\to k$ as $x\to a$ for all values of a. For example,

$$\lim_{x\to -25} 3 = 3, \qquad \lim_{x\to 0} 3 = 3, \qquad \lim_{x\to \pi} 3 = 3 \blacktriangleleft$$

▶ **Example 2** If $f(x) = x$, then as $x\to a$ it must also be true that $f(x)\to a$. For example,

$$\lim_{x\to 0} x = 0, \qquad \lim_{x\to -2} x = -2, \qquad \lim_{x\to \pi} x = \pi \blacktriangleleft$$

▶ **Example 3** You should know from your experience with fractions that for a fixed nonzero numerator, the closer the denominator is to zero, the larger the absolute value of the fraction. This fact and the data in Table 1.2.1 suggest why $1/x \to +\infty$ as $x \to 0^+$ and why $1/x \to -\infty$ as $x \to 0^-$. ◀

Table 1.2.1

	VALUES						CONCLUSION
x $1/x$	-1 -1	-0.1 -10	-0.01 -100	-0.001 -1000	-0.0001 $-10{,}000$	$\cdots$ $\cdots$	As $x \to 0^-$ the value of $1/x$ decreases without bound.
x $1/x$	1 1	0.1 10	0.01 100	0.001 1000	0.0001 $10{,}000$	$\cdots$ $\cdots$	As $x \to 0^+$ the value of $1/x$ increases without bound.

The following theorem, parts of which are proved in Appendix D, will be our basic tool for finding limits algebraically.

1.2.2 THEOREM *Let a be a real number, and suppose that*

$$\lim_{x\to a} f(x) = L_1 \quad \textit{and} \quad \lim_{x\to a} g(x) = L_2$$

That is, the limits exist and have values L_1 *and* L_2*, respectively. Then:*

(*a*) $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(*b*) $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$

(*c*) $\lim_{x\to a} [f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$

(*d*) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, *provided* $L_2 \neq 0$

(*e*) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$, *provided* $L_1 > 0$ *if n is even.*

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ *or as* $x \to a^+$.

Theorem 1.2.2(*e*) remains valid for n even and $L_1 = 0$, provided $f(x)$ is nonnegative for x near a with $x \neq a$.

This theorem can be stated informally as follows:

(*a*) *The limit of a sum is the sum of the limits.*

(*b*) *The limit of a difference is the difference of the limits.*

(*c*) *The limit of a product is the product of the limits.*

(*d*) *The limit of a quotient is the quotient of the limits, provided the limit of the denominator is not zero.*

(*e*) *The limit of an nth root is the nth root of the limit.*

For the special case of part (*c*) in which $f(x) = k$ is a constant function, we have

$$\lim_{x\to a} (kg(x)) = \lim_{x\to a} k \cdot \lim_{x\to a} g(x) = k \lim_{x\to a} g(x) \tag{1}$$

and similarly for one-sided limits. This result can be rephrased as follows:

A constant factor can be moved through a limit symbol.

Although parts (a) and (c) of Theorem 1.2.2 are stated for two functions, the results hold for any finite number of functions. Moreover, the various parts of the theorem can be used in combination to reformulate expressions involving limits.

▶ Example 4

$$\lim_{x\to a}[f(x)-g(x)+2h(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) + 2\lim_{x\to a} h(x)$$

$$\lim_{x\to a}[f(x)g(x)h(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a} h(x)\right)$$

$$\lim_{x\to a}[f(x)]^3 = \left(\lim_{x\to a} f(x)\right)^3 \qquad \boxed{\text{Take } g(x)=h(x)=f(x) \text{ in the last equation.}}$$

$$\lim_{x\to a}[f(x)]^n = \left(\lim_{x\to a} f(x)\right)^n \qquad \boxed{\begin{array}{l}\text{The extension of Theorem 1.2.2}(c)\text{ in which}\\ \text{there are } n \text{ factors, each of which is } f(x)\end{array}}$$

$$\lim_{x\to a} x^n = \left(\lim_{x\to a} x\right)^n = a^n \qquad \boxed{\text{Apply the previous result with } f(x)=x.} \;\blacktriangleleft$$

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS AS $x \to a$

▶ Example 5 Find $\lim\limits_{x\to 5}(x^2-4x+3)$.

Solution.

$$\begin{aligned}\lim_{x\to 5}(x^2-4x+3) &= \lim_{x\to 5} x^2 - \lim_{x\to 5} 4x + \lim_{x\to 5} 3 && \boxed{\text{Theorem 1.2.2}(a), (b)}\\ &= \lim_{x\to 5} x^2 - 4\lim_{x\to 5} x + \lim_{x\to 5} 3 && \boxed{\begin{array}{l}\text{A constant can be moved}\\ \text{through a limit symbol.}\end{array}}\\ &= 5^2 - 4(5) + 3 && \boxed{\text{The last part of Example 4}}\\ &= 8 \;\blacktriangleleft\end{aligned}$$

Observe that in Example 5 the limit of the polynomial $p(x) = x^2 - 4x + 3$ as $x \to 5$ turned out to be the same as $p(5)$. This is not an accident. The next result shows that, in general, the limit of a polynomial $p(x)$ as $x \to a$ is the same as the value of the polynomial at a. Knowing this fact allows us to reduce the computation of limits of polynomials to simply evaluating the polynomial at the appropriate point.

1.2.3 THEOREM *For any polynomial*

$$p(x) = c_0 + c_1x + \cdots + c_nx^n$$

and any real number a,

$$\lim_{x\to a} p(x) = c_0 + c_1a + \cdots + c_na^n = p(a)$$

PROOF

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a} \left(c_0 + c_1x + \cdots + c_nx^n\right)\\ &= \lim_{x\to a} c_0 + \lim_{x\to a} c_1x + \cdots + \lim_{x\to a} c_nx^n\\ &= \lim_{x\to a} c_0 + c_1 \lim_{x\to a} x + \cdots + c_n \lim_{x\to a} x^n\\ &= c_0 + c_1a + \cdots + c_na^n = p(a) \quad ■\end{aligned}$$

▶ **Example 6** Find $\lim_{x\to 1} (x^7 - 2x^5 + 1)^{35}$.

Solution. The function involved is a polynomial (why?), so the limit can be obtained by evaluating this polynomial at $x = 1$. This yields

$$\lim_{x\to 1} (x^7 - 2x^5 + 1)^{35} = 0 \quad ◀$$

Recall that a rational function is a ratio of two polynomials. The following example illustrates how Theorems 1.2.2(d) and 1.2.3 can sometimes be used in combination to compute limits of rational functions.

▶ **Example 7** Find $\lim_{x\to 2} \dfrac{5x^3+4}{x-3}$.

Solution.

$$\begin{aligned}\lim_{x\to 2} \frac{5x^3+4}{x-3} &= \frac{\lim_{x\to 2} (5x^3+4)}{\lim_{x\to 2} (x-3)} && \boxed{\text{Theorem 1.2.2}(d)}\\ &= \frac{5\cdot 2^3+4}{2-3} = -44 && \boxed{\text{Theorem 1.2.3}} \quad ◀\end{aligned}$$

The method used in the last example will not work for rational functions in which the limit of the denominator is zero because Theorem 1.2.2(d) is not applicable. There are two cases of this type to be considered—the case where the limit of the denominator is zero and the limit of the numerator is not, and the case where the limits of the numerator and denominator are both zero. If the limit of the denominator is zero but the limit of the numerator is not, then one can prove that the limit of the rational function does not exist and that one of the following situations occurs:

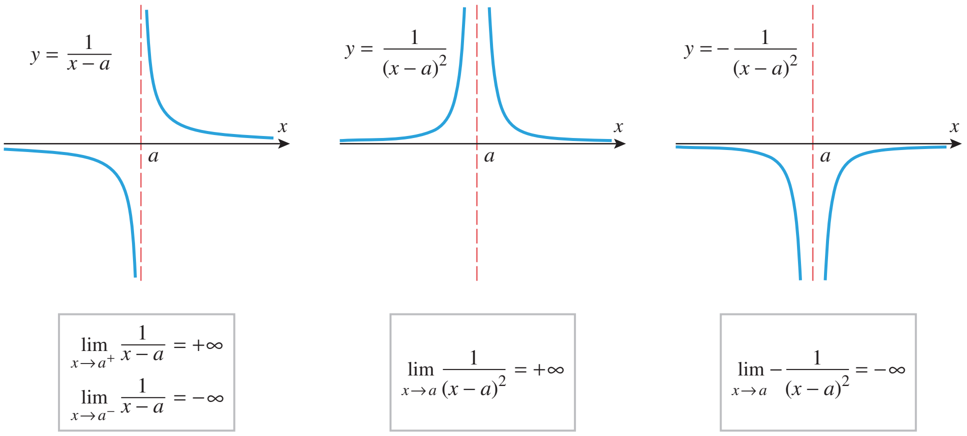
- The limit may be $-\infty$ from one side and $+\infty$ from the other.
- The limit may be $+\infty$.
- The limit may be $-\infty$.

Figure 1.2.2 illustrates these three possibilities graphically for rational functions of the form $1/(x-a)$, $1/(x-a)^2$, and $-1/(x-a)^2$.

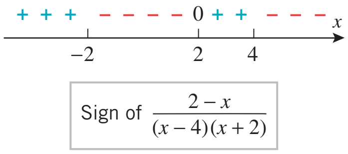
▶ **Example 8** Find

(a) $\lim_{x\to 4^+} \dfrac{2-x}{(x-4)(x+2)}$ (b) $\lim_{x\to 4^-} \dfrac{2-x}{(x-4)(x+2)}$ (c) $\lim_{x\to 4} \dfrac{2-x}{(x-4)(x+2)}$

Solution. In all three parts the limit of the numerator is -2, and the limit of the denominator is 0, so the limit of the ratio does not exist. To be more specific than this, we need



▲ **Figure 1.2.2**



▲ **Figure 1.2.3**

to analyze the sign of the ratio. The sign of the ratio, which is given in Figure 1.2.3, is determined by the signs of $2 - x$, $x - 4$, and $x + 2$. (The method of test points, discussed in Web Appendix E, provides a way of finding the sign of the ratio here.) It follows from this figure that as x approaches 4 from the right, the ratio is always negative; and as x approaches 4 from the left, the ratio is eventually positive. Thus,

$$\lim_{x\to 4^+} \frac{2-x}{(x-4)(x+2)} = -\infty \quad \text{and} \quad \lim_{x\to 4^-} \frac{2-x}{(x-4)(x+2)} = +\infty$$

Because the one-sided limits have opposite signs, all we can say about the two-sided limit is that it does not exist. ◀

In the case where $p(x)/q(x)$ is a rational function for which $p(a) = 0$ and $q(a) = 0$, the numerator and denominator must have one or more common factors of $x - a$. In this case the limit of $p(x)/q(x)$ as $x \to a$ can be found by canceling all common factors of $x - a$ and using one of the methods already considered to find the limit of the simplified function. Here is an example.

In Example 9(a), the simplified function $x - 3$ is defined at $x = 3$, but the original function is not. However, this has no effect on the limit as x *approaches* 3 since the two functions are identical if $x \neq 3$ (Exercise 50).

▶ **Example 9** Find

(a) $\lim_{x\to 3} \frac{x^2 - 6x + 9}{x - 3}$ (b) $\lim_{x\to -4} \frac{2x + 8}{x^2 + x - 12}$ (c) $\lim_{x\to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25}$

Solution (a). The numerator and the denominator both have a zero at $x = 3$, so there is a common factor of $x - 3$. Then

$$\lim_{x\to 3} \frac{x^2 - 6x + 9}{x - 3} = \lim_{x\to 3} \frac{(x-3)^2}{x-3} = \lim_{x\to 3} (x - 3) = 0$$

Solution (b). The numerator and the denominator both have a zero at $x = -4$, so there is a common factor of $x - (-4) = x + 4$. Then

$$\lim_{x\to -4} \frac{2x + 8}{x^2 + x - 12} = \lim_{x\to -4} \frac{2(x+4)}{(x+4)(x-3)} = \lim_{x\to -4} \frac{2}{x-3} = -\frac{2}{7}$$

Solution (c). The numerator and the denominator both have a zero at $x = 5$, so there is a common factor of $x - 5$. Then

$$\lim_{x\to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x\to 5} \frac{(x-5)(x+2)}{(x-5)(x-5)} = \lim_{x\to 5} \frac{x+2}{x-5}$$

However,

$$\lim_{x \to 5} (x+2) = 7 \neq 0 \quad \text{and} \quad \lim_{x \to 5} (x-5) = 0$$

so

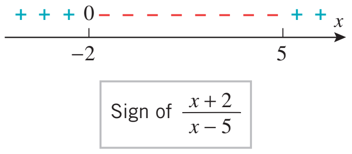
$$\lim_{x \to 5} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5} \frac{x+2}{x-5}$$

does not exist. More precisely, the sign analysis in Figure 1.2.4 implies that

$$\lim_{x \to 5^+} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5^+} \frac{x+2}{x-5} = +\infty$$

and

$$\lim_{x \to 5^-} \frac{x^2 - 3x - 10}{x^2 - 10x + 25} = \lim_{x \to 5^-} \frac{x+2}{x-5} = -\infty \blacktriangleleft$$



▲ **Figure 1.2.4**

Discuss the logical errors in the following statement: An indeterminate form of type $0/0$ must have a limit of zero because zero divided by anything is zero.

A quotient $f(x)/g(x)$ in which the numerator and denominator both have a limit of zero as $x \to a$ is called an ***indeterminate form of type*** $\mathbf{0/0}$. The problem with such limits is that it is difficult to tell by inspection whether the limit exists, and, if so, its value. Informally stated, this is because there are two conflicting influences at work. The value of $f(x)/g(x)$ would tend to zero as $f(x)$ approached zero if $g(x)$ were to remain at some fixed nonzero value, whereas the value of this ratio would tend to increase or decrease without bound as $g(x)$ approached zero if $f(x)$ were to remain at some fixed nonzero value. But with both $f(x)$ and $g(x)$ approaching zero, the behavior of the ratio depends on precisely how these conflicting tendencies offset one another for the particular f and g.

Sometimes, limits of indeterminate forms of type $0/0$ can be found by algebraic simplification, as in the last example, but frequently this will not work and other methods must be used. We will study such methods in later sections.

The following theorem summarizes our observations about limits of rational functions.

1.2.4 THEOREM *Let*

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(a) *If* $q(a) \neq 0$, *then* $\lim_{x \to a} f(x) = f(a)$.

(b) *If* $q(a) = 0$ *but* $p(a) \neq 0$, *then* $\lim_{x \to a} f(x)$ *does not exist.*

LIMITS INVOLVING RADICALS

▶ **Example 10** Find $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$.

Solution. In Example 2 of Section 1.1 we used numerical evidence to conjecture that this limit is 2. Here we will confirm this algebraically. Since this limit is an indeterminate form of type $0/0$, we will need to devise some strategy for making the limit (if it exists) evident. One such strategy is to rationalize the denominator of the function. This yields

$$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1 \quad (x \neq 1)$$

Confirm the limit in Example 10 by factoring the numerator.

Therefore,

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}(\sqrt{x}+1)=2 \blacktriangleleft$$

LIMITS OF PIECEWISE-DEFINED FUNCTIONS

For functions that are defined piecewise, a two-sided limit at a point where the formula changes is best obtained by first finding the one-sided limits at that point.

▶ Example 11 Let

$$f(x)=\begin{cases}1/(x+2), & x<-2\\ x^2-5, & -2<x\le 3\\ \sqrt{x+13}, & x>3\end{cases}$$

Find

$$\text{(a)}\ \lim_{x\to -2} f(x)\qquad \text{(b)}\ \lim_{x\to 0} f(x)\qquad \text{(c)}\ \lim_{x\to 3} f(x)$$

Solution (a). We will determine the stated two-sided limit by first considering the corresponding one-sided limits. For each one-sided limit, we must use that part of the formula that is applicable on the interval over which x varies. For example, as x approaches -2 from the left, the applicable part of the formula is

$$f(x)=\frac{1}{x+2}$$

and as x approaches -2 from the right, the applicable part of the formula near -2 is

$$f(x)=x^2-5$$

Thus,

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^-}\frac{1}{x+2}=-\infty$$

$$\lim_{x\to -2^+} f(x)=\lim_{x\to -2^+}(x^2-5)=(-2)^2-5=-1$$

from which it follows that $\lim_{x\to -2} f(x)$ does not exist.

Solution (b). The applicable part of the formula is $f(x)=x^2-5$ on both sides of 0, so there is no need to consider one-sided limits here. We see directly that

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}(x^2-5)=0^2-5=-5$$

Solution (c). Using the applicable parts of the formula for $f(x)$, we obtain

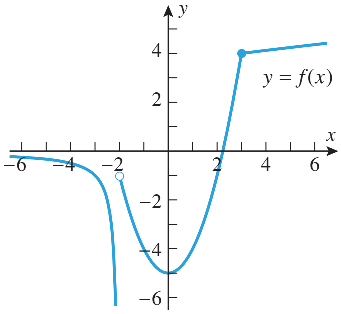
$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(x^2-5)=3^2-5=4$$

$$\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\sqrt{x+13}=\sqrt{\lim_{x\to 3^+}(x+13)}=\sqrt{3+13}=4$$

Since the one-sided limits are equal, we have

$$\lim_{x\to 3} f(x)=4$$

We note that the limit calculations in parts (a), (b), and (c) are consistent with the graph of f shown in Figure 1.2.5. ◀



▲ **Figure 1.2.5**

✔QUICK CHECK EXERCISES 1.2 *(See page 70 for answers.)*

1. In each part, find the limit by inspection.

(a) $\lim\limits_{x\to 8} 7 = \underline{\qquad\qquad}$ (b) $\lim\limits_{y\to 3^+} 12y = \underline{\qquad\qquad}$

(c) $\lim\limits_{x\to 0^-} \dfrac{x}{|x|} = \underline{\qquad\qquad}$ (d) $\lim\limits_{w\to 5} \dfrac{w}{|w|} = \underline{\qquad\qquad}$

(e) $\lim\limits_{z\to 1^-} \dfrac{1}{1-z} = \underline{\qquad\qquad}$

2. Given that $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = 2$, find the limits.

(a) $\lim\limits_{x\to a} [3f(x) + 2g(x)] = \underline{\qquad\qquad}$

(b) $\lim\limits_{x\to a} \dfrac{2f(x)+1}{1-f(x)g(x)} = \underline{\qquad\qquad}$

(c) $\lim\limits_{x\to a} \dfrac{\sqrt{f(x)+3}}{g(x)} = \underline{\qquad\qquad}$

3. Find the limits.

(a) $\lim\limits_{x\to -1} (x^3 + x^2 + x)^{101} = \underline{\qquad\qquad}$

(b) $\lim\limits_{x\to 2^-} \dfrac{(x-1)(x-2)}{x+1} = \underline{\qquad\qquad}$

(c) $\lim\limits_{x\to -1^+} \dfrac{(x-1)(x-2)}{x+1} = \underline{\qquad\qquad}$

(d) $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4} = \underline{\qquad\qquad}$

4. Let

$$f(x) = \begin{cases} x+1, & x \le 1 \\ x-1, & x > 1 \end{cases}$$

Find the limits that exist.

(a) $\lim\limits_{x\to 1^-} f(x) = \underline{\qquad\qquad}$

(b) $\lim\limits_{x\to 1^+} f(x) = \underline{\qquad\qquad}$

(c) $\lim\limits_{x\to 1} f(x) = \underline{\qquad\qquad}$

EXERCISE SET 1.2

1. Given that

$$\lim_{x\to a} f(x) = 2, \quad \lim_{x\to a} g(x) = -4, \quad \lim_{x\to a} h(x) = 0$$

find the limits.

(a) $\lim\limits_{x\to a} [f(x) + 2g(x)]$

(b) $\lim\limits_{x\to a} [h(x) - 3g(x) + 1]$

(c) $\lim\limits_{x\to a} [f(x)g(x)]$ (d) $\lim\limits_{x\to a} [g(x)]^2$

(e) $\lim\limits_{x\to a} \sqrt[3]{6 + f(x)}$ (f) $\lim\limits_{x\to a} \dfrac{2}{g(x)}$

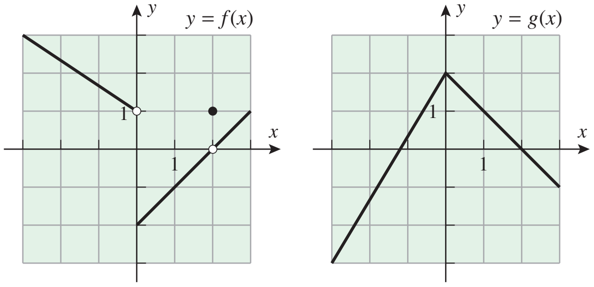
2. Use the graphs of f and g in the accompanying figure to find the limits that exist. If the limit does not exist, explain why.

(a) $\lim\limits_{x\to 2} [f(x) + g(x)]$ (b) $\lim\limits_{x\to 0} [f(x) + g(x)]$

(c) $\lim\limits_{x\to 0^+} [f(x) + g(x)]$ (d) $\lim\limits_{x\to 0^-} [f(x) + g(x)]$

(e) $\lim\limits_{x\to 2} \dfrac{f(x)}{1+g(x)}$ (f) $\lim\limits_{x\to 2} \dfrac{1+g(x)}{f(x)}$

(g) $\lim\limits_{x\to 0^+} \sqrt{f(x)}$ (h) $\lim\limits_{x\to 0^-} \sqrt{f(x)}$



▲ **Figure Ex-2**

3–30 Find the limits. ■

3. $\lim\limits_{x\to 2} x(x-1)(x+1)$

4. $\lim\limits_{x\to 3} x^3 - 3x^2 + 9x$

5. $\lim\limits_{x\to 3} \dfrac{x^2-2x}{x+1}$

6. $\lim\limits_{x\to 0} \dfrac{6x-9}{x^3-12x+3}$

7. $\lim\limits_{x\to 1^+} \dfrac{x^4-1}{x-1}$

8. $\lim\limits_{t\to -2} \dfrac{t^3+8}{t+2}$

9. $\lim\limits_{x\to -1} \dfrac{x^2+6x+5}{x^2-3x-4}$

10. $\lim\limits_{x\to 2} \dfrac{x^2-4x+4}{x^2+x-6}$

11. $\lim\limits_{x\to -1} \dfrac{2x^2+x-1}{x+1}$

12. $\lim\limits_{x\to 1} \dfrac{3x^2-x-2}{2x^2+x-3}$

13. $\lim\limits_{t\to 2} \dfrac{t^3+3t^2-12t+4}{t^3-4t}$

14. $\lim\limits_{t\to 1} \dfrac{t^3+t^2-5t+3}{t^3-3t+2}$

15. $\lim\limits_{x\to 3^+} \dfrac{x}{x-3}$

16. $\lim\limits_{x\to 3^-} \dfrac{x}{x-3}$

17. $\lim\limits_{x\to 3} \dfrac{x}{x-3}$

18. $\lim\limits_{x\to 2^+} \dfrac{x}{x^2-4}$

19. $\lim\limits_{x\to 2^-} \dfrac{x}{x^2-4}$

20. $\lim\limits_{x\to 2} \dfrac{x}{x^2-4}$

21. $\lim\limits_{y\to 6^+} \dfrac{y+6}{y^2-36}$

22. $\lim\limits_{y\to 6^-} \dfrac{y+6}{y^2-36}$

23. $\lim\limits_{y\to 6} \dfrac{y+6}{y^2-36}$

24. $\lim\limits_{x\to 4^+} \dfrac{3-x}{x^2-2x-8}$

25. $\lim\limits_{x\to 4^-} \dfrac{3-x}{x^2-2x-8}$

26. $\lim\limits_{x\to 4} \dfrac{3-x}{x^2-2x-8}$

27. $\lim\limits_{x\to 2^+} \dfrac{1}{|2-x|}$

28. $\lim\limits_{x\to 3^-} \dfrac{1}{|x-3|}$

29. $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

30. $\lim\limits_{y\to 4} \dfrac{4-y}{2-\sqrt{y}}$

31. Let

$$f(x) = \begin{cases} x-1, & x \le 3 \\ 3x-7, & x > 3 \end{cases}$$

(cont.)

Find
(a) $\lim_{x \to 3^-} f(x)$ (b) $\lim_{x \to 3^+} f(x)$ (c) $\lim_{x \to 3} f(x)$.

32. Let

$$g(t) = \begin{cases} t-2, & t<0 \\ t^2, & 0 \le t \le 2 \\ 2t, & t>2 \end{cases}$$

Find
(a) $\lim_{t \to 0} g(t)$ (b) $\lim_{t \to 1} g(t)$ (c) $\lim_{t \to 2} g(t)$.

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then so does $\lim_{x \to a}[f(x)+g(x)]$.

34. If $\lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a}[f(x)/g(x)]$ does not exist.

35. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and are equal, then $\lim_{x \to a}[f(x)/g(x)] = 1$.

36. If $f(x)$ is a rational function and $x = a$ is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.

37–38 First rationalize the numerator and then find the limit. ■

37. $\displaystyle\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$ **38.** $\displaystyle\lim_{x \to 0} \frac{\sqrt{x^2+4}-2}{x}$

39. Let

$$f(x) = \frac{x^3-1}{x-1}$$

(a) Find $\lim_{x \to 1} f(x)$.
(b) Sketch the graph of $y = f(x)$.

40. Let

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq -3 \\ k, & x = -3 \end{cases}$$

(a) Find k so that $f(-3) = \lim_{x \to -3} f(x)$.
(b) With k assigned the value $\lim_{x \to -3} f(x)$, show that $f(x)$ can be expressed as a polynomial.

FOCUS ON CONCEPTS

41. (a) Explain why the following calculation is incorrect.

$$\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x \to 0^+}\frac{1}{x} - \lim_{x \to 0^+}\frac{1}{x^2}$$
$$= +\infty - (+\infty) = 0$$

(b) Show that $\displaystyle\lim_{x \to 0^+}\left(\frac{1}{x} - \frac{1}{x^2}\right) = -\infty$.

42. (a) Explain why the following argument is incorrect.

$$\lim_{x \to 0}\left(\frac{1}{x} - \frac{2}{x^2+2x}\right) = \lim_{x \to 0}\frac{1}{x}\left(1 - \frac{2}{x+2}\right)$$
$$= \infty \cdot 0 = 0$$

(b) Show that $\displaystyle\lim_{x \to 0}\left(\frac{1}{x} - \frac{2}{x^2+2x}\right) = \frac{1}{2}$.

43. Find all values of a such that

$$\lim_{x \to 1}\left(\frac{1}{x-1} - \frac{a}{x^2-1}\right)$$

exists and is finite.

44. (a) Explain informally why

$$\lim_{x \to 0^-}\left(\frac{1}{x} + \frac{1}{x^2}\right) = +\infty$$

(b) Verify the limit in part (a) algebraically.

45. Let $p(x)$ and $q(x)$ be polynomials, with $q(x_0) = 0$. Discuss the behavior of the graph of $y = p(x)/q(x)$ in the vicinity of $x = x_0$. Give examples to support your conclusions.

46. Suppose that f and g are two functions such that $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a}[f(x)+g(x)]$ does not exist. Use Theorem 1.2.2. to prove that $\lim_{x \to a} g(x)$ does not exist.

47. Suppose that f and g are two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a}[f(x)+g(x)]$ exist. Use Theorem 1.2.2 to prove that $\lim_{x \to a} g(x)$ exists.

48. Suppose that f and g are two functions such that

$$\lim_{x \to a} g(x) = 0 \quad \text{and} \quad \lim_{x \to a} \frac{f(x)}{g(x)}$$

exists. Use Theorem 1.2.2 to prove that $\lim_{x \to a} f(x) = 0$.

49. **Writing** According to Newton's Law of Universal Gravitation, the gravitational force of attraction between two masses is inversely proportional to the square of the distance between them. What results of this section are useful in describing the gravitational force of attraction between the masses as they get closer and closer together?

50. **Writing** Suppose that f and g are two functions that are equal except at a finite number of points and that a denotes a real number. Explain informally why both

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

exist and are equal, or why both limits fail to exist. Write a short paragraph that explains the relationship of this result to the use of "algebraic simplification" in the evaluation of a limit.

✔ QUICK CHECK ANSWERS 1.2

1. (a) 7 (b) 36 (c) −1 (d) 1 (e) $+\infty$ **2.** (a) 7 (b) −3 (c) 1 **3.** (a) −1 (b) 0 (c) $+\infty$ (d) 8
4. (a) 2 (b) 0 (c) does not exist

1.3 LIMITS AT INFINITY; END BEHAVIOR OF A FUNCTION

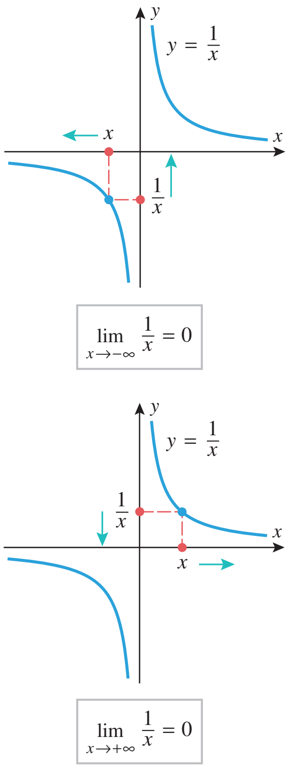
Up to now we have been concerned with limits that describe the behavior of a function $f(x)$ as x approaches some real number a. In this section we will be concerned with the behavior of $f(x)$ as x increases or decreases without bound.

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

If the values of a variable x increase without bound, then we write $x \to +\infty$, and if the values of x decrease without bound, then we write $x \to -\infty$. The behavior of a function $f(x)$ as x increases without bound or decreases without bound is sometimes called the ***end behavior*** of the function. For example,

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{1}{x} = 0 \tag{1–2}$$

are illustrated numerically in Table 1.3.1 and geometrically in Figure 1.3.1.



$\lim_{x\to-\infty} \frac{1}{x} = 0$

$\lim_{x\to+\infty} \frac{1}{x} = 0$

▲ **Figure 1.3.1**

Table 1.3.1

	VALUES						CONCLUSION
x $1/x$	−1 −1	−10 −0.1	−100 −0.01	−1000 −0.001	−10,000 −0.0001	· · · · · ·	As $x \to -\infty$ the value of $1/x$ increases toward zero.
x $1/x$	1 1	10 0.1	100 0.01	1000 0.001	10,000 0.0001	· · · · · ·	As $x \to +\infty$ the value of $1/x$ decreases toward zero.

In general, we will use the following notation.

1.3.1 LIMITS AT INFINITY (AN INFORMAL VIEW) If the values of $f(x)$ eventually get as close as we like to a number L as x increases without bound, then we write

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to +\infty \tag{3}$$

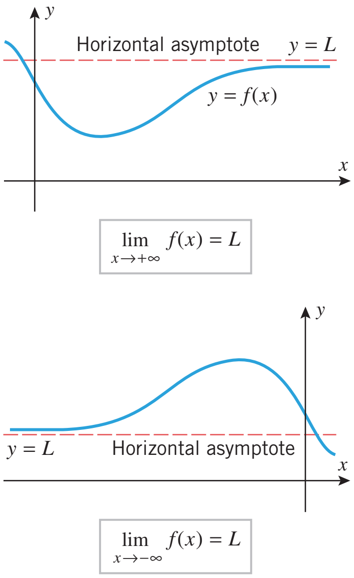
Similarly, if the values of $f(x)$ eventually get as close as we like to a number L as x decreases without bound, then we write

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to -\infty \tag{4}$$

Figure 1.3.2 illustrates the end behavior of a function f when

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

In the first case the graph of f eventually comes as close as we like to the line $y = L$ as x increases without bound, and in the second case it eventually comes as close as we like to the line $y = L$ as x decreases without bound. If either limit holds, we call the line $y = L$ a ***horizontal asymptote*** for the graph of f.



$\lim_{x\to+\infty} f(x) = L$

$\lim_{x\to-\infty} f(x) = L$

▲ **Figure 1.3.2**

▶ **Example 1** It follows from (1) and (2) that $y = 0$ is a horizontal asymptote for the graph of $f(x) = 1/x$ in both the positive and negative directions. This is consistent with the graph of $y = 1/x$ shown in Figure 1.3.1. ◀

LIMIT LAWS FOR LIMITS AT INFINITY

It can be shown that the limit laws in Theorem 1.2.2 carry over without change to limits at $+\infty$ and $-\infty$. Moreover, it follows by the same argument used in Section 1.2 that if n is a positive integer, then

$$\lim_{x\to+\infty}(f(x))^n = \left(\lim_{x\to+\infty} f(x)\right)^n \qquad \lim_{x\to-\infty}(f(x))^n = \left(\lim_{x\to-\infty} f(x)\right)^n \tag{5–6}$$

provided the indicated limit of $f(x)$ exists. It also follows that constants can be moved through the limit symbols for limits at infinity:

$$\lim_{x\to+\infty} kf(x) = k\lim_{x\to+\infty} f(x) \qquad \lim_{x\to-\infty} kf(x) = k\lim_{x\to-\infty} f(x) \tag{7–8}$$

provided the indicated limit of $f(x)$ exists.

Finally, if $f(x) = k$ is a constant function, then the values of f do not change as $x \to +\infty$ or as $x \to -\infty$, so

$$\lim_{x\to+\infty} k = k \qquad \lim_{x\to-\infty} k = k \tag{9–10}$$

▶ **Example 2** It follows from (1), (2), (5), and (6) that if n is a positive integer, then

$$\lim_{x\to+\infty}\frac{1}{x^n} = \left(\lim_{x\to+\infty}\frac{1}{x}\right)^n = 0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x^n} = \left(\lim_{x\to-\infty}\frac{1}{x}\right)^n = 0$$

INFINITE LIMITS AT INFINITY

Limits at infinity, like limits at a real number a, can fail to exist for various reasons. One such possibility is that the values of $f(x)$ increase or decrease without bound as $x \to +\infty$ or as $x \to -\infty$. We will use the following notation to describe this situation.

1.3.2 INFINITE LIMITS AT INFINITY (AN INFORMAL VIEW) If the values of $f(x)$ increase without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

$$\lim_{x\to+\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = +\infty$$

as appropriate; and if the values of $f(x)$ decrease without bound as $x \to +\infty$ or as $x \to -\infty$, then we write

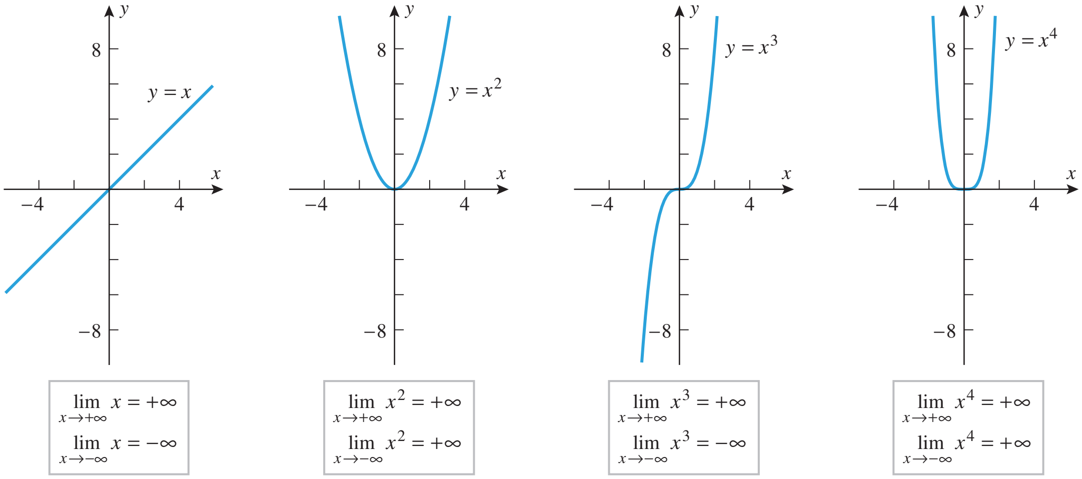
$$\lim_{x\to+\infty} f(x) = -\infty \quad \text{or} \quad \lim_{x\to-\infty} f(x) = -\infty$$

as appropriate.

LIMITS OF x^n AS $x \to \pm\infty$

Figure 1.3.3 illustrates the end behavior of the polynomials x^n for $n = 1, 2, 3,$ and 4. These are special cases of the following general results:

$$\lim_{x \to +\infty} x^n = +\infty, \quad n = 1, 2, 3, \ldots \qquad \lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases} \tag{11–12}$$



▲ **Figure 1.3.3**

Multiplying x^n by a positive real number does not affect limits (11) and (12), but multiplying by a negative real number reverses the sign.

▶ Example 3

$$\lim_{x \to +\infty} 2x^5 = +\infty, \qquad \lim_{x \to -\infty} 2x^5 = -\infty$$

$$\lim_{x \to +\infty} -7x^6 = -\infty, \qquad \lim_{x \to -\infty} -7x^6 = -\infty$$ ◀

LIMITS OF POLYNOMIALS AS $x \to \pm\infty$

There is a useful principle about polynomials which, expressed informally, states:

The end behavior of a polynomial matches the end behavior of its highest degree term.

More precisely, if $c_n \neq 0$, then

$$\lim_{x \to -\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to -\infty} c_n x^n \tag{13}$$

$$\lim_{x \to +\infty} \left(c_0 + c_1 x + \cdots + c_n x^n\right) = \lim_{x \to +\infty} c_n x^n \tag{14}$$

We can motivate these results by factoring out the highest power of x from the polynomial and examining the limit of the factored expression. Thus,

$$c_0 + c_1x + \cdots + c_nx^n = x^n\left(\frac{c_0}{x^n} + \frac{c_1}{x^{n-1}} + \cdots + c_n\right)$$

As $x \to -\infty$ or $x \to +\infty$, it follows from Example 2 that all of the terms with positive powers of x in the denominator approach 0, so (13) and (14) are certainly plausible.

▶ Example 4

$$\lim_{x\to-\infty}(7x^5 - 4x^3 + 2x - 9) = \lim_{x\to-\infty} 7x^5 = -\infty$$

$$\lim_{x\to-\infty}(-4x^8 + 17x^3 - 5x + 1) = \lim_{x\to-\infty} -4x^8 = -\infty \blacktriangleleft$$

LIMITS OF RATIONAL FUNCTIONS AS $x \to \pm\infty$

One technique for determining the end behavior of a rational function is to divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, after which the limiting behavior can be determined using results we have already established. Here are some examples.

▶ Example 5 Find $\displaystyle\lim_{x\to+\infty}\frac{3x+5}{6x-8}$.

Solution. Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, namely, $x^1 = x$. We obtain

$$\lim_{x\to+\infty}\frac{3x+5}{6x-8} = \lim_{x\to+\infty}\frac{3+\dfrac{5}{x}}{6-\dfrac{8}{x}}$$

Divide each term by x.

$$= \frac{\displaystyle\lim_{x\to+\infty}\left(3+\frac{5}{x}\right)}{\displaystyle\lim_{x\to+\infty}\left(6-\frac{8}{x}\right)}$$

Limit of a quotient is the quotient of the limits.

$$= \frac{\displaystyle\lim_{x\to+\infty}3 + \lim_{x\to+\infty}\frac{5}{x}}{\displaystyle\lim_{x\to+\infty}6 - \lim_{x\to+\infty}\frac{8}{x}}$$

Limit of a sum is the sum of the limits.

$$= \frac{\displaystyle 3 + 5\lim_{x\to+\infty}\frac{1}{x}}{\displaystyle 6 - 8\lim_{x\to+\infty}\frac{1}{x}} = \frac{3+0}{6-0} = \frac{1}{2}$$

A constant can be moved through a limit symbol; Formulas (2) and (9). ◀

▶ Example 6 Find

$$\text{(a)}\ \lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5} \qquad \text{(b)}\ \lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}$$

Solution (a). Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, namely, x^3. We obtain

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{\dfrac{4}{x}-\dfrac{1}{x^2}}{2-\dfrac{5}{x^3}}$$

Divide each term by x^3.

$$=\frac{\displaystyle\lim_{x\to-\infty}\left(\frac{4}{x}-\frac{1}{x^2}\right)}{\displaystyle\lim_{x\to-\infty}\left(2-\frac{5}{x^3}\right)}$$

Limit of a quotient is the quotient of the limits.

$$=\frac{\displaystyle\lim_{x\to-\infty}\frac{4}{x}-\lim_{x\to-\infty}\frac{1}{x^2}}{\displaystyle\lim_{x\to-\infty}2-\lim_{x\to-\infty}\frac{5}{x^3}}$$

Limit of a difference is the difference of the limits.

$$=\frac{\displaystyle 4\lim_{x\to-\infty}\frac{1}{x}-\lim_{x\to-\infty}\frac{1}{x^2}}{\displaystyle 2-5\lim_{x\to-\infty}\frac{1}{x^3}}=\frac{0-0}{2-0}=0$$

A constant can be moved through a limit symbol; Formula (10) and Example 2.

Solution (b). Divide each term in the numerator and denominator by the highest power of x that occurs in the denominator, namely, $x^1=x$. We obtain

$$\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^2-2x+\dfrac{1}{x}}{\dfrac{1}{x}-3}\tag{15}$$

In this case we cannot argue that the limit of the quotient is the quotient of the limits because the limit of the numerator does not exist. However, we have

$$\lim_{x\to+\infty}5x^2-2x=+\infty,\qquad \lim_{x\to+\infty}\frac{1}{x}=0,\qquad \lim_{x\to+\infty}\left(\frac{1}{x}-3\right)=-3$$

Thus, the numerator on the right side of (15) approaches $+\infty$ and the denominator has a finite *negative* limit. We conclude from this that the quotient approaches $-\infty$; that is,

$$\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^2-2x+\dfrac{1}{x}}{\dfrac{1}{x}-3}=-\infty \blacktriangleleft$$

■ A QUICK METHOD FOR FINDING LIMITS OF RATIONAL FUNCTIONS AS $x\to+\infty$ OR $x\to-\infty$

Since the end behavior of a polynomial matches the end behavior of its highest degree term, one can reasonably conclude:

> *The end behavior of a rational function matches the end behavior of the quotient of the highest degree term in the numerator divided by the highest degree term in the denominator.*

▶ **Example 7** Use the preceding observation to compute the limits in Examples 5 and 6.

Solution.

$$\lim_{x\to+\infty}\frac{3x+5}{6x-8}=\lim_{x\to+\infty}\frac{3x}{6x}=\lim_{x\to+\infty}\frac{1}{2}=\frac{1}{2}$$

$$\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4x^2}{2x^3}=\lim_{x\to-\infty}\frac{2}{x}=0$$

$$\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^3}{(-3x)}=\lim_{x\to+\infty}\left(-\frac{5}{3}x^2\right)=-\infty \quad ◀$$

LIMITS INVOLVING RADICALS

▶ **Example 8** Find

$$\text{(a)}\ \lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6} \qquad \text{(b)}\ \lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$$

In both parts it would be helpful to manipulate the function so that the powers of x are transformed to powers of $1/x$. This can be achieved in both cases by dividing the numerator and denominator by $|x|$ and using the fact that $\sqrt{x^2}=|x|$.

Solution (a). As $x\to+\infty$, the values of x under consideration are positive, so we can replace $|x|$ by x where helpful. We obtain

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to+\infty}\frac{\dfrac{\sqrt{x^2+2}}{|x|}}{\dfrac{3x-6}{|x|}}=\lim_{x\to+\infty}\frac{\dfrac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\dfrac{3x-6}{x}}$$

$$=\lim_{x\to+\infty}\frac{\sqrt{1+\dfrac{2}{x^2}}}{3-\dfrac{6}{x}}=\frac{\lim\limits_{x\to+\infty}\sqrt{1+\dfrac{2}{x^2}}}{\lim\limits_{x\to+\infty}\left(3-\dfrac{6}{x}\right)}$$

$$=\frac{\sqrt{\lim\limits_{x\to+\infty}\left(1+\dfrac{2}{x^2}\right)}}{\lim\limits_{x\to+\infty}\left(3-\dfrac{6}{x}\right)}=\frac{\sqrt{\left(\lim\limits_{x\to+\infty}1\right)+\left(2\lim\limits_{x\to+\infty}\dfrac{1}{x^2}\right)}}{\left(\lim\limits_{x\to+\infty}3\right)-\left(6\lim\limits_{x\to+\infty}\dfrac{1}{x}\right)}$$

$$=\frac{\sqrt{1+(2\cdot 0)}}{3-(6\cdot 0)}=\frac{1}{3}$$

Solution (b). As $x\to-\infty$, the values of x under consideration are negative, so we can replace $|x|$ by $-x$ where helpful. We obtain

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}=\lim_{x\to-\infty}\frac{\dfrac{\sqrt{x^2+2}}{|x|}}{\dfrac{3x-6}{|x|}}=\lim_{x\to-\infty}\frac{\dfrac{\sqrt{x^2+2}}{\sqrt{x^2}}}{\dfrac{3x-6}{(-x)}}$$

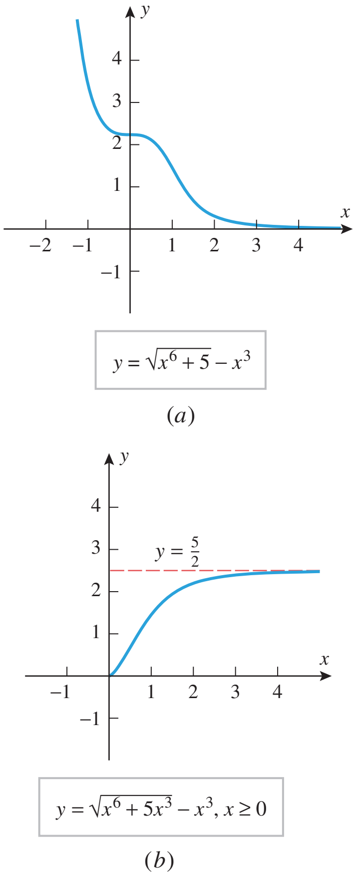
$$=\lim_{x\to-\infty}\frac{\sqrt{1+\dfrac{2}{x^2}}}{-3+\dfrac{6}{x}}=-\frac{1}{3} \quad ◀$$

TECHNOLOGY MASTERY

It follows from Example 8 that the function

$$f(x)=\frac{\sqrt{x^2+2}}{3x-6}$$

has an asymptote of $y=\frac{1}{3}$ in the positive direction and an asymptote of $y=-\frac{1}{3}$ in the negative direction. Confirm this using a graphing utility.



$y = \sqrt{x^6+5} - x^3$

(a)

$y = \sqrt{x^6+5x^3} - x^3,\ x \geq 0$

(b)

Figure 1.3.4

We noted in Section 1.1 that the standard rules of algebra do not apply to the symbols $+\infty$ and $-\infty$. Part (b) of Example 9 illustrates this. The terms $\sqrt{x^6+5x^3}$ and x^3 both approach $+\infty$ as $x \to +\infty$, but their difference does not approach 0.

Example 9 Find

$$\text{(a)} \lim_{x\to+\infty} (\sqrt{x^6+5} - x^3) \qquad \text{(b)} \lim_{x\to+\infty} (\sqrt{x^6+5x^3} - x^3)$$

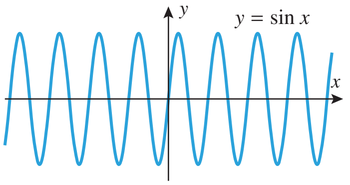
Solution. Graphs of the functions $f(x) = \sqrt{x^6+5} - x^3$, and $g(x) = \sqrt{x^6+5x^3} - x^3$ for $x \geq 0$, are shown in Figure 1.3.4. From the graphs we might conjecture that the requested limits are 0 and $\frac{5}{2}$, respectively. To confirm this, we treat each function as a fraction with a denominator of 1 and rationalize the numerator.

$$\begin{aligned}
\lim_{x\to+\infty} (\sqrt{x^6+5} - x^3) &= \lim_{x\to+\infty} (\sqrt{x^6+5} - x^3)\left(\frac{\sqrt{x^6+5}+x^3}{\sqrt{x^6+5}+x^3}\right) \\
&= \lim_{x\to+\infty} \frac{(x^6+5)-x^6}{\sqrt{x^6+5}+x^3} = \lim_{x\to+\infty} \frac{5}{\sqrt{x^6+5}+x^3} \\
&= \lim_{x\to+\infty} \frac{\frac{5}{x^3}}{\sqrt{1+\frac{5}{x^6}}+1} \qquad \boxed{\sqrt{x^6} = x^3 \text{ for } x > 0} \\
&= \frac{0}{\sqrt{1+0}+1} = 0
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to+\infty} (\sqrt{x^6+5x^3} - x^3) &= \lim_{x\to+\infty} (\sqrt{x^6+5x^3} - x^3)\left(\frac{\sqrt{x^6+5x^3}+x^3}{\sqrt{x^6+5x^3}+x^3}\right) \\
&= \lim_{x\to+\infty} \frac{(x^6+5x^3)-x^6}{\sqrt{x^6+5x^3}+x^3} = \lim_{x\to+\infty} \frac{5x^3}{\sqrt{x^6+5x^3}+x^3} \\
&= \lim_{x\to+\infty} \frac{5}{\sqrt{1+\frac{5}{x^3}}+1} \qquad \boxed{\sqrt{x^6} = x^3 \text{ for } x > 0} \\
&= \frac{5}{\sqrt{1+0}+1} = \frac{5}{2}
\end{aligned}$$

END BEHAVIOR OF TRIGONOMETRIC FUNCTIONS

Consider the function $f(x) = \sin x$ that is graphed in Figure 1.3.5. For this function the limits as $x \to +\infty$ and as $x \to -\infty$ fail to exist not because $f(x)$ increases or decreases without bound, but rather because the values vary between −1 and 1 without approaching some specific real number. In general, the trigonometric functions fail to have limits as $x \to +\infty$ and as $x \to -\infty$ because of periodicity. There is no specific notation to denote this kind of behavior.



There is no limit as $x \to +\infty$ or $x \to -\infty$.

Figure 1.3.5

✔QUICK CHECK EXERCISES 1.3 *(See page 80 for answers.)*

1. Find the limits.
   (a) $\lim_{x\to+\infty} (-2x) =$ ________
   (b) $\lim_{x\to-\infty} \frac{x}{|x|} =$ ________
   (c) $\lim_{x\to-\infty} (3 - x) =$ ________
   (d) $\lim_{x\to+\infty} \left(5 - \frac{1}{x}\right) =$ ________

2. Find the limits that exist.

(a) $\lim_{x\to-\infty} \dfrac{2x^2+x}{4x^2-3} =$ ________

(b) $\lim_{x\to+\infty} \dfrac{1}{2+\sin x} =$ ________

3. Given that

$$\lim_{x\to+\infty} f(x) = 2 \quad \text{and} \quad \lim_{x\to+\infty} g(x) = -3$$

find the limits that exist.

(a) $\lim_{x\to+\infty} [3f(x) - g(x)] =$ ________

(b) $\lim_{x\to+\infty} \dfrac{f(x)}{g(x)} =$ ________

(c) $\lim_{x\to+\infty} \dfrac{2f(x)+3g(x)}{3f(x)+2g(x)} =$ ________

(d) $\lim_{x\to+\infty} \sqrt{10 - f(x)g(x)} =$ ________

4. Consider the graphs of $y = 1/(x+1)$, $y = x/(x+1)$, and $y = x^2/(x+1)$. Which of these graphs has a horizontal asymptote?

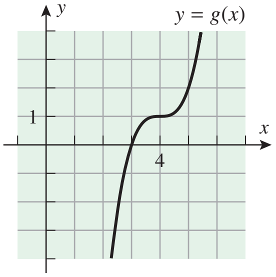
EXERCISE SET 1.3

Graphing Utility

1–4 In these exercises, make reasonable assumptions about the end behavior of the indicated function.

1. For the function g graphed in the accompanying figure, find

(a) $\lim_{x\to-\infty} g(x)$ (b) $\lim_{x\to+\infty} g(x)$.

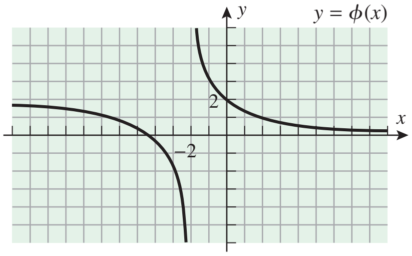


◀ Figure Ex-1

2. For the function ϕ graphed in the accompanying figure, find

(a) $\lim_{x\to-\infty} \phi(x)$

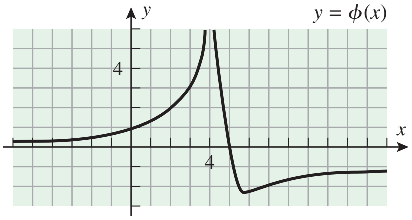
(b) $\lim_{x\to+\infty} \phi(x)$.



◀ Figure Ex-2

3. For the function ϕ graphed in the accompanying figure, find

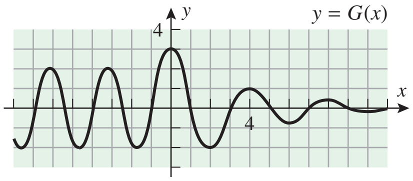
(a) $\lim_{x\to-\infty} \phi(x)$ (b) $\lim_{x\to+\infty} \phi(x)$.



◀ Figure Ex-3

4. For the function G graphed in the accompanying figure, find

(a) $\lim_{x\to-\infty} G(x)$ (b) $\lim_{x\to+\infty} G(x)$.



◀ Figure Ex-4

5. Given that

$$\lim_{x\to+\infty} f(x) = 3, \quad \lim_{x\to+\infty} g(x) = -5, \quad \lim_{x\to+\infty} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to+\infty} [f(x) + 3g(x)]$

(b) $\lim_{x\to+\infty} [h(x) - 4g(x) + 1]$

(c) $\lim_{x\to+\infty} [f(x)g(x)]$ (d) $\lim_{x\to+\infty} [g(x)]^2$

(e) $\lim_{x\to+\infty} \sqrt[3]{5 + f(x)}$ (f) $\lim_{x\to+\infty} \dfrac{3}{g(x)}$

(g) $\lim_{x\to+\infty} \dfrac{3h(x)+4}{x^2}$ (h) $\lim_{x\to+\infty} \dfrac{6f(x)}{5f(x)+3g(x)}$

6. Given that

$$\lim_{x\to-\infty} f(x) = 7 \quad \text{and} \quad \lim_{x\to-\infty} g(x) = -6$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x\to-\infty} [2f(x) - g(x)]$ (b) $\lim_{x\to-\infty} [6f(x) + 7g(x)]$

(c) $\lim_{x\to-\infty} [x^2 + g(x)]$ (d) $\lim_{x\to-\infty} [x^2 g(x)]$

(e) $\lim_{x\to-\infty} \sqrt[3]{f(x)g(x)}$ (f) $\lim_{x\to-\infty} \dfrac{g(x)}{f(x)}$

(g) $\lim_{x\to-\infty} \left[f(x) + \dfrac{g(x)}{x}\right]$ (h) $\lim_{x\to-\infty} \dfrac{xf(x)}{(2x+3)g(x)}$

7. Complete the table and make a guess about the limit indicated.

$$f(x) = \frac{\sqrt{x^2+x}}{x+1} \qquad \lim_{x\to+\infty} f(x)$$

x	10	100	1000	10,000	100,000	1,000,000
$f(x)$						

8. Complete the table and make a guess about the limit indicated.

$$f(x) = \frac{\sqrt{x^2+x}}{x+1} \qquad \lim_{x\to-\infty} f(x)$$

x	–10	–100	–1000	–10,000	–100,000	–1,000,000
$f(x)$						

9–32 Find the limits. ■

9. $\lim_{x\to+\infty} (1 + 2x - 3x^5)$

10. $\lim_{x\to+\infty} (2x^3 - 100x + 5)$

11. $\lim_{x\to+\infty} \sqrt{x}$

12. $\lim_{x\to-\infty} \sqrt{5-x}$

13. $\lim_{x\to+\infty} \frac{3x+1}{2x-5}$

14. $\lim_{x\to+\infty} \frac{5x^2-4x}{2x^2+3}$

15. $\lim_{y\to-\infty} \frac{3}{y+4}$

16. $\lim_{x\to+\infty} \frac{1}{x-12}$

17. $\lim_{x\to-\infty} \frac{x-2}{x^2+2x+1}$

18. $\lim_{x\to+\infty} \frac{5x^2+7}{3x^2-x}$

19. $\lim_{x\to+\infty} \frac{7-6x^5}{x+3}$

20. $\lim_{t\to-\infty} \frac{5-2t^3}{t^2+1}$

21. $\lim_{t\to+\infty} \frac{6-t^3}{7t^3+3}$

22. $\lim_{x\to-\infty} \frac{x+4x^3}{1-x^2+7x^3}$

23. $\lim_{x\to+\infty} \sqrt[3]{\frac{2+3x-5x^2}{1+8x^2}}$

24. $\lim_{s\to+\infty} \sqrt[3]{\frac{3s^7-4s^5}{2s^7+1}}$

25. $\lim_{x\to-\infty} \frac{\sqrt{5x^2-2}}{x+3}$

26. $\lim_{x\to+\infty} \frac{\sqrt{5x^2-2}}{x+3}$

27. $\lim_{y\to-\infty} \frac{2-y}{\sqrt{7+6y^2}}$

28. $\lim_{y\to+\infty} \frac{2-y}{\sqrt{7+6y^2}}$

29. $\lim_{x\to-\infty} \frac{\sqrt{3x^4+x}}{x^2-8}$

30. $\lim_{x\to+\infty} \frac{\sqrt{3x^4+x}}{x^2-8}$

31. $\lim_{x\to+\infty} (\sqrt{x^2+3} - x)$

32. $\lim_{x\to+\infty} (\sqrt{x^2-3x} - x)$

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. By subtraction, $\lim_{x\to+\infty} (x^2 - 1000x) = \infty - \infty = 0$.

34. If $y = L$ is a horizontal asymptote for the curve $y = f(x)$, then

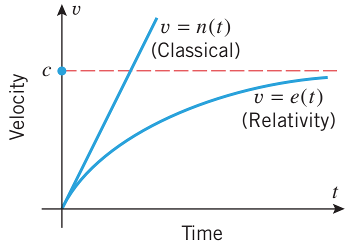
$$\lim_{x\to-\infty} f(x) = L \quad \text{and} \quad \lim_{x\to+\infty} f(x) = L$$

35. If $y = L$ is a horizontal asymptote for the curve $y = f(x)$, then it is possible for the graph of f to intersect the line $y = L$ infinitely many times.

36. If a rational function $p(x)/q(x)$ has a horizontal asymptote, then the degree of $p(x)$ must equal the degree of $q(x)$.

FOCUS ON CONCEPTS

37. Assume that a particle is accelerated by a constant force. The two curves $v = n(t)$ and $v = e(t)$ in the accompanying figure provide velocity versus time curves for the particle as predicted by classical physics and by the special theory of relativity, respectively. The parameter c represents the speed of light. Using the language of limits, describe the differences in the long-term predictions of the two theories.

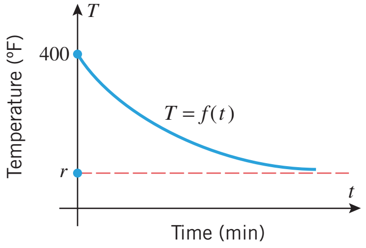


◀ **Figure Ex-37**

38. Let $T = f(t)$ denote the temperature of a baked potato t minutes after it has been removed from a hot oven. The accompanying figure shows the temperature versus time curve for the potato, where r is the temperature of the room.

(a) What is the physical significance of $\lim_{t\to 0^+} f(t)$?

(b) What is the physical significance of $\lim_{t\to+\infty} f(t)$?



◀ **Figure Ex-38**

39. Let

$$f(x) = \begin{cases} 2x^2+5, & x < 0 \\ \dfrac{3-5x^3}{1+4x+x^3}, & x \geq 0 \end{cases}$$

Find

(a) $\lim_{x\to-\infty} f(x)$ (b) $\lim_{x\to+\infty} f(x)$.

40. Let

$$g(t) = \begin{cases} \dfrac{2+3t}{5t^2+6}, & t < 1{,}000{,}000 \\ \dfrac{\sqrt{36t^2-100}}{5-t}, & t > 1{,}000{,}000 \end{cases}$$

Find

(a) $\lim_{t\to-\infty} g(t)$ (b) $\lim_{t\to+\infty} g(t)$.

41. Discuss the limits of $p(x) = (1 - x)^n$ as $x \to +\infty$ and $x \to -\infty$ for positive integer values of n.

42. In each part, find examples of polynomials $p(x)$ and $q(x)$ that satisfy the stated condition and such that $p(x) \to +\infty$ and $q(x) \to +\infty$ as $x \to +\infty$.

(a) $\displaystyle\lim_{x \to +\infty} \frac{p(x)}{q(x)} = 1$ (b) $\displaystyle\lim_{x \to +\infty} \frac{p(x)}{q(x)} = 0$

(c) $\displaystyle\lim_{x \to +\infty} \frac{p(x)}{q(x)} = +\infty$ (d) $\displaystyle\lim_{x \to +\infty} [p(x) - q(x)] = 3$

43. (a) Do any of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ have horizontal asymptotes?

(b) Do any of the trigonometric functions have vertical asymptotes? Where?

44. Find

$$\lim_{x \to +\infty} \frac{c_0 + c_1 x + \cdots + c_n x^n}{d_0 + d_1 x + \cdots + d_m x^m}$$

where $c_n \neq 0$ and $d_m \neq 0$. [*Hint:* Your answer will depend on whether $m < n$, $m = n$, or $m > n$.]

FOCUS ON CONCEPTS

45–46 These exercises develop some versions of the ***substitution principle***, a useful tool for the evaluation of limits.

45. (a) Suppose $g(x) \to +\infty$ as $x \to +\infty$. Given any function $f(x)$, explain why we can evaluate $\lim_{x \to +\infty} f[g(x)]$ by substituting $t = g(x)$ and writing

$$\lim_{x \to +\infty} f[g(x)] = \lim_{t \to +\infty} f(t)$$

(Here, "equality" is interpreted to mean that either both limits exist and are equal or that both limits fail to exist.)

(b) Why does the result in part (a) remain valid if $\lim_{x \to +\infty}$ is replaced everywhere by one of $\lim_{x \to -\infty}$, $\lim_{x \to c}$, $\lim_{x \to c^-}$, or $\lim_{x \to c^+}$?

46. (a) Suppose $g(x) \to -\infty$ as $x \to +\infty$. Given any function $f(x)$, explain why we can evaluate $\lim_{x \to +\infty} f[g(x)]$ by substituting $t = g(x)$ and writing

$$\lim_{x \to +\infty} f[g(x)] = \lim_{t \to -\infty} f(t)$$

(Here, "equality" is interpreted to mean that either both limits exist and are equal or that both limits fail to exist.)

(b) Why does the result in part (a) remain valid if $\lim_{x \to +\infty}$ is replaced everywhere by one of $\lim_{x \to -\infty}$, $\lim_{x \to c}$, $\lim_{x \to c^-}$, or $\lim_{x \to c^+}$?

47–50 Given that

$$\lim_{x \to -\infty} f(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} f(x) = +\infty$$

evaluate the limit using an appropriate substitution.

47. $\displaystyle\lim_{x \to 0^+} f(1/x)$ **48.** $\displaystyle\lim_{x \to 0^-} f(1/x)$

49. $\displaystyle\lim_{x \to 0^+} f(\csc x)$ **50.** $\displaystyle\lim_{x \to 0^-} f(\csc x)$

51–55 The notion of an asymptote can be extended to include curves as well as lines. Specifically, we say that curves $y = f(x)$ and $y = g(x)$ are ***asymptotic as $x \to +\infty$*** provided

$$\lim_{x \to +\infty} [f(x) - g(x)] = 0$$

and are ***asymptotic as $x \to -\infty$*** provided

$$\lim_{x \to -\infty} [f(x) - g(x)] = 0$$

In these exercises, determine a simpler function $g(x)$ such that $y = f(x)$ is asymptotic to $y = g(x)$ as $x \to +\infty$ or $x \to -\infty$. Use a graphing utility to generate the graphs of $y = f(x)$ and $y = g(x)$ and identify all vertical asymptotes.

51. $f(x) = \dfrac{x^2 - 2}{x - 2}$ [*Hint:* Divide $x - 2$ into $x^2 - 2$.]

52. $f(x) = \dfrac{x^3 - x + 3}{x}$ **53.** $f(x) = \dfrac{-x^3 + 3x^2 + x - 1}{x - 3}$

54. $f(x) = \dfrac{x^5 - x^3 + 3}{x^2 - 1}$ **55.** $f(x) = \sin x + \dfrac{1}{x - 1}$

56. Writing In some models for learning a skill (e.g., juggling), it is assumed that the skill level for an individual increases with practice but cannot become arbitrarily high. How do concepts of this section apply to such a model?

57. Writing In some population models it is assumed that a given ecological system possesses a ***carrying capacity*** L. Populations greater than the carrying capacity tend to decline toward L, while populations less than the carrying capacity tend to increase toward L. Explain why these assumptions are reasonable, and discuss how the concepts of this section apply to such a model.

✔ QUICK CHECK ANSWERS 1.3

1. (a) $-\infty$ (b) -1 (c) $+\infty$ (d) 5 **2.** (a) $\frac{1}{2}$ (b) does not exist **3.** (a) 9 (b) $-\frac{2}{3}$ (c) does not exist (d) 4

4. The graphs of $y = 1/(x + 1)$ and $y = x/(x + 1)$ have horizontal asymptotes.

(a) How much current flows if a voltage of 3.0 volts is applied across a resistance of 7.5 ohms?
(b) If the resistance varies by ±0.1 ohm, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?
(c) If temperature variations cause the resistance to vary by $\pm\delta$ from its value of 7.5 ohms, and the voltage remains constant at 3.0 volts, what is the resulting range of values for the current?
(d) If the current is not allowed to vary by more than $\epsilon = \pm 0.001$ ampere at a voltage of 3.0 volts, what variation of $\pm\delta$ from the value of 7.5 ohms is allowable?
(e) Certain alloys become ***superconductors*** as their temperature approaches absolute zero (-273°C), meaning that their resistance approaches zero. If the voltage remains constant, what happens to the current in a superconductor as $R \to 0^+$?

76. **Writing** Compare informal Definition 1.1.1 with Definition 1.4.1.
(a) What portions of Definition 1.4.1 correspond to the expression "values of $f(x)$ can be made as close as we like to L" in Definition 1.1.1? Explain.
(b) What portions of Definition 1.4.1 correspond to the expression "taking values of x sufficiently close to a (but not equal to a)" in Definition 1.1.1? Explain.

77. **Writing** Compare informal Definition 1.3.1 with Definition 1.4.2.
(a) What portions of Definition 1.4.2 correspond to the expression "values of $f(x)$ eventually get as close as we like to a number L" in Definition 1.3.1? Explain.
(b) What portions of Definition 1.4.2 correspond to the expression "as x increases without bound" in Definition 1.3.1? Explain.

✔QUICK CHECK ANSWERS 1.4

1. $\epsilon > 0$; $\delta > 0$; $0 < |x - a| < \delta$ **2.** $\lim_{x\to 1} f(x) = 5$ **3.** $\delta = \epsilon/5$ **4.** $\epsilon > 0$; N; $x > N$ **5.** $N = 10{,}000$

1.5 CONTINUITY

A thrown baseball cannot vanish at some point and reappear someplace else to continue its motion. Thus, we perceive the path of the ball as an unbroken curve. In this section, we translate "unbroken curve" into a precise mathematical formulation called continuity, and develop some fundamental properties of continuous curves.


Joseph Helfenberger/iStockphoto
A baseball moves along a "continuous" trajectory after leaving the pitcher's hand.

DEFINITION OF CONTINUITY

Intuitively, the graph of a function can be described as a "continuous curve" if it has no breaks or holes. To make this idea more precise we need to understand what properties of a function can cause breaks or holes. Referring to Figure 1.5.1, we see that the graph of a function has a break or hole if any of the following conditions occur:

- The function f is undefined at c (Figure 1.5.1*a*).
- The limit of $f(x)$ does not exist as x approaches c (Figures 1.5.1*b*, 1.5.1*c*).
- The value of the function and the value of the limit at c are different (Figure 1.5.1*d*).

This suggests the following definition.

1.5.1 DEFINITION A function f is said to be ***continuous at*** $x = c$ provided the following conditions are satisfied:

1. $f(c)$ is defined.

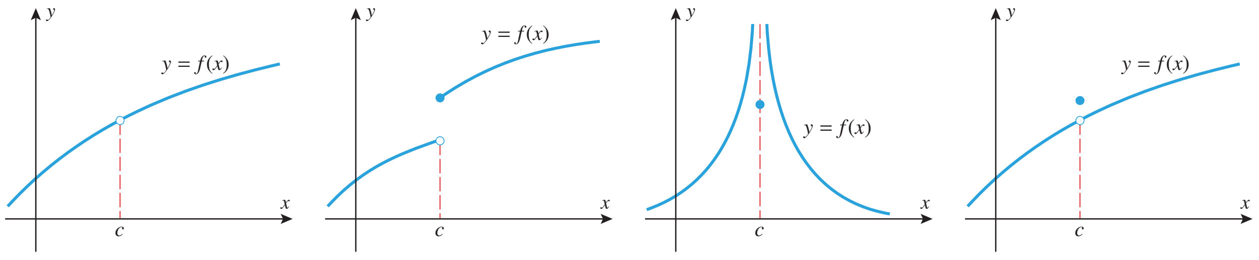
2. $\lim_{x\to c} f(x)$ exists.

3. $\lim_{x\to c} f(x) = f(c)$.

The third condition in Definition 1.5.1 actually implies the first two, since it is tacitly understood in the statement

$$\lim_{x\to c} f(x) = f(c)$$

that the limit exists and the function is defined at c. Thus, when we want to establish continuity at c our usual procedure will be to verify the third condition only.



(a) (b) (c) (d)

▲ **Figure 1.5.1**

If one or more of the conditions of this definition fails to hold, then we will say that f has a ***discontinuity at $x = c$***. Each function drawn in Figure 1.5.1 illustrates a discontinuity at $x = c$. In Figure 1.5.1*a*, the function is not defined at c, violating the first condition of Definition 1.5.1. In Figure 1.5.1*b*, the one-sided limits of $f(x)$ as x approaches c both exist but are not equal. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1*b* has a ***jump discontinuity*** at c. In Figure 1.5.1*c*, the one-sided limits of $f(x)$ as x approaches c are infinite. Thus, $\lim_{x\to c} f(x)$ does not exist, and this violates the second condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1*c* has an ***infinite discontinuity*** at c. In Figure 1.5.1*d*, the function is defined at c and $\lim_{x\to c} f(x)$ exists, but these two values are not equal, violating the third condition of Definition 1.5.1. We will say that a function like that in Figure 1.5.1*d* has a ***removable discontinuity*** at c. Exercises 33 and 34 help to explain why discontinuities of this type are given this name.

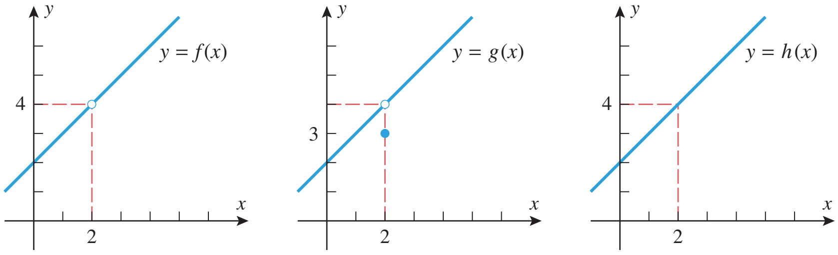
▶ **Example 1** Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2-4}{x-2}, \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

Solution. In each case we must determine whether the limit of the function as $x \to 2$ is the same as the value of the function at $x = 2$. In all three cases the functions are identical, except at $x = 2$, and hence all three have the same limit at $x = 2$, namely,

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 4$$

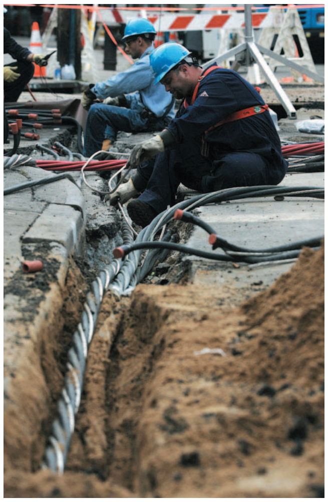
The function f is undefined at $x = 2$, and hence is not continuous at $x = 2$ (Figure 1.5.2*a*). The function g is defined at $x = 2$, but its value there is $g(2) = 3$, which is not the same as the limit as x approaches 2; hence, g is also not continuous at $x = 2$ (Figure 1.5.2*b*). The value of the function h at $x = 2$ is $h(2) = 4$, which is the same as the limit as x approaches 2; hence, h is continuous at $x = 2$ (Figure 1.5.2*c*). (Note that the function h could have been written more simply as $h(x) = x + 2$, but we wrote it in piecewise form to emphasize its relationship to f and g.) ◀



▶ **Figure 1.5.2** (a) (b) (c)

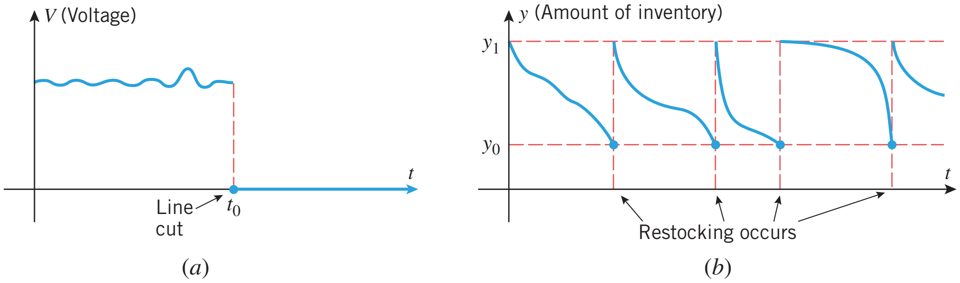
CONTINUITY IN APPLICATIONS

In applications, discontinuities often signal the occurrence of important physical events. For example, Figure 1.5.3*a* is a graph of voltage versus time for an underground cable that is accidentally cut by a work crew at time $t = t_0$ (the voltage drops to zero when the line is cut). Figure 1.5.3*b* shows the graph of inventory versus time for a company that restocks its warehouse to y_1 units when the inventory falls to y_0 units. The discontinuities occur at those times when restocking occurs.



Chris Hondros/Getty Images

A poor connection in a transmission cable can cause a discontinuity in the electrical signal it carries.



▲ **Figure 1.5.3**

CONTINUITY ON AN INTERVAL

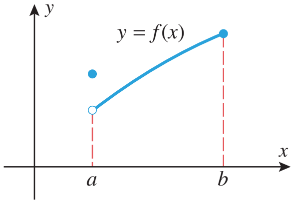
If a function f is continuous at each number in an open interval (a, b), then we say that f is ***continuous on* (*a, b*)**. This definition applies to infinite open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the case where f is continuous on $(-\infty, +\infty)$, we will say that f is ***continuous everywhere***.

Because Definition 1.5.1 involves a two-sided limit, that definition does not generally apply at the endpoints of a closed interval $[a, b]$ or at the endpoint of an interval of the form $[a, b)$, $(a, b]$, $(-\infty, b]$, or $[a, +\infty)$. To remedy this problem, we will agree that a function is continuous at an endpoint of an interval if its value at the endpoint is equal to the appropriate one-sided limit at that endpoint. For example, the function graphed in Figure 1.5.4 is continuous at the right endpoint of the interval $[a, b]$ because

$$\lim_{x \to b^-} f(x) = f(b)$$

but it is not continuous at the left endpoint because

$$\lim_{x \to a^+} f(x) \neq f(a)$$



▲ **Figure 1.5.4**

In general, we will say a function f is ***continuous from the left*** at c if

$$\lim_{x \to c^-} f(x) = f(c)$$

and is ***continuous from the right*** at c if

$$\lim_{x \to c^+} f(x) = f(c)$$

Using this terminology we define continuity on a closed interval as follows.

1.5.2 DEFINITION A function f is said to be ***continuous on a closed interval* [*a, b*]** if the following conditions are satisfied:

1. f is continuous on (a, b).

2. f is continuous from the right at a.

3. f is continuous from the left at b.

Modify Definition 1.5.2 appropriately so that it applies to intervals of the form $[a, +\infty)$, $(-\infty, b]$, $(a, b]$, and $[a, b)$.

Example 2 What can you say about the continuity of the function $f(x) = \sqrt{9 - x^2}$?

Solution. Because the natural domain of this function is the closed interval $[-3, 3]$, we will need to investigate the continuity of f on the open interval $(-3, 3)$ and at the two endpoints. If c is any point in the interval $(-3, 3)$, then it follows from Theorem 1.2.2(e) that

$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{9 - x^2} = \sqrt{\lim_{x\to c} (9 - x^2)} = \sqrt{9 - c^2} = f(c)$$

which proves f is continuous at each point in the interval $(-3, 3)$. The function f is also continuous at the endpoints since

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x\to 3^-} (9 - x^2)} = 0 = f(3)$$

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x\to -3^+} (9 - x^2)} = 0 = f(-3)$$

Thus, f is continuous on the closed interval $[-3, 3]$ (Figure 1.5.5). ◀

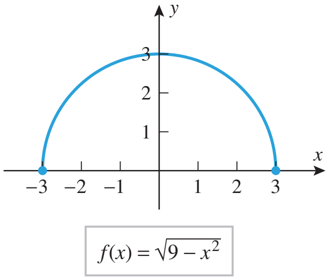


Figure 1.5.5

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

The following theorem, which is a consequence of Theorem 1.2.2, will enable us to reach conclusions about the continuity of functions that are obtained by adding, subtracting, multiplying, and dividing continuous functions.

1.5.3 THEOREM *If the functions f and g are continuous at c, then*

(a) *$f + g$ is continuous at c.*

(b) *$f - g$ is continuous at c.*

(c) *fg is continuous at c.*

(d) *f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.*

We will prove part (d). The remaining proofs are similar and will be left to the exercises.

PROOF First, consider the case where $g(c) = 0$. In this case $f(c)/g(c)$ is undefined, so the function f/g has a discontinuity at c.

Next, consider the case where $g(c) \neq 0$. To prove that f/g is continuous at c, we must show that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)} \tag{1}$$

Since f and g are continuous at c,

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c)$$

Thus, by Theorem 1.2.2(d)

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{f(c)}{g(c)}$$

which proves (1). ■

CONTINUITY OF POLYNOMIALS AND RATIONAL FUNCTIONS

The general procedure for showing that a function is continuous everywhere is to show that it is continuous at an *arbitrary* point. For example, we know from Theorem 1.2.3 that if

$p(x)$ is a polynomial and a is *any* real number, then

$$\lim_{x\to a} p(x) = p(a)$$

This shows that polynomials are continuous everywhere. Moreover, since rational functions are ratios of polynomials, it follows from part (d) of Theorem 1.5.3 that rational functions are continuous at points other than the zeros of the denominator, and at these zeros they have discontinuities. Thus, we have the following result.

1.5.4 THEOREM

(a) *A polynomial is continuous everywhere.*

(b) *A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.*

TECHNOLOGY MASTERY

If you use a graphing utility to generate the graph of the equation in Example 3, there is a good chance you will see the discontinuity at $x = 2$ but not at $x = 3$. Try it, and explain what you think is happening.

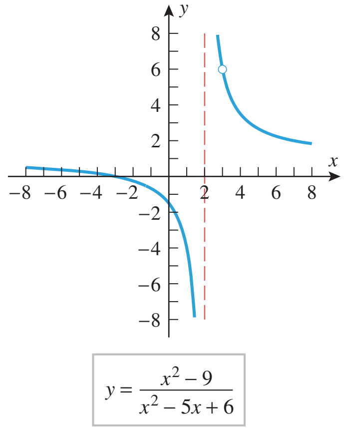
▶ Example 3 For what values of x is there a discontinuity in the graph of

$$y = \frac{x^2-9}{x^2-5x+6}?$$

Solution. The function being graphed is a rational function, and hence is continuous at every number where the denominator is nonzero. Solving the equation

$$x^2 - 5x + 6 = 0$$

yields discontinuities at $x = 2$ and at $x = 3$ (Figure 1.5.6). ◀



▲ **Figure 1.5.6**

▶ Example 4 Show that $|x|$ is continuous everywhere (Figure 0.1.9).

Solution. We can write $|x|$ as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

so $|x|$ is the same as the polynomial x on the interval $(0, +\infty)$ and is the same as the polynomial $-x$ on the interval $(-\infty, 0)$. But polynomials are continuous everywhere, so $x = 0$ is the only possible discontinuity for $|x|$. Since $|0| = 0$, to prove the continuity at $x = 0$ we must show that

$$\lim_{x\to 0} |x| = 0 \tag{2}$$

Because the piecewise formula for $|x|$ changes at 0, it will be helpful to consider the one-sided limits at 0 rather than the two-sided limit. We obtain

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 \quad \text{and} \quad \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$$

Thus, (2) holds and $|x|$ is continuous at $x = 0$. ◀

CONTINUITY OF COMPOSITIONS

The following theorem, whose proof is given in Appendix D, will be useful for calculating limits of compositions of functions.

1.5.5 THEOREM *If* $\lim_{x\to c} g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim_{x\to c} f(g(x)) = f(L)$. *That is,*

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$

This equality remains valid if $\lim_{x\to c}$ *is replaced everywhere by one of* $\lim_{x\to c^+}$, $\lim_{x\to c^-}$, $\lim_{x\to +\infty}$, *or* $\lim_{x\to -\infty}$.

In words, Theorem 1.5.5 states that a limit symbol can be moved through a function sign provided the limit of the expression inside the function sign exists and the function is continuous at this limit.

In the special case of this theorem where $f(x) = |x|$, the fact that $|x|$ is continuous everywhere allows us to write

$$\lim_{x\to c} |g(x)| = \left|\lim_{x\to c} g(x)\right| \tag{3}$$

provided $\lim_{x\to c} g(x)$ exists. Thus, for example,

$$\lim_{x\to 3} |5 - x^2| = \left|\lim_{x\to 3} (5 - x^2)\right| = |-4| = 4$$

The following theorem is concerned with the continuity of compositions of functions; the first part deals with continuity at a specific number and the second with continuity everywhere.

1.5.6 THEOREM

(*a*) *If the function* g *is continuous at* c, *and the function* f *is continuous at* $g(c)$, *then the composition* $f \circ g$ *is continuous at* c.

(*b*) *If the function* g *is continuous everywhere and the function* f *is continuous everywhere, then the composition* $f \circ g$ *is continuous everywhere.*

PROOF We will prove part (*a*) only; the proof of part (*b*) can be obtained by applying part (*a*) at an arbitrary number c. To prove that $f \circ g$ is continuous at c, we must show that the value of $f \circ g$ and the value of its limit are the same at $x = c$. But this is so, since we can write

$$\lim_{x\to c} (f \circ g)(x) = \lim_{x\to c} f(g(x)) \underset{\boxed{\text{Theorem 1.5.5}}}{=} f\left(\lim_{x\to c} g(x)\right) \underset{\boxed{g \text{ is continuous at } c.}}{=} f(g(c)) = (f \circ g)(c) \quad ■$$

Can the absolute value of a function that is not continuous everywhere be continuous everywhere? Justify your answer.

We know from Example 4 that the function $|x|$ is continuous everywhere. Thus, if $g(x)$ is continuous at c, then by part (*a*) of Theorem 1.5.6, the function $|g(x)|$ must also be continuous at c; and, more generally, if $g(x)$ is continuous everywhere, then so is $|g(x)|$. Stated informally:

The absolute value of a continuous function is continuous.

For example, the polynomial $g(x) = 4 - x^2$ is continuous everywhere, so we can conclude that the function $|4 - x^2|$ is also continuous everywhere (Figure 1.5.7).

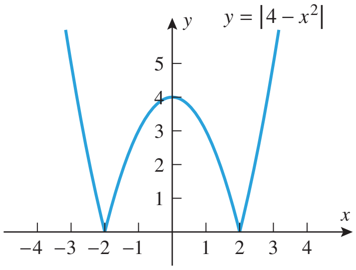


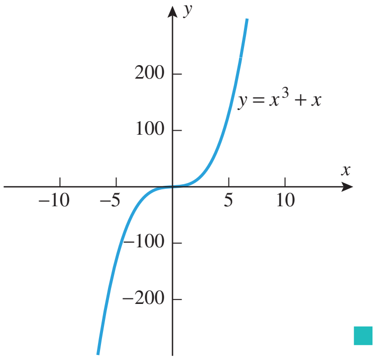
Figure 1.5.7

CONTINUITY OF INVERSE FUNCTIONS

Since the graphs of a one-to-one function f and its inverse f^{-1} are reflections of one another about the line $y = x$, it is clear geometrically that if the graph of f has no breaks or holes in it, then neither does the graph of f^{-1}. This, and the fact that the range of f is the domain of f^{-1}, suggests the following result, which we state without formal proof.

To paraphrase Theorem 1.5.7, *the inverse of a continuous function is continuous.*

1.5.7 THEOREM *If f is a one-to-one function that is continuous at each point of its domain, then f^{-1} is continuous at each point of its domain; that is, f^{-1} is continuous at each point in the range of f.*



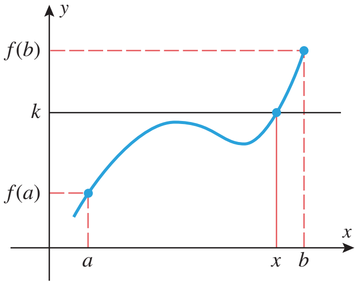
▲ **Figure 1.5.8**

▶ **Example 5** Show that the function $f(x) = x^3 + x$ has an inverse. Is f^{-1} continuous at each point in $(-\infty, +\infty)$?

Solution. Note that f is increasing for all x (why?), so f is one-to-one and has an inverse. Also notice that f is continuous everywhere since it is a polynomial. From Figure 1.5.8 we infer that the range of f is $(-\infty, +\infty)$, so the domain of f^{-1} is also $(-\infty, +\infty)$. Although a formula for $f^{-1}(x)$ cannot be found easily, we can use Theorem 1.5.7 to conclude that f^{-1} is continous on $(-\infty, +\infty)$. ◀

THE INTERMEDIATE-VALUE THEOREM

Figure 1.5.9 shows the graph of a function that is continuous on the closed interval $[a, b]$. The figure suggests that if we draw any horizontal line $y = k$, where k is between $f(a)$ and $f(b)$, then that line will cross the curve $y = f(x)$ at least once over the interval $[a, b]$. Stated in numerical terms, if f is continuous on $[a, b]$, then the function f must take on every value k between $f(a)$ and $f(b)$ at least once as x varies from a to b. For example, the polynomial $p(x) = x^5 - x + 3$ has a value of 3 at $x = 1$ and a value of 33 at $x = 2$. Thus, it follows from the continuity of p that the equation $x^5 - x + 3 = k$ has at least one solution in the interval $[1, 2]$ for every value of k between 3 and 33. This idea is stated more precisely in the following theorem.



▲ **Figure 1.5.9**

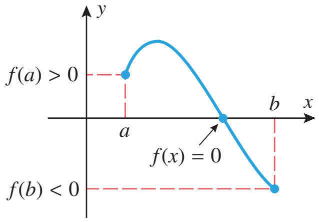
1.5.8 THEOREM (*Intermediate-Value Theorem*) *If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$.*

Although this theorem is intuitively obvious, its proof depends on a mathematically precise development of the real number system, which is beyond the scope of this text.

APPROXIMATING ROOTS USING THE INTERMEDIATE-VALUE THEOREM

A variety of problems can be reduced to solving an equation $f(x) = 0$ for its roots. Sometimes it is possible to solve for the roots exactly using algebra, but often this is not possible and one must settle for decimal approximations of the roots. One procedure for approximating roots is based on the following consequence of the Intermediate-Value Theorem.

1.5.9 THEOREM *If f is continuous on $[a, b]$, and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then there is at least one solution of the equation $f(x) = 0$ in the interval (a, b).*



▲ **Figure 1.5.10**

This result, which is illustrated in Figure 1.5.10, can be proved as follows.

PROOF Since $f(a)$ and $f(b)$ have opposite signs, 0 is between $f(a)$ and $f(b)$. Thus, by the Intermediate-Value Theorem there is at least one number x in the interval $[a, b]$ such that $f(x) = 0$. However, $f(a)$ and $f(b)$ are nonzero, so x must lie in the interval (a, b), which completes the proof. ■

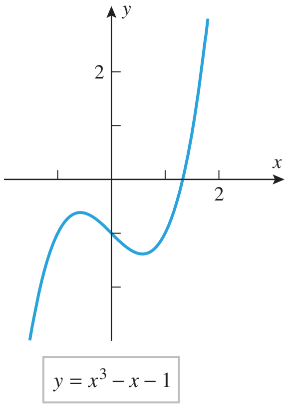
Before we illustrate how this theorem can be used to approximate roots, it will be helpful to discuss some standard terminology for describing errors in approximations. If x is an approximation to a quantity x_0, then we call

$$\epsilon = |x - x_0|$$

the ***absolute error*** or (less precisely) the ***error*** in the approximation. The terminology in Table 1.5.1 is used to describe the size of such errors.

Table 1.5.1

ERROR	DESCRIPTION
$\|x - x_0\| \leq 0.1$	x approximates x_0 with an error of at most 0.1.
$\|x - x_0\| \leq 0.01$	x approximates x_0 with an error of at most 0.01.
$\|x - x_0\| \leq 0.001$	x approximates x_0 with an error of at most 0.001.
$\|x - x_0\| \leq 0.0001$	x approximates x_0 with an error of at most 0.0001.
$\|x - x_0\| \leq 0.5$	x approximates x_0 to the nearest integer.
$\|x - x_0\| \leq 0.05$	x approximates x_0 to 1 decimal place (i.e., to the nearest tenth).
$\|x - x_0\| \leq 0.005$	x approximates x_0 to 2 decimal places (i.e., to the nearest hundredth).
$\|x - x_0\| \leq 0.0005$	x approximates x_0 to 3 decimal places (i.e., to the nearest thousandth).



▲ **Figure 1.5.11**

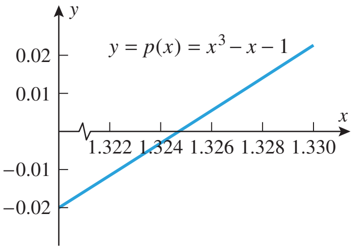
▶ **Example 6** The equation

$$x^3 - x - 1 = 0$$

cannot be solved algebraically very easily because the left side has no simple factors. However, if we graph $p(x) = x^3 - x - 1$ with a graphing utility (Figure 1.5.11), then we are led to conjecture that there is one real root and that this root lies inside the interval $[1, 2]$. The existence of a root in this interval is also confirmed by Theorem 1.5.9, since $p(1) = -1$ and $p(2) = 5$ have opposite signs. Approximate this root to two decimal-place accuracy.

Solution. Our objective is to approximate the unknown root x_0 with an error of at most 0.005. It follows that if we can find an interval of length 0.01 that contains the root, then the midpoint of that interval will approximate the root with an error of at most $\frac{1}{2}(0.01) = 0.005$, which will achieve the desired accuracy.

We know that the root x_0 lies in the interval $[1, 2]$. However, this interval has length 1, which is too large. We can pinpoint the location of the root more precisely by dividing the interval $[1, 2]$ into 10 equal parts and evaluating p at the points of subdivision using a calculating utility (Table 1.5.2). In this table $p(1.3)$ and $p(1.4)$ have opposite signs, so we know that the root lies in the interval $[1.3, 1.4]$. This interval has length 0.1, which is still too large, so we repeat the process by dividing the interval $[1.3, 1.4]$ into 10 parts and evaluating p at the points of subdivision; this yields Table 1.5.3, which tells us that the root is inside the interval $[1.32, 1.33]$ (Figure 1.5.12). Since this interval has length 0.01, its midpoint 1.325 will approximate the root with an error of at most 0.005. Thus, $x_0 \approx 1.325$ to two decimal-place accuracy. ◀



▲ **Figure 1.5.12**

Table 1.5.2

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$p(x)$	−1	−0.77	−0.47	−0.10	0.34	0.88	1.50	2.21	3.03	3.96	5

Table 1.5.3

x	1.3	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.4
$p(x)$	−0.103	−0.062	−0.020	0.023	0.066	0.110	0.155	0.201	0.248	0.296	0.344

REMARK

To say that x approximates x_0 to n decimal places does *not* mean that the first n decimal places of x and x_0 will be the same when the numbers are rounded to n decimal places. For example, $x = 1.084$ approximates $x_0 = 1.087$ to two decimal places because $|x - x_0| = 0.003$ (< 0.005). However, if we round these values to two decimal places, then we obtain $x \approx 1.08$ and $x_0 \approx 1.09$. Thus, if you approximate a number to n decimal places, then you should display that approximation to at least $n + 1$ decimal places to preserve the accuracy.

TECHNOLOGY MASTERY

Use a graphing or calculating utility to show that the root x_0 in Example 6 can be approximated as $x_0 \approx 1.3245$ to three decimal-place accuracy.

✔QUICK CHECK EXERCISES 1.5 *(See page 101 for answers.)*

1. What three conditions are satisfied if f is continuous at $x = c$?

2. Suppose that f and g are continuous functions such that $f(2) = 1$ and $\lim_{x\to 2}[f(x) + 4g(x)] = 13$. Find
(a) $g(2)$
(b) $\lim_{x\to 2} g(x)$.

3. Suppose that f and g are continuous functions such that $\lim_{x\to 3} g(x) = 5$ and $f(3) = -2$. Find $\lim_{x\to 3}[f(x)/g(x)]$.

4. For what values of x, if any, is the function

$$f(x) = \frac{x^2 - 16}{x^2 - 5x + 4}$$

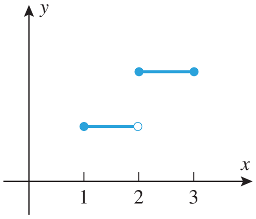
discontinuous?

5. Suppose that a function f is continuous everywhere and that $f(-2) = 3$, $f(-1) = -1$, $f(0) = -4$, $f(1) = 1$, and $f(2) = 5$. Does the Intermediate-Value Theorem guarantee that f has a root on the following intervals?
(a) $[-2, -1]$ (b) $[-1, 0]$ (c) $[-1, 1]$ (d) $[0, 2]$

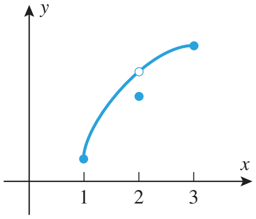
EXERCISE SET 1.5 Graphing Utility

1–4 Let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?
(a) $[1, 3]$ (b) $(1, 3)$ (c) $[1, 2]$
(d) $(1, 2)$ (e) $[2, 3]$ (f) $(2, 3)$
For each interval on which f is not continuous, indicate which conditions for the continuity of f do not hold.

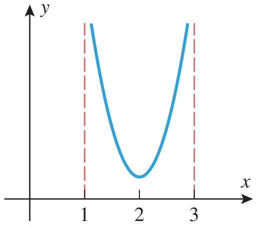
1.



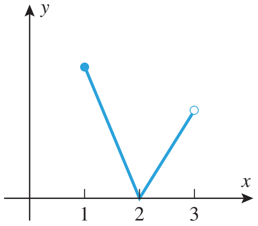
2.



3.



4.



5. Consider the functions

$$f(x) = \begin{cases} 1, & x \neq 4 \\ -1, & x = 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 4x - 10, & x \neq 4 \\ -6, & x = 4 \end{cases}$$

In each part, is the given function continuous at $x = 4$?
(a) $f(x)$ (b) $g(x)$ (c) $-g(x)$ (d) $|f(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $g(x) - 6f(x)$

6. Consider the functions

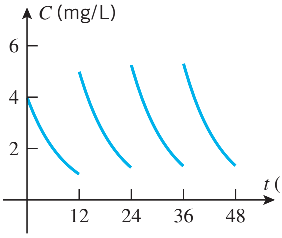
$$f(x) = \begin{cases} 1, & 0 \le x \\ 0, & x < 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & 0 \le x \\ 1, & x < 0 \end{cases}$$

In each part, is the given function continuous at $x = 0$?
(a) $f(x)$ (b) $g(x)$ (c) $f(-x)$ (d) $|g(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $f(x) + g(x)$

FOCUS ON CONCEPTS

7. In each part sketch the graph of a function f that satisfies the stated conditions.
(a) f is continuous everywhere except at $x = 3$, at which point it is continuous from the right.
(b) f has a two-sided limit at $x = 3$, but it is not continuous at $x = 3$.
(c) f is not continuous at $x = 3$, but if its value at $x = 3$ is changed from $f(3) = 1$ to $f(3) = 0$, it becomes continuous at $x = 3$.
(d) f is continuous on the interval $[0, 3)$ and is defined on the closed interval $[0, 3]$; but f is not continuous on the interval $[0, 3]$.

8. The accompanying figure models the concentration C of medication in the bloodstream of a patient over a 48-hour period of time. Discuss the significance of the discontinuities in the graph.



◀ **Figure Ex-8**

9. A student parking lot at a university charges \$2.00 for the first half hour (or any part) and \$1.00 for each subsequent half hour (or any part) up to a daily maximum of \$10.00.
(a) Sketch a graph of cost as a function of the time parked.
(b) Discuss the significance of the discontinuities in the graph to a student who parks there.

10. In each part determine whether the function is continuous or not, and explain your reasoning.
(a) The Earth's population as a function of time.
(b) Your exact height as a function of time.
(c) The cost of a taxi ride in your city as a function of the distance traveled.
(d) The volume of a melting ice cube as a function of time.

11–22 Find values of x, if any, at which f is not continuous. ■

11. $f(x) = 5x^4 - 3x + 7$ **12.** $f(x) = \sqrt[3]{x - 8}$

13. $f(x) = \dfrac{x + 2}{x^2 + 4}$ **14.** $f(x) = \dfrac{x + 2}{x^2 - 4}$

15. $f(x) = \dfrac{x}{2x^2 + x}$ **16.** $f(x) = \dfrac{2x + 1}{4x^2 + 4x + 5}$

17. $f(x) = \dfrac{3}{x} + \dfrac{x - 1}{x^2 - 1}$ **18.** $f(x) = \dfrac{5}{x} + \dfrac{2x}{x + 4}$

19. $f(x) = \dfrac{x^2 + 6x + 9}{|x| + 3}$ **20.** $f(x) = \left|4 - \dfrac{8}{x^4 + x}\right|$

21. $f(x) = \begin{cases} 2x + 3, & x \le 4 \\ 7 + \dfrac{16}{x}, & x > 4 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{3}{x - 1}, & x \ne 1 \\ 3, & x = 1 \end{cases}$

23–28 True–False Determine whether the statement is true or false. Explain your answer. ■

23. If $f(x)$ is continuous at $x = c$, then so is $|f(x)|$.

24. If $|f(x)|$ is continuous at $x = c$, then so is $f(x)$.

25. If f and g are discontinuous at $x = c$, then so is $f + g$.

26. If f and g are discontinuous at $x = c$, then so is fg.

27. If $\sqrt{f(x)}$ is continuous at $x = c$, then so is $f(x)$.

28. If $f(x)$ is continuous at $x = c$, then so is $\sqrt{f(x)}$.

29–30 Find a value of the constant k, if possible, that will make the function continuous everywhere. ■

29. (a) $f(x) = \begin{cases} 7x - 2, & x \le 1 \\ kx^2, & x > 1 \end{cases}$

(b) $f(x) = \begin{cases} kx^2, & x \le 2 \\ 2x + k, & x > 2 \end{cases}$

30. (a) $f(x) = \begin{cases} 9 - x^2, & x \ge -3 \\ k/x^2, & x < -3 \end{cases}$

(b) $f(x) = \begin{cases} 9 - x^2, & x \ge 0 \\ k/x^2, & x < 0 \end{cases}$

31. Find values of the constants k and m, if possible, that will make the function f continuous everywhere.

$$f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x + 1) + k, & -1 < x \le 2 \\ 2x^3 + x + 7, & x \le -1 \end{cases}$$

32. On which of the following intervals is

$$f(x) = \frac{1}{\sqrt{x - 2}}$$

continuous?
(a) $[2, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(2, +\infty)$ (d) $[1, 2)$

33–36 A function f is said to have a ***removable discontinuity*** at $x = c$ if $\lim_{x \to c} f(x)$ exists but f is not continuous at $x = c$, either because f is not defined at c or because the definition for $f(c)$ differs from the value of the limit. This terminology will be needed in these exercises. ■

33. (a) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is undefined.
(b) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is defined.

34. (a) The terminology *removable discontinuity* is appropriate because a removable discontinuity of a function f at $x = c$ can be "removed" by redefining the value of f appropriately at $x = c$. What value for $f(c)$ removes the discontinuity?
(b) Show that the following functions have removable discontinuities at $x = 1$, and sketch their graphs.

$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x = 1 \\ 1, & x < 1 \end{cases}$$

(c) What values should be assigned to $f(1)$ and $g(1)$ to remove the discontinuities?

35–36 Find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity. ■

35. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$
(c) $f(x) = \dfrac{x - 2}{|x| - 2}$

36. (a) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$ (b) $f(x) = \begin{cases} 2x - 3, & x \le 2 \\ x^2, & x > 2 \end{cases}$
(c) $f(x) = \begin{cases} 3x^2 + 5, & x \ne 1 \\ 6, & x = 1 \end{cases}$

37. (a) Use a graphing utility to generate the graph of the function $f(x) = (x + 3)/(2x^2 + 5x - 3)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Check your conjecture by factoring the denominator.

38. (a) Use a graphing utility to generate the graph of the function $f(x) = x/(x^3 - x + 2)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
(b) Use the Intermediate-Value Theorem to approximate the locations of all discontinuities to two decimal places.

39. Prove that $f(x) = x^{3/5}$ is continuous everywhere, carefully justifying each step.

40. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere, carefully justifying each step.

41. Prove:
(a) part (*a*) of Theorem 1.5.3
(b) part (*b*) of Theorem 1.5.3
(c) part (*c*) of Theorem 1.5.3.

42. Prove part (*b*) of Theorem 1.5.4.

43. (a) Use Theorem 1.5.5 to prove that if f is continuous at $x = c$, then $\lim_{h \to 0} f(c + h) = f(c)$.
(b) Prove that if $\lim_{h \to 0} f(c + h) = f(c)$, then f is continuous at $x = c$. [*Hint:* What does this limit tell you about the continuity of $g(h) = f(c + h)$?]
(c) Conclude from parts (a) and (b) that f is continuous at $x = c$ if and only if $\lim_{h \to 0} f(c + h) = f(c)$.

44. Prove: If f and g are continuous on $[a, b]$, and $f(a) > g(a)$, $f(b) < g(b)$, then there is at least one solution of the equation $f(x) = g(x)$ in (a, b). [*Hint:* Consider $f(x) - g(x)$.]

FOCUS ON CONCEPTS

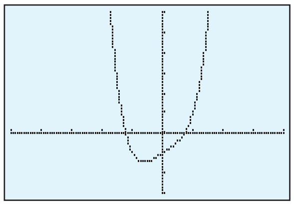
45. Give an example of a function f that is defined on a closed interval, and whose values at the endpoints have opposite signs, but for which the equation $f(x) = 0$ has no solution in the interval.

46. Let f be the function whose graph is shown in Exercise 2. For each interval, determine (i) whether the hypothesis of the Intermediate-Value Theorem is satisfied, and (ii) whether the conclusion of the Intermediate-Value Theorem is satisfied.
(a) $[1, 2]$ (b) $[2, 3]$ (c) $[1, 3]$

47. Show that the equation $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[-1, 1]$.

48. Prove: If $p(x)$ is a polynomial of odd degree, then the equation $p(x) = 0$ has at least one real solution.

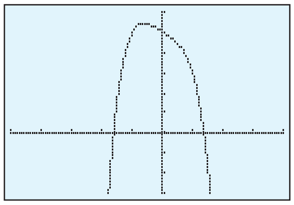
49. The accompanying figure shows the graph of the equation $y = x^4 + x - 1$. Use the method of Example 6 to approximate the x-intercepts with an error of at most 0.05.



$[-5, 4] \times [-3, 6]$
$x\text{Scl} = 1, y\text{Scl} = 1$

◀ **Figure Ex-49**

50. The accompanying figure shows the graph of the equation $y = 5 - x - x^4$. Use the method of Example 6 to approximate the roots of the equation $5 - x - x^4 = 0$ to two decimal-place accuracy.



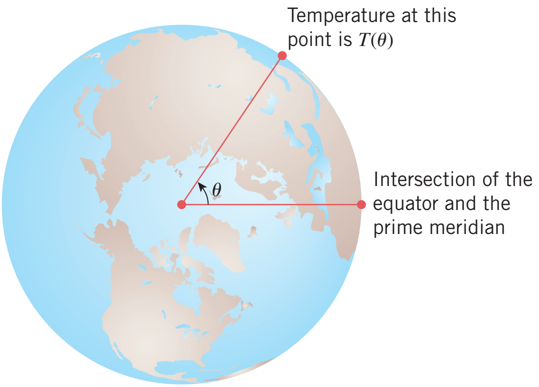
$[-5, 4] \times [-3, 6]$
$x\text{Scl} = 1, y\text{Scl} = 1$

◀ **Figure Ex-50**

51. Use the fact that $\sqrt{5}$ is a solution of $x^2 - 5 = 0$ to approximate $\sqrt{5}$ with an error of at most 0.005.

52. A sprinter, who is timed with a stopwatch, runs a hundred yard dash in 10 s. The stopwatch is reset to 0, and the sprinter is timed jogging back to the starting block. Show that there is at least one point on the track at which the reading on the stopwatch during the sprint is the same as the reading during the return jog. [*Hint:* Use the result in Exercise 44.]

53. Prove that there exist points on opposite sides of the equator that are at the same temperature. [*Hint:* Consider the accompanying figure, which shows a view of the equator from a point above the North Pole. Assume that the temperature $T(\theta)$ is a continuous function of the angle θ, and consider the function $f(\theta) = T(\theta + \pi) - T(\theta)$.]



▲ **Figure Ex-53**

54. Let R denote an elliptical region in the xy-plane, and define $f(z)$ to be the area within R that is on, or to the left of, the vertical line $x = z$. Prove that f is a continuous function of z. [*Hint:* Assume the ellipse is between the horizontal lines $y = a$ and $y = b$, $a < b$. Argue that $|f(z_1) - f(z_2)| \leq (b - a) \cdot |z_1 - z_2|$.]

55. Let R denote an elliptical region in the plane. For any line L, prove there is a line perpendicular to L that divides R in half by area. [*Hint:* Introduce coordinates so that L is the x-axis. Use the result in Exercise 54 and the Intermediate-Value Theorem.]

56. Suppose that f is continuous on the interval [0, 1] and that $0 \leq f(x) \leq 1$ for all x in this interval.
  (a) Sketch the graph of $y = x$ together with a possible graph for f over the interval [0, 1].
  (b) Use the Intermediate-Value Theorem to help prove that there is at least one number c in the interval [0, 1] such that $f(c) = c$.

57. Let $f(x) = x^6 + 3x + 5$, $x \geq 0$. Show that f is an invertible function and that f^{-1} is continuous on $[5, +\infty)$.

58. Suppose that f is an invertible function, $f(0) = 0$, f is continuous at 0, and $\lim_{x\to 0} f(x)/x$ exists. Given that $L = \lim_{x\to 0} f(x)/x$, show

$$\lim_{x\to 0} \frac{x}{f^{-1}(x)} = L$$

[*Hint:* Apply Theorem 1.5.5 to the composition $h \circ g$, where

$$h(x) = \begin{cases} f(x)/x, & x \neq 0 \\ L, & x = 0 \end{cases}$$

and $g(x) = f^{-1}(x)$.]

59. **Writing** It is often assumed that changing physical quantities such as the height of a falling object or the weight of a melting snowball, are continuous functions of time. Use specific examples to discuss the merits of this assumption.

60. **Writing** The Intermediate-Value Theorem is an example of what is known as an "existence theorem." In your own words, describe how to recognize an existence theorem, and discuss some of the ways in which an existence theorem can be useful.

✔ QUICK CHECK ANSWERS 1.5

1. $f(c)$ is defined; $\lim_{x\to c} f(x)$ exists; $\lim_{x\to c} f(x) = f(c)$ **2.** (a) 3 (b) 3 **3.** $-2/5$ **4.** $x = 1, 4$
5. (a) yes (b) no (c) yes (d) yes

1.6 CONTINUITY OF TRIGONOMETRIC FUNCTIONS

In this section we will discuss the continuity properties of trigonometric functions. We will also discuss some important limits involving such functions.

CONTINUITY OF TRIGONOMETRIC FUNCTIONS

Recall from trigonometry that the graphs of $\sin x$ and $\cos x$ are drawn as continuous curves. We will not formally prove that these functions are continuous, but we can motivate this fact by letting c be a fixed angle in radian measure and x a variable angle in radian measure. If, as illustrated in Figure 1.6.1, the angle x approaches the angle c, then the point $P(\cos x, \sin x)$

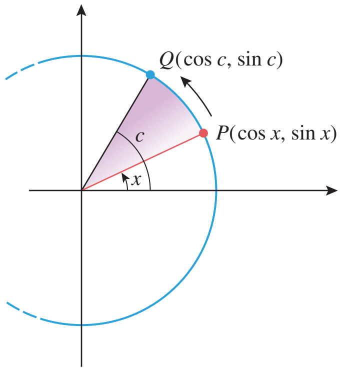


Figure 1.6.1

moves along the unit circle toward $Q(\cos c, \sin c)$, and the coordinates of P approach the corresponding coordinates of Q. This implies that

$$\lim_{x \to c} \sin x = \sin c \quad \text{and} \quad \lim_{x \to c} \cos x = \cos c \tag{1}$$

Thus, $\sin x$ and $\cos x$ are continuous at the arbitrary point c; that is, these functions are continuous everywhere.

The formulas in (1) can be used to find limits of the remaining trigonometric functions by expressing them in terms of $\sin x$ and $\cos x$; for example, if $\cos c \neq 0$, then

$$\lim_{x \to c} \tan x = \lim_{x \to c} \frac{\sin x}{\cos x} = \frac{\sin c}{\cos c} = \tan c$$

Thus, we are led to the following theorem.

Theorem 1.6.1 implies that the six basic trigonometric functions are continuous on their domains. In particular, $\sin x$ and $\cos x$ are continuous everywhere.

1.6.1 THEOREM *If c is any number in the natural domain of the stated trigonometric function, then*

$$\lim_{x \to c} \sin x = \sin c \qquad \lim_{x \to c} \cos x = \cos c \qquad \lim_{x \to c} \tan x = \tan c$$
$$\lim_{x \to c} \csc x = \csc c \qquad \lim_{x \to c} \sec x = \sec c \qquad \lim_{x \to c} \cot x = \cot c$$

Example 1 Find the limit

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right)$$

Solution. Since the cosine function is continuous everywhere, it follows from Theorem 1.5.5 that

$$\lim_{x \to 1} \cos(g(x)) = \cos\left(\lim_{x \to 1} g(x)\right)$$

provided $\lim_{x \to 1} g(x)$ exists. Thus,

$$\lim_{x \to 1} \cos\left(\frac{x^2 - 1}{x - 1}\right) = \lim_{x \to 1} \cos(x + 1) = \cos\left(\lim_{x \to 1} (x + 1)\right) = \cos 2 \ ◀$$

OBTAINING LIMITS BY SQUEEZING

In Section 1.1 we used numerical evidence to conjecture that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{2}$$

However, this limit is not easy to establish with certainty. The limit is an indeterminate form of type 0/0, and there is no simple algebraic manipulation that one can perform to obtain the limit. Later in the text we will develop general methods for finding limits of indeterminate forms, but in this particular case we can use a technique called *squeezing*.

The method of squeezing is used to prove that $f(x) \to L$ as $x \to c$ by "trapping" or "squeezing" f between two functions, g and h, whose limits as $x \to c$ are known with *certainty* to be L. As illustrated in Figure 1.6.2, this forces f to have a limit of L as well. This is the idea behind the following theorem, which we state without proof.

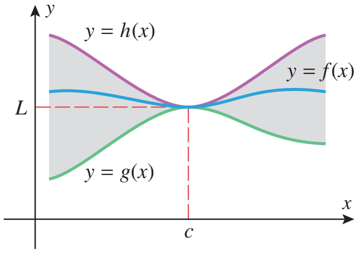


Figure 1.6.2

1.6.2 THEOREM (*The Squeezing Theorem*) *Let f, g, and h be functions satisfying*

$$g(x) \le f(x) \le h(x)$$

for all x in some open interval containing the number c, with the possible exception that the inequalities need not hold at c. If g and h have the same limit as x approaches c, say

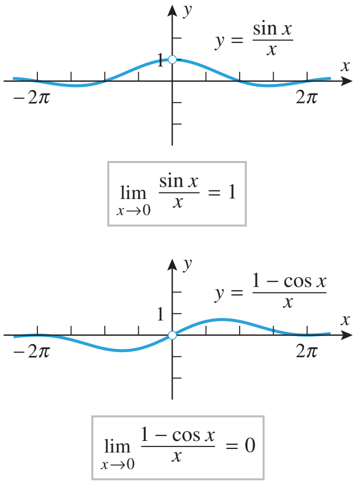
$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then f also has this limit as x approaches c, that is,

$$\lim_{x\to c} f(x) = L$$

The Squeezing Theorem also holds for one-sided limits and limits at $+\infty$ and $-\infty$. How do you think the hypotheses would change in those cases?

To illustrate how the Squeezing Theorem works, we will prove the following results, which are illustrated in Figure 1.6.3.



▲ **Figure 1.6.3**

1.6.3 THEOREM

$$(a)\quad \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad (b)\quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

PROOF (*a*) In this proof we will interpret x as an angle in radian measure, and we will assume to start that $0 < x < \pi/2$. As illustrated in Figure 1.6.4, the area of a sector with central angle x and radius 1 lies between the areas of two triangles, one with area $\frac{1}{2}\tan x$ and the other with area $\frac{1}{2}\sin x$. Since the sector has area $\frac{1}{2}x$ (see marginal note), it follows that

$$\frac{1}{2}\tan x \ge \frac{1}{2}x \ge \frac{1}{2}\sin x$$

Multiplying through by $2/(\sin x)$ and using the fact that $\sin x > 0$ for $0 < x < \pi/2$, we obtain

$$\frac{1}{\cos x} \ge \frac{x}{\sin x} \ge 1$$

Next, taking reciprocals reverses the inequalities, so we obtain

$$\cos x \le \frac{\sin x}{x} \le 1 \tag{3}$$

which squeezes the function $(\sin x)/x$ between the functions $\cos x$ and 1. Although we derived these inequalities by assuming that $0 < x < \pi/2$, they also hold for $-\pi/2 < x < 0$ [since replacing x by $-x$ and using the identities $\sin(-x) = -\sin x$, and $\cos(-x) = \cos x$ leaves (3) unchanged]. Finally, since

$$\lim_{x\to 0} \cos x = 1 \quad \text{and} \quad \lim_{x\to 0} 1 = 1$$

the Squeezing Theorem implies that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

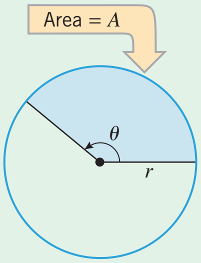
Recall that the area A of a sector of radius r and central angle θ is

$$A = \frac{1}{2}r^2\theta$$

This can be derived from the relationship

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

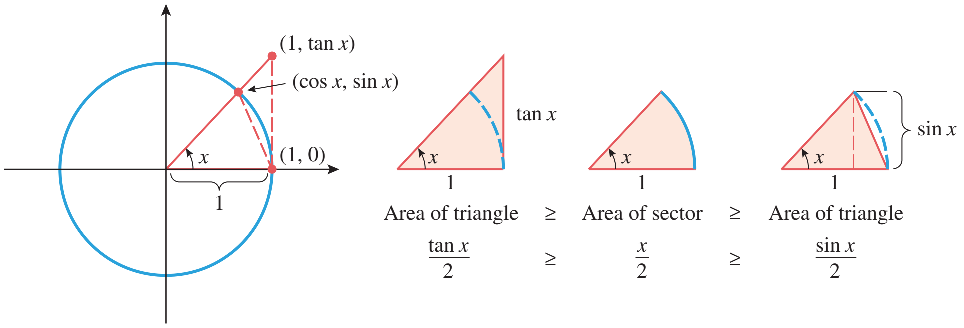
which states that the area of the sector is to the area of the circle as the central angle of the sector is to the central angle of the circle.



PROOF (*b*) For this proof we will use the limit in part (*a*), the continuity of the sine function, and the trigonometric identity $\sin^2 x = 1 - \cos^2 x$. We obtain

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0}\left[\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\right] = \lim_{x\to 0}\frac{\sin^2 x}{(1+\cos x)x}$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right) = (1)\left(\frac{0}{1+1}\right) = 0 \quad ■$$



▶ **Figure 1.6.4**

▶ Example 2 Find

$$\text{(a)}\ \lim_{x\to 0}\frac{\tan x}{x}\qquad \text{(b)}\ \lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}\qquad \text{(c)}\ \lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$$

Solution (a).

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right)=(1)(1)=1$$

Solution (b). The trick is to multiply and divide by 2, which will make the denominator the same as the argument of the sine function [just as in Theorem 1.6.3(*a*)]:

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=\lim_{\theta\to 0}2\cdot\frac{\sin 2\theta}{2\theta}=2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}$$

Now make the substitution $x=2\theta$, and use the fact that $x\to 0$ as $\theta\to 0$. This yields

$$\lim_{\theta\to 0}\frac{\sin 2\theta}{\theta}=2\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}=2\lim_{x\to 0}\frac{\sin x}{x}=2(1)=2$$

TECHNOLOGY MASTERY

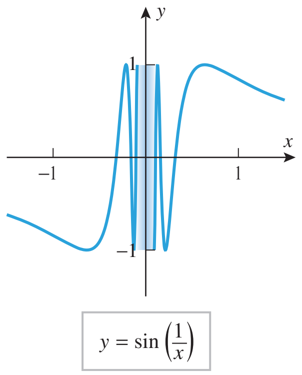
Use a graphing utility to confirm the limits in Example 2, and if you have a CAS, use it to obtain the limits.

Solution (c).

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim_{x\to 0}\frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 5x}{x}}=\lim_{x\to 0}\frac{3\cdot\dfrac{\sin 3x}{3x}}{5\cdot\dfrac{\sin 5x}{5x}}=\frac{3\cdot 1}{5\cdot 1}=\frac{3}{5}\ ◀$$

▶ Example 3 Discuss the limits

$$\text{(a)}\ \lim_{x\to 0}\sin\left(\frac{1}{x}\right)\qquad \text{(b)}\ \lim_{x\to 0}x\sin\left(\frac{1}{x}\right)$$



▲ **Figure 1.6.5**

Solution (a). Let us view $1/x$ as an angle in radian measure. As $x\to 0^+$, the angle $1/x$ approaches $+\infty$, so the values of $\sin(1/x)$ keep oscillating between -1 and 1 without approaching a limit. Similarly, as $x\to 0^-$, the angle $1/x$ approaches $-\infty$, so again the values of $\sin(1/x)$ keep oscillating between -1 and 1 without approaching a limit. These conclusions are consistent with the graph shown in Figure 1.6.5. Note that the oscillations become more and more rapid as $x\to 0$ because $1/x$ increases (or decreases) more and more rapidly as x approaches 0.

Solution (b). Since

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

it follows that if $x \neq 0$, then

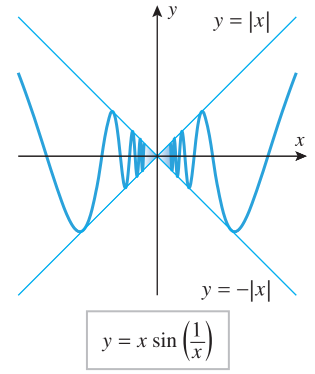
$$-|x| \le x \sin\left(\frac{1}{x}\right) \le |x| \tag{4}$$

Since $|x| \to 0$ as $x \to 0$, the inequalities in (4) and the Squeezing Theorem imply that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$

This is consistent with the graph shown in Figure 1.6.6. ◀

Confirm (4) by considering the cases $x > 0$ and $x < 0$ separately.



▲ **Figure 1.6.6**

REMARK It follows from part (b) of this example that the function

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$, since the value of the function and the value of the limit are the same at 0. This shows that the behavior of a function can be very complex in the vicinity of $x = c$, even though the function is continuous at c.

✔ QUICK CHECK EXERCISES 1.6 *(See page 107 for answers.)*

1. In each part, is the given function continuous on the interval $[0, \pi/2)$?
(a) $\sin x$ (b) $\cos x$ (c) $\tan x$ (d) $\csc x$

2. Evaluate
(a) $\lim_{x \to 0} \frac{\sin x}{x}$
(b) $\lim_{x \to 0} \frac{1 - \cos x}{x}$.

3. Suppose a function f has the property that for all real numbers x

$$3 - |x| \le f(x) \le 3 + |x|$$

From this we can conclude that $f(x) \to$ ________ as $x \to$ ________.

EXERCISE SET 1.6 Graphing Utility

1–8 Find the discontinuities, if any. ■

1. $f(x) = \sin(x^2 - 2)$

2. $f(x) = \cos\left(\frac{x}{x - \pi}\right)$

3. $f(x) = |\cot x|$

4. $f(x) = \sec x$

5. $f(x) = \csc x$

6. $f(x) = \frac{1}{1 + \sin^2 x}$

7. $f(x) = \frac{1}{1 - 2\sin x}$

8. $f(x) = \sqrt{2 + \tan^2 x}$

9–10 In each part, use Theorem 1.5.6(*b*) to show that the function is continuous everywhere. ■

9. (a) $\sin(x^3 + 7x + 1)$ (b) $|\sin x|$
(c) $\cos^3(x + 1)$

10. (a) $|3 + \sin 2x|$ (b) $\sin(\sin x)$
(c) $\cos^5 x - 2\cos^3 x + 1$

11–32 Find the limits. ■

11. $\lim_{x \to +\infty} \cos\left(\frac{1}{x}\right)$

12. $\lim_{x \to +\infty} \sin\left(\frac{\pi x}{2 - 3x}\right)$

13. $\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}$

14. $\lim_{h \to 0} \frac{\sin h}{2h}$

15. $\lim_{x \to 0} \frac{x^2 - 3\sin x}{x}$

16. $\lim_{x \to 0} \frac{2 - \cos 3x - \cos 4x}{x}$

17. $\lim_{\theta \to 0^+} \frac{\sin \theta}{\theta^2}$

18. $\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta}$

19. $\lim_{x \to 0} \dfrac{\tan 7x}{\sin 3x}$

20. $\lim_{x \to 0} \dfrac{\sin 6x}{\sin 8x}$

21. $\lim_{x \to 0^+} \dfrac{\sin x}{5\sqrt{x}}$

22. $\lim_{x \to 0} \dfrac{\sin^2 x}{3x^2}$

23. $\lim_{x \to 0} \dfrac{\sin x^2}{x}$

24. $\lim_{h \to 0} \dfrac{\sin h}{1 - \cos h}$

25. $\lim_{t \to 0} \dfrac{t^2}{1 - \cos^2 t}$

26. $\lim_{x \to 0} \dfrac{x}{\cos\left(\frac{1}{2}\pi - x\right)}$

27. $\lim_{\theta \to 0} \dfrac{\theta^2}{1 - \cos \theta}$

28. $\lim_{h \to 0} \dfrac{1 - \cos 3h}{\cos^2 5h - 1}$

29. $\lim_{x \to 0^+} \sin\left(\dfrac{1}{x}\right)$

30. $\lim_{x \to 0} \dfrac{\tan 3x^2 + \sin^2 5x}{x^2}$

31. $\lim_{x \to 0} \dfrac{\tan ax}{\sin bx}, \quad (a \neq 0, b \neq 0)$

32. $\lim_{x \to 0} \dfrac{\sin^2(kx)}{x}, \quad k \neq 0$

33–34 (a) Complete the table and make a guess about the limit indicated. (b) Find the exact value of the limit. ■

33. $f(x) = \dfrac{\sin(x-5)}{x^2 - 25}; \ \lim_{x \to 5} f(x)$

x	4	4.5	4.9	5.1	5.5	6
$f(x)$						

◀ **Table Ex-33**

34. $f(x) = \dfrac{\sin(x^2 + 3x + 2)}{x + 2}; \ \lim_{x \to -2} f(x)$

x	−2.1	−2.01	−2.001	−1.999	−1.99	−1.9
$f(x)$						

▲ **Table Ex-34**

35–38 True–False Determine whether the statement is true or false. Explain your answer. ■

35. Suppose that for all real numbers x, a function f satisfies

$$|f(x) + 5| \leq |x + 1|$$

Then $\lim_{x \to -1} f(x) = -5$.

36. For $0 < x < \pi/2$, the graph of $y = \sin x$ lies below the graph of $y = x$ and above the graph of $y = x \cos x$.

37. The function

$$f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous everywhere.

38. Suppose that M is a positive number and that for all real numbers x, a function f satisfies

$$-M \leq f(x) \leq M$$

Then

$$\lim_{x \to 0} x f(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} \frac{f(x)}{x} = 0$$

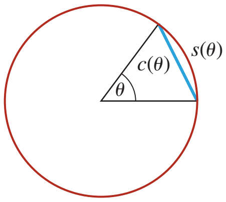
FOCUS ON CONCEPTS

39. In an attempt to verify that $\lim_{x \to 0} (\sin x)/x = 1$, a student constructs the accompanying table.
(a) What mistake did the student make?
(b) What is the exact value of the limit illustrated by this table?

x	−0.01	−0.001	0.001	0.01
$\sin x/x$	0.017453	0.017453	0.017453	0.017453

▲ **Table Ex-39**

40. In the circle in the accompanying figure, a central angle of measure θ radians subtends a chord of length $c(\theta)$ and a circular arc of length $s(\theta)$. Based on your intuition, what would you conjecture is the value of $\lim_{\theta \to 0^+} c(\theta)/s(\theta)$? Verify your conjecture by computing the limit.



◀ **Figure Ex-40**

41. Find a nonzero value for the constant k that makes

$$f(x) = \begin{cases} \dfrac{\tan kx}{x}, & x < 0 \\ 3x + 2k^2, & x \geq 0 \end{cases}$$

continuous at $x = 0$.

42. Is

$$f(x) = \begin{cases} \dfrac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

continuous at $x = 0$? Explain.

43. In parts (a)–(c), find the limit by making the indicated substitution.

(a) $\lim_{x \to +\infty} x \sin \dfrac{1}{x}; \quad t = \dfrac{1}{x}$

(b) $\lim_{x \to -\infty} x \left(1 - \cos \dfrac{1}{x}\right); \quad t = \dfrac{1}{x}$

(c) $\lim_{x \to \pi} \dfrac{\pi - x}{\sin x}; \quad t = \pi - x$

44. Find $\lim_{x \to 2} \dfrac{\cos(\pi/x)}{x - 2}$. $\left[\textit{Hint:} \text{ Let } t = \dfrac{\pi}{2} - \dfrac{\pi}{x}.\right]$

45. Find $\lim_{x \to 1} \dfrac{\sin(\pi x)}{x - 1}$.

46. Find $\lim_{x \to \pi/4} \dfrac{\tan x - 1}{x - \pi/4}$.

FOCUS ON CONCEPTS

47. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x\cos\frac{50\pi}{x}=0$$

and illustrate the principle involved by using a graphing utility to graph the equations $y=|x|$, $y=-|x|$, and $y=x\cos(50\pi/x)$ on the same screen in the window $[-1, 1]\times[-1, 1]$.

48. Use the Squeezing Theorem to show that

$$\lim_{x\to 0} x^2\sin\left(\frac{50\pi}{\sqrt[3]{x}}\right)=0$$

and illustrate the principle involved by using a graphing utility to graph the equations $y=x^2$, $y=-x^2$, and $y=x^2\sin(50\pi/\sqrt[3]{x})$ on the same screen in the window $[-0.5, 0.5]\times[-0.25, 0.25]$.

49. In Example 3 we used the Squeezing Theorem to prove that

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=0$$

Why couldn't we have obtained the same result by writing

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=\lim_{x\to 0} x\cdot\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$$
$$=0\cdot\lim_{x\to 0}\sin\left(\frac{1}{x}\right)=0?$$

50. Sketch the graphs of the curves $y=1-x^2$, $y=\cos x$, and $y=f(x)$, where f is a function that satisfies the inequalities

$$1-x^2\le f(x)\le\cos x$$

for all x in the interval $(-\pi/2, \pi/2)$. What can you say about the limit of $f(x)$ as $x\to 0$? Explain.

51. Sketch the graphs of the curves $y=1/x$, $y=-1/x$, and $y=f(x)$, where f is a function that satisfies the inequalities

$$-\frac{1}{x}\le f(x)\le\frac{1}{x}$$

for all x in the interval $[1, +\infty)$. What can you say about the limit of $f(x)$ as $x\to+\infty$? Explain your reasoning.

52. Draw pictures analogous to Figure 1.6.2 that illustrate the Squeezing Theorem for limits of the forms $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

53. (a) Use the Intermediate-Value Theorem to show that the equation $x=\cos x$ has at least one solution in the interval $[0, \pi/2]$.
(b) Show graphically that there is exactly one solution in the interval.
(c) Approximate the solution to three decimal places.

54. (a) Use the Intermediate-Value Theorem to show that the equation $x+\sin x=1$ has at least one solution in the interval $[0, \pi/6]$.
(b) Show graphically that there is exactly one solution in the interval.
(c) Approximate the solution to three decimal places.

55. In the study of falling objects near the surface of the Earth, the ***acceleration g due to gravity*** is commonly taken to be a constant 9.8 m/s^2. However, the elliptical shape of the Earth and other factors cause variations in this value that depend on latitude. The following formula, known as the World Geodetic System 1984 (WGS 84) Ellipsoidal Gravity Formula, is used to predict the value of g at a latitude of ϕ degrees (either north or south of the equator):

$$g=9.7803253359\frac{1+0.0019318526461\sin^2\phi}{\sqrt{1-0.0066943799901\sin^2\phi}}\ \text{m/s}^2$$

(a) Use a graphing utility to graph the curve $y=g(\phi)$ for $0°\le\phi\le 90°$. What do the values of g at $\phi=0°$ and at $\phi=90°$ tell you about the WGS 84 ellipsoid model for the Earth?
(b) Show that $g=9.8$ m/s^2 somewhere between latitudes of 38° and 39°.

56. Writing In your own words, explain the *practical value* of the Squeezing Theorem.

57. Writing A careful examination of the proof of Theorem 1.6.3 raises the issue of whether the proof might actually be a circular argument! Read the article "A Circular Argument" by Fred Richman in the March 1993 issue of *The College Mathematics Journal*, and write a short report on the author's principal points.

✔QUICK CHECK ANSWERS 1.6

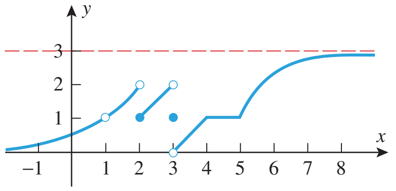
1. (a) yes (b) yes (c) yes (d) no **2.** (a) 1 (b) 0 **3.** 3; 0

CHAPTER 1 REVIEW EXERCISES

Graphing Utility

1. For the function f graphed in the accompanying figure, find the limit if it exists.

(a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$ (c) $\lim_{x\to 3} f(x)$
(d) $\lim_{x\to 4} f(x)$ (e) $\lim_{x\to +\infty} f(x)$ (f) $\lim_{x\to -\infty} f(x)$
(g) $\lim_{x\to 3^+} f(x)$ (h) $\lim_{x\to 3^-} f(x)$ (i) $\lim_{x\to 0} f(x)$



◀ **Figure Ex-1**

2. In each part, complete the table and make a conjecture about the value of the limit indicated. Confirm your conjecture by finding the limit analytically.

(a) $f(x) = \dfrac{x-2}{x^2-4}$; $\displaystyle\lim_{x\to 2^+} f(x)$

x	2.00001	2.0001	2.001	2.01	2.1	2.5
$f(x)$						

(b) $f(x) = \dfrac{\tan 4x}{x}$; $\displaystyle\lim_{x\to 0} f(x)$

x	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(x)$						

3–8 Find the limits. ■

3. $\displaystyle\lim_{x\to -1} \frac{x^3-x^2}{x-1}$

4. $\displaystyle\lim_{x\to 1} \frac{x^3-x^2}{x-1}$

5. $\displaystyle\lim_{x\to -3} \frac{3x+9}{x^2+4x+3}$

6. $\displaystyle\lim_{x\to 2^-} \frac{x+2}{x-2}$

7. $\displaystyle\lim_{x\to +\infty} \frac{(2x-1)^5}{(3x^2+2x-7)(x^3-9x)}$

8. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2}$

9. In each part, find the horizontal asymptotes, if any.

(a) $y = \dfrac{2x-7}{x^2-4x}$ (b) $y = \dfrac{x^3-x^2+10}{3x^2-4x}$

(c) $y = \dfrac{2x^2-6}{x^2+5x}$

10. In each part, find $\lim_{x\to a} f(x)$, if it exists, where a is replaced by $0, 5^+, -5^-, -5, 5, -\infty$, and $+\infty$.

(a) $f(x) = \sqrt{5-x}$

(b) $f(x) = \begin{cases} (x-5)/|x-5|, & x \neq 5 \\ 0, & x = 5 \end{cases}$

11–15 Find the limits. ■

11. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\tan 3x}$

12. $\displaystyle\lim_{x\to 0} \frac{x \sin x}{1-\cos x}$

13. $\displaystyle\lim_{x\to 0} \frac{3x-\sin(kx)}{x}, \quad k \neq 0$

14. $\displaystyle\lim_{\theta\to 0} \tan\left(\frac{1-\cos\theta}{\theta}\right)$

15. $\displaystyle\lim_{x\to -1} \frac{\sin(x+1)}{x^2-1}$

16. (a) Write a paragraph or two that describes how the limit of a function can fail to exist at $x = a$, and accompany your description with some specific examples.

(b) Write a paragraph or two that describes how the limit of a function can fail to exist as $x \to +\infty$ or $x \to -\infty$, and accompany your description with some specific examples.

(c) Write a paragraph or two that describes how a function can fail to be continuous at $x = a$, and accompany your description with some specific examples.

17. (a) Find a formula for a rational function that has a vertical asymptote at $x = 1$ and a horizontal asymptote at $y = 2$.

(b) Check your work by using a graphing utility to graph the function.

18. Paraphrase the ϵ-δ definition for $\lim_{x\to a} f(x) = L$ in terms of a graphing utility viewing window centered at the point (a, L).

19. Suppose that $f(x)$ is a function and that for any given $\epsilon > 0$, the condition $0 < |x-2| < \frac{3}{4}\epsilon$ guarantees that $|f(x)-5| < \epsilon$.

(a) What limit is described by this statement?

(b) Find a value of δ such that $0 < |x-2| < \delta$ guarantees that $|8f(x)-40| < 0.048$.

20. The limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

ensures that there is a number δ such that

$$\left|\frac{\sin x}{x} - 1\right| < 0.001$$

if $0 < |x| < \delta$. Estimate the largest such δ.

21. In each part, a positive number ϵ and the limit L of a function f at a are given. Find a number δ such that $|f(x)-L| < \epsilon$ if $0 < |x-a| < \delta$.

(a) $\displaystyle\lim_{x\to 2}(4x-7) = 1$; $\epsilon = 0.01$

(b) $\displaystyle\lim_{x\to 3/2} \frac{4x^2-9}{2x-3} = 6$; $\epsilon = 0.05$

(c) $\displaystyle\lim_{x\to 4} x^2 = 16$; $\epsilon = 0.001$

22. Use Definition 1.4.1 to prove the stated limits are correct.

(a) $\displaystyle\lim_{x\to 2}(4x-7) = 1$ (b) $\displaystyle\lim_{x\to 3/2} \frac{4x^2-9}{2x-3} = 6$

23. Suppose that f is continuous at x_0 and that $f(x_0) > 0$. Give either an ϵ-δ proof or a convincing verbal argument to show that there must be an open interval containing x_0 on which $f(x) > 0$.

24. (a) Let

$$f(x) = \frac{\sin x - \sin 1}{x-1}$$

Approximate $\lim_{x\to 1} f(x)$ by graphing f and calculating values for some appropriate choices of x.

(b) Use the identity

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

to find the exact value of $\displaystyle\lim_{x\to 1} f(x)$.

25. Find values of x, if any, at which the given function is not continuous.

(a) $f(x) = \dfrac{x}{x^2-1}$ (b) $f(x) = |x^3-2x^2|$

(c) $f(x) = \dfrac{x+3}{|x^2+3x|}$

26. Determine where f is continuous.

(a) $f(x) = \dfrac{x}{|x| - 3}$ (b) $f(x) = \cos^{-1}\left(\dfrac{1}{x}\right)$

(c) $f(x) = \dfrac{2x - 1}{2x^2 + 3x - 2}$

27. Suppose that

$$f(x) = \begin{cases} -x^4 + 3, & x \leq 2 \\ x^2 + 9, & x > 2 \end{cases}$$

Is f continuous everywhere? Justify your conclusion.

28. One dictionary describes a continuous function as "one whose value at each point is closely approached by its values at neighboring points."

(a) How would you explain the meaning of the terms "neighboring points" and "closely approached" to a nonmathematician?

(b) Write a paragraph that explains why the dictionary definition is consistent with Definition 1.5.1.

29. Show that the conclusion of the Intermediate-Value Theorem may be false if f is not continuous on the interval $[a, b]$.

30. Suppose that f is continuous on the interval $[0, 1]$, that $f(0) = 2$, and that f has no zeros in the interval. Prove that $f(x) > 0$ for all x in $[0, 1]$.

31. Show that the equation $x^4 + 5x^3 + 5x - 1 = 0$ has at least two real solutions in the interval $[-6, 2]$.

CHAPTER 1 MAKING CONNECTIONS

In Section 1.1 we developed the notion of a tangent line to a graph at a given point by considering it as a limiting position of secant lines through that point (Figure 1.1.4*a*). In these exercises we will develop an analogous idea in which secant lines are replaced by "secant circles" and the tangent line is replaced by a "tangent circle" (called the ***osculating circle***). We begin with the graph of $y = x^2$.

1. Recall that there is a unique circle through any three noncollinear points in the plane. For any positive real number x, consider the unique "secant circle" that passes through the fixed point $O(0, 0)$ and the variable points $Q(-x, x^2)$ and $P(x, x^2)$ (see the accompanying figure). Use plane geometry to explain why the center of this circle is the intersection of the y-axis and the perpendicular bisector of segment OP.

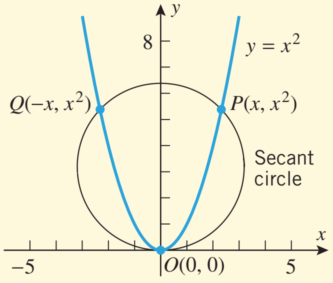


Figure Ex-1

2. (a) Let $(0, C(x))$ denote the center of the circle in Exercise 1 and show that

$$C(x) = \tfrac{1}{2}x^2 + \tfrac{1}{2}$$

(b) Show that as $x \to 0^+$, the secant circles approach a limiting position given by the circle that passes through the origin and is centered at $\left(0, \tfrac{1}{2}\right)$. This circle is the osculating circle to the graph of $y = x^2$ at the origin. (Figure Ex-2).

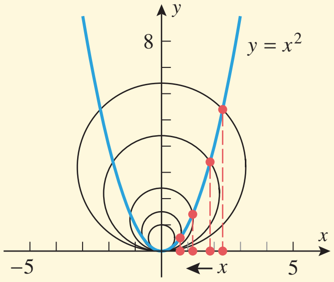


Figure Ex-2

3. Show that if we replace the curve $y = x^2$ by the curve $y = f(x)$, where f is an even function, then the formula for $C(x)$ becomes

$$C(x) = \frac{1}{2}\left[f(0) + f(x) + \frac{x^2}{f(x) - f(0)}\right]$$

[Here we assume that $f(x) \neq f(0)$ for positive values of x close to 0.] If $\lim_{x \to 0^+} C(x) = L \neq f(0)$, then we define the osculating circle to the curve $y = f(x)$ at $(0, f(0))$ to be the unique circle through $(0, f(0))$ with center $(0, L)$. If $C(x)$ does not have a finite limit different from $f(0)$ as $x \to 0^+$, then we say that the curve has no osculating circle at $(0, f(0))$.

4. In each part, determine the osculating circle to the curve $y = f(x)$ at $(0, f(0))$, if it exists.

(a) $f(x) = 4x^2$ (b) $f(x) = x^2 \cos x$

(c) $f(x) = |x|$ (d) $f(x) = x \sin x$

(e) $f(x) = \cos x$

(f) $f(x) = x^2 g(x)$, where $g(x)$ is an even continuous function with $g(0) \neq 0$

(g) $f(x) = x^4$

Chapter II

The Derivative

2

THE DERIVATIVE

Photo by Kirby Lee/WireImage/Getty Images

One of the crowning achievements of calculus is its ability to capture continuous motion mathematically, allowing that motion to be analyzed instant by instant.

Many real-world phenomena involve changing quantities—the speed of a rocket, the inflation of currency, the number of bacteria in a culture, the shock intensity of an earthquake, the voltage of an electrical signal, and so forth. In this chapter we will develop the concept of a "derivative," which is the mathematical tool for studying the rate at which one quantity changes relative to another. The study of rates of change is closely related to the geometric concept of a tangent line to a curve, so we will also be discussing the general definition of a tangent line and methods for finding its slope and equation. Later in the chapter, we will consider some applications of the derivative. These will include ways in which different rates of change can be related as well as the use of linear functions to approximate nonlinear functions.

2.1 TANGENT LINES AND RATES OF CHANGE

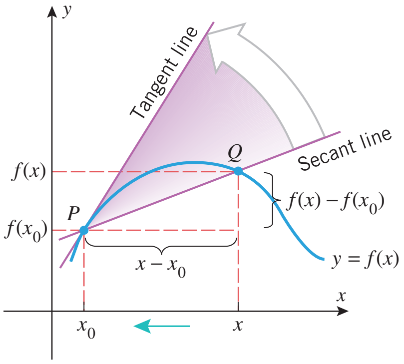
In this section we will discuss three ideas: tangent lines to curves, the velocity of an object moving along a line, and the rate at which one variable changes relative to another. Our goal is to show how these seemingly unrelated ideas are, in actuality, closely linked.

TANGENT LINES

In Example 1 of Section 1.1, we showed how the notion of a limit could be used to find an equation of a tangent line to a curve. At that stage in the text we did not have precise definitions of tangent lines and limits to work with, so the argument was intuitive and informal. However, now that limits have been defined precisely, we are in a position to give a mathematical definition of the tangent line to a curve $y = f(x)$ at a point $P(x_0, f(x_0))$ on the curve. As illustrated in Figure 2.1.1, consider a point $Q(x, f(x))$ on the curve that is distinct from P, and compute the slope m_{PQ} of the secant line through P and Q:

$$m_{PQ} = \frac{f(x) - f(x_0)}{x - x_0}$$

If we let x approach x_0, then the point Q will move along the curve and approach the point P. If the secant line through P and Q approaches a limiting position as $x \to x_0$, then we will regard that position to be the position of the tangent line at P. Stated another way, if the slope m_{PQ} of the secant line through P and Q approaches a limit as $x \to x_0$, then we regard that limit to be the slope $m_{\tan}$ of the tangent line at P. Thus, we make the following definition.



▶ **Figure 2.1.1**

2.1.1 DEFINITION Suppose that x_0 is in the domain of the function f. The ***tangent line*** to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the line with equation

$$y - f(x_0) = m_{\tan}(x - x_0)$$

where

$$m_{\tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

provided the limit exists. For simplicity, we will also call this the tangent line to $y = f(x)$ at x_0.

▶ **Example 1** Use Definition 2.1.1 to find an equation for the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$, and confirm the result agrees with that obtained in Example 1 of Section 1.1.

Solution. Applying Formula (1) with $f(x) = x^2$ and $x_0 = 1$, we have

$$\begin{aligned} m_{\tan} &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1}(x + 1) = 2 \end{aligned}$$

Thus, the tangent line to $y = x^2$ at $(1, 1)$ has equation

$$y - 1 = 2(x - 1) \quad \text{or equivalently} \quad y = 2x - 1$$

which agrees with Example 1 of Section 1.1. ◀

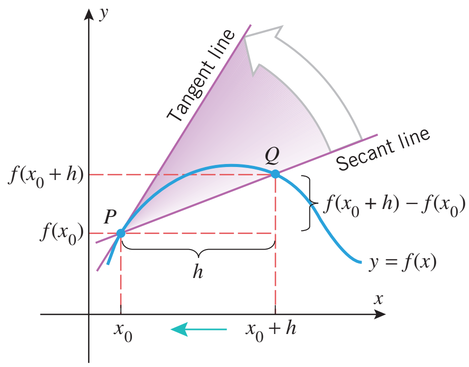
There is an alternative way of expressing Formula (1) that is commonly used. If we let h denote the difference

$$h = x - x_0$$

then the statement that $x \to x_0$ is equivalent to the statement $h \to 0$, so we can rewrite (1) in terms of x_0 and h as

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{2}$$

Figure 2.1.2 shows how Formula (2) expresses the slope of the tangent line as a limit of slopes of secant lines.



▶ **Figure 2.1.2**

▶ **Example 2** Compute the slope in Example 1 using Formula (2).

Formulas (1) and (2) for $m_{\tan}$ usually lead to indeterminate forms of type $0/0$, so you will generally need to perform algebraic simplifications or use other methods to determine limits of such indeterminate forms.

Solution. Applying Formula (2) with $f(x)=x^2$ and $x_0=1$, we obtain

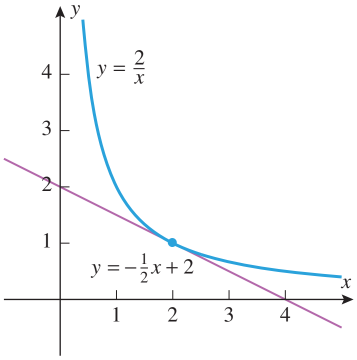
$$\begin{aligned} m_{\tan} &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0}\frac{(1+h)^2-1^2}{h} \\ &= \lim_{h\to 0}\frac{1+2h+h^2-1}{h} = \lim_{h\to 0}(2+h)=2 \end{aligned}$$

which agrees with the slope found in Example 1. ◀

▶ **Example 3** Find an equation for the tangent line to the curve $y=2/x$ at the point (2, 1) on this curve.

Solution. First, we will find the slope of the tangent line by applying Formula (2) with $f(x)=2/x$ and $x_0=2$. This yields

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0}\frac{\dfrac{2}{2+h}-1}{h} = \lim_{h\to 0}\frac{\left(\dfrac{2-(2+h)}{2+h}\right)}{h} \\ &= \lim_{h\to 0}\frac{-h}{h(2+h)} = -\left(\lim_{h\to 0}\frac{1}{2+h}\right) = -\frac{1}{2} \end{aligned}$$



▲ **Figure 2.1.3**

Thus, an equation of the tangent line at (2, 1) is

$$y-1=-\tfrac{1}{2}(x-2) \quad \text{or equivalently} \quad y=-\tfrac{1}{2}x+2$$

(see Figure 2.1.3). ◀

▶ **Example 4** Find the slopes of the tangent lines to the curve $y = \sqrt{x}$ at $x_0 = 1$, $x_0 = 4$, and $x_0 = 9$.

Solution. We could compute each of these slopes separately, but it will be more efficient to find the slope for a general value of x_0 and then substitute the specific numerical values. Proceeding in this way we obtain

$$
\begin{aligned}
m_{\tan} &= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x_0+h} - \sqrt{x_0}}{h} \cdot \frac{\sqrt{x_0+h} + \sqrt{x_0}}{\sqrt{x_0+h} + \sqrt{x_0}} \\
&= \lim_{h\to 0} \frac{x_0+h-x_0}{h(\sqrt{x_0+h} + \sqrt{x_0})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x_0+h} + \sqrt{x_0})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x_0+h} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}
\end{aligned}
$$

Rationalize the numerator to help eliminate the indeterminate form of the limit.

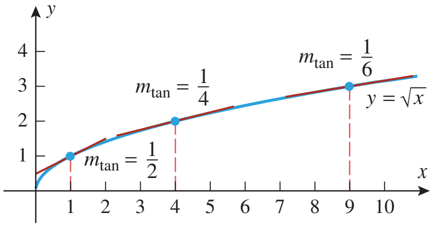
The slopes at $x_0 = 1, 4$, and 9 can now be obtained by substituting these values into our general formula for $m_{\tan}$. Thus,

$$\text{slope at } x_0 = 1\colon \ \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$\text{slope at } x_0 = 4\colon \ \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\text{slope at } x_0 = 9\colon \ \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

(see Figure 2.1.4). ◀



▶ **Figure 2.1.4**

VELOCITY

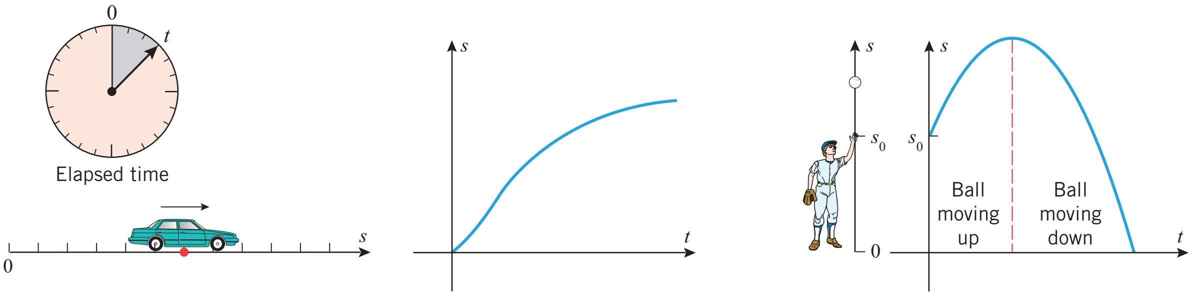
Carlos Santa Maria/iStockphoto

The velocity of an airplane describes its speed and direction.

One of the important themes in calculus is the study of motion. To describe the motion of an object completely, one must specify its ***speed*** (how fast it is going) and the direction in which it is moving. The speed and the direction of motion together comprise what is called the ***velocity*** of the object. For example, knowing that the speed of an aircraft is 500 mi/h tells us how fast it is going, but not which way it is moving. In contrast, knowing that the velocity of the aircraft is 500 mi/h *due south* pins down the speed and the direction of motion.

Later, we will study the motion of objects that move along curves in two- or three-dimensional space, but for now we will only consider motion along a line; this is called ***rectilinear motion***. Some examples are a piston moving up and down in a cylinder, a race car moving along a straight track, an object dropped from the top of a building and falling straight down, a ball thrown straight up and then falling down along the same line, and so forth.

For computational purposes, we will assume that a particle in rectilinear motion moves along a coordinate line, which we will call the s-axis. A graphical description of rectilinear motion along an s-axis can be obtained by making a plot of the s-coordinate of the particle versus the elapsed time t from starting time $t = 0$. This is called the ***position versus time curve*** for the particle. Figure 2.1.5 shows two typical position versus time curves. The first is for a car that starts at the origin and moves only in the positive direction of the s-axis. In this case s increases as t increases. The second is for a ball that is thrown straight up in the positive direction of an s-axis from some initial height s_0 and then falls straight down in the negative direction. In this case s increases as the ball moves up and decreases as it moves down.



Car moves only in the positive direction.

Position versus time curve

Position versus time curve

▲ **Figure 2.1.5**

If a particle in rectilinear motion moves along an s-axis so that its position coordinate function of the elapsed time t is

$$s = f(t) \tag{3}$$

then f is called the ***position function of the particle***; the graph of (3) is the position versus time curve. The ***average velocity*** of the particle over a time interval $[t_0, t_0 + h]$, $h > 0$, is defined to be

$$v_{\text{ave}} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0 + h) - f(t_0)}{h} \tag{4}$$

Show that (4) is also correct for a time interval $[t_0 + h, t_0]$, $h < 0$.

▶ Example 5 Suppose that $s = f(t) = 1 + 5t - 2t^2$ is the position function of a particle, where s is in meters and t is in seconds. Find the average velocities of the particle over the time intervals (a) $[0, 2]$ and (b) $[2, 3]$.

The change in position

$$f(t_0 + h) - f(t_0)$$

is also called the ***displacement*** of the particle over the time interval between t_0 and $t_0 + h$.

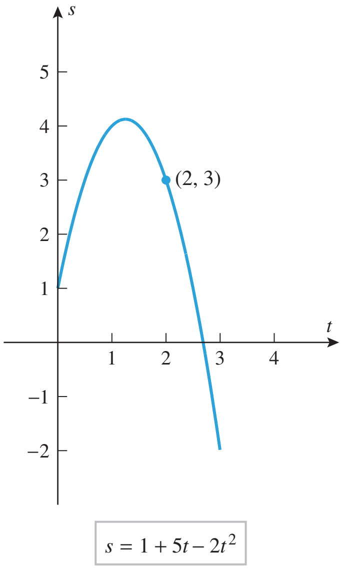
Solution (a). Applying (4) with $t_0 = 0$ and $h = 2$, we see that the average velocity is

$$v_{\text{ave}} = \frac{f(t_0 + h) - f(t_0)}{h} = \frac{f(2) - f(0)}{2} = \frac{3 - 1}{2} = \frac{2}{2} = 1 \text{ m/s}$$

Solution (b). Applying (4) with $t_0 = 2$ and $h = 1$, we see that the average velocity is

$$v_{\text{ave}} = \frac{f(t_0 + h) - f(t_0)}{h} = \frac{f(3) - f(2)}{1} = \frac{-2 - 3}{1} = \frac{-5}{1} = -5 \text{ m/s} \blacktriangleleft$$

For a particle in rectilinear motion, average velocity describes its behavior over an *interval* of time. We are interested in the particle's "instantaneous velocity," which describes its behavior at a specific *instant* in time. Formula (4) is not directly applicable for computing instantaneous velocity because the "time elapsed" at a specific instant is zero, so (4) is undefined. One way to circumvent this problem is to compute average velocities for small time intervals between $t = t_0$ and $t = t_0 + h$. These average velocities may be viewed as approximations to the "instantaneous velocity" of the particle at time t_0. If these average velocities have a limit as h approaches zero, then we can take that limit to be the *instantaneous velocity* of the particle at time t_0. Here is an example.



▲ **Figure 2.1.6**

Table 2.1.1

TIME INTERVAL	AVERAGE VELOCITY (m/s)
$2.0 \le t \le 3.0$	−5
$2.0 \le t \le 2.1$	−3.2
$2.0 \le t \le 2.01$	−3.02
$2.0 \le t \le 2.001$	−3.002
$2.0 \le t \le 2.0001$	−3.0002

Note the negative values for the velocities in Example 6. This is consistent with the fact that the object is moving in the negative direction along the s-axis.

▶ **Example 6** Consider the particle in Example 5, whose position function is

$$s = f(t) = 1 + 5t - 2t^2$$

The position of the particle at time $t = 2$ s is $s = 3$ m (Figure 2.1.6). Find the particle's instantaneous velocity at time $t = 2$ s.

Solution. As a first approximation to the particle's instantaneous velocity at time $t = 2$ s, let us recall from Example 5(b) that the average velocity over the time interval from $t = 2$ to $t = 3$ is $v_{\text{ave}} = -5$ m/s. To improve on this initial approximation we will compute the average velocity over a succession of smaller and smaller time intervals. We leave it to you to verify the results in Table 2.1.1. The average velocities in this table appear to be approaching a limit of -3 m/s, providing strong evidence that the instantaneous velocity at time $t = 2$ s is -3 m/s. To confirm this analytically, we start by computing the object's average velocity over a general time interval between $t = 2$ and $t = 2 + h$ using Formula (4):

$$v_{\text{ave}} = \frac{f(2+h) - f(2)}{h} = \frac{[1 + 5(2+h) - 2(2+h)^2] - 3}{h}$$

The object's instantaneous velocity at time $t = 2$ is calculated as a limit as $h \to 0$:

$$\begin{aligned}
\text{instantaneous velocity} &= \lim_{h\to 0} \frac{[1 + 5(2+h) - 2(2+h)^2] - 3}{h} \\
&= \lim_{h\to 0} \frac{-2 + (10 + 5h) - (8 + 8h + 2h^2)}{h} \\
&= \lim_{h\to 0} \frac{-3h - 2h^2}{h} = \lim_{h\to 0}(-3 - 2h) = -3
\end{aligned}$$

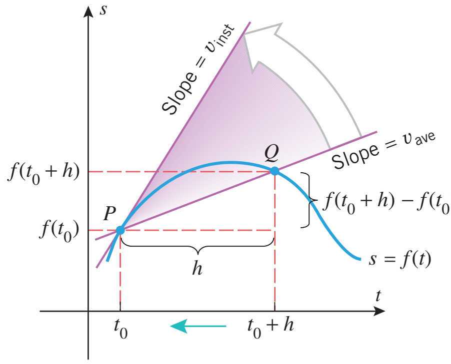
This confirms our numerical conjecture that the instantaneous velocity after 2 s is -3 m/s. ◀

Consider a particle in rectilinear motion with position function $s = f(t)$. Motivated by Example 6, we define the instantaneous velocity v_{inst} of the particle at time t_0 to be the limit as $h \to 0$ of its average velocities v_{ave} over time intervals between $t = t_0$ and $t = t_0 + h$. Thus, from (4) we obtain

$$v_{\text{inst}} = \lim_{h\to 0} \frac{f(t_0 + h) - f(t_0)}{h} \tag{5}$$

Confirm the solution to Example 5(b) by computing the slope of an appropriate secant line.

Geometrically, the average velocity v_{ave} between $t = t_0$ and $t = t_0 + h$ is the slope of the secant line through points $P(t_0, f(t_0))$ and $Q(t_0 + h, f(t_0 + h))$ on the position versus time curve, and the instantaneous velocity v_{inst} at time t_0 is the slope of the tangent line to the position versus time curve at the point $P(t_0, f(t_0))$ (Figure 2.1.7).

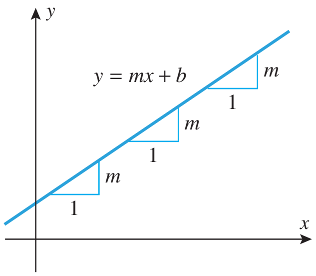


▶ **Figure 2.1.7**

SLOPES AND RATES OF CHANGE

Velocity can be viewed as *rate of change*—the rate of change of position with respect to time. Rates of change occur in other applications as well. For example:

- A microbiologist might be interested in the rate at which the number of bacteria in a colony changes with time.
- An engineer might be interested in the rate at which the length of a metal rod changes with temperature.
- An economist might be interested in the rate at which production cost changes with the quantity of a product that is manufactured.
- A medical researcher might be interested in the rate at which the radius of an artery changes with the concentration of alcohol in the bloodstream.



A 1-unit increase in x always produces an m-unit change in y.

▲ **Figure 2.1.8**

Our next objective is to define precisely what is meant by the "rate of change of y with respect to x" when y is a function of x. In the case where y is a linear function of x, say $y = mx + b$, the slope m is the natural measure of the rate of change of y with respect to x. As illustrated in Figure 2.1.8, each 1-unit increase in x anywhere along the line produces an m-unit change in y, so we see that y changes at a constant rate with respect to x along the line and that m measures this rate of change.

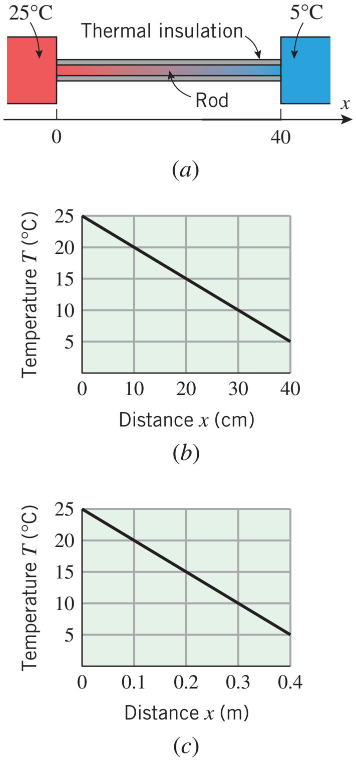
▶ **Example 7** Find the rate of change of y with respect to x if

$$\text{(a)}\ \ y = 2x - 1 \qquad \text{(b)}\ \ y = -5x + 1$$

Solution. In part (a) the rate of change of y with respect to x is $m = 2$, so each 1-unit increase in x produces a 2-unit increase in y. In part (b) the rate of change of y with respect to x is $m = -5$, so each 1-unit increase in x produces a 5-unit decrease in y. ◀

In applied problems, changing the units of measurement can change the slope of a line, so it is essential to include the units when calculating the slope and describing rates of change. The following example illustrates this.

▶ **Example 8** Suppose that a uniform rod of length 40 cm (= 0.4 m) is thermally insulated around the lateral surface and that the exposed ends of the rod are held at constant temperatures of 25°C and 5°C, respectively (Figure 2.1.9*a*). It is shown in physics that under appropriate conditions the graph of the temperature T versus the distance x from the left-hand end of the rod will be a straight line. Parts (*b*) and (*c*) of Figure 2.1.9 show two



(a)

(b)

(c)

Figure 2.1.9

such graphs: one in which x is measured in centimeters and one in which it is measured in meters. The slopes in the two cases are

$$m = \frac{5-25}{40-0} = \frac{-20}{40} = -0.5 \tag{6}$$

$$m = \frac{5-25}{0.4-0} = \frac{-20}{0.4} = -50 \tag{7}$$

The slope in (6) implies that the temperature *decreases* at a rate of 0.5°C per centimeter of distance from the left end of the rod, and the slope in (7) implies that the temperature decreases at a rate of 50°C per meter of distance from the left end of the rod. The two statements are equivalent physically, even though the slopes differ. ◀

Although the rate of change of y with respect to x is constant along a nonvertical line $y = mx + b$, this is not true for a general curve $y = f(x)$. For example, in Figure 2.1.10 the change in y that results from a 1-unit increase in x tends to have greater magnitude in regions where the curve rises or falls rapidly than in regions where it rises or falls slowly. As with velocity, we will distinguish between the average rate of change over an interval and the instantaneous rate of change at a specific point.

If $y = f(x)$, then we define the ***average rate of change of y with respect to x over the interval*** $[x_0, x_1]$ to be

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{8}$$

and we define the ***instantaneous rate of change of y with respect to x at*** x_0 to be

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \tag{9}$$

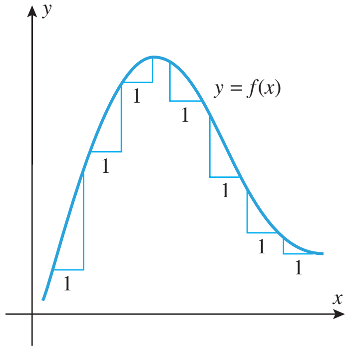


Figure 2.1.10

Geometrically, the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is the slope of the secant line through the points $P(x_0, f(x_0))$ and $Q(x_1, f(x_1))$ (Figure 2.1.11), and the instantaneous rate of change of y with respect to x at x_0 is the slope of the tangent line at the point $P(x_0, f(x_0))$ (since it is the limit of the slopes of the secant lines through P).

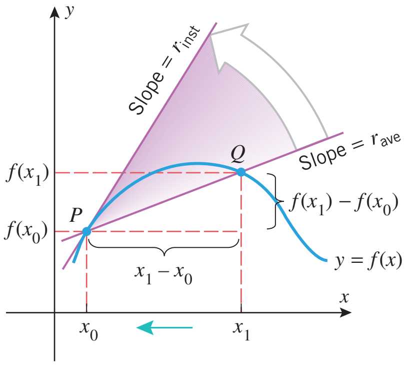


Figure 2.1.11

If desired, we can let $h = x_1 - x_0$, and rewrite (8) and (9) as

$$r_{\text{ave}} = \frac{f(x_0 + h) - f(x_0)}{h} \tag{10}$$

$$r_{\text{inst}} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{11}$$

▶ **Example 9** Let $y = x^2 + 1$.

(a) Find the average rate of change of y with respect to x over the interval [3, 5].

(b) Find the instantaneous rate of change of y with respect to x when $x = -4$.

Solution (a). We will apply Formula (8) with $f(x) = x^2 + 1$, $x_0 = 3$, and $x_1 = 5$. This yields

$$r_{\text{ave}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(5) - f(3)}{5 - 3} = \frac{26 - 10}{2} = 8$$

Thus, y increases an average of 8 units per unit increase in x over the interval [3, 5].

Solution (b). We will apply Formula (9) with $f(x) = x^2 + 1$ and $x_0 = -4$. This yields

$$r_{\text{inst}} = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to -4} \frac{f(x_1) - f(-4)}{x_1 - (-4)} = \lim_{x_1 \to -4} \frac{(x_1^2 + 1) - 17}{x_1 + 4}$$

$$= \lim_{x_1 \to -4} \frac{x_1^2 - 16}{x_1 + 4} = \lim_{x_1 \to -4} \frac{(x_1 + 4)(x_1 - 4)}{x_1 + 4} = \lim_{x_1 \to -4} (x_1 - 4) = -8$$

Thus, a small increase in x from $x = -4$ will produce approximately an 8-fold decrease in y. ◀

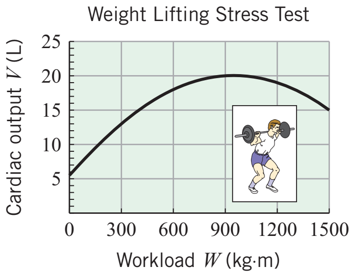
Perform the calculations in Example 9 using Formulas (10) and (11).

RATES OF CHANGE IN APPLICATIONS

In applied problems, average and instantaneous rates of change must be accompanied by appropriate units. In general, the units for a rate of change of y with respect to x are obtained by "dividing" the units of y by the units of x and then simplifying according to the standard rules of algebra. Here are some examples:

- If y is in degrees Fahrenheit (°F) and x is in inches (in), then a rate of change of y with respect to x has units of degrees Fahrenheit per inch (°F/in).
- If y is in feet per second (ft/s) and x is in seconds (s), then a rate of change of y with respect to x has units of feet per second per second (ft/s/s), which would usually be written as ft/s^2.
- If y is in newton-meters (N·m) and x is in meters (m), then a rate of change of y with respect to x has units of newtons (N), since N·m/m = N.
- If y is in foot-pounds (ft·lb) and x is in hours (h), then a rate of change of y with respect to x has units of foot-pounds per hour (ft·lb/h).

▶ **Example 10** The limiting factor in athletic endurance is cardiac output, that is, the volume of blood that the heart can pump per unit of time during an athletic competition. Figure 2.1.12 shows a stress-test graph of cardiac output V in liters (L) of blood versus workload W in kilogram-meters (kg·m) for 1 minute of weight lifting. This graph illustrates the known medical fact that cardiac output increases with the workload, but after reaching a peak value begins to decrease.



▲ **Figure 2.1.12**

(a) Use the secant line shown in Figure 2.1.13*a* to estimate the average rate of change of cardiac output with respect to workload as the workload increases from 300 to 1200 kg·m.

(b) Use the line segment shown in Figure 2.1.13*b* to estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 300 kg·m.

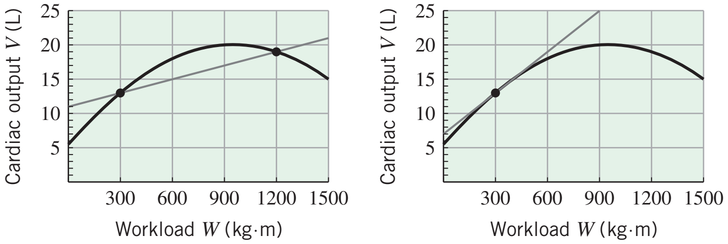
Solution (a). Using the estimated points (300, 13) and (1200, 19) to find the slope of the secant line, we obtain

$$r_{\text{ave}} \approx \frac{19-13}{1200-300} \approx 0.0067 \frac{\text{L}}{\text{kg}\cdot\text{m}}$$

This means that on average a 1-unit increase in workload produced a 0.0067 L increase in cardiac output over the interval.

Solution (b). We estimate the slope of the cardiac output curve at $W = 300$ by sketching a line that appears to meet the curve at $W = 300$ with slope equal to that of the curve (Figure 2.1.13*b*). Estimating points (0, 7) and (900, 25) on this line, we obtain

$$r_{\text{inst}} \approx \frac{25-7}{900-0} = 0.02 \frac{\text{L}}{\text{kg}\cdot\text{m}} \blacktriangleleft$$



(*a*) (*b*)

▶ **Figure 2.1.13**

✔QUICK CHECK EXERCISES 2.1 *(See page 122 for answers.)*

1. The slope $m_{\tan}$ of the tangent line to the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is given by

$$m_{\tan} = \lim_{x \to x_0} \underline{\hspace{2cm}} = \lim_{h \to 0} \underline{\hspace{2cm}}$$

2. The tangent line to the curve $y = (x-1)^2$ at the point $(-1, 4)$ has equation $4x + y = 0$. Thus, the value of the limit

$$\lim_{x \to -1} \frac{x^2 - 2x - 3}{x+1}$$

is ________.

3. A particle is moving along an s-axis, where s is in feet. During the first 5 seconds of motion, the position of the particle is given by

$$s = 10 - (3-t)^2, \quad 0 \le t \le 5$$

Use this position function to complete each part.

(a) Initially, the particle moves a distance of ________ ft in the (positive/negative) ________ direction; then it reverses direction, traveling a distance of ________ ft during the remainder of the 5-second period.

(b) The average velocity of the particle over the 5-second period is ________.

4. Let $s = f(t)$ be the equation of a position versus time curve for a particle in rectilinear motion, where s is in meters and t is in seconds. Assume that $s = -1$ when $t = 2$ and that the instantaneous velocity of the particle at this instant is 3 m/s. The equation of the tangent line to the position versus time curve at time $t = 2$ is ________.

5. Suppose that $y = x^2 + x$.

(a) The average rate of change of y with respect to x over the interval $2 \le x \le 5$ is ________.

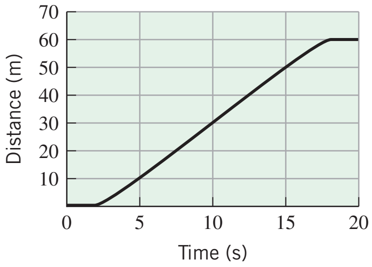
(b) The instantaneous rate of change of y with respect to x at $x = 2$, r_{inst}, is given by the limit ________.

EXERCISE SET 2.1

1. The accompanying figure on the next page shows the position versus time curve for an elevator that moves upward a distance of 60 m and then discharges its passengers.

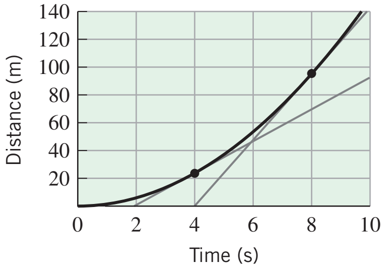
(a) Estimate the instantaneous velocity of the elevator at $t = 10$ s.

(b) Sketch a velocity versus time curve for the motion of the elevator for $0 \le t \le 20$.



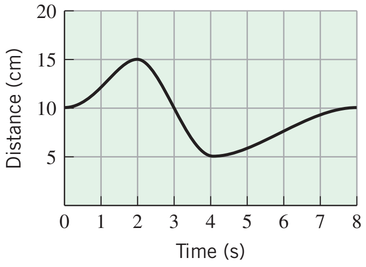
◀ **Figure Ex-1**

2. The accompanying figure shows the position versus time curve for an automobile over a period of time of 10 s. Use the line segments shown in the figure to estimate the instantaneous velocity of the automobile at time $t = 4$ s and again at time $t = 8$ s.



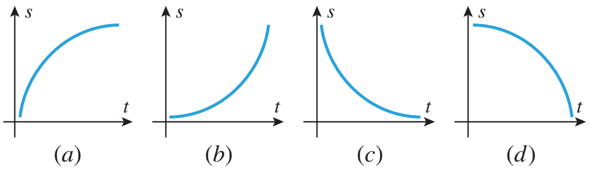
◀ **Figure Ex-2**

3. The accompanying figure shows the position versus time curve for a certain particle moving along a straight line. Estimate each of the following from the graph:
   (a) the average velocity over the interval $0 \le t \le 3$
   (b) the values of t at which the instantaneous velocity is zero
   (c) the values of t at which the instantaneous velocity is either a maximum or a minimum
   (d) the instantaneous velocity when $t = 3$ s.



◀ **Figure Ex-3**

4. The accompanying figure shows the position versus time curves of four different particles moving on a straight line. For each particle, determine whether its instantaneous velocity is increasing or decreasing with time.



(*a*) (*b*) (*c*) (*d*)

▲ **Figure Ex-4**

FOCUS ON CONCEPTS

5. If a particle moves at constant velocity, what can you say about its position versus time curve?

6. An automobile, initially at rest, begins to move along a straight track. The velocity increases steadily until suddenly the driver sees a concrete barrier in the road and applies the brakes sharply at time t_0. The car decelerates rapidly, but it is too late—the car crashes into the barrier at time t_1 and instantaneously comes to rest. Sketch a position versus time curve that might represent the motion of the car. Indicate how characteristics of your curve correspond to the events of this scenario.

7–10 For each exercise, sketch a curve and a line L satisfying the stated conditions. ■

7. L is tangent to the curve and intersects the curve in at least two points.

8. L intersects the curve in exactly one point, but L is not tangent to the curve.

9. L is tangent to the curve at two different points.

10. L is tangent to the curve at two different points and intersects the curve at a third point.

11–14 A function $y = f(x)$ and values of x_0 and x_1 are given.
(a) Find the average rate of change of y with respect to x over the interval $[x_0, x_1]$.
(b) Find the instantaneous rate of change of y with respect to x at the specified value of x_0.
(c) Find the instantaneous rate of change of y with respect to x at an arbitrary value of x_0.
(d) The average rate of change in part (a) is the slope of a certain secant line, and the instantaneous rate of change in part (b) is the slope of a certain tangent line. Sketch the graph of $y = f(x)$ together with those two lines. ■

11. $y = 2x^2;\ x_0 = 0,\ x_1 = 1$ **12.** $y = x^3;\ x_0 = 1,\ x_1 = 2$

13. $y = 1/x;\ x_0 = 2,\ x_1 = 3$ **14.** $y = 1/x^2;\ x_0 = 1,\ x_1 = 2$

15–18 A function $y = f(x)$ and an x-value x_0 are given.
(a) Find a formula for the slope of the tangent line to the graph of f at a general point $x = x_0$.
(b) Use the formula obtained in part (a) to find the slope of the tangent line for the given value of x_0. ■

15. $f(x) = x^2 - 1;\ x_0 = -1$

16. $f(x) = x^2 + 3x + 2;\ x_0 = 2$

17. $f(x) = x + \sqrt{x};\ x_0 = 1$

18. $f(x) = 1/\sqrt{x};\ x_0 = 4$

19–22 True–False Determine whether the statement is true or false. Explain your answer. ■

19. If $\lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = 3$, then $\lim\limits_{h \to 0} \dfrac{f(1 + h) - f(1)}{h} = 3$.

20. A tangent line to a curve $y = f(x)$ is a particular kind of secant line to the curve.

21. The velocity of an object represents a change in the object's position.

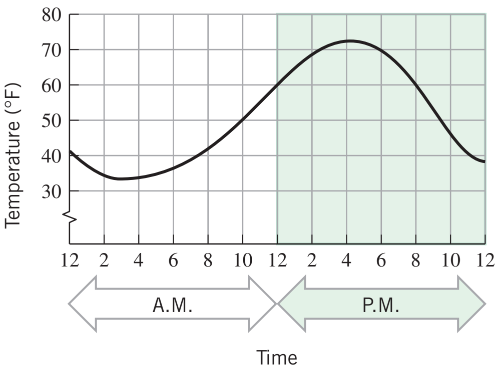
22. A 50-foot horizontal metal beam is supported on either end by concrete pillars and a weight is placed on the middle of the beam. If $f(x)$ models how many inches the center of the beam sags when the weight measures x tons, then the units of the rate of change of $y = f(x)$ with respect to x are inches/ton.

23. Suppose that the outside temperature versus time curve over a 24-hour period is as shown in the accompanying figure.

(a) Estimate the maximum temperature and the time at which it occurs.

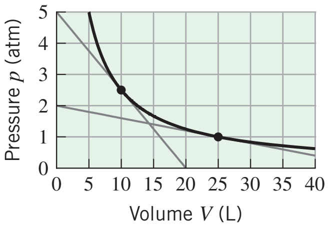
(b) The temperature rise is fairly linear from 8 A.M. to 2 P.M. Estimate the rate at which the temperature is increasing during this time period.

(c) Estimate the time at which the temperature is decreasing most rapidly. Estimate the instantaneous rate of change of temperature with respect to time at this instant.



▲ **Figure Ex-23**

24. The accompanying figure shows the graph of the pressure p in atmospheres (atm) versus the volume V in liters (L) of 1 mole of an ideal gas at a constant temperature of 300 K (kelvins). Use the line segments shown in the figure to estimate the rate of change of pressure with respect to volume at the points where $V = 10$ L and $V = 25$ L.



◀ **Figure Ex-24**

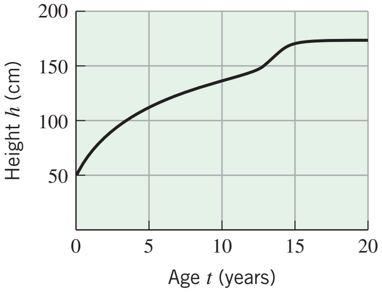
25. The accompanying figure shows the graph of the height h in centimeters versus the age t in years of an individual from birth to age 20.

(a) When is the growth rate greatest?

(b) Estimate the growth rate at age 5.

(c) At approximately what age between 10 and 20 is the growth rate greatest? Estimate the growth rate at this age.

(d) Draw a rough graph of the growth rate versus age.



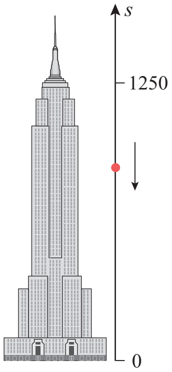
◀ **Figure Ex-25**

26. An object is released from rest (its initial velocity is zero) from the Empire State Building at a height of 1250 ft above street level (Figure Ex-26). The height of the object can be modeled by the position function $s = f(t) = 1250 - 16t^2$.

(a) Verify that the object is still falling at $t = 5$ s.

(b) Find the average velocity of the object over the time interval from $t = 5$ to $t = 6$ s.

(c) Find the object's instantaneous velocity at time $t = 5$ s.



◀ **Figure Ex-26**

27. During the first 40 s of a rocket flight, the rocket is propelled straight up so that in t seconds it reaches a height of $s = 0.3t^3$ ft.

(a) How high does the rocket travel in 40 s?

(b) What is the average velocity of the rocket during the first 40 s?

(c) What is the average velocity of the rocket during the first 1000 ft of its flight?

(d) What is the instantaneous velocity of the rocket at the end of 40 s?

28. An automobile is driven down a straight highway such that after $0 \le t \le 12$ seconds it is $s = 4.5t^2$ feet from its initial position. (cont.)

(a) Find the average velocity of the car over the interval [0, 12].
(b) Find the instantaneous velocity of the car at $t = 6$.

29. A robot moves in the positive direction along a straight line so that after t minutes its distance is $s = 6t^4$ feet from the origin.
(a) Find the average velocity of the robot over the interval [2, 4].
(b) Find the instantaneous velocity at $t = 2$.

30. Writing Discuss how the tangent line to the graph of a function $y = f(x)$ at a point $P(x_0, f(x_0))$ is defined in terms of secant lines to the graph through point P.

31. Writing A particle is in rectilinear motion during the time interval $0 \le t \le 2$. Explain the connection between the instantaneous velocity of the particle at time $t = 1$ and the average velocities of the particle during portions of the interval $0 \le t \le 2$.

✓QUICK CHECK ANSWERS 2.1

1. $\dfrac{f(x) - f(x_0)}{x - x_0}$; $\dfrac{f(x_0 + h) - f(x_0)}{h}$ **2.** -4 **3.** (a) 9; positive; 4 (b) 1 ft/s **4.** $s = 3t - 7$

5. (a) 8 (b) $\lim\limits_{x \to 2} \dfrac{(x^2 + x) - 6}{x - 2}$ or $\lim\limits_{h \to 0} \dfrac{[(2 + h)^2 + (2 + h)] - 6}{h}$

2.2 THE DERIVATIVE FUNCTION

In this section we will discuss the concept of a "derivative," which is the primary mathematical tool that is used to calculate and study rates of change.

DEFINITION OF THE DERIVATIVE FUNCTION

In the last section we showed that if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists, then it can be interpreted either as the slope of the tangent line to the curve $y = f(x)$ at $x = x_0$ or as the instantaneous rate of change of y with respect to x at $x = x_0$ [see Formulas (2) and (11) of that section]. This limit is so important that it has a special notation:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

You can think of f' (read "f prime") as a function whose input is x_0 and whose output is the number $f'(x_0)$ that represents either the slope of the tangent line to $y = f(x)$ at $x = x_0$ or the instantaneous rate of change of y with respect to x at $x = x_0$. To emphasize this function point of view, we will replace x_0 by x in (1) and make the following definition.

2.2.1 DEFINITION The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{2}$$

is called the ***derivative of f with respect to x***. The domain of f' consists of all x in the domain of f for which the limit exists.

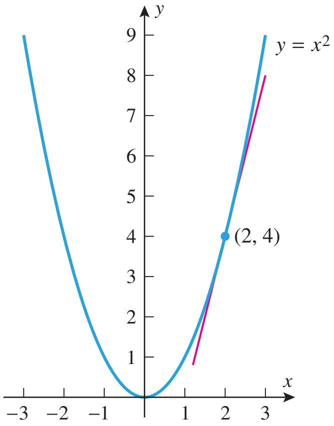
The expression

$$\frac{f(x + h) - f(x)}{h}$$

that appears in (2) is commonly called the ***difference quotient***.

The term "derivative" is used because the function f' is *derived* from the function f by a limiting process.

▶ **Example 1** Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to $y = x^2$ at $x = 2$.



▲ **Figure 2.2.1**

Solution. It follows from (2) that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

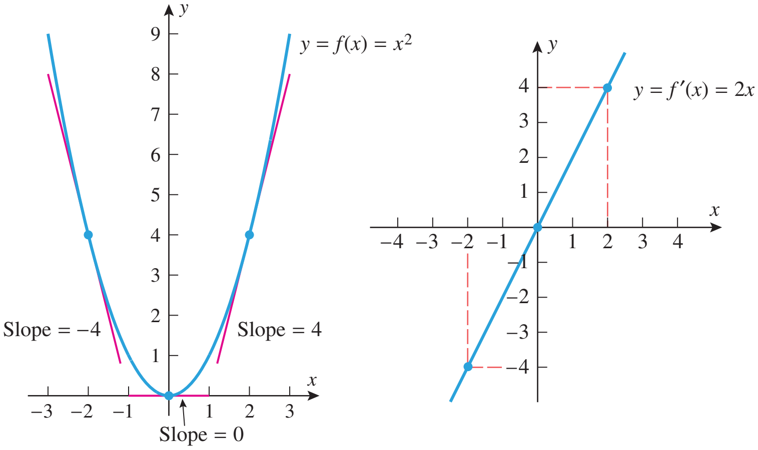
$$= \lim_{h\to 0} (2x + h) = 2x$$

Thus, the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 4$. Since $y = 4$ if $x = 2$, the point-slope form of the tangent line is

$$y - 4 = 4(x - 2)$$

which we can rewrite in slope-intercept form as $y = 4x - 4$ (Figure 2.2.1). ◀

You can think of f' as a "slope-producing function" in the sense that the value of $f'(x)$ at $x = x_0$ is the slope of the tangent line to the graph of f at $x = x_0$. This aspect of the derivative is illustrated in Figure 2.2.2, which shows the graphs of $f(x) = x^2$ and its derivative $f'(x) = 2x$ (obtained in Example 1). The figure illustrates that the values of $f'(x) = 2x$ at $x = -2$, 0, and 2 correspond to the slopes of the tangent lines to the graph of $f(x) = x^2$ at those values of x.



▶ **Figure 2.2.2**

In general, if $f'(x)$ is defined at $x = x_0$, then the point-slope form of the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$ may be found using the following steps.

Finding an Equation for the Tangent Line to $y = f(x)$ at $x = x_0$.

Step 1. Evaluate $f(x_0)$; the point of tangency is $(x_0, f(x_0))$.

Step 2. Find $f'(x)$ and evaluate $f'(x_0)$, which is the slope m of the line.

Step 3. Substitute the value of the slope m and the point $(x_0, f(x_0))$ into the point-slope form of the line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

or, equivalently,

$$y = f(x_0) + f'(x_0)(x - x_0) \tag{3}$$

▶ Example 2

(a) Find the derivative with respect to x of $f(x) = x^3 - x$.

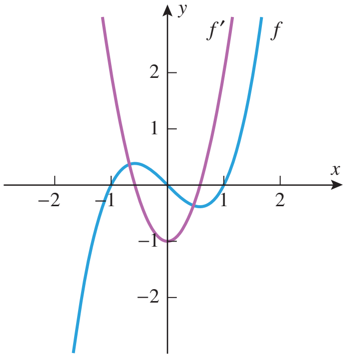
(b) Graph f and f' together, and discuss the relationship between the two graphs.

In Solution (a), the binomial formula is used to expand $(x+h)^3$. This formula may be found on the front endpaper.

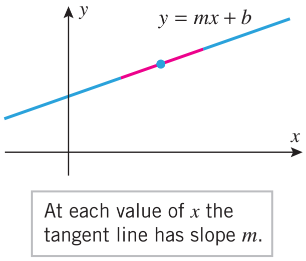
Solution (a).

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} \\
&= \lim_{h\to 0} [3x^2 + 3xh + h^2 - 1] = 3x^2 - 1
\end{aligned}
$$

Solution (b). Since $f'(x)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that $f'(x)$ is positive where the tangent line has positive slope, is negative where the tangent line has negative slope, and is zero where the tangent line is horizontal. We leave it for you to verify that this is consistent with the graphs of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$ shown in Figure 2.2.3. ◀



▲ **Figure 2.2.3**



At each value of x the tangent line has slope m.

▲ **Figure 2.2.4**

▶ Example 3 At each value of x, the tangent line to a line $y = mx + b$ coincides with the line itself (Figure 2.2.4), and hence all tangent lines have slope m. This suggests geometrically that if $f(x) = mx + b$, then $f'(x) = m$ for all x. This is confirmed by the following computations:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[m(x+h) + b] - [mx + b]}{h} \\
&= \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m \quad \blacktriangleleft
\end{aligned}
$$

The result in Example 3 is consistent with our earlier observation that the rate of change of y with respect to x along a line $y = mx + b$ is constant and that constant is m.

▶ Example 4

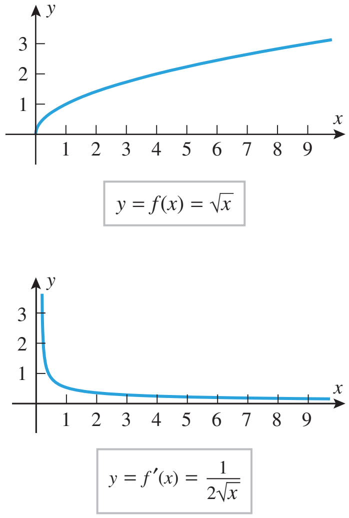
(a) Find the derivative with respect to x of $f(x) = \sqrt{x}$.

(b) Find the slope of the tangent line to $y = \sqrt{x}$ at $x = 9$.

(c) Find the limits of $f'(x)$ as $x \to 0^+$ and as $x \to +\infty$, and explain what those limits say about the graph of f.

Solution (a). Recall from Example 4 of Section 2.1 that the slope of the tangent line to $y = \sqrt{x}$ at $x = x_0$ is given by $m_{\tan} = 1/(2\sqrt{x_0})$. Thus, $f'(x) = 1/(2\sqrt{x})$.

Solution (b). The slope of the tangent line at $x = 9$ is $f'(9)$. From part (a), this slope is $f'(9) = 1/(2\sqrt{9}) = \frac{1}{6}$.



▲ **Figure 2.2.5**

Solution (c). The graphs of $f(x) = \sqrt{x}$ and $f'(x) = 1/(2\sqrt{x}\,)$ are shown in Figure 2.2.5. Observe that $f'(x) > 0$ if $x > 0$, which means that all tangent lines to the graph of $y = \sqrt{x}$ have positive slope at all points in this interval. Since

$$\lim_{x\to 0^+} \frac{1}{2\sqrt{x}} = +\infty \quad \text{and} \quad \lim_{x\to +\infty} \frac{1}{2\sqrt{x}} = 0$$

the graph of f becomes more and more vertical as $x \to 0^+$ and more and more horizontal as $x \to +\infty$. ◄

COMPUTING INSTANTANEOUS VELOCITY

It follows from Formula (5) of Section 2.1 (with t replacing t_0) that if $s = f(t)$ is the position function of a particle in rectilinear motion, then the instantaneous velocity at an arbitrary time t is given by

$$v_{\text{inst}} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}$$

Since the right side of this equation is the derivative of the function f (with t rather than x as the independent variable), it follows that if $f(t)$ is the position function of a particle in rectilinear motion, then the function

$$v(t) = f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \tag{4}$$

represents the instantaneous velocity of the particle at time t. Accordingly, we call (4) the ***instantaneous velocity function*** or, more simply, the ***velocity function*** of the particle.

► **Example 5** Recall the particle from Example 5 of Section 2.1 with position function $s = f(t) = 1 + 5t - 2t^2$. Here $f(t)$ is measured in meters and t is measured in seconds. Find the velocity function of the particle.

Solution. It follows from (4) that the velocity function is

$$\begin{aligned} v(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{[1 + 5(t+h) - 2(t+h)^2] - [1 + 5t - 2t^2]}{h} \\ &= \lim_{h\to 0} \frac{-2[t^2 + 2th + h^2 - t^2] + 5h}{h} = \lim_{h\to 0} \frac{-4th - 2h^2 + 5h}{h} \\ &= \lim_{h\to 0} (-4t - 2h + 5) = 5 - 4t \end{aligned}$$

where the units of velocity are meters per second. ◄

DIFFERENTIABILITY

It is possible that the limit that defines the derivative of a function f may not exist at certain points in the domain of f. At such points the derivative is undefined. To account for this possibility we make the following definition.

2.2.2 DEFINITION A function f is said to be ***differentiable at*** $\boldsymbol{x_0}$ if the limit

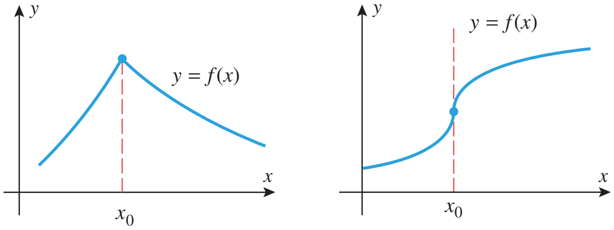
$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \tag{5}$$

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on (a, b)***, and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case we say that f is ***differentiable everywhere***.

Geometrically, a function f is differentiable at x_0 if the graph of f has a tangent line at x_0. Thus, f is not differentiable at any point x_0 where the secant lines from $P(x_0, f(x_0))$ to points $Q(x, f(x))$ distinct from P do not approach a unique *nonvertical* limiting position as $x \to x_0$. Figure 2.2.6 illustrates two common ways in which a function that is continuous at x_0 can fail to be differentiable at x_0. These can be described informally as

- corner points
- points of vertical tangency

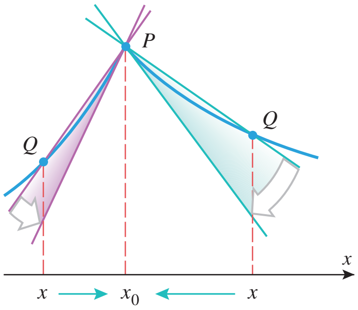
At a corner point, the slopes of the secant lines have different limits from the left and from the right, and hence the *two-sided* limit that defines the derivative does not exist (Figure 2.2.7). At a point of vertical tangency the slopes of the secant lines approach $+\infty$ or $-\infty$ from the left and from the right (Figure 2.2.8), so again the limit that defines the derivative does not exist.



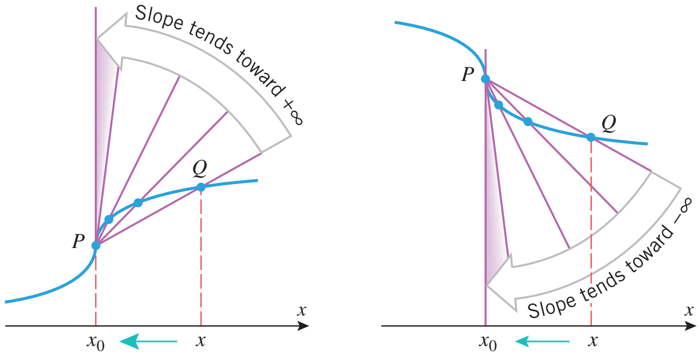
Corner point

Point of vertical tangency

▶ **Figure 2.2.6**



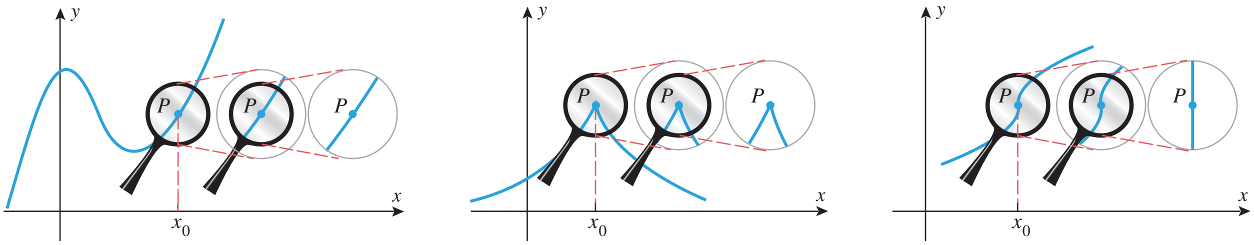
▲ **Figure 2.2.7**



▲ **Figure 2.2.8**

There are other less obvious circumstances under which a function may fail to be differentiable. (See Exercise 49, for example.)

Differentiability at x_0 can also be described informally in terms of the behavior of the graph of f under increasingly stronger magnification at the point $P(x_0, f(x_0))$ (Figure 2.2.9). If f is differentiable at x_0, then under sufficiently strong magnification at P the



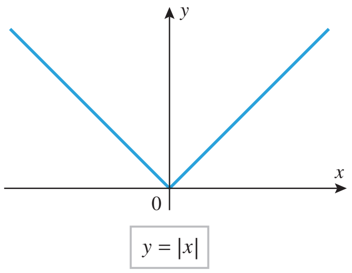
Differentiable at x_0

Not differentiable at x_0

Not differentiable at x_0

▲ **Figure 2.2.9**

graph looks like a nonvertical line (the tangent line); if a corner point occurs at x_0, then no matter how great the magnification at P the corner persists and the graph never looks like a nonvertical line; and if vertical tangency occurs at x_0, then the graph of f looks like a vertical line under sufficiently strong magnification at P.



▲ **Figure 2.2.10**

▶ **Example 6** The graph of $y = |x|$ in Figure 2.2.10 has a corner at $x = 0$, which implies that $f(x) = |x|$ is not differentiable at $x = 0$.

(a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$ by showing that the limit in Definition 2.2.2 does not exist at $x = 0$.

(b) Find a formula for $f'(x)$.

Solution (a). From Formula (5) with $x_0 = 0$, the value of $f'(0)$, if it were to exist, would be given by

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \tag{6}$$

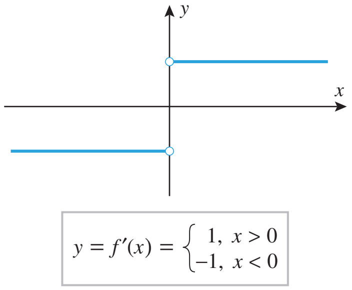
But

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0 \\ -1, & h < 0 \end{cases}$$

so that

$$\lim_{h \to 0^-} \frac{|h|}{h} = -1 \quad \text{and} \quad \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

Since these one-sided limits are not equal, the two-sided limit in (5) does not exist, and hence f is not differentiable at $x = 0$.



▲ **Figure 2.2.11**

Solution (b). A formula for the derivative of $f(x) = |x|$ can be obtained by writing $|x|$ in piecewise form and treating the cases $x > 0$ and $x < 0$ separately. If $x > 0$, then $f(x) = x$ and $f'(x) = 1$; if $x < 0$, then $f(x) = -x$ and $f'(x) = -1$. Thus,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

The graph of f' is shown in Figure 2.2.11. Observe that f' is not continuous at $x = 0$, so this example shows that a function that is continuous everywhere may have a derivative that fails to be continuous everywhere. ◀

THE RELATIONSHIP BETWEEN DIFFERENTIABILITY AND CONTINUITY

We already know that functions are not differentiable at corner points and points of vertical tangency. The next theorem shows that functions are not differentiable at points of discontinuity. We will do this by proving that if f is differentiable at a point, then it must be continuous at that point.

A theorem that says "If statement A is true, then statement B is true" is equivalent to the theorem that says "If statement B is not true, then statement A is not true." The two theorems are called ***contrapositive forms*** of one another. Thus, Theorem 2.2.3 can be rewritten in contrapositive form as "If a function f is not continuous at x_0, then f is not differentiable at x_0."

2.2.3 THEOREM *If a function f is differentiable at x_0, then f is continuous at x_0.*

PROOF We are given that f is differentiable at x_0, so it follows from (5) that $f'(x_0)$ exists and is given by

$$f'(x_0) = \lim_{h \to 0} \left[\frac{f(x_0+h) - f(x_0)}{h} \right] \tag{7}$$

To show that f is continuous at x_0, we must show that $\lim_{x\to x_0} f(x) = f(x_0)$ or, equivalently,

$$\lim_{x\to x_0} [f(x) - f(x_0)] = 0$$

Expressing this in terms of the variable $h = x - x_0$, we must prove that

$$\lim_{h\to 0} [f(x_0 + h) - f(x_0)] = 0$$

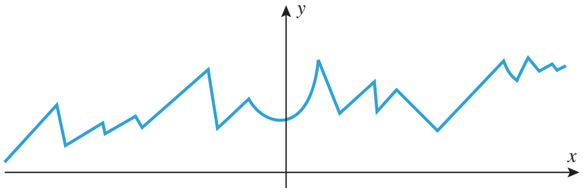
However, this can be proved using (7) as follows:

$$\begin{aligned}\lim_{h\to 0} [f(x_0 + h) - f(x_0)] &= \lim_{h\to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h} \cdot h\right] \\ &= \lim_{h\to 0} \left[\frac{f(x_0 + h) - f(x_0)}{h}\right] \cdot \lim_{h\to 0} h \\ &= f'(x_0) \cdot 0 = 0 \quad \blacksquare\end{aligned}$$

WARNING

The converse of Theorem 2.2.3 is false; that is, *a function may be continuous at a point but not differentiable at that point.* This occurs, for example, at corner points of continuous functions. For instance, $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there (Example 6).

The relationship between continuity and differentiability was of great historical significance in the development of calculus. In the early nineteenth century mathematicians believed that if a continuous function had many points of nondifferentiability, these points, like the tips of a sawblade, would have to be separated from one another and joined by smooth curve segments (Figure 2.2.12). This misconception was corrected by a series of discoveries beginning in 1834. In that year a Bohemian priest, philosopher, and mathematician named Bernhard Bolzano discovered a procedure for constructing a continuous function that is not differentiable at any point. Later, in 1860, the great German mathematician Karl Weierstrass (biography on p. 82) produced the first formula for such a function. The graphs of such functions are impossible to draw; it is as if the corners are so numerous that any segment of the curve, when suitably enlarged, reveals more corners. The discovery of these functions was important in that it made mathematicians distrustful of their geometric intuition and more reliant on precise mathematical proof. Recently, such functions have started to play a fundamental role in the study of geometric objects called ***fractals***. Fractals have revealed an order to natural phenomena that were previously dismissed as random and chaotic.



▶ **Figure 2.2.12**

Bernhard Bolzano (1781–1848) Bolzano, the son of an art dealer, was born in Prague, Bohemia (Czech Republic). He was educated at the University of Prague, and eventually won enough mathematical fame to be recommended for a mathematics chair there. However, Bolzano became an ordained Roman Catholic priest, and in 1805 he was appointed to a chair of Philosophy at the University of Prague. Bolzano was a man of great human compassion; he spoke out for educational reform, he voiced the right of individual conscience over government demands, and he lectured on the absurdity of war and militarism. His views so disenchanted Emperor Franz I of Austria that the emperor pressed the Archbishop of Prague to have Bolzano recant his statements. Bolzano refused and was then forced to retire in 1824 on a small pension. Bolzano's main contribution to mathematics was philosophical. His work helped convince mathematicians that sound mathematics must ultimately rest on rigorous proof rather than intuition. In addition to his work in mathematics, Bolzano investigated problems concerning space, force, and wave propagation.

[*Image: http://en.wikipedia.org/wiki/File:Bernard_Bolzano.jpg*]

DERIVATIVES AT THE ENDPOINTS OF AN INTERVAL

If a function f is defined on a closed interval $[a, b]$ but not outside that interval, then f' is not defined at the endpoints of the interval because derivatives are two-sided limits. To deal with this we define ***left-hand derivatives*** and ***right-hand derivatives*** by

$$f'_-(x) = \lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad f'_+(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$

respectively. These are called ***one-sided derivatives***. Geometrically, $f'_-(x)$ is the limit of the slopes of the secant lines as x is approached from the left and $f'_+(x)$ is the limit of the slopes of the secant lines as x is approached from the right. For a closed interval $[a, b]$, we will understand the derivative at the left endpoint to be $f'_+(a)$ and at the right endpoint to be $f'_-(b)$ (Figure 2.2.13).

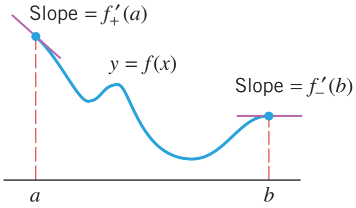


Figure 2.2.13

In general, we will say that f is ***differentiable*** on an interval of the form $[a, b]$, $[a, +\infty)$, $(-\infty, b]$, $[a, b)$, or $(a, b]$ if it is differentiable at all points inside the interval and the appropriate one-sided derivative exists at each included endpoint.

It can be proved that a function f is continuous from the left at those points where the left-hand derivative exists and is continuous from the right at those points where the right-hand derivative exists.

OTHER DERIVATIVE NOTATIONS

The process of finding a derivative is called ***differentiation***. You can think of differentiation as an *operation* on functions that associates a function f' with a function f. When the independent variable is x, the differentiation operation is also commonly denoted by

$$f'(x) = \frac{d}{dx}[f(x)] \quad \text{or} \quad f'(x) = D_x[f(x)]$$

In the case where there is a dependent variable $y = f(x)$, the derivative is also commonly denoted by

$$f'(x) = y'(x) \quad \text{or} \quad f'(x) = \frac{dy}{dx}$$

Later, the symbols dy and dx will be given specific meanings. However, for the time being do not regard dy/dx as a ratio, but rather as a single symbol denoting the derivative.

With the above notations, the value of the derivative at a point x_0 can be expressed as

$$f'(x_0) = \frac{d}{dx}[f(x)]\bigg|_{x=x_0}, \quad f'(x_0) = D_x[f(x)]\big|_{x=x_0}, \quad f'(x_0) = y'(x_0), \quad f'(x_0) = \frac{dy}{dx}\bigg|_{x=x_0}$$

If a variable w changes from some initial value w_0 to some final value w_1, then the final value minus the initial value is called an ***increment*** in w and is denoted by

$$\Delta w = w_1 - w_0 \tag{8}$$

Increments can be positive or negative, depending on whether the final value is larger or smaller than the initial value. The increment symbol in (8) should not be interpreted as a product; rather, Δw should be regarded as a single symbol representing the change in the value of w.

It is common to regard the variable h in the derivative formula

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{9}$$

as an increment Δx in x and write (9) as

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{10}$$

Moreover, if $y = f(x)$, then the numerator in (10) can be regarded as the increment

$$\Delta y = f(x + \Delta x) - f(x) \tag{11}$$

in which case

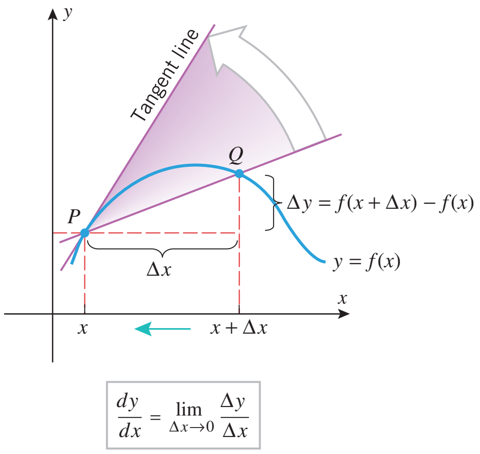
$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}} \tag{12}$$

The geometric interpretations of Δx and Δy are shown in Figure 2.2.14.

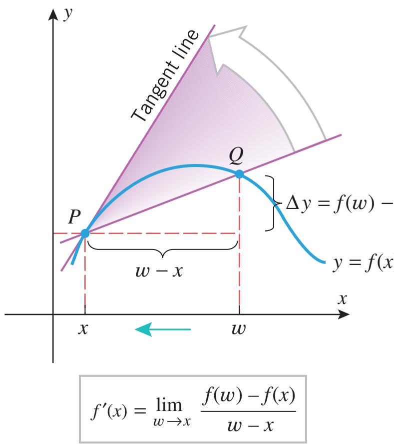
Sometimes it is desirable to express derivatives in a form that does not use increments at all. For example, if we let $w = x + h$ in Formula (9), then $w \to x$ as $h \to 0$, so we can rewrite that formula as

$$\boxed{f'(x) = \lim_{w \to x} \frac{f(w) - f(x)}{w - x}} \tag{13}$$

(Compare Figures 2.2.14 and 2.2.15.)



▲ **Figure 2.2.14**



▲ **Figure 2.2.15**

When letters other than x and y are used for the independent and dependent variables, the derivative notations must be adjusted accordingly. Thus, for example, if $s = f(t)$ is the position function for a particle in rectilinear motion, then the velocity function $v(t)$ in (4) can be expressed as

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{14}$$

✓ QUICK CHECK EXERCISES 2.2 *(See page 134 for answers.)*

1. The function $f'(x)$ is defined by the formula

$$f'(x) = \lim_{h \to 0} \underline{\hspace{2cm}}$$

2. (a) The derivative of $f(x) = x^2$ is $f'(x) =$ ________.
(b) The derivative of $f(x) = \sqrt{x}$ is $f'(x) =$ ________.

3. Suppose that the line $2x + 3y = 5$ is tangent to the graph of $y = f(x)$ at $x = 1$. The value of $f(1)$ is ________ and the value of $f'(1)$ is ________.

4. Which theorem guarantees us that if

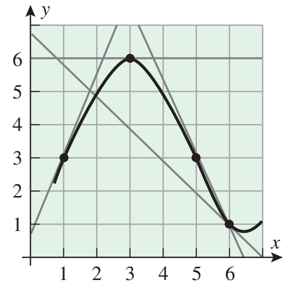
$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists, then $\lim_{x \to x_0} f(x) = f(x_0)$?

EXERCISE SET 2.2

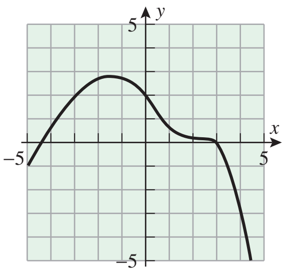
Graphing Utility

1. Use the graph of $y = f(x)$ in the accompanying figure to estimate the value of $f'(1)$, $f'(3)$, $f'(5)$, and $f'(6)$.



◀ **Figure Ex-1**

2. For the function graphed in the accompanying figure, arrange the numbers 0, $f'(-3)$, $f'(0)$, $f'(2)$, and $f'(4)$ in increasing order.



◀ **Figure Ex-2**

FOCUS ON CONCEPTS

3. (a) If you are given an equation for the tangent line at the point $(a, f(a))$ on a curve $y = f(x)$, how would you go about finding $f'(a)$?

(b) Given that the tangent line to the graph of $y = f(x)$ at the point $(2, 5)$ has the equation $y = 3x - 1$, find $f'(2)$.

(c) For the function $y = f(x)$ in part (b), what is the instantaneous rate of change of y with respect to x at $x = 2$?

4. Given that the tangent line to $y = f(x)$ at the point $(1, 2)$ passes through the point $(-1, -1)$, find $f'(1)$.

5. Sketch the graph of a function f for which $f(0) = -1$, $f'(0) = 0$, $f'(x) < 0$ if $x < 0$, and $f'(x) > 0$ if $x > 0$.

6. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 0$, and $f'(x) > 0$ if $x < 0$ or $x > 0$.

7. Given that $f(3) = -1$ and $f'(3) = 5$, find an equation for the tangent line to the graph of $y = f(x)$ at $x = 3$.

8. Given that $f(-2) = 3$ and $f'(-2) = -4$, find an equation for the tangent line to the graph of $y = f(x)$ at $x = -2$.

9–14 Use Definition 2.2.1 to find $f'(x)$, and then find the tangent line to the graph of $y = f(x)$ at $x = a$.

9. $f(x) = 2x^2;\ a = 1$

10. $f(x) = 1/x^2;\ a = -1$

11. $f(x) = x^3;\ a = 0$

12. $f(x) = 2x^3 + 1;\ a = -1$

13. $f(x) = \sqrt{x+1};\ a = 8$

14. $f(x) = \sqrt{2x+1};\ a = 4$

15–20 Use Formula (12) to find dy/dx.

15. $y = \dfrac{1}{x}$

16. $y = \dfrac{1}{x+1}$

17. $y = x^2 - x$

18. $y = x^4$

19. $y = \dfrac{1}{\sqrt{x}}$

20. $y = \dfrac{1}{\sqrt{x-1}}$

21–22 Use Definition 2.2.1 (with appropriate change in notation) to obtain the derivative requested.

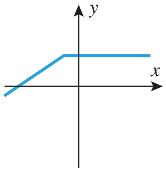
21. Find $f'(t)$ if $f(t) = 4t^2 + t$.

22. Find dV/dr if $V = \frac{4}{3}\pi r^3$.

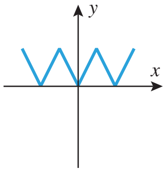
FOCUS ON CONCEPTS

23. Match the graphs of the functions shown in (a)–(f) with the graphs of their derivatives in (A)–(F).

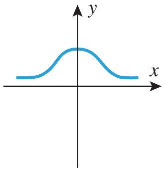
(a)



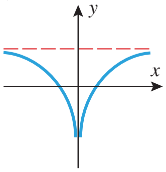
(b)



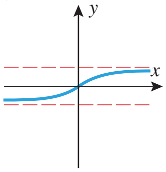
(c)



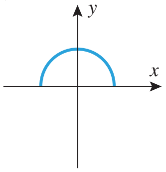
(d)



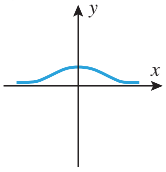
(e)



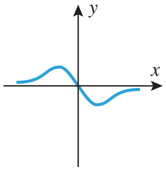
(f)



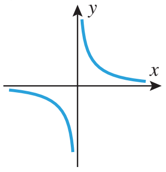
(A)



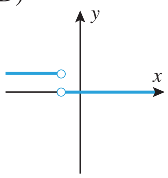
(B)



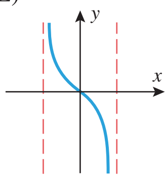
(C)



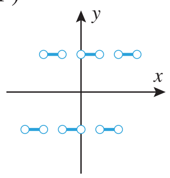
(D)



(E)



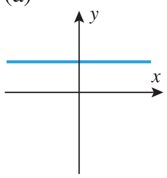
(F)



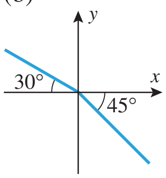
24. Let $f(x) = \sqrt{1 - x^2}$. Use a geometric argument to find $f'(\sqrt{2}/2)$.

25–26 Sketch the graph of the derivative of the function whose graph is shown.

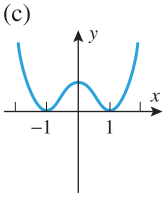
25. (a)



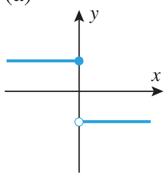
(b)



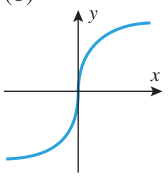
(c)



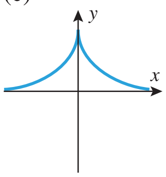
26. (a)



(b)



(c)



27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. If a curve $y = f(x)$ has a horizontal tangent line at $x = a$, then $f'(a)$ is not defined.

28. If the tangent line to the graph of $y = f(x)$ at $x = -2$ has negative slope, then $f'(-2) < 0$.

29. If a function f is continuous at $x = 0$, then f is differentiable at $x = 0$.

30. If a function f is differentiable at $x = 0$, then f is continuous at $x = 0$.

31–32 The given limit represents $f'(a)$ for some function f and some number a. Find $f(x)$ and a in each case.

31. (a) $\displaystyle\lim_{\Delta x \to 0} \frac{\sqrt{1 + \Delta x} - 1}{\Delta x}$ (b) $\displaystyle\lim_{x_1 \to 3} \frac{x_1^2 - 9}{x_1 - 3}$

32. (a) $\displaystyle\lim_{h \to 0} \frac{\cos(\pi + h) + 1}{h}$ (b) $\displaystyle\lim_{x \to 1} \frac{x^7 - 1}{x - 1}$

33. Find $dy/dx|_{x=1}$, given that $y = 1 - x^2$.

34. Find $dy/dx|_{x=-2}$, given that $y = (x + 2)/x$.

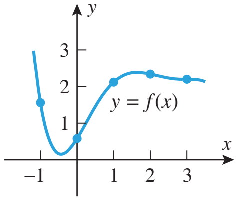
35. Find an equation for the line that is tangent to the curve $y = x^3 - 2x + 1$ at the point $(0, 1)$, and use a graphing utility to graph the curve and its tangent line on the same screen.

36. Use a graphing utility to graph the following on the same screen: the curve $y = x^2/4$, the tangent line to this curve at $x = 1$, and the secant line joining the points $(0, 0)$ and $(2, 1)$ on this curve.

37. Let $f(x) = 2^x$. Estimate $f'(1)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Formula (13) by making a table of values for a succession of values of w approaching 1.

38. Let $f(x) = \sin x$. Estimate $f'(\pi/4)$ by
(a) using a graphing utility to zoom in at an appropriate point until the graph looks like a straight line, and then estimating the slope
(b) using a calculating utility to estimate the limit in Formula (13) by making a table of values for a succession of values of w approaching $\pi/4$.

39–40 The function f whose graph is shown below has values as given in the accompanying table.



x	–1	0	1	2	3
$f(x)$	1.56	0.58	2.12	2.34	2.2

39. (a) Use data from the table to calculate the difference quotients

$$\frac{f(3) - f(1)}{3 - 1}, \quad \frac{f(2) - f(1)}{2 - 1}, \quad \frac{f(2) - f(0)}{2 - 0}$$

(b) Using the graph of $y = f(x)$, indicate which difference quotient in part (a) best approximates $f'(1)$ and which difference quotient gives the worst approximation to $f'(1)$.

40. Use data from the table to approximate the derivative values.
(a) $f'(0.5)$ (b) $f'(2.5)$

FOCUS ON CONCEPTS

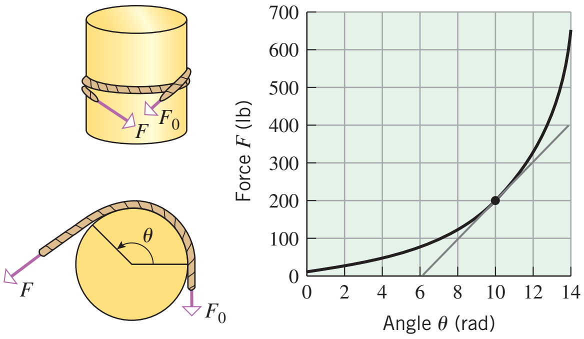
41. Suppose that the cost of drilling x feet for an oil well is $C = f(x)$ dollars.
(a) What are the units of $f'(x)$?
(b) In practical terms, what does $f'(x)$ mean in this case?
(c) What can you say about the sign of $f'(x)$?
(d) Estimate the cost of drilling an additional foot, starting at a depth of 300 ft, given that $f'(300) = 1000$.

42. A paint manufacturing company estimates that it can sell $g = f(p)$ gallons of paint at a price of p dollars per gallon.
(a) What are the units of dg/dp?
(b) In practical terms, what does dg/dp mean in this case?
(c) What can you say about the sign of dg/dp?
(d) Given that $dg/dp|_{p=10} = -100$, what can you say about the effect of increasing the price from \$10 per gallon to \$11 per gallon?

43. It is a fact that when a flexible rope is wrapped around a rough cylinder, a small force of magnitude F_0 at one end can resist a large force of magnitude F at the other end. The size of F depends on the angle θ through which the rope is wrapped around the cylinder (see the accompanying figure). The figure shows the graph of F (in pounds) versus θ (in radians), where F is the magnitude of the force that can be resisted by a force with magnitude $F_0 = 10$ lb for a certain rope and cylinder.

(a) Estimate the values of F and $dF/d\theta$ when the angle $\theta = 10$ radians.

(b) It can be shown that the force F satisfies the equation $dF/d\theta = \mu F$, where the constant μ is called the ***coefficient of friction***. Use the results in part (a) to estimate the value of μ.

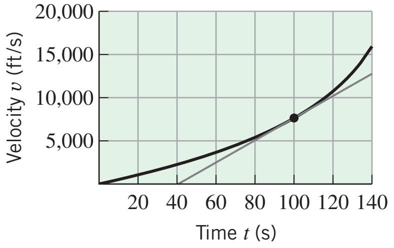


▲ **Figure Ex-43**

44. The accompanying figure shows the velocity versus time curve for a rocket in outer space where the only significant force on the rocket is from its engines. It can be shown that the mass $M(t)$ (in slugs) of the rocket at time t seconds satisfies the equation

$$M(t) = \frac{T}{dv/dt}$$

where T is the thrust (in lb) of the rocket's engines and v is the velocity (in ft/s) of the rocket. The thrust of the first stage of a *Saturn V* rocket is $T = 7{,}680{,}982$ lb. Use this value of T and the line segment in the figure to estimate the mass of the rocket at time $t = 100$.



◀ **Figure Ex-44**

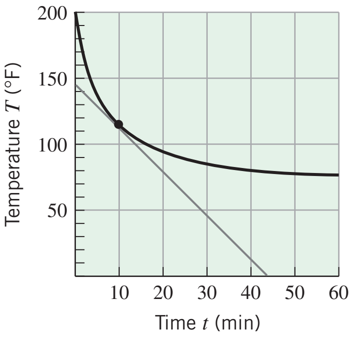
45. According to ***Newton's Law of Cooling***, the rate of change of an object's temperature is proportional to the difference between the temperature of the object and that of the surrounding medium. The accompanying figure shows the graph of the temperature T (in degrees Fahrenheit) versus time t (in minutes) for a cup of coffee, initially with a temperature of 200°F, that is allowed to cool in a room with a constant temperature of 75°F.

(a) Estimate T and dT/dt when $t = 10$ min.

(b) Newton's Law of Cooling can be expressed as

$$\frac{dT}{dt} = k(T - T_0)$$

where k is the constant of proportionality and T_0 is the temperature (assumed constant) of the surrounding medium. Use the results in part (a) to estimate the value of k.



◀ **Figure Ex-45**

46. Show that $f(x)$ is continuous but not differentiable at the indicated point. Sketch the graph of f.

(a) $f(x) = \sqrt[3]{x},\ x = 0$

(b) $f(x) = \sqrt[3]{(x-2)^2},\ x = 2$

47. Show that

$$f(x) = \begin{cases} x^2 + 1, & x \le 1 \\ 2x, & x > 1 \end{cases}$$

is continuous and differentiable at $x = 1$. Sketch the graph of f.

48. Show that

$$f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ x + 2, & x > 1 \end{cases}$$

is continuous but not differentiable at $x = 1$. Sketch the graph of f.

49. Show that

$$f(x) = \begin{cases} x \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$. Sketch the graph of f near $x = 0$. (See Figure 1.6.6 and the remark following Example 3 in Section 1.6.)

50. Show that

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is continuous and differentiable at $x = 0$. Sketch the graph of f near $x = 0$.

FOCUS ON CONCEPTS

51. Suppose that a function f is differentiable at x_0 and that $f'(x_0) > 0$. Prove that there exists an open interval containing x_0 such that if x_1 and x_2 are any two points in this interval with $x_1 < x_0 < x_2$, then $f(x_1) < f(x_0) < f(x_2)$.

52. Suppose that a function f is differentiable at x_0 and define $g(x) = f(mx + b)$, where m and b are constants. Prove that if x_1 is a point at which $mx_1 + b = x_0$, then $g(x)$ is differentiable at x_1 and $g'(x_1) = mf'(x_0)$.

53. Suppose that a function f is differentiable at $x = 0$ with $f(0) = f'(0) = 0$, and let $y = mx$, $m \neq 0$, denote any line of nonzero slope through the origin.

(a) Prove that there exists an open interval containing 0 such that for all nonzero x in this interval $|f(x)| < \left|\frac{1}{2}mx\right|$. [*Hint:* Let $\epsilon = \frac{1}{2}|m|$ and apply Definition 1.4.1 to (5) with $x_0 = 0$.]

(b) Conclude from part (a) and the triangle inequality that there exists an open interval containing 0 such that $|f(x)| < |f(x) - mx|$ for all x in this interval.

(c) Explain why the result obtained in part (b) may be interpreted to mean that the tangent line to the graph of f at the origin is the best *linear* approximation to f at that point.

54. Suppose that f is differentiable at x_0. Modify the argument of Exercise 53 to prove that the tangent line to the graph of f at the point $P(x_0, f(x_0))$ provides the best linear approximation to f at P. [*Hint:* Suppose that $y = f(x_0) + m(x - x_0)$ is any line through $P(x_0, f(x_0))$ with slope $m \neq f'(x_0)$. Apply Definition 1.4.1 to (5) with $x = x_0 + h$ and $\epsilon = \frac{1}{2}|f'(x_0) - m|$.]

55. **Writing** Write a paragraph that explains what it means for a function to be differentiable. Include examples of functions that are not differentiable as well as examples of functions that are differentiable.

56. **Writing** Explain the relationship between continuity and differentiability.

✔QUICK CHECK ANSWERS 2.2

1. $\dfrac{f(x+h) - f(x)}{h}$ **2.** (a) $2x$ (b) $\dfrac{1}{2\sqrt{x}}$ **3.** 1; $-\frac{2}{3}$

4. Theorem 2.2.3: If f is differentiable at x_0, then f is continuous at x_0.

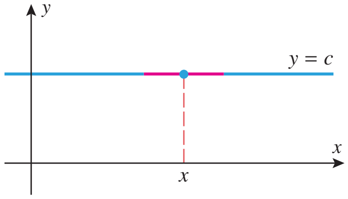
2.3 INTRODUCTION TO TECHNIQUES OF DIFFERENTIATION

In the last section we defined the derivative of a function f as a limit, and we used that limit to calculate a few simple derivatives. In this section we will develop some important theorems that will enable us to calculate derivatives more efficiently.

DERIVATIVE OF A CONSTANT

The simplest kind of function is a constant function $f(x) = c$. Since the graph of f is a horizontal line of slope 0, the tangent line to the graph of f has slope 0 for every x; and hence we can see geometrically that $f'(x) = 0$ (Figure 2.3.1). We can also see this algebraically since

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0$$



The tangent line to the graph of $f(x) = c$ has slope 0 for all x.

Figure 2.3.1

Thus, we have established the following result.

2.3.1 THEOREM *The derivative of a constant function is* 0; *that is, if c is any real number, then*

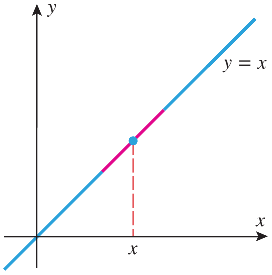
$$\frac{d}{dx}[c] = 0 \tag{1}$$

Example 1

$$\frac{d}{dx}[1] = 0, \quad \frac{d}{dx}[-3] = 0, \quad \frac{d}{dx}[\pi] = 0, \quad \frac{d}{dx}\left[-\sqrt{2}\right] = 0 \blacktriangleleft$$

DERIVATIVES OF POWER FUNCTIONS

The simplest power function is $f(x) = x$. Since the graph of f is a line of slope 1, it follows from Example 3 of Section 2.2 that $f'(x) = 1$ for all x (Figure 2.3.2). In other words,



The tangent line to the graph of $f(x) = x$ has slope 1 for all x.

Figure 2.3.2

$$\frac{d}{dx}[x] = 1 \qquad (2)$$

Example 1 of Section 2.2 shows that the power function $f(x) = x^2$ has derivative $f'(x) = 2x$. From Example 2 in that section one can infer that the power function $f(x) = x^3$ has derivative $f'(x) = 3x^2$. That is,

$$\frac{d}{dx}[x^2] = 2x \quad \text{and} \quad \frac{d}{dx}[x^3] = 3x^2 \qquad (3\text{–}4)$$

These results are special cases of the following more general result.

2.3.2 THEOREM (*The Power Rule*) *If n is a positive integer, then*

$$\frac{d}{dx}[x^n] = nx^{n-1} \qquad (5)$$

Verify that Formulas (2), (3), and (4) are the special cases of (5) in which $n = 1, 2,$ and 3.

PROOF Let $f(x) = x^n$. Thus, from the definition of a derivative and the binomial formula for expanding the expression $(x+h)^n$, we obtain

$$\begin{aligned}
\frac{d}{dx}[x^n] = f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}\\
&= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}\\
&= \lim_{h\to 0}\frac{nx^{n-1}h + \dfrac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}\\
&= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]\\
&= nx^{n-1} + 0 + \cdots + 0 + 0\\
&= nx^{n-1} \quad ■
\end{aligned}$$

The binomial formula can be found on the front endpaper of the text. Replacing y by h in this formula yields the identity used in the proof of Theorem 2.3.2.

▶ Example 2

$$\frac{d}{dx}[x^4] = 4x^3, \quad \frac{d}{dx}[x^5] = 5x^4, \quad \frac{d}{dt}[t^{12}] = 12t^{11} \;◀$$

Although our proof of the power rule in Formula (5) applies only to *positive* integer powers of x, it is not difficult to show that the same formula holds for all integer powers of x (Exercise 82). Also, we saw in Example 4 of Section 2.2 that

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \qquad (6)$$

which can be expressed as

$$\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{(1/2)-1}$$

Thus, Formula (5) is valid for $n = \frac{1}{2}$, as well. In fact, it can be shown that this formula holds for any real exponent. We state this more general result for our use now; the proof will follow in stages: Exercises 25–26 in Section 2.7 will show that the formula holds for rational exponents, and we will justify it for real exponents in Chapter 6.

2.3.3 THEOREM (*Extended Power Rule*) *If r is any real number, then*

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{7}$$

In words, *to differentiate a power function, decrease the constant exponent by one and multiply the resulting power function by the original exponent.*

▶ Example 3

$$\frac{d}{dx}[x^\pi] = \pi x^{\pi-1}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dw}\left[\frac{1}{w^{100}}\right] = \frac{d}{dw}[w^{-100}] = -100w^{-101} = -\frac{100}{w^{101}}$$

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}} \blacktriangleleft$$

DERIVATIVE OF A CONSTANT TIMES A FUNCTION

Formula (8) can also be expressed in function notation as

$$(cf)' = cf'$$

2.3.4 THEOREM (*Constant Multiple Rule*) *If f is differentiable at x and c is any real number, then cf is also differentiable at x and*

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \tag{8}$$

PROOF

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\ &= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]\\ &= c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= c\frac{d}{dx}[f(x)] \quad \blacksquare\end{aligned}$$

A constant factor can be moved through a limit sign.

In words, *a constant factor can be moved through a derivative sign.*

▶ Example 4

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{\pi}{x}\right] = \pi\frac{d}{dx}[x^{-1}] = \pi(-x^{-2}) = -\frac{\pi}{x^2} \blacktriangleleft$$

DERIVATIVES OF SUMS AND DIFFERENCES

Formulas (9) and (10) can also be expressed as

$$(f+g)' = f' + g'$$

$$(f-g)' = f' - g'$$

2.3.5 THEOREM (*Sum and Difference Rules*) *If f and g are differentiable at x, then so are $f+g$ and $f-g$ and*

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] \tag{9}$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)] \tag{10}$$

PROOF Formula (9) can be proved as follows:

$$\begin{aligned}\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]\end{aligned}$$

The limit of a sum is the sum of the limits.

Formula (10) can be proved in a similar manner or, alternatively, by writing $f(x)-g(x)$ as $f(x)+(-1)g(x)$ and then applying Formulas (8) and (9). ■

In words, *the derivative of a sum equals the sum of the derivatives*, and *the derivative of a difference equals the difference of the derivatives.*

▶ Example 5

$$\frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}] = 12x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{\sqrt{x}-2x}{\sqrt{x}}\right] &= \frac{d}{dx}[1-2\sqrt{x}\,]\\ &= \frac{d}{dx}[1] - \frac{d}{dx}[2\sqrt{x}\,] = 0 - 2\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{\sqrt{x}}\end{aligned}$$

See Formula (6). ◀

Although Formulas (9) and (10) are stated for sums and differences of two functions, they can be extended to any finite number of functions. For example, by grouping and applying Formula (9) twice we obtain

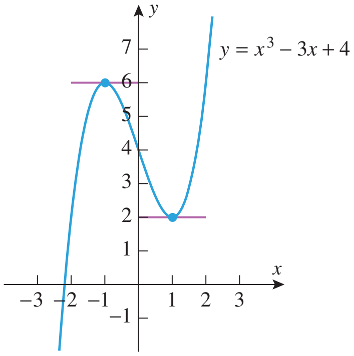
$$(f+g+h)' = [(f+g)+h]' = (f+g)'+h' = f'+g'+h'$$

As illustrated in the following example, the constant multiple rule together with the extended versions of the sum and difference rules can be used to differentiate any polynomial.

▶ **Example 6** Find dy/dx if $y = 3x^8 - 2x^5 + 6x + 1$.

Solution.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[3x^8 - 2x^5 + 6x + 1]\\ &= \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1]\\ &= 24x^7 - 10x^4 + 6 \quad ◀\end{aligned}$$



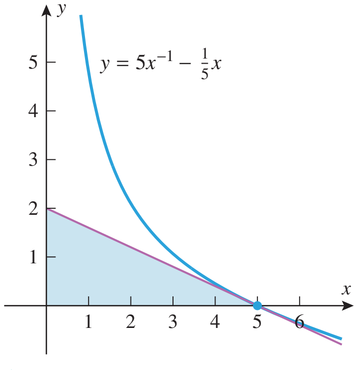
▲ **Figure 2.3.3**

▶ **Example 7** At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

Solution. Horizontal tangent lines have slope zero, so we must find those values of x for which $y'(x) = 0$. Differentiating yields

$$y'(x) = \frac{d}{dx}[x^3 - 3x + 4] = 3x^2 - 3$$

Thus, horizontal tangent lines occur at those values of x for which $3x^2 - 3 = 0$, that is, if $x = -1$ or $x = 1$. The corresponding points on the curve $y = x^3 - 3x + 4$ are $(-1, 6)$ and $(1, 2)$ (see Figure 2.3.3). ◀



▲ **Figure 2.3.4**

▶ **Example 8** Find the area of the triangle formed from the coordinate axes and the tangent line to the curve $y = 5x^{-1} - \frac{1}{5}x$ at the point $(5, 0)$.

Solution. Since the derivative of y with respect to x is

$$y'(x) = \frac{d}{dx}\left[5x^{-1} - \frac{1}{5}x\right] = \frac{d}{dx}[5x^{-1}] - \frac{d}{dx}\left[\frac{1}{5}x\right] = -5x^{-2} - \frac{1}{5}$$

the slope of the tangent line at the point $(5, 0)$ is $y'(5) = -\frac{2}{5}$. Thus, the equation of the tangent line at this point is

$$y - 0 = -\frac{2}{5}(x - 5) \quad \text{or equivalently} \quad y = -\frac{2}{5}x + 2$$

Since the y-intercept of this line is 2, the right triangle formed from the coordinate axes and the tangent line has legs of length 5 and 2, so its area is $\frac{1}{2}(5)(2) = 5$ (Figure 2.3.4). ◀

HIGHER DERIVATIVES

The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called the ***second derivative*** of f. As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, fifth, and even higher derivatives of f. These successive derivatives are denoted by

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

If $y = f(x)$, then successive derivatives can also be denoted by

$$y', \quad y'', \quad y''', \quad y^{(4)}, \quad y^{(5)}, \ldots$$

Other common notations are

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{d}{dx}[f(x)] \\
y'' &= \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right] = \frac{d^2}{dx^2}[f(x)] \\
y''' &= \frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2}{dx^2}[f(x)]\right] = \frac{d^3}{dx^3}[f(x)] \\
&\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

These are called, in succession, the *first derivative*, the *second derivative*, the *third derivative*, and so forth. The number of times that f is differentiated is called the ***order*** of the derivative. A general nth order derivative can be denoted by

$$\frac{d^ny}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)] \tag{11}$$

and the value of a general nth order derivative at a specific point $x = x_0$ can be denoted by

$$\left.\frac{d^ny}{dx^n}\right|_{x=x_0} = f^{(n)}(x_0) = \left.\frac{d^n}{dx^n}[f(x)]\right|_{x=x_0} \tag{12}$$

▶ **Example 9** If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then

$$\begin{aligned}
f'(x) &= 12x^3 - 6x^2 + 2x - 4 \\
f''(x) &= 36x^2 - 12x + 2 \\
f'''(x) &= 72x - 12 \\
f^{(4)}(x) &= 72 \\
f^{(5)}(x) &= 0 \\
&\ \vdots \\
f^{(n)}(x) &= 0 \quad (n \geq 5)
\end{aligned}$$
◀

We will discuss the significance of second derivatives and those of higher order in later sections.

✔QUICK CHECK EXERCISES 2.3 *(See page 142 for answers.)*

1. In each part, determine $f'(x)$.
 (a) $f(x) = \sqrt{6}$ (b) $f(x) = \sqrt{6}x$
 (c) $f(x) = 6\sqrt{x}$ (d) $f(x) = \sqrt{6x}$

2. In parts (a)–(d), determine $f'(x)$.
 (a) $f(x) = x^3 + 5$ (b) $f(x) = x^2(x^3 + 5)$
 (c) $f(x) = \dfrac{x^3 + 5}{2}$ (d) $f(x) = \dfrac{x^3 + 5}{x^2}$

3. The slope of the tangent line to the curve $y = x^2 + 4x + 7$ at $x = 1$ is ________.

4. If $f(x) = 3x^3 - 3x^2 + x + 1$, then $f''(x) =$ ________.

EXERCISE SET 2.3

Graphing Utility

1–8 Find dy/dx. ■

1. $y = 4x^7$　　**2.** $y = -3x^{12}$

3. $y = 3x^8 + 2x + 1$　　**4.** $y = \frac{1}{2}(x^4 + 7)$

5. $y = \pi^3$　　**6.** $y = \sqrt{2}x + (1/\sqrt{2})$

7. $y = -\frac{1}{3}(x^7 + 2x - 9)$　　**8.** $y = \dfrac{x^2+1}{5}$

9–16 Find $f'(x)$. ■

9. $f(x) = x^{-3} + \dfrac{1}{x^7}$　　**10.** $f(x) = \sqrt{x} + \dfrac{1}{x}$

11. $f(x) = -3x^{-8} + 2\sqrt{x}$　　**12.** $f(x) = 7x^{-6} - 5\sqrt{x}$

13. $f(x) = x^{\pi} + \dfrac{1}{x^{\sqrt{10}}}$　　**14.** $f(x) = \sqrt[3]{\dfrac{8}{x}}$

15. $f(x) = (3x^2 + 1)^2$

16. $f(x) = ax^3 + bx^2 + cx + d$　(a, b, c, d constant)

17–18 Find $y'(1)$. ■

17. $y = 5x^2 - 3x + 1$　　**18.** $y = \dfrac{x^{3/2} + 2}{x}$

19–20 Find dx/dt. ■

19. $x = t^2 - t$　　**20.** $x = \dfrac{t^2 + 1}{3t}$

21–24 Find $dy/dx|_{x=1}$. ■

21. $y = 1 + x + x^2 + x^3 + x^4 + x^5$

22. $y = \dfrac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{x^3}$

23. $y = (1 - x)(1 + x)(1 + x^2)(1 + x^4)$

24. $y = x^{24} + 2x^{12} + 3x^8 + 4x^6$

25–26 Approximate $f'(1)$ by considering the difference quotient

$$\frac{f(1+h) - f(1)}{h}$$

for values of h near 0, and then find the exact value of $f'(1)$ by differentiating. ■

25. $f(x) = x^3 - 3x + 1$　　**26.** $f(x) = \dfrac{1}{x^2}$

27–28 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating. ■

27. $f(x) = \dfrac{x^2 + 1}{x}$　　**28.** $f(x) = \dfrac{x + 2x^{3/2}}{\sqrt{x}}$

29–32 Find the indicated derivative. ■

29. $\dfrac{d}{dt}[16t^2]$　　**30.** $\dfrac{dC}{dr}$, where $C = 2\pi r$

31. $V'(r)$, where $V = \pi r^3$　　**32.** $\dfrac{d}{d\alpha}[2\alpha^{-1} + \alpha]$

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. If f and g are differentiable at $x = 2$, then

$$\frac{d}{dx}[f(x) - 8g(x)]\bigg|_{x=2} = f'(2) - 8g'(2)$$

34. If $f(x)$ is a cubic polynomial, then $f'(x)$ is a quadratic polynomial.

35. If $f'(2) = 5$, then

$$\frac{d}{dx}[4f(x) + x^3]\bigg|_{x=2} = \frac{d}{dx}[4f(x) + 8]\bigg|_{x=2} = 4f'(2) = 20$$

36. If $f(x) = x^2(x^4 - x)$, then

$$f''(x) = \frac{d}{dx}[x^2] \cdot \frac{d}{dx}[x^4 - x] = 2x(4x^3 - 1)$$

37. A spherical balloon is being inflated.

(a) Find a general formula for the instantaneous rate of change of the volume V with respect to the radius r, given that $V = \frac{4}{3}\pi r^3$.

(b) Find the rate of change of V with respect to r at the instant when the radius is $r = 5$.

38. Find $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right]$　(λ_0 is constant).

39. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = -3$ if $f(-3) = 2$ and $f'(-3) = 5$.

40. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 2$ if $f(2) = -2$ and $f'(2) = -1$.

41–42 Find d^2y/dx^2. ■

41. (a) $y = 7x^3 - 5x^2 + x$　　(b) $y = 12x^2 - 2x + 3$

(c) $y = \dfrac{x + 1}{x}$　　(d) $y = (5x^2 - 3)(7x^3 + x)$

42. (a) $y = 4x^7 - 5x^3 + 2x$　　(b) $y = 3x + 2$

(c) $y = \dfrac{3x - 2}{5x}$　　(d) $y = (x^3 - 5)(2x + 3)$

43–44 Find y'''. ■

43. (a) $y = x^{-5} + x^5$　　(b) $y = 1/x$

(c) $y = ax^3 + bx + c$　(a, b, c constant)

44. (a) $y = 5x^2 - 4x + 7$　　(b) $y = 3x^{-2} + 4x^{-1} + x$

(c) $y = ax^4 + bx^2 + c$　(a, b, c constant)

45. Find

(a) $f'''(2)$, where $f(x) = 3x^2 - 2$

(b) $\dfrac{d^2y}{dx^2}\bigg|_{x=1}$, where $y = 6x^5 - 4x^2$

(c) $\dfrac{d^4}{dx^4}[x^{-3}]\bigg|_{x=1}$.

46. Find
(a) $y'''(0)$, where $y = 4x^4 + 2x^3 + 3$
(b) $\left.\dfrac{d^4y}{dx^4}\right|_{x=1}$, where $y = \dfrac{6}{x^4}$.

47. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$.

48. Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

49–50 Use a graphing utility to make rough estimates of the locations of all horizontal tangent lines, and then find their exact locations by differentiating.

49. $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$ **50.** $y = \dfrac{x^2 + 9}{x}$

FOCUS ON CONCEPTS

51. Find a function $y = ax^2 + bx + c$ whose graph has an x-intercept of 1, a y-intercept of -2, and a tangent line with a slope of -1 at the y-intercept.

52. Find k if the curve $y = x^2 + k$ is tangent to the line $y = 2x$.

53. Find the x-coordinate of the point on the graph of $y = x^2$ where the tangent line is parallel to the secant line that cuts the curve at $x = -1$ and $x = 2$.

54. Find the x-coordinate of the point on the graph of $y = \sqrt{x}$ where the tangent line is parallel to the secant line that cuts the curve at $x = 1$ and $x = 4$.

55. Find the coordinates of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.

56. Show that any two tangent lines to the parabola $y = ax^2$, $a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

57. Suppose that L is the tangent line at $x = x_0$ to the graph of the cubic equation $y = ax^3 + bx$. Find the x-coordinate of the point where L intersects the graph a second time.

58. Show that the segment of the tangent line to the graph of $y = 1/x$ that is cut off by the coordinate axes is bisected by the point of tangency.

59. Show that the triangle that is formed by any tangent line to the graph of $y = 1/x$, $x > 0$, and the coordinate axes has an area of 2 square units.

60. Find conditions on a, b, c, and d so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has
(a) exactly two horizontal tangents
(b) exactly one horizontal tangent
(c) no horizontal tangents.

61. Newton's Law of Universal Gravitation states that the magnitude F of the force exerted by a point with mass M on a point with mass m is

$$F = \frac{GmM}{r^2}$$

where G is a constant and r is the distance between the bodies. Assuming that the points are moving, find a formula for the instantaneous rate of change of F with respect to r.

62. In the temperature range between 0°C and 700°C the resistance R [in ohms (Ω)] of a certain platinum resistance thermometer is given by

$$R = 10 + 0.04124T - 1.779 \times 10^{-5}T^2$$

where T is the temperature in degrees Celsius. Where in the interval from 0°C to 700°C is the resistance of the thermometer most sensitive and least sensitive to temperature changes? [*Hint:* Consider the size of dR/dT in the interval $0 \leq T \leq 700$.]

63–64 Use a graphing utility to make rough estimates of the intervals on which $f'(x) > 0$, and then find those intervals exactly by differentiating.

63. $f(x) = x - \dfrac{1}{x}$ **64.** $f(x) = x^3 - 3x$

65–68 You are asked in these exercises to determine whether a piecewise-defined function f is differentiable at a value $x = x_0$, where f is defined by different formulas on different sides of x_0. You may use without proof the following result, which is a consequence of the Mean-Value Theorem (discussed in Section 3.8). ***Theorem.*** *Let f be continuous at x_0 and suppose that $\lim_{x\to x_0} f'(x)$ exists. Then f is differentiable at x_0, and $f'(x_0) = \lim_{x\to x_0} f'(x)$.*

65. Show that

$$f(x) = \begin{cases} x^2 + x + 1, & x \leq 1 \\ 3x, & x > 1 \end{cases}$$

is continuous at $x = 1$. Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there. Sketch the graph of f.

66. Let

$$f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ \sqrt{x}, & x \geq 9 \end{cases}$$

Is f continuous at $x = 9$? Determine whether f is differentiable at $x = 9$. If so, find the value of the derivative there.

67. Let

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there.

68. Let

$$f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < \frac{1}{2} \\ \frac{3}{4}x^2, & x \geq \frac{1}{2} \end{cases}$$

Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

69. Find all points where f fails to be differentiable. Justify your answer.
(a) $f(x) = |3x - 2|$ (b) $f(x) = |x^2 - 4|$

70. In each part, compute f', f'', f''', and then state the formula for $f^{(n)}$.
(a) $f(x) = 1/x$ (b) $f(x) = 1/x^2$
[*Hint:* The expression $(-1)^n$ has a value of 1 if n is even and -1 if n is odd. Use this expression in your answer.]

71. (a) Prove:

$$\frac{d^2}{dx^2}[cf(x)] = c\frac{d^2}{dx^2}[f(x)]$$

$$\frac{d^2}{dx^2}[f(x)+g(x)] = \frac{d^2}{dx^2}[f(x)] + \frac{d^2}{dx^2}[g(x)]$$

(b) Do the results in part (a) generalize to nth derivatives? Justify your answer.

72. Let $f(x) = x^8 - 2x + 3$; find

$$\lim_{w\to 2} \frac{f'(w) - f'(2)}{w-2}$$

73. (a) Find $f^{(n)}(x)$ if $f(x) = x^n$, $n = 1, 2, 3, \ldots$.
(b) Find $f^{(n)}(x)$ if $f(x) = x^k$ and $n > k$, where k is a positive integer.
(c) Find $f^{(n)}(x)$ if

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

74. (a) Prove: If $f''(x)$ exists for each x in (a, b), then both f and f' are continuous on (a, b).
(b) What can be said about the continuity of f and its derivatives if $f^{(n)}(x)$ exists for each x in (a, b)?

75. Let $f(x) = (mx + b)^n$, where m and b are constants and n is an integer. Use the result of Exercise 52 in Section 2.2 to prove that $f'(x) = nm(mx + b)^{n-1}$.

76–77 Verify the result of Exercise 75 for $f(x)$. ■

76. $f(x) = (2x + 3)^2$ **77.** $f(x) = (3x - 1)^3$

78–81 Use the result of Exercise 75 to compute the derivative of the given function $f(x)$. ■

78. $f(x) = \dfrac{1}{x-1}$

79. $f(x) = \dfrac{3}{(2x+1)^2}$

80. $f(x) = \dfrac{x}{x+1}$

81. $f(x) = \dfrac{2x^2 + 4x + 3}{x^2 + 2x + 1}$

82. The purpose of this exercise is to extend the power rule (Theorem 2.3.2) to any integer exponent. Let $f(x) = x^n$, where n is any integer. If $n > 0$, then $f'(x) = nx^{n-1}$ by Theorem 2.3.2.
(a) Show that the conclusion of Theorem 2.3.2 holds in the case $n = 0$.
(b) Suppose that $n < 0$ and set $m = -n$ so that

$$f(x) = x^n = x^{-m} = \frac{1}{x^m}$$

Use Definition 2.2.1 and Theorem 2.3.2 to show that

$$\frac{d}{dx}\left[\frac{1}{x^m}\right] = -mx^{m-1}\cdot\frac{1}{x^{2m}}$$

and conclude that $f'(x) = nx^{n-1}$.

✔ QUICK CHECK ANSWERS 2.3

1. (a) 0 (b) $\sqrt{6}$ (c) $3/\sqrt{x}$ (d) $\sqrt{6}/(2\sqrt{x})$ **2.** (a) $3x^2$ (b) $5x^4 + 10x$ (c) $\frac{3}{2}x^2$ (d) $1 - 10x^{-3}$ **3.** 6 **4.** $18x - 6$

2.4 THE PRODUCT AND QUOTIENT RULES

In this section we will develop techniques for differentiating products and quotients of functions whose derivatives are known.

DERIVATIVE OF A PRODUCT

You might be tempted to conjecture that the derivative of a product of two functions is the product of their derivatives. However, a simple example will show this to be false. Consider the functions

$$f(x) = x \quad \text{and} \quad g(x) = x^2$$

The product of their derivatives is

$$f'(x)g'(x) = (1)(2x) = 2x$$

but their product is $h(x) = f(x)g(x) = x^3$, so the derivative of the product is

$$h'(x) = 3x^2$$

Thus, the derivative of the product is not equal to the product of the derivatives. The correct relationship, which is credited to Leibniz, is given by the following theorem.

2.4.1 THEOREM (*The Product Rule*) *If f and g are differentiable at x, then so is the product $f \cdot g$, and*

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)] \qquad (1)$$

Formula (1) can also be expressed as

$$(f \cdot g)' = f \cdot g' + g \cdot f'$$

PROOF Whereas the proofs of the derivative rules in the last section were straightforward applications of the derivative definition, a key step in this proof involves adding and subtracting the quantity $f(x+h)g(x)$ to the numerator in the derivative definition. This yields

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + g(x)\cdot\frac{f(x+h)-f(x)}{h}\right] \\
&= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \left[\lim_{h\to 0} f(x+h)\right]\frac{d}{dx}[g(x)] + \left[\lim_{h\to 0} g(x)\right]\frac{d}{dx}[f(x)] \\
&= f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]
\end{aligned}$$

[*Note:* In the last step $f(x+h) \to f(x)$ as $h \to 0$ because f is continuous at x by Theorem 2.2.3. Also, $g(x) \to g(x)$ as $h \to 0$ because $g(x)$ does not involve h and hence is treated as constant for the limit.] ■

In words, *the derivative of a product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.*

▶ **Example 1** Find dy/dx if $y = (4x^2 - 1)(7x^3 + x)$.

Solution. There are two methods that can be used to find dy/dx. We can either use the product rule or we can multiply out the factors in y and then differentiate. We will give both methods.

Method 1. (*Using the Product Rule*)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[(4x^2-1)(7x^3+x)] \\ &= (4x^2-1)\frac{d}{dx}[7x^3+x] + (7x^3+x)\frac{d}{dx}[4x^2-1] \\ &= (4x^2-1)(21x^2+1) + (7x^3+x)(8x) = 140x^4 - 9x^2 - 1\end{aligned}$$

Method 2. (*Multiplying First*)

$$y = (4x^2-1)(7x^3+x) = 28x^5 - 3x^3 - x$$

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[28x^5 - 3x^3 - x] = 140x^4 - 9x^2 - 1$$

which agrees with the result obtained using the product rule. ◀

▶ **Example 2** Find ds/dt if $s = (1+t)\sqrt{t}$.

Solution. Applying the product rule yields

$$\begin{aligned}\frac{ds}{dt} &= \frac{d}{dt}[(1+t)\sqrt{t}] \\ &= (1+t)\frac{d}{dt}[\sqrt{t}] + \sqrt{t}\frac{d}{dt}[1+t] \\ &= \frac{1+t}{2\sqrt{t}} + \sqrt{t} = \frac{1+3t}{2\sqrt{t}}\end{aligned}$$ ◀

DERIVATIVE OF A QUOTIENT

Just as the derivative of a product is not generally the product of the derivatives, so the derivative of a quotient is not generally the quotient of the derivatives. The correct relationship is given by the following theorem.

2.4.2 THEOREM (*The Quotient Rule*) *If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \tag{2}$$

Formula (2) can also be expressed as

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

PROOF

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h \cdot g(x) \cdot g(x+h)}$$

Adding and subtracting $f(x) \cdot g(x)$ in the numerator yields

$$\begin{aligned}
\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) - f(x)\cdot g(x+h) + f(x)\cdot g(x)}{h\cdot g(x)\cdot g(x+h)} \\
&= \lim_{h\to 0} \frac{\left[g(x)\cdot \dfrac{f(x+h)-f(x)}{h}\right] - \left[f(x)\cdot \dfrac{g(x+h)-g(x)}{h}\right]}{g(x)\cdot g(x+h)} \\
&= \frac{\lim\limits_{h\to 0} g(x)\cdot \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} - \lim\limits_{h\to 0} f(x)\cdot \lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}}{\lim\limits_{h\to 0} g(x)\cdot \lim\limits_{h\to 0} g(x+h)} \\
&= \frac{\left[\lim\limits_{h\to 0} g(x)\right]\cdot \dfrac{d}{dx}[f(x)] - \left[\lim\limits_{h\to 0} f(x)\right]\cdot \dfrac{d}{dx}[g(x)]}{\lim\limits_{h\to 0} g(x)\cdot \lim\limits_{h\to 0} g(x+h)} \\
&= \frac{g(x)\dfrac{d}{dx}[f(x)] - f(x)\dfrac{d}{dx}[g(x)]}{[g(x)]^2}
\end{aligned}$$

[See the note at the end of the proof of Theorem 2.4.1 for an explanation of the last step.] ■

In words, *the derivative of a quotient of two functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the denominator squared.*

Sometimes it is better to simplify a function first than to apply the quotient rule immediately. For example, it is easier to differentiate

$$f(x) = \frac{x^{3/2} + x}{\sqrt{x}}$$

by rewriting it as

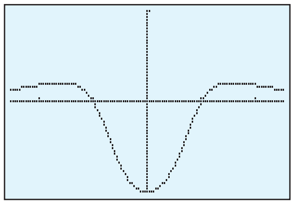
$$f(x) = x + \sqrt{x}$$

as opposed to using the quotient rule.

▶ **Example 3** Find $y'(x)$ for $y = \dfrac{x^3 + 2x^2 - 1}{x + 5}$.

Solution. Applying the quotient rule yields

$$\begin{aligned}
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^3+2x^2-1}{x+5}\right] &= \frac{(x+5)\dfrac{d}{dx}[x^3+2x^2-1] - (x^3+2x^2-1)\dfrac{d}{dx}[x+5]}{(x+5)^2} \\
&= \frac{(x+5)(3x^2+4x) - (x^3+2x^2-1)(1)}{(x+5)^2} \\
&= \frac{(3x^3+19x^2+20x) - (x^3+2x^2-1)}{(x+5)^2} \\
&= \frac{2x^3+17x^2+20x+1}{(x+5)^2}
\end{aligned}$$ ◀



[−2.5, 2.5] × [−1, 1]
xScl = 1, yScl = 1

$$y = \frac{x^2 - 1}{x^4 + 1}$$

▲ **Figure 2.4.1**

▶ **Example 4** Let $f(x) = \dfrac{x^2 - 1}{x^4 + 1}$.

(a) Graph $y = f(x)$, and use your graph to make rough estimates of the locations of all horizontal tangent lines.

(b) By differentiating, find the exact locations of the horizontal tangent lines.

Solution (a). In Figure 2.4.1 we have shown the graph of the equation $y = f(x)$ in the window $[-2.5, 2.5] \times [-1, 1]$. This graph suggests that horizontal tangent lines occur at $x = 0$, $x \approx 1.5$, and $x \approx -1.5$.

Solution (b). To find the exact locations of the horizontal tangent lines, we must find the points where $dy/dx = 0$. We start by finding dy/dx:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2-1}{x^4+1}\right] = \frac{(x^4+1)\dfrac{d}{dx}[x^2-1] - (x^2-1)\dfrac{d}{dx}[x^4+1]}{(x^4+1)^2}$$

$$= \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2}$$

The differentiation is complete. The rest is simplification.

$$= \frac{-2x^5+4x^3+2x}{(x^4+1)^2} = -\frac{2x(x^4-2x^2-1)}{(x^4+1)^2}$$

Now we will set $dy/dx = 0$ and solve for x. We obtain

$$-\frac{2x(x^4-2x^2-1)}{(x^4+1)^2} = 0$$

The solutions of this equation are the values of x for which the numerator is 0, that is,

$$2x(x^4-2x^2-1) = 0$$

The first factor yields the solution $x = 0$. Other solutions can be found by solving the equation

$$x^4 - 2x^2 - 1 = 0$$

This can be treated as a quadratic equation in x^2 and solved by the quadratic formula. This yields

$$x^2 = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

The minus sign yields imaginary values of x, which we ignore since they are not relevant to the problem. The plus sign yields the solutions

$$x = \pm\sqrt{1+\sqrt{2}}$$

In summary, horizontal tangent lines occur at

$$x = 0, \quad x = \sqrt{1+\sqrt{2}} \approx 1.55, \quad \text{and} \quad x = -\sqrt{1+\sqrt{2}} \approx -1.55$$

which is consistent with the rough estimates that we obtained graphically in part (a). ◀

Derive the following rule for differentiating a reciprocal:

$$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$$

Use it to find the derivative of

$$f(x) = \frac{1}{x^2+1}$$

SUMMARY OF DIFFERENTIATION RULES

The following table summarizes the differentiation rules that we have encountered thus far.

Table 2.4.1

RULES FOR DIFFERENTIATION

$\frac{d}{dx}[c] = 0$	$(f+g)' = f' + g'$	$(f \cdot g)' = f \cdot g' + g \cdot f'$	$\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
$(cf)' = cf'$	$(f-g)' = f' - g'$	$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$	$\frac{d}{dx}[x^r] = rx^{r-1}$

✔ QUICK CHECK EXERCISES 2.4 *(See page 148 for answers.)*

1. (a) $\dfrac{d}{dx}[x^2 f(x)] =$ _______ (b) $\dfrac{d}{dx}\left[\dfrac{f(x)}{x^2+1}\right] =$ _______

(c) $\dfrac{d}{dx}\left[\dfrac{x^2+1}{f(x)}\right] =$ _______

2. Find $F'(1)$ given that $f(1) = -1$, $f'(1) = 2$, $g(1) = 3$, and $g'(1) = -1$.

(a) $F(x) = 2f(x) - 3g(x)$ (b) $F(x) = [f(x)]^2$

(c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

EXERCISE SET 2.4

Graphing Utility

1–4 Compute the derivative of the given function $f(x)$ by (a) multiplying and then differentiating and (b) using the product rule. Verify that (a) and (b) yield the same result. ■

1. $f(x) = (x+1)(2x-1)$ **2.** $f(x) = (3x^2-1)(x^2+2)$

3. $f(x) = (x^2+1)(x^2-1)$

4. $f(x) = (x+1)(x^2-x+1)$

5–20 Find $f'(x)$. ■

5. $f(x) = (3x^2+6)\left(2x-\frac{1}{4}\right)$

6. $f(x) = (2-x-3x^3)(7+x^5)$

7. $f(x) = (x^3+7x^2-8)(2x^{-3}+x^{-4})$

8. $f(x) = \left(\dfrac{1}{x}+\dfrac{1}{x^2}\right)(3x^3+27)$

9. $f(x) = (x-2)(x^2+2x+4)$

10. $f(x) = (x^2+x)(x^2-x)$

11. $f(x) = \dfrac{3x+4}{x^2+1}$ **12.** $f(x) = \dfrac{x-2}{x^4+x+1}$

13. $f(x) = \dfrac{x^2}{3x-4}$ **14.** $f(x) = \dfrac{2x^2+5}{3x-4}$

15. $f(x) = \dfrac{(2\sqrt{x}+1)(x-1)}{x+3}$

16. $f(x) = (2\sqrt{x}+1)\left(\dfrac{2-x}{x^2+3x}\right)$

17. $f(x) = (2x+1)\left(1+\dfrac{1}{x}\right)(x^{-3}+7)$

18. $f(x) = x^{-5}(x^2+2x)(4-3x)(2x^9+1)$

19. $f(x) = (x^7+2x-3)^3$ **20.** $f(x) = (x^2+1)^4$

21–24 Find $dy/dx|_{x=1}$. ■

21. $y = \dfrac{2x-1}{x+3}$ **22.** $y = \dfrac{4x+1}{x^2-5}$

23. $y = \left(\dfrac{3x+2}{x}\right)(x^{-5}+1)$ **24.** $y = (2x^7-x^2)\left(\dfrac{x-1}{x+1}\right)$

25–26 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating. ■

25. $f(x) = \dfrac{x}{x^2+1}$ **26.** $f(x) = \dfrac{x^2-1}{x^2+1}$

27. Find $g'(4)$ given that $f(4) = 3$ and $f'(4) = -5$.

(a) $g(x) = \sqrt{x}f(x)$ (b) $g(x) = \dfrac{f(x)}{x}$

28. Find $g'(3)$ given that $f(3) = -2$ and $f'(3) = 4$.

(a) $g(x) = 3x^2 - 5f(x)$ (b) $g(x) = \dfrac{2x+1}{f(x)}$

29. In parts (a)–(d), $F(x)$ is expressed in terms of $f(x)$ and $g(x)$. Find $F'(2)$ given that $f(2) = -1$, $f'(2) = 4$, $g(2) = 1$, and $g'(2) = -5$.

(a) $F(x) = 5f(x) + 2g(x)$ (b) $F(x) = f(x) - 3g(x)$

(c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

30. Find $F'(\pi)$ given that $f(\pi) = 10$, $f'(\pi) = -1$, $g(\pi) = -3$, and $g'(\pi) = 2$.

(a) $F(x) = 6f(x) - 5g(x)$ (b) $F(x) = x(f(x)+g(x))$

(c) $F(x) = 2f(x)g(x)$ (d) $F(x) = \dfrac{f(x)}{4+g(x)}$

31–36 Find all values of x at which the tangent line to the given curve satisfies the stated property. ■

31. $y = \dfrac{x^2-1}{x+2}$; horizontal **32.** $y = \dfrac{x^2+1}{x-1}$; horizontal

33. $y = \dfrac{x^2+1}{x+1}$; parallel to the line $y = x$

34. $y = \dfrac{x+3}{x+2}$; perpendicular to the line $y = x$

35. $y = \dfrac{1}{x+4}$; passes through the origin

36. $y = \dfrac{2x+5}{x+2}$; y-intercept 2

FOCUS ON CONCEPTS

37. (a) What should it mean to say that two curves intersect at right angles?

(b) Show that the curves $y = 1/x$ and $y = 1/(2-x)$ intersect at right angles.

38. Find all values of a such that the curves $y = a/(x-1)$ and $y = x^2 - 2x + 1$ intersect at right angles.

39. Find a general formula for $F''(x)$ if $F(x) = xf(x)$ and f and f' are differentiable at x.

40. Suppose that the function f is differentiable everywhere and $F(x) = xf(x)$.

(a) Express $F'''(x)$ in terms of x and derivatives of f.

(b) For $n \geq 2$, conjecture a formula for $F^{(n)}(x)$.

41. A manufacturer of athletic footwear finds that the sales of their ZipStride brand running shoes is a function $f(p)$ of the selling price p (in dollars) for a pair of shoes. Suppose that $f(120) = 9000$ pairs of shoes and $f'(120) = -60$ pairs of shoes per dollar. The revenue that the manufacturer will receive for selling $f(p)$ pairs of shoes at p dollars per pair is $R(p) = p \cdot f(p)$. Find $R'(120)$. What impact would a small increase in price have on the manufacturer's revenue?

42. Solve the problem in Exercise 41 under the assumption that $f(120) = 9000$ and $f'(120) = -80$.

43. Use the quotient rule (Theorem 2.4.2) to derive the formula for the derivative of $f(x) = x^{-n}$, where n is a positive integer.

✔QUICK CHECK ANSWERS 2.4

1. (a) $x^2 f'(x) + 2xf(x)$ (b) $\dfrac{(x^2+1)f'(x) - 2xf(x)}{(x^2+1)^2}$ (c) $\dfrac{2xf(x) - (x^2+1)f'(x)}{[f(x)^2]}$ **2.** (a) 7 (b) −4 (c) 7 (d) $\frac{5}{9}$

2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

The main objective of this section is to obtain formulas for the derivatives of the six basic trigonometric functions. If needed, you will find a review of trigonometric functions in Appendix B.

We will assume in this section that the variable x in the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ is measured in radians. Also, we will need the limits in Theorem 1.6.3, but restated as follows using h rather than x as the variable:

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0}\frac{1-\cos h}{h} = 0 \tag{1–2}$$

Let us start with the problem of differentiating $f(x) = \sin x$. Using the definition of the derivative we obtain

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} && \boxed{\text{By the addition formula for sine}} \\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right] \\
&= \lim_{h\to 0}\left[\cos x\left(\frac{\sin h}{h}\right) - \sin x\left(\frac{1-\cos h}{h}\right)\right] && \boxed{\text{Algebraic reorganization}} \\
&= \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} - \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{1-\cos h}{h} \\
&= \left(\lim_{h\to 0}\cos x\right)(1) - \left(\lim_{h\to 0}\sin x\right)(0) && \boxed{\text{Formulas (1) and (2)}} \\
&= \lim_{h\to 0}\cos x = \cos x && \boxed{\begin{array}{l}\cos x \text{ does not involve the variable } h \text{ and hence}\\ \text{is treated as a constant in the limit computation.}\end{array}}
\end{aligned}$$

Formulas (1) and (2) and the derivation of Formulas (3) and (4) are only valid if h and x are in radians. See Exercise 49 for how Formulas (3) and (4) change when x is measured in degrees.

Thus, we have shown that

$$\frac{d}{dx}[\sin x] = \cos x \tag{3}$$

In the exercises we will ask you to use the same method to derive the following formula for the derivative of $\cos x$:

$$\frac{d}{dx}[\cos x] = -\sin x \tag{4}$$

▶ Example 1 Find dy/dx if $y = x \sin x$.

Solution. Using Formula (3) and the product rule we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[x \sin x] \\ &= x\frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x] \\ &= x\cos x + \sin x \quad \blacktriangleleft\end{aligned}$$

▶ Example 2 Find dy/dx if $y = \dfrac{\sin x}{1+\cos x}$.

Solution. Using the quotient rule together with Formulas (3) and (4) we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\cos x)\cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[1+\cos x]}{(1+\cos x)^2} \\ &= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x} \quad \blacktriangleleft\end{aligned}$$

Since Formulas (3) and (4) are only valid if x is in radians, the same is true for Formulas (5)–(8).

The derivatives of the remaining trigonometric functions are

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x \tag{5–6}$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x \tag{7–8}$$

These can all be obtained using the definition of the derivative, but it is easier to use Formulas (3) and (4) and apply the quotient rule to the relationships

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

For example,

$$\begin{aligned}\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] &= \frac{\cos x \cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[\cos x]}{\cos^2 x} \\ &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

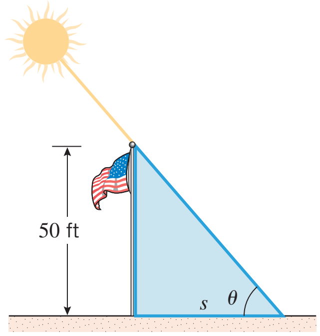
When finding the value of a derivative at a specific point $x = x_0$, it is important to substitute x_0 *after* the derivative is obtained. Thus, in Example 3 we made the substitution $x = \pi/4$ after f'' was calculated. What would have happened had we *incorrectly* substituted $x = \pi/4$ into $f'(x)$ before calculating f''?

▶ Example 3 Find $f''(\pi/4)$ if $f(x) = \sec x$.

$$\begin{aligned}f'(x) &= \sec x \tan x \\ f''(x) &= \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x] \\ &= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x \\ &= \sec^3 x + \sec x \tan^2 x\end{aligned}$$

Thus,

$$\begin{aligned} f''(\pi/4) &= \sec^3(\pi/4) + \sec(\pi/4)\tan^2(\pi/4) \\ &= (\sqrt{2})^3 + (\sqrt{2})(1)^2 = 3\sqrt{2} \end{aligned}$$ ◄



▲ **Figure 2.5.1**

▶ **Example 4** On a sunny day, a 50 ft flagpole casts a shadow that changes with the angle of elevation of the Sun. Let s be the length of the shadow and θ the angle of elevation of the Sun (Figure 2.5.1). Find the rate at which the length of the shadow is changing with respect to θ when $\theta = 45°$. Express your answer in units of feet/degree.

Solution. The variables s and θ are related by $\tan\theta = 50/s$ or, equivalently,

$$s = 50\cot\theta \tag{9}$$

If θ is measured in radians, then Formula (7) is applicable, which yields

$$\frac{ds}{d\theta} = -50\csc^2\theta$$

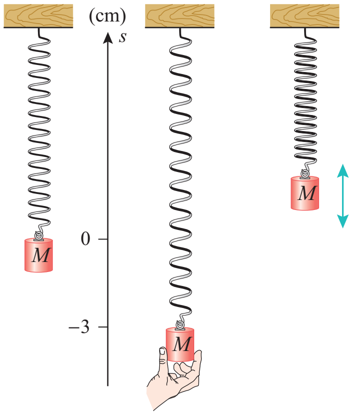
which is the rate of change of shadow length with respect to the elevation angle θ in units of feet/radian. When $\theta = 45°$ (or equivalently $\theta = \pi/4$ radians), we obtain

$$\left.\frac{ds}{d\theta}\right|_{\theta=\pi/4} = -50\csc^2(\pi/4) = -100 \text{ feet/radian}$$

Converting radians (rad) to degrees (deg) yields

$$-100\,\frac{\text{ft}}{\text{rad}} \cdot \frac{\pi}{180}\,\frac{\text{rad}}{\text{deg}} = -\frac{5}{9}\pi\,\frac{\text{ft}}{\text{deg}} \approx -1.75 \text{ ft/deg}$$

Thus, when $\theta = 45°$, the shadow length is decreasing (because of the minus sign) at an approximate rate of 1.75 ft/deg increase in the angle of elevation. ◄



▲ **Figure 2.5.2**

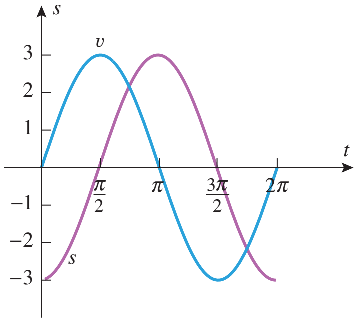
▶ **Example 5** As illustrated in Figure 2.5.2, suppose that a spring with an attached mass is stretched 3 cm beyond its rest position and released at time $t = 0$. Assuming that the position function of the top of the attached mass is

$$s = -3\cos t \tag{10}$$

where s is in centimeters and t is in seconds, find the velocity function and discuss the motion of the attached mass.

Solution. The velocity function is

$$v = \frac{ds}{dt} = \frac{d}{dt}[-3\cos t] = 3\sin t$$



▲ **Figure 2.5.3**

Figure 2.5.3 shows the graphs of the position and velocity functions. The position function tells us that the top of the mass oscillates between a low point of $s = -3$ and a high point of $s = 3$ with one complete oscillation occuring every 2π seconds [the period of (10)]. The top of the mass is moving up (the positive s-direction) when v is positive, is moving down when v is negative, and is at a high or low point when $v = 0$. Thus, for example, the top of the mass moves up from time $t = 0$ to time $t = \pi$, at which time it reaches the high point $s = 3$ and then moves down until time $t = 2\pi$, at which time it reaches the low point of $s = -3$. The motion then repeats periodically. ◄

In Example 5, the top of the mass has its maximum speed when it passes through its rest position. Why? What is that maximum speed?

✔QUICK CHECK EXERCISES 2.5 *(See page 153 for answers.)*

1. Find dy/dx.

(a) $y = \sin x$ (b) $y = \cos x$

(c) $y = \tan x$ (d) $y = \sec x$

2. Find $f'(x)$ and $f'(\pi/3)$ if $f(x) = \sin x \cos x$.

3. Use a derivative to evaluate each limit.

(a) $\lim\limits_{h\to 0} \dfrac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$ (b) $\lim\limits_{h\to 0} \dfrac{\csc(x+h)-\csc x}{h}$

EXERCISE SET 2.5 Graphing Utility

1–18 Find $f'(x)$. ■

1. $f(x) = 4\cos x + 2\sin x$

2. $f(x) = \dfrac{5}{x^2} + \sin x$

3. $f(x) = -4x^2 \cos x$

4. $f(x) = 2\sin^2 x$

5. $f(x) = \dfrac{5-\cos x}{5+\sin x}$

6. $f(x) = \dfrac{\sin x}{x^2+\sin x}$

7. $f(x) = \sec x - \sqrt{2}\tan x$

8. $f(x) = (x^2+1)\sec x$

9. $f(x) = 4\csc x - \cot x$

10. $f(x) = \cos x - x\csc x$

11. $f(x) = \sec x \tan x$

12. $f(x) = \csc x \cot x$

13. $f(x) = \dfrac{\cot x}{1+\csc x}$

14. $f(x) = \dfrac{\sec x}{1+\tan x}$

15. $f(x) = \sin^2 x + \cos^2 x$

16. $f(x) = \sec^2 x - \tan^2 x$

17. $f(x) = \dfrac{\sin x \sec x}{1+x\tan x}$

18. $f(x) = \dfrac{(x^2+1)\cot x}{3-\cos x \csc x}$

19–24 Find d^2y/dx^2. ■

19. $y = x\cos x$

20. $y = \csc x$

21. $y = x\sin x - 3\cos x$

22. $y = x^2\cos x + 4\sin x$

23. $y = \sin x \cos x$

24. $y = \tan x$

25. Find the equation of the line tangent to the graph of $\tan x$ at

(a) $x = 0$ (b) $x = \pi/4$ (c) $x = -\pi/4$.

26. Find the equation of the line tangent to the graph of $\sin x$ at

(a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$.

27. (a) Show that $y = x\sin x$ is a solution to $y'' + y = 2\cos x$.

(b) Show that $y = x\sin x$ is a solution of the equation $y^{(4)} + y'' = -2\cos x$.

28. (a) Show that $y = \cos x$ and $y = \sin x$ are solutions of the equation $y'' + y = 0$.

(b) Show that $y = A\sin x + B\cos x$ is a solution of the equation $y'' + y = 0$ for all constants A and B.

29. Find all values in the interval $[-2\pi, 2\pi]$ at which the graph of f has a horizontal tangent line.

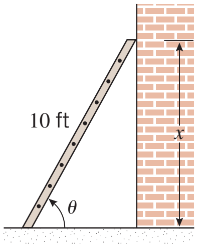
(a) $f(x) = \sin x$ (b) $f(x) = x + \cos x$

(c) $f(x) = \tan x$ (d) $f(x) = \sec x$

30. (a) Use a graphing utility to make rough estimates of the values in the interval $[0, 2\pi]$ at which the graph of $y = \sin x \cos x$ has a horizontal tangent line.

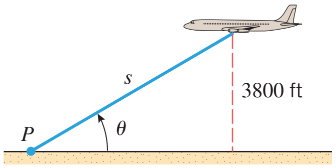
(b) Find the exact locations of the points where the graph has a horizontal tangent line.

31. A 10 ft ladder leans against a wall at an angle θ with the horizontal, as shown in the accompanying figure. The top of the ladder is x feet above the ground. If the bottom of the ladder is pushed toward the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.



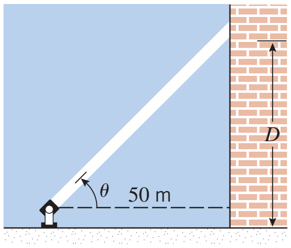
◀ **Figure Ex-31**

32. An airplane is flying on a horizontal path at a height of 3800 ft, as shown in the accompanying figure. At what rate is the distance s between the airplane and the fixed point P changing with respect to θ when $\theta = 30°$? Express the answer in units of feet/degree.



◀ **Figure Ex-32**

33. A searchlight is trained on the side of a tall building. As the light rotates, the spot it illuminates moves up and down the side of the building. That is, the distance D between ground level and the illuminated spot on the side of the building is a function of the angle θ formed by the light beam and the horizontal (see the accompanying figure). If the searchlight is located 50 m from the building, find the rate at which D is changing with respect to θ when $\theta = 45°$. Express your answer in units of meters/degree.



◀ **Figure Ex-33**

34. An Earth-observing satellite can see only a portion of the Earth's surface. The satellite has horizon sensors that can detect the angle θ shown in the accompanying figure. Let r be the radius of the Earth (assumed spherical) and h the distance of the satellite from the Earth's surface.

(a) Show that $h = r(\csc\theta - 1)$.

(b) Using $r = 6378$ km, find the rate at which h is changing with respect to θ when $\theta = 30°$. Express the answer in units of kilometers/degree.

Source: Adapted from *Space Mathematics*, NASA, 1985.

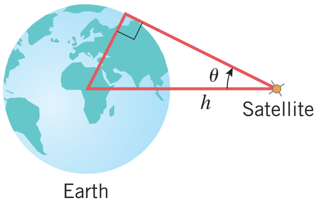


Figure Ex-34

35–38 True–False Determine whether the statement is true or false. Explain your answer. ■

35. If $g(x) = f(x)\sin x$, then $g'(x) = f'(x)\cos x$.

36. If $g(x) = f(x)\sin x$, then $g'(0) = f(0)$.

37. If $f(x)\cos x = \sin x$, then $f'(x) = \sec^2 x$.

38. Suppose that $g(x) = f(x)\sec x$, where $f(0) = 8$ and $f'(0) = -2$. Then

$$\begin{aligned} g'(0) &= \lim_{h\to 0}\frac{f(h)\sec h - f(0)}{h} = \lim_{h\to 0}\frac{8(\sec h - 1)}{h} \\ &= 8\cdot\frac{d}{dx}[\sec x]\bigg|_{x=0} = 8\sec 0\tan 0 = 0 \end{aligned}$$

39–40 Make a conjecture about the derivative by calculating the first few derivatives and observing the resulting pattern. ■

39. $\dfrac{d^{87}}{dx^{87}}[\sin x]$ **40.** $\dfrac{d^{100}}{dx^{100}}[\cos x]$

41. Let $f(x) = \cos x$. Find all positive integers n for which $f^{(n)}(x) = \sin x$.

42. Let $f(x) = \sin x$. Find all positive integers n for which $f^{(n)}(x) = \sin x$.

FOCUS ON CONCEPTS

43. In each part, determine where f is differentiable.

(a) $f(x) = \sin x$ (b) $f(x) = \cos x$
(c) $f(x) = \tan x$ (d) $f(x) = \cot x$
(e) $f(x) = \sec x$ (f) $f(x) = \csc x$
(g) $f(x) = \dfrac{1}{1+\cos x}$ (h) $f(x) = \dfrac{1}{\sin x\cos x}$
(i) $f(x) = \dfrac{\cos x}{2 - \sin x}$

44. (a) Derive Formula (4) using the definition of a derivative.

(b) Use Formulas (3) and (4) to obtain (7).

(c) Use Formula (4) to obtain (6).

(d) Use Formula (3) to obtain (8).

45. Use Formula (1), the alternative form for the definition of derivative given in Formula (13) of Section 2.2, that is,

$$f'(x) = \lim_{w\to x}\frac{f(w) - f(x)}{w - x}$$

and the difference identity

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

to show that $\dfrac{d}{dx}[\sin x] = \cos x$.

46. Follow the directions of Exercise 45 using the difference identity

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$

to show that $\dfrac{d}{dx}[\cos x] = -\sin x$.

47. (a) Show that $\displaystyle\lim_{h\to 0}\frac{\tan h}{h} = 1$.

(b) Use the result in part (a) to help derive the formula for the derivative of $\tan x$ directly from the definition of a derivative.

48. Without using any trigonometric identities, find

$$\lim_{x\to 0}\frac{\tan(x+y) - \tan y}{x}$$

[*Hint:* Relate the given limit to the definition of the derivative of an appropriate function of y.]

49. The derivative formulas for $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ were obtained under the assumption that x is measured in radians. If x is measured in degrees, then

$$\lim_{x\to 0}\frac{\sin x}{x} = \frac{\pi}{180}$$

(See Exercise 39 of Section 1.6). Use this result to prove that if x is measured in degrees, then

(a) $\dfrac{d}{dx}[\sin x] = \dfrac{\pi}{180}\cos x$

(b) $\dfrac{d}{dx}[\cos x] = -\dfrac{\pi}{180}\sin x$.

50. Writing Suppose that f is a function that is differentiable everywhere. Explain the relationship, if any, between the periodicity of f and that of f'. That is, if f is periodic, must f' also be periodic? If f' is periodic, must f also be periodic?

✔QUICK CHECK ANSWERS 2.5

1. (a) $\cos x$ (b) $-\sin x$ (c) $\sec^2 x$ (d) $\sec x \tan x$ **2.** $f'(x) = \cos^2 x - \sin^2 x$, $f'(\pi/3) = -\frac{1}{2}$

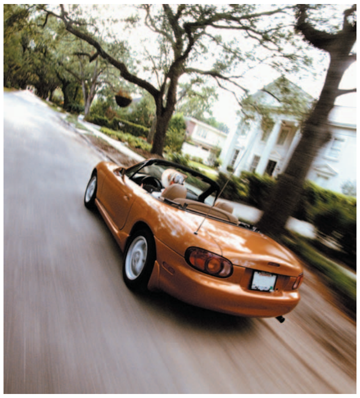
3. (a) $\left.\dfrac{d}{dx}[\sin x]\right|_{x=\pi/2} = 0$ (b) $\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

2.6 THE CHAIN RULE

In this section we will derive a formula that expresses the derivative of a composition $f \circ g$ in terms of the derivatives of f and g. This formula will enable us to differentiate complicated functions using known derivatives of simpler functions.

DERIVATIVES OF COMPOSITIONS

Suppose you are traveling to school in your car, which gets 20 miles per gallon of gasoline. The number of miles you can travel in your car without refueling is a function of the number of gallons of gas you have in the gas tank. In symbols, if y is the number of miles you can travel and u is the number of gallons of gas you have initially, then y is a function of u, or $y = f(u)$. As you continue your travels, you note that your local service station is selling gasoline for \$4 per gallon. The number of gallons of gas you have initially is a function of the amount of money you spend for that gas. If x is the number of dollars you spend on gas, then $u = g(x)$. Now 20 miles per gallon is the rate at which your mileage changes with respect to the amount of gasoline you use, so



Mike Brinson/Getty Images

The cost of a car trip is a combination of fuel efficiency and the cost of gasoline.

$$f'(u) = \frac{dy}{du} = 20 \text{ miles per gallon}$$

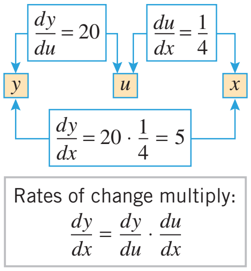
Similarly, since gasoline costs \$4 per gallon, each dollar you spend will give you 1/4 of a gallon of gas, and

$$g'(x) = \frac{du}{dx} = \frac{1}{4} \text{ gallons per dollar}$$

Notice that the number of miles you can travel is also a function of the number of dollars you spend on gasoline. This fact is expressible as the composition of functions

$$y = f(u) = f(g(x))$$

You might be interested in how many miles you can travel per dollar, which is dy/dx. Intuition suggests that rates of change multiply in this case (see Figure 2.6.1), so



▲ **Figure 2.6.1**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{20 \text{ miles}}{1 \text{ gallon}} \cdot \frac{1 \text{ gallon}}{4 \text{ dollars}} = \frac{20 \text{ miles}}{4 \text{ dollars}} = 5 \text{ miles per dollar}$$

The following theorem, the proof of which is given in Appendix D, formalizes the preceding ideas.

The name "chain rule" is appropriate because the desired derivative is obtained by a two-link "chain" of simpler derivatives.

2.6.1 THEOREM (*The Chain Rule*) *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if*

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

Formula (1) is easy to remember because the left side is exactly what results if we "cancel" the du's on the right side. This "canceling" device provides a good way of deducing the correct form of the chain rule when different variables are used. For example, if w is a function of x and x is a function of t, then the chain rule takes the form

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$

▶ Example 1 Find dy/dx if $y = \cos(x^3)$.

Solution. Let $u = x^3$ and express y as $y = \cos u$. Applying Formula (1) yields

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}[\cos u] \cdot \frac{d}{dx}[x^3] \\
&= (-\sin u) \cdot (3x^2) \\
&= (-\sin(x^3)) \cdot (3x^2) = -3x^2 \sin(x^3)
\end{aligned}$$ ◀

▶ Example 2 Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution. In this case the chain rule computations take the form

$$\begin{aligned}
\frac{dw}{dt} &= \frac{dw}{dx} \cdot \frac{dx}{dt} \\
&= \frac{d}{dx}[\tan x] \cdot \frac{d}{dt}[4t^3 + t] \\
&= (\sec^2 x) \cdot (12t^2 + 1) \\
&= [\sec^2(4t^3 + t)] \cdot (12t^2 + 1) = (12t^2 + 1)\sec^2(4t^3 + t)
\end{aligned}$$ ◀

AN ALTERNATIVE VERSION OF THE CHAIN RULE

Formula (1) for the chain rule can be unwieldy in some problems because it involves so many variables. As you become more comfortable with the chain rule, you may want to dispense with writing out the dependent variables by expressing (1) in the form

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x) = f'(g(x))g'(x) \tag{2}$$

Confirm that (2) is an alternative version of (1) by letting $y = f(g(x))$ and $u = g(x)$.

A convenient way to remember this formula is to call f the "outside function" and g the "inside function" in the composition $f(g(x))$ and then express (2) in words as:

The derivative of $f(g(x))$ is the derivative of the outside function evaluated at the inside function times the derivative of the inside function.

$$\frac{d}{dx}[f(g(x))] = \underbrace{f'(g(x))}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside function}}$$

▶ **Example 3** (*Example 1 revisited*) Find $h'(x)$ if $h(x) = \cos(x^3)$.

Solution. We can think of h as a composition $f(g(x))$ in which $g(x) = x^3$ is the inside function and $f(x) = \cos x$ is the outside function. Thus, Formula (2) yields

$$h'(x) = \underbrace{f'(g(x))}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{g'(x)}_{\text{Derivative of the inside function}}$$

$$= f'(x^3) \cdot 3x^2$$

$$= -\sin(x^3) \cdot 3x^2 = -3x^2 \sin(x^3)$$

which agrees with the result obtained in Example 1. ◀

▶ **Example 4**

$$\frac{d}{dx}[\tan^2 x] = \frac{d}{dx}[(\tan x)^2] = \underbrace{(2\tan x)}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{(\sec^2 x)}_{\text{Derivative of the inside function}} = 2\tan x \sec^2 x$$

$$\frac{d}{dx}\left[\sqrt{x^2+1}\right] = \underbrace{\frac{1}{2\sqrt{x^2+1}}}_{\text{Derivative of the outside function evaluated at the inside function}} \cdot \underbrace{2x}_{\text{Derivative of the inside function}} = \frac{x}{\sqrt{x^2+1}}$$

See Formula (6) of Section 2.3. ◀

GENERALIZED DERIVATIVE FORMULAS

There is a useful third variation of the chain rule that strikes a middle ground between Formulas (1) and (2). If we let $u = g(x)$ in (2), then we can rewrite that formula as

$$\boxed{\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}} \tag{3}$$

This result, called the ***generalized derivative formula*** for f, provides a way of using the derivative of $f(x)$ to produce the derivative of $f(u)$, where u is a function of x. Table 2.6.1 gives some examples of this formula.

Table 2.6.1

GENERALIZED DERIVATIVE FORMULAS

$\frac{d}{dx}[u^r] = ru^{r-1}\frac{du}{dx}$	
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u \tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u \cot u\frac{du}{dx}$

▶ Example 5 Find

$$\text{(a)}\ \frac{d}{dx}[\sin(2x)] \qquad \text{(b)}\ \frac{d}{dx}[\tan(x^2+1)] \qquad \text{(c)}\ \frac{d}{dx}\left[\sqrt{x^3+\csc x}\right]$$

$$\text{(d)}\ \frac{d}{dx}[x^2-x+2]^{3/4} \qquad \text{(e)}\ \frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right]$$

Solution (a). Taking $u = 2x$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2 = 2\cos 2x$$

Solution (b). Taking $u = x^2+1$ in the generalized derivative formula for $\tan u$ yields

$$\begin{aligned}\frac{d}{dx}[\tan(x^2+1)] &= \frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}\\ &= \sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1] = \sec^2(x^2+1)\cdot 2x\\ &= 2x\sec^2(x^2+1)\end{aligned}$$

Solution (c). Taking $u = x^3+\csc x$ in the generalized derivative formula for $\sqrt{u}$ yields

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right] &= \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{x^3+\csc x}}\cdot\frac{d}{dx}[x^3+\csc x]\\ &= \frac{1}{2\sqrt{x^3+\csc x}}\cdot(3x^2-\csc x\cot x) = \frac{3x^2-\csc x\cot x}{2\sqrt{x^3+\csc x}}\end{aligned}$$

Solution (d). Taking $u = x^2-x+2$ in the generalized derivative formula for $u^{3/4}$ yields

$$\begin{aligned}\frac{d}{dx}[x^2-x+2]^{3/4} &= \frac{d}{dx}[u^{3/4}] = \frac{3}{4}u^{-1/4}\frac{du}{dx}\\ &= \frac{3}{4}(x^2-x+2)^{-1/4}\cdot\frac{d}{dx}[x^2-x+2]\\ &= \frac{3}{4}(x^2-x+2)^{-1/4}(2x-1)\end{aligned}$$

Solution (e). Taking $u = 1+x^5\cot x$ in the generalized derivative formula for u^{-8} yields

$$\begin{aligned}\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right] &= \frac{d}{dx}[u^{-8}] = -8u^{-9}\frac{du}{dx}\\ &= -8(1+x^5\cot x)^{-9}\cdot\frac{d}{dx}[1+x^5\cot x]\\ &= -8(1+x^5\cot x)^{-9}\cdot\left[x^5(-\csc^2 x)+5x^4\cot x\right]\\ &= (8x^5\csc^2 x - 40x^4\cot x)(1+x^5\cot x)^{-9}\end{aligned}$$ ◀

Sometimes you will have to make adjustments in notation or apply the chain rule more than once to calculate a derivative.

▶ Example 6 Find

$$\text{(a)}\ \frac{d}{dx}\left[\sin(\sqrt{1+\cos x}\,)\right] \qquad \text{(b)}\ \frac{d\mu}{dt}\ \text{if}\ \mu = \sec\sqrt{\omega t} \qquad (\omega\ \text{constant})$$

Solution (a). Taking $u = \sqrt{1+\cos x}$ in the generalized derivative formula for $\sin u$ yields

$$\begin{aligned}\frac{d}{dx}\left[\sin(\sqrt{1+\cos x}\,)\right] &= \frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}\\ &= \cos(\sqrt{1+\cos x}\,)\cdot\frac{d}{dx}\left[\sqrt{1+\cos x}\right]\\ &= \cos(\sqrt{1+\cos x}\,)\cdot\frac{-\sin x}{2\sqrt{1+\cos x}}\\ &= -\frac{\sin x\cos(\sqrt{1+\cos x}\,)}{2\sqrt{1+\cos x}}\end{aligned}$$

We used the generalized derivative formula for $\sqrt{u}$ with $u = 1+\cos x$.

Solution (b).

$$\begin{aligned}\frac{d\mu}{dt} &= \frac{d}{dt}[\sec\sqrt{\omega t}\,] = \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{d}{dt}[\sqrt{\omega t}\,]\\ &= \sec\sqrt{\omega t}\tan\sqrt{\omega t}\frac{\omega}{2\sqrt{\omega t}}\end{aligned}$$

We used the generalized derivative formula for $\sec u$ with $u = \sqrt{\omega t}$.

We used the generalized derivative formula for $\sqrt{u}$ with $u = \omega t$. ◀

■ DIFFERENTIATING USING COMPUTER ALGEBRA SYSTEMS

Even with the chain rule and other differentiation rules, some derivative computations can be tedious to perform. For complicated derivatives, engineers and scientists often use computer algebra systems such as *Mathematica*, *Maple*, or *Sage*. For example, although we have all the mathematical tools to compute

$$\frac{d}{dx}\left[\frac{(x^2+1)^{10}\sin^3(\sqrt{x}\,)}{\sqrt{1+\csc x}}\right] \tag{4}$$

by hand, the computation is sufficiently involved that it may be more efficient (and less error-prone) to use a computer algebra system.

TECHNOLOGY MASTERY

If you have a CAS, use it to perform the differentiation in (4).

✔ QUICK CHECK EXERCISES 2.6 *(See page 160 for answers.)*

1. The chain rule states that the derivative of the composition of two functions is the derivative of the ________ function evaluated at the ________ function times the derivative of the ________ function.

2. If y is a differentiable function of u, and u is a differentiable function of x, then

$$\frac{dy}{dx} = \underline{\qquad\qquad}\cdot\underline{\qquad\qquad}$$

3. Find dy/dx.
(a) $y = (x^2+5)^{10}$ (b) $y = \sqrt{1+6x}$

4. Find dy/dx.
(a) $y = \sin(3x+2)$ (b) $y = (x^2\tan x)^4$

5. Suppose that $f(2) = 3$, $f'(2) = 4$, $g(3) = 6$, and $g'(3) = -5$. Evaluate
(a) $h'(2)$, where $h(x) = g(f(x))$
(b) $k'(3)$, where $k(x) = f\left(\frac{1}{3}g(x)\right)$.

EXERCISE SET 2.6 ⛶ Graphing Utility C CAS

1. Given that

$$f'(0) = 2,\ g(0) = 0 \quad\text{and}\quad g'(0) = 3$$

find $(f\circ g)'(0)$.

2. Given that

$$f'(9) = 5,\ g(2) = 9 \quad\text{and}\quad g'(2) = -3$$

find $(f\circ g)'(2)$.

3. Let $f(x) = x^5$ and $g(x) = 2x-3$.
(a) Find $(f\circ g)(x)$ and $(f\circ g)'(x)$.
(b) Find $(g\circ f)(x)$ and $(g\circ f)'(x)$.

4. Let $f(x) = 5\sqrt{x}$ and $g(x) = 4+\cos x$.
(a) Find $(f\circ g)(x)$ and $(f\circ g)'(x)$.
(b) Find $(g\circ f)(x)$ and $(g\circ f)'(x)$.

FOCUS ON CONCEPTS

5. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	−2	5	7
5	3	−1	12	4

(a) $F'(3)$, where $F(x) = f(g(x))$
(b) $G'(3)$, where $G(x) = g(f(x))$

6. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
−1	2	3	2	−3
2	0	4	1	−5

(a) $F'(-1)$, where $F(x) = f(g(x))$
(b) $G'(-1)$, where $G(x) = g(f(x))$

7–26 Find $f'(x)$. ■

7. $f(x) = (x^3 + 2x)^{37}$

8. $f(x) = (3x^2 + 2x - 1)^6$

9. $f(x) = \left(x^3 - \dfrac{7}{x}\right)^{-2}$

10. $f(x) = \dfrac{1}{(x^5 - x + 1)^9}$

11. $f(x) = \dfrac{4}{(3x^2 - 2x + 1)^3}$

12. $f(x) = \sqrt{x^3 - 2x + 5}$

13. $f(x) = \sqrt{4 + \sqrt{3x}}$

14. $f(x) = \sqrt[3]{12 + \sqrt{x}}$

15. $f(x) = \sin\left(\dfrac{1}{x^2}\right)$

16. $f(x) = \tan\sqrt{x}$

17. $f(x) = 4\cos^5 x$

18. $f(x) = 4x + 5\sin^4 x$

19. $f(x) = \cos^2(3\sqrt{x})$

20. $f(x) = \tan^4(x^3)$

21. $f(x) = 2\sec^2(x^7)$

22. $f(x) = \cos^3\left(\dfrac{x}{x+1}\right)$

23. $f(x) = \sqrt{\cos(5x)}$

24. $f(x) = \sqrt{3x - \sin^2(4x)}$

25. $f(x) = [x + \csc(x^3 + 3)]^{-3}$

26. $f(x) = [x^4 - \sec(4x^2 - 2)]^{-4}$

27–40 Find dy/dx. ■

27. $y = x^3 \sin^2(5x)$

28. $y = \sqrt{x}\tan^3(\sqrt{x})$

29. $y = x^5 \sec(1/x)$

30. $y = \dfrac{\sin x}{\sec(3x + 1)}$

31. $y = \cos(\cos x)$

32. $y = \sin(\tan 3x)$

33. $y = \cos^3(\sin 2x)$

34. $y = \dfrac{1 + \csc(x^2)}{1 - \cot(x^2)}$

35. $y = (5x + 8)^7 \left(1 - \sqrt{x}\right)^6$

36. $y = (x^2 + x)^5 \sin^8 x$

37. $y = \left(\dfrac{x - 5}{2x + 1}\right)^3$

38. $y = \left(\dfrac{1 + x^2}{1 - x^2}\right)^{17}$

39. $y = \dfrac{(2x + 3)^3}{(4x^2 - 1)^8}$

40. $y = [1 + \sin^3(x^5)]^{12}$

C **41–42** Use a CAS to find dy/dx. ■

41. $y = [x \sin 2x + \tan^4(x^7)]^5$

42. $y = \tan^4\left(2 + \dfrac{(7 - x)\sqrt{3x^2 + 5}}{x^3 + \sin x}\right)$

43–50 Find an equation for the tangent line to the graph at the specified value of x. ■

43. $y = x\cos 3x, \; x = \pi$

44. $y = \sin(1 + x^3), \; x = -3$

45. $y = \sec^3\left(\dfrac{\pi}{2} - x\right), \; x = -\dfrac{\pi}{2}$

46. $y = \left(x - \dfrac{1}{x}\right)^3, \; x = 2$

47. $y = \tan(4x^2), \; x = \sqrt{\pi}$

48. $y = 3\cot^4 x, \; x = \dfrac{\pi}{4}$

49. $y = x^2\sqrt{5 - x^2}, \; x = 1$

50. $y = \dfrac{x}{\sqrt{1 - x^2}}, \; x = 0$

51–54 Find d^2y/dx^2. ■

51. $y = x\cos(5x) - \sin^2 x$

52. $y = \sin(3x^2)$

53. $y = \dfrac{1 + x}{1 - x}$

54. $y = x\tan\left(\dfrac{1}{x}\right)$

55–58 Find the indicated derivative. ■

55. $y = \cot^3(\pi - \theta)$; find $\dfrac{dy}{d\theta}$.

56. $\lambda = \left(\dfrac{au + b}{cu + d}\right)^6$; find $\dfrac{d\lambda}{du}$ $(a, b, c, d$ constants).

57. $\dfrac{d}{d\omega}[a\cos^2 \pi\omega + b\sin^2 \pi\omega]$ $(a, b$ constants)

58. $x = \csc^2\left(\dfrac{\pi}{3} - y\right)$; find $\dfrac{dx}{dy}$.

59. (a) Use a graphing utility to obtain the graph of the function $f(x) = x\sqrt{4 - x^2}$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f'.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f'.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together.

60. (a) Use a graphing utility to obtain the graph of the function $f(x) = \sin x^2 \cos x$ over the interval $[-\pi/2, \pi/2]$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f' over the interval.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f' over the interval.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together over the interval.

61–64 True–False Determine whether the statement is true or false. Explain your answer. ■

61. If $y = f(x)$, then $\dfrac{d}{dx}[\sqrt{y}] = \sqrt{f'(x)}$.

62. If $y = f(u)$ and $u = g(x)$, then $dy/dx = f'(x) \cdot g'(x)$.

63. If $y = \cos[g(x)]$, then $dy/dx = -\sin[g'(x)]$.

64. If $y = \sin^3(3x^3)$, then $dy/dx = 27x^2 \sin^2(3x^3)\cos(3x^3)$.

65. If an object suspended from a spring is displaced vertically from its equilibrium position by a small amount and released, and if the air resistance and the mass of the spring are ignored, then the resulting oscillation of the object is called ***simple harmonic motion***. Under appropriate conditions the displacement y from equilibrium in terms of time t is given by

$$y = A\cos\omega t$$

where A is the initial displacement at time $t = 0$, and ω is a constant that depends on the mass of the object and the stiffness of the spring (see the accompanying figure). The constant $|A|$ is called the ***amplitude*** of the motion and ω the ***angular frequency***.

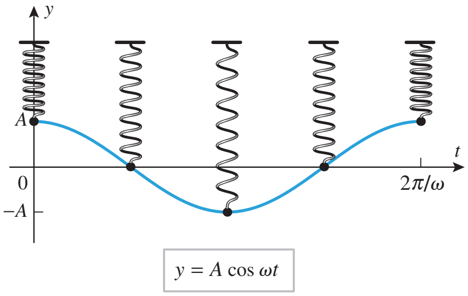
(a) Show that

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

(b) The ***period*** T is the time required to make one complete oscillation. Show that $T = 2\pi/\omega$.

(c) The ***frequency*** f of the vibration is the number of oscillations per unit time. Find f in terms of the period T.

(d) Find the amplitude, period, and frequency of an object that is executing simple harmonic motion given by $y = 0.6\cos 15t$, where t is in seconds and y is in centimeters.



▲ **Figure Ex-65**

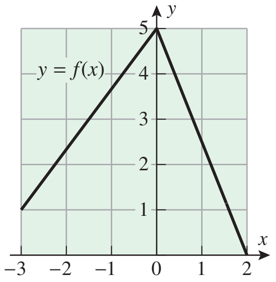
66. Find the value of the constant A so that $y = A\sin 3t$ satisfies the equation

$$\frac{d^2y}{dt^2} + 2y = 4\sin 3t$$

FOCUS ON CONCEPTS

67. Use the graph of the function f in the accompanying figure to evaluate

$$\frac{d}{dx}\left[\sqrt{x + f(x)}\right]\bigg|_{x=-1}$$



◀ **Figure Ex-67**

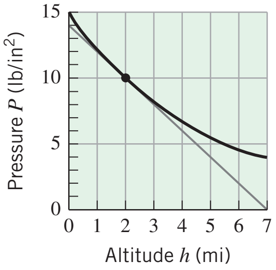
68. Using the function f in Exercise 67, evaluate

$$\frac{d}{dx}[f(2\sin x)]\bigg|_{x=\pi/6}$$

69. The accompanying figure shows the graph of atmospheric pressure p (lb/in^2) versus the altitude h (mi) above sea level.

(a) From the graph and the tangent line at $h = 2$ shown on the graph, estimate the values of p and dp/dh at an altitude of 2 mi.

(b) If the altitude of a space vehicle is increasing at the rate of 0.3 mi/s at the instant when it is 2 mi above sea level, how fast is the pressure changing with time at this instant?



◀ **Figure Ex-69**

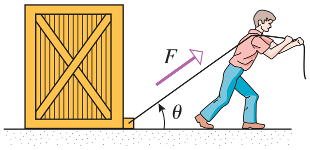
70. The force F (in pounds) acting at an angle θ with the horizontal that is needed to drag a crate weighing W pounds along a horizontal surface at a constant velocity is given by

$$F = \frac{\mu W}{\cos\theta + \mu\sin\theta}$$

where μ is a constant called the ***coefficient of sliding friction*** between the crate and the surface (see the accompanying figure). Suppose that the crate weighs 150 lb and that $\mu = 0.3$.

(a) Find $dF/d\theta$ when $\theta = 30°$. Express the answer in units of pounds/degree.

(b) Find dF/dt when $\theta = 30°$ if θ is decreasing at the rate of $0.5°$/s at this instant.



◀ **Figure Ex-70**

71. Recall that

$$\frac{d}{dx}(|x|) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Use this result and the chain rule to find

$$\frac{d}{dx}(|\sin x|)$$

for nonzero x in the interval $(-\pi, \pi)$.

72. Use the derivative formula for $\sin x$ and the identity

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to obtain the derivative formula for $\cos x$.

73. Let

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Show that f is continuous at $x = 0$.
(b) Use Definition 2.2.1 to show that $f'(0)$ does not exist.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Determine whether $\lim_{x \to 0} f'(x)$ exists.

74. Let

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Show that f is continuous at $x = 0$.
(b) Use Definition 2.2.1 to find $f'(0)$.
(c) Find $f'(x)$ for $x \neq 0$.
(d) Show that f' is not continuous at $x = 0$.

75. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$
2	1	7
8	5	−3

(a) $g'(2)$, where $g(x) = [f(x)]^3$
(b) $h'(2)$, where $h(x) = f(x^3)$

76. Given that $f'(x) = \sqrt{3x+4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.

77. Given that $f'(x) = \dfrac{x}{x^2+1}$ and $g(x) = \sqrt{3x-1}$, find $F'(x)$ if $F(x) = f(g(x))$.

78. Find $f'(x^2)$ if $\dfrac{d}{dx}[f(x^2)] = x^2$.

79. Find $\dfrac{d}{dx}[f(x)]$ if $\dfrac{d}{dx}[f(3x)] = 6x$.

80. Recall that a function f is ***even*** if $f(-x) = f(x)$ and ***odd*** if $f(-x) = -f(x)$, for all x in the domain of f. Assuming that f is differentiable, prove:
(a) f' is odd if f is even
(b) f' is even if f is odd.

81. Draw some pictures to illustrate the results in Exercise 80, and write a paragraph that gives an informal explanation of why the results are true.

82. Let $y = f_1(u)$, $u = f_2(v)$, $v = f_3(w)$, and $w = f_4(x)$. Express dy/dx in terms of dy/du, dw/dx, du/dv, and dv/dw.

83. Find a formula for

$$\frac{d}{dx}[f(g(h(x)))]$$

84. **Writing** The "co" in "cosine" comes from "complementary," since the cosine of an angle is the sine of the complementary angle, and vice versa:

$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \quad \text{and} \quad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

Suppose that we define a function g to be a *cofunction* of a function f if

$$g(x) = f\left(\frac{\pi}{2} - x\right) \quad \text{for all } x$$

Thus, cosine and sine are cofunctions of each other, as are cotangent and tangent, and also cosecant and secant. If g is the cofunction of f, state a formula that relates g' and the cofunction of f'. Discuss how this relationship is exhibited by the derivatives of the cosine, cotangent, and cosecant functions.

✔QUICK CHECK ANSWERS 2.6

1. outside; inside; inside **2.** $\dfrac{dy}{du} \cdot \dfrac{du}{dx}$ **3.** (a) $10(x^2+5)^9 \cdot 2x = 20x(x^2+5)^9$ (b) $\dfrac{1}{2\sqrt{1+6x}} \cdot 6 = \dfrac{3}{\sqrt{1+6x}}$

4. (a) $3\cos(3x+2)$ (b) $4(x^2 \tan x)^3(2x \tan x + x^2 \sec^2 x)$ **5.** (a) $g'(f(2))f'(2) = -20$ (b) $f'\left(\dfrac{1}{3}g(3)\right) \cdot \dfrac{1}{3}g'(3) = -\dfrac{20}{3}$

2.7 IMPLICIT DIFFERENTIATION

Up to now we have been concerned with differentiating functions that are given by equations of the form $y = f(x)$. In this section we will consider methods for differentiating functions for which it is inconvenient or impossible to express them in this form.

FUNCTIONS DEFINED EXPLICITLY AND IMPLICITLY

An equation of the form $y = f(x)$ is said to define y ***explicitly*** as a function of x because the variable y appears alone on one side of the equation and does not appear at all on the other side. However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$yx + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$, but it still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1}$$

Thus, we say that (1) defines y ***implicitly*** as a function of x, the function being

$$f(x) = \frac{x-1}{x+1}$$

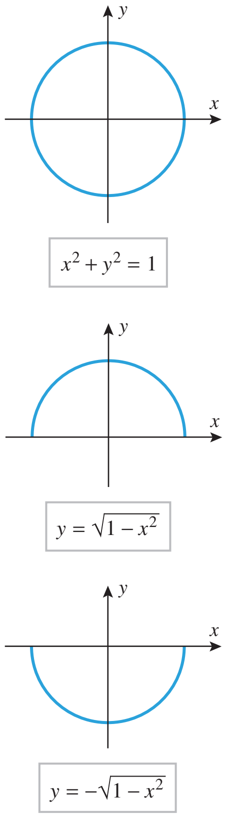
An equation in x and y can implicitly define more than one function of x. This can occur when the graph of the equation fails the vertical line test, so it is not the graph of a function of x. For example, if we solve the equation of the circle

$$x^2 + y^2 = 1 \tag{2}$$

for y in terms of x, we obtain $y = \pm\sqrt{1-x^2}$, so we have found two functions that are defined implicitly by (2), namely,

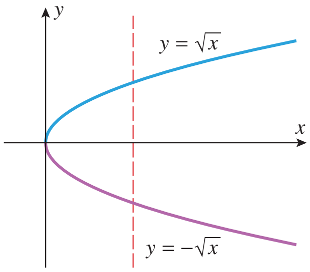
$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2} \tag{3}$$

The graphs of these functions are the upper and lower semicircles of the circle $x^2 + y^2 = 1$ (Figure 2.7.1). This leads us to the following definition.



▲ **Figure 2.7.1**

2.7.1 DEFINITION We will say that a given equation in x and y defines the function f ***implicitly*** if the graph of $y = f(x)$ coincides with a portion of the graph of the equation.



▲ **Figure 2.7.2** The graph of $x = y^2$ does not pass the vertical line test, but the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ do.

▶ **Example 1** The graph of $x = y^2$ is not the graph of a function of x, since it does not pass the vertical line test (Figure 2.7.2). However, if we solve this equation for y in terms of x, we obtain the equations $y = \sqrt{x}$ and $y = -\sqrt{x}$, whose graphs pass the vertical line test and are portions of the graph of $x = y^2$ (Figure 2.7.2). Thus, the equation $x = y^2$ implicitly defines the functions

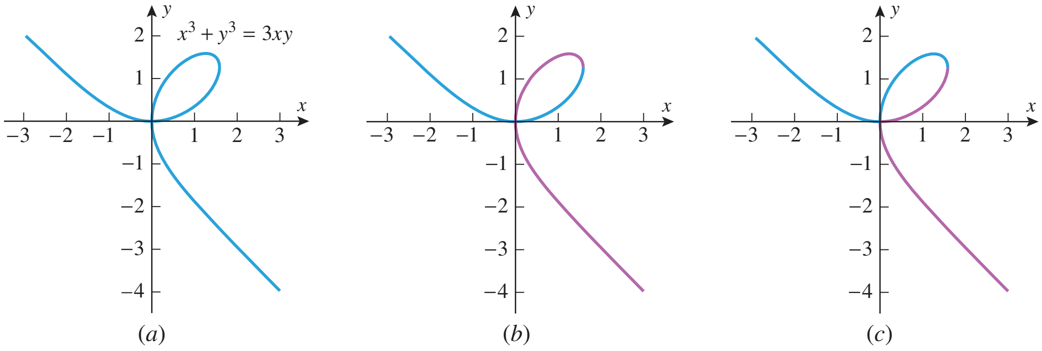
$$f_1(x) = \sqrt{x} \quad \text{and} \quad f_2(x) = -\sqrt{x}$$ ◀

Although it was a trivial matter in the last example to solve the equation $x = y^2$ for y in terms of x, it is difficult or impossible to do this for some equations. For example, the equation

$$x^3 + y^3 = 3xy \tag{4}$$

can be solved for y in terms of x, but the resulting formulas are too complicated to be practical. Other equations, such as $\sin(xy) = y$, cannot be solved for y by any elementary method. Thus, even though an equation may define one or more functions of x, it may not be possible or practical to find explicit formulas for those functions.

Fortunately, CAS programs, such as *Mathematica* and *Maple*, have "implicit plotting" capabilities that can graph equations such as (4). The graph of this equation, which is called the ***Folium of Descartes***, is shown in Figure 2.7.3a. Parts (b) and (c) of the figure show the graphs (in blue) of two functions that are defined implicitly by (4).



▲ **Figure 2.7.3**

IMPLICIT DIFFERENTIATION

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. To illustrate this, let us consider the simple equation

$$xy = 1 \tag{5}$$

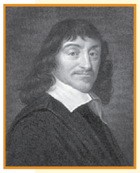
One way to find dy/dx is to rewrite this equation as

$$y = \frac{1}{x} \tag{6}$$

from which it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2} \tag{7}$$

Another way to obtain this derivative is to differentiate both sides of (5) *before* solving for y in terms of x, treating y as a (temporarily unspecified) differentiable function of x. With



René Descartes (1596–1650) Descartes, a French aristocrat, was the son of a government official. He graduated from the University of Poitiers with a law degree at age 20. After a brief probe into the pleasures of Paris he became a military engineer, first for the Dutch Prince of Nassau and then for the German Duke of Bavaria. It was during his service as a soldier that Descartes began to pursue mathematics seriously and develop his analytic geometry. After the wars, he returned to Paris where he stalked the city as an eccentric, wearing a sword in his belt and a plumed hat. He lived in leisure, seldom arose before 11 A.M., and dabbled in the study of human physiology, philosophy, glaciers, meteors, and rainbows. He eventually moved to Holland, where he published his *Discourse on the Method*, and finally to Sweden where he died while serving as tutor to Queen Christina. Descartes is regarded as a genius of the first magnitude. In addition to major contributions in mathematics and philosophy he is considered, along with William Harvey, to be a founder of modern physiology.

[*Image: http://en.wikipedia.org/wiki/File:Frans_Hals_-_Portret_van_Ren%C3%A9_Descartes.jpg*] this approach we obtain

$$\begin{aligned}\frac{d}{dx}[xy] &= \frac{d}{dx}[1]\\ x\frac{d}{dx}[y] + y\frac{d}{dx}[x] &= 0\\ x\frac{dy}{dx} + y &= 0\\ \frac{dy}{dx} &= -\frac{y}{x}\end{aligned}$$

If we now substitute (6) into the last expression, we obtain

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

which agrees with Equation (7). This method of obtaining derivatives is called ***implicit differentiation***.

▶ **Example 2** Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

$$\begin{aligned}\frac{d}{dx}[5y^2 + \sin y] &= \frac{d}{dx}[x^2]\\ 5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] &= 2x\\ 5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} &= 2x\\ 10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} &= 2x\end{aligned}$$

The chain rule was used here because y is a function of x.

Solving for dy/dx we obtain

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y} \tag{8}$$

Note that this formula involves both x and y. In order to obtain a formula for dy/dx that involves x alone, we would have to solve the original equation for y in terms of x and then substitute in (8). However, it is impossible to do this, so we are forced to leave the formula for dy/dx in terms of x and y. ◀

▶ **Example 3** Use implicit differentiation to find d^2y/dx^2 if $4x^2 - 2y^2 = 9$.

Solution. Differentiating both sides of $4x^2 - 2y^2 = 9$ with respect to x yields

$$8x - 4y\frac{dy}{dx} = 0$$

from which we obtain

$$\frac{dy}{dx} = \frac{2x}{y} \tag{9}$$

Differentiating both sides of (9) yields

$$\frac{d^2y}{dx^2} = \frac{(y)(2) - (2x)(dy/dx)}{y^2} \tag{10}$$

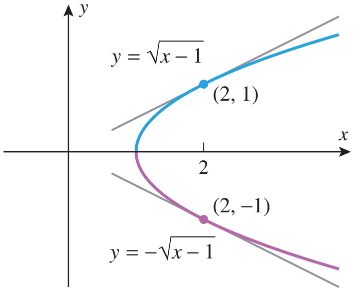
Substituting (9) into (10) and simplifying using the original equation, we obtain

$$\frac{d^2y}{dx^2} = \frac{2y - 2x(2x/y)}{y^2} = \frac{2y^2 - 4x^2}{y^3} = -\frac{9}{y^3}$$ ◀

In Examples 2 and 3, the resulting formulas for dy/dx involved both x and y. Although it is usually more desirable to have the formula for dy/dx expressed in terms of x alone, having the formula in terms of x and y is not an impediment to finding slopes and equations of tangent lines provided the x- and y-coordinates of the point of tangency are known. This is illustrated in the following example.

▶ Example 4 Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

Solution. We could proceed by solving the equation for y in terms of x, and then evaluating the derivative of $y = \sqrt{x-1}$ at $(2, 1)$ and the derivative of $y = -\sqrt{x-1}$ at $(2, -1)$ (Figure 2.7.4). However, implicit differentiation is more efficient since it can be used for the slopes of *both* tangent lines. Differentiating implicitly yields



▲ **Figure 2.7.4**

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$

$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$

$$2y\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the tangent lines to the curve at those points are

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2} \blacktriangleleft$$

▶ Example 5

(a) Use implicit differentiation to find dy/dx for the Folium of Descartes $x^3 + y^3 = 3xy$.

(b) Find an equation for the tangent line to the Folium of Descartes at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(c) At what point(s) in the first quadrant is the tangent line to the Folium of Descartes horizontal?

Solution (a). Differentiating implicitly yields

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[3xy]$$

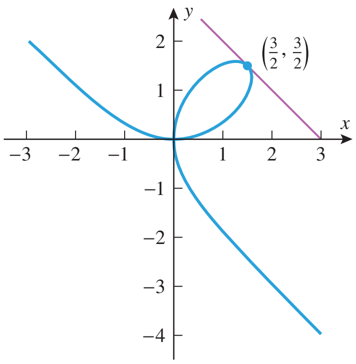
$$3x^2 + 3y^2\frac{dy}{dx} = 3x\frac{dy}{dx} + 3y$$

$$x^2 + y^2\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x} \tag{11}$$

Formula (11) cannot be evaluated at $(0, 0)$ and hence provides no information about the nature of the Folium of Descartes at the origin. Based on the graphs in Figure 2.7.3, what can you say about the differentiability of the implicitly defined functions graphed in blue in parts (*b*) and (*c*) of the figure?



▲ **Figure 2.7.5**

Solution (*b*). At the point $\left(\frac{3}{2}, \frac{3}{2}\right)$, we have $x = \frac{3}{2}$ and $y = \frac{3}{2}$, so from (11) the slope $m_{\tan}$ of the tangent line at this point is

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{\substack{x=3/2\\y=3/2}} = \frac{(3/2) - (3/2)^2}{(3/2)^2 - (3/2)} = -1$$

Thus, the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$ is

$$y - \tfrac{3}{2} = -1\left(x - \tfrac{3}{2}\right) \quad \text{or} \quad x + y = 3$$

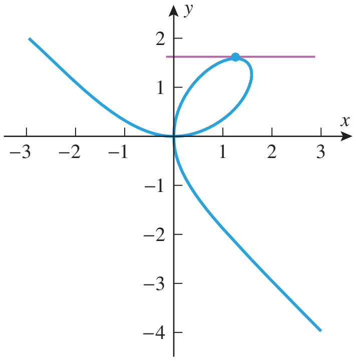
which is consistent with Figure 2.7.5.

Solution (*c*). The tangent line is horizontal at the points where $dy/dx = 0$, and from (11) this occurs only where $y - x^2 = 0$ or

$$y = x^2 \tag{12}$$

Substituting this expression for y in the equation $x^3 + y^3 = 3xy$ for the curve yields

$$\begin{aligned} x^3 + (x^2)^3 &= 3x^3 \\ x^6 - 2x^3 &= 0 \\ x^3(x^3 - 2) &= 0 \end{aligned}$$



▲ **Figure 2.7.6**

whose solutions are $x = 0$ and $x = 2^{1/3}$. From (12), the solutions $x = 0$ and $x = 2^{1/3}$ yield the points $(0, 0)$ and $(2^{1/3}, 2^{2/3})$, respectively. Of these two, only $(2^{1/3}, 2^{2/3})$ is in the first quadrant. Substituting $x = 2^{1/3}$, $y = 2^{2/3}$ into (11) yields

$$\left.\frac{dy}{dx}\right|_{\substack{x=2^{1/3}\\y=2^{2/3}}} = \frac{0}{2^{4/3} - 2^{2/3}} = 0$$

We conclude that $(2^{1/3}, 2^{2/3}) \approx (1.26, 1.59)$ is the only point on the Folium of Descartes in the first quadrant at which the tangent line is horizontal (Figure 2.7.6). ◀

DIFFERENTIABILITY OF FUNCTIONS DEFINED IMPLICITLY

When differentiating implicitly, it is assumed that y represents a differentiable function of x. If this is not so, then the resulting calculations may be nonsense. For example, if we differentiate the equation

$$x^2 + y^2 + 1 = 0 \tag{13}$$

we obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}$$

However, this derivative is meaningless because there are no real values of x and y that satisfy (13) (why?); and hence (13) does not define any real functions implicitly.

The nonsensical conclusion of these computations conveys the importance of knowing whether an equation in x and y that is to be differentiated implicitly actually defines some differentiable function of x implicitly. Unfortunately, this can be a difficult problem, so we will leave the discussion of such matters for more advanced courses in analysis.

✔QUICK CHECK EXERCISES 2.7 *(See page 167 for answers.)*

1. The equation $xy + 2y = 1$ defines implicitly the function $y =$ ________.
2. Use implicit differentiation to find dy/dx for $x^2 - y^3 = xy$.
3. The slope of the tangent line to the graph of $x + y + xy = 3$ at $(1, 1)$ is ________.
4. Use implicit differentiation to find d^2y/dx^2 for $\sin y = x$.

EXERCISE SET 2.7 C CAS

1–2
(a) Find dy/dx by differentiating implicitly.
(b) Solve the equation for y as a function of x, and find dy/dx from that equation.
(c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone. ■

1. $x + xy - 2x^3 = 2$ **2.** $\sqrt{y} - \sin x = 2$

3–12 Find dy/dx by implicit differentiation. ■

3. $x^2 + y^2 = 100$
4. $x^3 + y^3 = 3xy^2$
5. $x^2y + 3xy^3 - x = 3$
6. $x^3y^2 - 5x^2y + x = 1$
7. $\dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}} = 1$
8. $x^2 = \dfrac{x+y}{x-y}$
9. $\sin(x^2y^2) = x$
10. $\cos(xy^2) = y$
11. $\tan^3(xy^2 + y) = x$
12. $\dfrac{xy^3}{1 + \sec y} = 1 + y^4$

13–18 Find d^2y/dx^2 by implicit differentiation. ■

13. $2x^2 - 3y^2 = 4$
14. $x^3 + y^3 = 1$
15. $x^3y^3 - 4 = 0$
16. $xy + y^2 = 2$
17. $y + \sin y = x$
18. $x \cos y = y$

19–20 Find the slope of the tangent line to the curve at the given points in two ways: first by solving for y in terms of x and differentiating and then by implicit differentiation. ■

19. $x^2 + y^2 = 1$; $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$
20. $y^2 - x + 1 = 0$; $(10, 3)$, $(10, -3)$

21–24 True–False Determine whether the statement is true or false. Explain your answer. ■

21. If an equation in x and y defines a function $y = f(x)$ implicitly, then the graph of the equation and the graph of f are identical.

22. The function
$$f(x) = \begin{cases} \sqrt{1-x^2}, & 0 < x \le 1 \\ -\sqrt{1-x^2}, & -1 \le x \le 0 \end{cases}$$
is defined implicitly by the equation $x^2 + y^2 = 1$.

23. The function $|x|$ is not defined implicitly by the equation $(x+y)(x-y) = 0$.

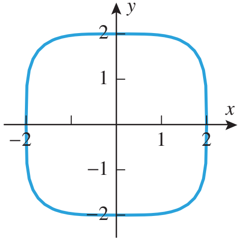
24. If y is defined implicitly as a function of x by the equation $x^2 + y^2 = 1$, then $dy/dx = -x/y$.

25. This exercise extends the Power Rule (Theorem 2.3.2) from positive integer exponents to arbitrary integer exponents. Suppose that $m < 0$ is an integer. Use Theorem 2.3.2 and implicit differentiation to prove that
$$\frac{d}{dx}(x^m) = mx^{m-1}$$

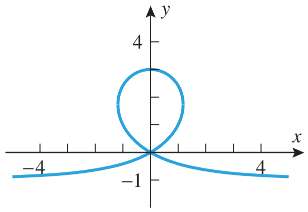
26. This exercise extends the Power Rule for integer exponents (Exercise 25) to rational exponents. Suppose that $r = m/n$ where m and n are integers with $n > 0$. Use implicit differentiation and the result of Exercise 25 to prove that
$$\frac{d}{dx}(x^r) = rx^{r-1}$$

27–30 Use implicit differentiation to find the slope of the tangent line to the curve at the specified point, and check that your answer is consistent with the accompanying graph. ■

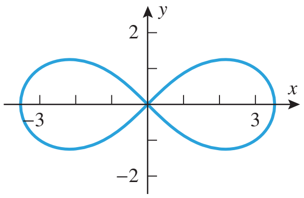
27. $x^4 + y^4 = 16$; $(1, \sqrt[4]{15})$ [***Lamé's special quartic***]
28. $y^3 + yx^2 + x^2 - 3y^2 = 0$; $(0, 3)$ [***trisectrix***]
29. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$; $(3, 1)$ [***lemniscate***]
30. $x^{2/3} + y^{2/3} = 4$; $(-1, 3\sqrt{3})$ [***four-cusped hypocycloid***]



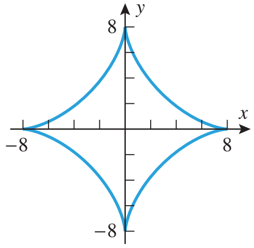
▲ **Figure Ex-27**



▲ **Figure Ex-28**



▲ **Figure Ex-29**



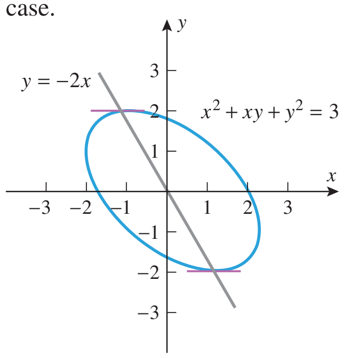
▲ **Figure Ex-30**

31–34 Use implicit differentiation to find the specified derivative. ■

31. $a^4 - t^4 = 6a^2t$; da/dt
32. $\sqrt{u} + \sqrt{v} = 5$; du/dv
33. $a^2\omega^2 + b^2\lambda^2 = 1$ (a, b constants); $d\omega/d\lambda$
34. $y = \sin x$; dx/dy

FOCUS ON CONCEPTS

35. In the accompanying figure, it appears that the ellipse $x^2 + xy + y^2 = 3$ has horizontal tangent lines at the points of intersection of the ellipse and the line $y = -2x$. Use implicit differentiation to explain why this is the case.



◀ **Figure Ex-35**

36. (a) A student claims that the ellipse $x^2 - xy + y^2 = 1$ has a horizontal tangent line at the point $(1, 1)$. Without doing any computations, explain why the student's claim must be incorrect.
(b) Find all points on the ellipse $x^2 - xy + y^2 = 1$ at which the tangent line is horizontal.

C **37.** (a) Use the implicit plotting capability of a CAS to graph the equation $y^4 + y^2 = x(x - 1)$.
(b) Use implicit differentiation to help explain why the graph in part (a) has no horizontal tangent lines.
(c) Solve the equation $y^4 + y^2 = x(x - 1)$ for x in terms of y and explain why the graph in part (a) consists of two parabolas.

38. Use implicit differentiation to find all points on the graph of $y^4 + y^2 = x(x - 1)$ at which the tangent line is vertical.

39. Find the values of a and b for the curve $x^2y + ay^2 = b$ if the point $(1, 1)$ is on its graph and the tangent line at $(1, 1)$ has the equation $4x + 3y = 7$.

40. At what point(s) is the tangent line to the curve $y^3 = 2x^2$ perpendicular to the line $x + 2y - 2 = 0$?

41–42 These exercises deal with the rotated ellipse C whose equation is $x^2 - xy + y^2 = 4$. ■

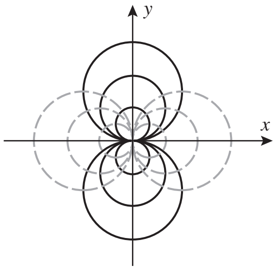
41. Show that the line $y = x$ intersects C at two points P and Q and that the tangent lines to C at P and Q are parallel.

42. Prove that if $P(a, b)$ is a point on C, then so is $Q(-a, -b)$ and that the tangent lines to C through P and through Q are parallel.

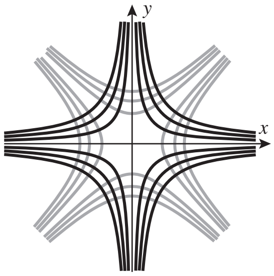
43–44 Two curves are said to be ***orthogonal*** if their tangent lines are perpendicular at each point of intersection, and two families of curves are said to be ***orthogonal trajectories*** of one another if each member of one family is orthogonal to each member of the other family. This terminology is used in these exercises. ■

43. The accompanying figure shows some typical members of the families of circles $x^2 + (y - c)^2 = c^2$ (black curves) and $(x - k)^2 + y^2 = k^2$ (gray curves). Show that these families are orthogonal trajectories of one another. [*Hint:* For the tangent lines to be perpendicular at a point of intersection, the slopes of those tangent lines must be negative reciprocals of one another.]

44. The accompanying figure shows some typical members of the families of hyperbolas $xy = c$ (black curves) and $x^2 - y^2 = k$ (gray curves), where $c \neq 0$ and $k \neq 0$. Use the hint in Exercise 43 to show that these families are orthogonal trajectories of one another.



▲ **Figure Ex-43**



▲ **Figure Ex-44**

C **45.** (a) Use the implicit plotting capability of a CAS to graph the curve C whose equation is $x^3 - 2xy + y^3 = 0$.
(b) Use the graph in part (a) to estimate the x-coordinates of a point in the first quadrant that is on C and at which the tangent line to C is parallel to the x-axis.
(c) Find the exact value of the x-coordinate in part (b).

C **46.** (a) Use the implicit plotting capability of a CAS to graph the curve C whose equation is $x^3 - 2xy + y^3 = 0$.
(b) Use the graph to guess the coordinates of a point in the first quadrant that is on C and at which the tangent line to C is parallel to the line $y = -x$.
(c) Use implicit differentiation to verify your conjecture in part (b).

47. Find dy/dx if

$$2y^3t + t^3y = 1 \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{\cos t}$$

48. Find equations for two lines through the origin that are tangent to the ellipse $2x^2 - 4x + y^2 + 1 = 0$.

49. **Writing** Write a paragraph that compares the concept of an *explicit* definition of a function with that of an *implicit* definition of a function.

50. **Writing** A student asks: "Suppose implicit differentiation yields an undefined expression at a point. Does this mean that dy/dx is undefined at that point?" Using the equation $x^2 - 2xy + y^2 = 0$ as a basis for your discussion, write a paragraph that answers the student's question.

✔ QUICK CHECK ANSWERS 2.7

1. $\dfrac{1}{x+2}$ **2.** $\dfrac{dy}{dx} = \dfrac{2x - y}{x + 3y^2}$ **3.** -1 **4.** $\dfrac{d^2y}{dx^2} = \sec^2 y \tan y$

2.8 RELATED RATES

In this section we will study related rates problems. In such problems one tries to find the rate at which some quantity is changing by relating the quantity to other quantities whose rates of change are known.

DIFFERENTIATING EQUATIONS TO RELATE RATES

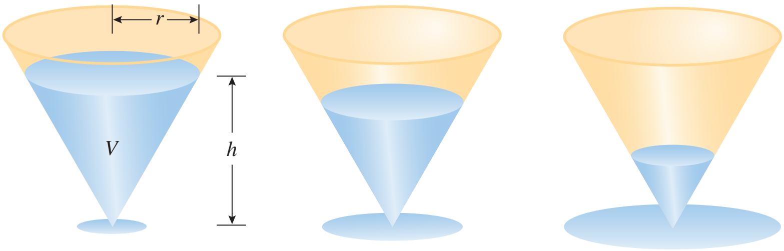
Figure 2.8.1 shows a liquid draining through a conical filter. As the liquid drains, its volume V, height h, and radius r are functions of the elapsed time t, and at each instant these variables are related by the equation

$$V = \frac{\pi}{3}r^2h$$

If we were interested in finding the rate of change of the volume V with respect to the time t, we could begin by differentiating both sides of this equation with respect to t to obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left[r^2\frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Thus, to find dV/dt at a specific time t from this equation we would need to have values for r, h, dh/dt, and dr/dt at that time. This is called a ***related rates problem*** because the goal is to find an unknown rate of change by *relating* it to other variables whose values and whose rates of change at time t are known or can be found in some way. Let us begin with a simple example.



▲ **Figure 2.8.1**

Arni Katz/Phototake

Oil spill from a ruptured tanker.

▶ **Example 1** Suppose that x and y are differentiable functions of t and are related by the equation $y = x^3$. Find dy/dt at time $t = 1$ if $x = 2$ and $dx/dt = 4$ at time $t = 1$.

Solution. Using the chain rule to differentiate both sides of the equation $y = x^3$ with respect to t yields

$$\frac{dy}{dt} = \frac{d}{dt}[x^3] = 3x^2\frac{dx}{dt}$$

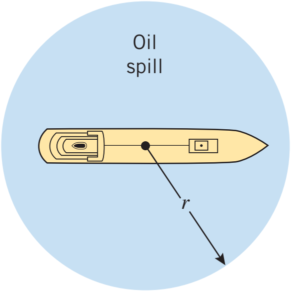
Thus, the value of dy/dt at time $t = 1$ is

$$\left.\frac{dy}{dt}\right|_{t=1} = 3(2)^2\left.\frac{dx}{dt}\right|_{t=1} = 12 \cdot 4 = 48 \blacktriangleleft$$

▶ **Example 2** Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

Solution. Let

t = number of seconds elapsed from the time of the spill
r = radius of the spill in feet after t seconds
A = area of the spill in square feet after t seconds



▲ **Figure 2.8.2**

(Figure 2.8.2). We know the rate at which the radius is increasing, and we want to find the rate at which the area is increasing at the instant when $r = 60$; that is, we want to find

$$\left.\frac{dA}{dt}\right|_{r=60} \quad \text{given that} \quad \frac{dr}{dt} = 2 \text{ ft/s}$$

This suggests that we look for an equation relating A and r that we can differentiate with respect to t to produce a relationship between dA/dt and dr/dt. But A is the area of a circle of radius r, so

$$A = \pi r^2 \tag{1}$$

Differentiating both sides of (1) with respect to t yields

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \tag{2}$$

Thus, when $r = 60$ the area of the spill is increasing at the rate of

$$\left.\frac{dA}{dt}\right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2/\text{s} \approx 754 \text{ ft}^2/\text{s} \ ◂$$

With some minor variations, the method used in Example 2 can be used to solve a variety of related rates problems. We can break the method down into five steps.

A Strategy for Solving Related Rates Problems

Step 1. Assign letters to all quantities that vary with time and any others that seem relevant to the problem. Give a definition for each letter.

Step 2. Identify the rates of change that are known and the rate of change that is to be found. Interpret each rate as a derivative.

Step 3. Find an equation that relates the variables whose rates of change were identified in Step 2. To do this, it will often be helpful to draw an appropriately labeled figure that illustrates the relationship.

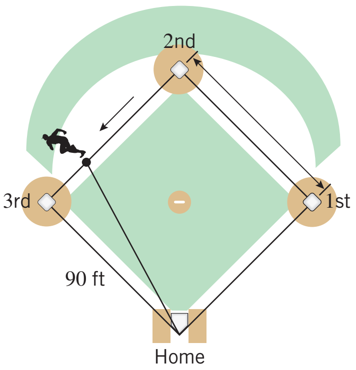
Step 4. Differentiate both sides of the equation obtained in Step 3 with respect to time to produce a relationship between the known rates of change and the unknown rate of change.

Step 5. *After* completing Step 4, substitute all known values for the rates of change and the variables, and then solve for the unknown rate of change.

WARNING

We have italicized the word "After" in Step 5 because it is a common error to substitute numerical values before performing the differentiation. For instance, in Example 2 had we substituted the known value of $r = 60$ in (1) before differentiating, we would have obtained $dA/dt = 0$, which is obviously incorrect.

▶ **Example 3** A baseball diamond is a square whose sides are 90 ft long (Figure 2.8.3). Suppose that a player running from second base to third base has a speed of 30 ft/s at the instant when he is 20 ft from third base. At what rate is the player's distance from home plate changing at that instant?



▲ **Figure 2.8.3**

Solution. We are given a constant speed with which the player is approaching third base, and we want to find the rate of change of the distance between the player and home plate at a particular instant. Thus, let

t = number of seconds since the player left second base
x = distance in feet from the player to third base
y = distance in feet from the player to home plate

(Figure 2.8.4). Thus, we want to find

$$\left.\frac{dy}{dt}\right|_{x=20} \quad \text{given that} \quad \left.\frac{dx}{dt}\right|_{x=20} = -30 \text{ ft/s}$$

The quantity

$$\left.\frac{dx}{dt}\right|_{x=20}$$

is negative because x is decreasing with respect to t.

As suggested by Figure 2.8.4, an equation relating the variables x and y can be obtained using the Theorem of Pythagoras:

$$x^2 + 90^2 = y^2 \tag{3}$$

Differentiating both sides of this equation with respect to t yields

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

from which we obtain

$$\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \tag{4}$$

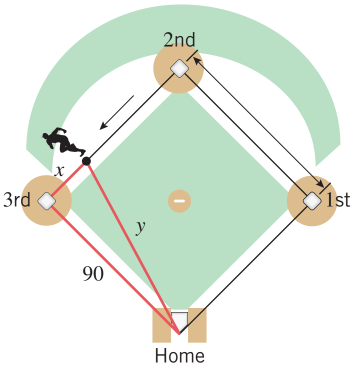
When $x = 20$, it follows from (3) that

$$y = \sqrt{20^2 + 90^2} = \sqrt{8500} = 10\sqrt{85}$$

so that (4) yields

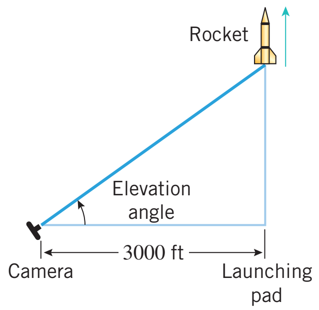
$$\left.\frac{dy}{dt}\right|_{x=20} = \frac{20}{10\sqrt{85}}(-30) = -\frac{60}{\sqrt{85}} \approx -6.51 \text{ ft/s}$$

The negative sign in the answer tells us that y is decreasing, which makes sense physically from Figure 2.8.4. ◀



▲ **Figure 2.8.4**

▶ **Example 4** In Figure 2.8.5 we have shown a camera mounted at a point 3000 ft from the base of a rocket launching pad. If the rocket is rising vertically at 880 ft/s when it is 4000 ft above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?



▲ **Figure 2.8.5**

Solution. Let

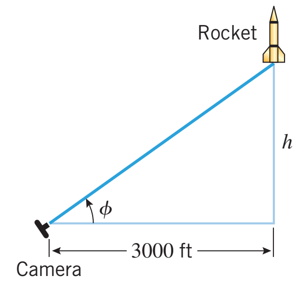
t = number of seconds elapsed from the time of launch
ϕ = camera elevation angle in radians after t seconds
h = height of the rocket in feet after t seconds

(Figure 2.8.6). At each instant the rate at which the camera elevation angle must change is $d\phi/dt$, and the rate at which the rocket is rising is dh/dt. We want to find

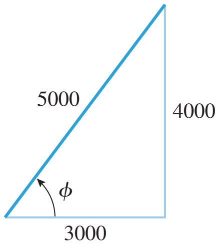
$$\left.\frac{d\phi}{dt}\right|_{h=4000} \quad \text{given that} \quad \left.\frac{dh}{dt}\right|_{h=4000} = 880 \text{ ft/s}$$

From Figure 2.8.6 we see that

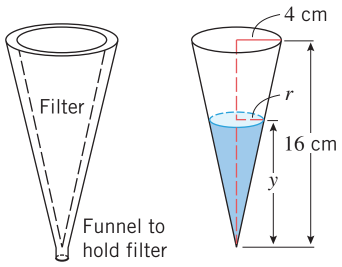
$$\tan\phi = \frac{h}{3000} \tag{5}$$



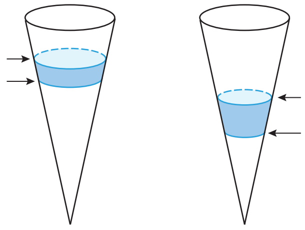
▲ **Figure 2.8.6**



▲ **Figure 2.8.7**



▲ **Figure 2.8.8**



The same volume has drained, but the change in height is greater near the bottom than near the top.

▲ **Figure 2.8.9**

Differentiating both sides of (5) with respect to t yields

$$(\sec^2\phi)\frac{d\phi}{dt} = \frac{1}{3000}\frac{dh}{dt} \tag{6}$$

When $h = 4000$, it follows that

$$(\sec\phi)\Big|_{h=4000} = \frac{5000}{3000} = \frac{5}{3}$$

(see Figure 2.8.7), so that from (6)

$$\left(\frac{5}{3}\right)^2 \frac{d\phi}{dt}\Big|_{h=4000} = \frac{1}{3000}\cdot 880 = \frac{22}{75}$$

$$\frac{d\phi}{dt}\Big|_{h=4000} = \frac{22}{75}\cdot\frac{9}{25} = \frac{66}{625} \approx 0.11 \text{ rad/s} \approx 6.05 \text{ deg/s} \blacktriangleleft$$

▶ **Example 5** Suppose that liquid is to be cleared of sediment by allowing it to drain through a conical filter that is 16 cm high and has a radius of 4 cm at the top (Figure 2.8.8). Suppose also that the liquid is forced out of the cone at a constant rate of 2 cm^3/min.

(a) Do you think that the depth of the liquid will decrease at a constant rate? Give a verbal argument that justifies your conclusion.

(b) Find a formula that expresses the rate at which the depth of the liquid is changing in terms of the depth, and use that formula to determine whether your conclusion in part (a) is correct.

(c) At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep?

Solution (a). For the volume of liquid to decrease by a *fixed amount*, it requires a greater decrease in depth when the cone is close to empty than when it is almost full (Figure 2.8.9). This suggests that for the volume to decrease at a constant rate, the depth must decrease at an increasing rate.

Solution (b). Let

t = time elapsed from the initial observation (min)
V = volume of liquid in the cone at time t (cm^3)
y = depth of the liquid in the cone at time t (cm)
r = radius of the liquid surface at time t (cm)

(Figure 2.8.8). At each instant the rate at which the volume of liquid is changing is dV/dt, and the rate at which the depth is changing is dy/dt. We want to express dy/dt in terms of y given that dV/dt has a constant value of $dV/dt = -2$. (We must use a minus sign here because V *decreases* as t increases.)

From the formula for the volume of a cone, the volume V, the radius r, and the depth y are related by

$$V = \tfrac{1}{3}\pi r^2 y \tag{7}$$

If we differentiate both sides of (7) with respect to t, the right side will involve the quantity dr/dt. Since we have no direct information about dr/dt, it is desirable to eliminate r from (7) before differentiating. This can be done using similar triangles. From Figure 2.8.8 we see that

$$\frac{r}{y} = \frac{4}{16} \quad \text{or} \quad r = \frac{1}{4}y$$

Substituting this expression in (7) gives

$$V = \frac{\pi}{48} y^3 \tag{8}$$

Differentiating both sides of (8) with respect to t we obtain

$$\frac{dV}{dt} = \frac{\pi}{48}\left(3y^2 \frac{dy}{dt}\right)$$

or

$$\frac{dy}{dt} = \frac{16}{\pi y^2}\frac{dV}{dt} = \frac{16}{\pi y^2}(-2) = -\frac{32}{\pi y^2} \tag{9}$$

which expresses dy/dt in terms of y. The minus sign tells us that y is decreasing with time, and

$$\left|\frac{dy}{dt}\right| = \frac{32}{\pi y^2}$$

tells us how fast y is decreasing. From this formula we see that $|dy/dt|$ increases as y decreases, which confirms our conjecture in part (a) that the depth of the liquid decreases more quickly as the liquid drains through the filter.

Solution (c). The rate at which the depth is changing when the depth is 8 cm can be obtained from (9) with $y = 8$:

$$\left.\frac{dy}{dt}\right|_{y=8} = -\frac{32}{\pi(8^2)} = -\frac{1}{2\pi} \approx -0.16 \text{ cm/min}$$ ◀

✓QUICK CHECK EXERCISES 2.8

(See page 175 for answers.)

1. If $A = x^2$ and $\dfrac{dx}{dt} = 3$, find $\left.\dfrac{dA}{dt}\right|_{x=10}$.
2. If $A = x^2$ and $\dfrac{dA}{dt} = 3$, find $\left.\dfrac{dx}{dt}\right|_{x=10}$.
3. A 10-foot ladder stands on a horizontal floor and leans against a vertical wall. Use x to denote the distance along the floor from the wall to the foot of the ladder, and use y to denote the distance along the wall from the floor to the top of the ladder. If the foot of the ladder is dragged away from the wall, find an equation that relates rates of change of x and y with respect to time.
4. Suppose that a block of ice in the shape of a right circular cylinder melts so that it retains its cylindrical shape. Find an equation that relates the rates of change of the volume (V), height (h), and radius (r) of the block of ice.

EXERCISE SET 2.8

1–4 Both x and y denote functions of t that are related by the given equation. Use this equation and the given derivative information to find the specified derivative. ■

1. Equation: $y = 3x + 5$.
   (a) Given that $dx/dt = 2$, find dy/dt when $x = 1$.
   (b) Given that $dy/dt = -1$, find dx/dt when $x = 0$.
2. Equation: $x + 4y = 3$.
   (a) Given that $dx/dt = 1$, find dy/dt when $x = 2$.
   (b) Given that $dy/dt = 4$, find dx/dt when $x = 3$.
3. Equation: $4x^2 + 9y^2 = 1$.
   (a) Given that $dx/dt = 3$, find dy/dt when $(x, y) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{3\sqrt{2}}\right)$.
   (b) Given that $dy/dt = 8$, find dx/dt when $(x, y) = \left(\frac{1}{3}, -\frac{\sqrt{5}}{9}\right)$.
4. Equation: $x^2 + y^2 = 2x + 4y$.
   (a) Given that $dx/dt = -5$, find dy/dt when $(x, y) = (3, 1)$.
   (b) Given that $dy/dt = 6$, find dx/dt when $(x, y) = (1 + \sqrt{2}, 2 + \sqrt{3})$.

FOCUS ON CONCEPTS

5. Let A be the area of a square whose sides have length x, and assume that x varies with the time t.
   (a) Draw a picture of the square with the labels A and x placed appropriately.
   (b) Write an equation that relates A and x. *(cont.)*

(c) Use the equation in part (b) to find an equation that relates dA/dt and dx/dt.

(d) At a certain instant the sides are 3 ft long and increasing at a rate of 2 ft/min. How fast is the area increasing at that instant?

6. In parts (a)–(d), let A be the area of a circle of radius r, and assume that r increases with the time t.

(a) Draw a picture of the circle with the labels A and r placed appropriately.

(b) Write an equation that relates A and r.

(c) Use the equation in part (b) to find an equation that relates dA/dt and dr/dt.

(d) At a certain instant the radius is 5 cm and increasing at the rate of 2 cm/s. How fast is the area increasing at that instant?

7. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time.

(a) How are dV/dt, dh/dt, and dr/dt related?

(b) At a certain instant, the height is 6 in and increasing at 1 in/s, while the radius is 10 in and decreasing at 1 in/s. How fast is the volume changing at that instant? Is the volume increasing or decreasing at that instant?

8. Let l be the length of a diagonal of a rectangle whose sides have lengths x and y, and assume that x and y vary with time.

(a) How are dl/dt, dx/dt, and dy/dt related?

(b) If x increases at a constant rate of $\frac{1}{2}$ ft/s and y decreases at a constant rate of $\frac{1}{4}$ ft/s, how fast is the size of the diagonal changing when $x = 3$ ft and $y = 4$ ft? Is the diagonal increasing or decreasing at that instant?

9. Let θ (in radians) be an acute angle in a right triangle, and let x and y, respectively, be the lengths of the sides adjacent to and opposite θ. Suppose also that x and y vary with time.

(a) How are $d\theta/dt$, dx/dt, and dy/dt related?

(b) At a certain instant, $x = 2$ units and is increasing at 1 unit/s, while $y = 2$ units and is decreasing at $\frac{1}{4}$ unit/s. How fast is θ changing at that instant? Is θ increasing or decreasing at that instant?

10. Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

11. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

12. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

13. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

14. A spherical balloon is inflated so that its volume is increasing at the rate of 3 ft^3/min. How fast is the diameter of the balloon increasing when the radius is 1 ft?

15. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

16. A 17 ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

17. A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

18. A 10 ft plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being pushed toward the wall at the rate of 6 in/s, how fast is the acute angle that the plank makes with the ground increasing?

19. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

20. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 mi from the launchpad. How fast is the rocket rising when it is 4 mi high and its distance from the radar station is increasing at a rate of 2000 mi/h?

21. For the camera and rocket shown in Figure 2.8.5, at what rate is the camera-to-rocket distance changing when the rocket is 4000 ft up and rising vertically at 880 ft/s?

22. For the camera and rocket shown in Figure 2.8.5, at what rate is the rocket rising when the elevation angle is $\pi/4$ radians and increasing at a rate of 0.2 rad/s?

23. A satellite is in an elliptical orbit around the Earth. Its distance r (in miles) from the center of the Earth is given by

$$r = \frac{4995}{1 + 0.12\cos\theta}$$

where θ is the angle measured from the point on the orbit nearest the Earth's surface (see the accompanying figure on the next page).

(a) Find the altitude of the satellite at ***perigee*** (the point nearest the surface of the Earth) and at ***apogee*** (the point farthest from the surface of the Earth). Use 3960 mi as the radius of the Earth.

(b) At the instant when θ is 120°, the angle θ is increasing at the rate of 2.7°/min. Find the altitude of the satellite and the rate at which the altitude is changing at this instant. Express the rate in units of mi/min.

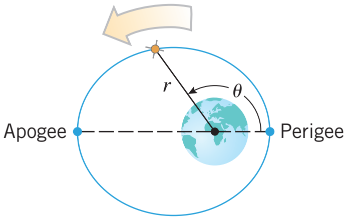


Figure Ex-23

24. An aircraft is flying horizontally at a constant height of 4000 ft above a fixed observation point (see the accompanying figure). At a certain instant the angle of elevation θ is 30° and decreasing, and the speed of the aircraft is 300 mi/h.

(a) How fast is θ decreasing at this instant? Express the result in units of deg/s.

(b) How fast is the distance between the aircraft and the observation point changing at this instant? Express the result in units of ft/s. Use 1 mi = 5280 ft.

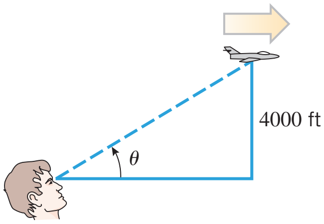


Figure Ex-24

25. A conical water tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows into the tank at a rate of 20 ft^3/min, how fast is the depth of the water increasing when the water is 16 ft deep?

26. Grain pouring from a chute at the rate of 8 ft^3/min forms a conical pile whose height is always twice its radius. How fast is the height of the pile increasing at the instant when the pile is 6 ft high?

27. Sand pouring from a chute forms a conical pile whose height is always equal to the diameter. If the height increases at a constant rate of 5 ft/min, at what rate is sand pouring from the chute when the pile is 10 ft high?

28. Wheat is poured through a chute at the rate of 10 ft^3/min and falls in a conical pile whose bottom radius is always half the altitude. How fast will the circumference of the base be increasing when the pile is 8 ft high?

29. An aircraft is climbing at a 30° angle to the horizontal. How fast is the aircraft gaining altitude if its speed is 500 mi/h?

30. A boat is pulled into a dock by means of a rope attached to a pulley on the dock (see the accompanying figure). The rope is attached to the bow of the boat at a point 10 ft below the pulley. If the rope is pulled through the pulley at a rate of 20 ft/min, at what rate will the boat be approaching the dock when 125 ft of rope is out?

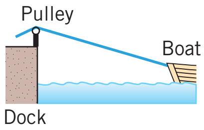


Figure Ex-30

31. For the boat in Exercise 30, how fast must the rope be pulled if we want the boat to approach the dock at a rate of 12 ft/min at the instant when 125 ft of rope is out?

32. A man 6 ft tall is walking at the rate of 3 ft/s toward a streetlight 18 ft high (see the accompanying figure).

(a) At what rate is his shadow length changing?

(b) How fast is the tip of his shadow moving?

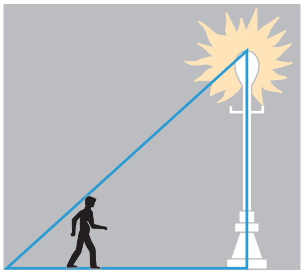


Figure Ex-32

33. A beacon that makes one revolution every 10 s is located on a ship anchored 4 kilometers from a straight shoreline. How fast is the beam moving along the shoreline when it makes an angle of 45° with the shore?

34. An aircraft is flying at a constant altitude with a constant speed of 600 mi/h. An antiaircraft missile is fired on a straight line perpendicular to the flight path of the aircraft so that it will hit the aircraft at a point P (see the accompanying figure). At the instant the aircraft is 2 mi from the impact point P the missile is 4 mi from P and flying at 1200 mi/h. At that instant, how rapidly is the distance between missile and aircraft decreasing?

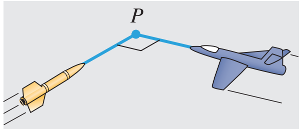


Figure Ex-34

35. Solve Exercise 34 under the assumption that the angle between the flight paths is 120° instead of the assumption that the paths are perpendicular. [*Hint:* Use the law of cosines.]

36. A police helicopter is flying due north at 100 mi/h and at a constant altitude of $\frac{1}{2}$ mi. Below, a car is traveling west on a highway at 75 mi/h. At the moment the helicopter crosses over the highway the car is 2 mi east of the helicopter.

(a) How fast is the distance between the car and helicopter changing at the moment the helicopter crosses the highway?

(b) Is the distance between the car and helicopter increasing or decreasing at that moment?

37. A particle is moving along the curve whose equation is
$$\frac{xy^3}{1+y^2} = \frac{8}{5}$$
Assume that the x-coordinate is increasing at the rate of 6 units/s when the particle is at the point $(1, 2)$.
(a) At what rate is the y-coordinate of the point changing at that instant?
(b) Is the particle rising or falling at that instant?

38. A point P is moving along the curve whose equation is $y = \sqrt{x^3+17}$. When P is at $(2, 5)$, y is increasing at the rate of 2 units/s. How fast is x changing?

39. A point P is moving along the line whose equation is $y = 2x$. How fast is the distance between P and the point $(3, 0)$ changing at the instant when P is at $(3, 6)$ if x is decreasing at the rate of 2 units/s at that instant?

40. A point P is moving along the curve whose equation is $y = \sqrt{x}$. Suppose that x is increasing at the rate of 4 units/s when $x = 3$.
(a) How fast is the distance between P and the point $(2, 0)$ changing at this instant?
(b) How fast is the angle of inclination of the line segment from P to $(2, 0)$ changing at this instant?

41. A particle is moving along the curve $y = x/(x^2+1)$. Find all values of x at which the rate of change of x with respect to time is three times that of y. [Assume that dx/dt is never zero.]

42. A particle is moving along the curve $16x^2 + 9y^2 = 144$. Find all points (x, y) at which the rates of change of x and y with respect to time are equal. [Assume that dx/dt and dy/dt are never both zero at the same point.]

43. The ***thin lens equation*** in physics is
$$\frac{1}{s} + \frac{1}{S} = \frac{1}{f}$$
where s is the object distance from the lens, S is the image distance from the lens, and f is the focal length of the lens. Suppose that a certain lens has a focal length of 6 cm and that an object is moving toward the lens at the rate of 2 cm/s. How fast is the image distance changing at the instant when the object is 10 cm from the lens? Is the image moving away from the lens or toward the lens?

44. Water is stored in a cone-shaped reservoir (vertex down). Assuming the water evaporates at a rate proportional to the surface area exposed to the air, show that the depth of the water will decrease at a constant rate that does not depend on the dimensions of the reservoir.

45. A meteor enters the Earth's atmosphere and burns up at a rate that, at each instant, is proportional to its surface area. Assuming that the meteor is always spherical, show that the radius decreases at a constant rate.

46. On a certain clock the minute hand is 4 in long and the hour hand is 3 in long. How fast is the distance between the tips of the hands changing at 9 o'clock?

47. Coffee is poured at a uniform rate of 20 cm^3/s into a cup whose inside is shaped like a truncated cone (see the accompanying figure). If the upper and lower radii of the cup are 4 cm and 2 cm and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up? [*Hint:* Extend the cup downward to form a cone.]

◀ **Figure Ex-47**

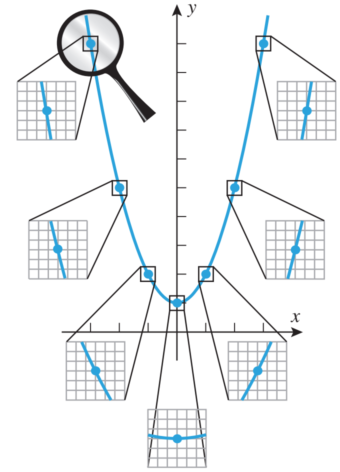
✔ QUICK CHECK ANSWERS 2.8

1. 60 **2.** $\frac{3}{20}$ **3.** $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$ **4.** $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}$

2.9 LOCAL LINEAR APPROXIMATION; DIFFERENTIALS

In this section we will show how derivatives can be used to approximate nonlinear functions by linear functions. Also, up to now we have been interpreting dy/dx as a single entity representing the derivative. In this section we will define the quantities dx and dy themselves, thereby allowing us to interpret dy/dx as an actual ratio.

Recall from Section 2.2 that if a function f is differentiable at x_0, then a sufficiently magnified portion of the graph of f centered at the point $P(x_0, f(x_0))$ takes on the appearance of a straight line segment. Figure 2.9.1 illustrates this at several points on the graph of $y = x^2 + 1$. For this reason, a function that is differentiable at x_0 is sometimes said to be ***locally linear*** at x_0.



Magnifying portions of the graph of $y = x^2 + 1$

▲ **Figure 2.9.1**

The line that best approximates the graph of f in the vicinity of $P(x_0, f(x_0))$ is the tangent line to the graph of f at x_0, given by the equation

$$y = f(x_0) + f'(x_0)(x - x_0)$$

[see Formula (3) of Section 2.2]. Thus, for values of x near x_0 we can approximate values of $f(x)$ by

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{1}$$

This is called the ***local linear approximation*** of f at x_0. This formula can also be expressed in terms of the increment $\Delta x = x - x_0$ as

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \tag{2}$$

▶ Example 1

(a) Find the local linear approximation of $f(x) = \sqrt{x}$ at $x_0 = 1$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{1.1}$, and compare your approximation to the result produced directly by a calculating utility.

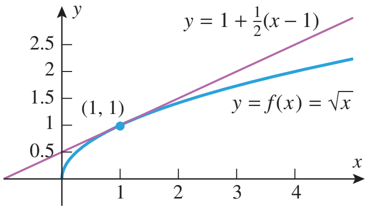
Solution (a). Since $f'(x) = 1/(2\sqrt{x})$, it follows from (1) that the local linear approximation of $\sqrt{x}$ at a point x_0 is

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}(x - x_0)$$

Thus, the local linear approximation at $x_0 = 1$ is

$$\sqrt{x} \approx 1 + \tfrac{1}{2}(x - 1) \tag{3}$$

The graphs of $y = \sqrt{x}$ and the local linear approximation $y = 1 + \frac{1}{2}(x - 1)$ are shown in Figure 2.9.2.



▲ **Figure 2.9.2**

Solution (b). Applying (3) with $x = 1.1$ yields

$$\sqrt{1.1} \approx 1 + \tfrac{1}{2}(1.1 - 1) = 1.05$$

Since the tangent line $y = 1 + \frac{1}{2}(x - 1)$ in Figure 2.9.2 lies above the graph of $f(x) = \sqrt{x}$, we would expect this approximation to be slightly too large. This expectation is confirmed by the calculator approximation $\sqrt{1.1} \approx 1.04881$. ◀

Examples 1 and 2 illustrate important ideas and are not meant to suggest that you should use local linear approximations for computations that your calculating utility can perform. The main application of local linear approximation is in modeling problems where it is useful to replace complicated functions by simpler ones.

▶ Example 2

(a) Find the local linear approximation of $f(x) = \sin x$ at $x_0 = 0$.

(b) Use the local linear approximation obtained in part (a) to approximate $\sin 2°$, and compare your approximation to the result produced directly by your calculating device.

Solution (a). Since $f'(x) = \cos x$, it follows from (1) that the local linear approximation of $\sin x$ at a point x_0 is

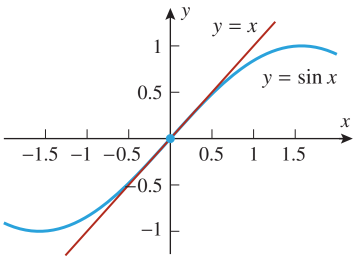
$$\sin x \approx \sin x_0 + (\cos x_0)(x - x_0)$$

Thus, the local linear approximation at $x_0 = 0$ is

$$\sin x \approx \sin 0 + (\cos 0)(x - 0)$$

which simplifies to

$$\sin x \approx x \tag{4}$$



▲ **Figure 2.9.3**

Solution (b). The variable x in (4) is in radian measure, so we must first convert 2° to radians before we can apply this approximation. Since

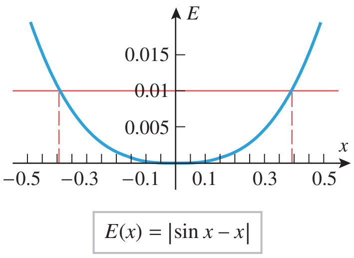
$$2^\circ = 2\left(\frac{\pi}{180}\right) = \frac{\pi}{90} \approx 0.0349066 \text{ radian}$$

it follows from (4) that $\sin 2^\circ \approx 0.0349066$. Comparing the two graphs in Figure 2.9.3, we would expect this approximation to be slightly larger than the exact value. The calculator approximation $\sin 2^\circ \approx 0.0348995$ shows that this is indeed the case. ◀

ERROR IN LOCAL LINEAR APPROXIMATIONS

As a general rule, the accuracy of the local linear approximation to $f(x)$ at x_0 will deteriorate as x gets progressively farther from x_0. To illustrate this for the approximation $\sin x \approx x$ in Example 2, let us graph the function

$$E(x) = |\sin x - x|$$



▲ **Figure 2.9.4**

which is the absolute value of the error in the approximation (Figure 2.9.4).

In Figure 2.9.4, the graph shows how the absolute error in the local linear approximation of $\sin x$ increases as x moves progressively farther from 0 in either the positive or negative direction. The graph also tells us that for values of x between the two vertical lines, the absolute error does not exceed 0.01. Thus, for example, we could use the local linear approximation $\sin x \approx x$ for all values of x in the interval $-0.35 < x < 0.35$ (radians) with confidence that the approximation is within ±0.01 of the exact value.

DIFFERENTIALS

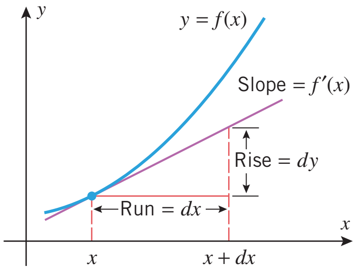
Newton and Leibniz each used a different notation when they published their discoveries of calculus, thereby creating a notational divide between Britain and the European continent that lasted for more than 50 years. The ***Leibniz notation*** dy/dx eventually prevailed because it suggests correct formulas in a natural way, the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

being a good example.

Up to now we have interpreted dy/dx as a single entity representing the derivative of y with respect to x; the symbols "dy" and "dx," which are called ***differentials***, have had no meanings attached to them. Our next goal is to define these symbols in such a way that dy/dx can be treated as an actual ratio. To do this, assume that f is differentiable at a point x, *define* dx to be an independent variable that can have any real value, and *define* dy by the formula

$$dy = f'(x)\,dx \tag{5}$$



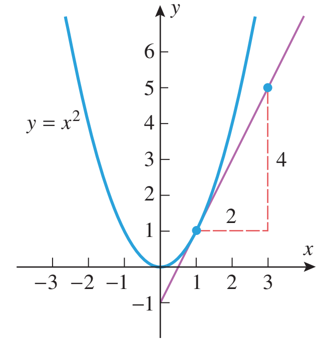
▲ **Figure 2.9.5**

If $dx \neq 0$, then we can divide both sides of (5) by dx to obtain

$$\frac{dy}{dx} = f'(x) \tag{6}$$

Thus, we have achieved our goal of defining dy and dx so their ratio is $f'(x)$. Formula (5) is said to express (6) in ***differential form***.

To interpret (5) geometrically, note that $f'(x)$ is the slope of the tangent line to the graph of f at x. The differentials dy and dx can be viewed as a corresponding rise and run of this tangent line (Figure 2.9.5).



▲ **Figure 2.9.6**

▶ **Example 3** Express the derivative with respect to x of $y = x^2$ in differential form, and discuss the relationship between dy and dx at $x = 1$.

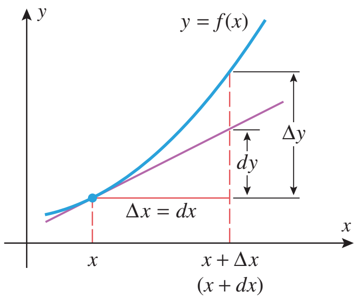
Solution. The derivative of y with respect to x is $dy/dx = 2x$, which can be expressed in differential form as

$$dy = 2x\,dx$$

When $x = 1$ this becomes

$$dy = 2\,dx$$

This tells us that if we travel along the tangent line to the curve $y = x^2$ at $x = 1$, then a change of dx units in x produces a change of $2\,dx$ units in y. Thus, for example, a run of $dx = 2$ units produces a rise of $dy = 4$ units along the tangent line (Figure 2.9.6). ◀



▲ **Figure 2.9.7**

It is important to understand the distinction between the increment Δy and the differential dy. To see the difference, let us assign the independent variables dx and Δx the same value, so $dx = \Delta x$. Then Δy represents the change in y that occurs when we start at x and travel *along the curve* $y = f(x)$ until we have moved Δx $(= dx)$ units in the x-direction, while dy represents the change in y that occurs if we start at x and travel *along the tangent line* until we have moved dx $(= \Delta x)$ units in the x-direction (Figure 2.9.7).

▶ **Example 4** Let $y = \sqrt{x}$.

(a) Find formulas for Δy and dy.

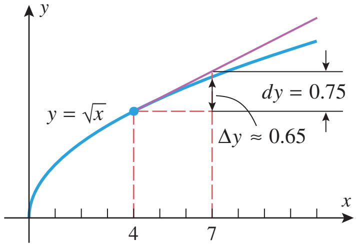
(b) Evaluate Δy and dy at $x = 4$ with $dx = \Delta x = 3$. Then make a sketch of $y = \sqrt{x}$ showing the values of Δy and dy in the picture.

Solution (a). With $y = f(x) = \sqrt{x}$ we obtain

$$\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}$$

and

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad dy = \frac{1}{2\sqrt{x}}\,dx$$



▲ **Figure 2.9.8**

Solution (b). At $x = 4$ with $dx = \Delta x = 3$,

$$\Delta y = \sqrt{7} - \sqrt{4} \approx 0.65$$

and

$$dy = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4} = 0.75$$

Figure 2.9.8 shows the curve $y = \sqrt{x}$ together with Δy and dy. ◀

LOCAL LINEAR APPROXIMATION FROM THE DIFFERENTIAL POINT OF VIEW

Although Δy and dy are generally different, the differential dy will nonetheless be a good approximation of Δy provided $dx = \Delta x$ is close to 0. To see this, recall from Section 2.2 that

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

It follows that if Δx is close to 0, then we will have $f'(x) \approx \Delta y/\Delta x$ or, equivalently,

$$\Delta y \approx f'(x)\Delta x$$

If we agree to let $dx = \Delta x$, then we can rewrite this as

$$\Delta y \approx f'(x)\,dx = dy \tag{7}$$

In words, this states that for values of dx near zero the differential dy closely approximates the increment Δy (Figure 2.9.7). But this is to be expected since the graph of the tangent line at x is the local linear approximation of the graph of f.

ERROR PROPAGATION

In real-world applications, small errors in measured quantities will invariably occur. These measurement errors are of importance in scientific research—all scientific measurements come with measurement errors included. For example, your height might be measured as 170 ± 0.5 cm, meaning that your exact height lies somewhere between 169.5 and 170.5 cm. Researchers often must use these inexactly measured quantities to compute other quantities, thereby *propagating* the errors from the measured quantitites to the computed quantities. This phenomenon is called ***error propagation***. Researchers must be able to estimate errors in the computed quantities. Our goal is to show how to estimate these errors using local linear approximation and differentials. For this purpose, suppose

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Real-world measurements inevitably have small errors.

x_0 is the exact value of the quantity being measured
$y_0 = f(x_0)$ is the exact value of the quantity being computed
x is the measured value of x_0
$y = f(x)$ is the computed value of y

We define $dx\ (= \Delta x) = x - x_0$ to be the ***measurement error*** of x
$\Delta y = f(x) - f(x_0)$ to be the ***propagated error*** of y

It follows from (7) with x_0 replacing x that the propagated error Δy can be approximated by

$$\Delta y \approx dy = f'(x_0)\, dx \tag{8}$$

Unfortunately, there is a practical difficulty in applying this formula since the value of x_0 is unknown. (Keep in mind that only the measured value x is known to the researcher.) This being the case, it is standard practice in research to use the measured value x in place of x_0 in (8) and use the approximation

$$\Delta y \approx dy = f'(x)\, dx \tag{9}$$

for the propagated error.

Note that measurement error is positive if the measured value is greater than the exact value and is negative if it is less than the exact value. The sign of the propagated error conveys similar information.

Example 5 Suppose that the side of a square is measured with a ruler to be 10 inches with a measurement error of at most $\pm\frac{1}{32}$ in. Estimate the error in the computed area of the square.

Explain why an error estimate of at most $\pm\frac{1}{32}$ inch is reasonable for a ruler that is calibrated in sixteenths of an inch.

Solution. Let x denote the exact length of a side and y the exact area so that $y = x^2$. It follows from (9) with $f(x) = x^2$ that if dx is the measurement error, then the propagated error Δy can be approximated as

$$\Delta y \approx dy = 2x\, dx$$

Substituting the measured value $x = 10$ into this equation yields

$$dy = 20\, dx \tag{10}$$

But to say that the measurement error is at most $\pm\frac{1}{32}$ means that

$$-\frac{1}{32} \le dx \le \frac{1}{32}$$

Multiplying these inequalities through by 20 and applying (10) yields

$$20\left(-\tfrac{1}{32}\right) \le dy \le 20\left(\tfrac{1}{32}\right) \quad \text{or equivalently} \quad -\tfrac{5}{8} \le dy \le \tfrac{5}{8}$$

Thus, the propagated error in the area is estimated to be within $\pm\frac{5}{8}$ in^2. ◀

If the true value of a quantity is q and a measurement or calculation produces an error Δq, then $\Delta q/q$ is called the ***relative error*** in the measurement or calculation; when expressed as a percentage, $\Delta q/q$ is called the ***percentage error***. As a practical matter, the true value q is usually unknown, so that the measured or calculated value of q is used instead; and the relative error is approximated by dq/q.

▶ **Example 6** The diameter of a polyurethane sphere is measured with percentage error within ±0.4%. Estimate the percentage error in the calculated volume of the sphere.

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Polyurethane spheres are used in the maintenance of oil and gas pipelines.

Solution. A sphere of diameter x has radius $r = x/2$ and volume

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{x}{2}\right)^3 = \frac{1}{6}\pi x^3$$

Therefore

$$\frac{dV}{dx} = \frac{1}{2}\pi x^2 \quad \text{and} \quad dV = \frac{1}{2}\pi x^2\,dx$$

The relative error in V is then approximately

$$\frac{dV}{V} = \frac{\frac{1}{2}\pi x^2\,dx}{\frac{1}{6}\pi x^3} = 3\,\frac{dx}{x} \tag{11}$$

We are given that the percentage error in the measured value of x is within ±0.4%, which means that

$$-0.004 \le \frac{dx}{x} \le 0.004$$

Multiplying these inequalities through by 3 and applying (11) yields

$$-0.012 \le \frac{dV}{V} \le 0.012$$

Thus, we estimate the percentage error in the calculated value of V to be within ±1.2%. ◀

MORE NOTATION; DIFFERENTIAL FORMULAS

The symbol df is another common notation for the differential of a function $y = f(x)$. For example, if $f(x) = \sin x$, then we can write $df = \cos x\,dx$. We can also view the symbol "d" as an *operator* that acts on a function to produce the corresponding differential. For example, $d[x^2] = 2x\,dx$, $d[\sin x] = \cos x\,dx$, and so on. All of the general rules of differentiation then have corresponding differential versions:

DERIVATIVE FORMULA	DIFFERENTIAL FORMULA
$\frac{d}{dx}[c] = 0$	$d[c] = 0$
$\frac{d}{dx}[cf] = c\frac{df}{dx}$	$d[cf] = c\,df$
$\frac{d}{dx}[f+g] = \frac{df}{dx} + \frac{dg}{dx}$	$d[f+g] = df + dg$
$\frac{d}{dx}[fg] = f\frac{dg}{dx} + g\frac{df}{dx}$	$d[fg] = f\,dg + g\,df$
$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$	$d\left[\frac{f}{g}\right] = \frac{g\,df - f\,dg}{g^2}$

For example,

$$\begin{aligned} d[x^2 \sin x] &= (x^2 \cos x + 2x \sin x)\,dx \\ &= x^2(\cos x\,dx) + (2x\,dx)\sin x \\ &= x^2 d[\sin x] + (\sin x)\,d[x^2] \end{aligned}$$

illustrates the differential version of the product rule.

✔QUICK CHECK EXERCISES 2.9 *(See page 183 for answers.)*

1. The local linear approximation of f at x_0 uses the ________ line to the graph of $y = f(x)$ at $x = x_0$ to approximate values of ________ for values of x near ________.
2. Find an equation for the local linear approximation to $y = 5 - x^2$ at $x_0 = 2$.
3. Let $y = 5 - x^2$. Find dy and Δy at $x = 2$ with $dx = \Delta x = 0.1$.
4. The intensity of light from a light source is a function $I = f(x)$ of the distance x from the light source. Suppose that a small gemstone is measured to be 10 m from a light source, $f(10) = 0.2$ W/m^2, and $f'(10) = -0.04$ W/m^3. If the distance $x = 10$ m was obtained with a measurement error within ± 0.05 m, estimate the percentage error in the calculated intensity of the light on the gemstone.

EXERCISE SET 2.9 Graphing Utility

1. (a) Use Formula (1) to obtain the local linear approximation of x^3 at $x_0 = 1$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $(1.02)^3$, and confirm that the formula obtained in part (b) produces the same result.

2. (a) Use Formula (1) to obtain the local linear approximation of $1/x$ at $x_0 = 2$.
   (b) Use Formula (2) to rewrite the approximation obtained in part (a) in terms of Δx.
   (c) Use the result obtained in part (a) to approximate $1/2.05$, and confirm that the formula obtained in part (b) produces the same result.

FOCUS ON CONCEPTS

3. (a) Find the local linear approximation of the function $f(x) = \sqrt{1+x}$ at $x_0 = 0$, and use it to approximate $\sqrt{0.9}$ and $\sqrt{1.1}$.
   (b) Graph f and its tangent line at x_0 together, and use the graphs to illustrate the relationship between the exact values and the approximations of $\sqrt{0.9}$ and $\sqrt{1.1}$.

4. A student claims that whenever a local linear approximation is used to approximate the square root of a number, the approximation is too large.
   (a) Write a few sentences that make the student's claim precise, and justify this claim geometrically.
   (b) Verify the student's claim algebraically using approximation (1).

5–10 Confirm that the stated formula is the local linear approximation at $x_0 = 0$. ■

5. $(1+x)^{15} \approx 1 + 15x$

6. $\dfrac{1}{\sqrt{1-x}} \approx 1 + \dfrac{1}{2}x$

7. $\tan x \approx x$

8. $\dfrac{1}{1+x} \approx 1 - x$

9–12 Confirm that the stated formula is the local linear approximation of f at $x_0 = 1$, where $\Delta x = x - 1$. ■

9. $f(x) = x^4;\ (1+\Delta x)^4 \approx 1 + 4\Delta x$

10. $f(x) = \sqrt{x};\ \sqrt{1+\Delta x} \approx 1 + \frac{1}{2}\Delta x$

11. $f(x) = \dfrac{1}{2+x};\ \dfrac{1}{3+\Delta x} \approx \dfrac{1}{3} - \dfrac{1}{9}\Delta x$

12. $f(x) = (4+x)^3;\ (5+\Delta x)^3 \approx 125 + 75\Delta x$

13–16 Confirm that the formula is the local linear approximation at $x_0 = 0$, and use a graphing utility to estimate an interval of x-values on which the error is at most ± 0.1. ■

13. $\sqrt{x+3} \approx \sqrt{3} + \dfrac{1}{2\sqrt{3}}x$

14. $\dfrac{1}{\sqrt{9-x}} \approx \dfrac{1}{3} + \dfrac{1}{54}x$

15. $\tan 2x \approx 2x$

16. $\dfrac{1}{(1+2x)^5} \approx 1 - 10x$

17. (a) Use the local linear approximation of $\sin x$ at $x_0 = 0$ obtained in Example 2 to approximate $\sin 1°$, and compare the approximation to the result produced directly by your calculating device.
   (b) How would you choose x_0 to approximate $\sin 44°$?
   (c) Approximate $\sin 44°$; compare the approximation to the result produced directly by your calculating device.

18. (a) Use the local linear approximation of $\tan x$ at $x_0 = 0$ to approximate $\tan 2°$, and compare the approximation to the result produced directly by your calculating device.
   (b) How would you choose x_0 to approximate $\tan 61°$?
   (c) Approximate $\tan 61°$; compare the approximation to the result produced directly by your calculating device.

19–27 Use an appropriate local linear approximation to estimate the value of the given quantity. ■

19. $(3.02)^4$ **20.** $(1.97)^3$ **21.** $\sqrt{65}$

22. $\sqrt{24}$ **23.** $\sqrt{80.9}$ **24.** $\sqrt{36.03}$

25. $\sin 0.1$ **26.** $\tan 0.2$ **27.** $\cos 31°$

FOCUS ON CONCEPTS

28. The approximation $(1+x)^k \approx 1 + kx$ is commonly used by engineers for quick calculations.
   (a) Derive this result, and use it to make a rough estimate of $(1.001)^{37}$.
   (b) Compare your estimate to that produced directly by your calculating device. *(cont.)*

(c) If k is a positive integer, how is the approximation $(1+x)^k \approx 1+kx$ related to the expansion of $(1+x)^k$ using the binomial theorem?

29. Use the approximation $(1+x)^k \approx 1+kx$, along with some mental arithmetic to show that $\sqrt[3]{8.24} \approx 2.02$ and $4.08^{3/2} \approx 8.24$.

30. Referring to the accompanying figure, suppose that the angle of elevation of the top of the building, as measured from a point 500 ft from its base, is found to be $\theta = 6°$. Use an appropriate local linear approximation, along with some mental arithmetic to show that the building is about 52 ft high.

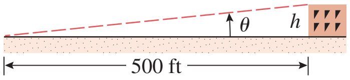


Figure Ex-30

31. (a) Let $y = 1/x$. Find dy and Δy at $x = 1$ with $dx = \Delta x = -0.5$.
(b) Sketch the graph of $y = 1/x$, showing dy and Δy in the picture.

32. (a) Let $y = \sqrt{x}$. Find dy and Δy at $x = 9$ with $dx = \Delta x = -1$.
(b) Sketch the graph of $y = \sqrt{x}$, showing dy and Δy in the picture.

33–36 Find formulas for dy and Δy. ■

33. $y = x^3$

34. $y = 8x - 4$

35. $y = x^2 - 2x + 1$

36. $y = \sin x$

37–40 Find the differential dy. ■

37. (a) $y = 4x^3 - 7x^2$ (b) $y = x \cos x$

38. (a) $y = 1/x$ (b) $y = 5 \tan x$

39. (a) $y = x\sqrt{1-x}$ (b) $y = (1+x)^{-17}$

40. (a) $y = \dfrac{1}{x^3 - 1}$ (b) $y = \dfrac{1 - x^3}{2 - x}$

41–44 True–False Determine whether the statement is true or false. Explain your answer. ■

41. A differential dy is defined to be a very small change in y.

42. The error in approximation (2) is the same as the error in approximation (7).

43. A local linear approximation to a function can never be identically equal to the function.

44. A local linear approximation to a nonconstant function can never be constant.

45–48 Use the differential dy to approximate Δy when x changes as indicated. ■

45. $y = \sqrt{3x - 2}$; from $x = 2$ to $x = 2.03$

46. $y = \sqrt{x^2 + 8}$; from $x = 1$ to $x = 0.97$

47. $y = \dfrac{x}{x^2 + 1}$; from $x = 2$ to $x = 1.96$

48. $y = x\sqrt{8x + 1}$; from $x = 3$ to $x = 3.05$

49. The side of a square is measured to be 10 ft, with a possible error of ±0.1 ft.
(a) Use differentials to estimate the error in the calculated area.
(b) Estimate the percentage errors in the side and the area.

50. The side of a cube is measured to be 25 cm, with a possible error of ±1 cm.
(a) Use differentials to estimate the error in the calculated volume.
(b) Estimate the percentage errors in the side and volume.

51. The hypotenuse of a right triangle is known to be 10 in exactly, and one of the acute angles is measured to be 30°, with a possible error of ±1°.
(a) Use differentials to estimate the errors in the sides opposite and adjacent to the measured angle.
(b) Estimate the percentage errors in the sides.

52. One side of a right triangle is known to be 25 cm exactly. The angle opposite to this side is measured to be 60°, with a possible error of ±0.5°.
(a) Use differentials to estimate the errors in the adjacent side and the hypotenuse.
(b) Estimate the percentage errors in the adjacent side and hypotenuse.

53. The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of ±5%, use differentials to estimate the percentage error in R. (Assume k is exact.)

54. A 12-foot ladder leaning against a wall makes an angle θ with the floor. If the top of the ladder is h feet up the wall, express h in terms of θ and then use dh to estimate the change in h if θ changes from 60° to 59°.

55. The area of a right triangle with a hypotenuse of H is calculated using the formula $A = \frac{1}{4}H^2 \sin 2\theta$, where θ is one of the acute angles. Use differentials to approximate the error in calculating A if $H = 4$ cm (exactly) and θ is measured to be 30°, with a possible error of ±15′.

56. The side of a square is measured with a possible percentage error of ±1%. Use differentials to estimate the percentage error in the area.

57. The side of a cube is measured with a possible percentage error of ±2%. Use differentials to estimate the percentage error in the volume.

58. The volume of a sphere is to be computed from a measured value of its radius. Estimate the maximum permissible percentage error in the measurement if the percentage error in the volume must be kept within ±3%. ($V = \frac{4}{3}\pi r^3$ is the volume of a sphere of radius r.)

59. The area of a circle is to be computed from a measured value of its diameter. Estimate the maximum permissible percentage error in the measurement if the percentage error in the area must be kept within ±1%.

60. A steel cube with 1-inch sides is coated with 0.01 inch of copper. Use differentials to estimate the volume of copper in the coating. [*Hint:* Let ΔV be the change in the volume of the cube.]

61. A metal rod 15 cm long and 5 cm in diameter is to be covered (except for the ends) with insulation that is 0.1 cm thick. Use differentials to estimate the volume of insulation. [*Hint:* Let ΔV be the change in volume of the rod.]

62. The time required for one complete oscillation of a pendulum is called its ***period***. If L is the length of the pendulum and the oscillation is small, then the period is given by $P = 2\pi\sqrt{L/g}$, where g is the constant acceleration due to gravity. Use differentials to show that the percentage error in P is approximately half the percentage error in L.

63. If the temperature T of a metal rod of length L is changed by an amount ΔT, then the length will change by the amount $\Delta L = \alpha L \Delta T$, where α is called the ***coefficient of linear expansion***. For moderate changes in temperature α is taken as constant.

(a) Suppose that a rod 40 cm long at 20°C is found to be 40.006 cm long when the temperature is raised to 30°C. Find α.

(b) If an aluminum pole is 180 cm long at 15°C, how long is the pole if the temperature is raised to 40°C? [Take $\alpha = 2.3 \times 10^{-5}/°\mathrm{C}$.]

64. If the temperature T of a solid or liquid of volume V is changed by an amount ΔT, then the volume will change by the amount $\Delta V = \beta V \Delta T$, where β is called the ***coefficient of volume expansion***. For moderate changes in temperature β is taken as constant. Suppose that a tank truck loads 4000 gallons of ethyl alcohol at a temperature of 35°C and delivers its load sometime later at a temperature of 15°C. Using $\beta = 7.5 \times 10^{-4}/°\mathrm{C}$ for ethyl alcohol, find the number of gallons delivered.

65. **Writing** Explain why the local linear approximation of a function value is equivalent to the use of a differential to approximate a change in the function.

66. **Writing** The local linear approximation

$$\sin x \approx x$$

is known as the *small angle approximation* and has both practical and theoretical applications. Do some research on some of these applications, and write a short report on the results of your investigations.

✔QUICK CHECK ANSWERS 2.9

1. tangent; $f(x)$; x_0 **2.** $y = 1 + (-4)(x-2)$ or $y = -4x + 9$ **3.** $dy = -0.4$, $\Delta y = -0.41$ **4.** within ±1%

CHAPTER 2 REVIEW EXERCISES

∿ Graphing Utility C CAS

1. Explain the difference between average and instantaneous rates of change, and discuss how they are calculated.

2. In parts (a)–(d), use the function $y = \frac{1}{2}x^2$.

(a) Find the average rate of change of y with respect to x over the interval [3, 4].

(b) Find the instantaneous rate of change of y with respect to x at $x = 3$.

(c) Find the instantaneous rate of change of y with respect to x at a general x-value.

(d) Sketch the graph of $y = \frac{1}{2}x^2$ together with the secant line whose slope is given by the result in part (a), and indicate graphically the slope of the tangent line that corresponds to the result in part (b).

3. Complete each part for the function $f(x) = x^2 + 1$.

(a) Find the slope of the tangent line to the graph of f at a general x-value.

(b) Find the slope of the tangent line to the graph of f at $x = 2$.

4. A car is traveling on a straight road that is 120 mi long. For the first 100 mi the car travels at an average velocity of 50 mi/h. Show that no matter how fast the car travels for the final 20 mi it cannot bring the average velocity up to 60 mi/h for the entire trip.

5. At time $t = 0$ a car moves into the passing lane to pass a slow-moving truck. The average velocity of the car from $t = 1$ to $t = 1 + h$ is

$$v_{\text{ave}} = \frac{3(h+1)^{2.5} + 580h - 3}{10h}$$

Estimate the instantaneous velocity of the car at $t = 1$, where time is in seconds and distance is in feet.

∿ **6.** A skydiver jumps from an airplane. Suppose that the distance she falls during the first t seconds before her parachute opens is $s(t) = 976((0.835)^t - 1) + 176t$, where s is in feet. Graph s versus t for $0 \le t \le 20$, and use your graph to estimate the instantaneous velocity at $t = 15$.

7. A particle moves on a line away from its initial position so that after t hours it is $s = 3t^2 + t$ miles from its initial position.

(cont.)

(a) Find the average velocity of the particle over the interval [1, 3].
(b) Find the instantaneous velocity at $t = 1$.

8. State the definition of a derivative, and give two interpretations of it.

9. Use the definition of a derivative to find dy/dx, and check your answer by calculating the derivative using appropriate derivative formulas.
(a) $y = \sqrt{9 - 4x}$ (b) $y = \dfrac{x}{x+1}$

10. Suppose that $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ k(x-1), & x > 1. \end{cases}$
For what values of k is f
(a) continuous? (b) differentiable?

11. The accompanying figure shows the graph of $y = f'(x)$ for an unspecified function f.
(a) For what values of x does the curve $y = f(x)$ have a horizontal tangent line?
(b) Over what intervals does the curve $y = f(x)$ have tangent lines with positive slope?
(c) Over what intervals does the curve $y = f(x)$ have tangent lines with negative slope?
(d) Given that $g(x) = f(x)\sin x$, find $g''(0)$.

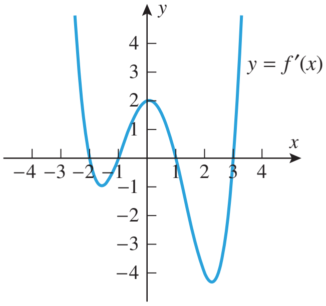


Figure Ex-11

12. Sketch the graph of a function f for which $f(0) = 1$, $f'(0) = 0$, $f'(x) > 0$ if $x < 0$, and $f'(x) < 0$ if $x > 0$.

13. According to the U.S. Bureau of the Census, the estimated and projected midyear world population, N, in billions for the years 1950, 1975, 2000, 2025, and 2050 was 2.555, 4.088, 6.080, 7.841, and 9.104, respectively. Although the increase in population is not a continuous function of the time t, we can apply the ideas in this section if we are willing to approximate the graph of N versus t by a continuous curve, as shown in the accompanying figure.
(a) Use the tangent line at $t = 2000$ shown in the figure to approximate the value of dN/dt there. Interpret your result as a rate of change.
(b) The instantaneous ***growth rate*** is defined as

$$\frac{dN/dt}{N}$$

Use your answer to part (a) to approximate the instantaneous growth rate at the start of the year 2000. Express the result as a percentage and include the proper units.

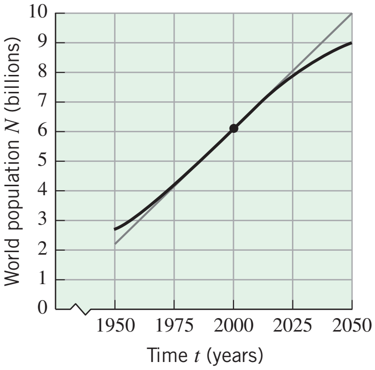


Figure Ex-13

14. Use a graphing utility to graph the function

$$f(x) = |x^4 - x - 1| - x$$

and estimate the values of x where the derivative of this function does not exist.

C **15–18** (a) Use a CAS to find $f'(x)$ via Definition 2.2.1; (b) check the result by finding the derivative by hand; (c) use the CAS to find $f''(x)$. ■

15. $f(x) = x^2 \sin x$ **16.** $f(x) = \sqrt{x} + \cos^2 x$

17. $f(x) = \dfrac{2x^2 - x + 5}{3x + 2}$ **18.** $f(x) = \dfrac{\tan x}{1 + x^2}$

19. The amount of water in a tank t minutes after it has started to drain is given by $W = 100(t - 15)^2$ gal.
(a) At what rate is the water running out at the end of 5 min?
(b) What is the average rate at which the water flows out during the first 5 min?

20. Use the formula $V = l^3$ for the volume of a cube of side l to find
(a) the average rate at which the volume of a cube changes with l as l increases from $l = 2$ to $l = 4$
(b) the instantaneous rate at which the volume of a cube changes with l when $l = 5$.

21–22 Zoom in on the graph of f on an interval containing $x = x_0$ until the graph looks like a straight line. Estimate the slope of this line and then check your answer by finding the exact value of $f'(x_0)$. ■

21. (a) $f(x) = x^2 - 1,\ x_0 = 1.8$
(b) $f(x) = \dfrac{x^2}{x-2},\ x_0 = 3.5$

22. (a) $f(x) = x^3 - x^2 + 1,\ x_0 = 2.3$
(b) $f(x) = \dfrac{x}{x^2+1},\ x_0 = -0.5$

23. Suppose that a function f is differentiable at $x = 1$ and

$$\lim_{h \to 0} \frac{f(1+h)}{h} = 5$$

Find $f(1)$ and $f'(1)$.

24. Suppose that a function f is differentiable at $x = 2$ and

$$\lim_{x\to 2} \frac{x^3 f(x) - 24}{x - 2} = 28$$

Find $f(2)$ and $f'(2)$.

25. Find the equations of all lines through the origin that are tangent to the curve $y = x^3 - 9x^2 - 16x$.

26. Find all values of x for which the tangent line to the curve $y = 2x^3 - x^2$ is perpendicular to the line $x + 4y = 10$.

27. Let $f(x) = x^2$. Show that for any distinct values of a and b, the slope of the tangent line to $y = f(x)$ at $x = \frac{1}{2}(a + b)$ is equal to the slope of the secant line through the points (a, a^2) and (b, b^2). Draw a picture to illustrate this result.

28. In each part, evaluate the expression given that $f(1) = 1$, $g(1) = -2$, $f'(1) = 3$, and $g'(1) = -1$.

(a) $\left.\dfrac{d}{dx}[f(x)g(x)]\right|_{x=1}$ (b) $\left.\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]\right|_{x=1}$

(c) $\left.\dfrac{d}{dx}\left[\sqrt{f(x)}\right]\right|_{x=1}$ (d) $\dfrac{d}{dx}[f(1)g'(1)]$

29–32 Find $f'(x)$. ■

29. (a) $f(x) = x^8 - 3\sqrt{x} + 5x^{-3}$
(b) $f(x) = (2x + 1)^{101}(5x^2 - 7)$

30. (a) $f(x) = \sin x + 2\cos^3 x$
(b) $f(x) = (1 + \sec x)(x^2 - \tan x)$

31. (a) $f(x) = \sqrt{3x + 1}(x - 1)^2$
(b) $f(x) = \left(\dfrac{3x + 1}{x^2}\right)^3$

32. (a) $f(x) = \cot\left(\dfrac{\csc 2x}{x^3 + 5}\right)$ (b) $f(x) = \dfrac{1}{2x + \sin^3 x}$

33–34 Find the values of x at which the curve $y = f(x)$ has a horizontal tangent line. ■

33. $f(x) = (2x + 7)^6(x - 2)^5$ **34.** $f(x) = \dfrac{(x - 3)^4}{x^2 + 2x}$

35. Find all lines that are simultaneously tangent to the graph of $y = x^2 + 1$ and to the graph of $y = -x^2 - 1$.

36. (a) Let n denote an even positive integer. Generalize the result of Exercise 35 by finding all lines that are simultaneously tangent to the graph of $y = x^n + n - 1$ and to the graph of $y = -x^n - n + 1$.
(b) Let n denote an odd positive integer. Are there any lines that are simultaneously tangent to the graph of $y = x^n + n - 1$ and to the graph of $y = -x^n - n + 1$? Explain.

37. Find all values of x for which the line that is tangent to $y = 3x - \tan x$ is parallel to the line $y - x = 2$.

38. Approximate the values of x at which the tangent line to the graph of $y = x^3 - \sin x$ is horizontal.

39. Suppose that $f(x) = M\sin x + N\cos x$ for some constants M and N. If $f(\pi/4) = 3$ and $f'(\pi/4) = 1$, find an equation for the tangent line to $y = f(x)$ at $x = 3\pi/4$.

40. Suppose that $f(x) = M\tan x + N\sec x$ for some constants M and N. If $f(\pi/4) = 2$ and $f'(\pi/4) = 0$, find an equation for the tangent line to $y = f(x)$ at $x = 0$.

41. Suppose that $f'(x) = 2x \cdot f(x)$ and $f(2) = 5$.
(a) Find $g'(\pi/3)$ if $g(x) = f(\sec x)$.
(b) Find $h'(2)$ if $h(x) = [f(x)/(x - 1)]^4$.

42–44 Find dy/dx. ■

42. $y = \sqrt[4]{6x - 5}$ **43.** $y = \sqrt[3]{x^2 + x}$

44. $y = \dfrac{(3 - 2x)^{4/3}}{x^2}$

45–46 (a) Find dy/dx by differentiating implicitly. (b) Solve the equation for y as a function of x, and find dy/dx from that equation. (c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone. ■

45. $x^3 + xy - 2x = 1$ **46.** $xy = x - y$

47–50 Find dy/dx by implicit differentiation. ■

47. $\dfrac{1}{y} + \dfrac{1}{x} = 1$ **48.** $x^3 - y^3 = 6xy$

49. $\sec(xy) = y$ **50.** $x^2 = \dfrac{\cot y}{1 + \csc y}$

51–52 Find d^2y/dx^2 by implicit differentiation. ■

51. $3x^2 - 4y^2 = 7$ **52.** $2xy - y^2 = 3$

53. Use implicit differentiation to find the slope of the tangent line to the curve $y = x\tan(\pi y/2)$, $x > 0$, $y > 0$ (***the quadratrix of Hippias***) at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

54. At what point(s) is the tangent line to the curve $y^2 = 2x^3$ perpendicular to the line $4x - 3y + 1 = 0$?

55. Prove that if P and Q are two distinct points on the rotated ellipse $x^2 + xy + y^2 = 4$ such that P, Q, and the origin are collinear, then the tangent lines to the ellipse at P and Q are parallel.

56. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the x-axis.

57. Find the coordinates of the point in the first quadrant at which the tangent line to the curve $x^3 - xy + y^3 = 0$ is parallel to the y-axis.

58. Use implicit differentiation to show that the equation of the tangent line to the curve $y^2 = kx$ at (x_0, y_0) is

$$y_0 y = \tfrac{1}{2}k(x + x_0)$$

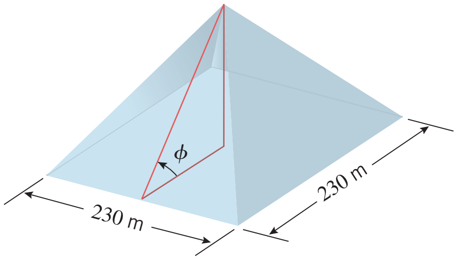
59. An oil slick on a lake is surrounded by a floating circular containment boom. As the boom is pulled in, the circular containment area shrinks. If the boom is pulled in at the rate of 5 m/min, at what rate is the containment area shrinking when the containment area has a diameter of 100 m?

60. The hypotenuse of a right triangle is growing at a constant rate of a centimeters per second and one leg is decreasing at a constant rate of b centimeters per second. How fast is the acute angle between the hypotenuse and the other leg changing at the instant when both legs are 1 cm?

61. In each part, use the given information to find Δx, Δy, and dy.
- (a) $y = 1/(x-1)$; x decreases from 2 to 1.5.
- (b) $y = \tan x$; x increases from $-\pi/4$ to 0.
- (c) $y = \sqrt{25 - x^2}$; x increases from 0 to 3.

62. Use an appropriate local linear approximation to estimate the value of $\cot 46°$, and compare your answer to the value obtained with a calculating device.

63. The base of the Great Pyramid at Giza is a square that is 230 m on each side.
- (a) As illustrated in the accompanying figure, suppose that an archaeologist standing at the center of a side measures the angle of elevation of the apex to be $\phi = 51°$ with an error of $\pm 0.5°$. What can the archaeologist reasonably say about the height of the pyramid?
- (b) Use differentials to estimate the allowable error in the elevation angle that will ensure that the error in calculating the height is at most ±5 m.



▲ **Figure Ex-63**

CHAPTER 2 MAKING CONNECTIONS

1. Suppose that f is a function with the properties (i) f is differentiable everywhere, (ii) $f(x+y) = f(x)f(y)$ for all values of x and y, (iii) $f(0) \neq 0$, and (iv) $f'(0) = 1$.
- (a) Show that $f(0) = 1$. [*Hint:* Consider $f(0+0)$.]
- (b) Show that $f(x) > 0$ for all values of x. [*Hint:* First show that $f(x) \neq 0$ for any x by considering $f(x-x)$.]
- (c) Use the definition of derivative (Definition 2.2.1) to show that $f'(x) = f(x)$ for all values of x.

2. Suppose that f and g are functions each of which has the properties (i)–(iv) in Exercise 1.
- (a) Show that $y = f(2x)$ satisfies the equation $y' = 2y$ in two ways: using property (ii), and by directly applying the chain rule (Theorem 2.6.1).
- (b) If k is any constant, show that $y = f(kx)$ satisfies the equation $y' = ky$.
- (c) Find a value of k such that $y = f(x)g(x)$ satisfies the equation $y' = ky$.
- (d) If $h = f/g$, find $h'(x)$. Make a conjecture about the relationship between f and g.

3. (a) Apply the product rule (Theorem 2.4.1) twice to show that if f, g, and h are differentiable functions, then $f \cdot g \cdot h$ is differentiable and

$$(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

- (b) Suppose that f, g, h, and k are differentiable functions. Derive a formula for $(f \cdot g \cdot h \cdot k)'$.
- (c) Based on the result in part (a), make a conjecture about a formula differentiating a product of n functions. Prove your formula using induction.

4. (a) Apply the quotient rule (Theorem 2.4.2) twice to show that if f, g, and h are differentiable functions, then $(f/g)/h$ is differentiable where it is defined and

$$[(f/g)/h]' = \frac{f' \cdot g \cdot h - f \cdot g' \cdot h - f \cdot g \cdot h'}{g^2h^2}$$

- (b) Derive the derivative formula of part (a) by first simplifying $(f/g)/h$ and then applying the quotient and product rules.
- (c) Apply the quotient rule (Theorem 2.4.2) twice to derive a formula for $[f/(g/h)]'$.
- (d) Derive the derivative formula of part (c) by first simplifying $f/(g/h)$ and then applying the quotient and product rules.

5. Assume that $h(x) = f(x)/g(x)$ is differentiable. Derive the quotient rule formula for $h'(x)$ (Theorem 2.4.2) in two ways:
- (a) Write $h(x) = f(x) \cdot [g(x)]^{-1}$ and use the product and chain rules (Theorems 2.4.1 and 2.6.1) to differentiate h.
- (b) Write $f(x) = h(x) \cdot g(x)$ and use the product rule to derive a formula for $h'(x)$.

EXPANDING THE CALCULUS HORIZON

To learn how derivatives can be used in the field of robotics, see the module entitled **Robotics** at:

www.wiley.com/college/anton

Chapter III

The Derivative in Graphing and Applications

Stone/Getty Images

3

THE DERIVATIVE IN GRAPHING AND APPLICATIONS

Derivatives can help to find the most cost-effective location for an offshore oil-drilling rig.

In this chapter we will study various applications of the derivative. For example, we will use methods of calculus to analyze functions and their graphs. In the process, we will show how calculus and graphing utilities, working together, can provide most of the important information about the behavior of functions. Another important application of the derivative will be in the solution of optimization problems. For example, if time is the main consideration in a problem, we might be interested in finding the quickest way to perform a task, and if cost is the main consideration, we might be interested in finding the least expensive way to perform a task. Mathematically, optimization problems can be reduced to finding the largest or smallest value of a function on some interval, and determining where the largest or smallest value occurs. Using the derivative, we will develop the mathematical tools necessary for solving such problems. We will also use the derivative to study the motion of a particle moving along a line, and we will show how the derivative can help us to approximate solutions of equations.

3.1 ANALYSIS OF FUNCTIONS I: INCREASE, DECREASE, AND CONCAVITY

Although graphing utilities are useful for determining the general shape of a graph, many problems require more precision than graphing utilities are capable of producing. The purpose of this section is to develop mathematical tools that can be used to determine the exact shape of a graph and the precise locations of its key features.

INCREASING AND DECREASING FUNCTIONS

The terms *increasing*, *decreasing*, and *constant* are used to describe the behavior of a function as we travel left to right along its graph. For example, the function graphed in Figure 3.1.1 can be described as increasing to the left of $x = 0$, decreasing from $x = 0$ to $x = 2$, increasing from $x = 2$ to $x = 4$, and constant to the right of $x = 4$.

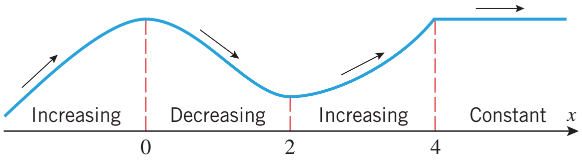


Figure 3.1.1

The following definition, which is illustrated in Figure 3.1.2, expresses these intuitive ideas precisely.

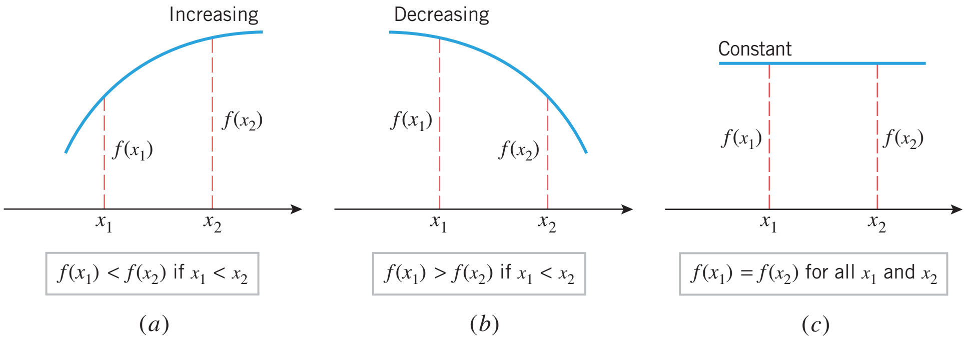
The definitions of "increasing," "decreasing," and "constant" describe the behavior of a function on an *interval* and not at a point. In particular, it is not inconsistent to say that the function in Figure 3.1.1 is decreasing on the interval $[0, 2]$ and increasing on the interval $[2, 4]$.

3.1.1 DEFINITION Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

(a) f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

(b) f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

(c) f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.



(*a*) (*b*) (*c*)

Figure 3.1.2

Figure 3.1.3 suggests that a differentiable function f is increasing on any interval where each tangent line to its graph has positive slope, is decreasing on any interval where each tangent line to its graph has negative slope, and is constant on any interval where each tangent line to its graph has zero slope. This intuitive observation suggests the following important theorem that will be proved in Section 3.8.

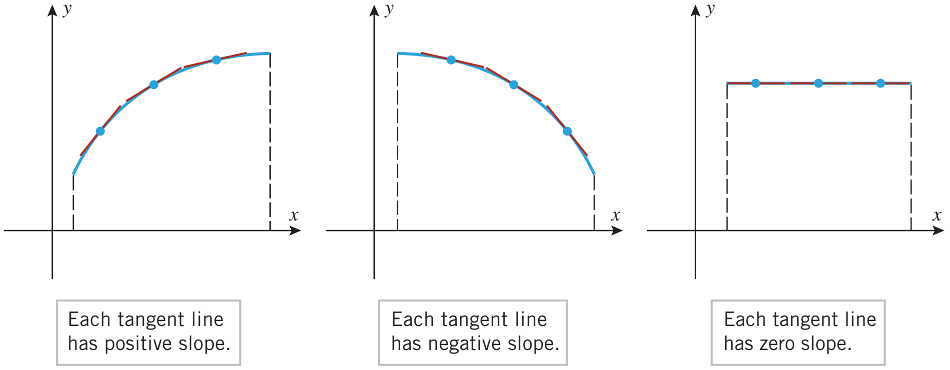


Figure 3.1.3

Observe that the derivative conditions in Theorem 3.1.2 are only required to hold *inside* the interval $[a, b]$, even though the conclusions apply to the entire interval.

3.1.2 THEOREM *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

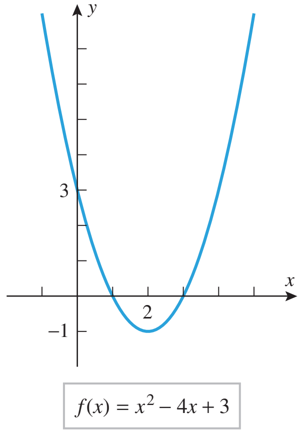
(*a*) *If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.*

(*b*) *If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.*

(*c*) *If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.*

Although stated for closed intervals, Theorem 3.1.2 is applicable on any interval on which f is continuous. For example, if f is continuous on $[a, +\infty)$ and $f'(x) > 0$ on $(a, +\infty)$, then f is increasing on $[a, +\infty)$; and if f is continuous on $(-\infty, +\infty)$ and $f'(x) < 0$ on $(-\infty, +\infty)$, then f is decreasing on $(-\infty, +\infty)$.

► Example 1 Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.



$f(x) = x^2 - 4x + 3$

▲ Figure 3.1.4

Solution. The graph of f in Figure 3.1.4 suggests that f is decreasing for $x \leq 2$ and increasing for $x \geq 2$. To confirm this, we analyze the sign of f'. The derivative of f is

$$f'(x) = 2x - 4 = 2(x - 2)$$

It follows that

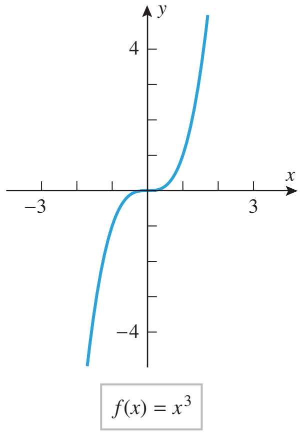
$$\begin{aligned} f'(x) &< 0 \quad \text{if} \quad x < 2 \\ f'(x) &> 0 \quad \text{if} \quad 2 < x \end{aligned}$$

Since f is continuous everywhere, it follows from the comment after Theorem 3.1.2 that

$$\begin{aligned} &f \text{ is decreasing on } (-\infty, 2] \\ &f \text{ is increasing on } [2, +\infty) \end{aligned}$$

These conclusions are consistent with the graph of f in Figure 3.1.4. ◄

► Example 2 Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.



$f(x) = x^3$

▲ Figure 3.1.5

Solution. The graph of f in Figure 3.1.5 suggests that f is increasing over the entire x-axis. To confirm this, we differentiate f to obtain $f'(x) = 3x^2$. Thus,

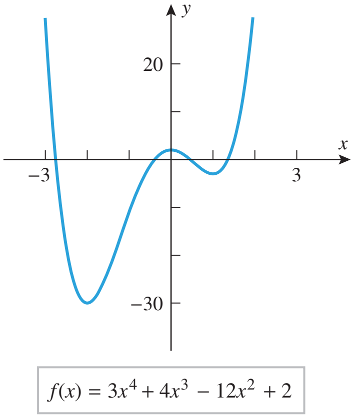
$$\begin{aligned} f'(x) &> 0 \quad \text{if} \quad x < 0 \\ f'(x) &> 0 \quad \text{if} \quad 0 < x \end{aligned}$$

Since f is continuous everywhere,

$$\begin{aligned} &f \text{ is increasing on } (-\infty, 0] \\ &f \text{ is increasing on } [0, +\infty) \end{aligned}$$

Since f is increasing on the adjacent intervals $(-\infty, 0]$ and $[0, +\infty)$, it follows that f is increasing on their union $(-\infty, +\infty)$ (see Exercise 51). ◄

► Example 3



$f(x) = 3x^4 + 4x^3 - 12x^2 + 2$

▲ Figure 3.1.6

(a) Use the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 + 2$ in Figure 3.1.6 to make a conjecture about the intervals on which f is increasing or decreasing.

(b) Use Theorem 3.1.2 to determine whether your conjecture is correct.

Solution (a). The graph suggests that the function f is decreasing if $x \leq -2$, increasing if $-2 \leq x \leq 0$, decreasing if $0 \leq x \leq 1$, and increasing if $x \geq 1$.

Solution (b). Differentiating f we obtain

$$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x + 2)(x - 1)$$

The sign analysis of f' in Table 3.1.1 can be obtained using the method of test points discussed in Web Appendix E. The conclusions in Table 3.1.1 confirm the conjecture in part (a). ◄

Table 3.1.1

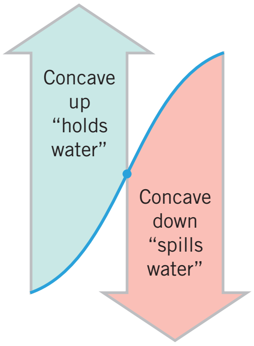
INTERVAL	$(12x)(x+2)(x-1)$	$f'(x)$	CONCLUSION
$x < -2$	$(-)(-)(-)$	$-$	f is decreasing on $(-\infty, -2]$
$-2 < x < 0$	$(-)(+)(-)$	$+$	f is increasing on $[-2, 0]$
$0 < x < 1$	$(+)(+)(-)$	$-$	f is decreasing on $[0, 1]$
$1 < x$	$(+)(+)(+)$	$+$	f is increasing on $[1, +\infty)$

CONCAVITY

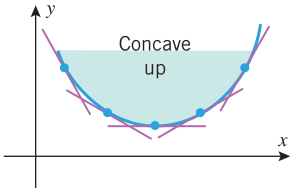
Although the sign of the derivative of f reveals where the graph of f is increasing or decreasing, it does not reveal the direction of *curvature*. For example, the graph is increasing on both sides of the point in Figure 3.1.7, but on the left side it has an upward curvature ("holds water") and on the right side it has a downward curvature ("spills water"). On intervals where the graph of f has upward curvature we say that f is *concave up*, and on intervals where the graph has downward curvature we say that f is *concave down*.

Figure 3.1.8 suggests two ways to characterize the concavity of a differentiable function f on an open interval:

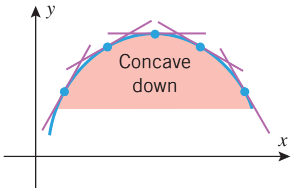
- f is concave up on an open interval if its tangent lines have increasing slopes on that interval and is concave down if they have decreasing slopes.
- f is concave up on an open interval if its graph lies above its tangent lines on that interval and is concave down if it lies below its tangent lines.



▲ **Figure 3.1.7**



Increasing slopes



Decreasing slopes

▲ **Figure 3.1.8**

Our formal definition for "concave up" and "concave down" corresponds to the first of these characterizations.

3.1.3 DEFINITION If f is differentiable on an open interval, then f is said to be ***concave up*** on the open interval if f' is increasing on that interval, and f is said to be ***concave down*** on the open interval if f' is decreasing on that interval.

Since the slopes of the tangent lines to the graph of a differentiable function f are the values of its derivative f', it follows from Theorem 3.1.2 (applied to f' rather than f) that f' will be increasing on intervals where f'' is positive and that f' will be decreasing on intervals where f'' is negative. Thus, we have the following theorem.

3.1.4 THEOREM *Let f be twice differentiable on an open interval.*

(a) If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.

(b) If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.

▶ **Example 4** Figure 3.1.4 suggests that the function $f(x) = x^2 - 4x + 3$ is concave up on the interval $(-\infty, +\infty)$. This is consistent with Theorem 3.1.4, since $f'(x) = 2x - 4$ and $f''(x) = 2$, so

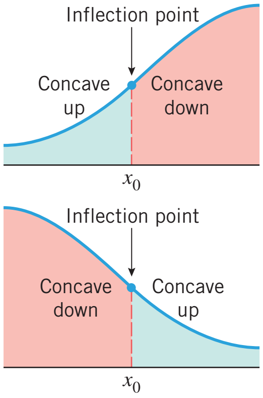
$$f''(x) > 0 \quad \text{on the interval } (-\infty, +\infty)$$

Also, Figure 3.1.5 suggests that $f(x) = x^3$ is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, +\infty)$. This agrees with Theorem 3.1.4, since $f'(x) = 3x^2$ and $f''(x) = 6x$, so

$$f''(x) < 0 \quad \text{if } x < 0 \quad \text{and} \quad f''(x) > 0 \quad \text{if } x > 0$$ ◄

INFLECTION POINTS

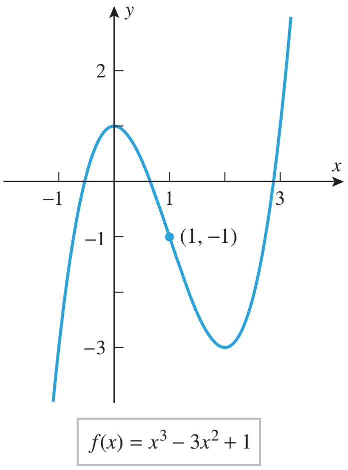
We see from Example 4 and Figure 3.1.5 that the graph of $f(x) = x^3$ changes from concave down to concave up at $x = 0$. Points where a curve changes from concave up to concave down or vice versa are of special interest, so there is some terminology associated with them.



▲ **Figure 3.1.9**

3.1.5 DEFINITION If f is continuous on an open interval containing a value x_0, and if f changes the direction of its concavity at the point $(x_0, f(x_0))$, then we say that f has an ***inflection point at*** $\boldsymbol{x_0}$, and we call the point $(x_0, f(x_0))$ on the graph of f an ***inflection point*** of f (Figure 3.1.9).

▶ **Example 5** Figure 3.1.10 shows the graph of the function $f(x) = x^3 - 3x^2 + 1$. Use the first and second derivatives of f to determine the intervals on which f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.



▲ **Figure 3.1.10**

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

$$f''(x) = 6x - 6 = 6(x - 1)$$

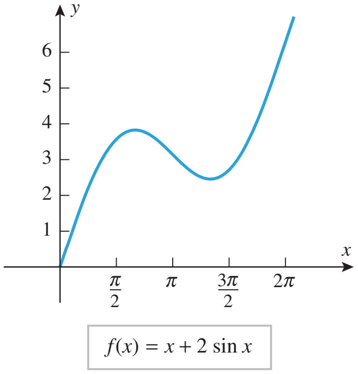
The sign analysis of these derivatives is shown in the following tables:

INTERVAL	$(3x)(x-2)$	$f'(x)$	CONCLUSION
$x < 0$	$(-)(-)$	$+$	f is increasing on $(-\infty, 0]$
$0 < x < 2$	$(+)(-)$	$-$	f is decreasing on $[0, 2]$
$x > 2$	$(+)(+)$	$+$	f is increasing on $[2, +\infty)$

INTERVAL	$6(x-1)$	$f''(x)$	CONCLUSION
$x < 1$	$(-)$	$-$	f is concave down on $(-\infty, 1)$
$x > 1$	$(+)$	$+$	f is concave up on $(1, +\infty)$

The second table shows that there is an inflection point at $x = 1$, since f changes from concave down to concave up at that point. The inflection point is $(1, f(1)) = (1, -1)$. All of these conclusions are consistent with the graph of f. ◄

One can correctly guess from Figure 3.1.10 that the function $f(x) = x^3 - 3x^2 + 1$ has an inflection point at $x = 1$ without actually computing derivatives. However, sometimes changes in concavity are so subtle that calculus is essential to confirm their existence and identify their location.



▲ **Figure 3.1.11**

▶ **Example 6** Figure 3.1.11 shows the graph of the function $f(x) = x + 2\sin x$ over the interval $[0, 2\pi]$. Use the first and second derivatives of f to determine where f is increasing, decreasing, concave up, and concave down. Locate all inflection points and confirm that your conclusions are consistent with the graph.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 1 + 2\cos x$$
$$f''(x) = -2\sin x$$

Since f' is a continuous function, it changes sign on the interval $(0, 2\pi)$ only at points where $f'(x) = 0$ (why?). These values are solutions of the equation

$$1 + 2\cos x = 0 \quad \text{or equivalently} \quad \cos x = -\tfrac{1}{2}$$

There are two solutions of this equation in the interval $(0, 2\pi)$, namely, $x = 2\pi/3$ and $x = 4\pi/3$ (verify). Similarly, f'' is a continuous function, so its sign changes in the interval $(0, 2\pi)$ will occur only at values of x for which $f''(x) = 0$. These values are solutions of the equation

$$-2\sin x = 0$$

There is one solution of this equation in the interval $(0, 2\pi)$, namely, $x = \pi$. With the help of these "sign transition points" we obtain the sign analysis shown in the following tables:

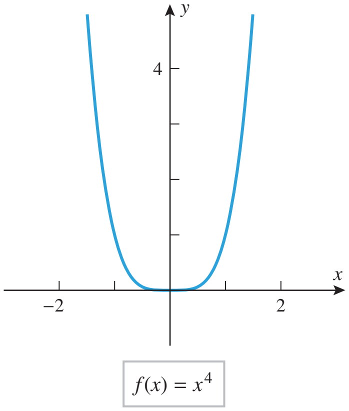
INTERVAL	$f'(x) = 1 + 2\cos x$	CONCLUSION
$0 < x < 2\pi/3$	+	f is increasing on $[0, 2\pi/3]$
$2\pi/3 < x < 4\pi/3$	–	f is decreasing on $[2\pi/3, 4\pi/3]$
$4\pi/3 < x < 2\pi$	+	f is increasing on $[4\pi/3, 2\pi]$

INTERVAL	$f''(x) = -2\sin x$	CONCLUSION
$0 < x < \pi$	–	f is concave down on $(0, \pi)$
$\pi < x < 2\pi$	+	f is concave up on $(\pi, 2\pi)$

The signs in the two tables of Example 6 can be obtained either using the method of test points or using the unit circle definition of the sine and cosine functions.

The second table shows that there is an inflection point at $x = \pi$, since f changes from concave down to concave up at that point. All of these conclusions are consistent with the graph of f. ◀

In the preceding examples the inflection points of f occurred wherever $f''(x) = 0$. However, this is not always the case. Here is a specific example.



▲ **Figure 3.1.12**

▶ **Example 7** Find the inflection points, if any, of $f(x) = x^4$.

Solution. Calculating the first two derivatives of f we obtain

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$

Since $f''(x)$ is positive for $x < 0$ and for $x > 0$, the function f is concave up on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$. Thus, there is no change in concavity and hence no inflection point at $x = 0$, even though $f''(0) = 0$ (Figure 3.1.12). ◀

Give an argument to show that the function $f(x) = x^4$ graphed in Figure 3.1.12 is concave up on the interval $(-\infty, +\infty)$.

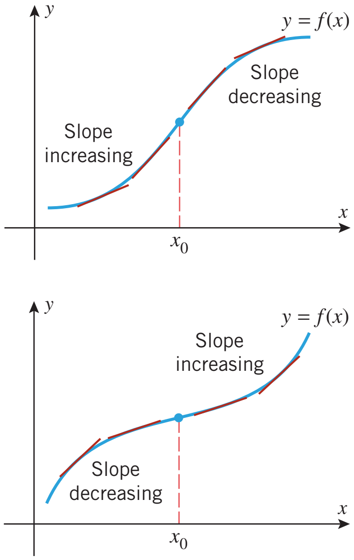
We will see later that if a function f has an inflection point at $x = x_0$ and $f''(x_0)$ exists, then $f''(x_0) = 0$. Further, we will see in Section 3.3 that an inflection point may also occur where $f''(x)$ is not defined.

INFLECTION POINTS IN APPLICATIONS

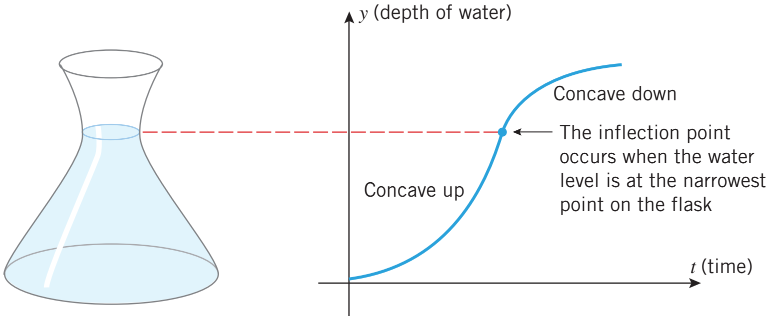
Inflection points of a function f are those points on the graph of $y = f(x)$ where the slopes of the tangent lines change from increasing to decreasing or vice versa (Figure 3.1.13). Since the slope of the tangent line at a point on the graph of $y = f(x)$ can be interpreted as the rate of change of y with respect to x at that point, we can interpret inflection points in the following way:

> *Inflection points mark the places on the curve $y = f(x)$ where the rate of change of y with respect to x changes from increasing to decreasing, or vice versa.*

This is a subtle idea, since we are dealing with a change in a rate of change. It can help with your understanding of this idea to realize that inflection points may have interpretations in more familiar contexts. For example, consider the statement "Oil prices rose sharply during the first half of the year but have since begun to level off." If the price of oil is plotted as a function of time of year, this statement suggests the existence of an inflection point on the graph near the end of June. (Why?) To give a more visual example, consider the flask shown in Figure 3.1.14. Suppose that water is added to the flask so that the volume increases at a constant rate with respect to the time t, and let us examine the rate at which the water level y rises with respect to t. Initially, the level y will rise at a slow rate because of the wide base. However, as the diameter of the flask narrows, the rate at which the level y rises will increase until the level is at the narrow point in the neck. From that point on the rate at which the level rises will decrease as the diameter gets wider and wider. Thus, the narrow point in the neck is the point at which the rate of change of y with respect to t changes from increasing to decreasing.



▲ **Figure 3.1.13**



▶ **Figure 3.1.14**

✔QUICK CHECK EXERCISES 3.1 *(See page 196 for answers.)*

1. (a) A function f is increasing on (a, b) if ________ whenever $a < x_1 < x_2 < b$.
   (b) A function f is decreasing on (a, b) if ________ whenever $a < x_1 < x_2 < b$.
   (c) A function f is concave up on (a, b) if f' is ________ on (a, b).
   (d) If $f''(a)$ exists and f has an inflection point at $x = a$, then $f''(a)$ ________.

2. Let $f(x) = 0.1(x^3 - 3x^2 - 9x)$. Then

$$f'(x) = 0.1(3x^2 - 6x - 9) = 0.3(x + 1)(x - 3)$$

$$f''(x) = 0.6(x - 1)$$

   (a) Solutions to $f'(x) = 0$ are $x =$ ________.
   (b) The function f is increasing on the interval(s) ________.

(cont.)

(c) The function f is concave down on the interval(s) ________.

(d) ________ is an inflection point on the graph of f.

3. Suppose that $f(x)$ has derivative $f'(x) = x(x-4)^2$. Then $f''(x) = (x-4)(3x-4)$.
(a) The function f is increasing on the interval(s) ________.
(b) The function f is concave up on the interval(s) ________.
(c) The function f is concave down on the interval(s) ________.

4. Consider the statement "The rise in the cost of living slowed during the first half of the year." If we graph the cost of living versus time for the first half of the year, how does the graph reflect this statement?

EXERCISE SET 3.1

Graphing Utility C CAS

FOCUS ON CONCEPTS

1. In each part, sketch the graph of a function f with the stated properties, and discuss the signs of f' and f''.
(a) The function f is concave up and increasing on the interval $(-\infty, +\infty)$.
(b) The function f is concave down and increasing on the interval $(-\infty, +\infty)$.
(c) The function f is concave up and decreasing on the interval $(-\infty, +\infty)$.
(d) The function f is concave down and decreasing on the interval $(-\infty, +\infty)$.

2. In each part, sketch the graph of a function f with the stated properties.
(a) f is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave up on $(0, +\infty)$.
(b) f is increasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave down on $(0, +\infty)$.
(c) f is decreasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave up on $(0, +\infty)$.
(d) f is decreasing on $(-\infty, +\infty)$, has an inflection point at the origin, and is concave down on $(0, +\infty)$.

3. Use the graph of the equation $y = f(x)$ in the accompanying figure to find the signs of dy/dx and d^2y/dx^2 at the points A, B, and C.

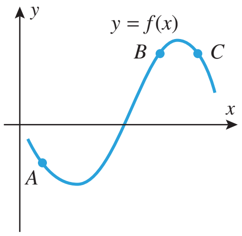


Figure Ex-3

4. Use the graph of the equation $y = f'(x)$ in the accompanying figure to find the signs of dy/dx and d^2y/dx^2 at the points A, B, and C.

5. Use the graph of $y = f''(x)$ in the accompanying figure to determine the x-coordinates of all inflection points of f. Explain your reasoning.

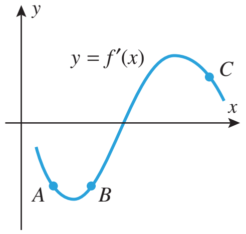


Figure Ex-4

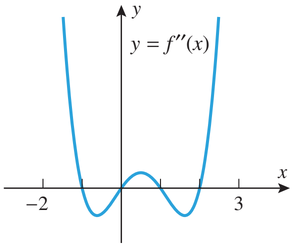


Figure Ex-5

6. Use the graph of $y = f'(x)$ in the accompanying figure to replace the question mark with $<$, $=$, or $>$, as appropriate. Explain your reasoning.
(a) $f(0)\ ?\ f(1)$ (b) $f(1)\ ?\ f(2)$ (c) $f'(0)\ ?\ 0$
(d) $f'(1)\ ?\ 0$ (e) $f''(0)\ ?\ 0$ (f) $f''(2)\ ?\ 0$

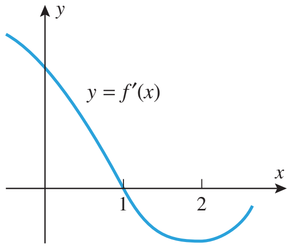


Figure Ex-6

7. In each part, use the graph of $y = f(x)$ in the accompanying figure to find the requested information.
(a) Find the intervals on which f is increasing.
(b) Find the intervals on which f is decreasing.
(c) Find the open intervals on which f is concave up.
(d) Find the open intervals on which f is concave down.
(e) Find all values of x at which f has an inflection point.

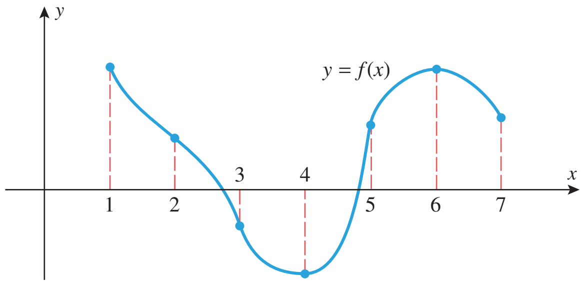


Figure Ex-7

8. Use the graph in Exercise 7 to make a table that shows the signs of f' and f'' over the intervals $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 5)$, $(5, 6)$, and $(6, 7)$.

9–10 A sign chart is presented for the first and second derivatives of a function f. Assuming that f is continuous everywhere, find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points. ■

9.

INTERVAL	SIGN OF $f'(x)$	SIGN OF $f''(x)$
$x < 1$	$-$	$+$
$1 < x < 2$	$+$	$+$
$2 < x < 3$	$+$	$-$
$3 < x < 4$	$-$	$-$
$4 < x$	$-$	$+$

10.

INTERVAL	SIGN OF $f'(x)$	SIGN OF $f''(x)$
$x < 1$	$+$	$+$
$1 < x < 3$	$+$	$-$
$3 < x$	$+$	$+$

11–14 True–False Assume that f is differentiable everywhere. Determine whether the statement is true or false. Explain your answer. ■

11. If f is decreasing on $[0, 2]$, then $f(0) > f(1) > f(2)$.

12. If $f'(1) > 0$, then f is increasing on $[0, 2]$.

13. If f is increasing on $[0, 2]$, then $f'(1) > 0$.

14. If f' is increasing on $[0, 1]$ and f' is decreasing on $[1, 2]$, then f has an inflection point at $x = 1$.

15–26 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points. ■

15. $f(x) = x^2 - 3x + 8$

16. $f(x) = 5 - 4x - x^2$

17. $f(x) = (2x + 1)^3$

18. $f(x) = 5 + 12x - x^3$

19. $f(x) = 3x^4 - 4x^3$

20. $f(x) = x^4 - 5x^3 + 9x^2$

21. $f(x) = \dfrac{x - 2}{(x^2 - x + 1)^2}$

22. $f(x) = \dfrac{x}{x^2 + 2}$

23. $f(x) = \sqrt[3]{x^2 + x + 1}$

24. $f(x) = x^{4/3} - x^{1/3}$

25. $f(x) = (x^{2/3} - 1)^2$

26. $f(x) = x^{2/3} - x$

27–32 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility. ■

27. $f(x) = \sin x - \cos x;\ [-\pi, \pi]$

28. $f(x) = \sec x \tan x;\ (-\pi/2, \pi/2)$

29. $f(x) = 1 - \tan(x/2);\ (-\pi, \pi)$

30. $f(x) = 2x + \cot x;\ (0, \pi)$

31. $f(x) = (\sin x + \cos x)^2;\ [-\pi, \pi]$

32. $f(x) = \sin^2 2x;\ [0, \pi]$

FOCUS ON CONCEPTS

33. In parts (a)–(c), sketch a continuous curve $y = f(x)$ with the stated properties.
  (a) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for all x
  (b) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for $x < 2,\ f''(x) > 0$ for $x > 2$
  (c) $f(2) = 4,\ f''(x) < 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = +\infty,\ \lim_{x\to 2^-} f'(x) = -\infty$

34. In each part sketch a continuous curve $y = f(x)$ with the stated properties.
  (a) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for all x
  (b) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for $x < 2,\ f''(x) < 0$ for $x > 2$
  (c) $f(2) = 4,\ f''(x) > 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = -\infty,\ \lim_{x\to 2^-} f'(x) = +\infty$

35–38 If f is increasing on an interval $[0, b)$, then it follows from Definition 3.1.1 that $f(0) < f(x)$ for each x in the interval $(0, b)$. Use this result in these exercises. ■

35. Show that $\sqrt[3]{1 + x} < 1 + \frac{1}{3}x$ if $x > 0$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = 1 + \frac{1}{3}x - \sqrt[3]{1 + x}$ is increasing on $[0, +\infty)$.]

36. Show that $x < \tan x$ if $0 < x < \pi/2$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = \tan x - x$ is increasing on $[0, \pi/2)$.]

37. Use a graphing utility to make a conjecture about the relative sizes of x and $\sin x$ for $x \geq 0$, and prove your conjecture.

38. Use a graphing utility to make a conjecture about the relative sizes of $1 - x^2/2$ and $\cos x$ for $x \geq 0$, and prove your conjecture. [*Hint:* Use the result of Exercise 37.]

39–40 Use a graphing utility to generate the graphs of f' and f'' over the stated interval; then use those graphs to estimate the x-coordinates of the inflection points of f, the intervals on which f is concave up or down, and the intervals on which f is increasing or decreasing. Check your estimates by graphing f. ■

39. $f(x) = x^4 - 24x^2 + 12x,\quad -5 \leq x \leq 5$

40. $f(x) = \dfrac{1}{1 + x^2},\quad -5 \leq x \leq 5$

C **41–42** Use a CAS to find f'' and to approximate the x-coordinates of the inflection points to six decimal places. Confirm that your answer is consistent with the graph of f. ■

41. $f(x) = \dfrac{10x - 3}{3x^2 - 5x + 8}$ **42.** $f(x) = \dfrac{x^3 - 8x + 7}{\sqrt{x^2 + 1}}$

43. Use Definition 3.1.1 to prove that $f(x) = x^2$ is increasing on $[0, +\infty)$.

44. Use Definition 3.1.1 to prove that $f(x) = 1/x$ is decreasing on $(0, +\infty)$.

FOCUS ON CONCEPTS

45–48 Determine whether the statements are true or false. If a statement is false, find functions for which the statement fails to hold. ■

45. (a) If f and g are increasing on an interval, then so is $f + g$.
(b) If f and g are increasing on an interval, then so is $f \cdot g$.

46. (a) If f and g are concave up on an interval, then so is $f + g$.
(b) If f and g are concave up on an interval, then so is $f \cdot g$.

47. In each part, find functions f and g that are increasing on $(-\infty, +\infty)$ and for which $f - g$ has the stated property.
(a) $f - g$ is decreasing on $(-\infty, +\infty)$.
(b) $f - g$ is constant on $(-\infty, +\infty)$.
(c) $f - g$ is increasing on $(-\infty, +\infty)$.

48. In each part, find functions f and g that are positive and increasing on $(-\infty, +\infty)$ and for which f/g has the stated property.
(a) f/g is decreasing on $(-\infty, +\infty)$.
(b) f/g is constant on $(-\infty, +\infty)$.
(c) f/g is increasing on $(-\infty, +\infty)$.

49. (a) Prove that a general cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$

has exactly one inflection point.
(b) Prove that if a cubic polynomial has three x-intercepts, then the inflection point occurs at the average value of the intercepts.
(c) Use the result in part (b) to find the inflection point of the cubic polynomial $f(x) = x^3 - 3x^2 + 2x$, and check your result by using f'' to determine where f is concave up and concave down.

50. From Exercise 49, the polynomial $f(x) = x^3 + bx^2 + 1$ has one inflection point. Use a graphing utility to reach a conclusion about the effect of the constant b on the location of the inflection point. Use f'' to explain what you have observed graphically.

51. Use Definition 3.1.1 to prove:
(a) If f is increasing on the intervals $(a, c]$ and $[c, b)$, then f is increasing on (a, b).
(b) If f is decreasing on the intervals $(a, c]$ and $[c, b)$, then f is decreasing on (a, b).

52. Use part (a) of Exercise 51 to show that $f(x) = x + \sin x$ is increasing on the interval $(-\infty, +\infty)$.

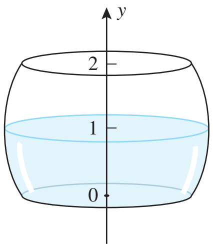
53. Use part (b) of Exercise 51 to show that $f(x) = \cos x - x$ is decreasing on the interval $(-\infty, +\infty)$.

54. Let $y = 1/(1 + x^2)$. Find the values of x for which y is increasing most rapidly or decreasing most rapidly.

FOCUS ON CONCEPTS

55–58 Suppose that water is flowing at a constant rate into the container shown. Make a rough sketch of the graph of the water level y versus the time t. Make sure that your sketch conveys where the graph is concave up and concave down, and label the y-coordinates of the inflection points. ■

55.



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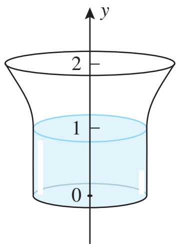

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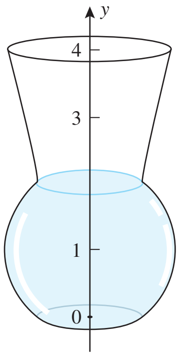

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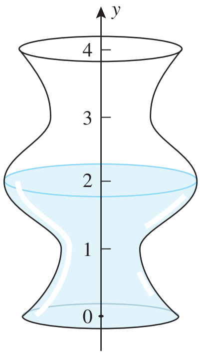


59. Writing An approaching storm causes the air temperature to fall. Make a statement that indicates there is an inflection point in the graph of temperature versus time. Explain how the existence of an inflection point follows from your statement.

60. Writing Explain what the sign analyses of $f'(x)$ and $f''(x)$ tell us about the graph of $y = f(x)$.

✔ QUICK CHECK ANSWERS 3.1

1. (a) $f(x_1) < f(x_2)$ (b) $f(x_1) > f(x_2)$ (c) increasing (d) $= 0$ **2.** (a) $-1, 3$ (b) $(-\infty, -1]$ and $[3, +\infty)$ (c) $(-\infty, 1)$ (d) $(1, -1.1)$ **3.** (a) $[0, +\infty)$ (b) $\left(-\infty, \frac{4}{3}\right), (4, +\infty)$ (c) $\left(\frac{4}{3}, 4\right)$ **4.** The graph is increasing and concave down.

3.2 ANALYSIS OF FUNCTIONS II: RELATIVE EXTREMA; GRAPHING POLYNOMIALS

In this section we will develop methods for finding the high and low points on the graph of a function and we will discuss procedures for analyzing the graphs of polynomials.

RELATIVE MAXIMA AND MINIMA

If we imagine the graph of a function f to be a two-dimensional mountain range with hills and valleys, then the tops of the hills are called "relative maxima," and the bottoms of the valleys are called "relative minima" (Figure 3.2.1). The relative maxima are the high points in their *immediate vicinity*, and the relative minima are the low points. A relative maximum need not be the highest point in the entire mountain range, and a relative minimum need not be the lowest point—they are just high and low points *relative* to the nearby terrain. These ideas are captured in the following definition.

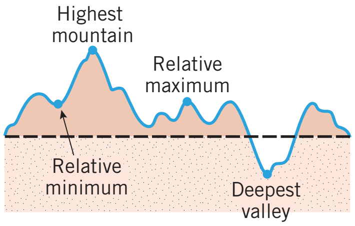
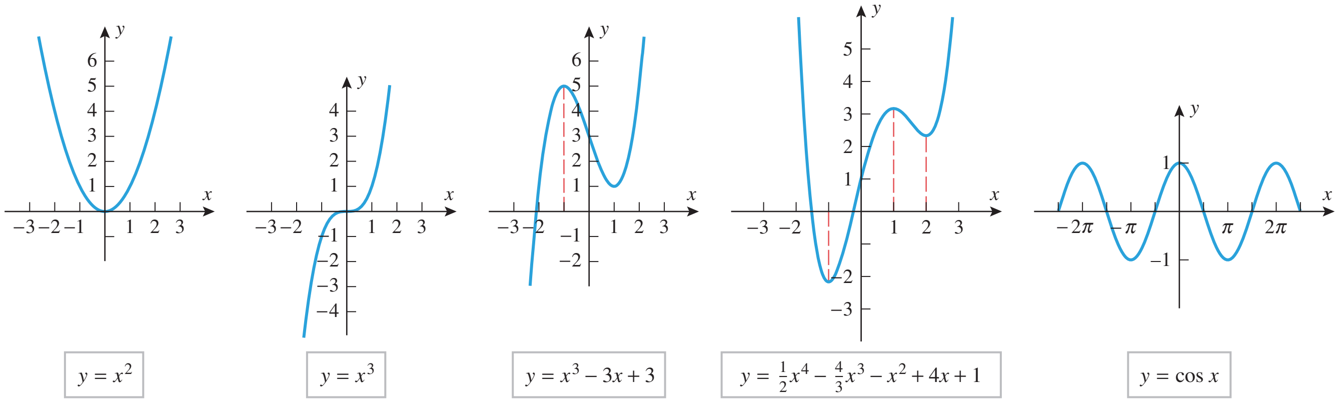


Figure 3.2.1

3.2.1 DEFINITION A function f is said to have a ***relative maximum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval. Similarly, f is said to have a ***relative minimum*** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval. If f has either a relative maximum or a relative minimum at x_0, then f is said to have a ***relative extremum*** at x_0.

Example 1 We can see from Figure 3.2.2 that:

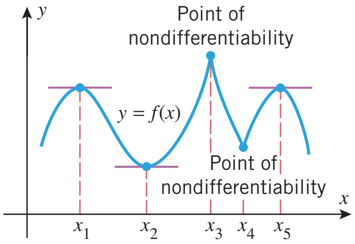
- $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.
- $f(x) = x^3$ has no relative extrema.
- $f(x) = x^3 - 3x + 3$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.
- $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ has relative minima at $x = -1$ and $x = 2$ and a relative maximum at $x = 1$.
- $f(x) = \cos x$ has relative maxima at all even multiples of π and relative minima at all odd multiples of π. ◀



$y = x^2$ $y = x^3$ $y = x^3 - 3x + 3$ $y = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$ $y = \cos x$

Figure 3.2.2

The relative extrema for the five functions in Example 1 occur at points where the graphs of the functions have horizontal tangent lines. Figure 3.2.3 illustrates that a relative extremum can also occur at a point where a function is not differentiable. In general, we define a ***critical point*** for a function f to be a point in the domain of f at which either the graph of f has a horizontal tangent line or f is not differentiable. To distinguish between the two types of critical points we call x a ***stationary point*** of f if $f'(x) = 0$. The following theorem, which is proved in Appendix D, states that the critical points for a function form a complete set of candidates for relative extrema on the interior of the domain of the function.



▲ **Figure 3.2.3** The points x_1, x_2, x_3, x_4, and x_5 are critical points. Of these, x_1, x_2, and x_5 are stationary points.

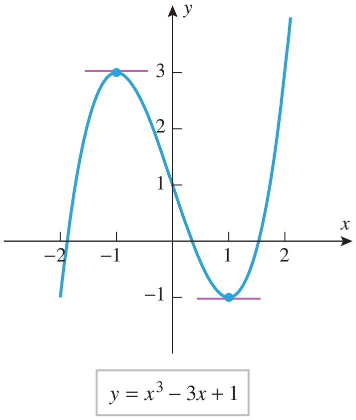
3.2.2 THEOREM *Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extremum at $x = x_0$, then $x = x_0$ is a critical point of f; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.*

▶ **Example 2** Find all critical points of $f(x) = x^3 - 3x + 1$.

Solution. The function f, being a polynomial, is differentiable everywhere, so its critical points are all stationary points. To find these points we must solve the equation $f'(x) = 0$. Since

$$f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$$

we conclude that the critical points occur at $x = -1$ and $x = 1$. This is consistent with the graph of f in Figure 3.2.4. ◀



▲ **Figure 3.2.4**

What is the maximum number of critical points that a polynomial of degree n can have? Why?

▶ **Example 3** Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

Solution. The function f is continuous everywhere and its derivative is

$$f'(x) = 5x^{2/3} - 10x^{-1/3} = 5x^{-1/3}(x-2) = \frac{5(x-2)}{x^{1/3}}$$

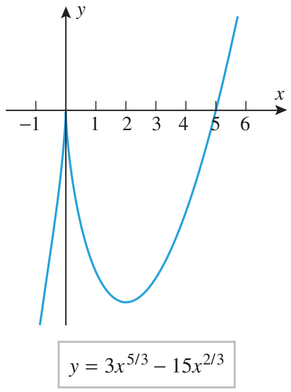
We see from this that $f'(x) = 0$ if $x = 2$ and $f'(x)$ is undefined if $x = 0$. Thus $x = 0$ and $x = 2$ are critical points and $x = 2$ is a stationary point. This is consistent with the graph of f shown in Figure 3.2.5. ◀

TECHNOLOGY MASTERY

Your graphing utility may have trouble producing portions of the graph in Figure 3.2.5 because of the fractional exponents. If this is the case for you, graph the function

$$y = 3(|x|/x)|x|^{5/3} - 15|x|^{2/3}$$

which is equivalent to $f(x)$ for $x \neq 0$. Appendix A explores the method suggested here in more detail.

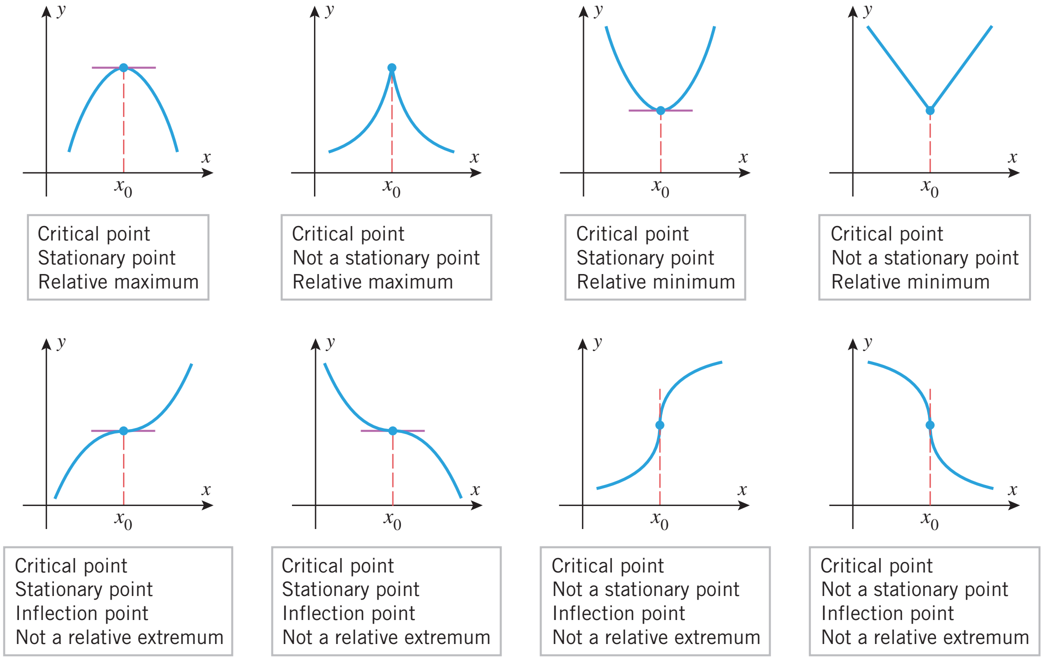


▲ **Figure 3.2.5**

FIRST DERIVATIVE TEST

Theorem 3.2.2 asserts that the relative extrema must occur at critical points, but it does *not* say that a relative extremum occurs at *every* critical point. For example, for the eight critical points in Figure 3.2.6, relative extrema occur at each x_0 in the top row but not at any x_0 in the bottom row. Moreover, at the critical points in the first row the derivatives have opposite signs on the two sides of x_0, whereas at the critical points in the second row the signs of the derivatives are the same on both sides. This suggests:

A function f has a relative extremum at those critical points where f' changes sign.



▲ **Figure 3.2.6**

We can actually take this a step further. At the two relative maxima in Figure 3.2.6 the derivative is positive on the left side and negative on the right side, and at the two relative minima the derivative is negative on the left side and positive on the right side. All of this is summarized more precisely in the following theorem.

3.2.3 THEOREM (*First Derivative Test*) *Suppose that f is continuous at a critical point x_0.*

(*a*) *If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.*

(*b*) *If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.*

(*c*) *If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.*

Informally stated, parts (*a*) and (*b*) of Theorem 3.2.3 tell us that for a continuous function, relative maxima occur at critical points where the derivative changes from + to − and relative minima where it changes from − to +.

Use the first derivative test to confirm the behavior at x_0 of each graph in Figure 3.2.6.

PROOF We will prove part (*a*) and leave parts (*b*) and (*c*) as exercises. We are assuming that $f'(x) > 0$ on the interval (a, x_0) and that $f'(x) < 0$ on the interval (x_0, b), and we want to show that

$$f(x_0) \geq f(x)$$

for all x in the interval (a, b). However, the two hypotheses, together with Theorem 3.1.2 and its associated marginal note imply that f is increasing on the interval $(a, x_0]$ and decreasing on the interval $[x_0, b)$. Thus, $f(x_0) \geq f(x)$ for all x in (a, b) with equality only at x_0. ■

▶ **Example 4** We showed in Example 3 that the function $f(x) = 3x^{5/3} - 15x^{2/3}$ has critical points at $x = 0$ and $x = 2$. Figure 3.2.5 suggests that f has a relative maximum at $x = 0$ and a relative minimum at $x = 2$. Confirm this using the first derivative test.

Solution. We showed in Example 3 that

$$f'(x) = \frac{5(x-2)}{x^{1/3}}$$

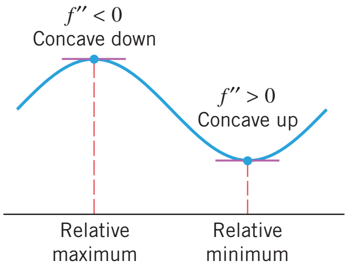
A sign analysis of this derivative is shown in Table 3.2.1. The sign of f' changes from + to − at $x = 0$, so there is a relative maximum at that point. The sign changes from − to + at $x = 2$, so there is a relative minimum at that point. ◀

Table 3.2.1

INTERVAL	$5(x-2)/x^{1/3}$	$f'(x)$
$x < 0$	$(-)/(-)$	+
$0 < x < 2$	$(-)/(+)$	−
$x > 2$	$(+)/(+)$	+

SECOND DERIVATIVE TEST

There is another test for relative extrema that is based on the following geometric observation: A function f has a relative maximum at a stationary point if the graph of f is concave down on an open interval containing that point, and it has a relative minimum if it is concave up (Figure 3.2.7).



▶ **Figure 3.2.7**

3.2.4 THEOREM (*Second Derivative Test*) *Suppose that f is twice differentiable at the point x_0.*

(*a*) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.*

(*b*) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.*

(*c*) *If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.*

We will prove parts (*a*) and (*c*) and leave part (*b*) as an exercise.

The second derivative test is often easier to apply than the first derivative test. However, the first derivative test can be used at any critical point of a continuous function, while the second derivative test applies only at stationary points where the second derivative exists.

PROOF (*a*) We are given that $f'(x_0) = 0$ and $f''(x_0) > 0$, and we want to show that f has a relative minimum at x_0. Expressing $f''(x_0)$ as a limit and using the two given conditions we obtain

$$f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0} > 0$$

This implies that for x sufficiently close to but different from x_0 we have

$$\frac{f'(x)}{x - x_0} > 0 \qquad (1)$$

Thus, there is an open interval extending left from x_0 and an open interval extending right from x_0 on which (1) holds. On the open interval extending left the denominator in (1) is negative, so $f'(x) < 0$, and on the open interval extending right the denominator is positive, so $f'(x) > 0$. It now follows from part (*b*) of the first derivative test (Theorem 3.2.3) that f has a relative minimum at x_0.

PROOF (*c*) To prove this part of the theorem we need only provide functions for which $f'(x_0) = 0$ and $f''(x_0) = 0$ at some point x_0, but with one having a relative minimum at x_0, one having a relative maximum at x_0, and one having neither at x_0. We leave it as an exercise for you to show that three such functions are $f(x) = x^4$ (relative minimum at $x = 0$), $f(x) = -x^4$ (relative maximum at $x = 0$), and $f(x) = x^3$ (neither a relative maximum nor a relative minimum at x_0). ■

▶ **Example 5** Find the relative extrema of $f(x) = 3x^5 - 5x^3$.

Solution. We have

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x + 1)(x - 1)$$
$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

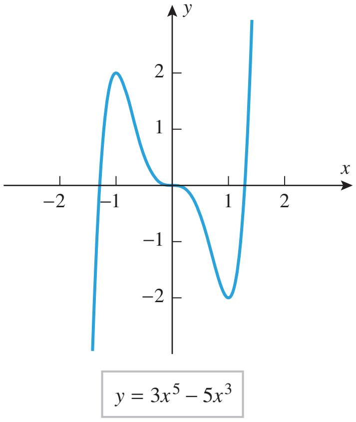
Solving $f'(x) = 0$ yields the stationary points $x = 0$, $x = -1$, and $x = 1$. As shown in the following table, we can conclude from the second derivative test that f has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

STATIONARY POINT	$30x(2x^2 - 1)$	$f''(x)$	SECOND DERIVATIVE TEST
$x = -1$	-30	$-$	f has a relative maximum
$x = 0$	0	0	Inconclusive
$x = 1$	30	$+$	f has a relative minimum

The test is inconclusive at $x = 0$, so we will try the first derivative test at that point. A sign analysis of f' is given in the following table:

INTERVAL	$15x^2(x + 1)(x - 1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	$-$
$0 < x < 1$	$(+)(+)(-)$	$-$

Since there is no sign change in f' at $x = 0$, there is neither a relative maximum nor a relative minimum at that point. All of this is consistent with the graph of f shown in Figure 3.2.8. ◀



▲ **Figure 3.2.8**

GEOMETRIC IMPLICATIONS OF MULTIPLICITY

Our final goal in this section is to outline a general procedure that can be used to analyze and graph polynomials. To do so, it will be helpful to understand how the graph of a polynomial behaves in the vicinity of its roots. For example, it would be nice to know what property of the polynomial in Example 5 produced the inflection point and horizontal tangent at the root $x = 0$.

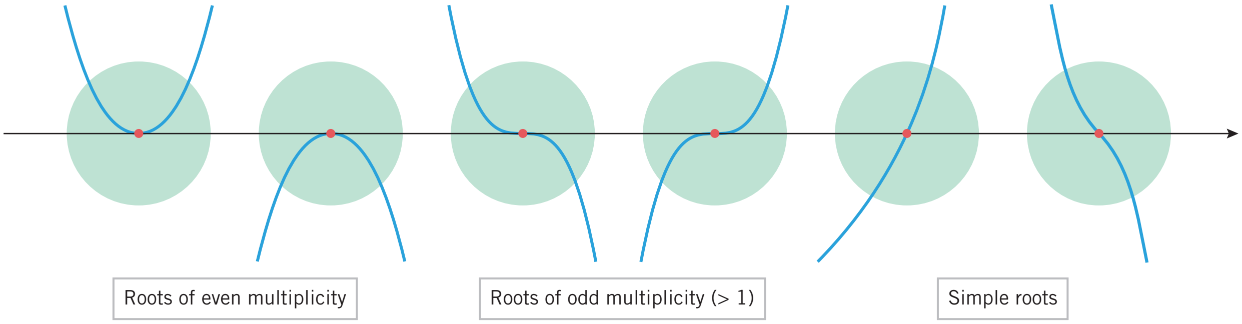
Recall that a root $x = r$ of a polynomial $p(x)$ has ***multiplicity m*** if $(x - r)^m$ divides $p(x)$ but $(x - r)^{m+1}$ does not. A root of multiplicity 1 is called a ***simple root***. Figure 3.2.9 and the following theorem show that the behavior of a polynomial in the vicinity of a real root is determined by the multiplicity of that root (we omit the proof).

3.2.5 THE GEOMETRIC IMPLICATIONS OF MULTIPLICITY *Suppose that $p(x)$ is a polynomial with a root of multiplicity m at $x = r$.*

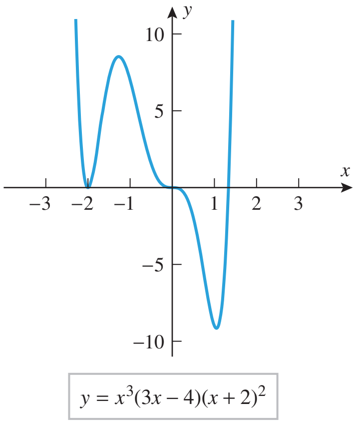
(a) *If m is even, then the graph of $y = p(x)$ is tangent to the x-axis at $x = r$, does not cross the x-axis there, and does not have an inflection point there.*

(b) *If m is odd and greater than 1, then the graph is tangent to the x-axis at $x = r$, crosses the x-axis there, and also has an inflection point there.*

(c) *If $m = 1$ (so that the root is simple), then the graph is not tangent to the x-axis at $x = r$, crosses the x-axis there, and may or may not have an inflection point there.*



▲ **Figure 3.2.9**



▲ **Figure 3.2.10**

▶ **Example 6** Make a conjecture about the behavior of the graph of

$$y = x^3(3x - 4)(x + 2)^2$$

in the vicinity of its x-intercepts, and test your conjecture by generating the graph.

Solution. The x-intercepts occur at $x = 0$, $x = \frac{4}{3}$, and $x = -2$. The root $x = 0$ has multiplicity 3, which is odd, so at that point the graph should be tangent to the x-axis, cross the x-axis, and have an inflection point there. The root $x = -2$ has multiplicity 2, which is even, so the graph should be tangent to but not cross the x-axis there. The root $x = \frac{4}{3}$ is simple, so at that point the curve should cross the x-axis without being tangent to it. All of this is consistent with the graph in Figure 3.2.10. ◀

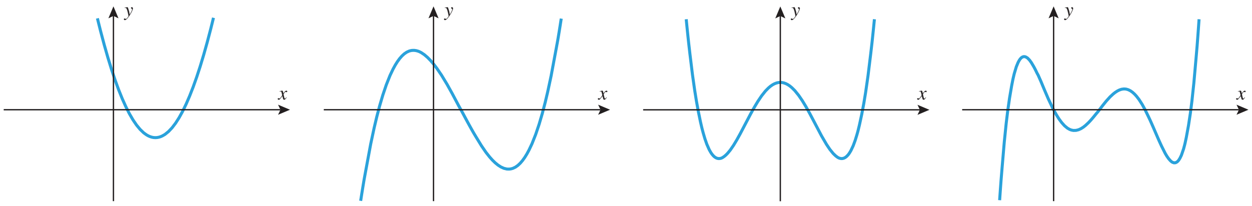
ANALYSIS OF POLYNOMIALS

Historically, the term "curve sketching" meant using calculus to help draw the graph of a function by hand—the graph was the goal. Since graphs can now be produced with great precision using calculators and computers, the purpose of curve sketching has changed. Today, we typically start with a graph produced by a calculator or computer, then use curve sketching to identify important features of the graph that the calculator or computer might have missed. Thus, the goal of curve sketching is no longer the graph itself, but rather the information it reveals about the function.

Polynomials are among the simplest functions to graph and analyze. Their significant features are symmetry, intercepts, relative extrema, inflection points, and the behavior as $x \to +\infty$ and as $x \to -\infty$. Figure 3.2.11 shows the graphs of four polynomials in x. The graphs in Figure 3.2.11 have properties that are common to all polynomials:

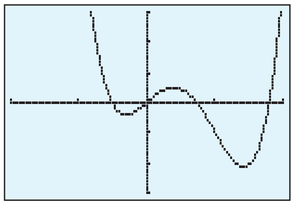
- The natural domain of a polynomial is $(-\infty, +\infty)$.
- Polynomials are continuous everywhere.
- Polynomials are differentiable everywhere, so their graphs have no corners or vertical tangent lines.
- The graph of a nonconstant polynomial eventually increases or decreases without bound as $x \to +\infty$ and as $x \to -\infty$. This is because the limit of a nonconstant polynomial as $x \to +\infty$ or as $x \to -\infty$ is $\pm\infty$, depending on the sign of the term of highest degree and whether the polynomial has even or odd degree [see Formulas (13) and (14) of Section 1.3 and the related discussion].
- The graph of a polynomial of degree $n\,(> 2)$ has at most n x-intercepts, at most $n-1$ relative extrema, and at most $n-2$ inflection points. This is because the x-intercepts, relative extrema, and inflection points of a polynomial $p(x)$ are among the real solutions of the equations $p(x) = 0$, $p'(x) = 0$, and $p''(x) = 0$, and the polynomials in these equations have degree n, $n-1$, and $n-2$, respectively. Thus, for example, the graph of a quadratic polynomial has at most two x-intercepts, one relative extremum, and no inflection points; and the graph of a cubic polynomial has at most three x-intercepts, two relative extrema, and one inflection point.

For each of the graphs in Figure 3.2.11, count the number of x-intercepts, relative extrema, and inflection points, and confirm that your count is consistent with the degree of the polynomial.



Degree 2 Degree 3 Degree 4 Degree 5

▲ **Figure 3.2.11**



[−2, 2] × [−3, 3]

$y = 3x^4 - 6x^3 + 2x$

▲ **Figure 3.2.12**

► **Example 7** Figure 3.2.12 shows the graph of

$$y = 3x^4 - 6x^3 + 2x$$

produced on a graphing calculator. Confirm that the graph is not missing any significant features.

Solution. We can be confident that the graph shows all significant features of the polynomial because the polynomial has degree 4 and we can account for four roots, three relative extrema, and two inflection points. Moreover, the graph suggests the correct behavior as $x \to +\infty$ and as $x \to -\infty$, since

$$\lim_{x\to+\infty}(3x^4 - 6x^3 + 2x) = \lim_{x\to+\infty} 3x^4 = +\infty$$

$$\lim_{x\to-\infty}(3x^4 - 6x^3 + 2x) = \lim_{x\to-\infty} 3x^4 = +\infty \blacktriangleleft$$

► **Example 8** Sketch the graph of the equation

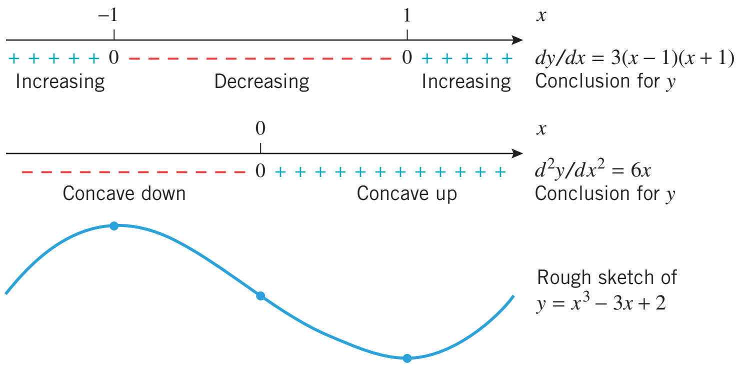
$$y = x^3 - 3x + 2$$

and identify the locations of the intercepts, relative extrema, and inflection points.

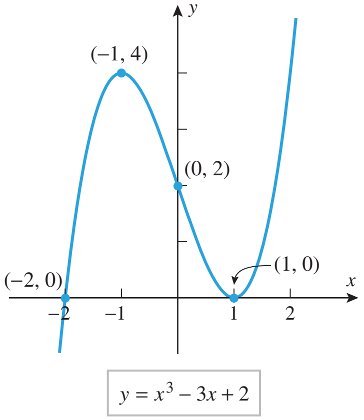
Solution. The following analysis will produce the information needed to sketch the graph:

A review of polynomial factoring is given in Appendix C.

- *x-intercepts:* Factoring the polynomial yields
$$x^3 - 3x + 2 = (x + 2)(x - 1)^2$$
which tells us that the x-intercepts are $x = -2$ and $x = 1$.
- *y-intercept:* Setting $x = 0$ yields $y = 2$.
- *End behavior:* We have
$$\lim_{x\to+\infty} (x^3 - 3x + 2) = \lim_{x\to+\infty} x^3 = +\infty$$
$$\lim_{x\to-\infty} (x^3 - 3x + 2) = \lim_{x\to-\infty} x^3 = -\infty$$
so the graph increases without bound as $x \to +\infty$ and decreases without bound as $x \to -\infty$.
- *Derivatives:*
$$\frac{dy}{dx} = 3x^2 - 3 = 3(x - 1)(x + 1)$$
$$\frac{d^2y}{dx^2} = 6x$$
- *Increase, decrease, relative extrema, inflection points:* Figure 3.2.13 gives a sign analysis of the first and second derivatives and indicates its geometric significance. There are stationary points at $x = -1$ and $x = 1$. Since the sign of dy/dx changes from $+$ to $-$ at $x = -1$, there is a relative maximum there, and since it changes from $-$ to $+$ at $x = 1$, there is a relative minimum there. The sign of d^2y/dx^2 changes from $-$ to $+$ at $x = 0$, so there is an inflection point there.
- *Final sketch:* Figure 3.2.14 shows the final sketch with the coordinates of the intercepts, relative extrema, and inflection point labeled. ◄



▲ **Figure 3.2.13**



▲ **Figure 3.2.14**

✔ QUICK CHECK EXERCISES 3.2 *(See page 207 for answers.)*

1. A function f has a relative maximum at x_0 if there is an open interval containing x_0 on which $f(x)$ is ________ $f(x_0)$ for every x in the interval.
2. Suppose that f is defined everywhere and $x = 2, 3, 5, 7$ are critical points for f. If $f'(x)$ is positive on the intervals $(-\infty, 2)$ and $(5, 7)$, and if $f'(x)$ is negative on the intervals $(2, 3)$, $(3, 5)$, and $(7, +\infty)$, then f has relative maxima at $x =$ ________ and f has relative minima at $x =$ ________.
3. Suppose that f is defined everywhere and $x = -2$ and $x = 1$ are critical points for f. If $f''(x) = 2x + 1$, then f has a relative ________ at $x = -2$ and f has a relative ________ at $x = 1$.

4. Let $f(x) = (x^2 - 4)^2$. Then $f'(x) = 4x(x^2 - 4)$ and $f''(x) = 4(3x^2 - 4)$. Identify the locations of the (a) relative maxima, (b) relative minima, and (c) inflection points on the graph of f.

EXERCISE SET 3.2

Graphing Utility C CAS

FOCUS ON CONCEPTS

1. In each part, sketch the graph of a continuous function f with the stated properties.

(a) f is concave up on the interval $(-\infty, +\infty)$ and has exactly one relative extremum.

(b) f is concave up on the interval $(-\infty, +\infty)$ and has no relative extrema.

(c) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to +\infty$ as $x \to +\infty$.

(d) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to -\infty$ as $x \to +\infty$.

2. In each part, sketch the graph of a continuous function f with the stated properties.

(a) f has exactly one relative extremum on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

(b) f has exactly two relative extrema on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

(c) f has exactly one inflection point and one relative extremum on $(-\infty, +\infty)$.

(d) f has infinitely many relative extrema, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

3. (a) Use both the first and second derivative tests to show that $f(x) = 3x^2 - 6x + 1$ has a relative minimum at $x = 1$.

(b) Use both the first and second derivative tests to show that $f(x) = x^3 - 3x + 3$ has a relative minimum at $x = 1$ and a relative maximum at $x = -1$.

4. (a) Use both the first and second derivative tests to show that $f(x) = \sin^2 x$ has a relative minimum at $x = 0$.

(b) Use both the first and second derivative tests to show that $g(x) = \tan^2 x$ has a relative minimum at $x = 0$.

(c) Give an informal verbal argument to explain without calculus why the functions in parts (a) and (b) have relative minima at $x = 0$.

5. (a) Show that both of the functions $f(x) = (x - 1)^4$ and $g(x) = x^3 - 3x^2 + 3x - 2$ have stationary points at $x = 1$.

(b) What does the second derivative test tell you about the nature of these stationary points?

(c) What does the first derivative test tell you about the nature of these stationary points?

6. (a) Show that $f(x) = 1 - x^5$ and $g(x) = 3x^4 - 8x^3$ both have stationary points at $x = 0$.

(b) What does the second derivative test tell you about the nature of these stationary points?

(c) What does the first derivative test tell you about the nature of these stationary points?

7–14 Locate the critical points and identify which critical points are stationary points. ■

7. $f(x) = 4x^4 - 16x^2 + 17$

8. $f(x) = 3x^4 + 12x$

9. $f(x) = \dfrac{x + 1}{x^2 + 3}$

10. $f(x) = \dfrac{x^2}{x^3 + 8}$

11. $f(x) = \sqrt[3]{x^2 - 25}$

12. $f(x) = x^2(x - 1)^{2/3}$

13. $f(x) = |\sin x|$

14. $f(x) = \sin |x|$

15–18 True–False Assume that f is continuous everywhere. Determine whether the statement is true or false. Explain your answer. ■

15. If f has a relative maximum at $x = 1$, then $f(1) \geq f(2)$.

16. If f has a relative maximum at $x = 1$, then $x = 1$ is a critical point for f.

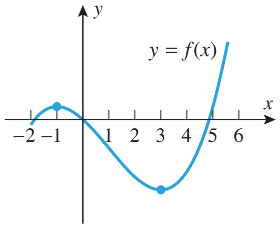
17. If $f''(x) > 0$, then f has a relative minimum at $x = 1$.

18. If $p(x)$ is a polynomial such that $p'(x)$ has a simple root at $x = 1$, then p has a relative extremum at $x = 1$.

FOCUS ON CONCEPTS

19–20 The graph of a function $f(x)$ is given. Sketch graphs of $y = f'(x)$ and $y = f''(x)$. ■

19.



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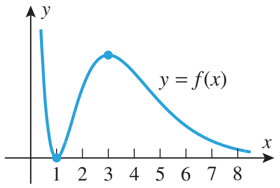
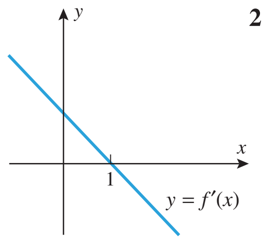


21–24 Use the graph of f' shown in the figure on the next page to estimate all values of x at which f has (a) relative minima, (b) relative maxima, and (c) inflection points. (d) Draw a rough sketch of the graph of a function f with the given derivative. ■

21.



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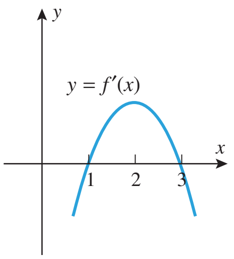

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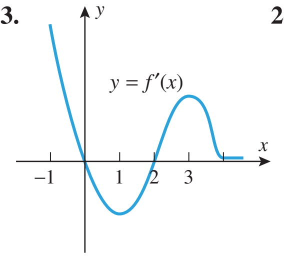

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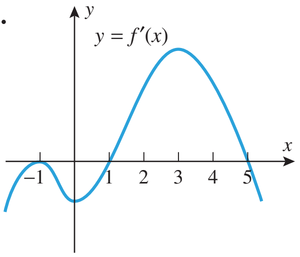


25–28 Use the given derivative to find all critical points of f, and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere.

25. $f'(x) = x^2(x^3 - 5)$ **26.** $f'(x) = 4x^3 - 9x$

27. $f'(x) = \dfrac{2 - 3x}{\sqrt[3]{x + 2}}$ **28.** $f'(x) = \dfrac{x^2 - 7}{\sqrt[3]{x^2 + 4}}$

29–32 Find the relative extrema using both first and second derivative tests.

29. $f(x) = 1 + 8x - 3x^2$ **30.** $f(x) = x^4 - 12x^3$

31. $f(x) = \sin 2x, \quad 0 < x < \pi$

32. $f(x) = x + \sin 2x, \quad 0 < x < \pi$

33–42 Use any method to find the relative extrema of the function f.

33. $f(x) = x^4 - 4x^3 + 4x^2$ **34.** $f(x) = x(x - 4)^3$

35. $f(x) = x^3(x + 1)^2$ **36.** $f(x) = x^2(x + 1)^3$

37. $f(x) = 2x + 3x^{2/3}$ **38.** $f(x) = 2x + 3x^{1/3}$

39. $f(x) = \dfrac{x + 3}{x - 2}$ **40.** $f(x) = \dfrac{x^2}{x^4 + 16}$

41. $f(x) = |3x - x^2|$ **42.** $f(x) = |1 + \sqrt[3]{x}|$

43–52 Give a graph of the polynomial and label the coordinates of the intercepts, stationary points, and inflection points. Check your work with a graphing utility.

43. $p(x) = x^2 - 3x - 4$ **44.** $p(x) = 1 + 8x - x^2$

45. $p(x) = 2x^3 - 3x^2 - 36x + 5$

46. $p(x) = 2 - x + 2x^2 - x^3$

47. $p(x) = (x + 1)^2(2x - x^2)$

48. $p(x) = x^4 - 6x^2 + 5$

49. $p(x) = x^4 - 2x^3 + 2x - 1$ **50.** $p(x) = 4x^3 - 9x^4$

51. $p(x) = x(x^2 - 1)^2$ **52.** $p(x) = x(x^2 - 1)^3$

53. In each part: (i) Make a conjecture about the behavior of the graph in the vicinity of its x-intercepts. (ii) Make a rough sketch of the graph based on your conjecture and the limits of the polynomial as $x \to +\infty$ and as $x \to -\infty$. (iii) Compare your sketch to the graph generated with a graphing utility.
(a) $y = x(x - 1)(x + 1)$ (b) $y = x^2(x - 1)^2(x + 1)^2$
(c) $y = x^2(x - 1)^2(x + 1)^3$ (d) $y = x(x - 1)^5(x + 1)^4$

54. Sketch the graph of $y = (x - a)^m(x - b)^n$ for the stated values of m and n, assuming that $a < b$ (six graphs in total).
(a) $m = 1, \ n = 1, 2, 3$ (b) $m = 2, \ n = 2, 3$
(c) $m = 3, \ n = 3$

55–58 Find the relative extrema in the interval $0 < x < 2\pi$, and confirm that your results are consistent with the graph of f generated with a graphing utility.

55. $f(x) = |\sin 2x|$ **56.** $f(x) = \sqrt{3}x + 2\sin x$

57. $f(x) = \cos^2 x$ **58.** $f(x) = \dfrac{\sin x}{2 - \cos x}$

59–60 Use a graphing utility to generate the graphs of f' and f'' over the stated interval, and then use those graphs to estimate the x-coordinates of the relative extrema of f. Check that your estimates are consistent with the graph of f.

59. $f(x) = x^4 - 24x^2 + 12x, \quad -5 \le x \le 5$

60. $f(x) = \sin \frac{1}{2}x \cos x, \quad -\pi/2 \le x \le \pi/2$

C **61–64** Use a CAS to graph f' and f'', and then use those graphs to estimate the x-coordinates of the relative extrema of f. Check that your estimates are consistent with the graph of f.

61. $f(x) = \dfrac{10x^3 - 3}{3x^2 - 5x + 8}$ **62.** $f(x) = \dfrac{x^3 - x^2}{x^2 + 1}$

63. $f(x) = \sqrt{x^4 + \cos^2 x}$ **64.** $f(x) = \dfrac{x^3 - 8x + 7}{\sqrt{x^2 + 1}}$

65. In each part, find k so that f has a relative extremum at the point where $x = 3$.
(a) $f(x) = x^2 + \dfrac{k}{x}$ (b) $f(x) = \dfrac{x}{x^2 + k}$

C **66.** (a) Use a CAS to graph the function

$$f(x) = \frac{x^4 + 1}{x^2 + 1}$$

and use the graph to estimate the x-coordinates of the relative extrema.
(b) Find the exact x-coordinates by using the CAS to solve the equation $f'(x) = 0$.

67. Let h and g have relative maxima at x_0. Prove or disprove:
(a) $h + g$ has a relative maximum at x_0
(b) $h - g$ has a relative maximum at x_0.

68. Sketch some curves that show that the three parts of the first derivative test (Theorem 3.2.3) can be false without the assumption that f is continuous at x_0.

69. Writing Discuss the relative advantages or disadvantages of using the first derivative test versus using the second derivative test to classify candidates for relative extrema on the interior of the domain of a function. Include specific examples to illustrate your points.

70. Writing If $p(x)$ is a polynomial, discuss the usefulness of knowing zeros for p, p', and p'' when determining information about the graph of p.

✔QUICK CHECK ANSWERS 3.2

1. less than or equal to **2.** 2, 7; 5 **3.** maximum; minimum **4.** (a) (0, 16) (b) (−2, 0) and (2, 0) (c) $(-2/\sqrt{3}, 64/9)$ and $(2/\sqrt{3}, 64/9)$

3.3 ANALYSIS OF FUNCTIONS III: RATIONAL FUNCTIONS, CUSPS, AND VERTICAL TANGENTS

In this section we will discuss procedures for graphing rational functions and other kinds of curves. We will also discuss the interplay between calculus and technology in curve sketching.

PROPERTIES OF GRAPHS

In many problems, the properties of interest in the graph of a function are:

- symmetries
- x-intercepts
- relative extrema
- intervals of increase and decrease
- asymptotes
- periodicity
- y-intercepts
- concavity
- inflection points
- behavior as $x \to +\infty$ or as $x \to -\infty$

Some of these properties may not be relevant in certain cases; for example, asymptotes are characteristic of rational functions but not of polynomials, and periodicity is characteristic of trigonometric functions but not of polynomial or rational functions. Thus, when analyzing the graph of a function f, it helps to know something about the general properties of the family to which it belongs.

In a given problem you will usually have a definite objective for your analysis of a graph. For example, you may be interested in showing all of the important characteristics of the function, you may only be interested in the behavior of the graph as $x \to +\infty$ or as $x \to -\infty$, or you may be interested in some specific feature such as a particular inflection point. Thus, your objectives in the problem will dictate those characteristics on which you want to focus.

GRAPHING RATIONAL FUNCTIONS

Recall that a rational function is a function of the form $f(x) = P(x)/Q(x)$ in which $P(x)$ and $Q(x)$ are polynomials. Graphs of rational functions are more complicated than those of polynomials because of the possibility of asymptotes and discontinuities (see Figure 0.3.11, for example). If $P(x)$ and $Q(x)$ have no common factors, then the information obtained in the following steps will usually be sufficient to obtain an accurate sketch of the graph of a rational function.

Graphing a Rational Function $f(x) = P(x)/Q(x)$ if $P(x)$ and $Q(x)$ have no Common Factors

Step 1. (symmetries). Determine whether there is symmetry about the y-axis or the origin.

Step 2. (x- and y-intercepts). Find the x- and y-intercepts.

Step 3. (vertical asymptotes). Find the values of x for which $Q(x) = 0$. The graph has a vertical asymptote at each such value.

Step 4. (sign of $f(x)$). The only places where $f(x)$ can change sign are at the x-intercepts or vertical asymptotes. Mark the points on the x-axis at which these occur and calculate a sample value of $f(x)$ in each of the open intervals determined by these points. This will tell you whether $f(x)$ is positive or negative over that interval.

Step 5. (end behavior). Determine the end behavior of the graph by computing the limits of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. If either limit has a finite value L, then the line $y = L$ is a horizontal asymptote.

Step 6. (derivatives). Find $f'(x)$ and $f''(x)$.

Step 7. (conclusions and graph). Analyze the sign changes of $f'(x)$ and $f''(x)$ to determine the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down. Determine the locations of all stationary points, relative extrema, and inflection points. Use the sign analysis of $f(x)$ to determine the behavior of the graph in the vicinity of the vertical asymptotes. Sketch a graph of f that exhibits these conclusions.

▶ **Example 1** Sketch a graph of the equation

$$y = \frac{2x^2 - 8}{x^2 - 16}$$

and identify the locations of the intercepts, relative extrema, inflection points, and asymptotes.

Solution. The numerator and denominator have no common factors, so we will use the procedure just outlined.

- *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.
- *x- and y-intercepts:* Setting $y = 0$ yields the x-intercepts $x = -2$ and $x = 2$. Setting $x = 0$ yields the y-intercept $y = \frac{1}{2}$.
- *Vertical asymptotes:* We observed above that the numerator and denominator of y have no common factors, so the graph has vertical asymptotes at the points where the denominator of y is zero, namely, at $x = -4$ and $x = 4$.
- *Sign of y:* The set of points where x-intercepts or vertical asymptotes occur is $\{-4, -2, 2, 4\}$. These points divide the x-axis into the open intervals

$$(-\infty, -4), \quad (-4, -2), \quad (-2, 2), \quad (2, 4), \quad (4, +\infty)$$

We can find the sign of y on each interval by choosing an arbitrary test point in the interval and evaluating $y = f(x)$ at the test point (Table 3.3.1). This analysis is summarized on the first line of Figure 3.3.1a.

Table 3.3.1

SIGN ANALYSIS OF $y = \dfrac{2x^2 - 8}{x^2 - 16}$

INTERVAL	TEST POINT	VALUE OF y	SIGN OF y
$(-\infty, -4)$	-5	$14/3$	+
$(-4, -2)$	-3	$-10/7$	–
$(-2, 2)$	0	$1/2$	+
$(2, 4)$	3	$-10/7$	–
$(4, +\infty)$	5	$14/3$	+

- *End behavior:* The limits

$$\lim_{x\to+\infty}\frac{2x^2-8}{x^2-16}=\lim_{x\to+\infty}\frac{2-(8/x^2)}{1-(16/x^2)}=2$$

$$\lim_{x\to-\infty}\frac{2x^2-8}{x^2-16}=\lim_{x\to-\infty}\frac{2-(8/x^2)}{1-(16/x^2)}=2$$

  yield the horizontal asymptote $y=2$.

- *Derivatives:*

$$\frac{dy}{dx}=\frac{(x^2-16)(4x)-(2x^2-8)(2x)}{(x^2-16)^2}=-\frac{48x}{(x^2-16)^2}$$

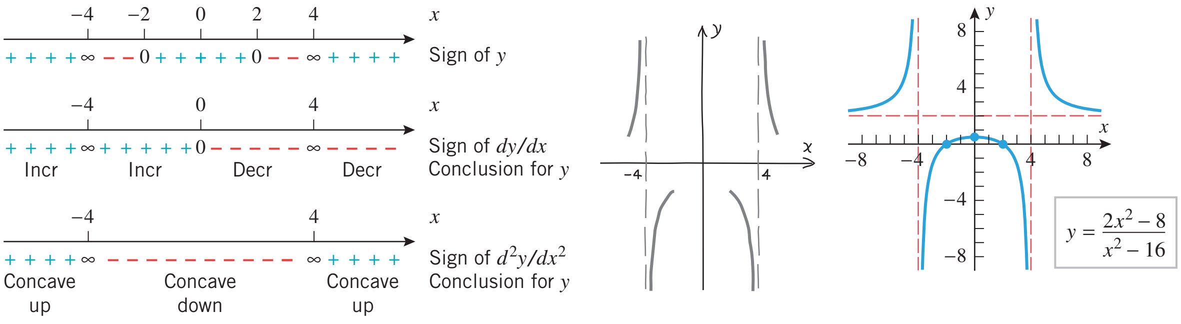
$$\frac{d^2y}{dx^2}=\frac{48(16+3x^2)}{(x^2-16)^3}\quad\text{(verify)}$$

Conclusions and graph:

- The sign analysis of y in Figure 3.3.1a reveals the behavior of the graph in the vicinity of the vertical asymptotes: The graph increases without bound as $x\to-4^-$ and decreases without bound as $x\to-4^+$; and the graph decreases without bound as $x\to4^-$ and increases without bound as $x\to4^+$ (Figure 3.3.1b).
- The sign analysis of dy/dx in Figure 3.3.1a shows that the graph is increasing to the left of $x=0$ and is decreasing to the right of $x=0$. Thus, there is a relative maximum at the stationary point $x=0$. There are no relative minima.
- The sign analysis of d^2y/dx^2 in Figure 3.3.1a shows that the graph is concave up to the left of $x=-4$, is concave down between $x=-4$ and $x=4$, and is concave up to the right of $x=4$. There are no inflection points.

The procedure we stated for graphing a rational function $P(x)/Q(x)$ applies only if the polynomials $P(x)$ and $Q(x)$ have no common factors. How would you find the graph if those polynomials have common factors?

The graph is shown in Figure 3.3.1c. ◄



(a) (b) (c)

▲ **Figure 3.3.1**

▶ **Example 2** Sketch a graph of

$$y=\frac{x^2-1}{x^3}$$

and identify the locations of all asymptotes, intercepts, relative extrema, and inflection points.

Solution. The numerator and denominator have no common factors, so we will use the procedure outlined previously.

- *Symmetries:* Replacing x by $-x$ and y by $-y$ yields an equation that simplifies to the original equation, so the graph is symmetric about the origin.

- *x- and y-intercepts:* Setting $y = 0$ yields the x-intercepts $x = -1$ and $x = 1$. Setting $x = 0$ leads to a division by zero, so there is no y-intercept.
- *Vertical asymptotes:* Setting $x^3 = 0$ yields the solution $x = 0$. This is not a root of $x^2 - 1$, so $x = 0$ is a vertical asymptote.
- *Sign of y:* The set of points where x-intercepts or vertical asymptotes occur is $\{-1, 0, 1\}$. These points divide the x-axis into the open intervals

$$(-\infty, -1), \quad (-1, 0), \quad (0, 1), \quad (1, +\infty)$$

Table 3.3.2 uses the method of test points to produce the sign of y on each of these intervals.

Table 3.3.2

SIGN ANALYSIS OF $y = \dfrac{x^2 - 1}{x^3}$

INTERVAL	TEST POINT	VALUE OF y	SIGN OF y
$(-\infty, -1)$	-2	$-\frac{3}{8}$	$-$
$(-1, 0)$	$-\frac{1}{2}$	6	$+$
$(0, 1)$	$\frac{1}{2}$	-6	$-$
$(1, +\infty)$	2	$\frac{3}{8}$	$+$

- *End behavior:* The limits

$$\lim_{x \to +\infty} \frac{x^2 - 1}{x^3} = \lim_{x \to +\infty} \left(\frac{1}{x} - \frac{1}{x^3}\right) = 0$$
$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^3} = \lim_{x \to -\infty} \left(\frac{1}{x} - \frac{1}{x^3}\right) = 0$$

yield the horizontal asymptote $y = 0$.

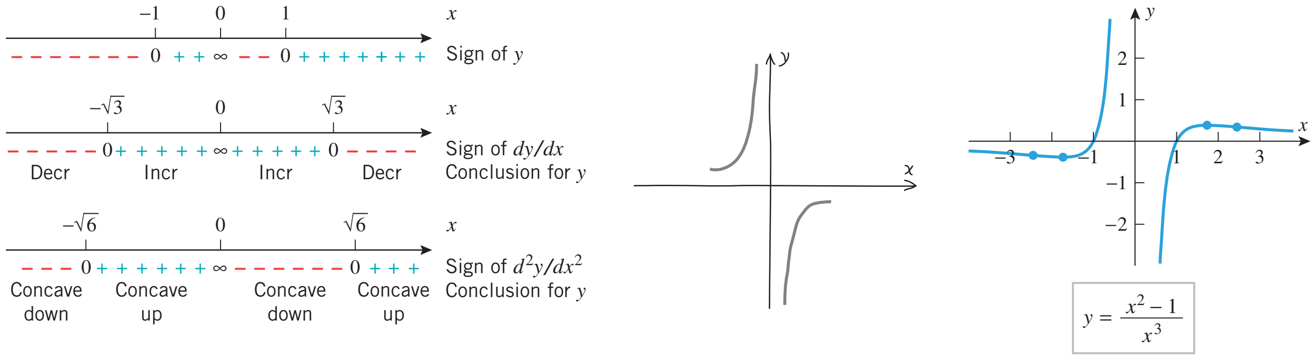
- *Derivatives:*

$$\frac{dy}{dx} = \frac{x^3(2x) - (x^2 - 1)(3x^2)}{(x^3)^2} = \frac{3 - x^2}{x^4} = \frac{(\sqrt{3} + x)(\sqrt{3} - x)}{x^4}$$
$$\frac{d^2y}{dx^2} = \frac{x^4(-2x) - (3 - x^2)(4x^3)}{(x^4)^2} = \frac{2(x^2 - 6)}{x^5} = \frac{2(x - \sqrt{6})(x + \sqrt{6})}{x^5}$$

Conclusions and graph:

- The sign analysis of y in Figure 3.3.2*a* reveals the behavior of the graph in the vicinity of the vertical asymptote $x = 0$: The graph increases without bound as $x \to 0^-$ and decreases without bound as $x \to 0^+$ (Figure 3.3.2*b*).
- The sign analysis of dy/dx in Figure 3.3.2*a* shows that there is a relative minimum at $x = -\sqrt{3}$ and a relative maximum at $x = \sqrt{3}$.
- The sign analysis of d^2y/dx^2 in Figure 3.3.2*a* shows that the graph changes concavity at the vertical asymptote $x = 0$ and that there are inflection points at $x = -\sqrt{6}$ and $x = \sqrt{6}$.

The graph is shown in Figure 3.3.2*c*. To produce a slightly more accurate sketch, we used a graphing utility to help plot the relative extrema and inflection points. You should confirm



(*a*) (*b*) (*c*)

▲ **Figure 3.3.2**

that the approximate coordinates of the inflection points are $(-2.45, -0.34)$ and $(2.45, 0.34)$ and that the approximate coordinates of the relative minimum and relative maximum are $(-1.73, -0.38)$ and $(1.73, 0.38)$, respectively. ◀

RATIONAL FUNCTIONS WITH OBLIQUE OR CURVILINEAR ASYMPTOTES

In the rational functions of Examples 1 and 2, the degree of the numerator did not exceed the degree of the denominator, and the asymptotes were either vertical or horizontal. If the numerator of a rational function has greater degree than the denominator, then other kinds of "asymptotes" are possible. For example, consider the rational functions

$$f(x) = \frac{x^2+1}{x} \quad \text{and} \quad g(x) = \frac{x^3-x^2-8}{x-1} \tag{1}$$

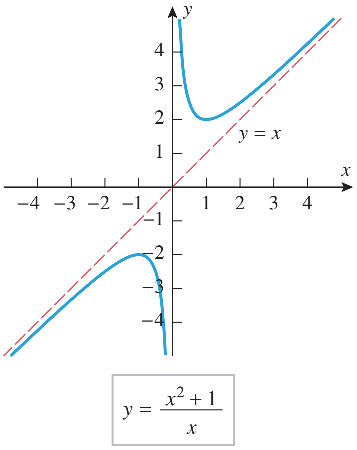
By division we can rewrite these as

$$f(x) = x + \frac{1}{x} \quad \text{and} \quad g(x) = x^2 - \frac{8}{x-1}$$

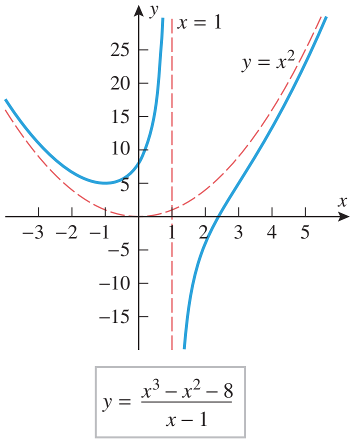
Since the second terms both approach 0 as $x \to +\infty$ or as $x \to -\infty$, it follows that

$$(f(x) - x) \to 0 \quad \text{as } x \to +\infty \text{ or as } x \to -\infty$$
$$(g(x) - x^2) \to 0 \quad \text{as } x \to +\infty \text{ or as } x \to -\infty$$

Geometrically, this means that the graph of $y = f(x)$ eventually gets closer and closer to the line $y = x$ as $x \to +\infty$ or as $x \to -\infty$. The line $y = x$ is called an ***oblique*** or ***slant asymptote*** of f. Similarly, the graph of $y = g(x)$ eventually gets closer and closer to the parabola $y = x^2$ as $x \to +\infty$ or as $x \to -\infty$. The parabola is called a ***curvilinear asymptote*** of g. The graphs of the functions in (1) are shown in Figures 3.3.3 and 3.3.4.



▲ **Figure 3.3.3**



▲ **Figure 3.3.4**

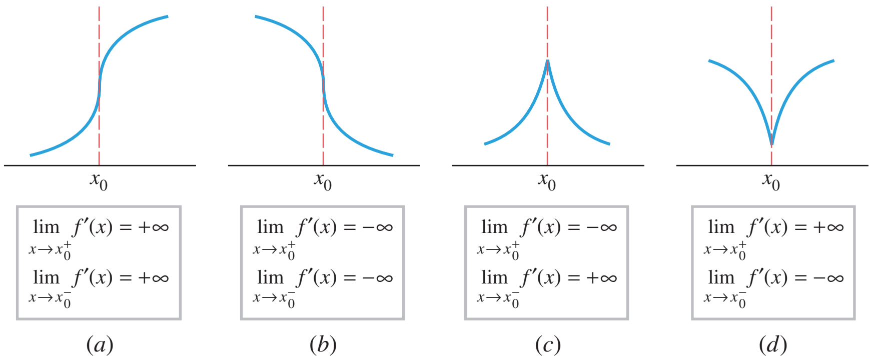
In general, if $f(x) = P(x)/Q(x)$ is a rational function, then we can find quotient and remainder polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x) + \frac{r(x)}{Q(x)}$$

and the degree of $r(x)$ is less than the degree of $Q(x)$. Then $r(x)/Q(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$, so $y = q(x)$ is an asymptote of f. This asymptote will be an oblique line if the degree of $P(x)$ is one greater than the degree of $Q(x)$, and it will be curvilinear if the degree of $P(x)$ exceeds that of $Q(x)$ by two or more. Problems involving these kinds of asymptotes are given in the exercises (Exercises 17 and 18).

GRAPHS WITH VERTICAL TANGENTS AND CUSPS

Figure 3.3.5 shows four curve elements that are commonly found in graphs of functions that involve radicals or fractional exponents. In all four cases, the function is not differentiable at x_0 because the secant line through $(x_0, f(x_0))$ and $(x, f(x))$ approaches a vertical position as x approaches x_0 from either side. Thus, in each case, the curve has a vertical tangent line



▶ **Figure 3.3.5**

at $(x_0, f(x_0))$. In parts (a) and (b) of the figure, there is an inflection point at x_0 because there is a change in concavity at that point. In parts (c) and (d), where $f'(x)$ approaches $+\infty$ from one side of x_0 and $-\infty$ from the other side, we say that the graph has a ***cusp*** at x_0.

The steps that are used to sketch the graph of a rational function can serve as guidelines for sketching graphs of other types of functions. This is illustrated in Examples 3 and 4.

▶ **Example 3** Sketch the graph of $y = (x-4)^{2/3}$.

- *Symmetries:* There are no symmetries about the coordinate axes or the origin (verify). However, the graph of $y = (x-4)^{2/3}$ is symmetric about the line $x = 4$ since it is a translation (4 units to the right) of the graph of $y = x^{2/3}$, which is symmetric about the y-axis.
- *x- and y-intercepts:* Setting $y = 0$ yields the x-intercept $x = 4$. Setting $x = 0$ yields the y-intercept $y = \sqrt[3]{16} \approx 2.5$.
- *Vertical asymptotes:* None, since $f(x) = (x-4)^{2/3}$ is continuous everywhere.
- *End behavior:* The graph has no horizontal asymptotes since

$$\lim_{x\to+\infty} (x-4)^{2/3} = +\infty \quad \text{and} \quad \lim_{x\to-\infty} (x-4)^{2/3} = +\infty$$

- *Derivatives:*

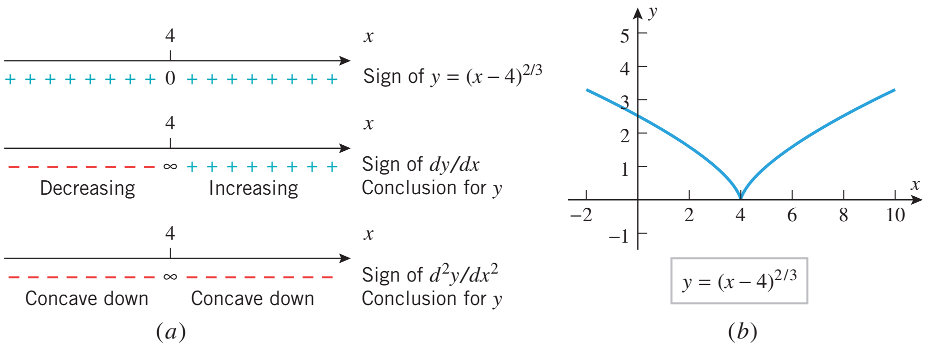
$$\frac{dy}{dx} = f'(x) = \frac{2}{3}(x-4)^{-1/3} = \frac{2}{3(x-4)^{1/3}}$$
$$\frac{d^2y}{dx^2} = f''(x) = -\frac{2}{9}(x-4)^{-4/3} = -\frac{2}{9(x-4)^{4/3}}$$

- *Vertical tangent lines:* There is a vertical tangent line and cusp at $x = 4$ of the type in Figure 3.3.5d since $f(x) = (x-4)^{2/3}$ is continuous at $x = 4$ and

$$\lim_{x\to4^+} f'(x) = \lim_{x\to4^+} \frac{2}{3(x-4)^{1/3}} = +\infty$$
$$\lim_{x\to4^-} f'(x) = \lim_{x\to4^-} \frac{2}{3(x-4)^{1/3}} = -\infty$$

Conclusions and graph:

- The function $f(x) = (x-4)^{2/3} = ((x-4)^{1/3})^2$ is nonnegative for all x. There is a zero for f at $x = 4$.
- There is a critical point at $x = 4$ since f is not differentiable there. We saw above that a cusp occurs at this point. The sign analysis of dy/dx in Figure 3.3.6a and the first derivative test show that there is a relative minimum at this cusp since $f'(x) < 0$ if $x < 4$ and $f'(x) > 0$ if $x > 4$.



▶ **Figure 3.3.6** (a) (b)

- The sign analysis of d^2y/dx^2 in Figure 3.3.6*a* shows that the graph is concave down on both sides of the cusp.

The graph is shown in Figure 3.3.6*b*. ◀

GRAPHING USING CALCULUS AND TECHNOLOGY TOGETHER

Thus far in this chapter we have used calculus to produce graphs of functions; the graph was the end result. Now we will work in the reverse direction by *starting* with a graph produced by a graphing utility. Our goal will be to use the tools of calculus to determine the exact locations of relative extrema, inflection points, and other features suggested by that graph and to determine whether the graph may be missing some important features that we would like to see.

▶ **Example 4** Use a graphing utility to generate the graph of $f(x) = 6x^{1/3} + 3x^{4/3}$, and discuss what it tells you about relative extrema, inflection points, asymptotes, and end behavior. Use calculus to find the exact locations of all key features of the graph.

Solution. Figure 3.3.7*b* shows a graph of f produced by a graphing utility. The graph suggests that there are x-intercepts at $x = 0$ and $x = -2$, a relative minimum between $x = -1$ and $x = 0$, no horizontal or vertical asymptotes, a vertical tangent at $x = 0$, and an inflection point at $x = 0$. It will help in our analysis to write

$$f(x) = 6x^{1/3} + 3x^{4/3} = 3x^{1/3}(2 + x)$$

- *Symmetries:* There are no symmetries about the coordinate axes or the origin (verify).
- *x- and y-intercepts:* Setting $3x^{1/3}(2 + x) = 0$ yields the x-intercepts $x = 0$ and $x = -2$. Setting $x = 0$ yields the y-intercept $y = 0$.
- *Vertical asymptotes:* None, since $f(x) = 6x^{1/3} + 3x^{4/3}$ is continuous everywhere.
- *End behavior:* The graph has no horizontal asymptotes since

$$\lim_{x\to+\infty} (6x^{1/3} + 3x^{4/3}) = \lim_{x\to+\infty} 3x^{1/3}(2 + x) = +\infty$$
$$\lim_{x\to-\infty} (6x^{1/3} + 3x^{4/3}) = \lim_{x\to-\infty} 3x^{1/3}(2 + x) = +\infty$$

- *Derivatives:*

$$\frac{dy}{dx} = f'(x) = 2x^{-2/3} + 4x^{1/3} = 2x^{-2/3}(1 + 2x) = \frac{2(2x + 1)}{x^{2/3}}$$
$$\frac{d^2y}{dx^2} = f''(x) = -\frac{4}{3}x^{-5/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-5/3}(-1 + x) = \frac{4(x - 1)}{3x^{5/3}}$$

- *Vertical tangent lines:* There is a vertical tangent line at $x = 0$ since f is continuous there and

$$\lim_{x\to0^+} f'(x) = \lim_{x\to0^+} \frac{2(2x + 1)}{x^{2/3}} = +\infty$$
$$\lim_{x\to0^-} f'(x) = \lim_{x\to0^-} \frac{2(2x + 1)}{x^{2/3}} = +\infty$$

  This and the change in concavity at $x = 0$ mean that $(0, 0)$ is an inflection point of the type in Figure 3.3.5*a*.

Conclusions and graph:

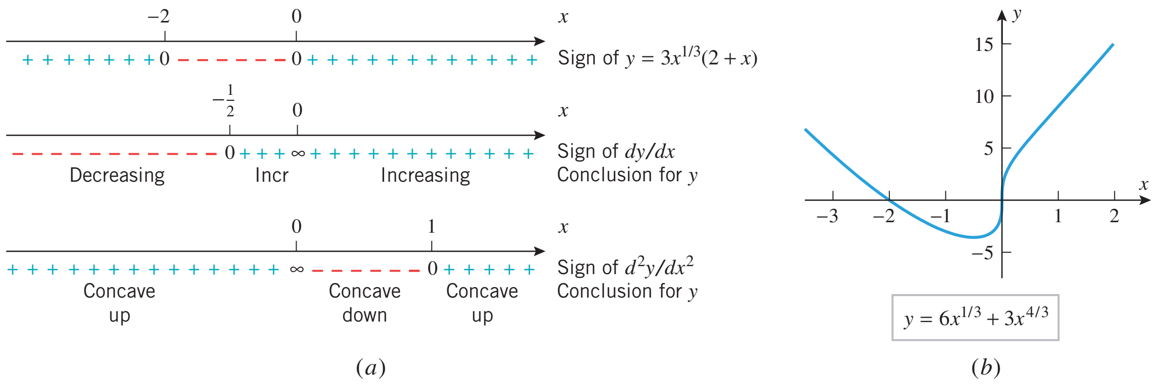
- From the sign analysis of y in Figure 3.3.7*a*, the graph is below the x-axis between the x-intercepts $x = -2$ and $x = 0$ and is above the x-axis if $x < -2$ or $x > 0$.

- From the formula for dy/dx we see that there is a stationary point at $x = -\frac{1}{2}$ and a critical point at $x = 0$ at which f is not differentiable. We saw above that a vertical tangent line and inflection point are at that critical point.
- The sign analysis of dy/dx in Figure 3.3.7a and the first derivative test show that there is a relative minimum at the stationary point at $x = -\frac{1}{2}$ (verify).
- The sign analysis of d^2y/dx^2 in Figure 3.3.7a shows that in addition to the inflection point at the vertical tangent there is an inflection point at $x = 1$ at which the graph changes from concave down to concave up.

These conclusions reinforce, and in some cases extend, the information we inferred about the graph of f from Figure 3.3.7b. ◄

TECHNOLOGY MASTERY

The graph in Figure 3.3.7b was generated with a graphing utility. However, the inflection point at $x = 1$ is so subtle that it is not evident from this graph. See if you can produce a version of this graph with your graphing utility that makes the inflection point evident.



▲ **Figure 3.3.7**

✔QUICK CHECK EXERCISES 3.3

(See page 216 for answers.)

1. Let $f(x) = \dfrac{3(x+1)(x-3)}{(x+2)(x-4)}$. Given that

$$f'(x) = \frac{-30(x-1)}{(x+2)^2(x-4)^2}, \qquad f''(x) = \frac{90(x^2-2x+4)}{(x+2)^3(x-4)^3}$$

determine the following properties of the graph of f.

(a) The x- and y-intercepts are ________.
(b) The vertical asymptotes are ________.
(c) The horizontal asymptote is ________.
(d) The graph is above the x-axis on the intervals ________.
(e) The graph is increasing on the intervals ________.
(f) The graph is concave up on the intervals ________.
(g) The relative maximum point on the graph is ________.

2. Let $f(x) = \dfrac{x^2-4}{x^{8/3}}$. Given that

$$f'(x) = \frac{-2(x^2-16)}{3x^{11/3}}, \qquad f''(x) = \frac{2(5x^2-176)}{9x^{14/3}}$$

determine the following properties of the graph of f.

(a) The x-intercepts are ________.
(b) The vertical asymptote is ________.
(c) The horizontal asymptote is ________.
(d) The graph is above the x-axis on the intervals ________.
(e) The graph is increasing on the intervals ________.
(f) The graph is concave up on the intervals ________.
(g) Inflection points occur at $x =$ ________.

EXERCISE SET 3.3

Graphing Utility

1–14 Give a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal and vertical asymptotes and label them with their equations. Label point(s), if any, where the graph crosses a horizontal asymptote. Check your work with a graphing utility.

1. $\dfrac{2x-6}{4-x}$
2. $\dfrac{8}{x^2-4}$
3. $\dfrac{x}{x^2-4}$
4. $\dfrac{x^2}{x^2-4}$
5. $\dfrac{x^2}{x^2+4}$
6. $\dfrac{(x^2-1)^2}{x^4+1}$
7. $\dfrac{x^3+1}{x^3-1}$
8. $2 - \dfrac{1}{3x^2+x^3}$
9. $\dfrac{4}{x^2} - \dfrac{2}{x} + 3$
10. $\dfrac{3(x+1)^2}{(x-1)^2}$
11. $\dfrac{(3x+1)^2}{(x-1)^2}$
12. $3 + \dfrac{x+1}{(x-1)^4}$
13. $\dfrac{x^2+x}{1-x^2}$
14. $\dfrac{x^2}{1-x^3}$

15–16 In each part, make a rough sketch of the graph using asymptotes and appropriate limits but no derivatives. Compare your graph to that generated with a graphing utility.

15. (a) $y = \dfrac{3x^2 - 8}{x^2 - 4}$ (b) $y = \dfrac{x^2 + 2x}{x^2 - 1}$

16. (a) $y = \dfrac{2x - x^2}{x^2 + x - 2}$ (b) $y = \dfrac{x^2}{x^2 - x - 2}$

17. Show that $y = x + 3$ is an oblique asymptote of the graph of $f(x) = x^2/(x - 3)$. Sketch the graph of $y = f(x)$ showing this asymptotic behavior.

18. Show that $y = 3 - x^2$ is a curvilinear asymptote of the graph of $f(x) = (2 + 3x - x^3)/x$. Sketch the graph of $y = f(x)$ showing this asymptotic behavior.

19–24 Sketch a graph of the rational function and label the coordinates of the stationary points and inflection points. Show the horizontal, vertical, oblique, and curvilinear asymptotes and label them with their equations. Label point(s), if any, where the graph crosses an asymptote. Check your work with a graphing utility.

19. $x^2 - \dfrac{1}{x}$ **20.** $\dfrac{x^2 - 2}{x}$ **21.** $\dfrac{(x - 2)^3}{x^2}$

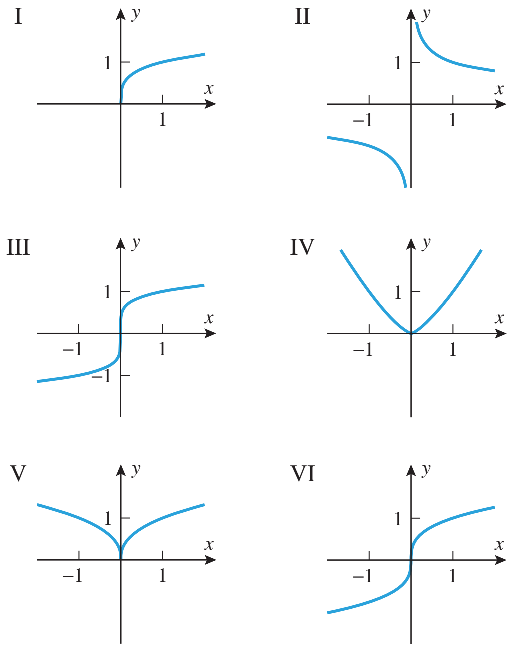
22. $x - \dfrac{1}{x} - \dfrac{1}{x^2}$ **23.** $\dfrac{x^3 - 4x - 8}{x + 2}$ **24.** $\dfrac{x^5}{x^2 + 1}$

FOCUS ON CONCEPTS

25. In each part, match the function with graphs I–VI.

(a) $x^{1/3}$ (b) $x^{1/4}$ (c) $x^{1/5}$

(d) $x^{2/5}$ (e) $x^{4/3}$ (f) $x^{-1/3}$



▲ **Figure Ex-25**

26. Sketch the general shape of the graph of $y = x^{1/n}$, and then explain in words what happens to the shape of the graph as n increases if

(a) n is a positive even integer

(b) n is a positive odd integer.

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. Suppose that $f(x) = P(x)/Q(x)$, where P and Q are polynomials with no common factors. If $y = 5$ is a horizontal asymptote for the graph of f, then P and Q have the same degree.

28. If the graph of f has a vertical asymptote at $x = 1$, then f cannot be continuous at $x = 1$.

29. If the graph of f' has a vertical asymptote at $x = 1$, then f cannot be continuous at $x = 1$.

30. If the graph of f has a cusp at $x = 1$, then f cannot have an inflection point at $x = 1$.

31–38 Give a graph of the function and identify the locations of all critical points and inflection points. Check your work with a graphing utility.

31. $\sqrt{4x^2 - 1}$ **32.** $\sqrt[3]{x^2 - 4}$

33. $2x + 3x^{2/3}$ **34.** $2x^2 - 3x^{4/3}$

35. $4x^{1/3} - x^{4/3}$ **36.** $5x^{2/3} + x^{5/3}$

37. $\dfrac{8 + x}{2 + \sqrt[3]{x}}$ **38.** $\dfrac{8(\sqrt{x} - 1)}{x}$

39–44 Give a graph of the function and identify the locations of all relative extrema and inflection points. Check your work with a graphing utility.

39. $x + \sin x$ **40.** $x - \tan x$

41. $\sqrt{3}\cos x + \sin x$ **42.** $\sin x + \cos x$

43. $\sin^2 x - \cos x, \quad -\pi \le x \le 3\pi$

44. $\sqrt{\tan x}, \quad 0 \le x < \pi/2$

FOCUS ON CONCEPTS

45. The accompanying figure on the next page shows the graph of the *derivative* of a function h that is defined and continuous on the interval $(-\infty, +\infty)$. Assume that the graph of h' has a vertical asymptote at $x = 3$ and that

$$h'(x) \to 0^+ \text{ as } x \to -\infty$$

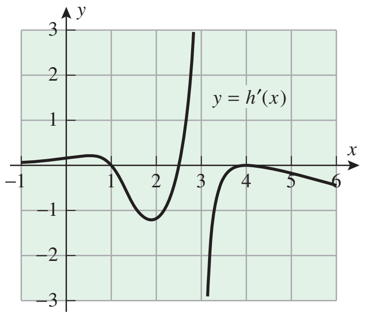
$$h'(x) \to -\infty \text{ as } x \to +\infty$$

(a) What are the critical points for $h(x)$?

(b) Identify the intervals on which $h(x)$ is increasing.

(c) Identify the x-coordinates of relative extrema for $h(x)$ and classify each as a relative maximum or relative minimum.

(d) Estimate the x-coordinates of inflection points for $h(x)$.



◀ **Figure Ex-45**

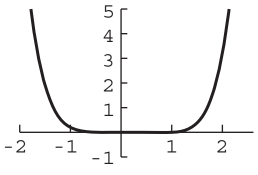
46. Let $f(x) = (1 - 2x)h(x)$, where $h(x)$ is as given in Exercise 45. Suppose that $x = 5$ is a critical point for $f(x)$.
(a) Estimate $h(5)$.
(b) Use the second derivative test to determine whether $f(x)$ has a relative maximum or a relative minimum at $x = 5$.

47. A rectangular plot of land is to be fenced off so that the area enclosed will be 400 ft^2. Let L be the length of fencing needed and x the length of one side of the rectangle. Show that $L = 2x + 800/x$ for $x > 0$, and sketch the graph of L versus x for $x > 0$.

48. A box with a square base and open top is to be made from sheet metal so that its volume is 500 in^3. Let S be the area of the surface of the box and x the length of a side of the square base. Show that $S = x^2 + 2000/x$ for $x > 0$, and sketch the graph of S versus x for $x > 0$.

49. The accompanying figure shows a computer-generated graph of the polynomial $y = 0.1x^5(x - 1)$ using a viewing window of $[-2, 2.5] \times [-1, 5]$. Show that the choice of the vertical scale caused the computer to miss important features of the graph. Find the features that were missed and make your own sketch of the graph that shows the missing features.

50. The accompanying figure shows a computer-generated graph of the polynomial $y = 0.1x^5(x + 1)^2$ using a viewing window of $[-2, 1.5] \times [-0.2, 0.2]$. Show that the choice of the vertical scale caused the computer to miss important features of the graph. Find the features that were missed and make your own sketch of the graph that shows the missing features.



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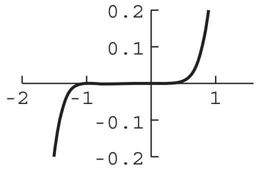
▲ **Figure Ex-49**



Generated by Mathematica

▲ **Figure Ex-50**

51. **Writing** Suppose that $x = x_0$ is a point at which a function f is continuous but not differentiable and that $f'(x)$ approaches different finite limits as x approaches x_0 from either side. Invent your own term to describe the graph of f at such a point and discuss the appropriateness of your term.

52. **Writing** Suppose that the graph of a function f is obtained using a graphing utility. Discuss the information that calculus techniques can provide about f to add to what can already be inferred about f from the graph as shown on your utility's display.

✔QUICK CHECK ANSWERS 3.3

1. (a) $(-1, 0), (3, 0), \left(0, \frac{9}{8}\right)$ (b) $x = -2$ and $x = 4$ (c) $y = 3$ (d) $(-\infty, -2), (-1, 3)$, and $(4, +\infty)$ (e) $(-\infty, -2)$ and $(-2, 1]$ (f) $(-\infty, -2)$ and $(4, +\infty)$ (g) $\left(1, \frac{4}{3}\right)$ **2.** (a) $(-2, 0), (2, 0)$ (b) $x = 0$ (c) $y = 0$ (d) $(-\infty, -2)$ and $(2, +\infty)$ (e) $(-\infty, -4]$ and $(0, 4]$ (f) $(-\infty, -4\sqrt{11/5})$ and $(4\sqrt{11/5}, +\infty)$ (g) $\pm 4\sqrt{11/5} \approx \pm 5.93$

3.4 ABSOLUTE MAXIMA AND MINIMA

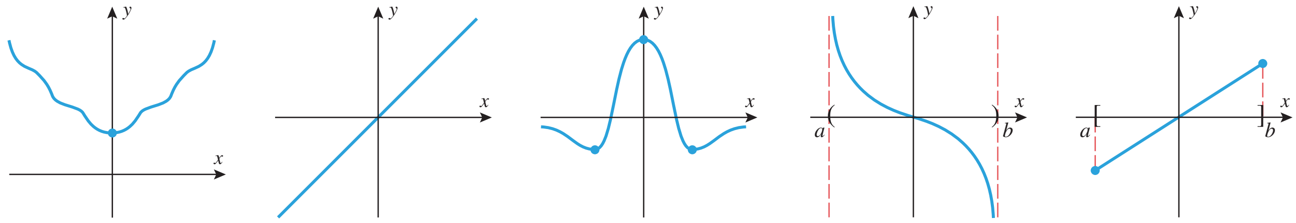
At the beginning of Section 3.2 we observed that if the graph of a function f is viewed as a two-dimensional mountain range (Figure 3.2.1), then the relative maxima and minima correspond to the tops of the hills and the bottoms of the valleys; that is, they are the high and low points in their immediate vicinity. In this section we will be concerned with the more encompassing problem of finding the highest and lowest points over the entire mountain range, that is, we will be looking for the top of the highest hill and the bottom of the deepest valley. In mathematical terms, we will be looking for the largest and smallest values of a function over an interval.

ABSOLUTE EXTREMA

We will begin with some terminology for describing the largest and smallest values of a function on an interval.

3.4.1 DEFINITION Consider an interval in the domain of a function f and a point x_0 in that interval. We say that f has an ***absolute maximum*** at x_0 if $f(x) \leq f(x_0)$ for all x in the interval, and we say that f has an ***absolute minimum*** at x_0 if $f(x_0) \leq f(x)$ for all x in the interval. We say that f has an ***absolute extremum*** at x_0 if it has either an absolute maximum or an absolute minimum at that point.

If f has an absolute maximum at the point x_0 on an interval, then $f(x_0)$ is the largest value of f on the interval, and if f has an absolute minimum at x_0, then $f(x_0)$ is the smallest value of f on the interval. In general, there is no guarantee that a function will actually have an absolute maximum or minimum on a given interval (Figure 3.4.1).



f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.
(*a*)

f has no absolute extrema on $(-\infty, +\infty)$.
(*b*)

f has an absolute maximum and minimum on $(-\infty, +\infty)$.
(*c*)

f has no absolute extrema on (a, b).
(*d*)

f has an absolute maximum and minimum on $[a, b]$.
(*e*)

▲ **Figure 3.4.1**

THE EXTREME VALUE THEOREM

Parts (*a*)–(*d*) of Figure 3.4.1 show that a continuous function may or may not have absolute maxima or minima on an infinite interval or on a finite open interval. However, the following theorem shows that a continuous function must have both an absolute maximum and an absolute minimum on every *finite closed* interval [see part (*e*) of Figure 3.4.1].

The hypotheses in the Extreme-Value Theorem are essential. That is, if either the interval is not closed or f is not continuous on the interval, then f need not have absolute extrema on the interval (Exercises 4–6).

3.4.2 THEOREM (*Extreme-Value Theorem*) *If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.*

REMARK Although the proof of this theorem is too difficult to include here, you should be able to convince yourself of its validity with a little experimentation—try graphing various continuous functions over the interval $[0, 1]$, and convince yourself that there is no way to avoid having a highest and lowest point on a graph. As a physical analogy, if you imagine the graph to be a roller-coaster track starting at $x = 0$ and ending at $x = 1$, the roller coaster will have to pass through a highest point and a lowest point during the trip.

The Extreme-Value Theorem is an example of what mathematicians call an ***existence theorem***. Such theorems state conditions under which certain objects exist, in this case absolute extrema. However, knowing that an object exists and finding it are two separate things. We will now address methods for determining the locations of absolute extrema under the conditions of the Extreme-Value Theorem.

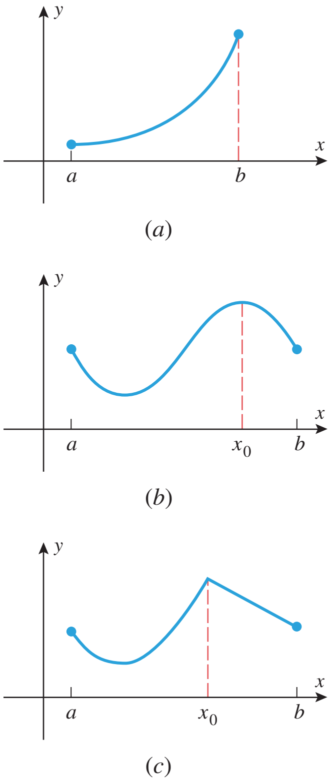
If f is continuous on the finite closed interval $[a, b]$, then the absolute extrema of f occur either at the endpoints of the interval or inside on the open interval (a, b). If the absolute extrema happen to fall inside, then the following theorem tells us that they must occur at critical points of f.

Theorem 3.4.3 is also valid on infinite open intervals, that is, intervals of the form $(-\infty, +\infty)$, $(a, +\infty)$, and $(-\infty, b)$.

3.4.3 THEOREM *If f has an absolute extremum on an open interval (a, b), then it must occur at a critical point of f.*

PROOF If f has an absolute maximum on (a, b) at x_0, then $f(x_0)$ is also a relative maximum for f; for if $f(x_0)$ is the largest value of f on all (a, b), then $f(x_0)$ is certainly the largest value for f in the immediate vicinity of x_0. Thus, x_0 is a critical point of f by Theorem 3.2.2. The proof for absolute minima is similar. ■

It follows from this theorem that if f is continuous on the finite closed interval $[a, b]$, then the absolute extrema occur either at the endpoints of the interval or at critical points inside the interval (Figure 3.4.2). Thus, we can use the following procedure to find the absolute extrema of a continuous function on a finite closed interval $[a, b]$.



▲ **Figure 3.4.2** In part (*a*) the absolute maximum occurs at an endpoint of $[a, b]$, in part (*b*) it occurs at a stationary point in (a, b), and in part (*c*) it occurs at a critical point in (a, b) where f is not differentiable.

***A Procedure for Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval* [a, b]**

Step 1. Find the critical points of f in (a, b).

Step 2. Evaluate f at all the critical points and at the endpoints a and b.

Step 3. The largest of the values in Step 2 is the absolute maximum value of f on $[a, b]$ and the smallest value is the absolute minimum.

▶ **Example 1** Find the absolute maximum and minimum values of the function $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$, and determine where these values occur.

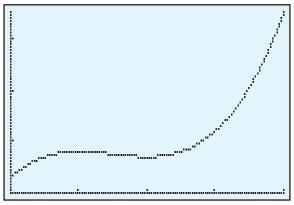
Solution. Since f is continuous and differentiable everywhere, the absolute extrema must occur either at endpoints of the interval or at solutions to the equation $f'(x) = 0$ in the open interval $(1, 5)$. The equation $f'(x) = 0$ can be written as

$$6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3) = 0$$

Thus, there are stationary points at $x = 2$ and at $x = 3$. Evaluating f at the endpoints, at $x = 2$, and at $x = 3$ yields

$$f(1) = 2(1)^3 - 15(1)^2 + 36(1) = 23$$
$$f(2) = 2(2)^3 - 15(2)^2 + 36(2) = 28$$
$$f(3) = 2(3)^3 - 15(3)^2 + 36(3) = 27$$
$$f(5) = 2(5)^3 - 15(5)^2 + 36(5) = 55$$

from which we conclude that the absolute minimum of f on $[1, 5]$ is 23, occurring at $x = 1$, and the absolute maximum of f on $[1, 5]$ is 55, occurring at $x = 5$. This is consistent with the graph of f in Figure 3.4.3. ◀



$[1, 5] \times [20, 55]$
$x\text{Scl} = 1,\ y\text{Scl} = 10$

$y = 2x^3 - 15x^2 + 36x$

▲ **Figure 3.4.3**

▶ **Example 2** Find the absolute extrema of $f(x) = 6x^{4/3} - 3x^{1/3}$ on the interval $[-1, 1]$, and determine where these values occur.

Solution. Note that f is continuous everywhere and therefore the Extreme-Value Theorem guarantees that f has a maximum and a minimum value in the interval $[-1, 1]$. Differentiating, we obtain

$$f'(x) = 8x^{1/3} - x^{-2/3} = x^{-2/3}(8x - 1) = \frac{8x - 1}{x^{2/3}}$$

Thus, $f'(x) = 0$ at $x = \frac{1}{8}$, and $f'(x)$ is undefined at $x = 0$. Evaluating f at these critical points and endpoints yields Table 3.4.1, from which we conclude that an absolute minimum value of $-\frac{9}{8}$ occurs at $x = \frac{1}{8}$, and an absolute maximum value of 9 occurs at $x = -1$. ◀

Table 3.4.1

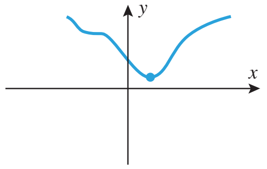
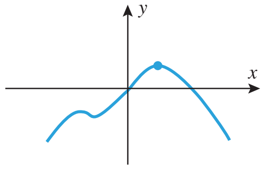
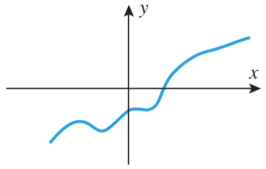
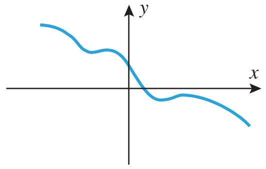
x	-1	0	$\frac{1}{8}$	1
$f(x)$	9	0	$-\frac{9}{8}$	3

ABSOLUTE EXTREMA ON INFINITE INTERVALS

We observed earlier that a continuous function may or may not have absolute extrema on an infinite interval (see Figure 3.4.1). However, certain conclusions about the existence of absolute extrema of a continuous function f on $(-\infty, +\infty)$ can be drawn from the behavior of $f(x)$ as $x \to -\infty$ and as $x \to +\infty$ (Table 3.4.2).

Table 3.4.2

ABSOLUTE EXTREMA ON INFINITE INTERVALS

LIMITS	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = -\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
CONCLUSION IF *f* IS CONTINUOUS EVERYWHERE	f has an absolute minimum but no absolute maximum on $(-\infty, +\infty)$.	f has an absolute maximum but no absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.	f has neither an absolute maximum nor an absolute minimum on $(-\infty, +\infty)$.
GRAPH				

▶ **Example 3** What can you say about the existence of absolute extrema on $(-\infty, +\infty)$ for polynomials?

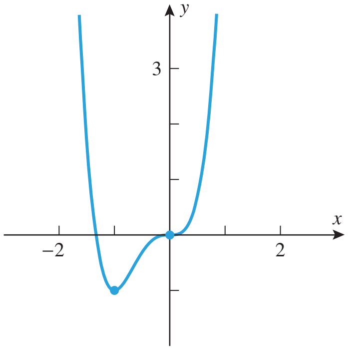
Solution. If $p(x)$ is a polynomial of odd degree, then

$$\lim_{x\to+\infty} p(x) \quad \text{and} \quad \lim_{x\to-\infty} p(x) \tag{1}$$

have opposite signs (one is $+\infty$ and the other is $-\infty$), so there are no absolute extrema. On the other hand, if $p(x)$ has even degree, then the limits in (1) have the same sign (both $+\infty$ or both $-\infty$). If the leading coefficient is positive, then both limits are $+\infty$, and there is an absolute minimum but no absolute maximum; if the leading coefficient is negative, then both limits are $-\infty$, and there is an absolute maximum but no absolute minimum. ◀

▶ **Example 4** Determine by inspection whether $p(x) = 3x^4 + 4x^3$ has any absolute extrema. If so, find them and state where they occur.

Solution. Since $p(x)$ has even degree and the leading coefficient is positive, $p(x) \to +\infty$ as $x \to \pm\infty$. Thus, there is an absolute minimum but no absolute maximum. From Theorem 3.4.3 [applied to the interval $(-\infty, +\infty)$], the absolute minimum must occur at a critical



$p(x) = 3x^4 + 4x^3$

▲ **Figure 3.4.4**

point of p. Since p is differentiable everywhere, we can find all critical points by solving the equation $p'(x) = 0$. This equation is

$$12x^3 + 12x^2 = 12x^2(x + 1) = 0$$

from which we conclude that the critical points are $x = 0$ and $x = -1$. Evaluating p at these critical points yields

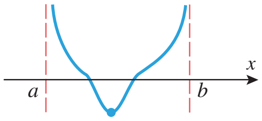
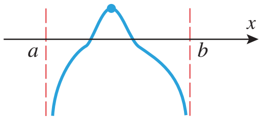
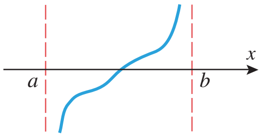
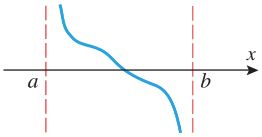
$$p(0) = 0 \quad \text{and} \quad p(-1) = -1$$

Therefore, p has an absolute minimum of -1 at $x = -1$ (Figure 3.4.4). ◀

ABSOLUTE EXTREMA ON OPEN INTERVALS

We know that a continuous function may or may not have absolute extrema on an open interval. However, certain conclusions about the existence of absolute extrema of a continuous function f on a finite open interval (a, b) can be drawn from the behavior of $f(x)$ as $x \to a^+$ and as $x \to b^-$ (Table 3.4.3). Similar conclusions can be drawn for intervals of the form $(-\infty, b)$ or $(a, +\infty)$.

Table 3.4.3
ABSOLUTE EXTREMA ON OPEN INTERVALS

LIMITS	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
CONCLUSION IF f IS CONTINUOUS ON (a, b)	f has an absolute minimum but no absolute maximum on (a, b).	f has an absolute maximum but no absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).	f has neither an absolute maximum nor an absolute minimum on (a, b).
GRAPH				

▶ **Example 5** Determine whether the function

$$f(x) = \frac{1}{x^2 - x}$$

has any absolute extrema on the interval $(0, 1)$. If so, find them and state where they occur.

Solution. Since f is continuous on the interval $(0, 1)$ and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x^2 - x} = \lim_{x\to 0^+} \frac{1}{x(x-1)} = -\infty$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x^2 - x} = \lim_{x\to 1^-} \frac{1}{x(x-1)} = -\infty$$

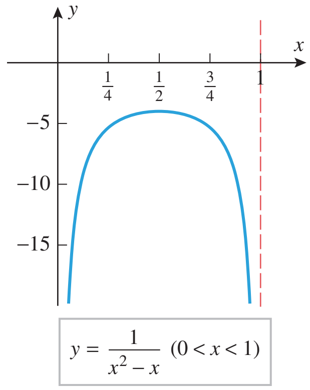
the function f has an absolute maximum but no absolute minimum on the interval $(0, 1)$. By Theorem 3.4.3 the absolute maximum must occur at a critical point of f in the interval $(0, 1)$. We have

$$f'(x) = -\frac{2x - 1}{(x^2 - x)^2}$$

so the only solution of the equation $f'(x) = 0$ is $x = \frac{1}{2}$. Although f is not differentiable at $x = 0$ or at $x = 1$, these values are doubly disqualified since they are neither in the domain of f nor in the interval $(0, 1)$. Thus, the absolute maximum occurs at $x = \frac{1}{2}$, and this absolute maximum is

$$f\left(\tfrac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2 - \frac{1}{2}} = -4$$

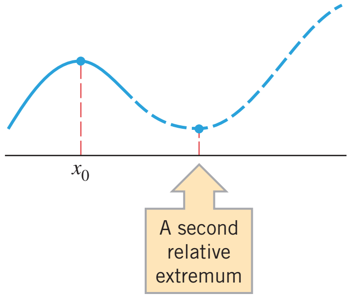
(Figure 3.4.5). ◀



▲ **Figure 3.4.5**

ABSOLUTE EXTREMA OF FUNCTIONS WITH ONE RELATIVE EXTREMUM

If a continuous function has only one relative extremum on a finite or infinite interval, then that relative extremum must of necessity also be an absolute extremum. To understand why this is so, suppose that f has a relative maximum at x_0 in an interval, and there are no other relative extrema of f on the interval. If $f(x_0)$ is *not* the absolute maximum of f on the interval, then the graph of f has to make an upward turn somewhere on the interval to rise above $f(x_0)$. However, this cannot happen because in the process of making an upward turn it would produce a second relative extremum (Figure 3.4.6). Thus, $f(x_0)$ must be the absolute maximum as well as a relative maximum. This idea is captured in the following theorem, which we state without proof.



▲ **Figure 3.4.6**

3.4.4 THEOREM *Suppose that f is continuous and has exactly one relative extremum on an interval, say at x_0.*

(*a*) *If f has a relative minimum at x_0, then $f(x_0)$ is the absolute minimum of f on the interval.*

(*b*) *If f has a relative maximum at x_0, then $f(x_0)$ is the absolute maximum of f the interval.*

This theorem is often helpful in situations where other methods are difficult or tedious to apply.

Example 6 Find the absolute extrema, if any, of the function $f(x) = x^3 - 3x^2 + 4$ on the interval $(0, +\infty)$.

Solution. We have

$$\lim_{x \to +\infty} f(x) = +\infty$$

(verify), so f does not have an absolute maximum on the interval $(0, +\infty)$. However, the continuity of f together with the fact that

$$\lim_{x \to 0^+} f(x) = f(0) = 4$$

is finite allow for the possibility that f has an absolute minimum on $(0, +\infty)$. If so, it would have to occur at a critical point of f, so we consider

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

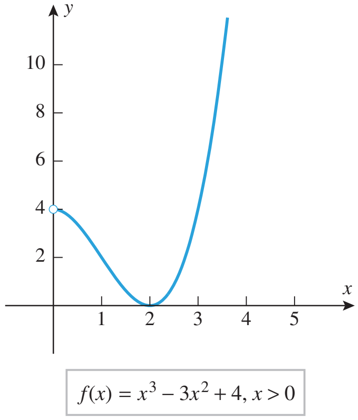
We see that $x = 0$ and $x = 2$ are the critical points of f. Of these, only $x = 2$ is in the interval $(0, +\infty)$, so this is the point at which an absolute minimum could occur. To see whether an absolute minimum actually does occur at this point, we can apply part (*a*) of Theorem 3.4.4. Since

$$f''(x) = 6x - 6$$

we have

$$f''(2) = 6 > 0$$

so a relative minimum occurs at $x = 2$ by the second derivative test. Thus, f has an absolute minimum at $x = 2$, and this absolute minimum is $f(2) = 0$ (Figure 3.4.7). ◀

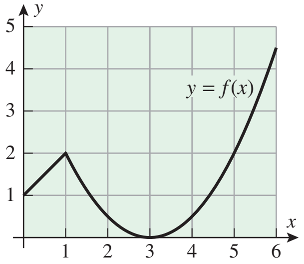


▲ **Figure 3.4.7**

Does the function in Example 6 have an absolute minimum on the interval $(-\infty, +\infty)$?

✔QUICK CHECK EXERCISES 3.4 *(See page 224 for answers.)*

1. Use the accompanying graph to find the x-coordinates of the relative extrema and absolute extrema of f on $[0, 6]$.



◀ **Figure Ex-1**

2. Suppose that a function f is continuous on $[-4, 4]$ and has critical points at $x = -3, 0, 2$. Use the accompanying table to determine the absolute maximum and absolute minimum values, if any, for f on the indicated intervals.
(a) $[1, 4]$ (b) $[-2, 2]$ (c) $[-4, 4]$ (d) $(-4, 4)$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	2224	-1333	0	1603	2096	2293	2400	2717	6064

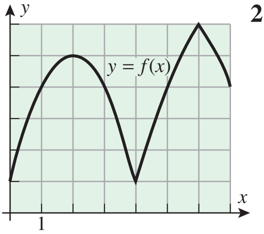
3. Let $f(x) = x^3 - 3x^2 - 9x + 25$. Use the derivative $f'(x) = 3(x + 1)(x - 3)$ to determine the absolute maximum and absolute minimum values, if any, for f on each of the given intervals.
(a) $[0, 4]$ (b) $[-2, 4]$ (c) $[-4, 2]$
(d) $[-5, 10]$ (e) $(-5, 4)$

EXERCISE SET 3.4 Graphing Utility CAS

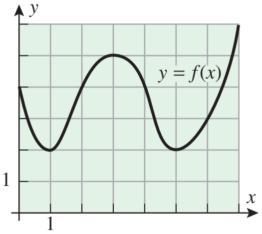
FOCUS ON CONCEPTS

1–2 Use the graph to find x-coordinates of the relative extrema and absolute extrema of f on $[0, 7]$.

1.



2.



3. In each part, sketch the graph of a continuous function f with the stated properties on the interval $[0, 10]$.
(a) f has an absolute minimum at $x = 0$ and an absolute maximum at $x = 10$.
(b) f has an absolute minimum at $x = 2$ and an absolute maximum at $x = 7$.
(c) f has relative minima at $x = 1$ and $x = 8$, has relative maxima at $x = 3$ and $x = 7$, has an absolute minimum at $x = 5$, and has an absolute maximum at $x = 10$.

4. In each part, sketch the graph of a continuous function f with the stated properties on the interval $(-\infty, +\infty)$.
(a) f has no relative extrema or absolute extrema.
(b) f has an absolute minimum at $x = 0$ but no absolute maximum.
(c) f has an absolute maximum at $x = -5$ and an absolute minimum at $x = 5$.

5. Let

$$f(x) = \begin{cases} \dfrac{1}{1-x}, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

Explain why f has a minimum value but no maximum value on the closed interval $[0, 1]$.

6. Let

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ \frac{1}{2}, & x = 0, 1 \end{cases}$$

Explain why f has neither a minimum value nor a maximum value on the closed interval $[0, 1]$.

7–16 Find the absolute maximum and minimum values of f on the given closed interval, and state where those values occur.

7. $f(x) = 4x^2 - 12x + 10$; $[1, 2]$

8. $f(x) = 8x - x^2$; $[0, 6]$

9. $f(x) = (x - 2)^3$; $[1, 4]$

10. $f(x) = 2x^3 + 3x^2 - 12x$; $[-3, 2]$

11. $f(x) = \dfrac{3x}{\sqrt{4x^2 + 1}}$; $[-1, 1]$

12. $f(x) = (x^2 + x)^{2/3}$; $[-2, 3]$

13. $f(x) = x - 2\sin x$; $[-\pi/4, \pi/2]$

14. $f(x) = \sin x - \cos x$; $[0, \pi]$

15. $f(x) = 1 + |9 - x^2|$; $[-5, 1]$

16. $f(x) = |6 - 4x|$; $[-3, 3]$

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. If a function f is continuous on $[a, b]$, then f has an absolute maximum on $[a, b]$.

18. If a function f is continuous on (a, b), then f has an absolute minimum on (a, b).

19. If a function f has an absolute minimum on (a, b), then there is a critical point of f in (a, b).

20. If a function f is continuous on $[a, b]$ and f has no relative extreme values in (a, b), then the absolute maximum value of f exists and occurs either at $x = a$ or at $x = b$.

21–28 Find the absolute maximum and minimum values of f, if any, on the given interval, and state where those values occur.

21. $f(x) = x^2 - x - 2;\ (-\infty, +\infty)$

22. $f(x) = 3 - 4x - 2x^2;\ (-\infty, +\infty)$

23. $f(x) = 4x^3 - 3x^4;\ (-\infty, +\infty)$

24. $f(x) = x^4 + 4x;\ (-\infty, +\infty)$

25. $f(x) = 2x^3 - 6x + 2;\ (-\infty, +\infty)$

26. $f(x) = x^3 - 9x + 1;\ (-\infty, +\infty)$

27. $f(x) = \dfrac{x^2+1}{x+1};\ (-5, -1)$

28. $f(x) = \dfrac{x-2}{x+1};\ (-1, 5]$

29–38 Use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.

29. $f(x) = (x^2 - 2x)^2;\ (-\infty, +\infty)$

30. $f(x) = (x-1)^2(x+2)^2;\ (-\infty, +\infty)$

31. $f(x) = x^{2/3}(20 - x);\ [-1, 20]$

32. $f(x) = \dfrac{x}{x^2+2};\ [-1, 4]$

33. $f(x) = 1 + \dfrac{1}{x};\ (0, +\infty)$

34. $f(x) = \dfrac{2x^2 - 3x + 3}{x^2 - 2x + 2};\ [1, +\infty)$

35. $f(x) = \dfrac{2 - \cos x}{\sin x};\ [\pi/4, 3\pi/4]$

36. $f(x) = \sin^2 x + \cos x;\ [-\pi, \pi]$

37. $f(x) = \sin(\cos x);\ [0, 2\pi]$

38. $f(x) = \cos(\sin x);\ [0, \pi]$

39. Find the absolute maximum and minimum values of

$$f(x) = \begin{cases} 4x - 2, & x < 1 \\ (x-2)(x-3), & x \geq 1 \end{cases}$$

on $\left[\frac{1}{2}, \frac{7}{2}\right]$.

40. Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on $[0, 2]$. Is this value a maximum or minimum?

41–42 If f is a periodic function, then the locations of all absolute extrema on the interval $(-\infty, +\infty)$ can be obtained by finding the locations of the absolute extrema for one period and using the periodicity to locate the rest. Use this idea in these exercises to find the absolute maximum and minimum values of the function, and state the x-values at which they occur.

41. $f(x) = 2\cos x + \cos 2x$ **42.** $f(x) = 3\cos\dfrac{x}{3} + 2\cos\dfrac{x}{2}$

43–44 One way of proving that $f(x) \leq g(x)$ for all x in a given interval is to show that $0 \leq g(x) - f(x)$ for all x in the interval; and one way of proving the latter inequality is to show that the absolute minimum value of $g(x) - f(x)$ on the interval is nonnegative. Use this idea to prove the inequalities in these exercises.

43. Prove that $\sin x \leq x$ for all x in the interval $[0, 2\pi]$.

44. Prove that $\cos x \geq 1 - (x^2/2)$ for all x in the interval $[0, 2\pi]$.

45. What is the smallest possible slope for a tangent to the graph of the equation $y = x^3 - 3x^2 + 5x$?

46. (a) Show that $f(x) = \sec x + \csc x$ has a minimum value but no maximum value on the interval $(0, \pi/2)$.
(b) Find the minimum value in part (a).

C **47.** Show that the absolute minimum value of

$$f(x) = x^2 + \frac{x^2}{(8-x)^2}, \quad x > 8$$

occurs at $x = 10$ by using a CAS to find $f'(x)$ and to solve the equation $f'(x) = 0$.

C **48.** The vertical displacement $f(t)$ of a cork bobbing up and down on the ocean's surface may be modeled by the function

$$f(t) = A\cos t + B\sin t$$

where $A > 0$ and $B > 0$. Use a CAS to find the maximum and minimum values of $f(t)$ in terms of A and B.

49. Suppose that the equations of motion of a paper airplane during the first 12 seconds of flight are

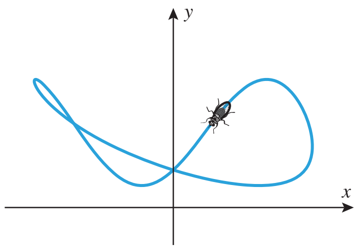
$$x = t - 2\sin t, \quad y = 2 - 2\cos t \qquad (0 \leq t \leq 12)$$

What are the highest and lowest points in the trajectory, and when is the airplane at those points?

50. The accompanying figure shows the path of a fly whose equations of motion are

$$x = \frac{\cos t}{2 + \sin t}, \quad y = 3 + \sin(2t) - 2\sin^2 t \qquad (0 \leq t \leq 2\pi)$$

(a) How high and low does it fly?
(b) How far left and right of the origin does it fly?



◀ **Figure Ex-50**

51. Let $f(x) = ax^2 + bx + c$, where $a > 0$. Prove that $f(x) \geq 0$ for all x if and only if $b^2 - 4ac \leq 0$. [*Hint:* Find the minimum of $f(x)$.]

52. Prove Theorem 3.4.3 in the case where the extreme value is a minimum.

53. **Writing** Suppose that f is continuous and positive-valued everywhere and that the x-axis is an asymptote for the graph of f, both as $x \to -\infty$ and as $x \to +\infty$. Explain why f cannot have an absolute minimum but may have a relative minimum.

54. **Writing** Explain the difference between a relative maximum and an absolute maximum. Sketch a graph that illustrates a function with a relative maximum that is not an absolute maximum, and sketch another graph illustrating an absolute maximum that is not a relative maximum. Explain how these graphs satisfy the given conditions.

✔QUICK CHECK ANSWERS 3.4

1. There is a relative minimum at $x = 3$, a relative maximum at $x = 1$, an absolute minimum at $x = 3$, and an absolute maximum at $x = 6$. **2.** (a) max, 6064; min, 2293 (b) max, 2400; min, 0 (c) max, 6064; min, −1333 (d) no max; min, −1333
3. (a) max, $f(0) = 25$; min, $f(3) = -2$ (b) max, $f(-1) = 30$; min, $f(3) = -2$ (c) max, $f(-1) = 30$; min, $f(-4) = -51$ (d) max, $f(10) = 635$; min, $f(-5) = -130$ (e) max, $f(-1) = 30$; no min

3.5 APPLIED MAXIMUM AND MINIMUM PROBLEMS

In this section we will show how the methods discussed in the last section can be used to solve various applied optimization problems.

CLASSIFICATION OF OPTIMIZATION PROBLEMS

The applied optimization problems that we will consider in this section fall into the following two categories:

- Problems that reduce to maximizing or minimizing a continuous function over a finite closed interval.
- Problems that reduce to maximizing or minimizing a continuous function over an infinite interval or a finite interval that is not closed.

For problems of the first type the Extreme-Value Theorem (3.4.2) guarantees that the problem has a solution, and we know that the solution can be obtained by examining the values of the function at the critical points and at the endpoints. However, for problems of the second type there may or may not be a solution. If the function is continuous and has exactly one relative extremum of the appropriate type on the interval, then Theorem 3.4.4 guarantees the existence of a solution and provides a method for finding it. In cases where this theorem is not applicable some ingenuity may be required to solve the problem.

PROBLEMS INVOLVING FINITE CLOSED INTERVALS

In his *On a Method for the Evaluation of Maxima and Minima*, the seventeenth century French mathematician Pierre de Fermat solved an optimization problem very similar to the one posed in our first example. Fermat's work on such optimization problems prompted the French mathematician Laplace to proclaim Fermat the "true inventor of the differential calculus." Although this honor must still reside with Newton and Leibniz, it is the case that Fermat developed procedures that anticipated parts of differential calculus.

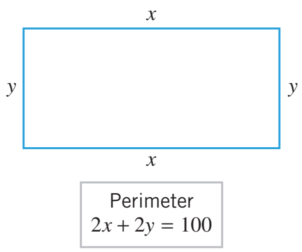
▶ **Example 1** A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

Solution. Let

$$\begin{aligned} x &= \text{length of the rectangle (ft)} \\ y &= \text{width of the rectangle (ft)} \\ A &= \text{area of the rectangle (ft}^2\text{)} \end{aligned}$$

Then

$$A = xy \tag{1}$$



▲ **Figure 3.5.1**

Since the perimeter of the rectangle is 100 ft, the variables x and y are related by the equation

$$2x + 2y = 100 \quad \text{or} \quad y = 50 - x \tag{2}$$

(See Figure 3.5.1.) Substituting (2) in (1) yields

$$A = x(50 - x) = 50x - x^2 \tag{3}$$

Because x represents a length, it cannot be negative, and because the two sides of length x cannot have a combined length exceeding the total perimeter of 100 ft, the variable x must satisfy

$$0 \le x \le 50 \tag{4}$$

Thus, we have reduced the problem to that of finding the value (or values) of x in $[0, 50]$, for which A is maximum. Since A is a polynomial in x, it is continuous on $[0, 50]$, and so the maximum must occur at an endpoint of this interval or at a critical point.

From (3) we obtain

$$\frac{dA}{dx} = 50 - 2x$$

Setting $dA/dx = 0$ we obtain

$$50 - 2x = 0$$

or $x = 25$. Thus, the maximum occurs at one of the values

$$x = 0, \quad x = 25, \quad x = 50$$

Pierre de Fermat (1601–1665) Fermat, the son of a successful French leather merchant, was a lawyer who practiced mathematics as a hobby. He received a Bachelor of Civil Laws degree from the University of Orleans in 1631 and subsequently held various government positions, including a post as councillor to the Toulouse parliament. Although he was apparently financially successful, confidential documents of that time suggest that his performance in office and as a lawyer was poor, perhaps because he devoted so much time to mathematics. Throughout his life, Fermat fought all efforts to have his mathematical results published. He had the unfortunate habit of scribbling his work in the margins of books and often sent his results to friends without keeping copies for himself. As a result, he never received credit for many major achievements until his name was raised from obscurity in the mid-nineteenth century. It is now known that Fermat, simultaneously and independently of Descartes, developed analytic geometry. Unfortunately, Descartes and Fermat argued bitterly over various problems so that there was never any real cooperation between these two great geniuses.

Fermat solved many fundamental calculus problems. He obtained the first procedure for differentiating polynomials, and solved many important maximization, minimization, area, and tangent problems. His work served to inspire Isaac Newton. Fermat is best known for his work in number theory, the study of properties of and relationships between whole numbers. He was the first mathematician to make substantial contributions to this field after the ancient Greek mathematician Diophantus. Unfortunately, none of Fermat's contemporaries appreciated his work in this area, a fact that eventually pushed Fermat into isolation and obscurity in later life. In addition to his work in calculus and number theory, Fermat was one of the founders of probability theory and made major contributions to the theory of optics. Outside mathematics, Fermat was a classical scholar of some note, was fluent in French, Italian, Spanish, Latin, and Greek, and he composed a considerable amount of Latin poetry.

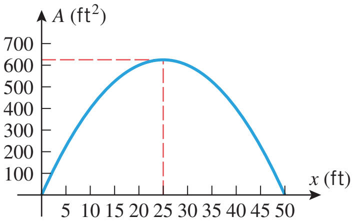
One of the great mysteries of mathematics is shrouded in Fermat's work in number theory. In the margin of a book by Diophantus, Fermat scribbled that for integer values of n greater than 2, the equation $x^n + y^n = z^n$ has no nonzero integer solutions for x, y, and z. He stated, "I have discovered a truly marvelous proof of this, which however the margin is not large enough to contain." This result, which became known as "Fermat's last theorem," appeared to be true, but its proof evaded the greatest mathematical geniuses for 300 years until Professor Andrew Wiles of Princeton University presented a proof in June 1993 in a dramatic series of three lectures that drew international media attention (see *New York Times*, June 27, 1993). As it turned out, that proof had a serious gap that Wiles and Richard Taylor fixed and published in 1995. A prize of 100,000 German marks was offered in 1908 for the solution, but it is worthless today because of inflation.

[*Image: http://en.wikipedia.org/wiki/File:Pierre_de_Fermat.png*]

Table 3.5.1

x	0	25	50
A	0	625	0

Substituting these values in (3) yields Table 3.5.1, which tells us that the maximum area of 625 ft^2 occurs at $x = 25$, which is consistent with the graph of (3) in Figure 3.5.2. From (2) the corresponding value of y is 25, so the rectangle of perimeter 100 ft with greatest area is a square with sides of length 25 ft. ◀



▲ **Figure 3.5.2**

In Example 1 we included $x = 0$ and $x = 50$ as possible values of x, even though these correspond to rectangles with two sides of length zero. If we view this as a purely mathematical problem, then there is nothing wrong with this. However, if we view this as an applied problem in which the rectangle will be formed from physical material, then it would make sense to exclude these values.

Example 1 illustrates the following five-step procedure that can be used for solving many applied maximum and minimum problems.

A Procedure for Solving Applied Maximum and Minimum Problems

Step 1. Draw an appropriate figure and label the quantities relevant to the problem.

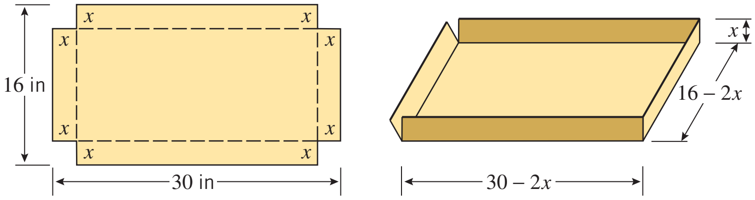
Step 2. Find a formula for the quantity to be maximized or minimized.

Step 3. Using the conditions stated in the problem to eliminate variables, express the quantity to be maximized or minimized as a function of one variable.

Step 4. Find the interval of possible values for this variable from the physical restrictions in the problem.

Step 5. If applicable, use the techniques of the preceding section to obtain the maximum or minimum.

▶ **Example 2** An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides (Figure 3.5.3). What size should the squares be to obtain a box with the largest volume?



▶ **Figure 3.5.3**

(*a*) (*b*)

Solution. For emphasis, we explicitly list the steps of the five-step problem-solving procedure given above as an outline for the solution of this problem. (In later examples we will follow these guidelines without listing the steps.)

- *Step 1:* Figure 3.5.3*a* illustrates the cardboard piece with squares removed from its corners. Let

  x = length (in inches) of the sides of the squares to be cut out
  V = volume (in cubic inches) of the resulting box

- *Step 2:* Because we are removing a square of side x from each corner, the resulting box will have dimensions $16 - 2x$ by $30 - 2x$ by x (Figure 3.5.3*b*). Since the volume of a box is the product of its dimensions, we have

$$V = (16 - 2x)(30 - 2x)x = 480x - 92x^2 + 4x^3 \tag{5}$$

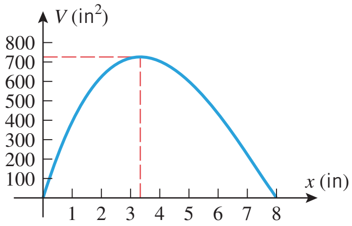
- *Step 3:* Note that our volume expression is already in terms of the single variable x.

- *Step 4:* The variable x in (5) is subject to certain restrictions. Because x represents a length, it cannot be negative, and because the width of the cardboard is 16 inches, we cannot cut out squares whose sides are more than 8 inches long. Thus, the variable x in (5) must satisfy
$$0 \le x \le 8$$
and hence we have reduced our problem to finding the value (or values) of x in the interval [0, 8] for which (5) is a maximum.

Table 3.5.2

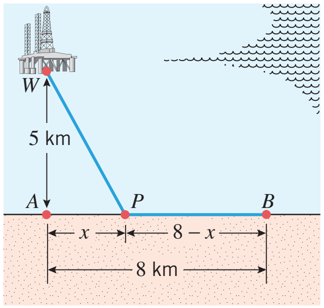
x	0	$\frac{10}{3}$	8
V	0	$\frac{19{,}600}{27} \approx 726$	0

- *Step 5:* From (5) we obtain
$$\begin{aligned}\frac{dV}{dx} &= 480 - 184x + 12x^2 = 4(120 - 46x + 3x^2)\\ &= 4(x - 12)(3x - 10)\end{aligned}$$
Setting $dV/dx = 0$ yields
$$x = \tfrac{10}{3} \quad \text{and} \quad x = 12$$
Since $x = 12$ falls outside the interval [0, 8], the maximum value of V occurs either at the critical point $x = \frac{10}{3}$ or at the endpoints $x = 0$, $x = 8$. Substituting these values into (5) yields Table 3.5.2, which tells us that the greatest possible volume $V = \frac{19{,}600}{27}\ \text{in}^3 \approx 726\ \text{in}^3$ occurs when we cut out squares whose sides have length $\frac{10}{3}$ inches. This is consistent with the graph of (5) shown in Figure 3.5.4. ◄



▲ **Figure 3.5.4**

► **Example 3** Figure 3.5.5 shows an offshore oil well located at a point W that is 5 km from the closest point A on a straight shoreline. Oil is to be piped from W to a shore point B that is 8 km from A by piping it on a straight line under water from W to some shore point P between A and B and then on to B via pipe along the shoreline. If the cost of laying pipe is \$1,000,000/km under water and \$500,000/km over land, where should the point P be located to minimize the cost of laying the pipe?



▲ **Figure 3.5.5**

Solution. Let

x = distance (in kilometers) between A and P

c = cost (in millions of dollars) for the entire pipeline

From Figure 3.5.5 the length of pipe under water is the distance between W and P. By the Theorem of Pythagoras that length is
$$\sqrt{x^2 + 25} \tag{6}$$
Also from Figure 3.5.5, the length of pipe over land is the distance between P and B, which is
$$8 - x \tag{7}$$
From (6) and (7) it follows that the total cost c (in millions of dollars) for the pipeline is
$$c = 1\left(\sqrt{x^2 + 25}\,\right) + \tfrac{1}{2}(8 - x) = \sqrt{x^2 + 25} + \tfrac{1}{2}(8 - x) \tag{8}$$
Because the distance between A and B is 8 km, the distance x between A and P must satisfy
$$0 \le x \le 8$$
We have thus reduced our problem to finding the value (or values) of x in the interval [0, 8] for which c is a minimum. Since c is a continuous function of x on the closed interval [0, 8], we can use the methods developed in the preceding section to find the minimum.

From (8) we obtain
$$\frac{dc}{dx} = \frac{x}{\sqrt{x^2 + 25}} - \frac{1}{2}$$

Setting $dc/dx = 0$ and solving for x yields

$$\frac{x}{\sqrt{x^2+25}} = \frac{1}{2} \tag{9}$$

$$x^2 = \frac{1}{4}(x^2+25)$$

$$x = \pm\frac{5}{\sqrt{3}}$$

The number $-5/\sqrt{3}$ is not a solution of (9) and must be discarded, leaving $x = 5/\sqrt{3}$ as the only critical point. Since this point lies in the interval $[0, 8]$, the minimum must occur at one of the values

$$x = 0, \quad x = 5/\sqrt{3}, \quad x = 8$$

Substituting these values into (8) yields Table 3.5.3, which tells us that the least possible cost of the pipeline (to the nearest dollar) is $c = \$8,330,127$, and this occurs when the point P is located at a distance of $5/\sqrt{3} \approx 2.89$ km from A. ◄

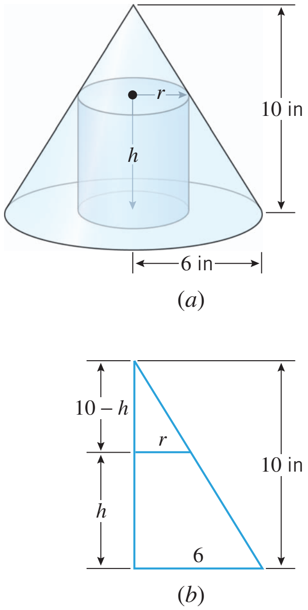
TECHNOLOGY MASTERY

If you have a CAS, use it to check all of the computations in Example 3. Specifically, differentiate c with respect to x, solve the equation $dc/dx = 0$, and perform all of the numerical calculations.

Table 3.5.3

x	0	$\frac{5}{\sqrt{3}}$	8
c	9	$\frac{10}{\sqrt{3}} + \left(4 - \frac{5}{2\sqrt{3}}\right) \approx 8.330127$	$\sqrt{89} \approx 9.433981$

► Example 4 Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches (Figure 3.5.6*a*).



(*a*)

(*b*)

▲ **Figure 3.5.6**

Solution. Let

r = radius (in inches) of the cylinder
h = height (in inches) of the cylinder
V = volume (in cubic inches) of the cylinder

The formula for the volume of the inscribed cylinder is

$$V = \pi r^2 h \tag{10}$$

To eliminate one of the variables in (10) we need a relationship between r and h. Using similar triangles (Figure 3.5.6*b*) we obtain

$$\frac{10-h}{r} = \frac{10}{6} \quad \text{or} \quad h = 10 - \frac{5}{3}r \tag{11}$$

Substituting (11) into (10) we obtain

$$V = \pi r^2\left(10 - \frac{5}{3}r\right) = 10\pi r^2 - \frac{5}{3}\pi r^3 \tag{12}$$

which expresses V in terms of r alone. Because r represents a radius, it cannot be negative, and because the radius of the inscribed cylinder cannot exceed the radius of the cone, the variable r must satisfy

$$0 \le r \le 6$$

Thus, we have reduced the problem to that of finding the value (or values) of r in $[0, 6]$ for which (12) is a maximum. Since V is a continuous function of r on $[0, 6]$, the methods developed in the preceding section apply.

From (12) we obtain

$$\frac{dV}{dr} = 20\pi r - 5\pi r^2 = 5\pi r(4 - r)$$

Setting $dV/dr = 0$ gives

$$5\pi r(4 - r) = 0$$

so $r = 0$ and $r = 4$ are critical points. Since these lie in the interval [0, 6], the maximum must occur at one of the values

$$r = 0, \quad r = 4, \quad r = 6$$

Table 3.5.4

r	0	4	6
V	0	$\frac{160}{3}\pi$	0

Substituting these values into (12) yields Table 3.5.4, which tells us that the maximum volume $V = \frac{160}{3}\pi \approx 168$ in^3 occurs when the inscribed cylinder has radius 4 in. When $r = 4$ it follows from (11) that $h = \frac{10}{3}$. Thus, the inscribed cylinder of largest volume has radius $r = 4$ in and height $h = \frac{10}{3}$ in. ◀

PROBLEMS INVOLVING INTERVALS THAT ARE NOT BOTH FINITE AND CLOSED

▶ **Example 5** A closed cylindrical can is to hold 1 liter (1000 cm^3) of liquid. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?

Solution. Let

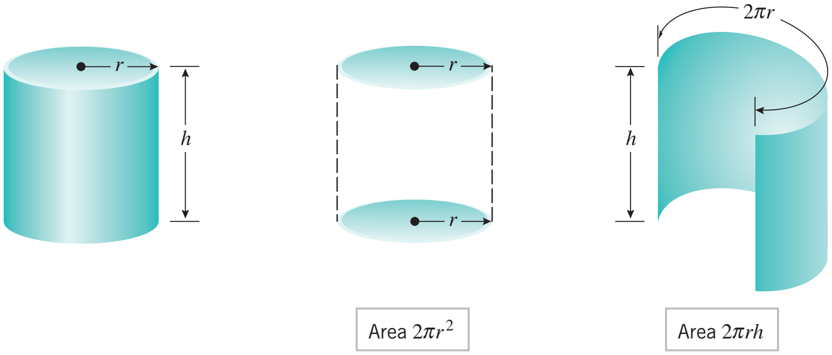
h = height (in cm) of the can
r = radius (in cm) of the can
S = surface area (in cm^2) of the can

Assuming there is no waste or overlap, the amount of material needed for manufacture will be the same as the surface area of the can. Since the can consists of two circular disks of radius r and a rectangular sheet with dimensions h by $2\pi r$ (Figure 3.5.7), the surface area will be

$$S = 2\pi r^2 + 2\pi rh \tag{13}$$

Since S depends on two variables, r and h, we will look for some condition in the problem that will allow us to express one of these variables in terms of the other. For this purpose, observe that the volume of the can is 1000 cm^3, so it follows from the formula $V = \pi r^2 h$ for the volume of a cylinder that

$$1000 = \pi r^2 h \quad \text{or} \quad h = \frac{1000}{\pi r^2} \tag{14–15}$$



▲ **Figure 3.5.7**

Substituting (15) in (13) yields

$$S = 2\pi r^2 + \frac{2000}{r} \tag{16}$$

Thus, we have reduced the problem to finding a value of r in the interval $(0, +\infty)$ for which S is minimum. Since S is a continuous function of r on the interval $(0, +\infty)$ and

$$\lim_{r \to 0^+} \left(2\pi r^2 + \frac{2000}{r}\right) = +\infty \quad \text{and} \quad \lim_{r \to +\infty} \left(2\pi r^2 + \frac{2000}{r}\right) = +\infty$$

the analysis in Table 3.4.3 implies that S does have a minimum on the interval $(0, +\infty)$. Since this minimum must occur at a critical point, we calculate

$$\frac{dS}{dr} = 4\pi r - \frac{2000}{r^2} \tag{17}$$

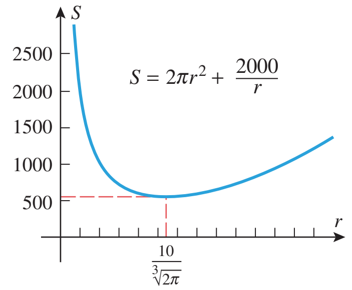
Setting $dS/dr = 0$ gives

$$r = \frac{10}{\sqrt[3]{2\pi}} \approx 5.4 \tag{18}$$

Since (18) is the only critical point in the interval $(0, +\infty)$, this value of r yields the minimum value of S. From (15) the value of h corresponding to this r is

$$h = \frac{1000}{\pi(10/\sqrt[3]{2\pi})^2} = \frac{20}{\sqrt[3]{2\pi}} = 2r$$

It is not an accident here that the minimum occurs when the height of the can is equal to the diameter of its base (Exercise 29).



▲ Figure 3.5.8

In Example 5, the surface area S has no absolute maximum, since S increases without bound as the radius r approaches 0 (Figure 3.5.8). Thus, had we asked for the dimensions of the can requiring the *maximum* amount of material for its manufacture, there would have been no solution to the problem. Optimization problems with no solution are sometimes called *ill posed*.

Second Solution. The conclusion that a minimum occurs at the value of r in (18) can be deduced from Theorem 3.4.4 and the second derivative test by noting that

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive if $r > 0$ and hence is positive if $r = 10/\sqrt[3]{2\pi}$. This implies that a relative minimum, and therefore a minimum, occurs at the critical point $r = 10/\sqrt[3]{2\pi}$.

Third Solution. An alternative justification that the critical point $r = 10/\sqrt[3]{2\pi}$ corresponds to a minimum for S is to view the graph of S versus r (Figure 3.5.8). ◀

▶ **Example 6** Find a point on the curve $y = x^2$ that is closest to the point $(18, 0)$.

Solution. The distance L between $(18, 0)$ and an arbitrary point (x, y) on the curve $y = x^2$ (Figure 3.5.9) is given by

$$L = \sqrt{(x-18)^2 + (y-0)^2}$$

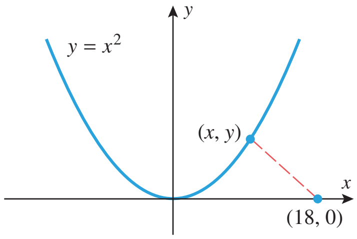
Since (x, y) lies on the curve, x and y satisfy $y = x^2$; thus,

$$L = \sqrt{(x-18)^2 + x^4} \tag{19}$$

Because there are no restrictions on x, the problem reduces to finding a value of x in $(-\infty, +\infty)$ for which (19) is a minimum. The distance L and the square of the distance L^2 are minimized at the same value (see Exercise 68). Thus, the minimum value of L in (19) and the minimum value of

$$S = L^2 = (x-18)^2 + x^4 \tag{20}$$

occur at the same x-value.



▲ Figure 3.5.9

From (20),

$$\frac{dS}{dx} = 2(x - 18) + 4x^3 = 4x^3 + 2x - 36 \tag{21}$$

so the critical points satisfy $4x^3 + 2x - 36 = 0$ or, equivalently,

$$2x^3 + x - 18 = 0 \tag{22}$$

To solve for x we will begin by checking the divisors of -18 to see whether the polynomial on the left side has any integer roots (see Appendix C). These divisors are ± 1, ± 2, ± 3, ± 6, ± 9, and ± 18. A check of these values shows that $x = 2$ is a root, so $x - 2$ is a factor of the polynomial. After dividing the polynomial by this factor we can rewrite (22) as

$$(x - 2)(2x^2 + 4x + 9) = 0$$

Thus, the remaining solutions of (22) satisfy the quadratic equation

$$2x^2 + 4x + 9 = 0$$

But this equation has no real solutions (using the quadratic formula), so $x = 2$ is the only critical point of S. To determine the nature of this critical point we will use the second derivative test. From (21),

$$\frac{d^2S}{dx^2} = 12x^2 + 2, \quad \text{so} \quad \left.\frac{d^2S}{dx^2}\right|_{x=2} = 50 > 0$$

which shows that a relative minimum occurs at $x = 2$. Since $x = 2$ yields the only relative extremum for L, it follows from Theorem 3.4.4 that an absolute minimum value of L also occurs at $x = 2$. Thus, the point on the curve $y = x^2$ closest to (18, 0) is

$$(x, y) = (x, x^2) = (2, 4) \blacktriangleleft$$

AN APPLICATION TO ECONOMICS

Three functions of importance to an economist or a manufacturer are

$C(x)$ = total cost of producing x units of a product during some time period
$R(x)$ = total revenue from selling x units of the product during the time period
$P(x)$ = total profit obtained by selling x units of the product during the time period

These are called, respectively, the ***cost function***, ***revenue function***, and ***profit function***. If all units produced are sold, then these are related by

$$\boxed{\begin{array}{l} P(x) = R(x) - C(x) \\ \text{[profit]} = \text{[revenue]} - \text{[cost]} \end{array}} \tag{23}$$

The total cost $C(x)$ of producing x units can be expressed as a sum

$$C(x) = a + M(x) \tag{24}$$

where a is a constant, called ***overhead***, and $M(x)$ is a function representing ***manufacturing cost***. The overhead, which includes such fixed costs as rent and insurance, does not depend on x; it must be paid even if nothing is produced. On the other hand, the manufacturing cost $M(x)$, which includes such items as cost of materials and labor, depends on the number of items manufactured. It is shown in economics that with suitable simplifying assumptions, $M(x)$ can be expressed in the form

$$M(x) = bx + cx^2$$

where b and c are constants. Substituting this in (24) yields

$$C(x) = a + bx + cx^2 \tag{25}$$

If a manufacturing firm can sell all the items it produces for p dollars apiece, then its total revenue $R(x)$ (in dollars) will be

$$R(x) = px \tag{26}$$

and its total profit $P(x)$ (in dollars) will be

$$P(x) = [\text{total revenue}] - [\text{total cost}] = R(x) - C(x) = px - C(x)$$

Thus, if the cost function is given by (25),

$$P(x) = px - (a + bx + cx^2) \tag{27}$$

Depending on such factors as number of employees, amount of machinery available, economic conditions, and competition, there will be some upper limit l on the number of items a manufacturer is capable of producing and selling. Thus, during a fixed time period the variable x in (27) will satisfy

$$0 \le x \le l$$

By determining the value or values of x in $[0, l]$ that maximize (27), the firm can determine how many units of its product must be manufactured and sold to yield the greatest profit. This is illustrated in the following numerical example.

▶ Example 7 A liquid form of antibiotic manufactured by a pharmaceutical firm is sold in bulk at a price of $200 per unit. If the total production cost (in dollars) for x units is

$$C(x) = 500{,}000 + 80x + 0.003x^2$$

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of antibiotic must be manufactured and sold in that time to maximize the profit?

Jim Karageorge/Getty Images

A pharmaceutical firm's profit is a function of the number of units produced.

Solution. Since the total revenue for selling x units is $R(x) = 200x$, the profit $P(x)$ on x units will be

$$P(x) = R(x) - C(x) = 200x - (500{,}000 + 80x + 0.003x^2) \tag{28}$$

Since the production capacity is at most 30,000 units, x must lie in the interval [0, 30,000]. From (28)

$$\frac{dP}{dx} = 200 - (80 + 0.006x) = 120 - 0.006x$$

Setting $dP/dx = 0$ gives

$$120 - 0.006x = 0 \quad \text{or} \quad x = 20{,}000$$

Since this critical point lies in the interval [0, 30,000], the maximum profit must occur at one of the values

$$x = 0, \quad x = 20{,}000, \quad \text{or} \quad x = 30{,}000$$

Substituting these values in (28) yields Table 3.5.5, which tells us that the maximum profit $P = \$700{,}000$ occurs when $x = 20{,}000$ units are manufactured and sold in the specified time. ◀

Table 3.5.5

x	0	20,000	30,000
$P(x)$	−500,000	700,000	400,000

MARGINAL ANALYSIS

Economists call $P'(x)$, $R'(x)$, and $C'(x)$ the ***marginal profit***, ***marginal revenue***, and ***marginal cost***, respectively; and they interpret these quantities as the *additional* profit, revenue, and cost that result from producing and selling one additional unit of the product when the production and sales levels are at x units. These interpretations follow from the local linear approximations of the profit, revenue, and cost functions. For example, it follows from Formula (2) of Section 2.9 that when the production and sales levels are at x units the local linear approximation of the profit function is

$$P(x + \Delta x) \approx P(x) + P'(x)\Delta x$$

Thus, if $\Delta x = 1$ (one additional unit produced and sold), this formula implies

$$P(x + 1) \approx P(x) + P'(x)$$

and hence the *additional* profit that results from producing and selling one additional unit can be approximated as

$$P(x + 1) - P(x) \approx P'(x)$$

Similarly, $R(x + 1) - R(x) \approx R'(x)$ and $C(x + 1) - C(x) \approx C'(x)$.

A BASIC PRINCIPLE OF ECONOMICS

It follows from (23) that $P'(x) = 0$ has the same solution as $C'(x) = R'(x)$, and this implies that the maximum profit must occur at a point where the marginal revenue is equal to the marginal cost; that is:

> *If profit is maximum, then the cost of manufacturing and selling an additional unit of a product is approximately equal to the revenue generated by the additional unit.*

In Example 7, the maximum profit occurs when $x = 20{,}000$ units. Note that

$$C(20{,}001) - C(20{,}000) = \$200.003 \quad \text{and} \quad R(20{,}001) - R(20{,}000) = \$200$$

which is consistent with this basic economic principle.

✔QUICK CHECK EXERCISES 3.5

(See page 238 for answers.)

1. A positive number x and its reciprocal are added together. The smallest possible value of this sum is obtained by minimizing $f(x) =$ ________ for x in the interval ________.

2. Two nonnegative numbers, x and y, have a sum equal to 10. The largest possible product of the two numbers is obtained by maximizing $f(x) =$ ________ for x in the interval ________.

3. A rectangle in the xy-plane has one corner at the origin, an adjacent corner at the point $(x, 0)$, and a third corner at a point on the line segment from $(0, 4)$ to $(3, 0)$. The largest possible area of the rectangle is obtained by maximizing $A(x) =$ ________ for x in the interval ________.

4. An open box is to be made from a 20-inch by 32-inch piece of cardboard by cutting out x-inch by x-inch squares from the four corners and bending up the sides. The largest possible volume of the box is obtained by maximizing $V(x) =$ ________ for x in the interval ________.

EXERCISE SET 3.5

1. Find a number in the closed interval $\left[\frac{1}{2}, \frac{3}{2}\right]$ such that the sum of the number and its reciprocal is
(a) as small as possible (b) as large as possible.

2. How should two nonnegative numbers be chosen so that their sum is 1 and the sum of their squares is
(a) as large as possible (b) as small as possible?

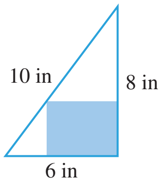
3. A rectangular field is to be bounded by a fence on three sides and by a straight stream on the fourth side. Find the dimensions of the field with maximum area that can be enclosed using 1000 ft of fence.

4. The boundary of a field is a right triangle with a straight stream along its hypotenuse and with fences along its other two sides. Find the dimensions of the field with maximum area that can be enclosed using 1000 ft of fence.

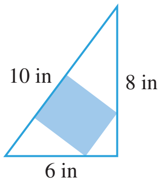
5. A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy-duty fencing selling for \$3 a foot, while the remaining two sides will use standard fencing selling for \$2 a foot. What are the dimensions of the rectangular plot of greatest area that can be fenced in at a cost of \$6000?

6. A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in. Find the dimensions of the rectangle with greatest area assuming the rectangle is positioned as in Figure Ex-6.

7. Solve the problem in Exercise 6 assuming the rectangle is positioned as in Figure Ex-7.



▲ **Figure Ex-6**



▲ **Figure Ex-7**

8. A rectangle has its two lower corners on the x-axis and its two upper corners on the curve $y = 16 - x^2$. For all such rectangles, what are the dimensions of the one with largest area?

9. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10.

10. Find the point P in the first quadrant on the curve $y = x^{-2}$ such that a rectangle with sides on the coordinate axes and a vertex at P has the smallest possible perimeter.

11. A rectangular area of 3200 ft^2 is to be fenced off. Two opposite sides will use fencing costing \$1 per foot and the remaining sides will use fencing costing \$2 per foot. Find the dimensions of the rectangle of least cost.

12. Show that among all rectangles with perimeter p, the square has the maximum area.

13. Show that among all rectangles with area A, the square has the minimum perimeter.

14. A wire of length 12 in can be bent into a circle, bent into a square, or cut into two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure(s) is to be
(a) a maximum (b) a minimum?

15. A rectangle R in the plane has corners at $(\pm 8, \pm 12)$, and a 100 by 100 square S is positioned in the plane so that its sides are parallel to the coordinate axes and the lower left corner of S is on the line $y = -3x$. What is the largest possible area of a region in the plane that is contained in both R and S?

16. Solve the problem in Exercise 15 if S is a 16 by 16 square.

17. Solve the problem in Exercise 15 if S is positioned with its lower left corner on the line $y = -6x$.

18. A rectangular page is to contain 42 square inches of printable area. The margins at the top and bottom of the page are each 1 inch, one side margin is 1 inch, and the other side margin is 2 inches. What should the dimensions of the page be so that the least amount of paper is used?

19. A box with a square base is taller than it is wide. In order to send the box through the U.S. mail, the height of the box and the perimeter of the base can sum to no more than 108 in. What is the maximum volume for such a box?

20. A box with a square base is wider than it is tall. In order to send the box through the U.S. mail, the width of the box and the perimeter of one of the (nonsquare) sides of the box can sum to no more than 108 in. What is the maximum volume for such a box?

21. An open box is to be made from a 3 ft by 8 ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum volume that the box can have.

22. A closed rectangular container with a square base is to have a volume of 2250 in^3. The material for the top and bottom of the container will cost \$2 per in^2, and the material for the sides will cost \$3 per in^2. Find the dimensions of the container of least cost.

23. A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimensions of the container of least cost.

24. A container with square base, vertical sides, and open top is to be made from 1000 ft^2 of material. Find the dimensions of the container with greatest volume.

25. A rectangular container with two square sides and an open top is to have a volume of V cubic units. Find the dimensions of the container with minimum surface area.

26. A church window consisting of a rectangle topped by a semicircle is to have a perimeter p. Find the radius of the semicircle if the area of the window is to be maximum.

27. Find the dimensions of the right circular cylinder of largest volume that can be inscribed in a sphere of radius R.

28. Find the dimensions of the right circular cylinder of greatest surface area that can be inscribed in a sphere of radius R.

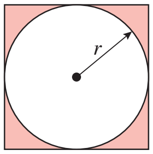
29. A closed, cylindrical can is to have a volume of V cubic units. Show that the can of minimum surface area is achieved when the height is equal to the diameter of the base.

30. A closed cylindrical can is to have a surface area of S square units. Show that the can of maximum volume is achieved when the height is equal to the diameter of the base.

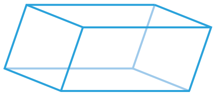
31. A cylindrical can, open at the top, is to hold 500 cm^3 of liquid. Find the height and radius that minimize the amount of material needed to manufacture the can.

32. A soup can in the shape of a right circular cylinder of radius r and height h is to have a prescribed volume V. The top and bottom are cut from squares as shown in Figure Ex-32 on the next page. If the shaded corners are wasted, but there is no other waste, find the ratio r/h for the can requiring the least material (including waste).

33. A box-shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length (Figure Ex-33 on the next page). If the frame is to be made from a wire of length L, what should the dimensions be to obtain a box of greatest volume?



▲ **Figure Ex-32**

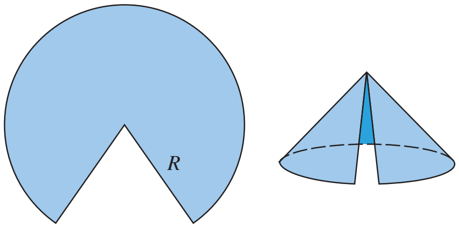


▲ **Figure Ex-33**

34. Suppose that the sum of the surface areas of a sphere and a cube is a constant.
- (a) Show that the sum of their volumes is smallest when the diameter of the sphere is equal to the length of an edge of the cube.
- (b) When will the sum of their volumes be greatest?

35. Find the height and radius of the cone of slant height L whose volume is as large as possible.

36. A cone is made from a circular sheet of radius R by cutting out a sector and gluing the cut edges of the remaining piece together (Figure Ex-36). What is the maximum volume attainable for the cone?



▲ **Figure Ex-36**

37. A cone-shaped paper drinking cup is to hold 100 cm^3 of water. Find the height and radius of the cup that will require the least amount of paper.

38. Find the dimensions of the isosceles triangle of least area that can be circumscribed about a circle of radius R.

39. Find the height and radius of the right circular cone with least volume that can be circumscribed about a sphere of radius R.

40. A commercial cattle ranch currently allows 20 steers per acre of grazing land; on the average its steers weigh 2000 lb at market. Estimates by the Agriculture Department indicate that the average market weight per steer will be reduced by 50 lb for each additional steer added per acre of grazing land. How many steers per acre should be allowed in order for the ranch to get the largest possible total market weight for its cattle?

41. A company mines low-grade nickel ore. If the company mines x tons of ore, it can sell the ore for $p = 225 - 0.25x$ dollars per ton. Find the revenue and marginal revenue functions. At what level of production would the company obtain the maximum revenue?

42. A fertilizer producer finds that it can sell its product at a price of $p = 300 - 0.1x$ dollars per unit when it produces x units of fertilizer. The total production cost (in dollars) for x units is

$$C(x) = 15{,}000 + 125x + 0.025x^2$$

If the production capacity of the firm is at most 1000 units of fertilizer in a specified time, how many units must be manufactured and sold in that time to maximize the profit?

43. (a) A chemical manufacturer sells sulfuric acid in bulk at a price of \$100 per unit. If the daily total production cost in dollars for x units is

$$C(x) = 100{,}000 + 50x + 0.0025x^2$$

and if the daily production capacity is at most 7000 units, how many units of sulfuric acid must be manufactured and sold daily to maximize the profit?
- (b) Would it benefit the manufacturer to expand the daily production capacity?
- (c) Use marginal analysis to approximate the effect on profit if daily production could be increased from 7000 to 7001 units.

44. A firm determines that x units of its product can be sold daily at p dollars per unit, where

$$x = 1000 - p$$

The cost of producing x units per day is

$$C(x) = 3000 + 20x$$

- (a) Find the revenue function $R(x)$.
- (b) Find the profit function $P(x)$.
- (c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.
- (d) Find the maximum profit.
- (e) What price per unit must be charged to obtain the maximum profit?

45. In a certain chemical manufacturing process, the daily weight y of defective chemical output depends on the total weight x of all output according to the empirical formula

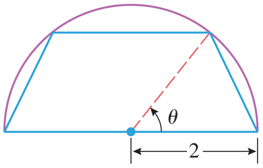
$$y = 0.01x + 0.00003x^2$$

where x and y are in pounds. If the profit is \$100 per pound of nondefective chemical produced and the loss is \$20 per pound of defective chemical produced, how many pounds of chemical should be produced daily to maximize the total daily profit?

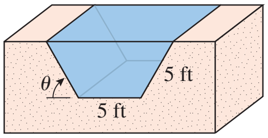
46. An independent truck driver charges a client \$15 for each hour of driving, plus the cost of fuel. At highway speeds of v miles per hour, the trucker's rig gets $10 - 0.07v$ miles per gallon of diesel fuel. If diesel fuel costs \$2.50 per gallon, what speed v will minimize the cost to the client?

47. A trapezoid is inscribed in a semicircle of radius 2 so that one side is along the diameter (Figure Ex-47 on the next page). Find the maximum possible area for the trapezoid. [*Hint:* Express the area of the trapezoid in terms of θ.]

48. A drainage channel is to be made so that its cross section is a trapezoid with equally sloping sides (Figure Ex-48 on the next page). If the sides and bottom all have a length of 5 ft, how should the angle θ $(0 \le \theta \le \pi/2)$ be chosen to yield the greatest cross-sectional area of the channel?



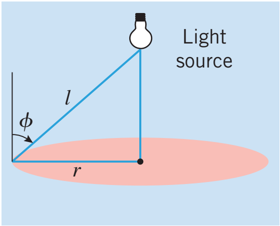
▲ **Figure Ex-47**



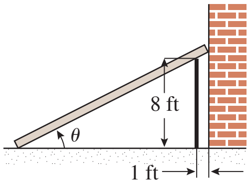
▲ **Figure Ex-48**

49. A lamp is suspended above the center of a round table of radius r. How high above the table should the lamp be placed to achieve maximum illumination at the edge of the table? [Assume that the illumination I is directly proportional to the cosine of the angle of incidence ϕ of the light rays and inversely proportional to the square of the distance l from the light source (Figure Ex-49).]

50. A plank is used to reach over a fence 8 ft high to support a wall that is 1 ft behind the fence (Figure Ex-50). What is the length of the shortest plank that can be used? [*Hint:* Express the length of the plank in terms of the angle θ shown in the figure.]



▲ **Figure Ex-49**



▲ **Figure Ex-50**

51. Two particles, A and B, are in motion in the xy-plane. Their coordinates at each instant of time t $(t \ge 0)$ are given by $x_A = t$, $y_A = 2t$, $x_B = 1 - t$, and $y_B = t$. Find the minimum distance between A and B.

52. Follow the directions of Exercise 51, with $x_A = t$, $y_A = t^2$, $x_B = 2t$, and $y_B = 2$.

53. Find the coordinates of the point P on the curve

$$y = \frac{1}{x^2} \quad (x > 0)$$

where the segment of the tangent line at P that is cut off by the coordinate axes has its shortest length.

54. Find the x-coordinate of the point P on the parabola

$$y = 1 - x^2 \quad (0 < x \le 1)$$

where the triangle that is enclosed by the tangent line at P and the coordinate axes has the smallest area.

55. Where on the curve $y = (1 + x^2)^{-1}$ does the tangent line have the greatest slope?

56. A rectangular water tank has a base of area four square meters. Water flows into the tank until the amount of water in the tank is 20 m^3, at which point any additional flow into the tank is diverted by an overflow valve. Suppose that the tank is initially empty, and water is pumped into the tank so that after t minutes, $(2t^3 + 7t)/(t^2 + 12)$ cubic meters of water has been pumped into the tank. At what time is the height of the water in the tank increasing most rapidly?

57. The shoreline of Circle Lake is a circle with diameter 2 mi. Nancy's training routine begins at point E on the eastern shore of the lake. She jogs along the north shore to a point P and then swims the straight line distance, if any, from P to point W diametrically opposite E (Figure Ex-57). Nancy swims at a rate of 2 mi/h and jogs at 8 mi/h. How far should Nancy jog in order to complete her training routine in

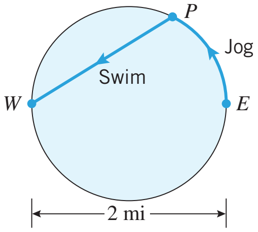
(a) the least amount of time

(b) the greatest amount of time?

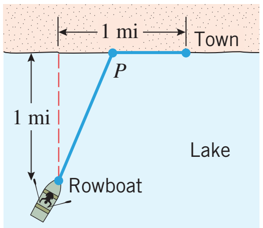
58. A man is floating in a rowboat 1 mile from the (straight) shoreline of a large lake. A town is located on the shoreline 1 mile from the point on the shoreline closest to the man. As suggested in Figure Ex-58, he intends to row in a straight line to some point P on the shoreline and then walk the remaining distance to the town. To what point should he row in order to reach his destination in the least time if

(a) he can walk 5 mi/h and row 3 mi/h

(b) he can walk 5 mi/h and row 4 mi/h?



▲ **Figure Ex-57**

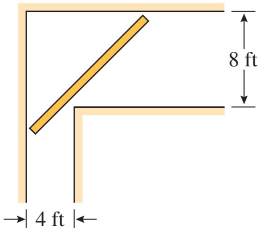


▲ **Figure Ex-58**

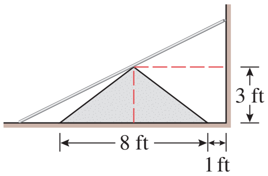
59. A pipe of negligible diameter is to be carried horizontally around a corner from a hallway 8 ft wide into a hallway 4 ft wide (Figure Ex-59). What is the maximum length that the pipe can have?

Source: An interesting discussion of this problem in the case where the diameter of the pipe is not neglected is given by Norman Miller in the *American Mathematical Monthly*, Vol. 56, 1949, pp. 177–179.

60. A concrete barrier whose cross section is an isosceles triangle runs parallel to a wall. The height of the barrier is 3 ft, the width of the base of a cross section is 8 ft, and the barrier is positioned on level ground with its base 1 ft from the wall. A straight, stiff metal rod of negligible diameter has one end on the ground, the other end against the wall, and touches the top of the barrier (Figure Ex-60). What is the minimum length the rod can have?



▲ **Figure Ex-59**



▲ **Figure Ex-60**

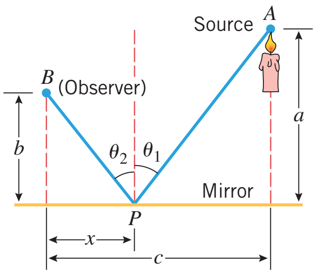
61. Suppose that the intensity of a point light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two point light sources with strengths of S and $8S$ are separated by a distance of 90 cm. Where on the line segment between the two sources is the total intensity a minimum?

62. Given points $A(2, 1)$ and $B(5, 4)$, find the point P in the interval $[2, 5]$ on the x-axis that maximizes angle APB.

63. The lower edge of a painting, 10 ft in height, is 2 ft above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?

FOCUS ON CONCEPTS

64. ***Fermat's principle*** (biography on p. 225) in optics states that light traveling from one point to another follows that path for which the total travel time is minimum. In a uniform medium, the paths of "minimum time" and "shortest distance" turn out to be the same, so that light, if unobstructed, travels along a straight line. Assume that we have a light source, a flat mirror, and an observer in a uniform medium. If a light ray leaves the source, bounces off the mirror, and travels on to the observer, then its path will consist of two line segments, as shown in Figure Ex-64. According to Fermat's principle, the path will be such that the total travel time t is minimum or, since the medium is uniform, the path will be such that the total distance traveled from A to P to B is as small as possible. Assuming the minimum occurs when $dt/dx = 0$, show that the light ray will strike the mirror at the point P where the "angle of incidence" θ_1 equals the "angle of reflection" θ_2.



◀ **Figure Ex-64**

65. Fermat's principle (Exercise 64) also explains why light rays traveling between air and water undergo bending (refraction). Imagine that we have two uniform media (such as air and water) and a light ray traveling from a source A in one medium to an observer B in the other medium (Figure Ex-65). It is known that light travels at a constant speed in a uniform medium, but more slowly in a dense medium (such as water) than in a thin medium (such as air). Consequently, the path of shortest time from A to B is not necessarily a straight line, but rather some broken line path A to P to B allowing the light to take greatest advantage of its higher speed through the thin medium. ***Snell's law of refraction*** (biography on p. 238) states that the path of the light ray will be such that

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

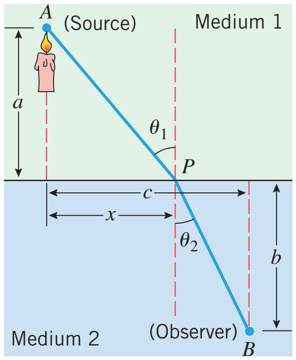
where v_1 is the speed of light in the first medium, v_2 is the speed of light in the second medium, and θ_1 and θ_2 are the angles shown in Figure Ex-65. Show that this follows from the assumption that the path of minimum time occurs when $dt/dx = 0$.

66. A farmer wants to walk at a constant rate from her barn to a straight river, fill her pail, and carry it to her house in the least time.

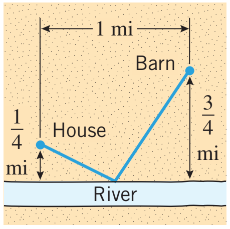
(a) Explain how this problem relates to Fermat's principle and the light-reflection problem in Exercise 64.

(b) Use the result of Exercise 64 to describe geometrically the best path for the farmer to take.

(c) Use part (b) to determine where the farmer should fill her pail if her house and barn are located as in Figure Ex-66.



▲ **Figure Ex-65**



▲ **Figure Ex-66**

67. If an unknown physical quantity x is measured n times, the measurements $x_1, x_2, \ldots, x_n$ often vary because of uncontrollable factors such as temperature, atmospheric pressure, and so forth. Thus, a scientist is often faced with the problem of using n different observed measurements to obtain an estimate $\bar{x}$ of an unknown quantity x. One method for making such an estimate is based on the ***least squares principle***, which states that the estimate $\bar{x}$ should be chosen to minimize

$$s = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2$$

which is the sum of the squares of the deviations between the estimate $\bar{x}$ and the measured values. Show that the estimate resulting from the least squares principle is

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

that is, $\bar{x}$ is the arithmetic average of the observed values.

68. Prove: If $f(x) \geq 0$ on an interval and if $f(x)$ has a maximum value on that interval at x_0, then $\sqrt{f(x)}$ also has a maximum value at x_0. Similarly for minimum values. [*Hint:* Use the fact that $\sqrt{x}$ is an increasing function on the interval $[0, +\infty)$.]

69. **Writing** Discuss the importance of finding intervals of possible values imposed by physical restrictions on variables in an applied maximum or minimum problem.

✔QUICK CHECK ANSWERS 3.5

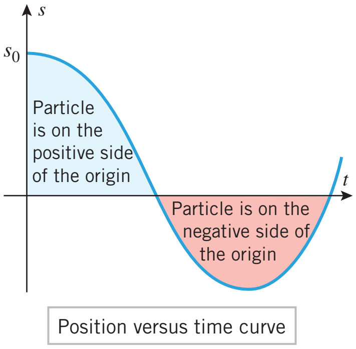
1. $x + \dfrac{1}{x}$; $(0, +\infty)$ **2.** $x(10 - x)$; $[0, 10]$ **3.** $x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x$; $[0, 3]$
4. $x(20 - 2x)(32 - 2x) = 4x^3 - 104x^2 + 640x$; $[0, 10]$

3.6 RECTILINEAR MOTION

In this section we will continue the study of rectilinear motion that we began in Section 2.1. We will define the notion of "acceleration" mathematically, and we will show how the tools of calculus developed earlier in this chapter can be used to analyze rectilinear motion in more depth.

REVIEW OF TERMINOLOGY

Recall from Section 2.1 that a particle that can move in either direction along a coordinate line is said to be in ***rectilinear motion***. The line might be an x-axis, a y-axis, or a coordinate line inclined at some angle. In general discussions we will designate the coordinate line as the s-axis. We will assume that units are chosen for measuring distance and time and that we begin observing the motion of the particle at time $t = 0$. As the particle moves along the s-axis, its coordinate s will be some function of time, say $s = s(t)$. We call $s(t)$ the ***position function*** of the particle,[*] and we call the graph of s versus t the ***position versus time curve***. If the coordinate of a particle at time t_1 is $s(t_1)$ and the coordinate at a later time t_2 is $s(t_2)$, then $s(t_2) - s(t_1)$ is called the ***displacement*** of the particle over the time interval $[t_1, t_2]$. The displacement describes the change in position of the particle.



▲ **Figure 3.6.1**

Figure 3.6.1 shows a typical position versus time curve for a particle in rectilinear motion. We can tell from that graph that the coordinate of the particle at time $t = 0$ is s_0, and we can tell from the sign of s when the particle is on the negative or the positive side of the origin as it moves along the coordinate line.

▶ **Example 1** Figure 3.6.2*a* shows the position versus time curve for a particle moving along an s-axis. In words, describe how the position of the particle changes with time.

Solution. The particle is at $s = -3$ at time $t = 0$. It moves in the positive direction until time $t = 4$, since s is increasing. At time $t = 4$ the particle is at position $s = 3$. At that time it turns around and travels in the negative direction until time $t = 7$, since s is decreasing. At time $t = 7$ the particle is at position $s = -1$, and it remains stationary thereafter, since s is constant for $t > 7$. This is illustrated schematically in Figure 3.6.2*b*. ◀

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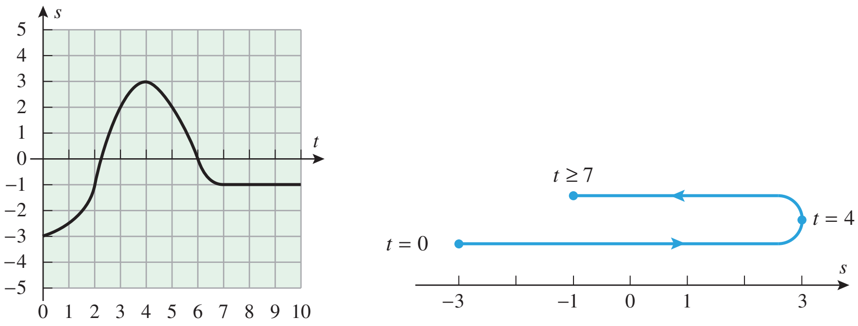
[*] In writing $s = s(t)$, rather than the more familiar $s = f(t)$, we are using the letter s both as the dependent variable and the name of the function. This is common practice in engineering and physics.

Willebrord van Roijen Snell (1591–1626) Dutch mathematician. Snell, who succeeded his father to the post of Professor of Mathematics at the University of Leiden in 1613, is most famous for the result of light refraction that bears his name. Although this phenomenon was studied as far back as the ancient Greek astronomer Ptolemy, until Snell's work the relationship was incorrectly thought to be $\theta_1/v_1 = \theta_2/v_2$. Snell's law was published by Descartes in 1638 without giving proper credit to Snell. Snell also discovered a method for determining distances by triangulation that founded the modern technique of mapmaking.



(*a*) (*b*)

▲ **Figure 3.6.2**

VELOCITY AND SPEED

Recall from Formula (5) of Section 2.1 and Formula (4) of Section 2.2 that the instantaneous velocity of a particle in rectilinear motion is the derivative of the position function. Thus, if a particle in rectilinear motion has position function $s(t)$, then we define its ***velocity function*** $v(t)$ to be

$$v(t) = s'(t) = \frac{ds}{dt} \tag{1}$$

We should more properly call $v(t)$ the ***instantaneous velocity function*** to distinguish instantaneous velocity from average velocity. However, we will follow the standard practice of referring to it as the "velocity function," leaving it understood that it describes instantaneous velocity.

The sign of the velocity tells which way the particle is moving—a positive value for $v(t)$ means that s is increasing with time, so the particle is moving in the positive direction, and a negative value for $v(t)$ means that s is decreasing with time, so the particle is moving in the negative direction. If $v(t) = 0$, then the particle has momentarily stopped.

For a particle in rectilinear motion it is important to distinguish between its ***velocity***, which describes how fast and in what direction the particle is moving, and its ***speed***, which describes only how fast the particle is moving. We make this distinction by defining speed to be the absolute value of velocity. Thus a particle with a velocity of 2 m/s has a speed of 2 m/s and is moving in the positive direction, while a particle with a velocity of −2 m/s also has a speed of 2 m/s but is moving in the negative direction.

Since the instantaneous speed of a particle is the absolute value of its instantaneous velocity, we define its ***speed function*** to be

$$|v(t)| = |s'(t)| = \left|\frac{ds}{dt}\right| \tag{2}$$

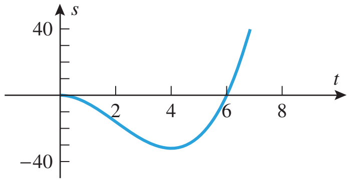
The speed function, which is always nonnegative, tells us how fast the particle is moving but not its direction of motion.

▶ **Example 2** Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis, where s is in meters and t is in seconds. Find the velocity and speed functions, and show the graphs of position, velocity, and speed versus time.

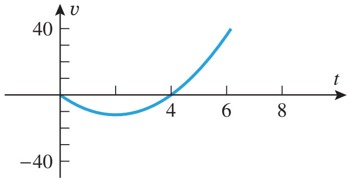
Solution. From (1) and (2), the velocity and speed functions are given by

$$v(t) = \frac{ds}{dt} = 3t^2 - 12t \quad \text{and} \quad |v(t)| = |3t^2 - 12t|$$

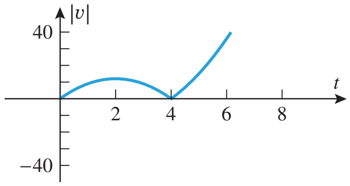
The graphs of position, velocity, and speed versus time are shown in Figure 3.6.3. Observe that velocity and speed both have units of meters per second (m/s), since s is in meters (m) and time is in seconds (s). ◀



Position versus time



Velocity versus time



Speed versus time

▲ **Figure 3.6.3**

The graphs in Figure 3.6.3 provide a wealth of visual information about the motion of the particle. For example, the position versus time curve tells us that the particle is on the negative side of the origin for $0 < t < 6$, is on the positive side of the origin for $t > 6$, and is at the origin at times $t = 0$ and $t = 6$. The velocity versus time curve tells us that the particle is moving in the negative direction if $0 < t < 4$, is moving in the positive direction if $t > 4$, and is momentarily stopped at times $t = 0$ and $t = 4$ (the velocity is zero at those times). The speed versus time curve tells us that the speed of the particle is increasing for $0 < t < 2$, decreasing for $2 < t < 4$, and increasing again for $t > 4$.

ACCELERATION

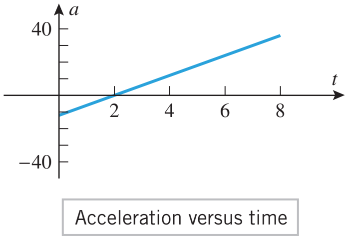
In rectilinear motion, the rate at which the instantaneous velocity of a particle changes with time is called its ***instantaneous acceleration***. Thus, if a particle in rectilinear motion has velocity function $v(t)$, then we define its ***acceleration function*** to be

$$a(t) = v'(t) = \frac{dv}{dt} \tag{3}$$

Alternatively, we can use the fact that $v(t) = s'(t)$ to express the acceleration function in terms of the position function as

$$a(t) = s''(t) = \frac{d^2s}{dt^2} \tag{4}$$

▶ Example 3 Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis, where s is in meters and t is in seconds. Find the acceleration function $a(t)$, and show the graph of acceleration versus time.



▲ **Figure 3.6.4**

Solution. From Example 2, the velocity function of the particle is $v(t) = 3t^2 - 12t$, so the acceleration function is

$$a(t) = \frac{dv}{dt} = 6t - 12$$

and the acceleration versus time curve is the line shown in Figure 3.6.4. Note that in this example the acceleration has units of m/s^2, since v is in meters per second (m/s) and time is in seconds (s). ◀

SPEEDING UP AND SLOWING DOWN

We will say that a particle in rectilinear motion is ***speeding up*** when its speed is increasing and is ***slowing down*** when its speed is decreasing. In everyday language an object that is speeding up is said to be "accelerating" and an object that is slowing down is said to be "decelerating"; thus, one might expect that a particle in rectilinear motion will be speeding up when its acceleration is positive and slowing down when it is negative. Although this is true for a particle moving in the positive direction, it is *not* true for a particle moving in the negative direction—a particle with negative velocity is speeding up when its acceleration is negative and slowing down when its acceleration is positive. This is because a positive acceleration implies an increasing velocity, and increasing a negative velocity decreases its absolute value; similarly, a negative acceleration implies a decreasing velocity, and decreasing a negative velocity increases its absolute value.

The preceding informal discussion can be summarized as follows (Exercise 39):

If $a(t) = 0$ over a certain time interval, what does this tell you about the motion of the particle during that time?

INTERPRETING THE SIGN OF ACCELERATION *A particle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.*

▶ **Example 4** In Examples 2 and 3 we found the velocity versus time curve and the acceleration versus time curve for a particle with position function $s(t) = t^3 - 6t^2$. Use those curves to determine when the particle is speeding up and slowing down, and confirm that your results are consistent with the speed versus time curve obtained in Example 2.

Solution. Over the time interval $0 < t < 2$ the velocity and acceleration are negative, so the particle is speeding up. This is consistent with the speed versus time curve, since the speed is increasing over this time interval. Over the time interval $2 < t < 4$ the velocity is negative and the acceleration is positive, so the particle is slowing down. This is also consistent with the speed versus time curve, since the speed is decreasing over this time interval. Finally, on the time interval $t > 4$ the velocity and acceleration are positive, so the particle is speeding up, which again is consistent with the speed versus time curve. ◀

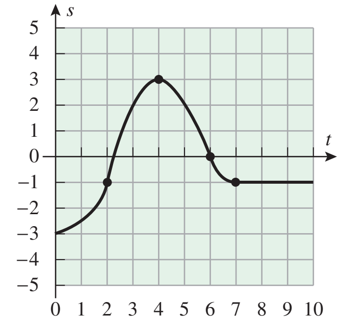
ANALYZING THE POSITION VERSUS TIME CURVE

The position versus time curve contains all of the significant information about the position and velocity of a particle in rectilinear motion:

- If $s(t) > 0$, the particle is on the positive side of the s-axis.
- If $s(t) < 0$, the particle is on the negative side of the s-axis.
- The slope of the curve at any time is equal to the instantaneous velocity at that time.
- Where the curve has positive slope, the velocity is positive and the particle is moving in the positive direction.
- Where the curve has negative slope, the velocity is negative and the particle is moving in the negative direction.
- Where the slope of the curve is zero, the velocity is zero, and the particle is momentarily stopped.

Information about the acceleration of a particle in rectilinear motion can also be deduced from the position versus time curve by examining its concavity. For example, we know that the position versus time curve will be concave up on intervals where $s''(t) > 0$ and will be concave down on intervals where $s''(t) < 0$. But we know from (4) that $s''(t)$ is the acceleration, so that on intervals where the position versus time curve is concave up the particle has a positive acceleration, and on intervals where it is concave down the particle has a negative acceleration.

Table 3.6.1 summarizes our observations about the position versus time curve.

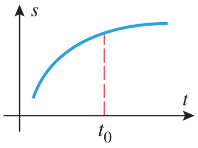
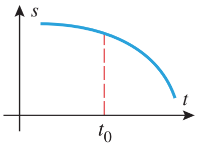
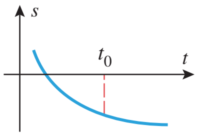
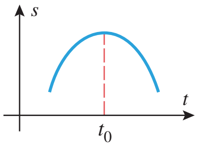


▲ **Figure 3.6.5**

▶ **Example 5** Use the position versus time curve in Figure 3.6.5 to determine when the particle in Example 1 is speeding up and slowing down.

Solution. From $t = 0$ to $t = 2$, the acceleration and velocity are positive, so the particle is speeding up. From $t = 2$ to $t = 4$, the acceleration is negative and the velocity is positive, so the particle is slowing down. At $t = 4$, the velocity is zero, so the particle has momentarily stopped. From $t = 4$ to $t = 6$, the acceleration is negative and the velocity is negative, so the particle is speeding up. From $t = 6$ to $t = 7$, the acceleration is positive and the velocity is negative, so the particle is slowing down. Thereafter, the velocity is zero, so the particle has stopped. ◀

Table 3.6.1
ANALYSIS OF PARTICLE MOTION

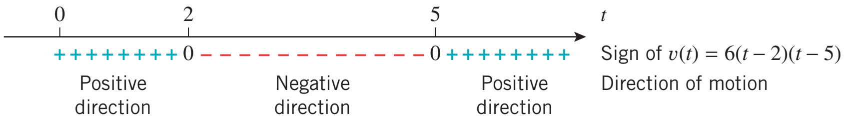
POSITION VERSUS TIME CURVE	CHARACTERISTICS OF THE CURVE AT $t = t_0$	BEHAVIOR OF THE PARTICLE AT TIME $t = t_0$
	• $s(t_0) > 0$ • Curve has positive slope. • Curve is concave down.	• Particle is on the positive side of the origin. • Particle is moving in the positive direction. • Velocity is decreasing. • Particle is slowing down.
	• $s(t_0) > 0$ • Curve has negative slope. • Curve is concave down.	• Particle is on the positive side of the origin. • Particle is moving in the negative direction. • Velocity is decreasing. • Particle is speeding up.
	• $s(t_0) < 0$ • Curve has negative slope. • Curve is concave up.	• Particle is on the negative side of the origin. • Particle is moving in the negative direction. • Velocity is increasing. • Particle is slowing down.
	• $s(t_0) > 0$ • Curve has zero slope. • Curve is concave down.	• Particle is on the positive side of the origin. • Particle is momentarily stopped. • Velocity is decreasing.

▶ Example 6 Suppose that the position function of a particle moving on a coordinate line is given by $s(t) = 2t^3 - 21t^2 + 60t + 3$. Analyze the motion of the particle for $t \geq 0$.

Solution. The velocity and acceleration functions are

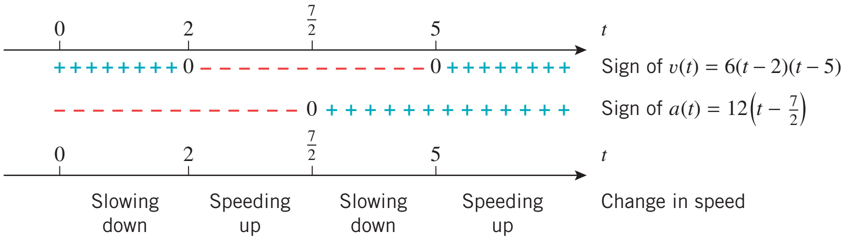
$$v(t) = s'(t) = 6t^2 - 42t + 60 = 6(t-2)(t-5)$$
$$a(t) = v'(t) = 12t - 42 = 12\left(t - \tfrac{7}{2}\right)$$

- *Direction of motion:* The sign analysis of the velocity function in Figure 3.6.6 shows that the particle is moving in the positive direction over the time interval $0 \leq t < 2$, stops momentarily at time $t = 2$, moves in the negative direction over the time interval $2 < t < 5$, stops momentarily at time $t = 5$, and then moves in the positive direction thereafter.



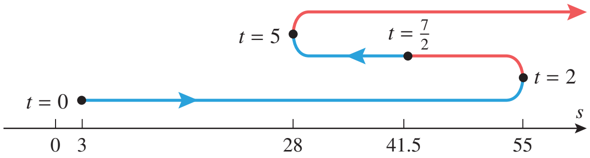
▲ **Figure 3.6.6**

- *Change in speed:* A comparison of the signs of the velocity and acceleration functions is shown in Figure 3.6.7. Since the particle is speeding up when the signs are the same and is slowing down when they are opposite, we see that the particle is slowing down over the time interval $0 \le t < 2$ and stops momentarily at time $t = 2$. It is then speeding up over the time interval $2 < t < \frac{7}{2}$. At time $t = \frac{7}{2}$ the instantaneous acceleration is zero, so the particle is neither speeding up nor slowing down. It is then slowing down over the time interval $\frac{7}{2} < t < 5$ and stops momentarily at time $t = 5$. Thereafter, it is speeding up.



▲ **Figure 3.6.7**

Conclusions: The diagram in Figure 3.6.8 summarizes the above information schematically. The curved line is descriptive only; the actual path is back and forth on the coordinate line. The coordinates of the particle at times $t = 0$, $t = 2$, $t = \frac{7}{2}$, and $t = 5$ were computed from $s(t)$. Segments in red indicate that the particle is speeding up and segments in blue indicate that it is slowing down. ◀



▶ **Figure 3.6.8**

✔QUICK CHECK EXERCISES 3.6 *(See page 246 for answers.)*

1. For a particle in rectilinear motion, the velocity and position functions $v(t)$ and $s(t)$ are related by the equation ________, and the acceleration and velocity functions $a(t)$ and $v(t)$ are related by the equation ________.

2. Suppose that a particle moving along the s-axis has position function $s(t) = 7t - 2t^2$. At time $t = 3$, the particle's position is ________, its velocity is ________, its speed is ________, and its acceleration is ________.

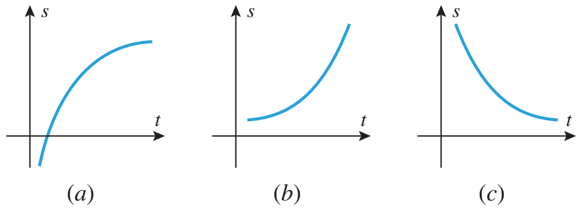
3. A particle in rectilinear motion is speeding up if the signs of its velocity and acceleration are ________, and it is slowing down if these signs are ________.

4. Suppose that a particle moving along the s-axis has position function $s(t) = t^4 - 24t^2$ over the time interval $t \ge 0$. The particle slows down over the time interval(s) ________.

EXERCISE SET 3.6 Graphing Utility

FOCUS ON CONCEPTS

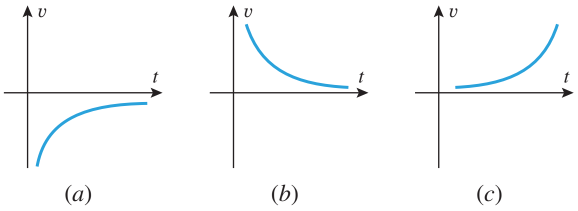
1. The graphs of three position functions are shown in the accompanying figure. In each case determine the signs of the velocity and acceleration, and then determine whether the particle is speeding up or slowing down.



(*a*) (*b*) (*c*)

▲ **Figure Ex-1**

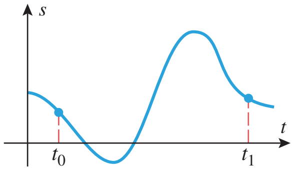
2. The graphs of three velocity functions are shown in the accompanying figure. In each case determine the sign of the acceleration, and then determine whether the particle is speeding up or slowing down.



(a) (b) (c)

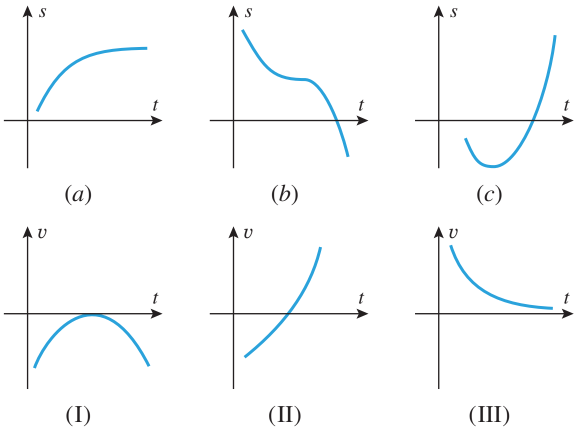
▲ **Figure Ex-2**

3. The graph of the position function of a particle moving on a horizontal line is shown in the accompanying figure.
(a) Is the particle moving left or right at time t_0?
(b) Is the acceleration positive or negative at time t_0?
(c) Is the particle speeding up or slowing down at time t_0?
(d) Is the particle speeding up or slowing down at time t_1?



◀ **Figure Ex-3**

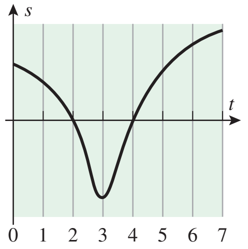
4. For the graphs in the accompanying figure, match the position functions (a)–(c) with their corresponding velocity functions (I)–(III).



▲ **Figure Ex-4**

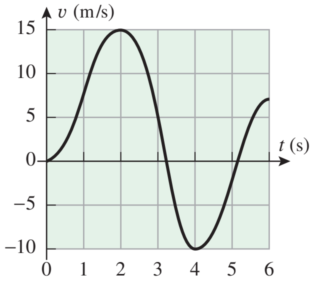
5. Sketch a reasonable graph of s versus t for a mouse that is trapped in a narrow corridor (an s-axis with the positive direction to the right) and scurries back and forth as follows. It runs right with a constant speed of 1.2 m/s for a while, then gradually slows down to 0.6 m/s, then quickly speeds up to 2.0 m/s, then gradually slows to a stop but immediately reverses direction and quickly speeds up to 1.2 m/s.

6. The accompanying figure shows the position versus time curve for an ant that moves along a narrow vertical pipe, where t is measured in seconds and the s-axis is along the pipe with the positive direction up.
(a) When, if ever, is the ant above the origin?
(b) When, if ever, does the ant have velocity zero?
(c) When, if ever, is the ant moving down the pipe?



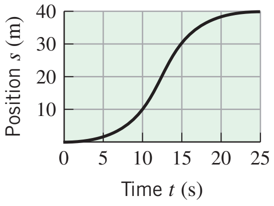
◀ **Figure Ex-6**

7. The accompanying figure shows the graph of velocity versus time for a particle moving along a coordinate line. Make a rough sketch of the graphs of speed versus time and acceleration versus time.



◀ **Figure Ex-7**

8. The accompanying figure shows the position versus time graph for an elevator that ascends 40 m from one stop to the next.
(a) Estimate the velocity when the elevator is halfway up to the top.
(b) Sketch rough graphs of the velocity versus time curve and the acceleration versus time curve.



◀ **Figure Ex-8**

9–12 True–False Determine whether the statement is true or false. Explain your answer. ■

9. A particle is speeding up when its position versus time graph is increasing.

10. Velocity is the derivative of position with respect to time.

11. Acceleration is the absolute value of velocity.

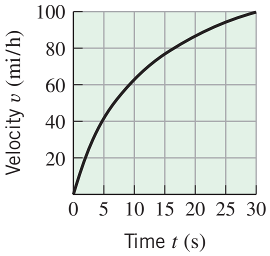
12. If the position versus time curve is increasing and concave down, then the particle is slowing down.

13. The accompanying figure shows the velocity versus time graph for a test run on a Chevrolet Volt. Using this graph, estimate
(a) the acceleration at 60 mi/h (in ft/s^2)
(b) the time at which the maximum acceleration occurs.

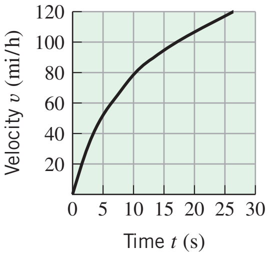
Source: Data from *Car and Driver Magazine*, December 2010.

14. The accompanying figure shows the velocity versus time graph for a test run on a Dodge Challenger. Using this graph, estimate
(a) the acceleration at 60 mi/h (in ft/s^2)
(b) the time at which the maximum acceleration occurs.

Source: Data from *Car and Driver Magazine*, March 2011.



▲ **Figure Ex-13**



▲ **Figure Ex-14**

15–16 The function $s(t)$ describes the position of a particle moving along a coordinate line, where s is in meters and t is in seconds.
(a) Make a table showing the position, velocity, and acceleration to two decimal places at times $t = 1, 2, 3, 4, 5$.
(b) At each of the times in part (a), determine whether the particle is stopped; if it is not, state its direction of motion.
(c) At each of the times in part (a), determine whether the particle is speeding up, slowing down, or neither. ■

15. $s(t) = \sin \dfrac{\pi t}{4}$ **16.** $s(t) = 2\cos\left(\dfrac{\pi}{3}t - \dfrac{2\pi}{3}\right)$

17–20 The function $s(t)$ describes the position of a particle moving along a coordinate line, where s is in feet and t is in seconds.
(a) Find the velocity and acceleration functions.
(b) Find the position, velocity, speed, and acceleration at time $t = 1$.
(c) At what times is the particle stopped?
(d) When is the particle speeding up? Slowing down?
(e) Find the total distance traveled by the particle from time $t = 0$ to time $t = 5$. ■

17. $s(t) = t^3 - 3t^2, \quad t \geq 0$

18. $s(t) = t^4 - 4t^2 + 4, \quad t \geq 0$

19. $s(t) = 9 - 9\cos(\pi t/3), \quad 0 \leq t \leq 5$

20. $s(t) = \dfrac{t}{t^2 + 4}, \quad t \geq 0$

21. Let $s(t) = t/(t^2 + 5)$ be the position function of a particle moving along a coordinate line, where s is in meters and t is in seconds. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for $t \geq 0$, and use those graphs where needed.
(a) Use the appropriate graph to make a rough estimate of the time at which the particle first reverses the direction of its motion; and then find the time exactly.
(b) Find the exact position of the particle when it first reverses the direction of its motion.
(c) Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

22. Let $s(t) = 4t^2/(2t^4 + 3)$ be the position function of a particle moving along a coordinate line, where s is in meters and t is in seconds. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for $t \geq 0$, and use those graphs where needed.
(a) Use the appropriate graph to make a rough estimate of the time at which the particle first reverses the direction of its motion; and then find the time exactly.
(b) Find the exact position of the particle when it first reverses the direction of its motion.
(c) Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

23–28 A position function of a particle moving along a coordinate line is given. Use the method of Example 6 to analyze the motion of the particle for $t \geq 0$, and give a schematic picture of the motion (as in Figure 3.6.8). ■

23. $s = -4t + 3$ **24.** $s = 5t^2 - 20t$

25. $s = t^3 - 9t^2 + 24t$ **26.** $s = t + \dfrac{25}{t + 2}$

27. $s = \begin{cases} \cos t, & 0 \leq t < 2\pi \\ 1, & t \geq 2\pi \end{cases}$

28. $s = \begin{cases} 2t(t-2)^2, & 0 \leq t < 3 \\ 13 - 7(t-4)^2, & t \geq 3 \end{cases}$

29. Let $s(t) = 5t^2 - 22t$ be the position function of a particle moving along a coordinate line, where s is in feet and t is in seconds.
(a) Find the maximum speed of the particle during the time interval $1 \leq t \leq 3$.
(b) When, during the time interval $1 \leq t \leq 3$, is the particle farthest from the origin? What is its position at that instant?

30. Let $s = 100/(t^2 + 12)$ be the position function of a particle moving along a coordinate line, where s is in feet and t is in seconds. Find the maximum speed of the particle for $t \geq 0$, and find the direction of motion of the particle when it has its maximum speed.

31–32 A position function of a particle moving along a coordinate line is provided. (a) Evaluate s and v when $a = 0$. (b) Evaluate s and a when $v = 0$.

31. $s = \sin 2t, \quad 0 \leq t \leq \pi/2$

32. $s = t^3 - 6t^2 + 1$

33. Let $s = \sqrt{2t^2 + 1}$ be the position function of a particle moving along a coordinate line.
  (a) Use a graphing utility to generate the graph of v versus t, and make a conjecture about the velocity of the particle as $t \to +\infty$.
  (b) Check your conjecture by finding $\lim_{t \to +\infty} v$.

34. (a) Use the chain rule to show that for a particle in rectilinear motion $a = v(dv/ds)$.
  (b) Let $s = \sqrt{3t + 7}, t \geq 0$. Find a formula for v in terms of s and use the equation in part (a) to find the acceleration when $s = 5$.

35. Suppose that the position functions of two particles, P_1 and P_2, in motion along the same line are

$$s_1 = \tfrac{1}{2}t^2 - t + 3 \quad \text{and} \quad s_2 = -\tfrac{1}{4}t^2 + t + 1$$

respectively, for $t \geq 0$.
  (a) Prove that P_1 and P_2 do not collide.
  (b) How close do P_1 and P_2 get to each other?
  (c) During what intervals of time are they moving in opposite directions?

36. Let $s_A = 15t^2 + 10t + 20$ and $s_B = 5t^2 + 40t, t \geq 0$, be the position functions of cars A and B that are moving along parallel straight lanes of a highway.
  (a) How far is car A ahead of car B when $t = 0$?
  (b) At what instants of time are the cars next to each other?
  (c) At what instant of time do they have the same velocity? Which car is ahead at this instant?

37. Prove that a particle is speeding up if the velocity and acceleration have the same sign, and slowing down if they have opposite signs. [*Hint:* Let $r(t) = |v(t)|$ and find $r'(t)$ using the chain rule.]

38. Writing A speedometer on a bicycle calculates the bicycle's speed by measuring the time per rotation for one of the bicycle's wheels. Explain how this measurement can be used to calculate an average velocity for the bicycle, and discuss how well it approximates the instantaneous velocity for the bicycle.

39. Writing A toy rocket is launched into the air and falls to the ground after its fuel runs out. Describe the rocket's acceleration and when the rocket is speeding up or slowing down during its flight. Accompany your description with a sketch of a graph of the rocket's acceleration versus time.

✔QUICK CHECK ANSWERS 3.6

1. $v(t) = s'(t);\ a(t) = v'(t)$ **2.** 3; -5; 5; -4 **3.** the same; opposite **4.** $2 < t < 2\sqrt{3}$

3.7 NEWTON'S METHOD

In Section 1.5 we showed how to approximate the roots of an equation $f(x) = 0$ using the Intermediate-Value Theorem. In this section we will study a technique, called "Newton's Method," that is usually more efficient than that method. Newton's Method is the technique used by many commercial and scientific computer programs for finding roots.

NEWTON'S METHOD

In beginning algebra one learns that the solution of a first-degree equation $ax + b = 0$ is given by the formula $x = -b/a$, and the solutions of a second-degree equation

$$ax^2 + bx + c = 0$$

are given by the quadratic formula. Formulas also exist for the solutions of all third- and fourth-degree equations, although they are too complicated to be of practical use. In 1826 it was shown by the Norwegian mathematician Niels Henrik Abel that it is impossible to construct a similar formula for the solutions of a *general* fifth-degree equation or higher. Thus, for a *specific* fifth-degree polynomial equation such as

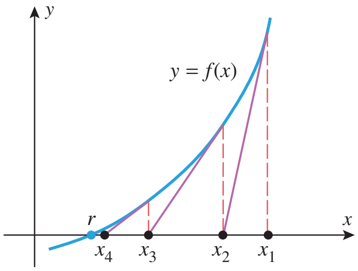
$$x^5 - 9x^4 + 2x^3 - 5x^2 + 17x - 8 = 0$$

it may be difficult or impossible to find exact values for all of the solutions. Similar difficulties occur for nonpolynomial equations such as

$$x - \cos x = 0$$

For such equations the solutions are generally approximated in some way, often by the method we will now discuss.

Suppose that we are trying to find a root r of the equation $f(x) = 0$, and suppose that by some method we are able to obtain an initial rough estimate, x_1, of r, say by generating the graph of $y = f(x)$ with a graphing utility and examining the x-intercept. If $f(x_1) = 0$, then $r = x_1$. If $f(x_1) \neq 0$, then we consider an easier problem, that of finding a root to a linear equation. The best linear approximation to $y = f(x)$ near $x = x_1$ is given by the tangent line to the graph of f at x_1, so it might be reasonable to expect that the x-intercept to this tangent line provides an improved approximation to r. Call this intercept x_2 (Figure 3.7.1). We can now treat x_2 in the same way we did x_1. If $f(x_2) = 0$, then $r = x_2$. If $f(x_2) \neq 0$, then construct the tangent line to the graph of f at x_2, and take x_3 to be the x-intercept of this tangent line. Continuing in this way we can generate a succession of values $x_1, x_2, x_3, x_4, \ldots$ that will usually approach r. This procedure for approximating r is called ***Newton's Method***.



▲ **Figure 3.7.1**

To implement Newton's Method analytically, we must derive a formula that will tell us how to calculate each improved approximation from the preceding approximation. For this purpose, we note that the point-slope form of the tangent line to $y = f(x)$ at the initial approximation x_1 is

$$y - f(x_1) = f'(x_1)(x - x_1) \tag{1}$$

If $f'(x_1) \neq 0$, then this line is not parallel to the x-axis and consequently it crosses the x-axis at some point $(x_2, 0)$. Substituting the coordinates of this point in (1) yields

$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

Niels Henrik Abel (1802–1829) Norwegian mathematician. Abel was the son of a poor Lutheran minister and a remarkably beautiful mother from whom he inherited strikingly good looks. In his brief life of 26 years Abel lived in virtual poverty and suffered a succession of adversities, yet he managed to prove major results that altered the mathematical landscape forever. At the age of thirteen he was sent away from home to a school whose better days had long passed. By a stroke of luck the school had just hired a teacher named Bernt Michael Holmboe, who quickly discovered that Abel had extraordinary mathematical ability. Together, they studied the calculus texts of Euler and works of Newton and the later French mathematicians. By the time he graduated, Abel was familar with most of the great mathematical literature. In 1820 his father died, leaving the family in dire financial straits. Abel was able to enter the University of Christiania in Oslo only because he was granted a free room and several professors supported him directly from their salaries. The University had no advanced courses in mathematics, so Abel took a preliminary degree in 1822 and then continued to study mathematics on his own. In 1824 he published at his own expense the proof that it is impossible to solve the general fifth-degree polynomial equation algebraically. With the hope that this landmark paper would lead to his recognition and acceptance by the European mathematical community, Abel sent the paper to the great German mathematician Gauss, who casually declared it to be a "monstrosity" and tossed it aside. However, in 1826 Abel's paper on the fifth-degree equation and other work was published in the first issue of a new journal, founded by his friend, Leopold Crelle. In the summer of 1826 he completed a landmark work on transcendental functions, which he submitted to the French Academy of Sciences. He hoped to establish himself as a major mathematician, for many young mathematicians had gained quick distinction by having their work accepted by the Academy. However, Abel waited in vain because the paper was either ignored or misplaced by one of the referees, and it did not surface again until two years after his death. That paper was later described by one major mathematician as "…the most important mathematical discovery that has been made in our century…." After submitting his paper, Abel returned to Norway, ill with tuberculosis and in heavy debt. While eking out a meager living as a tutor, he continued to produce great work and his fame spread. Soon great efforts were being made to secure a suitable mathematical position for him. Fearing that his great work had been lost by the Academy, he mailed a proof of the main results to Crelle in January of 1829. In April he suffered a violent hemorrhage and died. Two days later Crelle wrote to inform him that an appointment had been secured for him in Berlin and his days of poverty were over! Abel's great paper was finally published by the Academy twelve years after his death.

[*Image: http://en.wikipedia.org/wiki/File:Niels_Henrik_Abel2.jpg*]

Solving for x_2 we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

The next approximation can be obtained more easily. If we view x_2 as the starting approximation and x_3 the new approximation, we can simply apply (2) with x_2 in place of x_1 and x_3 in place of x_2. This yields

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

provided $f'(x_2) \neq 0$. In general, if x_n is the nth approximation, then it is evident from the pattern in (2) and (3) that the improved approximation x_{n+1} is given by

Newton's Method

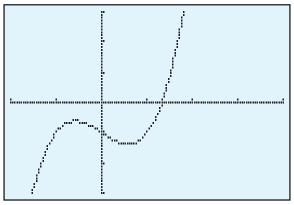
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots \tag{4}$$

▶ Example 1 Use Newton's Method to approximate the real solutions of

$$x^3 - x - 1 = 0$$

Solution. Let $f(x) = x^3 - x - 1$, so $f'(x) = 3x^2 - 1$ and (4) becomes

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} \tag{5}$$



[−2, 4] × [−3, 3]
xScl = 1, yScl = 1

$y = x^3 - x - 1$

▲ **Figure 3.7.2**

From the graph of f in Figure 3.7.2, we see that the given equation has only one real solution. This solution lies between 1 and 2 because $f(1) = -1 < 0$ and $f(2) = 5 > 0$. We will use $x_1 = 1.5$ as our first approximation ($x_1 = 1$ or $x_1 = 2$ would also be reasonable choices).

Letting $n = 1$ in (5) and substituting $x_1 = 1.5$ yields

$$x_2 = 1.5 - \frac{(1.5)^3 - 1.5 - 1}{3(1.5)^2 - 1} \approx 1.34782609 \tag{6}$$

(We used a calculator that displays nine digits.) Next, we let $n = 2$ in (5) and substitute x_2 to obtain

$$x_3 = x_2 - \frac{x_2^3 - x_2 - 1}{3x_2^2 - 1} \approx 1.32520040 \tag{7}$$

If we continue this process until two identical approximations are generated in succession, we obtain

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.34782609 \\ x_3 &\approx 1.32520040 \\ x_4 &\approx 1.32471817 \\ x_5 &\approx 1.32471796 \\ x_6 &\approx 1.32471796 \end{aligned}$$

At this stage there is no need to continue further because we have reached the display accuracy limit of our calculator, and all subsequent approximations that the calculator generates will likely be the same. Thus, the solution is approximately $x \approx 1.32471796$. ◀

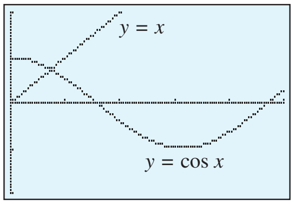
TECHNOLOGY MASTERY

Many calculators and computer programs calculate internally with more digits than they display. Where possible, you should use stored calculated values rather than values displayed from earlier calculations. Thus, in Example 1 the value of x_2 used in (7) should be the stored value, not the value in (6).

▶ Example 2 It is evident from Figure 3.7.3 that if x is in radians, then the equation

$$\cos x = x$$

has a solution between 0 and 1. Use Newton's Method to approximate it.



[0, 5] × [−2, 2]
xScl = 1, yScl = 1

▲ **Figure 3.7.3**

Solution. Rewrite the equation as

$$x - \cos x = 0$$

and apply (4) with $f(x) = x - \cos x$. Since $f'(x) = 1 + \sin x$, (4) becomes

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n} \tag{8}$$

From Figure 3.7.3, the solution seems closer to $x = 1$ than $x = 0$, so we will use $x_1 = 1$ (radian) as our initial approximation. Letting $n = 1$ in (8) and substituting $x_1 = 1$ yields

$$x_2 = 1 - \frac{1 - \cos 1}{1 + \sin 1} \approx 0.750363868$$

Next, letting $n = 2$ in (8) and substituting this value of x_2 yields

$$x_3 = x_2 - \frac{x_2 - \cos x_2}{1 + \sin x_2} \approx 0.739112891$$

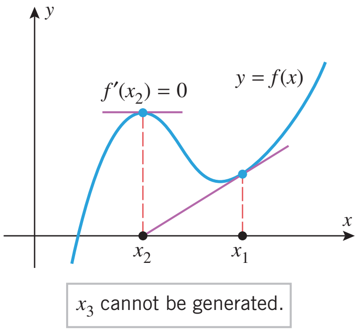
If we continue this process until two identical approximations are generated in succession, we obtain

$$\begin{aligned} x_1 &= 1 \\ x_2 &\approx 0.750363868 \\ x_3 &\approx 0.739112891 \\ x_4 &\approx 0.739085133 \\ x_5 &\approx 0.739085133 \end{aligned}$$

Thus, to the accuracy limit of our calculator, the solution of the equation $\cos x = x$ is $x \approx 0.739085133$. ◀

SOME DIFFICULTIES WITH NEWTON'S METHOD

When Newton's Method works, the approximations usually converge toward the solution with dramatic speed. However, there are situations in which the method fails. For example, if $f'(x_n) = 0$ for some n, then (4) involves a division by zero, making it impossible to generate x_{n+1}. However, this is to be expected because the tangent line to $y = f(x)$ is parallel to the x-axis where $f'(x_n) = 0$, and hence this tangent line does not cross the x-axis to generate the next approximation (Figure 3.7.4).



x_3 cannot be generated.

▲ **Figure 3.7.4**

Newton's Method can fail for other reasons as well; sometimes it may overlook the root you are trying to find and converge to a different root, and sometimes it may fail to converge altogether. For example, consider the equation

$$x^{1/3} = 0$$

which has $x = 0$ as its only solution, and try to approximate this solution by Newton's Method with a starting value of $x_0 = 1$. Letting $f(x) = x^{1/3}$, Formula (4) becomes

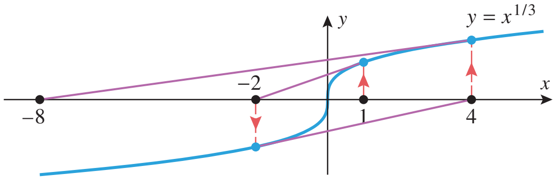
$$x_{n+1} = x_n - \frac{(x_n)^{1/3}}{\frac{1}{3}(x_n)^{-2/3}} = x_n - 3x_n = -2x_n$$

Beginning with $x_1 = 1$, the successive values generated by this formula are

$$x_1 = 1, \quad x_2 = -2, \quad x_3 = 4, \quad x_4 = -8, \ldots$$

which obviously do not converge to $x = 0$. Figure 3.7.5 illustrates what is happening geometrically in this situation.

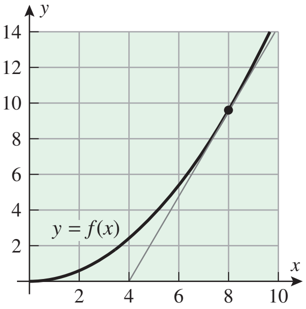
To learn more about the conditions under which Newton's Method converges and for a discussion of error questions, you should consult a book on numerical analysis. For a more in-depth discussion of Newton's Method and its relationship to contemporary studies of chaos and fractals, you may want to read the article, "Newton's Method and Fractal



▶ **Figure 3.7.5**

Patterns," by Philip Straffin, which appears in *Applications of Calculus*, MAA Notes, Vol. 3, No. 29, 1993, published by the Mathematical Association of America.

✔QUICK CHECK EXERCISES 3.7 *(See page 252 for answers.)*



◀ **Figure Ex-1**

1. Use the accompanying graph to estimate x_2 and x_3 if Newton's Method is applied to the equation $y = f(x)$ with $x_1 = 8$.
2. Suppose that $f(1) = 2$ and $f'(1) = 4$. If Newton's Method is applied to $y = f(x)$ with $x_1 = 1$, then $x_2 =$ ________.
3. Suppose we are given that $f(0) = 3$ and that $x_2 = 3$ when Newton's Method is applied to $y = f(x)$ with $x_1 = 0$. Then $f'(0) =$ ________.
4. If Newton's Method is applied to $y = x^5 - 2$ with $x_1 = 1$, then $x_2 =$ ________.

EXERCISE SET 3.7 Graphing Utility

In this exercise set, express your answers with as many decimal digits as your calculating utility can display, but use the procedure in the Technology Mastery on p. 248.

1. Approximate $\sqrt{2}$ by applying Newton's Method to the equation $x^2 - 2 = 0$.
2. Approximate $\sqrt{5}$ by applying Newton's Method to the equation $x^2 - 5 = 0$.
3. Approximate $\sqrt[3]{6}$ by applying Newton's Method to the equation $x^3 - 6 = 0$.
4. To what equation would you apply Newton's Method to approximate the nth root of a?

5–8 The given equation has one real solution. Approximate it by Newton's Method.

5. $x^3 - 2x - 2 = 0$
6. $x^3 + x - 1 = 0$
7. $x^5 + x^4 - 5 = 0$
8. $x^5 - 3x + 3 = 0$

9–14 Use a graphing utility to determine how many solutions the equation has, and then use Newton's Method to approximate the solution that satisfies the stated condition.

9. $x^4 + x^2 - 4 = 0;\ x < 0$
10. $x^5 - 5x^3 - 2 = 0;\ x > 0$
11. $2\cos x = x;\ x > 0$
12. $\sin x = x^2;\ x > 0$
13. $x - \tan x = 0;\ \pi/2 < x < 3\pi/2$
14. $1 + x^2 \sin x = 0;\ \pi/2 < x < 3\pi/2$

15–20 Use a graphing utility to determine the number of times the curves intersect; and then apply Newton's Method, where needed, to approximate the x-coordinates of all intersections.

15. $y = x^3$ and $y = 1 - x$
16. $y = \sin x$ and $y = x^3 - 2x^2 + 1$
17. $y = x^2$ and $y = \sqrt{2x + 1}$
18. $y = \frac{1}{8}x^3 - 1$ and $y = \cos x - 2$

19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. Newton's Method uses the tangent line to $y = f(x)$ at $x = x_n$ to compute x_{n+1}.
20. Newton's Method is a process to find exact solutions to $f(x) = 0$.
21. If $f(x) = 0$ has a root, then Newton's Method starting at $x = x_1$ will approximate the root nearest x_1.
22. Newton's Method can be used to appoximate a point of intersection of two curves.

23. The ***mechanic's rule*** for approximating square roots states that $\sqrt{a} \approx x_{n+1}$, where

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3, \ldots$$

and x_1 is any positive approximation to $\sqrt{a}$.

(a) Apply Newton's Method to

$$f(x) = x^2 - a$$

to derive the mechanic's rule.

(b) Use the mechanic's rule to approximate $\sqrt{10}$.

24. Many calculators compute reciprocals using the approximation $1/a \approx x_{n+1}$, where

$$x_{n+1} = x_n(2 - ax_n), \quad n = 1, 2, 3, \ldots$$

and x_1 is an initial approximation to $1/a$. This formula makes it possible to perform divisions using multiplications and subtractions, which is a faster procedure than dividing directly.

(a) Apply Newton's Method to

$$f(x) = \frac{1}{x} - a$$

to derive this approximation.

(b) Use the formula to approximate $\frac{1}{17}$.

25. Use Newton's Method to approximate the absolute minimum of $f(x) = \frac{1}{4}x^4 + x^2 - 5x$.

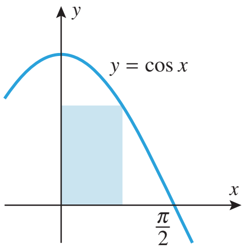
26. Use Newton's Method to approximate the absolute maximum of $f(x) = x \sin x$ on the interval $[0, \pi]$.

27. Use Newton's Method to approximate the coordinates of the the point on the parabola $y = x^2$ that is closest to the point $(1, 0)$.

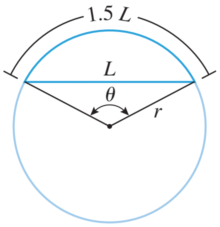
28. Use Newton's Method to approximate the dimensions of the rectangle of largest area that can be inscribed under the curve $y = \cos x$ for $0 \le x \le \pi/2$ (Figure Ex-28).

29. (a) Show that on a circle of radius r, the central angle θ that subtends an arc whose length is 1.5 times the length L of its chord satisfies the equation $\theta = 3\sin(\theta/2)$ (Figure Ex-29).

(b) Use Newton's Method to approximate θ.

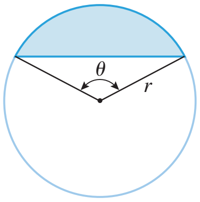


▲ **Figure Ex-28**



▲ **Figure Ex-29**

30. A ***segment*** of a circle is the region enclosed by an arc and its chord (Figure Ex-34). If r is the radius of the circle and θ the angle subtended at the center of the circle, then it can be shown that the area A of the segment is $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. Find the value of θ for which the area of the segment is one-fourth the area of the circle. Give θ to the nearest degree.



◀ **Figure Ex-30**

31–32 Use Newton's Method to approximate all real values of y satisfying the given equation for the indicated value of x.

31. $xy^4 + x^3y = 1;\ x = 1$ **32.** $xy - \cos\left(\frac{1}{2}xy\right) = 0;\ x = 2$

33. An ***annuity*** is a sequence of equal payments that are paid or received at regular time intervals. For example, you may want to deposit equal amounts at the end of each year into an interest-bearing account for the purpose of accumulating a lump sum at some future time. If, at the end of each year, interest of $i \times 100\%$ on the account balance for that year is added to the account, then the account is said to pay $i \times 100\%$ interest, ***compounded annually***. It can be shown that if payments of Q dollars are deposited at the end of each year into an account that pays $i \times 100\%$ compounded annually, then at the time when the nth payment and the accrued interest for the past year are deposited, the amount $S(n)$ in the account is given by the formula

$$S(n) = \frac{Q}{i}[(1 + i)^n - 1]$$

Suppose that you can invest \$5000 in an interest-bearing account at the end of each year, and your objective is to have \$250,000 on the 25th payment. Approximately what annual compound interest rate must the account pay for you to achieve your goal? [*Hint:* Show that the interest rate i satisfies the equation $50i = (1 + i)^{25} - 1$, and solve it using Newton's Method.]

FOCUS ON CONCEPTS

34. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{x}{x^2 + 1}$$

and use it to explain what happens if you apply Newton's Method with a starting value of $x_1 = 2$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

(b) Use the graph generated in part (a) to explain what happens if you apply Newton's Method with a starting value of $x_1 = 0.5$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

35. (a) Apply Newton's Method to $f(x) = x^2 + 1$ with a starting value of $x_1 = 0.5$, and determine if the values of $x_2, \ldots, x_{10}$ appear to converge.

(b) Explain what is happening.

36. In each part, explain what happens if you apply Newton's Method to a function f when the given condition is satisfied for some value of n.

(a) $f(x_n) = 0$ (b) $x_{n+1} = x_n$

(c) $x_{n+2} = x_n \neq x_{n+1}$

37. **Writing** Compare Newton's Method and the Intermediate-Value Theorem (1.5.8; see Example 6 in Section 1.5) as methods to locate solutions to $f(x) = 0$.

38. **Writing** Newton's Method uses a local linear approximation to $y = f(x)$ at $x = x_n$ to find an "improved" approximation x_{n+1} to a zero of f. Your friend proposes a process that uses a local quadratic approximation to $y = f(x)$ at $x = x_n$ (that is, matching values for the function and its first two derivatives) to obtain x_{n+1}. Discuss the pros and cons of this proposal. Support your statements with some examples.

✔QUICK CHECK ANSWERS 3.7

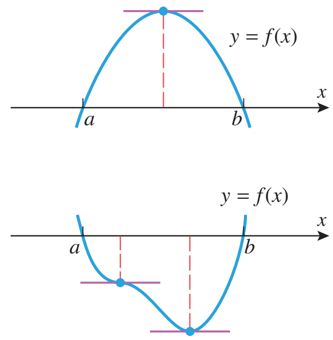
1. $x_2 \approx 4$, $x_3 \approx 2$ **2.** $\frac{1}{2}$ **3.** -1 **4.** 1.2

3.8 ROLLE'S THEOREM; MEAN-VALUE THEOREM

In this section we will discuss a result called the Mean-Value Theorem. This theorem has so many important consequences that it is regarded as one of the major principles in calculus.

ROLLE'S THEOREM

We will begin with a special case of the Mean-Value Theorem, called Rolle's Theorem, in honor of the mathematician Michel Rolle. This theorem states the geometrically obvious fact that if the graph of a differentiable function intersects the x-axis at two places, a and b, then somewhere between a and b there must be at least one place where the tangent line is horizontal (Figure 3.8.1). The precise statement of the theorem is as follows.



▲ **Figure 3.8.1**

3.8.1 THEOREM (*Rolle's Theorem*) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If*

$$f(a) = 0 \quad \textit{and} \quad f(b) = 0$$

then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

Michel Rolle (1652–1719) French mathematician. Rolle, the son of a shopkeeper, received only an elementary education. He married early and as a young man struggled hard to support his family on the meager wages of a transcriber for notaries and attorneys. In spite of his financial problems and minimal education, Rolle studied algebra and Diophantine analysis (a branch of number theory) on his own. Rolle's fortune changed dramatically in 1682 when he published an elegant solution of a difficult, unsolved problem in Diophantine analysis. The public recognition of his achievement led to a patronage under minister Louvois, a job as an elementary mathematics teacher, and eventually to a short-term administrative post in the Ministry of War. In 1685 he joined the Académie des Sciences in a low-level position for which he received no regular salary until 1699. He stayed at the Académie until he died of apoplexy in 1719.

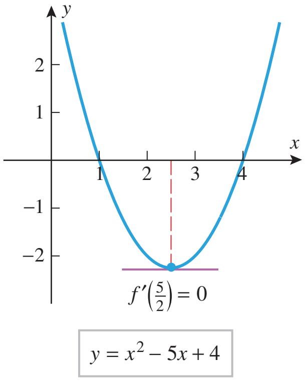
While Rolle's forte was always Diophantine analysis, his most important work was a book on the algebra of equations, called *Traité d'algèbre*, published in 1690. In that book Rolle firmly established the notation $\sqrt[n]{a}$ [earlier written as $\sqrt{\textcircled{n}\ a}$] for the nth root of a, and proved a polynomial version of the theorem that today bears his name. (Rolle's Theorem was named by Giusto Bellavitis in 1846.) Ironically, Rolle was one of the most vocal early antagonists of calculus. He strove intently to demonstrate that it gave erroneous results and was based on unsound reasoning. He quarreled so vigorously on the subject that the Académie des Sciences was forced to intervene on several occasions. Among his several achievements, Rolle helped advance the currently accepted size order for negative numbers. Descartes, for example, viewed -2 as smaller than -5. Rolle preceded most of his contemporaries by adopting the current convention in 1691.

PROOF We will divide the proof into three cases: the case where $f(x) = 0$ for all x in (a, b), the case where $f(x) > 0$ at some point in (a, b), and the case where $f(x) < 0$ at some point in (a, b).

CASE I If $f(x) = 0$ for all x in (a, b), then $f'(c) = 0$ at every point c in (a, b) because f is a constant function on that interval.

CASE 2 Assume that $f(x) > 0$ at some point in (a, b). Since f is continuous on $[a, b]$, it follows from the Extreme-Value Theorem (3.4.2) that f has an absolute maximum on $[a, b]$. The absolute maximum value cannot occur at an endpoint of $[a, b]$ because we have assumed that $f(a) = f(b) = 0$, and that $f(x) > 0$ at some point in (a, b). Thus, the absolute maximum must occur at some point c in (a, b). It follows from Theorem 3.4.3 that c is a critical point of f, and since f is differentiable on (a, b), this critical point must be a stationary point; that is, $f'(c) = 0$.

CASE 3 Assume that $f(x) < 0$ at some point in (a, b). The proof of this case is similar to Case 2 and will be omitted. ■



▲ **Figure 3.8.2**

▶ **Example 1** Find the two x-intercepts of the function $f(x) = x^2 - 5x + 4$ and confirm that $f'(c) = 0$ at some point c between those intercepts.

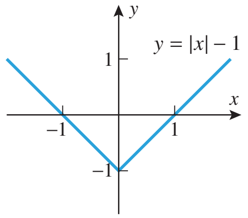
Solution. The function f can be factored as

$$x^2 - 5x + 4 = (x - 1)(x - 4)$$

so the x-intercepts are $x = 1$ and $x = 4$. Since the polynomial f is continuous and differentiable everywhere, the hypotheses of Rolle's Theorem are satisfied on the interval $[1, 4]$. Thus, we are guaranteed the existence of at least one point c in the interval $(1, 4)$ such that $f'(c) = 0$. Differentiating f yields

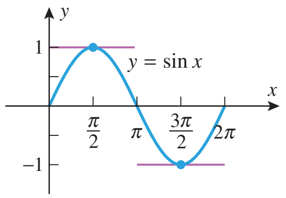
$$f'(x) = 2x - 5$$

Solving the equation $f'(x) = 0$ yields $x = \frac{5}{2}$, so $c = \frac{5}{2}$ is a point in the interval $(1, 4)$ at which $f'(c) = 0$ (Figure 3.8.2). ◀



▲ **Figure 3.8.3**

▶ **Example 2** The differentiability requirement in Rolle's Theorem is critical. If f fails to be differentiable at even one place in the interval (a, b), then the conclusion of the theorem may not hold. For example, the function $f(x) = |x| - 1$ graphed in Figure 3.8.3 has roots at $x = -1$ and $x = 1$, yet there is no horizontal tangent to the graph of f over the interval $(-1, 1)$. ◀



▲ **Figure 3.8.4**

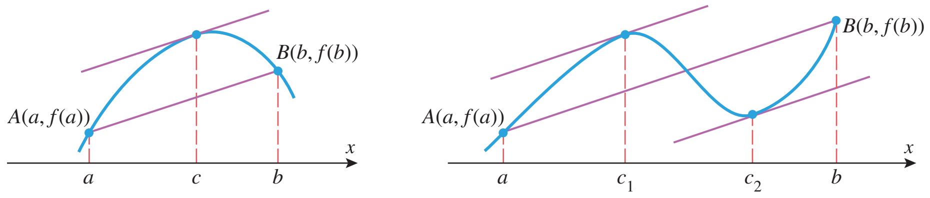
▶ **Example 3** If f satisfies the conditions of Rolle's Theorem on $[a, b]$, then the theorem guarantees the existence of *at least* one point c in (a, b) at which $f'(c) = 0$. There may, however, be more than one such c. For example, the function $f(x) = \sin x$ is continuous and differentiable everywhere, so the hypotheses of Rolle's Theorem are satisfied on the interval $[0, 2\pi]$ whose endpoints are roots of f. As indicated in Figure 3.8.4, there are two points in the interval $[0, 2\pi]$ at which the graph of f has a horizontal tangent, $c_1 = \pi/2$ and $c_2 = 3\pi/2$. ◀

In Examples 1 and 3 we were able to find exact values of c because the equation $f'(x) = 0$ was easy to solve. However, in the applications of Rolle's Theorem it is usually the *existence of* c that is important and not its actual value.

THE MEAN-VALUE THEOREM

Rolle's Theorem is a special case of a more general result, called the ***Mean-Value Theorem***. Geometrically, this theorem states that between any two points $A(a, f(a))$ and $B(b, f(b))$

on the graph of a differentiable function f, there is at least one place where the tangent line to the graph is parallel to the secant line joining A and B (Figure 3.8.5).



(a) (b)

▲ **Figure 3.8.5**

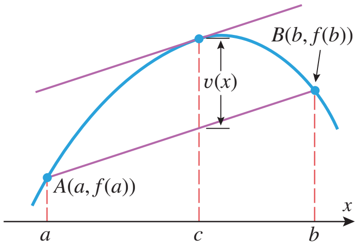
Note that the slope of the secant line joining $A(a, f(a))$ and $B(b, f(b))$ is

$$\frac{f(b) - f(a)}{b - a}$$

and that the slope of the tangent line at c in Figure 3.8.5a is $f'(c)$. Similarly, in Figure 3.8.5b the slopes of the tangent lines at c_1 and c_2 are $f'(c_1)$ and $f'(c_2)$, respectively. Since nonvertical parallel lines have the same slope, the Mean-Value Theorem can be stated precisely as follows.

3.8.2 THEOREM (*Mean-Value Theorem*) *Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$



The tangent line is parallel to the secant line where the vertical distance $v(x)$ between the secant line and the graph of f is maximum.

▲ **Figure 3.8.6**

MOTIVATION FOR THE PROOF OF THEOREM 3.8.2 Figure 3.8.6 suggests that (1) will hold (i.e., the tangent line will be parallel to the secant line) at a point c where the vertical distance between the curve and the secant line is maximum. Thus, to prove the Mean-Value Theorem it is natural to begin by looking for a formula for the vertical distance $v(x)$ between the curve $y = f(x)$ and the secant line joining $(a, f(a))$ and $(b, f(b))$.

PROOF OF THEOREM 3.8.2 Since the two-point form of the equation of the secant line joining $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or, equivalently,

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

the difference $v(x)$ between the height of the graph of f and the height of the secant line is

$$v(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right] \tag{2}$$

Since $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), so is $v(x)$. Moreover,

$$v(a) = 0 \quad \text{and} \quad v(b) = 0$$

so that $v(x)$ satisfies the hypotheses of Rolle's Theorem on the interval $[a, b]$. Thus, there is a point c in (a, b) such that $v'(c) = 0$. But from Equation (2)

$$v'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

so

$$v'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

Since $v'(c) = 0$, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \blacksquare$$

▶ **Example 4** Show that the function $f(x) = \frac{1}{4}x^3 + 1$ satisfies the hypotheses of the Mean-Value Theorem over the interval $[0, 2]$, and find all values of c in the interval $(0, 2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

Solution. The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[0, 2]$ and differentiable on $(0, 2)$, so the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 2$. But

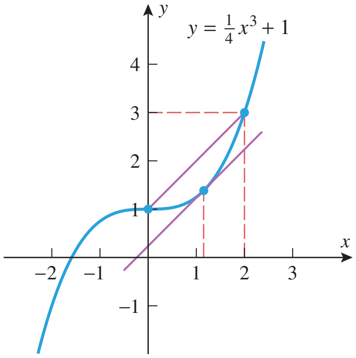
$$f(a) = f(0) = 1, \qquad f(b) = f(2) = 3$$

$$f'(x) = \frac{3x^2}{4}, \qquad f'(c) = \frac{3c^2}{4}$$

so in this case Equation (1) becomes

$$\frac{3c^2}{4} = \frac{3-1}{2-0} \quad \text{or} \quad 3c^2 = 4$$

which has the two solutions $c = \pm 2/\sqrt{3} \approx \pm 1.15$. However, only the positive solution lies in the interval $(0, 2)$; this value of c is consistent with Figure 3.8.7. ◀



▲ Figure 3.8.7

VELOCITY INTERPRETATION OF THE MEAN-VALUE THEOREM

There is a nice interpretation of the Mean-Value Theorem in the situation where $x = f(t)$ is the position versus time curve for a car moving along a straight road. In this case, the right side of (1) is the average velocity of the car over the time interval from $a \le t \le b$, and the left side is the instantaneous velocity at time $t = c$. Thus, the Mean-Value Theorem implies that at least once during the time interval the instantaneous velocity must equal the average velocity. This agrees with our real-world experience—if the average velocity for a trip is 40 mi/h, then sometime during the trip the speedometer has to read 40 mi/h.

▶ **Example 5** You are driving on a straight highway on which the speed limit is 55 mi/h. At 8:05 A.M. a police car clocks your velocity at 50 mi/h and at 8:10 A.M. a second police car posted 5 mi down the road clocks your velocity at 55 mi/h. Explain why the police have a right to charge you with a speeding violation.

Solution. You traveled 5 mi in 5 min $\left(= \frac{1}{12} \text{ h}\right)$, so your average velocity was 60 mi/h. Therefore, the Mean-Value Theorem guarantees the police that your instantaneous velocity was 60 mi/h at least once over the 5 mi section of highway. ◀

CONSEQUENCES OF THE MEAN-VALUE THEOREM

We stated at the beginning of this section that the Mean-Value Theorem is the starting point for many important results in calculus. As an example of this, we will use it to prove Theorem 3.1.2, which was one of our fundamental tools for analyzing graphs of functions.

3.1.2 THEOREM (Revisited) *Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).*

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

PROOF (a) Suppose that x_1 and x_2 are points in $[a, b]$ such that $x_1 < x_2$. We must show that $f(x_1) < f(x_2)$. Because the hypotheses of the Mean-Value Theorem are satisfied on the entire interval $[a, b]$, they are satisfied on the subinterval $[x_1, x_2]$. Thus, there is some point c in the open interval (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

or, equivalently,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \tag{3}$$

Since c is in the open interval (x_1, x_2), it follows that $a < c < b$; thus, $f'(c) > 0$. However, $x_2 - x_1 > 0$ since we assumed that $x_1 < x_2$. It follows from (3) that $f(x_2) - f(x_1) > 0$ or, equivalently, $f(x_1) < f(x_2)$, which is what we were to prove. The proofs of parts (b) and (c) are similar and are left as exercises. ■

THE CONSTANT DIFFERENCE THEOREM

We know from our earliest study of derivatives that the derivative of a constant is zero. Part (c) of Theorem 3.1.2 is the converse of that result; that is, a function whose derivative is zero on an interval must be constant on that interval. If we apply this to the difference of two functions, we obtain the following useful theorem.

3.8.3 THEOREM (Constant Difference Theorem) *If f and g are differentiable on an interval, and if $f'(x) = g'(x)$ for all x in that interval, then $f - g$ is constant on the interval; that is, there is a constant k such that $f(x) - g(x) = k$ or, equivalently,*

$$f(x) = g(x) + k$$

for all x in the interval.

PROOF Let x_1 and x_2 be any points in the interval such that $x_1 < x_2$. Since the functions f and g are differentiable on the interval, they are continuous on the interval. Since $[x_1, x_2]$ is a subinterval, it follows that f and g are continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Moreover, it follows from the basic properties of derivatives and continuity that the same is true of the function

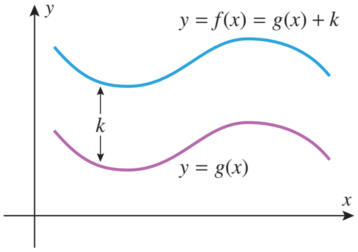
$$F(x) = f(x) - g(x)$$

Since

$$F'(x) = f'(x) - g'(x) = 0$$

it follows from part (c) of Theorem 3.1.2 that $F(x) = f(x) - g(x)$ is constant on the interval $[x_1, x_2]$. This means that $f(x) - g(x)$ has the same value at any two points x_1 and x_2 in the interval, and this implies that $f - g$ is constant on the interval. ■

Geometrically, the Constant Difference Theorem tells us that if f and g have the same derivative on an interval, then the graphs of f and g are vertical translations of each other over that interval (Figure 3.8.8).



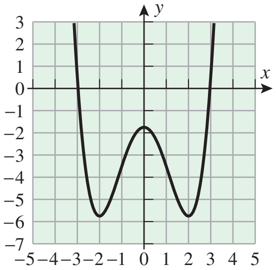
If $f'(x) = g'(x)$ on an interval, then the graphs of f and g are vertical translations of each other.

▲ **Figure 3.8.8**

✔QUICK CHECK EXERCISES 3.8 *(See page 259 for answers.)*

1. Let $f(x) = x^2 - x$.
  (a) An interval on which f satisfies the hypotheses of Rolle's Theorem is ________.
  (b) Find all values of c that satisfy the conclusion of Rolle's Theorem for the function f on the interval in part (a).

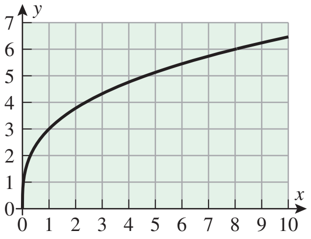
2. Use the accompanying graph of f to find an interval $[a, b]$ on which Rolle's Theorem applies, and find all values of c in that interval that satisfy the conclusion of the theorem.



◀ **Figure Ex-2**

3. Let $f(x) = x^2 - x$.
  (a) Find a point b such that the slope of the secant line through $(0, 0)$ and $(b, f(b))$ is 1.
  (b) Find all values of c that satisfy the conclusion of the Mean-Value Theorem for the function f on the interval $[0, b]$, where b is the point found in part (a).

4. Use the graph of f in the accompanying figure to estimate all values of c that satisfy the conclusion of the Mean-Value Theorem on the interval
  (a) $[0, 8]$ (b) $[0, 4]$.



◀ **Figure Ex-4**

5. Find a function f such that the graph of f contains the point $(1, 5)$ and such that for every value of x_0 the tangent line to the graph of f at x_0 is parallel to the tangent line to the graph of $y = x^2$ at x_0.

EXERCISE SET 3.8 Graphing Utility

1–4 Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem. ■

1. $f(x) = x^2 - 8x + 15;\ [3, 5]$

2. $f(x) = \frac{1}{2}x - \sqrt{x};\ [0, 4]$

3. $f(x) = \cos x;\ [\pi/2, 3\pi/2]$

4. $f(x) = (x^2 - 1)/(x - 2);\ [-1, 1]$

5–8 Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem. ■

5. $f(x) = x^2 - x;\ [-3, 5]$

6. $f(x) = x^3 + x - 4;\ [-1, 2]$

7. $f(x) = \sqrt{25 - x^2};\ [-5, 3]$

8. $f(x) = x - \dfrac{1}{x};\ [3, 4]$

9. (a) Find an interval $[a, b]$ on which

$$f(x) = x^4 + x^3 - x^2 + x - 2$$

satisfies the hypotheses of Rolle's Theorem.
  (b) Generate the graph of $f'(x)$, and use it to make rough estimates of all values of c in the interval obtained in part (a) that satisfy the conclusion of Rolle's Theorem.
  (c) Use Newton's Method to improve on the rough estimates obtained in part (b).

10. Let $f(x) = x^3 - 4x$.
  (a) Find the equation of the secant line through the points $(-2, f(-2))$ and $(1, f(1))$.
  (b) Show that there is only one point c in the interval $(-2, 1)$ that satisfies the conclusion of the Mean-Value Theorem for the secant line in part (a).
  (c) Find the equation of the tangent line to the graph of f at the point $(c, f(c))$.
  (d) Use a graphing utility to generate the secant line in part (a) and the tangent line in part (c) in the same coordinate system, and confirm visually that the two lines seem parallel.

11–14 True–False Determine whether the statement is true or false. Explain your answer. ■

11. Rolle's Theorem says that if f is a continuous function on $[a, b]$ and $f(a) = f(b)$, then there is a point between a and b at which the curve $y = f(x)$ has a horizontal tangent line.

12. If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is a point between a and b at which the instantaneous rate of change of f matches the average rate of change of f over $[a, b]$.

13. The Constant Difference Theorem says that if two functions have derivatives that differ by a constant on an interval, then the functions are equal on the interval.

14. One application of the Mean-Value Theorem is to prove that a function with positive derivative on an interval must be increasing on that interval.

FOCUS ON CONCEPTS

15. Let $f(x) = \tan x$.
(a) Show that there is no point c in the interval $(0, \pi)$ such that $f'(c) = 0$, even though $f(0) = f(\pi) = 0$.
(b) Explain why the result in part (a) does not contradict Rolle's Theorem.

16. Let $f(x) = x^{2/3}$, $a = -1$, and $b = 8$.
(a) Show that there is no point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(b) Explain why the result in part (a) does not contradict the Mean-Value Theorem.

17. (a) Show that if f is differentiable on $(-\infty, +\infty)$, and if $y = f(x)$ and $y = f'(x)$ are graphed in the same coordinate system, then between any two x-intercepts of f there is at least one x-intercept of f'.
(b) Give some examples that illustrate this.

18. Review Formulas (8) and (9) in Section 2.1 and use the Mean-Value Theorem to show that if f is differentiable on $(-\infty, +\infty)$, then for any interval $[x_0, x_1]$ there is at least one point in (x_0, x_1) where the instantaneous rate of change of y with respect to x is equal to the average rate of change over the interval.

19–21 Use the result of Exercise 18 in these exercises. ■

19. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer reads exactly 48 mi/h at least once during the trip.

20. At 11 A.M. on a certain morning the outside temperature was 76°F. At 11 P.M. that evening it had dropped to 52°F.
(a) Show that at some instant during this period the temperature was decreasing at the rate of 2°F/h.
(b) Suppose that you know the temperature reached a high of 88°F sometime between 11 A.M. and 11 P.M. Show that at some instant during this period the temperature was decreasing at a rate greater than 3°F/h.

21. Suppose that two runners in a 100 m dash finish in a tie. Show that they had the same velocity at least once during the race.

22. Use the fact that

$$\frac{d}{dx}(3x^4 + x^2 - 4x) = 12x^3 + 2x - 4$$

to show that the equation $12x^3 + 2x - 4 = 0$ has at least one solution in the interval $(0, 1)$.

23. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in the interval $(-\infty, +\infty)$, and if f and g have the same value at some point x_0, then $f(x) = g(x)$ for all x in $(-\infty, +\infty)$.
(b) Use the result in part (a) to confirm the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

24. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in $(-\infty, +\infty)$, and if $f(x_0) - g(x_0) = c$ at some point x_0, then

$$f(x) - g(x) = c$$

for all x in $(-\infty, +\infty)$.
(b) Use the result in part (a) to show that the function

$$h(x) = (x - 1)^3 - (x^2 + 3)(x - 3)$$

is constant for all x in $(-\infty, +\infty)$, and find the constant.
(c) Check the result in part (b) by multiplying out and simplifying the formula for $h(x)$.

FOCUS ON CONCEPTS

25. (a) Use the Mean-Value Theorem to show that if f is differentiable on an interval, and if $|f'(x)| \leq M$ for all values of x in the interval, then

$$|f(x) - f(y)| \leq M|x - y|$$

for all values of x and y in the interval.
(b) Use the result in part (a) to show that

$$|\sin x - \sin y| \leq |x - y|$$

for all real values of x and y.

26. (a) Use the Mean-Value Theorem to show that if f is differentiable on an open interval, and if $|f'(x)| \geq M$ for all values of x in the interval, then

$$|f(x) - f(y)| \geq M|x - y|$$

for all values of x and y in the interval.
(b) Use the result in part (a) to show that

$$|\tan x - \tan y| \geq |x - y|$$

for all values of x and y in the interval $(-\pi/2, \pi/2)$.
(c) Use the result in part (b) to show that

$$|\tan x + \tan y| \geq |x + y|$$

for all values of x and y in the interval $(-\pi/2, \pi/2)$.

27. (a) Use the Mean-Value Theorem to show that

$$\sqrt{y} - \sqrt{x} < \frac{y - x}{2\sqrt{x}}$$

if $0 < x < y$.
(b) Use the result in part (a) to show that if $0 < x < y$, then $\sqrt{xy} < \frac{1}{2}(x + y)$.

28. Show that if f is differentiable on an open interval and $f'(x) \neq 0$ on the interval, the equation $f(x) = 0$ can have at most one real root in the interval.

29. Use the result in Exercise 28 to show the following:
(a) The equation $x^3 + 4x - 1 = 0$ has exactly one real root.
(b) If $b^2 - 3ac < 0$ and if $a \neq 0$, then the equation

$$ax^3 + bx^2 + cx + d = 0$$

has exactly one real root.

30. Use the inequality $\sqrt{3} < 1.8$ to prove that

$$1.7 < \sqrt{3} < 1.75$$

[*Hint:* Let $f(x) = \sqrt{x}$, $a = 3$, and $b = 4$ in the Mean-Value Theorem.]

31. Use the Mean-Value Theorem to prove that

$$x - \frac{x^3}{6} < \sin x < x \quad (x > 0)$$

32. Show that if f and g are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = f(x)$$

for all x, then $f^2(x) - g^2(x)$ is a constant.

33. (a) Show that if f and g are functions for which

$$f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x)$$

for all x, then $f^2(x) + g^2(x)$ is a constant.

(b) Give an example of functions f and g with this property.

FOCUS ON CONCEPTS

34. Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Prove: If $f(a) = g(a)$ and $f(b) = g(b)$, then there is a point c in (a, b) such that $f'(c) = g'(c)$.

35. Illustrate the result in Exercise 36 by drawing an appropriate picture.

36. (a) Prove that if $f''(x) > 0$ for all x in (a, b), then $f'(x) = 0$ at most once in (a, b).

(b) Give a geometric interpretation of the result in (a).

37. (a) Prove part (*b*) of Theorem 3.1.2.

(b) Prove part (*c*) of Theorem 3.1.2.

38. Use the Mean-Value Theorem to prove the following result: Let f be continuous at x_0 and suppose that $\lim_{x \to x_0} f'(x)$ exists. Then f is differentiable at x_0, and

$$f'(x_0) = \lim_{x \to x_0} f'(x)$$

[*Hint:* The derivative $f'(x_0)$ is given by

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

provided this limit exists.]

FOCUS ON CONCEPTS

39. Let

$$f(x) = \begin{cases} 3x^2, & x \le 1 \\ ax + b, & x > 1 \end{cases}$$

Find the values of a and b so that f will be differentiable at $x = 1$.

40. (a) Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$$

Show that

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$$

but that $f'(0)$ does not exist.

(b) Let

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$$

Show that $f'(0)$ exists but $f''(0)$ does not.

41. Use the Mean-Value Theorem to prove the following result: The graph of a function f has a point of vertical tangency at $(x_0, f(x_0))$ if f is continuous at x_0 and $f'(x)$ approaches either $+\infty$ or $-\infty$ as $x \to x_0^+$ and as $x \to x_0^-$.

42. **Writing** Suppose that $p(x)$ is a nonconstant polynomial with zeros at $x = a$ and $x = b$. Explain how both the Extreme-Value Theorem (3.4.2) and Rolle's Theorem can be used to show that p has a critical point between a and b.

43. **Writing** Find and describe a physical situation that illustrates the Mean-Value Theorem.

✔ QUICK CHECK ANSWERS 3.8

1. (a) $[0, 1]$ (b) $c = \frac{1}{2}$ **2.** $[-3, 3]$; $c = -2, 0, 2$ **3.** (a) $b = 2$ (b) $c = 1$ **4.** (a) 1.5 (b) 0.8 **5.** $f(x) = x^2 + 4$

CHAPTER 3 REVIEW EXERCISES

☐ Graphing Utility 🄲 CAS

1. (a) If $x_1 < x_2$, what relationship must hold between $f(x_1)$ and $f(x_2)$ if f is increasing on an interval containing x_1 and x_2? Decreasing? Constant?

(b) What condition on f' ensures that f is increasing on an interval $[a, b]$? Decreasing? Constant?

2. (a) What condition on f' ensures that f is concave up on an open interval? Concave down?

(b) What condition on f'' ensures that f is concave up on an open interval? Concave down?

(c) In words, what is an inflection point of f?

3–8 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points. ■

3. $f(x) = x^2 - 5x + 6$

4. $f(x) = x^4 - 8x^2 + 16$

5. $f(x) = \dfrac{x^2}{x^2 + 2}$

6. $f(x) = \sqrt[3]{x + 2}$

7. $f(x) = x^{1/3}(x + 4)$

8. $f(x) = x^{4/3} - x^{1/3}$

9–12 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility. ■

9. $f(x) = \cos x;\ [0, 2\pi]$

10. $f(x) = \tan x;\ (-\pi/2, \pi/2)$

11. $f(x) = \sin x \cos x;\ [0, \pi]$

12. $f(x) = \cos^2 x - 2\sin x;\ [0, 2\pi]$

13. In each part, sketch a continuous curve $y = f(x)$ with the stated properties.

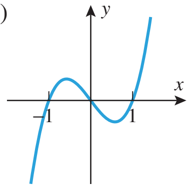
(a) $f(2) = 4,\ f'(2) = 1,\ f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$

(b) $f(2) = 4,\ f''(x) > 0$ for $x < 2,\ f''(x) < 0$ for $x > 2$, $\lim_{x\to 2^-} f'(x) = +\infty,\ \lim_{x\to 2^+} f'(x) = +\infty$

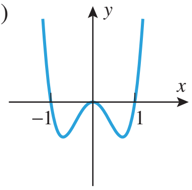
(c) $f(2) = 4,\ f''(x) < 0$ for $x \neq 2,\ \lim_{x\to 2^-} f'(x) = 1$, $\lim_{x\to 2^+} f'(x) = -1$

14. In parts (a)–(d), the graph of a polynomial with degree at most 6 is given. Find equations for polynomials that produce graphs with these shapes, and check your answers with a graphing utility.

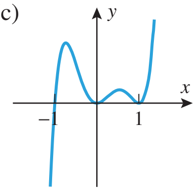
(a)



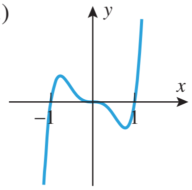
(b)



(c)



(d)



15. For a general quadratic polynomial

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

find conditions on a, b, and c to ensure that f is always increasing or always decreasing on $[0, +\infty)$.

16. For the general cubic polynomial

$$f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$

find conditions on a, b, c, and d to ensure that f is always increasing or always decreasing on $(-\infty, +\infty)$.

17. (a) Where on the graph of $y = f(x)$ would you expect y to be increasing or decreasing most rapidly with respect to x?

(b) In words, what is a relative extremum?

(c) State a procedure for determining where the relative extrema of f occur.

18. Determine whether the statement is true or false. If it is false, give an example for which the statement fails.

(a) If f has a relative maximum at x_0, then $f(x_0)$ is the largest value that $f(x)$ can have.

(b) If the largest value for f on the interval (a, b) is at x_0, then f has a relative maximum at x_0.

(c) A function f has a relative extremum at each of its critical points.

19. (a) According to the first derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?

(b) According to the second derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?

20–22 Locate the critical points and identify which critical points correspond to stationary points. ■

20. (a) $f(x) = x^3 + 3x^2 - 9x + 1$
(b) $f(x) = x^4 - 6x^2 - 3$

21. (a) $f(x) = \dfrac{x}{x^2 + 2}$ (b) $f(x) = \dfrac{x^2 - 3}{x^2 + 1}$

22. (a) $f(x) = x^{1/3}(x - 4)$ (b) $f(x) = x^{4/3} - 6x^{1/3}$

23. In each part, find all critical points, and use the first derivative test to classify them as relative maxima, relative minima, or neither.

(a) $f(x) = x^{1/3}(x - 7)^2$

(b) $f(x) = 2\sin x - \cos 2x,\quad 0 \leq x \leq 2\pi$

(c) $f(x) = 3x - (x - 1)^{3/2}$

24. In each part, find all critical points, and use the second derivative test (where possible) to classify them as relative maxima, relative minima, or neither.

(a) $f(x) = x^{-1/2} + \frac{1}{9}x^{1/2}$

(b) $f(x) = x^2 + 8/x$

(c) $f(x) = \sin^2 x - \cos x,\quad 0 \leq x \leq 2\pi$

25–32 Give a graph of the function f, and identify the limits as $x \to \pm\infty$, as well as locations of all relative extrema, inflection points, and asymptotes (as appropriate). ■

25. $f(x) = x^4 - 3x^3 + 3x^2 + 1$

26. $f(x) = x^5 - 4x^4 + 4x^3$

27. $f(x) = \tan(x^2 + 1)$ **28.** $f(x) = x - \cos x$

29. $f(x) = \dfrac{x^2}{x^2 + 2x + 5}$ **30.** $f(x) = \dfrac{25 - 9x^2}{x^3}$

31. $f(x) = \begin{cases} \frac{1}{2}x^2, & x \leq 0 \\ -x^2, & x > 0 \end{cases}$

32. $f(x) = (1 + x)^{2/3}(3 - x)^{1/3}$

33–38 Use any method to find the relative extrema of the function f. ■

33. $f(x) = x^3 + 5x - 2$ **34.** $f(x) = x^4 - 2x^2 + 7$

35. $f(x) = x^{4/5}$ **36.** $f(x) = 2x + x^{2/3}$

37. $f(x) = \dfrac{x^2}{x^2+1}$ **38.** $f(x) = \dfrac{x}{x+2}$

39–40 When using a graphing utility, important features of a graph may be missed if the viewing window is not chosen appropriately. This is illustrated in Exercises 39 and 40. ■

39. (a) Generate the graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{400}x$ over the interval $[-5, 5]$, and make a conjecture about the locations and nature of all critical points.
(b) Find the exact locations of all the critical points, and classify them as relative maxima, relative minima, or neither.
(c) Confirm the results in part (b) by graphing f over an appropriate interval.

40. (a) Generate the graph of

$$f(x) = \tfrac{1}{5}x^5 - \tfrac{7}{8}x^4 + \tfrac{1}{3}x^3 + \tfrac{7}{2}x^2 - 6x$$

over the interval $[-5, 5]$, and make a conjecture about the locations and nature of all critical points.
(b) Find the exact locations of all the critical points, and classify them as relative maxima, relative minima, or neither.
(c) Confirm the results in part (b) by graphing portions of f over appropriate intervals. [*Note:* It will not be possible to find a single window in which all of the critical points are discernible.]

41. (a) Use a graphing utility to generate the graphs of $y = x$ and $y = (x^3 - 8)/(x^2 + 1)$ together over the interval $[-5, 5]$, and make a conjecture about the relationship between the two graphs.
(b) Confirm your conjecture in part (a).

42. Use implicit differentiation to show that a function defined implicitly by $\sin x + \cos y = 2y$ has a critical point whenever $\cos x = 0$. Then use either the first or second derivative test to classify these critical points as relative maxima or minima.

43. Let

$$f(x) = \frac{2x^3 + x^2 - 15x + 7}{(2x-1)(3x^2 + x - 1)}$$

Graph $y = f(x)$, and find the equations of all horizontal and vertical asymptotes. Explain why there is no vertical asymptote at $x = \frac{1}{2}$, even though the denominator of f is zero at that point.

C **44.** Let

$$f(x) = \frac{x^5 - x^4 - 3x^3 + 2x + 4}{x^7 - 2x^6 - 3x^5 + 6x^4 + 4x - 8}$$

(a) Use a CAS to factor the numerator and denominator of f, and use the results to determine the locations of all vertical asymptotes.
(b) Confirm that your answer is consistent with the graph of f.

45. (a) What inequality must $f(x)$ satisfy for the function f to have an absolute maximum on an interval I at x_0?
(b) What inequality must $f(x)$ satisfy for f to have an absolute minimum on an interval I at x_0?
(c) What is the difference between an absolute extremum and a relative extremum?

46. According to the Extreme-Value Theorem, what conditions on a function f and a given interval guarantee that f will have both an absolute maximum and an absolute minimum on the interval?

47. In each part, determine whether the statement is true or false, and justify your answer.
(a) If f is differentiable on the open interval (a, b), and if f has an absolute extremum on that interval, then it must occur at a stationary point of f.
(b) If f is continuous on the open interval (a, b), and if f has an absolute extremum on that interval, then it must occur at a stationary point of f.

48–50 In each part, find the absolute minimum m and the absolute maximum M of f on the given interval (if they exist), and state where the absolute extrema occur. ■

48. (a) $f(x) = 1/x;\ [-2, -1]$
(b) $f(x) = x^3 - x^4;\ \left[-1, \frac{3}{2}\right]$
(c) $f(x) = x - \tan x;\ [-\pi/4, \pi/4]$

49. (a) $f(x) = x^2 - 3x - 1;\ (-\infty, +\infty)$
(b) $f(x) = x^3 - 3x - 2;\ (-\infty, +\infty)$
(c) $f(x) = -|x^2 - 2x|;\ [1, 3]$

50. (a) $f(x) = 2x^5 - 5x^4 + 7;\ (-1, 3)$
(b) $f(x) = (3 - x)/(2 - x);\ (0, 2)$
(c) $f(x) = 2x/(x^2 + 3);\ (0, 2]$
(d) $f(x) = x^2(x - 2)^{1/3};\ (0, 3]$

51. In each part, use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values.
(a) $f(x) = (x^2 - 1)^2;\ (-\infty, +\infty)$
(b) $f(x) = x/(x^2 + 1);\ [0, +\infty)$
(c) $f(x) = 2\sec x - \tan x;\ [0, \pi/4]$

52. Prove that $\tan x > x$ for all x in $(0, \pi/2)$.

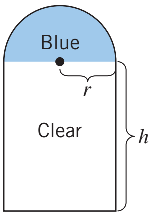
C **53.** Let

$$f(x) = \frac{x^3 + 2}{x^4 + 1}$$

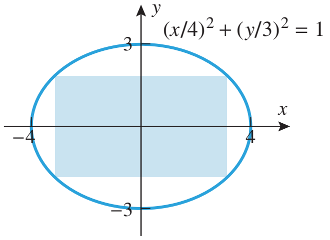
(a) Generate the graph of $y = f(x)$, and use the graph to make rough estimates of the coordinates of the absolute extrema.
(b) Use a CAS to solve the equation $f'(x) = 0$ and then use it to make more accurate approximations of the coordinates in part (a).

54. A church window consists of a blue semicircular section surmounting a clear rectangular section as shown in the accompanying figure on the next page. The blue glass lets through half as much light per unit area as the clear glass. Find the radius r of the window that admits the most light if the perimeter of the entire window is to be P feet.

55. Find the dimensions of the rectangle of maximum area that can be inscribed inside the ellipse $(x/4)^2 + (y/3)^2 = 1$ (see the accompanying figure).



▲ **Figure Ex-54**



▲ **Figure Ex-55**

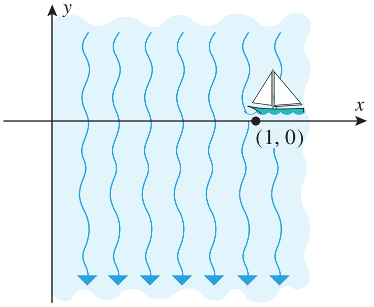
56. As shown in the accompanying figure, suppose that a boat enters the river at the point $(1, 0)$ and maintains a heading toward the origin. As a result of the strong current, the boat follows the path

$$y = \frac{x^{10/3} - 1}{2x^{2/3}}$$

where x and y are in miles.

(a) Graph the path taken by the boat.

(b) Can the boat reach the origin? If not, discuss its fate and find how close it comes to the origin.



◀ **Figure Ex-56**

57. A sheet of cardboard 12 in square is used to make an open box by cutting squares of equal size from the four corners and folding up the sides. What size squares should be cut to obtain a box with largest possible volume?

58. Is it true or false that a particle in rectilinear motion is speeding up when its velocity is increasing and slowing down when its velocity is decreasing? Justify your answer.

59. (a) Can an object in rectilinear motion reverse direction if its acceleration is constant? Justify your answer using a velocity versus time curve.

(b) Can an object in rectilinear motion have increasing speed and decreasing acceleration? Justify your answer using a velocity versus time curve.

60. Suppose that the position function of a particle in rectilinear motion is given by the formula $s(t) = t/(2t^2 + 8)$ for $t \geq 0$.

(a) Use a graphing utility to generate the position, velocity, and acceleration versus time curves.

(b) Use the appropriate graph to make a rough estimate of the time when the particle reverses direction, and then find that time exactly.

(c) Find the position, velocity, and acceleration at the instant when the particle reverses direction.

(d) Use the appropriate graphs to make rough estimates of the time intervals on which the particle is speeding up and the time intervals on which it is slowing down, and then find those time intervals exactly.

(e) When does the particle have its maximum and minimum velocities?

61. For parts (a)–(f), suppose that the position function of a particle in rectilinear motion is given by the formula

$$s(t) = \frac{t^2 + 1}{t^4 + 1}, \quad t \geq 0$$

(a) Use a CAS to find simplified formulas for the velocity function $v(t)$ and the acceleration function $a(t)$.

(b) Graph the position, velocity, and acceleration versus time curves.

(c) Use the appropriate graph to make a rough estimate of the time at which the particle is farthest from the origin and its distance from the origin at that time.

(d) Use the appropriate graph to make a rough estimate of the time interval during which the particle is moving in the positive direction.

(e) Use the appropriate graphs to make rough estimates of the time intervals during which the particle is speeding up and the time intervals during which it is slowing down.

(f) Use the appropriate graph to make a rough estimate of the maximum speed of the particle and the time at which the maximum speed occurs.

62. Draw an appropriate picture, and describe the basic idea of Newton's Method without using any formulas.

63. Use Newton's Method to approximate all three solutions of $x^3 - 4x + 1 = 0$.

64. Use Newton's Method to approximate the smallest positive solution of $\sin x + \cos x = 0$.

65. Use a graphing utility to determine the number of times the curve $y = x^3$ intersects the curve $y = (x/2) - 1$. Then apply Newton's Method to approximate the x-coordinates of all intersections.

66. According to ***Kepler's law***, the planets in our solar system move in elliptical orbits around the Sun. If a planet's closest approach to the Sun occurs at time $t = 0$, then the distance r from the center of the planet to the center of the Sun at some later time t can be determined from the equation

$$r = a(1 - e\cos\phi)$$

where a is the average distance between centers, e is a positive constant that measures the "flatness" of the elliptical orbit, and ϕ is the solution of *Kepler's equation*

$$\frac{2\pi t}{T} = \phi - e\sin\phi$$

in which T is the time it takes for one complete orbit of the planet. Estimate the distance from the Earth to the Sun when $t = 90$ days. [First find ϕ from Kepler's equation, and then use this value of ϕ to find the distance. Use $a = 150 \times 10^6$ km, $e = 0.0167$, and $T = 365$ days.]

67. Using the formulas in Exercise 66, find the distance from the planet Mars to the Sun when $t = 1$ year. For Mars use $a = 228 \times 10^6$ km, $e = 0.0934$, and $T = 1.88$ years.

68. Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and suppose that $f(a) = f(b)$. Is it true or false that f must have at least one stationary point in (a, b)? Justify your answer.

69. In each part, determine whether all of the hypotheses of Rolle's Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of c guaranteed in the conclusion of the theorem.

(a) $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$
(b) $f(x) = x^{2/3} - 1$ on $[-1, 1]$
(c) $f(x) = \sin(x^2)$ on $[0, \sqrt{\pi}\,]$

70. In each part, determine whether all of the hypotheses of the Mean-Value Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of c guaranteed in the conclusion of the theorem.

(a) $f(x) = |x - 1|$ on $[-2, 2]$
(b) $f(x) = \dfrac{x+1}{x-1}$ on $[2, 3]$
(c) $f(x) = \begin{cases} 3 - x^2 & \text{if } x \le 1 \\ 2/x & \text{if } x > 1 \end{cases}$ on $[0, 2]$

71. Use the fact that

$$\frac{d}{dx}(x^6 - 2x^2 + x) = 6x^5 - 4x + 1$$

to show that the equation $6x^5 - 4x + 1 = 0$ has at least one solution in the interval $(0, 1)$.

72. Let $g(x) = x^3 - 4x + 6$. Find $f(x)$ so that $f'(x) = g'(x)$ and $f(1) = 2$.

CHAPTER 3 MAKING CONNECTIONS

1. Suppose that $g(x)$ is a function that is defined and differentiable for all real numbers x and that $g(x)$ has the following properties:

(i) $g(0) = 2$ and $g'(0) = -\frac{2}{3}$.
(ii) $g(4) = 3$ and $g'(4) = 3$.
(iii) $g(x)$ is concave up for $x < 4$ and concave down for $x > 4$.
(iv) $g(x) \ge -10$ for all x.

Use these properties to answer the following questions.

(a) How many zeros does g have?
(b) How many zeros does g' have?
(c) Exactly one of the following limits is possible:

$$\lim_{x\to+\infty} g'(x) = -5, \qquad \lim_{x\to+\infty} g'(x) = 0, \qquad \lim_{x\to+\infty} g'(x) = 5$$

Identify which of these results is possible and draw a rough sketch of the graph of such a function $g(x)$. Explain why the other two results are impossible.

2. The two graphs in the accompanying figure depict a function $r(x)$ and its derivative $r'(x)$.

(a) Approximate the coordinates of each inflection point on the graph of $y = r(x)$.
(b) Suppose that $f(x)$ is a function that is continuous everywhere and whose *derivative* satisfies

$$f'(x) = (x^2 - 4) \cdot r(x)$$

What are the critical points for $f(x)$? At each critical point, identify whether $f(x)$ has a (relative) maximum, minimum, or neither a maximum or minimum. Approximate $f''(1)$.

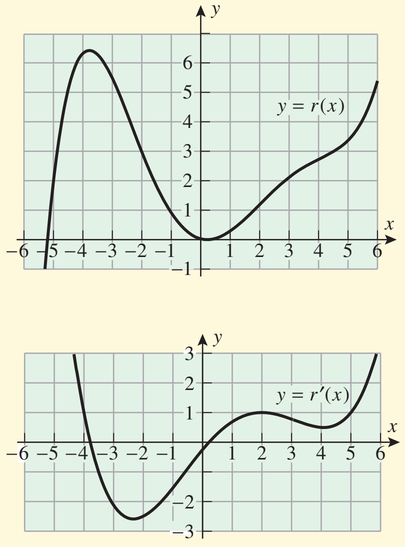


Figure Ex-2

3. With the function $r(x)$ as provided in Exercise 2, let $g(x)$ be a function that is continuous everywhere such that $g'(x) = x - r(x)$. For which values of x does $g(x)$ have an inflection point?

4. Suppose that f is a function whose derivative is continuous everywhere. Assume that there exists a real number c such that when Newton's Method is applied to f, the inequality

$$|x_n - c| < \frac{1}{n}$$

is satisfied for all values of $n = 1, 2, 3, \ldots$.

(a) Explain why
$$|x_{n+1} - x_n| < \frac{2}{n}$$
for all values of $n = 1, 2, 3, \ldots$.

(b) Show that there exists a positive constant M such that
$$|f(x_n)| \leq M|x_{n+1} - x_n| < \frac{2M}{n}$$
for all values of $n = 1, 2, 3, \ldots$.

(c) Prove that if $f(c) \neq 0$, then there exists a positive integer N such that
$$\frac{|f(c)|}{2} < |f(x_n)|$$
if $n > N$. [*Hint:* Argue that $f(x) \to f(c)$ as $x \to c$ and then apply Definition 1.4.1 with $\epsilon = \frac{1}{2}|f(c)|$.]

(d) What can you conclude from parts (b) and (c)?

5. What are the important elements in the argument suggested by Exercise 4? Can you extend this argument to a wider collection of functions?

6. A bug crawling on a linoleum floor along the edge of a plush carpet encounters an irregularity in the form of a 2 in by 3 in rectangular section of carpet that juts out into the linoleum as illustrated in Figure Ex-6a. The bug crawls at 0.7 in/s on the linoleum, but only at 0.3 in/s through the carpet, and its goal is to travel from point A to point B. Four possible routes from A to B are as follows: (i) crawl on linoleum along the edge of the carpet; (ii) crawl through the carpet to a point on the wider side of the rectangle, and finish the journey on linoleum along the edge of the carpet; (iii) crawl through the carpet to a point on the shorter side of the rectangle, and finish the journey on linoleum along the edge of the carpet; or (iv) crawl through the carpet directly to point B. (See Figure Ex-6b.)

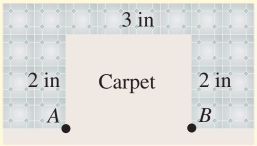


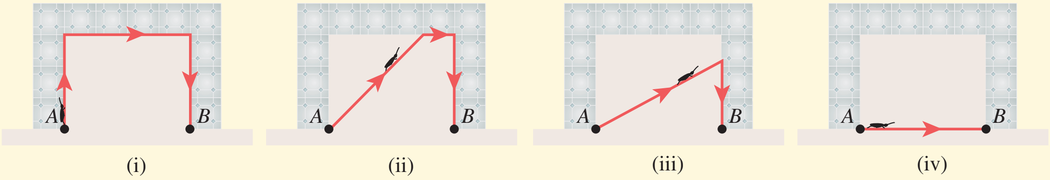
Figure Ex-6a

(a) Calculate the times it would take the bug to crawl from A to B via routes (i) and (iv).

(b) Suppose the bug follows route (ii) and use x to represent the total distance the bug crawls on linoleum. Identify the appropriate interval for x in this case, and determine the shortest time for the bug to complete the journey using route (ii).

(c) Suppose the bug follows route (iii) and again use x to represent the total distance the bug crawls on linoleum. Identify the appropriate interval for x in this case, and determine the shortest time for the bug to complete the journey using route (iii).

(d) Which of routes (i), (ii), (iii), or (iv) is quickest? What is the shortest time for the bug to complete the journey?



(i) (ii) (iii) (iv)

Figure Ex-6b

Chapter IV

Integration

Jon Ferrey/Allsport/Getty Images

4

INTEGRATION

If a dragster moves with varying velocity over a certain time interval, it is possible to find the distance it travels during that time interval using techniques of calculus.

In this chapter we will begin with an overview of the problem of finding areas—we will discuss what the term "area" means, and we will outline two approaches to defining and calculating areas. Following this overview, we will discuss the Fundamental Theorem of Calculus, which is the theorem that relates the problems of finding tangent lines and areas, and we will discuss techniques for calculating areas. We will then use the ideas in this chapter to define the average value of a function, to continue our study of rectilinear motion, and to examine some consequences of the chain rule in integral calculus.

4.1 AN OVERVIEW OF THE AREA PROBLEM

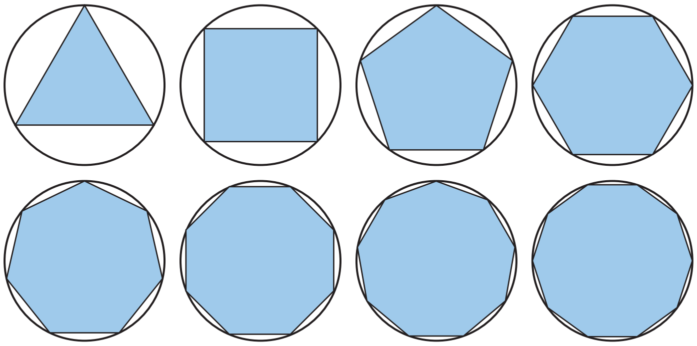
In this introductory section we will consider the problem of calculating areas of plane regions with curvilinear boundaries. All of the results in this section will be reexamined in more detail later in this chapter. Our purpose here is simply to introduce and motivate the fundamental concepts.

THE AREA PROBLEM

Formulas for the areas of polygons, such as squares, rectangles, triangles, and trapezoids, were well known in many early civilizations. However, the problem of finding formulas for regions with curved boundaries (a circle being the simplest example) caused difficulties for early mathematicians.

The first real progress in dealing with the general area problem was made by the Greek mathematician Archimedes, who obtained areas of regions bounded by circular arcs, parabolas, spirals, and various other curves using an ingenious procedure that was later called the *method of exhaustion*. The method, when applied to a circle, consists of inscribing a succession of regular polygons in the circle and allowing the number of sides to increase indefinitely (Figure 4.1.1). As the number of sides increases, the polygons tend to "exhaust" the region inside the circle, and the areas of the polygons become better and better approximations of the exact area of the circle.

To see how this works numerically, let $A(n)$ denote the area of a regular n-sided polygon inscribed in a circle of radius 1. Table 4.1.1 shows the values of $A(n)$ for various choices of n. Note that for large values of n the area $A(n)$ appears to be close to π (square units),



▲ **Figure 4.1.1**

Table 4.1.1

n	$A(n)$
100	3.13952597647
200	3.14107590781
300	3.14136298250
400	3.14146346236
500	3.14150997084
1000	3.14157198278
2000	3.14158748588
3000	3.14159035683
4000	3.14159136166
5000	3.14159182676
10,000	3.14159244688

Archimedes **(287 B.C.–212 B.C.)** Greek mathematician and scientist. Born in Syracuse, Sicily, Archimedes was the son of the astronomer Pheidias and possibly related to Heiron II, king of Syracuse. Most of the facts about his life come from the Roman biographer, Plutarch, who inserted a few tantalizing pages about him in the massive biography of the Roman soldier, Marcellus. In the words of one writer, "the account of Archimedes is slipped like a tissue-thin shaving of ham in a bull-choking sandwich."

Archimedes ranks with Newton and Gauss as one of the three greatest mathematicians who ever lived, and he is certainly the greatest mathematician of antiquity. His mathematical work is so modern in spirit and technique that it is barely distinguishable from that of a seventeenth-century mathematician, yet it was all done without benefit of algebra or a convenient number system. Among his mathematical achievements, Archimedes developed a general method (exhaustion) for finding areas and volumes, and he used the method to find areas bounded by parabolas and spirals and to find volumes of cylinders, paraboloids, and segments of spheres. He gave a procedure for approximating π and bounded its value between $3\frac{10}{71}$ and $3\frac{1}{7}$. In spite of the limitations of the Greek numbering system, he devised methods for finding square roots and invented a method based on the Greek myriad (10,000) for representing numbers as large as 1 followed by 80 million billion zeros.

Of all his mathematical work, Archimedes was most proud of his discovery of a method for finding the volume of a sphere—he showed that the volume of a sphere is two-thirds the volume of the smallest cylinder that can contain it. At his request, the figure of a sphere and cylinder was engraved on his tombstone.

In addition to mathematics, Archimedes worked extensively in mechanics and hydrostatics. Nearly every schoolchild knows Archimedes as the absent-minded scientist who, on realizing that a floating object displaces its weight of liquid, leaped from his bath and ran naked through the streets of Syracuse shouting, "Eureka, Eureka!"—(meaning, "I have found it!"). Archimedes actually created the discipline of hydrostatics and used it to find equilibrium positions for various floating bodies. He laid down the fundamental postulates of mechanics, discovered the laws of levers, and calculated centers of gravity for various flat surfaces and solids. In the excitement of discovering the mathematical laws of the lever, he is said to have declared, "Give me a place to stand and I will move the earth."

Although Archimedes was apparently more interested in pure mathematics than its applications, he was an engineering genius. During the second Punic war, when Syracuse was attacked by the Roman fleet under the command of Marcellus, it was reported by Plutarch that Archimedes' military inventions held the fleet at bay for three years. He invented super catapults that showered the Romans with rocks weighing a quarter ton or more, and fearsome mechanical devices with iron "beaks and claws" that reached over the city walls, grasped the ships, and spun them against the rocks. After the first repulse, Marcellus called Archimedes a "geometrical Briareus (a hundred-armed mythological monster) who uses our ships like cups to ladle water from the sea."

Eventually the Roman army was victorious and contrary to Marcellus' specific orders the 75-year-old Archimedes was killed by a Roman soldier. According to one report of the incident, the soldier cast a shadow across the sand in which Archimedes was working on a mathematical problem. When the annoyed Archimedes yelled, "Don't disturb my circles," the soldier flew into a rage and cut the old man down.

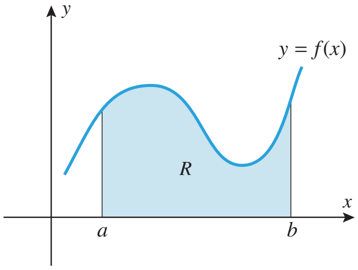
Although there is no known likeness or statue of this great man, nine works of Archimedes have survived to the present day. Especially important is his treatise, *The Method of Mechanical Theorems*, which was part of a palimpsest found in Constantinople in 1906. In this treatise Archimedes explains how he made some of his discoveries, using reasoning that anticipated ideas of the integral calculus. Thought to be lost, the Archimedes palimpsest later resurfaced in 1998, when it was purchased by an anonymous private collector for two million dollars.

[Image: http://commons.wikimedia.org/wiki/File:Archimedes_%28Idealportrait%29.jpg]

as one would expect. This suggests that for a circle of radius 1, the method of exhaustion is equivalent to an equation of the form

$$\lim_{n \to \infty} A(n) = \pi$$

Since Greek mathematicians were suspicious of the concept of "infinity," they avoided its use in mathematical arguments. As a result, computation of area using the method of exhaustion was a very cumbersome procedure. It remained for Newton and Leibniz to obtain a general method for finding areas that explicitly used the notion of a limit. We will discuss their method in the context of the following problem.

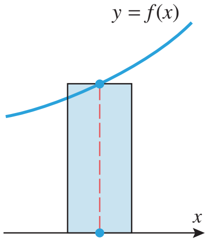


▲ **Figure 4.1.2**

4.1.1 THE AREA PROBLEM Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis (Figure 4.1.2).

THE RECTANGLE METHOD FOR FINDING AREAS

One approach to the area problem is to use Archimedes' method of exhaustion in the following way:



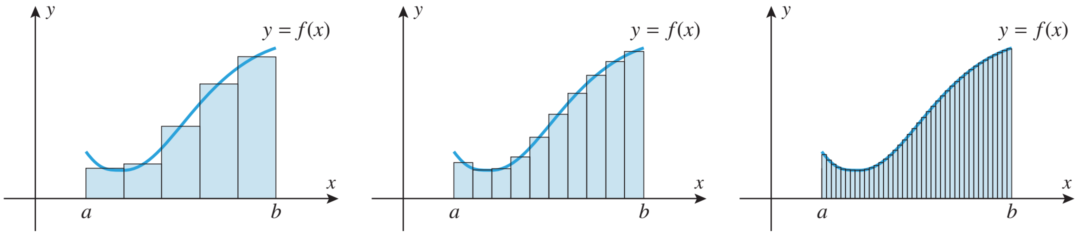
▲ **Figure 4.1.3**

Logically speaking, we cannot really talk about computing areas without a precise mathematical definition of the term "area." Later in this chapter we will give such a definition, but for now we will treat the concept intuitively.

- Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter—it can be above the center, above an endpoint, or above any other point in the subinterval. In Figure 4.1.3 it is above the center.
- For each n, the total area of the rectangles can be viewed as an *approximation* to the exact area under the curve over the interval $[a, b]$. Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit (Figure 4.1.4). That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n \to +\infty} A_n$$

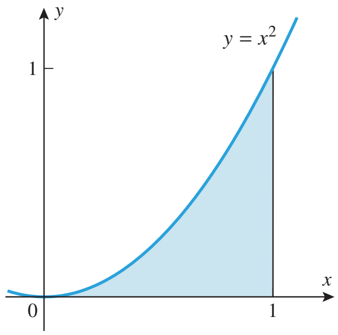
We will call this the ***rectangle method*** for computing A.



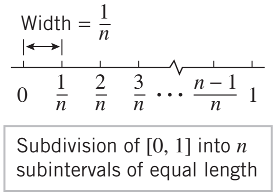
▲ **Figure 4.1.4**

To illustrate this idea, we will use the rectangle method to approximate the area under the curve $y = x^2$ over the interval [0, 1] (Figure 4.1.5). We will begin by dividing the interval [0, 1] into n equal subintervals, from which it follows that each subinterval has length $1/n$; the endpoints of the subintervals occur at

$$0,\ \frac{1}{n},\ \frac{2}{n},\ \frac{3}{n}, \ldots,\ \frac{n-1}{n},\ 1$$



▲ **Figure 4.1.5**



▲ **Figure 4.1.6**

(Figure 4.1.6). We want to construct a rectangle over each of these subintervals whose height is the value of the function $f(x) = x^2$ at some point in the subinterval. To be specific, let us use the right endpoints, in which case the heights of our rectangles will be

$$\left(\frac{1}{n}\right)^2, \ \left(\frac{2}{n}\right)^2, \ \left(\frac{3}{n}\right)^2, \ldots, \ 1^2$$

and since each rectangle has a base of width $1/n$, the total area A_n of the n rectangles will be

$$A_n = \left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + 1^2\right]\left(\frac{1}{n}\right) \tag{1}$$

For example, if $n = 4$, then the total area of the four approximating rectangles would be

$$A_4 = \left[\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2 + 1^2\right]\left(\tfrac{1}{4}\right) = \tfrac{15}{32} = 0.46875$$

Table 4.1.2 shows the result of evaluating (1) on a computer for some increasingly large values of n. These computations suggest that the exact area is close to $\frac{1}{3}$. Later in this chapter we will prove that this area is exactly $\frac{1}{3}$ by showing that

$$\lim_{n\to\infty} A_n = \tfrac{1}{3}$$

Table 4.1.2

n	4	10	100	1000	10,000	100,000
A_n	0.468750	0.385000	0.338350	0.333834	0.333383	0.333338

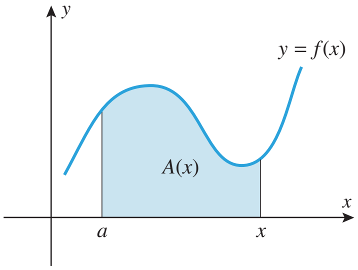
TECHNOLOGY MASTERY

Use a calculating utility to compute the value of A_{10} in Table 4.1.2. Some calculating utilities have special commands for computing sums such as that in (1) for any specified value of n. If your utility has this feature, use it to compute A_{100} as well.

THE ANTIDERIVATIVE METHOD FOR FINDING AREAS

Although the rectangle method is appealing intuitively, the limits that result can only be evaluated in certain cases. For this reason, progress on the area problem remained at a rudimentary level until the latter part of the seventeenth century when Isaac Newton and Gottfried Leibniz independently discovered a fundamental relationship between areas and derivatives. Briefly stated, they showed that if f is a nonnegative continuous function on the interval $[a, b]$, and if $A(x)$ denotes the area under the graph of f over the interval $[a, x]$, where x is any point in the interval $[a, b]$ (Figure 4.1.7), then

$$A'(x) = f(x) \tag{2}$$



▲ **Figure 4.1.7**

The following example confirms Formula (2) in some cases where a formula for $A(x)$ can be found using elementary geometry.

▶ **Example 1** For each of the functions f, find the area $A(x)$ between the graph of f and the interval $[a, x] = [-1, x]$, and find the derivative $A'(x)$ of this area function.

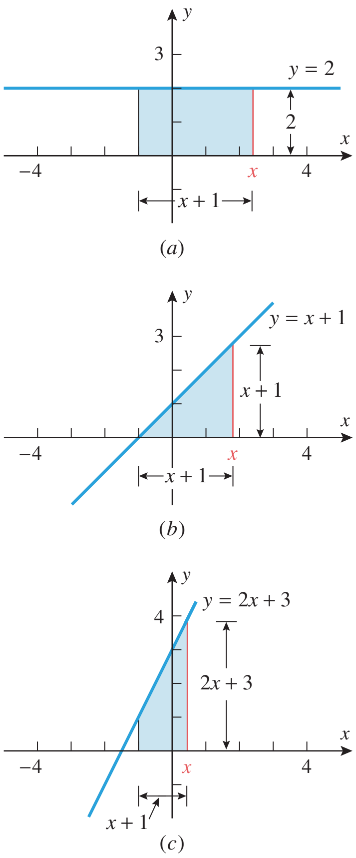
(a) $f(x) = 2$ (b) $f(x) = x + 1$ (c) $f(x) = 2x + 3$

Solution (a). From Figure 4.1.8*a* we see that

$$A(x) = 2(x - (-1)) = 2(x + 1) = 2x + 2$$

is the area of a rectangle of height 2 and base $x + 1$. For this area function,

$$A'(x) = 2 = f(x)$$



▲ **Figure 4.1.8**

How does the solution to Example 2 change if the interval $[0, 1]$ is replaced by the interval $[-1, 1]$?

Solution (b). From Figure 4.1.8*b* we see that

$$A(x) = \frac{1}{2}(x+1)(x+1) = \frac{x^2}{2} + x + \frac{1}{2}$$

is the area of an isosceles right triangle with base and height equal to $x + 1$. For this area function,

$$A'(x) = x + 1 = f(x)$$

Solution (c). Recall that the formula for the area of a trapezoid is $A = \frac{1}{2}(b + b')h$, where b and b' denote the lengths of the parallel sides of the trapezoid, and the altitude h denotes the distance between the parallel sides. From Figure 4.1.8*c* we see that

$$A(x) = \tfrac{1}{2}((2x+3)+1)(x-(-1)) = x^2 + 3x + 2$$

is the area of a trapezoid with parallel sides of lengths 1 and $2x + 3$ and with altitude $x - (-1) = x + 1$. For this area function,

$$A'(x) = 2x + 3 = f(x) \blacktriangleleft$$

Formula (2) is important because it relates the area function A and the region-bounding function f. Although a formula for $A(x)$ may be difficult to obtain directly, its derivative, $f(x)$, is given. If a formula for $A(x)$ can be recovered from the given formula for $A'(x)$, then the area under the graph of f over the interval $[a, b]$ can be obtained by computing $A(b)$.

The process of finding a function from its derivative is called ***antidifferentiation***, and a procedure for finding areas via antidifferentiation is called the ***antiderivative method***. To illustrate this method, let us revisit the problem of finding the area in Figure 4.1.5.

▶ **Example 2** Use the antiderivative method to find the area under the graph of $y = x^2$ over the interval $[0, 1]$.

Solution. Let x be any point in the interval $[0, 1]$, and let $A(x)$ denote the area under the graph of $f(x) = x^2$ over the interval $[0, x]$. It follows from (2) that

$$A'(x) = x^2 \tag{3}$$

To find $A(x)$ we must look for a function whose derivative is x^2. By guessing, we see that one such function is $\frac{1}{3}x^3$, so by Theorem 3.8.3

$$A(x) = \tfrac{1}{3}x^3 + C \tag{4}$$

for some real constant C. We can determine the specific value for C by considering the case where $x = 0$. In this case (4) implies that

$$A(0) = C \tag{5}$$

But if $x = 0$, then the interval $[0, x]$ reduces to a single point. If we agree that the area above a single point should be taken as zero, then $A(0) = 0$ and (5) implies that $C = 0$. Thus, it follows from (4) that

$$A(x) = \tfrac{1}{3}x^3$$

is the area function we are seeking. This implies that the area under the graph of $y = x^2$ over the interval $[0, 1]$ is

$$A(1) = \tfrac{1}{3}(1^3) = \tfrac{1}{3}$$

This is consistent with the result that we previously obtained numerically. ◀

As Example 2 illustrates, antidifferentiation is a process in which one tries to "undo" a differentiation. One of the objectives in this chapter is to develop efficient antidifferentiation procedures.

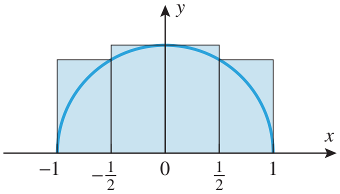
■ THE RECTANGLE METHOD AND THE ANTIDERIVATIVE METHOD COMPARED

The rectangle method and the antiderivative method provide two very different approaches to the area problem, each of which is important. The antiderivative method is usually the more efficient way to *compute* areas, but it is the rectangle method that is used to formally *define* the notion of area, thereby allowing us to prove mathematical results about areas. The underlying idea of the rectangle approach is also important because it can be adapted readily to such diverse problems as finding the volume of a solid, the length of a curve, the mass of an object, and the work done in pumping water out of a tank, to name a few.

✔QUICK CHECK EXERCISES 4.1

(See page 271 for answers.)

1. Let R denote the region below the graph of $f(x) = \sqrt{1-x^2}$ and above the interval $[-1, 1]$.
  (a) Use a geometric argument to find the area of R.
  (b) What estimate results if the area of R is approximated by the total area within the rectangles of the accompanying figure?



◀ **Figure Ex-1**

2. Suppose that when the area A between the graph of a function $y = f(x)$ and an interval $[a, b]$ is approximated by the areas of n rectangles, the total area of the rectangles is $A_n = 2 + (2/n)$, $n = 1, 2, \ldots$. Then, $A =$ ________.

3. The area under the graph of $y = x^2$ over the interval $[0, 3]$ is ________.

4. Find a formula for the area $A(x)$ between the graph of the function $f(x) = x$ and the interval $[0, x]$, and verify that $A'(x) = f(x)$.

5. The area under the graph of $y = f(x)$ over the interval $[0, x]$ is $A(x) = x + \sin x$. It follows that $f(x) =$ ________.

EXERCISE SET 4.1

1–8 Estimate the area between the graph of the function f and the interval $[a, b]$. Use an approximation scheme with n rectangles similar to our treatment of $f(x) = x^2$ in this section. If your calculating utility will perform automatic summations, estimate the specified area using $n = 10$, 50, and 100 rectangles. Otherwise, estimate this area using $n = 2$, 5, and 10 rectangles. ■

1. $f(x) = \sqrt{x}$; $[a, b] = [0, 1]$

2. $f(x) = \dfrac{1}{x+1}$; $[a, b] = [0, 1]$

3. $f(x) = \sin x$; $[a, b] = [0, \pi]$

4. $f(x) = \cos x$; $[a, b] = [0, \pi/2]$

5. $f(x) = \dfrac{1}{x}$; $[a, b] = [1, 2]$

6. $f(x) = \cos x$; $[a, b] = [-\pi/2, \pi/2]$

7. $f(x) = \sqrt{1-x^2}$; $[a, b] = [0, 1]$

8. $f(x) = \sqrt{1-x^2}$; $[a, b] = [-1, 1]$

9–14 Graph each function over the specified interval. Then use simple area formulas from geometry to find the area function $A(x)$ that gives the area between the graph of the specified function f and the interval $[a, x]$. Confirm that $A'(x) = f(x)$ in every case. ■

9. $f(x) = 3$; $[a, x] = [1, x]$

10. $f(x) = 5$; $[a, x] = [2, x]$

11. $f(x) = 2x + 2$; $[a, x] = [0, x]$

12. $f(x) = 3x - 3$; $[a, x] = [1, x]$ .25

13. $f(x) = 2x + 2$; $[a, x] = [1, x]$

14. $f(x) = 3x - 3$; $[a, x] = [2, x]$

15–18 True–False Determine whether the statement is true or false. Explain your answer. ■

15. If $A(n)$ denotes the area of a regular n-sided polygon inscribed in a circle of radius 2, then $\lim_{n\to+\infty} A(n) = 2\pi$.

16. If the area under the curve $y = x^2$ over an interval is approximated by the total area of a collection of rectangles, the approximation will be too large.

17. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A'(x) = f(x)$.

18. If $A(x)$ is the area under the graph of a nonnegative continuous function f over an interval $[a, x]$, then $A(x)$ will be a continuous function.

FOCUS ON CONCEPTS

19. Explain how to use the formula for $A(x)$ found in the solution to Example 2 to determine the area between the graph of $y = x^2$ and the interval $[3, 6]$.

20. Repeat Exercise 19 for the interval $[-3, 9]$.

21. Let A denote the area between the graph of $f(x) = \sqrt{x}$ and the interval $[0, 1]$, and let B denote the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$. Explain geometrically why $A + B = 1$.

22. Let A denote the area between the graph of $f(x) = 1/x$ and the interval $[1, 2]$, and let B denote the area between the graph of f and the interval $\left[\frac{1}{2}, 1\right]$. Explain geometrically why $A = B$.

23–24 The area $A(x)$ under the graph of f and over the interval $[a, x]$ is given. Find the function f and the value of a. ■

23. $A(x) = x^2 - 4$ **24.** $A(x) = x^2 - x$

25. Writing Compare and contrast the rectangle method and the antiderivative method.

26. Writing Suppose that f is a nonnegative continuous function on an interval $[a, b]$ and that $g(x) = f(x) + C$, where C is a positive constant. What will be the area of the region *between* the graphs of f and g?

✔QUICK CHECK ANSWERS 4.1

1. (a) $\frac{\pi}{2}$ (b) $1 + \frac{\sqrt{3}}{2}$ **2.** 2 **3.** 9 **4.** $A(x) = \frac{x^2}{2};\ A'(x) = \frac{2x}{2} = x = f(x)$ **5.** $\cos x + 1$

4.2 THE INDEFINITE INTEGRAL

In the last section we saw how antidifferentiation could be used to find exact areas. In this section we will develop some fundamental results about antidifferentiation.

■ ANTIDERIVATIVES

4.2.1 DEFINITION A function F is called an ***antiderivative*** of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,

$$\tfrac{1}{3}x^3, \quad \tfrac{1}{3}x^3 + 2, \quad \tfrac{1}{3}x^3 - 5, \quad \tfrac{1}{3}x^3 + \sqrt{2}$$

are all antiderivatives of $f(x) = x^2$.

It is reasonable to ask if there are antiderivatives of a function f that cannot be obtained by adding some constant to a known antiderivative F. The answer is *no*—once a single antiderivative of f on an open interval is known, all other antiderivatives on that interval are obtainable by adding constants to the known antiderivative. This is so because Theorem

3.8.3 tells us that if two functions have the same derivative on an open interval, then the functions differ by a constant on the interval. The following theorem summarizes these observations.

4.2.2 THEOREM *If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant C the function $F(x)+C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x)+C$ by choosing the constant C appropriately.*

THE INDEFINITE INTEGRAL

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x)+C$. To emphasize this process, Equation (1) is recast using ***integral notation***,

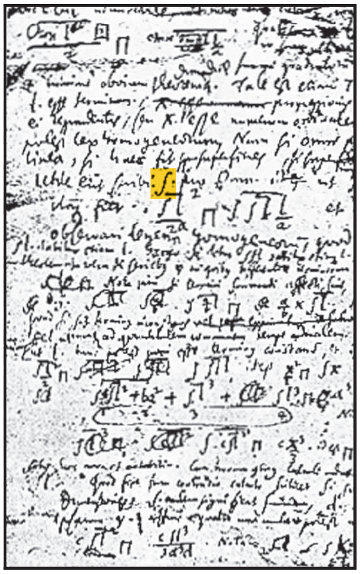
$$\int f(x)\,dx = F(x) + C \tag{2}$$

where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus,

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x) \tag{3}$$



Reproduced from C. I. Gerhardt's "Briefwechsel von G. W. Leibniz mit Mathematikern (1899)."

Extract from the manuscript of Leibniz dated October 29, 1675 in which the integral sign first appeared (see yellow highlight).

The expression $\int f(x)\,dx$ is called an ***indefinite integral***. The adjective "indefinite" emphasizes that the result of antidifferentiation is a "generic" function, described only up to a constant term. The "elongated s" that appears on the left side of (2) is called an ***integral sign***,[*] the function $f(x)$ is called the ***integrand***, and the constant C is called the ***constant of integration***. Equation (2) should be read as:

The integral of $f(x)$ with respect to x is equal to $F(x)$ plus a constant.

The differential symbol, dx, in the differentiation and antidifferentiation operations

$$\frac{d}{dx}[\ \ ] \quad \text{and} \quad \int [\ \ ]\,dx$$

---

[*] This notation was devised by Leibniz. In his early papers Leibniz used the notation "omn." (an abbreviation for the Latin word "omnes") to denote integration. Then on October 29, 1675 he wrote, "It will be useful to write $\int$ for omn., thus $\int l$ for omn. l . . . ." Two or three weeks later he refined the notation further and wrote $\int[\ \ ]\,dx$ rather than $\int$ alone. This notation is so useful and so powerful that its development by Leibniz must be regarded as a major milestone in the history of mathematics and science.

serves to identify the independent variable. If an independent variable other than x is used, say t, then the notation must be adjusted appropriately. Thus,

$$\frac{d}{dt}[F(t)] = f(t) \quad \text{and} \quad \int f(t)\,dt = F(t) + C$$

are equivalent statements. Here are some examples of derivative formulas and their equivalent integration formulas:

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2\,dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + C$
$\frac{d}{du}[u^{3/2}] = \frac{3}{2}u^{1/2}$	$\int \frac{3}{2}u^{1/2}\,du = u^{3/2} + C$

For simplicity, the dx is sometimes absorbed into the integrand. For example,

$$\int 1\,dx \quad \text{can be written as} \quad \int dx$$

$$\int \frac{1}{x^2}\,dx \quad \text{can be written as} \quad \int \frac{dx}{x^2}$$

INTEGRATION FORMULAS

Integration is essentially educated guesswork—given the derivative f of a function F, one tries to guess what the function F is. However, many basic integration formulas can be obtained directly from their companion differentiation formulas. Some of the most important are given in Table 4.2.1.

Table 4.2.1
INTEGRATION FORMULAS

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$	5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$	6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

▶ **Example 1** The second integration formula in Table 4.2.1 will be easier to remember if you express it in words:

To integrate a power of x (other than -1), add 1 to the exponent and divide by the new exponent.

Formula 2 in Table 4.2.1 is not applicable to integrating x^{-1}; we will see how to integrate this function in Chapter 6.

Here are some examples:

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \boxed{r = 2}$$

$$\int x^3\,dx = \frac{x^4}{4} + C \qquad \boxed{r = 3}$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx = \frac{x^{-5+1}}{-5+1} + C = -\frac{1}{4x^4} + C \qquad \boxed{r = -5}$$

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C = \tfrac{2}{3}(\sqrt{x}\,)^3 + C \qquad \boxed{r = \tfrac{1}{2}}$$

PROPERTIES OF THE INDEFINITE INTEGRAL

Our first properties of antiderivatives follow directly from the simple constant factor, sum, and difference rules for derivatives.

4.2.3 THEOREM *Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then:*

(*a*) *A constant factor can be moved through an integral sign; that is,*

$$\int cf(x)\,dx = cF(x) + C$$

(*b*) *An antiderivative of a sum is the sum of the antiderivatives; that is,*

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(*c*) *An antiderivative of a difference is the difference of the antiderivatives; that is,*

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

PROOF In general, to establish the validity of an equation of the form

$$\int h(x)\,dx = H(x) + C$$

one must show that

$$\frac{d}{dx}[H(x)] = h(x)$$

We are given that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, so we know that

$$\frac{d}{dx}[F(x)] = f(x) \quad \text{and} \quad \frac{d}{dx}[G(x)] = g(x)$$

Thus,

$$\frac{d}{dx}[cF(x)] = c\frac{d}{dx}[F(x)] = cf(x)$$

$$\frac{d}{dx}[F(x) + G(x)] = \frac{d}{dx}[F(x)] + \frac{d}{dx}[G(x)] = f(x) + g(x)$$

$$\frac{d}{dx}[F(x) - G(x)] = \frac{d}{dx}[F(x)] - \frac{d}{dx}[G(x)] = f(x) - g(x)$$

which proves the three statements of the theorem. ■

The statements in Theorem 4.2.3 can be summarized by the following formulas:

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{4}$$

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{5}$$

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx \tag{6}$$

However, these equations must be applied carefully to avoid errors and unnecessary complexities arising from the constants of integration. For example, if you use (4) to integrate $2x$ by writing

$$\int 2x\,dx = 2\int x\,dx = 2\left(\frac{x^2}{2}+C\right) = x^2+2C$$

then you will have an unnecessarily complicated form of the arbitrary constant. This kind of problem can be avoided by inserting the constant of integration in the final result rather than in intermediate calculations. Exercises 59 and 60 explore how careless application of these formulas can lead to errors.

▶ Example 2 Evaluate

$$\text{(a)}\ \int 4\cos x\,dx \qquad \text{(b)}\ \int (x+x^2)\,dx$$

Solution (a). Since $F(x) = \sin x$ is an antiderivative for $f(x) = \cos x$ (Table 4.2.1), we obtain

$$\int 4\cos x\,dx \underset{\boxed{(4)}}{=} 4\int \cos x\,dx = 4\sin x + C$$

Solution (b). From Table 4.2.1 we obtain

$$\int (x+x^2)\,dx \underset{\boxed{(5)}}{=} \int x\,dx + \int x^2\,dx = \frac{x^2}{2}+\frac{x^3}{3}+C \quad ◄$$

Parts (b) and (c) of Theorem 4.2.3 can be extended to more than two functions, which in combination with part (a) results in the following general formula:

$$\begin{aligned}&\int [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)]\,dx \\ &\qquad = c_1\int f_1(x)\,dx + c_2\int f_2(x)\,dx + \cdots + c_n\int f_n(x)\,dx\end{aligned} \tag{7}$$

▶ Example 3

$$\begin{aligned}\int (3x^6 - 2x^2 + 7x + 1)\,dx &= 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx \\ &= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C \quad ◄\end{aligned}$$

Sometimes it is useful to rewrite an integrand in a different form before performing the integration. This is illustrated in the following example.

▶ Example 4 Evaluate

$$\text{(a)}\ \int \frac{\cos x}{\sin^2 x}\,dx \qquad \text{(b)}\ \int \frac{t^2 - 2t^4}{t^4}\,dt$$

Solution (a).

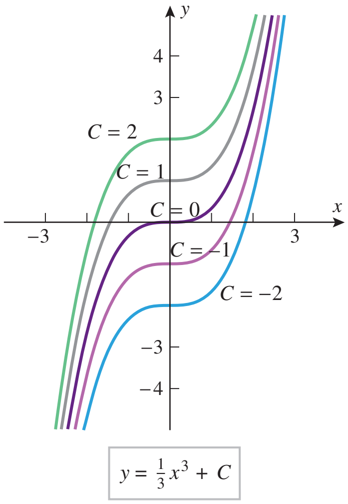
$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\,\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx = -\csc x + C$$

Formula 8 in Table 4.2.1

Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt = \int (t^{-2} - 2)\,dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$



▲ **Figure 4.2.1**

INTEGRAL CURVES

Graphs of antiderivatives of a function f are called ***integral curves*** of f. We know from Theorem 4.2.2 that if $y = F(x)$ is any integral curve of $f(x)$, then all other integral curves are vertical translations of this curve, since they have equations of the form $y = F(x) + C$. For example, $y = \frac{1}{3}x^3$ is one integral curve for $f(x) = x^2$, so all the other integral curves have equations of the form $y = \frac{1}{3}x^3 + C$; conversely, the graph of any equation of this form is an integral curve (Figure 4.2.1).

In many problems one is interested in finding a function whose derivative satisfies specified conditions. The following example illustrates a geometric problem of this type.

In Example 5, the requirement that the graph of f pass through the point $(2, 1)$ selects the single integral curve $y = \frac{1}{3}x^3 - \frac{5}{3}$ from the family of curves $y = \frac{1}{3}x^3 + C$ (Figure 4.2.2).

▶ Example 5 Suppose that a curve $y = f(x)$ in the xy-plane has the property that at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve given that it passes through the point $(2, 1)$.

Solution. Since the slope of the line tangent to $y = f(x)$ is dy/dx, we have $dy/dx = x^2$, and

$$y = \int x^2\,dx = \tfrac{1}{3}x^3 + C$$

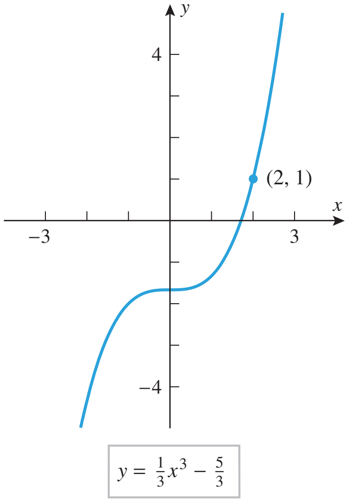
Since the curve passes through $(2, 1)$, a specific value for C can be found by using the fact that $y = 1$ if $x = 2$. Substituting these values in the above equation yields

$$1 = \tfrac{1}{3}(2^3) + C \quad \text{or} \quad C = -\tfrac{5}{3}$$

so an equation of the curve is

$$y = \tfrac{1}{3}x^3 - \tfrac{5}{3}$$

(Figure 4.2.2). ◀



▲ **Figure 4.2.2**

INTEGRATION FROM THE VIEWPOINT OF DIFFERENTIAL EQUATIONS

We will now consider another way of looking at integration that will be useful in our later work. Suppose that $f(x)$ is a known function and we are interested in finding a function $F(x)$ such that $y = F(x)$ satisfies the equation

$$\frac{dy}{dx} = f(x) \tag{8}$$

The solutions of this equation are the antiderivatives of $f(x)$, and we know that these can be obtained by integrating $f(x)$. For example, the solutions of the equation

$$\frac{dy}{dx} = x^2 \tag{9}$$

are

$$y = \int x^2\,dx = \frac{x^3}{3} + C$$

Equation (8) is called a ***differential equation*** because it involves a derivative of an unknown function. Differential equations are different from the kinds of equations we have encountered so far in that the unknown is a *function* and not a *number* as in an equation such as $x^2 + 5x - 6 = 0$.

Sometimes we will not be interested in finding all of the solutions of (8), but rather we will want only the solution whose graph passes through a specified point (x_0, y_0). For example, in Example 5 we solved (9) for the integral curve that passed through the point (2, 1).

For simplicity, it is common in the study of differential equations to denote a solution of $dy/dx = f(x)$ as $y(x)$ rather than $F(x)$, as earlier. With this notation, the problem of finding a function $y(x)$ whose derivative is $f(x)$ and whose graph passes through the point (x_0, y_0) is expressed as

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{10}$$

This is called an ***initial-value problem***, and the requirement that $y(x_0) = y_0$ is called the ***initial condition*** for the problem.

▶ **Example 6** Solve the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

Solution. The solution of the differential equation is

$$y = \int \cos x\,dx = \sin x + C \tag{11}$$

The initial condition $y(0) = 1$ implies that $y = 1$ if $x = 0$; substituting these values in (11) yields

$$1 = \sin(0) + C \quad \text{or} \quad C = 1$$

Thus, the solution of the initial-value problem is $y = \sin x + 1$. ◀

SLOPE FIELDS

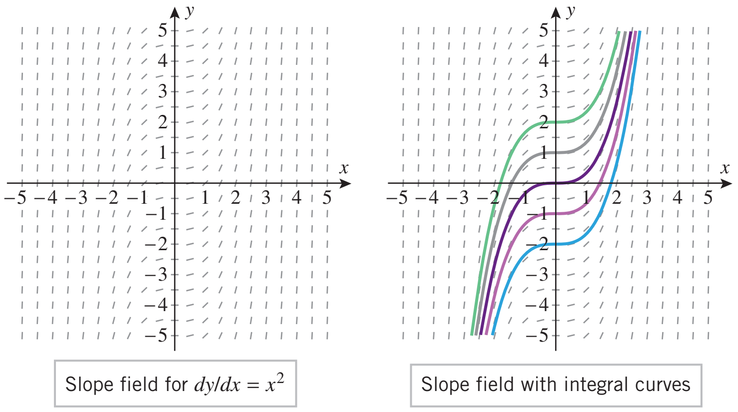
If we interpret dy/dx as the slope of a tangent line, then at a point (x, y) on an integral curve of the equation $dy/dx = f(x)$, the slope of the tangent line is $f(x)$. What is interesting about this is that the slopes of the tangent lines to the integral curves can be obtained without actually solving the differential equation. For example, if

$$\frac{dy}{dx} = \sqrt{x^2 + 1}$$

then we know without solving the equation that at the point where $x = 1$ the tangent line to an integral curve has slope $\sqrt{1^2 + 1} = \sqrt{2}$; and more generally, at a point where $x = a$, the tangent line to an integral curve has slope $\sqrt{a^2 + 1}$.

A geometric description of the integral curves of a differential equation $dy/dx = f(x)$ can be obtained by choosing a rectangular grid of points in the xy-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small portions of the tangent lines through those points. The resulting picture, which is called a ***slope field*** or ***direction field*** for the equation, shows the "direction" of the integral curves at the gridpoints. With sufficiently many gridpoints it is often possible to visualize the integral curves themselves; for example, Figure 4.2.3*a* shows a slope field for the differential equation $dy/dx = x^2$, and Figure 4.2.3*b* shows that same field with the integral curves imposed on it—the more gridpoints that are used, the more completely the slope field reveals the shape of the integral curves. However, the amount of computation can be considerable, so computers are usually used when slope fields with many gridpoints are needed.

Slope fields will be studied in more detail later in the text.



(*a*) (*b*)

Figure 4.2.3

✔QUICK CHECK EXERCISES 4.2 (*See page 281 for answers.*)

1. A function F is an antiderivative of a function f on an interval if ________ for all x in the interval.

2. Write an equivalent integration formula for each given derivative formula.

(a) $\dfrac{d}{dx}[\sqrt{x}] = \dfrac{1}{2\sqrt{x}}$ (b) $\dfrac{d}{dx}[\sin x] = \cos x$

3. Evaluate the integrals.

(a) $\displaystyle\int [x^3 + x + 5]\,dx$ (b) $\displaystyle\int [\sec^2 x - \csc x \cot x]\,dx$

4. The graph of $y = x^2 + x$ is an integral curve for the function $f(x) =$ ________. If G is a function whose graph is also an integral curve for f, and if $G(1) = 5$, then $G(x) =$ ________.

5. A slope field for the differential equation

$$\frac{dy}{dx} = \frac{2x}{x^2 - 4}$$

has a line segment with slope ________ through the point $(0, 5)$ and has a line segment with slope ________ through the point $(-4, 1)$.

EXERCISE SET 4.2 Graphing Utility C CAS

1. In each part, confirm that the formula is correct, and state a corresponding integration formula.

(a) $\dfrac{d}{dx}[\sqrt{1+x^2}\,] = \dfrac{x}{\sqrt{1+x^2}}$

(b) $\dfrac{d}{dx}\left[\dfrac{1}{3}\sin(1+x^3)\right] = x^2\cos(1+x^3)$

2. In each part, confirm that the stated formula is correct by differentiating.

(a) $\displaystyle\int x \sin x\,dx = \sin x - x\cos x + C$

(b) $\displaystyle\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} + C$

FOCUS ON CONCEPTS

3. What is a *constant of integration*? Why does an answer to an integration problem involve a constant of integration?

4. What is an *integral curve* of a function f? How are two integral curves of a function f related?

5–8 Find the derivative and state a corresponding integration formula. ■

5. $\dfrac{d}{dx}[\sqrt{x^3+5}\,]$

6. $\dfrac{d}{dx}\left[\dfrac{x}{x^2+3}\right]$

7. $\dfrac{d}{dx}[\sin(2\sqrt{x})]$

8. $\dfrac{d}{dx}[\sin x - x\cos x]$

9–10 Evaluate the integral by rewriting the integrand appropriately, if required, and applying the power rule (Formula 2 in Table 4.2.1). ■

9. (a) $\int x^8\,dx$ (b) $\int x^{5/7}\,dx$ (c) $\int x^3\sqrt{x}\,dx$

10. (a) $\int \sqrt[3]{x^2}\,dx$ (b) $\int \frac{1}{x^6}\,dx$ (c) $\int x^{-7/8}\,dx$

11–14 Evaluate each integral by applying Theorem 4.2.3 and Formula 2 in Table 4.2.1 appropriately. ■

11. $\int \left[5x + \frac{2}{3x^5}\right] dx$ **12.** $\int \left[x^{-1/2} - 3x^{7/5} + \frac{1}{9}\right] dx$

13. $\int [x^{-3} - 3x^{1/4} + 8x^2]\,dx$

14. $\int \left[\frac{10}{y^{3/4}} - \sqrt[3]{y} + \frac{4}{\sqrt{y}}\right] dy$

15–30 Evaluate the integral and check your answer by differentiating. ■

15. $\int x(1 + x^3)\,dx$ **16.** $\int (2 + y^2)^2\,dy$

17. $\int x^{1/3}(2 - x)^2\,dx$ **18.** $\int (1 + x^2)(2 - x)\,dx$

19. $\int \frac{x^5 + 2x^2 - 1}{x^4}\,dx$ **20.** $\int \frac{1 - 2t^3}{t^3}\,dt$

21. $\int [3\sin x - 2\sec^2 x]\,dx$ **22.** $\int [\csc^2 t - \sec t \tan t]\,dt$

23. $\int \sec x(\sec x + \tan x)\,dx$ **24.** $\int \csc x(\sin x + \cot x)\,dx$

25. $\int \frac{\sec\theta}{\cos\theta}\,d\theta$ **26.** $\int \frac{dy}{\csc y}$

27. $\int \frac{\sin x}{\cos^2 x}\,dx$ **28.** $\int \left[\phi + \frac{2}{\sin^2\phi}\right] d\phi$

29. $\int [1 + \sin^2\theta \csc\theta]\,d\theta$ **30.** $\int \frac{\sec x + \cos x}{2\cos x}\,dx$

31. Evaluate the integral

$$\int \frac{1}{1 + \sin x}\,dx$$

by multiplying the numerator and denominator by an appropriate expression.

32. Use the double-angle formula $\cos 2x = 2\cos^2 x - 1$ to evaluate the integral

$$\int \frac{1}{1 + \cos 2x}\,dx$$

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

34. If C denotes a constant of integration, the two formulas

$$\int \cos x\,dx = \sin x + C$$

$$\int \cos x\,dx = (\sin x + \pi) + C$$

are both correct equations.

35. The function $f(x) = \sec x + 1$ is a solution to the initial-value problem

$$\frac{dy}{dx} = \sec x \tan x, \quad y(0) = 1$$

36. Every integral curve of the slope field

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

is the graph of an increasing function of x.

37. Use a graphing utility to generate some representative integral curves of the function $f(x) = 5x^4 - \sec^2 x$ over the interval $(-\pi/2, \pi/2)$.

38. Use a graphing utility to generate some representative integral curves of the function $f(x) = (x^2 - 1)/x^2$ over the interval $(0, 5)$.

39–40 Solve the initial-value problems. ■

39. (a) $\frac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

(b) $\frac{dy}{dt} = \sin t + 1,\ y\left(\frac{\pi}{3}\right) = \frac{1}{2}$

(c) $\frac{dy}{dx} = \frac{x + 1}{\sqrt{x}},\ y(1) = 0$

40. (a) $\frac{dy}{dx} = \frac{1}{(2x)^3},\ y(1) = 0$

(b) $\frac{dy}{dt} = \sec^2 t - \sin t,\ y\left(\frac{\pi}{4}\right) = 1$

(c) $\frac{dy}{dx} = x^2\sqrt{x^3},\ y(0) = 0$

41–44 A particle moves along an s-axis with position function $s = s(t)$ and velocity function $v(t) = s'(t)$. Use the given information to find $s(t)$. ■

41. $v(t) = 32t;\ \ s(0) = 20$ **42.** $v(t) = \cos t;\ \ s(0) = 2$

43. $v(t) = 3\sqrt{t};\ \ s(4) = 1$ **44.** $v(t) = \sin t;\ \ s(0) = 0$

45. Find the general form of a function whose second derivative is $\sqrt{x}$. [*Hint:* Solve the equation $f''(x) = \sqrt{x}$ for $f(x)$ by integrating both sides twice.]

46. Find a function f such that $f''(x) = x + \cos x$ and such that $f(0) = 1$ and $f'(0) = 2$. [*Hint:* Integrate both sides of the equation twice.]

47–51 Find an equation of the curve that satisfies the given conditions. ■

47. At each point (x, y) on the curve the slope is $2x + 1$; the curve passes through the point $(-3, 0)$.

48. At each point (x, y) on the curve the slope is $(x + 1)^2$; the curve passes through the point $(-2, 8)$.

49. At each point (x, y) on the curve the slope is $-\sin x$; the curve passes through the point $(0, 2)$.

50. At each point (x, y) on the curve the slope equals the square of the distance between the point and the y-axis; the point $(-1, 2)$ is on the curve.

51. At each point (x, y) on the curve, y satisfies the condition $d^2y/dx^2 = 6x$; the line $y = 5 - 3x$ is tangent to the curve at the point where $x = 1$.

52. In each part, use a CAS to solve the initial-value problem.

(a) $\dfrac{dy}{dx} = x^2 \cos 3x, \quad y(\pi/2) = -1$

(b) $\dfrac{dy}{dx} = \dfrac{x^3}{(4+x^2)^{3/2}}, \quad y(0) = -2$

53. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = x$ in the region $-5 \le x \le 5$ and $-5 \le y \le 5$.

(b) Graph some representative integral curves of the function $f(x) = x$.

(c) Find an equation for the integral curve that passes through the point $(2, 1)$.

54. (a) Use a graphing utility to generate a slope field for the differential equation $dy/dx = \sqrt{x}$ in the region $0 \le x \le 10$ and $-5 \le y \le 5$.

(b) Graph some representative integral curves of the function $f(x) = \sqrt{x}$ for $x > 0$.

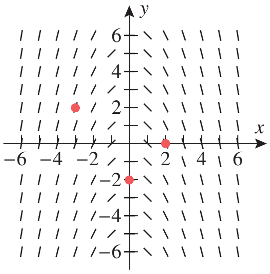
(c) Find an equation for the integral curve that passes through the point $(0, 1)$.

55–58 The given slope field figure corresponds to one of the differential equations below. Identify the differential equation that matches the figure, and sketch solution curves through the highlighted points.

(a) $\dfrac{dy}{dx} = 2$ (b) $\dfrac{dy}{dx} = -x$

(c) $\dfrac{dy}{dx} = x^2 - 4$ (d) $\dfrac{dy}{dx} = \sin x$ ■

55.



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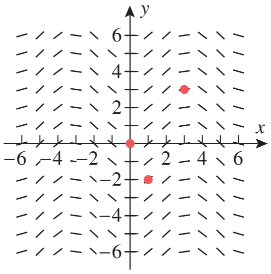

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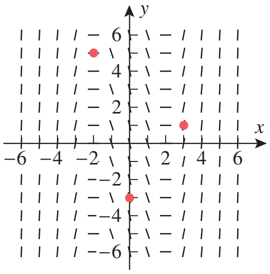

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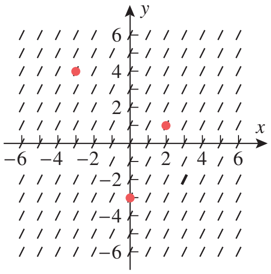


FOCUS ON CONCEPTS

59. Critique the following "proof" that an arbitrary constant must be zero:

$$C = \int 0\,dx = \int 0 \cdot 0\,dx = 0 \int 0\,dx = 0$$

60. Critique the following "proof" that an arbitrary constant must be zero:

$$0 = \left(\int x\,dx\right) - \left(\int x\,dx\right)$$

$$= \int (x - x)\,dx = \int 0\,dx = C$$

61. Let F and G be the functions defined by

$$F(x) = \frac{x \sin x}{x} \quad \text{and} \quad G(x) = \begin{cases} 2 + \sin x, & x > 0 \\ -1 + \sin x, & x < 0 \end{cases}$$

(a) Show that F and G have the same derivative.

(b) Show that $G(x) \ne F(x) + C$ for any constant C.

(c) Do parts (a) and (b) contradict Theorem 4.2.2? Explain.

62. Follow the directions of Exercise 61 using

$$F(x) = \frac{x^2 + 3x}{x} \quad \text{and} \quad G(x) = \begin{cases} x + 3, & x > 0 \\ x, & x < 0 \end{cases}$$

63–64 Use a trigonometric identity to evaluate the integral. ■

63. $\displaystyle\int \tan^2 x\,dx$ **64.** $\displaystyle\int \cot^2 x\,dx$

65–66 Use the identities $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$ to help evaluate the integrals ■

65. $\displaystyle\int \sin^2(x/2)\,dx$ **66.** $\displaystyle\int \cos^2(x/2)\,dx$

67. The speed of sound in air at 0°C (or 273 K on the Kelvin scale) is 1087 ft/s, but the speed v increases as the temperature T rises. Experimentation has shown that the rate of change of v with respect to T is

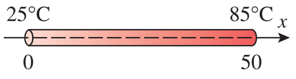
$$\frac{dv}{dT} = \frac{1087}{2\sqrt{273}} T^{-1/2}$$

where v is in feet per second and T is in kelvins (K). Find a formula that expresses v as a function of T.

68. Suppose that a uniform metal rod 50 cm long is insulated laterally, and the temperatures at the exposed ends are maintained at 25°C and 85°C, respectively. Assume that an x-axis is chosen as in the accompanying figure and that the temperature $T(x)$ satisfies the equation

$$\frac{d^2T}{dx^2} = 0$$

Find $T(x)$ for $0 \le x \le 50$.



◀ **Figure Ex-68**

69. **Writing** What is an *initial-value problem*? Describe the sequence of steps for solving an initial-value problem.

70. **Writing** What is a *slope field*? How are slope fields and integral curves related?

✔QUICK CHECK ANSWERS 4.2

1. $F'(x) = f(x)$ **2.** (a) $\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$ (b) $\int \cos x\,dx = \sin x + C$
3. (a) $\frac{1}{4}x^4 + \frac{1}{2}x^2 + 5x + C$ (b) $\tan x + \csc x + C$ **4.** $2x + 1;\ x^2 + x + 3$ **5.** $0;\ -\frac{2}{3}$

4.3 INTEGRATION BY SUBSTITUTION

In this section we will study a technique, called ***substitution****, that can often be used to transform complicated integration problems into simpler ones.*

■ *u*-SUBSTITUTION

The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C \tag{1}$$

or since F is an antiderivative of f,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \tag{2}$$

For our purposes it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x)\,dx$. With this notation (2) can be expressed as

$$\int f(u)\,du = F(u) + C \tag{3}$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

is called the ***method of u-substitution***. Here our emphasis is *not* on the interpretation of the expression $du = g'(x)\,dx$. Rather, the differential notation serves primarily as a useful "bookkeeping" device for the method of u-substitution. The following example illustrates how the method works.

▶ **Example 1** Evaluate $\int (x^2 + 1)^{50} \cdot 2x\,dx$.

Solution. If we let $u = x^2 + 1$, then $du/dx = 2x$, which implies that $du = 2x\,dx$. Thus, the given integral can be written as

$$\int (x^2 + 1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2 + 1)^{51}}{51} + C \ ◀$$

It is important to realize that in the method of u-substitution you have control over the choice of u, but once you make that choice you have no control over the resulting expression for du. Thus, in the last example we *chose* $u = x^2 + 1$ but $du = 2x\,dx$ was *computed*.

Fortunately, our choice of u, combined with the computed du, worked out perfectly to produce an integral involving u that was easy to evaluate. However, in general, the method of u-substitution will fail if the chosen u and the computed du cannot be used to produce an integrand in which no expressions involving x remain, or if you cannot evaluate the resulting integral. Thus, for example, the substitution $u = x^2$, $du = 2x\,dx$ will not work for the integral

$$\int 2x \sin x^4\,dx$$

because this substitution results in the integral

$$\int \sin u^2\,du$$

which still cannot be evaluated in terms of familiar functions.

In general, there are no hard and fast rules for choosing u, and in some problems no choice of u will work. In such cases other methods need to be used, some of which will be discussed later. Making appropriate choices for u will come with experience, but you may find the following guidelines, combined with a mastery of the basic integrals in Table 4.2.1, helpful.

Guidelines for u-Substitution

Step 1. Look for some composition $f(g(x))$ within the integrand for which the substitution

$$u = g(x), \qquad du = g'(x)\,dx$$

produces an integral that is expressed entirely in terms of u and its differential du. This may or may not be possible.

Step 2. If you are successful in Step 1, then try to evaluate the resulting integral in terms of u. Again, this may or may not be possible.

Step 3. If you are successful in Step 2, then replace u by $g(x)$ to express your final answer in terms of x.

EASY TO RECOGNIZE SUBSTITUTIONS

The easiest substitutions occur when the integrand is the derivative of a known function, except for a constant added to or subtracted from the independent variable.

▶ Example 2

$$\int \sin(x+9)\,dx \underset{\boxed{\begin{array}{c}u = x+9\\ du = 1\cdot dx = dx\end{array}}}{=} \int \sin u\,du = -\cos u + C = -\cos(x+9) + C$$

$$\int (x-8)^{23}\,dx \underset{\boxed{\begin{array}{c}u = x-8\\ du = 1\cdot dx = dx\end{array}}}{=} \int u^{23}\,du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$ ◄

Another easy u-substitution occurs when the integrand is the derivative of a known function, except for a constant that multiplies or divides the independent variable. The following example illustrates two ways to evaluate such integrals.

▶ Example 3 Evaluate $\int \cos 5x\,dx$.

Solution.

$$\int \cos 5x\,dx = \int (\cos u)\cdot \frac{1}{5}\,du = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$

$$\boxed{\begin{array}{c} u = 5x \\ du = 5\,dx \text{ or } dx = \frac{1}{5}\,du \end{array}}$$

Alternative Solution. There is a variation of the preceding method that some people prefer. The substitution $u = 5x$ requires $du = 5\,dx$. If there were a factor of 5 in the integrand, then we could group the 5 and dx together to form the du required by the substitution. Since there is no factor of 5, we will insert one and compensate by putting a factor of $\frac{1}{5}$ in front of the integral. The computations are as follows:

$$\int \cos 5x\,dx = \frac{1}{5}\int \cos 5x \cdot 5\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C \blacktriangleleft$$

$$\boxed{\begin{array}{c} u = 5x \\ du = 5\,dx \end{array}}$$

More generally, if the integrand is a composition of the form $f(ax+b)$, where $f(x)$ is an easy to integrate function, then the substitution $u = ax+b$, $du = a\,dx$ will work.

▶ Example 4

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5} = \int \frac{3\,du}{u^5} = 3\int u^{-5}\,du = -\frac{3}{4}u^{-4} + C = -\frac{3}{4}\left(\frac{1}{3}x - 8\right)^{-4} + C \blacktriangleleft$$

$$\boxed{\begin{array}{c} u = \frac{1}{3}x - 8 \\ du = \frac{1}{3}\,dx \text{ or } dx = 3\,du \end{array}}$$

With the help of Theorem 4.2.3, a complicated integral can sometimes be computed by expressing it as a sum of simpler integrals.

▶ Example 5

$$\begin{aligned}
\int \left(\frac{1}{x^2} + \sec^2 \pi x\right) dx &= \int \frac{dx}{x^2} + \int \sec^2 \pi x\,dx \\
&= -\frac{1}{x} + \int \sec^2 \pi x\,dx \\
&= -\frac{1}{x} + \frac{1}{\pi}\int \sec^2 u\,du \qquad \boxed{\begin{array}{c} u = \pi x \\ du = \pi\,dx \text{ or } dx = \frac{1}{\pi}\,du \end{array}} \\
&= -\frac{1}{x} + \frac{1}{\pi}\tan u + C = -\frac{1}{x} + \frac{1}{\pi}\tan \pi x + C \blacktriangleleft
\end{aligned}$$

The next three examples illustrate a substitution $u = g(x)$ where $g(x)$ is a nonlinear function.

▶ **Example 6** Evaluate $\int \sin^2 x \cos x \, dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x \, dx$$

Thus,

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C \quad ◀$$

▶ **Example 7** Evaluate $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$.

Solution. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}} \, dx \quad \text{or} \quad 2\,du = \frac{1}{\sqrt{x}} \, dx$$

Thus,

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx = \int 2 \cos u \, du = 2 \int \cos u \, du = 2 \sin u + C = 2 \sin \sqrt{x} + C \quad ◀$$

▶ **Example 8** Evaluate $\int t^4 \sqrt[3]{3 - 5t^5} \, dt$.

Solution.

$$\int t^4 \sqrt[3]{3 - 5t^5} \, dt = -\frac{1}{25} \int \sqrt[3]{u} \, du = -\frac{1}{25} \int u^{1/3} \, du$$

$$\boxed{\begin{gathered} u = 3 - 5t^5 \\ du = -25t^4 \, dt \text{ or } -\tfrac{1}{25} \, du = t^4 \, dt \end{gathered}}$$

$$= -\frac{1}{25} \frac{u^{4/3}}{4/3} + C = -\frac{3}{100} \left(3 - 5t^5\right)^{4/3} + C \quad ◀$$

LESS APPARENT SUBSTITUTIONS

The method of substitution is relatively straightforward, provided the integrand contains an easily recognized composition $f(g(x))$ and the remainder of the integrand is a constant multiple of $g'(x)$. If this is not the case, the method may still apply but may require more computation.

▶ **Example 9** Evaluate $\int x^2 \sqrt{x - 1} \, dx$.

Solution. The composition $\sqrt{x - 1}$ suggests the substitution

$$u = x - 1 \quad \text{so that} \quad du = dx \tag{4}$$

From the first equality in (4)

$$x^2 = (u + 1)^2 = u^2 + 2u + 1$$

so that

$$\int x^2\sqrt{x-1}\,dx = \int (u^2+2u+1)\sqrt{u}\,du = \int (u^{5/2}+2u^{3/2}+u^{1/2})\,du$$

$$= \tfrac{2}{7}u^{7/2} + \tfrac{4}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{2}{7}(x-1)^{7/2} + \tfrac{4}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C \blacktriangleleft$$

▶ **Example 10** Evaluate $\int \cos^3 x\,dx$.

Solution. The only compositions in the integrand that suggest themselves are

$$\cos^3 x = (\cos x)^3 \quad \text{and} \quad \cos^2 x = (\cos x)^2$$

However, neither the substitution $u = \cos x$ nor the substitution $u = \cos^2 x$ work (verify). In this case, an appropriate substitution is not suggested by the composition contained in the integrand. On the other hand, note from Equation (2) that the derivative $g'(x)$ appears as a factor in the integrand. This suggests that we write

$$\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx$$

and solve the equation $du = \cos x\,dx$ for $u = \sin x$. Since $\sin^2 x + \cos^2 x = 1$, we then have

$$\int \cos^3 x\,dx = \int \cos^2 x\cos x\,dx = \int (1-\sin^2 x)\cos x\,dx = \int (1-u^2)\,du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C \blacktriangleleft$$

TECHNOLOGY MASTERY

If you have a CAS, use it to calculate the integrals in the examples in this section. If your CAS produces an answer that is different from the one in the text, then confirm algebraically that the two answers agree. Also, explore the effect of using the CAS to simplify the expressions it produces for the integrals.

INTEGRATION USING COMPUTER ALGEBRA SYSTEMS

The advent of computer algebra systems has made it possible to evaluate many kinds of integrals that would be laborious to evaluate by hand. For example, a handheld calculator evaluated the integral

$$\int \frac{5x^2}{(1+x)^{1/3}}\,dx = \frac{3(x+1)^{2/3}(5x^2-6x+9)}{8} + C$$

in about a second. The computer algebra system *Mathematica*, running on a personal computer, required even less time to evaluate this same integral. However, just as one would not want to rely on a calculator to compute $2+2$, so one would not want to use a CAS to integrate a simple function such as $f(x) = x^2$. Thus, even if you have a CAS, you will want to develop a reasonable level of competence in evaluating basic integrals. Moreover, the mathematical techniques that we will introduce for evaluating basic integrals are precisely the techniques that computer algebra systems use to evaluate more complicated integrals.

✔QUICK CHECK EXERCISES 4.3 *(See page 287 for answers.)*

1. Indicate the u-substitution.

(a) $\int 3x^2(1+x^3)^{25}\,dx = \int u^{25}\,du$ if $u =$ ________ and $du =$ ________.

(b) $\int 2x\sin x^2\,dx = \int \sin u\,du$ if $u =$ ________ and $du =$ ________.

(c) $\int \frac{18x}{\sqrt{1+9x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,du$ if $u =$ ________ and $du =$ ________.

2. Supply the missing integrand corresponding to the indicated u-substitution.

(a) $\displaystyle\int 5(5x-3)^{-1/3}\,dx = \int \rule{2cm}{0.4pt}\,du;\ u = 5x-3$

(b) $\displaystyle\int (3-\tan x)\sec^2 x\,dx = \int \rule{2cm}{0.4pt}\,du;$
$u = 3-\tan x$

(c) $\displaystyle\int \frac{\sqrt[3]{8+\sqrt{x}}}{\sqrt{x}}\,dx = \int \rule{2cm}{0.4pt}\,du;\ u = 8+\sqrt{x}$

EXERCISE SET 4.3

Graphing Utility   C CAS

1–8 Evaluate the integrals using the indicated substitutions.

1. (a) $\displaystyle\int 2x(x^2+1)^{23}\,dx;\ u = x^2+1$

(b) $\displaystyle\int \cos^3 x \sin x\,dx;\ u = \cos x$

2. (a) $\displaystyle\int \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx;\ u = \sqrt{x}$

(b) $\displaystyle\int \frac{3x\,dx}{\sqrt{4x^2+5}};\ u = 4x^2+5$

3. (a) $\displaystyle\int \sec^2(4x+1)\,dx;\ u = 4x+1$

(b) $\displaystyle\int y\sqrt{1+2y^2}\,dy;\ u = 1+2y^2$

4. (a) $\displaystyle\int \sqrt{\sin\pi\theta}\cos\pi\theta\,d\theta;\ u = \sin\pi\theta$

(b) $\displaystyle\int (2x+7)(x^2+7x+3)^{4/5}\,dx;\ u = x^2+7x+3$

5. (a) $\displaystyle\int \cot x\csc^2 x\,dx;\ u = \cot x$

(b) $\displaystyle\int (1+\sin t)^9\cos t\,dt;\ u = 1+\sin t$

6. (a) $\displaystyle\int \cos 2x\,dx;\ u = 2x$ (b) $\displaystyle\int x\sec^2 x^2\,dx;\ u = x^2$

7. (a) $\displaystyle\int x^2\sqrt{1+x}\,dx;\ u = 1+x$

(b) $\displaystyle\int [\csc(\sin x)]^2\cos x\,dx;\ u = \sin x$

8. (a) $\displaystyle\int \sin(x-\pi)\,dx;\ u = x-\pi$

(b) $\displaystyle\int \frac{5x^4}{(x^5+1)^2}\,dx;\ u = x^5+1$

FOCUS ON CONCEPTS

9. Explain the connection between the chain rule for differentiation and the method of u-substitution for integration.

10. Explain how the substitution $u = ax+b$ helps to perform an integration in which the integrand is $f(ax+b)$, where $f(x)$ is an easy to integrate function.

11–36 Evaluate the integrals using appropriate substitutions.

11. $\displaystyle\int (4x-3)^9\,dx$

12. $\displaystyle\int x^3\sqrt{5+x^4}\,dx$

13. $\displaystyle\int \sin 7x\,dx$

14. $\displaystyle\int \cos\frac{x}{3}\,dx$

15. $\displaystyle\int \sec 4x\tan 4x\,dx$

16. $\displaystyle\int \sec^2 5x\,dx$

17. $\displaystyle\int t\sqrt{7t^2+12}\,dt$

18. $\displaystyle\int \frac{x}{\sqrt{4-5x^2}}\,dx$

19. $\displaystyle\int \frac{6}{(1-2x)^3}\,dx$

20. $\displaystyle\int \frac{x^2+1}{\sqrt{x^3+3x}}\,dx$

21. $\displaystyle\int \frac{x^3}{(5x^4+2)^3}\,dx$

22. $\displaystyle\int \frac{\sin(1/x)}{3x^2}\,dx$

23. $\displaystyle\int \frac{\sin(5/x)}{x^2}\,dx$

24. $\displaystyle\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$

25. $\displaystyle\int \cos^4 3t\sin 3t\,dt$

26. $\displaystyle\int \cos 2t\sin^5 2t\,dt$

27. $\displaystyle\int x\sec^2(x^2)\,dx$

28. $\displaystyle\int \frac{\cos 4\theta}{(1+2\sin 4\theta)^4}\,d\theta$

29. $\displaystyle\int \cos 4\theta\sqrt{2-\sin 4\theta}\,d\theta$

30. $\displaystyle\int \tan^3 5x\sec^2 5x\,dx$

31. $\displaystyle\int \sec^3 2x\tan 2x\,dx$

32. $\displaystyle\int [\sin(\sin\theta)]\cos\theta\,d\theta$

33. $\displaystyle\int \frac{y}{\sqrt{2y+1}}\,dy$

34. $\displaystyle\int x\sqrt{4-x}\,dx$

35. $\displaystyle\int \sin^3 2\theta\,d\theta$

36. $\displaystyle\int \sec^4 3\theta\,d\theta$ [*Hint:* Apply a trigonometric identity.]

37–39 Evaluate the integrals assuming that n is a positive integer and $b \neq 0$.

37. $\displaystyle\int (a+bx)^n\,dx$

38. $\displaystyle\int \sqrt[n]{a+bx}\,dx$

39. $\displaystyle\int \sin^n(a+bx)\cos(a+bx)\,dx$

C **40.** Use a CAS to check the answers you obtained in Exercises 37–39. If the answer produced by the CAS does not match yours, show that the two answers are equivalent. [*Suggestion: Mathematica* users may find it helpful to apply the Simplify command to the answer.]

FOCUS ON CONCEPTS

41. (a) Evaluate the integral $\int \sin x \cos x\, dx$ by two methods: first by letting $u = \sin x$, and then by letting $u = \cos x$.

(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

42. (a) Evaluate the integral $\int (5x-1)^2\, dx$ by two methods: first square and integrate, then let $u = 5x - 1$.

(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

43–44 Solve the initial-value problems. ■

43. $\dfrac{dy}{dx} = \sqrt{5x+1},\;\; y(3) = -2$

44. $\dfrac{dy}{dx} = 2 + \sin 3x,\;\; y(\pi/3) = 0$

45. (a) Evaluate $\int [x/\sqrt{x^2+1}]\, dx$.

(b) Use a graphing utility to generate some typical integral curves of $f(x) = x/\sqrt{x^2+1}$ over the interval $(-5, 5)$.

46. (a) Evaluate $\int 2x \sin(25 - x^2)\, dx$.

(b) Use a graphing utility to generate some typical integral curves of $f(x) = 2x \sin(25 - x^2)$ over the interval $(-5, 5)$.

47. Find a function f such that the slope of the tangent line at a point (x, y) on the curve $y = f(x)$ is $\sqrt{3x+1}$ and the curve passes through the point $(0, 1)$.

48. A population of minnows in a lake is estimated to be 100,000 at the beginning of the year 2010. Suppose that t years after the beginning of 2010 the rate of growth of the population $p(t)$ (in thousands) is given by $p'(t) = (3 + 0.12t)^{3/2}$. Estimate the projected population at the beginning of the year 2015.

49. Let $y(t)$ denote the number of *E. coli* cells in a container of nutrient solution t minutes after the start of an experiment. Assume that $y(t)$ is modeled by the initial-value problem

$$\frac{dy}{dt} = 0.95(0.79 + 0.024t)^{3/2}, \qquad y(0) = 20$$

Use this model to estimate the number of *E. coli* cells in the container 20 minutes after the start of the experiment.

50. **Writing** If you want to evaluate an integral by u-substitution, how do you decide what part of the integrand to choose for u?

51. **Writing** The evaluation of an integral can sometimes result in apparently different answers (Exercises 41 and 42). Explain why this occurs and give an example. How might you show that two apparently different answers are actually equivalent?

QUICK CHECK ANSWERS 4.3

1. (a) $1 + x^3$; $3x^2\, dx$ (b) x^2; $2x\, dx$ (c) $1 + 9x^2$; $18x\, dx$ **2.** (a) $u^{-1/3}$ (b) $-u$ (c) $2\sqrt[3]{u}$

4.4 THE DEFINITION OF AREA AS A LIMIT; SIGMA NOTATION

Our main goal in this section is to use the rectangle method to give a precise mathematical definition of the "area under a curve."

■ SIGMA NOTATION

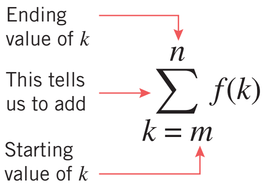
To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called ***sigma notation*** or ***summation notation*** because it uses the uppercase Greek letter Σ (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

in which each term is of the form k^2, where k is one of the integers from 1 to 5. In sigma notation this sum can be written as

$$\sum_{k=1}^{5} k^2$$

which is read "the summation of k^2, where k runs from 1 to 5." The notation tells us to form the sum of the terms that result when we substitute successive integers for k in the expression k^2, starting with $k = 1$ and ending with $k = 5$.



▲ **Figure 4.4.1**

More generally, if $f(k)$ is a function of k, and if m and n are integers such that $m \le n$, then

$$\sum_{k=m}^{n} f(k) \tag{1}$$

denotes the sum of the terms that result when we substitute successive integers for k, starting with $k = m$ and ending with $k = n$ (Figure 4.4.1).

► Example 1

$$\sum_{k=4}^{8} k^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3$$

$$\sum_{k=1}^{5} 2k = 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 2\cdot 4 + 2\cdot 5 = 2 + 4 + 6 + 8 + 10$$

$$\sum_{k=0}^{5} (2k+1) = 1 + 3 + 5 + 7 + 9 + 11$$

$$\sum_{k=0}^{5} (-1)^k(2k+1) = 1 - 3 + 5 - 7 + 9 - 11$$

$$\sum_{k=-3}^{1} k^3 = (-3)^3 + (-2)^3 + (-1)^3 + 0^3 + 1^3 = -27 - 8 - 1 + 0 + 1$$

$$\sum_{k=1}^{3} k \sin\left(\frac{k\pi}{5}\right) = \sin\frac{\pi}{5} + 2\sin\frac{2\pi}{5} + 3\sin\frac{3\pi}{5} \quad ◄$$

The numbers m and n in (1) are called, respectively, the ***lower*** and ***upper limits of summation***; and the letter k is called the ***index of summation***. It is not essential to use k as the index of summation; any letter not reserved for another purpose will do. For example,

$$\sum_{i=1}^{6} \frac{1}{i}, \quad \sum_{j=1}^{6} \frac{1}{j}, \quad \text{and} \quad \sum_{n=1}^{6} \frac{1}{n}$$

all denote the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

If the upper and lower limits of summation are the same, then the "sum" in (1) reduces to a single term. For example,

$$\sum_{k=2}^{2} k^3 = 2^3 \quad \text{and} \quad \sum_{i=1}^{1} \frac{1}{i+2} = \frac{1}{1+2} = \frac{1}{3}$$

In the sums

$$\sum_{i=1}^{5} 2 \quad \text{and} \quad \sum_{j=0}^{2} x^3$$

the expression to the right of the Σ sign does not involve the index of summation. In such cases, we take all the terms in the sum to be the same, with one term for each allowable value of the summation index. Thus,

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 \quad \text{and} \quad \sum_{j=0}^{2} x^3 = x^3 + x^3 + x^3$$

CHANGING THE LIMITS OF SUMMATION

A sum can be written in more than one way using sigma notation with different limits of summation and correspondingly different summands. For example,

$$\sum_{i=1}^{5} 2i = 2+4+6+8+10 = \sum_{j=0}^{4}(2j+2) = \sum_{k=3}^{7}(2k-4)$$

On occasion we will want to change the sigma notation for a given sum to a sigma notation with different limits of summation.

PROPERTIES OF SUMS

When stating general properties of sums it is often convenient to use a subscripted letter such as a_k in place of the function notation $f(k)$. For example,

$$\sum_{k=1}^{5} a_k = a_1+a_2+a_3+a_4+a_5 = \sum_{j=1}^{5} a_j = \sum_{k=-1}^{3} a_{k+2}$$

$$\sum_{k=1}^{n} a_k = a_1+a_2+\cdots+a_n = \sum_{j=1}^{n} a_j = \sum_{k=-1}^{n-2} a_{k+2}$$

Our first properties provide some basic rules for manipulating sums.

4.4.1 THEOREM

(*a*) $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ (*if c does not depend on k*)

(*b*) $\displaystyle\sum_{k=1}^{n}(a_k+b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(*c*) $\displaystyle\sum_{k=1}^{n}(a_k-b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

We will prove parts (*a*) and (*b*) and leave part (*c*) as an exercise.

PROOF (*a*)

$$\sum_{k=1}^{n} ca_k = ca_1+ca_2+\cdots+ca_n = c(a_1+a_2+\cdots+a_n) = c\sum_{k=1}^{n} a_k$$

PROOF (*b*)

$$\sum_{k=1}^{n}(a_k+b_k) = (a_1+b_1)+(a_2+b_2)+\cdots+(a_n+b_n)$$

$$= (a_1+a_2+\cdots+a_n)+(b_1+b_2+\cdots+b_n) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \quad ■$$

Restating Theorem 4.4.1 in words:

(*a*) *A constant factor can be moved through a sigma sign.*

(*b*) *Sigma distributes across sums.*

(*c*) *Sigma distributes across differences.*

SUMMATION FORMULAS

The following theorem lists some useful formulas for sums of powers of integers. The derivations of these formulas are given in Appendix D.

TECHNOLOGY MASTERY

If you have access to a CAS, it will provide a method for finding closed forms such as those in Theorem 4.4.2. Use your CAS to confirm the formulas in that theorem, and then find closed forms for

$$\sum_{k=1}^{n} k^4 \quad \text{and} \quad \sum_{k=1}^{n} k^5$$

4.4.2 THEOREM

(*a*) $$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

(*b*) $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(*c*) $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

▶ **Example 2** Evaluate $\sum_{k=1}^{30} k(k+1)$.

Solution.

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30} (k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

$$= \frac{30(31)(61)}{6} + \frac{30(31)}{2} = 9920$$

Theorem 4.4.2(*a*), (*b*) ◀

TECHNOLOGY MASTERY

Many calculating utilities provide some way of evaluating sums expressed in sigma notation. If your utility has this capability, use it to confirm that the result in Example 2 is correct.

In formulas such as

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{or} \quad 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

the left side of the equality is said to express the sum in ***open form*** and the right side is said to express it in ***closed form***. The open form indicates the summands and the closed form is an explicit formula for the sum.

▶ **Example 3** Express $\sum_{k=1}^{n} (3+k)^2$ in closed form.

Solution.

$$\sum_{k=1}^{n} (3+k)^2 = 4^2 + 5^2 + \cdots + (3+n)^2$$

$$= [1^2 + 2^2 + 3^3 + 4^2 + 5^2 + \cdots + (3+n)^2] - [1^2 + 2^2 + 3^2]$$

$$= \left(\sum_{k=1}^{3+n} k^2\right) - 14$$

$$= \frac{(3+n)(4+n)(7+2n)}{6} - 14 = \frac{1}{6}(73n + 21n^2 + 2n^3)$$ ◀

A DEFINITION OF AREA

We now turn to the problem of giving a precise definition of what is meant by the "area under a curve." Specifically, suppose that the function f is continuous and nonnegative on the interval $[a, b]$, and let R denote the region bounded below by the x-axis, bounded on the sides by the vertical lines $x = a$ and $x = b$, and bounded above by the curve $y = f(x)$ (Figure 4.4.2). Using the rectangle method of Section 4.1, we can motivate a definition for the area of R as follows:

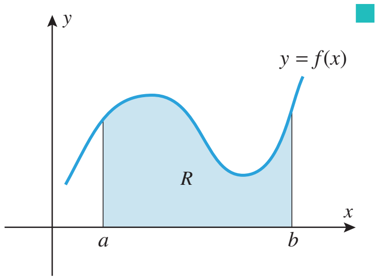


Figure 4.4.2

- Divide the interval $[a, b]$ into n equal subintervals by inserting $n - 1$ equally spaced points between a and b, and denote those points by

$$x_1, x_2, \ldots, x_{n-1}$$

(Figure 4.4.3). Each of these subintervals has width $(b - a)/n$, which is customarily denoted by

$$\Delta x = \frac{b-a}{n}$$

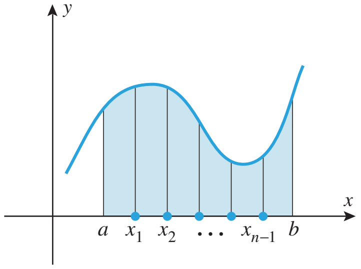


Figure 4.4.3

- Over each subinterval construct a rectangle whose height is the value of f at an arbitrarily selected point in the subinterval. Thus, if

$$x_1^*, x_2^*, \ldots, x_n^*$$

denote the points selected in the subintervals, then the rectangles will have heights $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ and areas

$$f(x_1^*)\Delta x, \quad f(x_2^*)\Delta x, \ldots, \quad f(x_n^*)\Delta x$$

(Figure 4.4.4).

- The union of the rectangles forms a region R_n whose area can be regarded as an approximation to the area A of the region R; that is,

$$A = \text{area}(R) \approx \text{area}(R_n) = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

(Figure 4.4.5). This can be expressed more compactly in sigma notation as

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$$

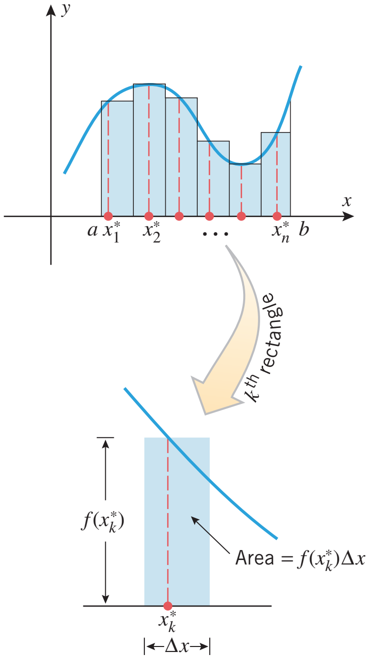


Figure 4.4.4

- Repeat the process using more and more subdivisions, and define the area of R to be the "limit" of the areas of the approximating regions R_n as n increases without bound. That is, we define the area A as

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

In summary, we make the following definition.

4.4.3 DEFINITION (*Area Under a Curve*) If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the ***area*** A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{2}$$

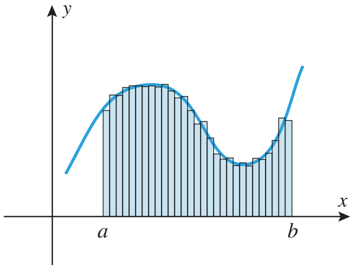


Figure 4.4.5 area$(R_n) \approx$ area(R)

REMARK There is a difference in interpretation between $\lim_{n\to+\infty}$ and $\lim_{x\to+\infty}$, where n represents a positive integer and x represents a real number. Later we will study limits of the type $\lim_{n\to+\infty}$ in detail, but for now suffice it to say that the computational techniques we have used for limits of type $\lim_{x\to+\infty}$ will also work for $\lim_{n\to+\infty}$.

The limit in (2) is interpreted to mean that given any number $\epsilon > 0$ the inequality

$$\left| A - \sum_{k=1}^{n} f(x_k^*)\Delta x \right| < \epsilon$$

holds when n is sufficiently large, no matter how the points x_k^* are selected.

The values of $x_1^*, x_2^*, \ldots, x_n^*$ in (2) can be chosen arbitrarily, so it is conceivable that different choices of these values might produce different values of A. Were this to happen, then Definition 4.4.3 would not be an acceptable definition of area. Fortunately, this does not happen; it is proved in advanced courses that if f is continuous (as we have assumed), then the same value of A results no matter how the x_k^* are chosen. In practice they are chosen in some systematic fashion, some common choices being

- the left endpoint of each subinterval
- the right endpoint of each subinterval
- the midpoint of each subinterval

To be more specific, suppose that the interval $[a, b]$ is divided into n equal parts of length $\Delta x = (b - a)/n$ by the points $x_1, x_2, \ldots, x_{n-1}$, and let $x_0 = a$ and $x_n = b$ (Figure 4.4.6). Then,

$$x_k = a + k\Delta x \quad \text{for } k = 0, 1, 2, \ldots, n$$

Thus, the left endpoint, right endpoint, and midpoint choices for $x_1^*, x_2^*, \ldots, x_n^*$ are given by

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \boxed{\text{Left endpoint}} \tag{3}$$

$$x_k^* = x_k = a + k\Delta x \qquad \boxed{\text{Right endpoint}} \tag{4}$$

$$x_k^* = \tfrac{1}{2}(x_{k-1} + x_k) = a + \left(k - \tfrac{1}{2}\right)\Delta x \qquad \boxed{\text{Midpoint}} \tag{5}$$

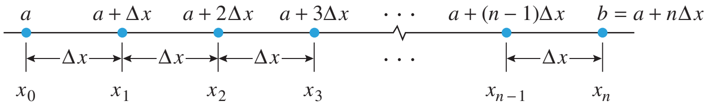


Figure 4.4.6

When applicable, the antiderivative method will be the method of choice for finding exact areas. However, the following examples will help to reinforce the ideas that we have just discussed.

Example 4 Use Definition 4.4.3 with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval $[0, 1]$.

Solution. The length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

so it follows from (4) that

$$x_k^* = a + k\Delta x = \frac{k}{n}$$

Thus,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}(x_k^*)^2\Delta x = \sum_{k=1}^{n}\left(\frac{k}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n}k^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \boxed{\text{Part } (b) \text{ of Theorem 4.4.2}}$$

$$= \frac{1}{6}\left(\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right) = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

from which it follows that

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left[\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{3}$$

Observe that this is consistent with the results in Table 4.1.2 and the related discussion in Section 4.1. ◀

In the solution to Example 4 we made use of one of the "closed form" summation formulas from Theorem 4.4.2. The next result collects some consequences of Theorem 4.4.2 that can facilitate computations of area using Definition 4.4.3.

What pattern is revealed by parts (*b*)–(*d*) of Theorem 4.4.4? Does part (*a*) also fit this pattern? What would you conjecture to be the value of

$$\lim_{n\to+\infty}\frac{1}{n^m}\sum_{k=1}^{n}k^{m-1}?$$

4.4.4 THEOREM

(*a*) $\displaystyle\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}1 = 1$ (*b*) $\displaystyle\lim_{n\to+\infty}\frac{1}{n^2}\sum_{k=1}^{n}k = \frac{1}{2}$

(*c*) $\displaystyle\lim_{n\to+\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2 = \frac{1}{3}$ (*d*) $\displaystyle\lim_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n}k^3 = \frac{1}{4}$

The proof of Theorem 4.4.4 is left as an exercise for the reader.

▶ **Example 5** Use Definition 4.4.3 with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = f(x) = 9 - x^2$ and over the interval [0, 3].

Solution. Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

so it follows from (5) that

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k-\frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$f(x_k^*)\Delta x = [9-(x_k^*)^2]\Delta x = \left[9 - \left(k-\frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right)$$

$$= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}$$

from which it follows that

$$\begin{aligned}
A &= \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \lim_{n\to+\infty} \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}\right) \\
&= \lim_{n\to+\infty} 27\left[\frac{1}{n}\sum_{k=1}^{n} 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n}\left(\frac{1}{n^2}\sum_{k=1}^{n} k\right) - \frac{1}{4n^2}\left(\frac{1}{n}\sum_{k=1}^{n} 1\right)\right] \\
&= 27\left[1 - \frac{1}{3} + 0\cdot\frac{1}{2} - 0\cdot 1\right] = 18 \qquad \boxed{\text{Theorem 4.4.4}}
\end{aligned}$$

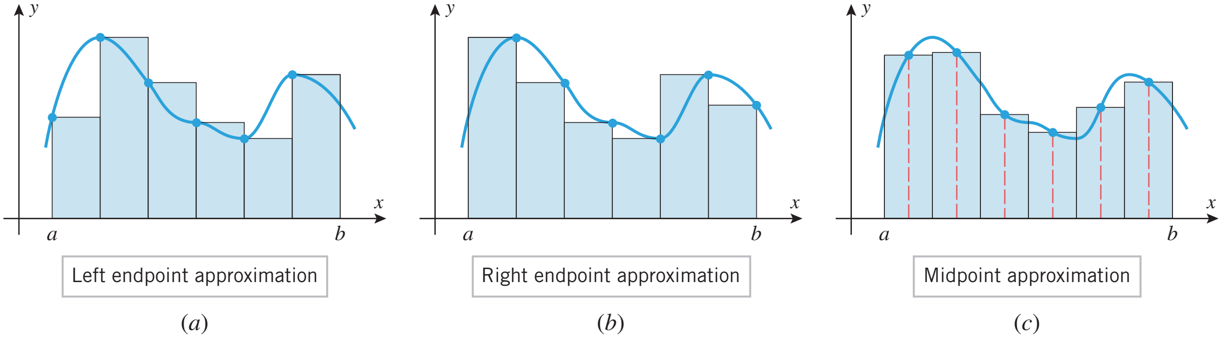
◀

NUMERICAL APPROXIMATIONS OF AREA

The antiderivative method discussed in Section 4.1 (and to be studied in more detail later) is an appropriate tool for finding the exact area under a curve when an antiderivative of the integrand can be found. However, if an antiderivative cannot be found, then we must resort to *approximating* the area. Definition 4.4.3 provides a way of doing this. It follows from this definition that if n is large, then

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k^*) = \Delta x[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)] \tag{6}$$

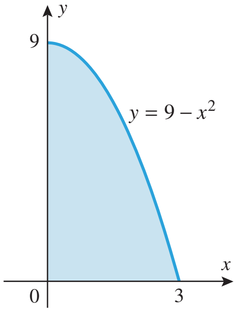
will be a good approximation to the area A. If one of Formulas (3), (4), or (5) is used to choose the x_k^* in (6), then the result is called the ***left endpoint approximation***, the ***right endpoint approximation***, or the ***midpoint approximation***, respectively (Figure 4.4.7).



(a) (b) (c)

▲ **Figure 4.4.7**

▶ Example 6 Find the left endpoint, right endpoint, and midpoint approximations of the area under the curve $y = 9 - x^2$ over the interval [0, 3] with $n = 10$, $n = 20$, and $n = 50$ (Figure 4.4.8). Compare the accuracies of these three methods.



▲ **Figure 4.4.8**

Solution. Details of the computations for the case $n = 10$ are shown to six decimal places in Table 4.4.1 and the results of all the computations are given in Table 4.4.2. We showed in Example 5 that the exact area is 18 (i.e., 18 square units), so in this case the midpoint approximation is more accurate than the endpoint approximations. This is also evident geometrically from Figure 4.4.9. You can also see from the figure that in this case the left endpoint approximation overestimates the area and the right endpoint approximation underestimates it. Later in the text we will investigate the error that results when an area is approximated by the midpoint rule. ◀

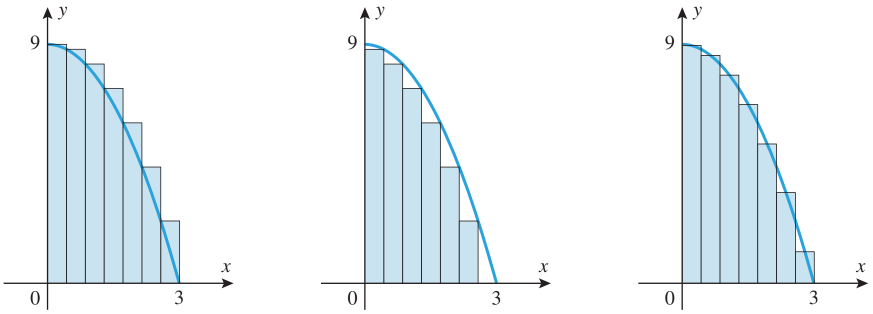
Table 4.4.1

$n = 10,\ \Delta x = (b - a)/n = (3 - 0)/10 = 0.3$

	LEFT ENDPOINT APPROXIMATION		RIGHT ENDPOINT APPROXIMATION		MIDPOINT APPROXIMATION	
k	x_k^*	$9 - (x_k^*)^2$	x_k^*	$9 - (x_k^*)^2$	x_k^*	$9 - (x_k^*)^2$
1	0.0	9.000000	0.3	8.910000	0.15	8.977500
2	0.3	8.910000	0.6	8.640000	0.45	8.797500
3	0.6	8.640000	0.9	8.190000	0.75	8.437500
4	0.9	8.190000	1.2	7.560000	1.05	7.897500
5	1.2	7.560000	1.5	6.750000	1.35	7.177500
6	1.5	6.750000	1.8	5.760000	1.65	6.277500
7	1.8	5.760000	2.1	4.590000	1.95	5.197500
8	2.1	4.590000	2.4	3.240000	2.25	3.937500
9	2.4	3.240000	2.7	1.710000	2.55	2.497500
10	2.7	1.710000	3.0	0.000000	2.85	0.877500
		64.350000		55.350000		60.075000
$\Delta x \sum_{k=1}^{n} f(x_k^*)$	(0.3)(64.350000) = 19.305000		(0.3)(55.350000) = 16.605000		(0.3)(60.075000) = 18.022500	

Table 4.4.2

n	LEFT ENDPOINT APPROXIMATION	RIGHT ENDPOINT APPROXIMATION	MIDPOINT APPROXIMATION
10	19.305000	16.605000	18.022500
20	18.663750	17.313750	18.005625
50	18.268200	17.728200	18.000900



The left endpoint approximation overestimates the area.

The right endpoint approximation underestimates the area.

The midpoint approximation is better than the endpoint approximations.

▲ **Figure 4.4.9**

NET SIGNED AREA

In Definition 4.4.3 we assumed that f is continuous and nonnegative on the interval $[a, b]$. If f is continuous and attains both positive and negative values on $[a, b]$, then the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{7}$$

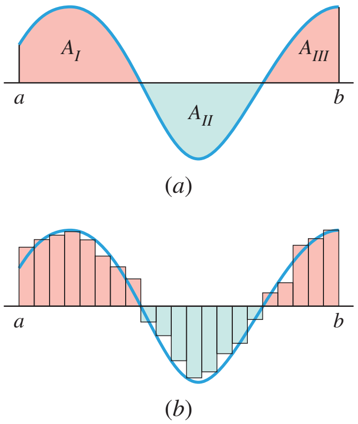
no longer represents the area between the curve $y = f(x)$ and the interval $[a, b]$ on the x-axis; rather, it represents a difference of areas—the area of the region that is above the interval $[a, b]$ and below the curve $y = f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y = f(x)$. We call this the ***net signed area*** between the graph of $y = f(x)$ and the interval $[a, b]$. For example, in Figure 4.4.10a, the net signed area between the curve $y = f(x)$ and the interval $[a, b]$ is

$$(A_I + A_{III}) - A_{II} = \big[\text{area above } [a, b]\big] - \big[\text{area below } [a, b]\big]$$

To explain why the limit in (7) represents this net signed area, let us subdivide the interval $[a, b]$ in Figure 4.4.10a into n equal subintervals and examine the terms in the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \tag{8}$$

If $f(x_k^*)$ is positive, then the product $f(x_k^*)\Delta x$ represents the area of the rectangle with height $f(x_k^*)$ and base Δx (the pink rectangles in Figure 4.4.10b). However, if $f(x_k^*)$ is negative, then the product $f(x_k^*)\Delta x$ is the *negative* of the area of the rectangle with height $|f(x_k^*)|$ and base Δx (the green rectangles in Figure 4.4.10b). Thus, (8) represents the total area of the pink rectangles minus the total area of the green rectangles. As n increases, the pink rectangles fill out the regions with areas A_I and A_{III} and the green rectangles fill out the region with area A_{II}, which explains why the limit in (7) represents the signed area between $y = f(x)$ and the interval $[a, b]$. We formalize this in the following definition.



(a)

(b)

Figure 4.4.10

As with Definition 4.4.3, it can be proved that the limit in (9) always exists and that the same value of A results no matter how the points in the subintervals are chosen.

4.4.5 DEFINITION (*Net Signed Area*) If the function f is continuous on $[a, b]$, then the ***net signed area*** A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \tag{9}$$

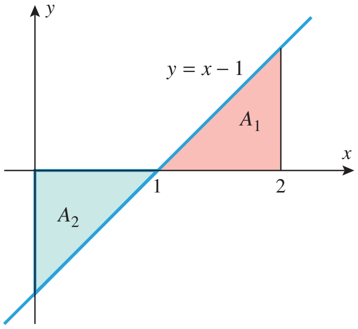


Figure 4.4.11

Figure 4.4.11 shows the graph of $f(x) = x - 1$ over the interval $[0, 2]$. It is geometrically evident that the areas A_1 and A_2 in that figure are equal, so we expect the net signed area between the graph of f and the interval $[0, 2]$ to be zero.

▶ **Example 7** Confirm that the net signed area between the graph of $f(x) = x - 1$ and the interval $[0, 2]$ is zero by using Definition 4.4.5 with x_k^* chosen to be the left endpoint of each subinterval.

Solution. Each subinterval has length

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

so it follows from (3) that

$$x_k^* = a + (k-1)\Delta x = (k-1)\left(\frac{2}{n}\right)$$

Thus,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}(x_k^* - 1)\Delta x = \sum_{k=1}^{n}\left[(k-1)\left(\frac{2}{n}\right) - 1\right]\left(\frac{2}{n}\right)$$
$$= \sum_{k=1}^{n}\left[\left(\frac{4}{n^2}\right)k - \frac{4}{n^2} - \frac{2}{n}\right]$$

from which it follows that

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to+\infty}\left[4\left(\frac{1}{n^2}\sum_{k=1}^{n}k\right) - \frac{4}{n}\left(\frac{1}{n}\sum_{k=1}^{n}1\right) - 2\left(\frac{1}{n}\sum_{k=1}^{n}1\right)\right]$$
$$= 4\left(\frac{1}{2}\right) - 0\cdot 1 - 2\cdot 1 = 0 \qquad \boxed{\text{Theorem 4.4.4}}$$

This confirms that the net signed area is zero. ◀

✔QUICK CHECK EXERCISES 4.4 *(See page 299 for answers.)*

1. (a) Write the sum in two ways:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8} = \sum_{k=1}^{4}\underline{\hspace{2cm}} = \sum_{j=0}^{3}\underline{\hspace{2cm}}$$

(b) Express the sum $10 + 10^2 + 10^3 + 10^4 + 10^5$ using sigma notation.

2. Express the sums in closed form.

(a) $\sum_{k=1}^{n} k$ (b) $\sum_{k=1}^{n}(6k+1)$ (c) $\sum_{k=1}^{n} k^2$

3. Divide the interval $[1, 3]$ into $n = 4$ subintervals of equal length.

(a) Each subinterval has width ______.

(b) The left endpoints of the subintervals are ______.

(c) The midpoints of the subintervals are ______.

(d) The right endpoints of the subintervals are ______.

4. Find the left endpoint approximation for the area between the curve $y = x^2$ and the interval $[1, 3]$ using $n = 4$ equal subdivisions of the interval.

5. The right endpoint approximation for the net signed area between $y = f(x)$ and an interval $[a, b]$ is given by

$$\sum_{k=1}^{n}\frac{6k+1}{n^2}$$

Find the exact value of this net signed area.

EXERCISE SET 4.4 $\boxed{\text{C}}$ CAS

1. Evaluate.

(a) $\sum_{k=1}^{3} k^3$ (b) $\sum_{j=2}^{6}(3j-1)$ (c) $\sum_{i=-4}^{1}(i^2 - i)$

(d) $\sum_{n=0}^{5} 1$ (e) $\sum_{k=0}^{4}(-2)^k$ (f) $\sum_{n=1}^{6}\sin n\pi$

2. Evaluate.

(a) $\sum_{k=1}^{4} k\sin\frac{k\pi}{2}$ (b) $\sum_{j=0}^{5}(-1)^j$ (c) $\sum_{i=7}^{20}\pi^2$

(d) $\sum_{m=3}^{5} 2^{m+1}$ (e) $\sum_{n=1}^{6}\sqrt{n}$ (f) $\sum_{k=0}^{10}\cos k\pi$

3–8 Write each expression in sigma notation but do not evaluate. ■

3. $1 + 2 + 3 + \cdots + 10$

4. $3\cdot 1 + 3\cdot 2 + 3\cdot 3 + \cdots + 3\cdot 20$

5. $2 + 4 + 6 + 8 + \cdots + 20$

6. $1 + 3 + 5 + 7 + \cdots + 15$

7. $1 - 3 + 5 - 7 + 9 - 11$ **8.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

9. (a) Express the sum of the even integers from 2 to 100 in sigma notation.

(b) Express the sum of the odd integers from 1 to 99 in sigma notation.

10. Express in sigma notation.

(a) $a_1 - a_2 + a_3 - a_4 + a_5$

(b) $-b_0 + b_1 - b_2 + b_3 - b_4 + b_5$

(c) $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

(d) $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$

11–16 Use Theorem 4.4.2 to evaluate the sums. Check your answers using the summation feature of a calculating utility. ■

11. $\displaystyle\sum_{k=1}^{100} k$ **12.** $\displaystyle\sum_{k=1}^{100} (7k+1)$ **13.** $\displaystyle\sum_{k=1}^{20} k^2$

14. $\displaystyle\sum_{k=4}^{20} k^2$ **15.** $\displaystyle\sum_{k=1}^{30} k(k-2)(k+2)$

16. $\displaystyle\sum_{k=1}^{6} (k-k^3)$

17–20 Express the sums in closed form. ■

17. $\displaystyle\sum_{k=1}^{n} \frac{3k}{n}$ **18.** $\displaystyle\sum_{k=1}^{n-1} \frac{k^2}{n}$ **19.** $\displaystyle\sum_{k=1}^{n-1} \frac{k^3}{n^2}$

20. $\displaystyle\sum_{k=1}^{n} \left(\frac{5}{n} - \frac{2k}{n}\right)$

21–24 True–False Determine whether the statement is true or false. Explain your answer. ■

21. For all positive integers n

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

22. The midpoint approximation is the average of the left endpoint approximation and the right endpoint approximation.

23. Every right endpoint approximation for the area under the graph of $y = x^2$ over an interval $[a, b]$ will be an overestimate.

24. For any continuous function f, the area between the graph of f and an interval $[a, b]$ (on which f is defined) is equal to the absolute value of the net signed area between the graph of f and the interval $[a, b]$.

FOCUS ON CONCEPTS

25. (a) Write the first three and final two summands in the sum

$$\sum_{k=1}^{n} \left(2 + k \cdot \frac{3}{n}\right)^4 \frac{3}{n}$$

Explain why this sum gives the right endpoint approximation for the area under the curve $y = x^4$ over the interval $[2, 5]$.

(b) Show that a change in the index range of the sum in part (a) can produce the left endpoint approximation for the area under the curve $y = x^4$ over the interval $[2, 5]$.

26. For a function f that is continuous on $[a, b]$, Definition 4.4.5 says that the net signed area A between $y = f(x)$ and the interval $[a, b]$ is

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Give geometric interpretations for the symbols n, x_k^*, and Δx. Explain how to interpret the limit in this definition.

27–30 Divide the specified interval into $n = 4$ subintervals of equal length and then compute

$$\sum_{k=1}^{4} f(x_k^*)\Delta x$$

with x_k^* as (a) the left endpoint of each subinterval, (b) the midpoint of each subinterval, and (c) the right endpoint of each subinterval. Illustrate each part with a graph of f that includes the rectangles whose areas are represented in the sum. ■

27. $f(x) = 3x + 1$; $[2, 6]$ **28.** $f(x) = 1/x$; $[1, 9]$

29. $f(x) = \cos x$; $[0, \pi]$ **30.** $f(x) = 2x - x^2$; $[-1, 3]$

C **31–34** Use a calculating utility with summation capabilities or a CAS to obtain an approximate value for the area between the curve $y = f(x)$ and the specified interval with $n = 10$, 20, and 50 subintervals using the (a) left endpoint, (b) midpoint, and (c) right endpoint approximations. ■

31. $f(x) = 1/x$; $[1, 2]$ **32.** $f(x) = 1/x^2$; $[1, 3]$

33. $f(x) = \sqrt{x}$; $[0, 4]$ **34.** $f(x) = \sin x$; $[0, \pi/2]$

35–40 Use Definition 4.4.3 with x_k^* as the *right* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

35. $f(x) = x/2$; $[1, 4]$ **36.** $f(x) = 5 - x$; $[0, 5]$

37. $f(x) = 9 - x^2$; $[0, 3]$ **38.** $f(x) = 4 - \frac{1}{4}x^2$; $[0, 3]$

39. $f(x) = x^3$; $[2, 6]$ **40.** $f(x) = 1 - x^3$; $[-3, -1]$

41–44 Use Definition 4.4.3 with x_k^* as the *left* endpoint of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

41. $f(x) = x/2$; $[1, 4]$ **42.** $f(x) = 5 - x$; $[0, 5]$

43. $f(x) = 9 - x^2$; $[0, 3]$ **44.** $f(x) = 4 - \frac{1}{4}x^2$; $[0, 3]$

45–48 Use Definition 4.4.3 with x_k^* as the *midpoint* of each subinterval to find the area under the curve $y = f(x)$ over the specified interval. ■

45. $f(x) = 2x$; $[0, 4]$ **46.** $f(x) = 6 - x$; $[1, 5]$

47. $f(x) = x^2$; $[0, 1]$ **48.** $f(x) = x^2$; $[-1, 1]$

49–52 Use Definition 4.4.5 with x_k^* as the *right* endpoint of each subinterval to find the net signed area between the curve $y = f(x)$ and the specified interval. ■

49. $f(x) = x$; $[-1, 1]$. Verify your answer with a simple geometric argument.

50. $f(x) = x$; $[-1, 2]$. Verify your answer with a simple geometric argument.

51. $f(x) = x^2 - 1$; $[0, 2]$ **52.** $f(x) = x^3$; $[-1, 1]$

53. (a) Show that the area under the graph of $y = x^3$ and over the interval $[0, b]$ is $b^4/4$.

(b) Find a formula for the area under $y = x^3$ over the interval $[a, b]$, where $a \geq 0$.

54. Find the area between the graph of $y = \sqrt{x}$ and the interval $[0, 1]$. [*Hint:* Use the result of Exercise 21 of Section 4.1.]

55. An artist wants to create a rough triangular design using uniform square tiles glued edge to edge. She places n tiles in a row to form the base of the triangle and then makes each successive row two tiles shorter than the preceding row. Find a formula for the number of tiles used in the design. [*Hint:* Your answer will depend on whether n is even or odd.]

56. An artist wants to create a sculpture by gluing together uniform spheres. She creates a rough rectangular base that has 50 spheres along one edge and 30 spheres along the other. She then creates successive layers by gluing spheres in the grooves of the preceding layer. How many spheres will there be in the sculpture?

57–60 Consider the sum

$$\begin{aligned}\sum_{k=1}^{4}[(k+1)^3-k^3] &= [5^3-4^3]+[4^3-3^3]\\ &\quad +[3^3-2^3]+[2^3-1^3]\\ &= 5^3-1^3 = 124\end{aligned}$$

For convenience, the terms are listed in reverse order. Note how cancellation allows the entire sum to collapse like a telescope. A sum is said to ***telescope*** when part of each term cancels part of an adjacent term, leaving only portions of the first and last terms uncanceled. Evaluate the telescoping sums in these exercises.

57. $\displaystyle\sum_{k=5}^{17}(3^k-3^{k-1})$

58. $\displaystyle\sum_{k=1}^{50}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

59. $\displaystyle\sum_{k=2}^{20}\left(\frac{1}{k^2}-\frac{1}{(k-1)^2}\right)$

60. $\displaystyle\sum_{k=1}^{100}(2^{k+1}-2^k)$

61. (a) Show that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

$$\left[\textit{Hint: } \frac{1}{(2n-1)(2n+1)}=\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$

(b) Use the result in part (a) to find

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}$$

62. (a) Show that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

$$\left[\textit{Hint: } \frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}\right]$$

(b) Use the result in part (a) to find

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)}$$

63. Let $\bar{x}$ denote the arithmetic average of the n numbers $x_1, x_2, \ldots, x_n$. Use Theorem 4.4.1 to prove that

$$\sum_{i=1}^{n}(x_i-\bar{x})=0$$

64. Let

$$S=\sum_{k=0}^{n}ar^k$$

Show that $S-rS=a-ar^{n+1}$ and hence that

$$\sum_{k=0}^{n}ar^k=\frac{a-ar^{n+1}}{1-r}\quad (r\neq 1)$$

(A sum of this form is called a ***geometric sum***.)

65. By writing out the sums, determine whether the following are valid identities.

(a) $\displaystyle\int\left[\sum_{i=1}^{n}f_i(x)\right]dx=\sum_{i=1}^{n}\left[\int f_i(x)\,dx\right]$

(b) $\displaystyle\frac{d}{dx}\left[\sum_{i=1}^{n}f_i(x)\right]=\sum_{i=1}^{n}\left[\frac{d}{dx}[f_i(x)]\right]$

66. Which of the following are valid identities?

(a) $\displaystyle\sum_{i=1}^{n}a_ib_i=\sum_{i=1}^{n}a_i\sum_{i=1}^{n}b_i$ (b) $\displaystyle\sum_{i=1}^{n}a_i^2=\left(\sum_{i=1}^{n}a_i\right)^2$

(c) $\displaystyle\sum_{i=1}^{n}\frac{a_i}{b_i}=\frac{\sum_{i=1}^{n}a_i}{\sum_{i=1}^{n}b_i}$ (d) $\displaystyle\sum_{i=1}^{n}a_i=\sum_{j=0}^{n-1}a_{j+1}$

67. Prove part (c) of Theorem 4.4.1.

68. Prove Theorem 4.4.4.

69. **Writing** What is *net signed area*? How does this concept expand our application of the rectangle method?

70. **Writing** Based on Example 6, one might conjecture that the midpoint approximation always provides a better approximation than either endpoint approximation. Discuss the merits of this conjecture.

QUICK CHECK ANSWERS 4.4

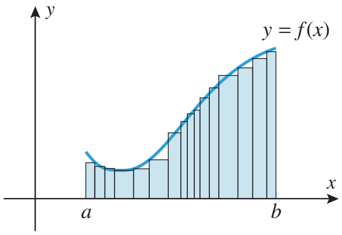
1. (a) $\dfrac{1}{2k}$; $\dfrac{1}{2(j+1)}$ (b) $\displaystyle\sum_{k=1}^{5}10^k$ **2.** (a) $\dfrac{n(n+1)}{2}$ (b) $3n(n+1)+n$ (c) $\dfrac{n(n+1)(2n+1)}{6}$ **3.** (a) 0.5 (b) 1, 1.5, 2, 2.5 (c) 1.25, 1.75, 2.25, 2.75 (d) 1.5, 2, 2.5, 3 **4.** 6.75 **5.** $\displaystyle\lim_{n\to+\infty}\frac{3n^2+4n}{n^2}=3$

4.5 THE DEFINITE INTEGRAL

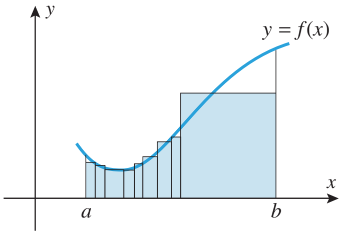
In this section we will introduce the concept of a "definite integral," which will link the concept of area to other important concepts such as length, volume, density, probability, and work.

RIEMANN SUMS AND THE DEFINITE INTEGRAL

In our definition of net signed area (Definition 4.4.5), we assumed that for each positive number n, the interval $[a, b]$ was subdivided into n subintervals of equal length to create bases for the approximating rectangles. For some functions it may be more convenient to use rectangles with different widths (see Making Connections Exercises 2 and 3); however, if we are to "exhaust" an area with rectangles of different widths, then it is important that successive subdivisions be constructed in such a way that the widths of all the rectangles approach zero as n increases (Figure 4.5.1). Thus, we must preclude the kind of situation that occurs in Figure 4.5.2 in which the right half of the interval is never subdivided. If this kind of subdivision were allowed, the error in the approximation would not approach zero as n increased.



▲ **Figure 4.5.1**



▲ **Figure 4.5.2**

A ***partition*** of the interval $[a, b]$ is a collection of points

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

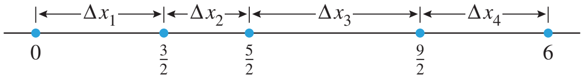
that divides $[a, b]$ into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1, \quad \Delta x_3 = x_3 - x_2, \ldots, \quad \Delta x_n = x_n - x_{n-1}$$

The partition is said to be ***regular*** provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

For a regular partition, the widths of the approximating rectangles approach zero as n is made large. Since this need not be the case for a general partition, we need some way to measure the "size" of these widths. One approach is to let $\max \Delta x_k$ denote the largest of the subinterval widths. The magnitude $\max \Delta x_k$ is called the ***mesh size*** of the partition. For example, Figure 4.5.3 shows a partition of the interval $[0, 6]$ into four subintervals with a mesh size of 2.



$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2$

▶ **Figure 4.5.3**

If we are to generalize Definition 4.4.5 so that it allows for unequal subinterval widths, we must replace the constant length Δx by the variable length Δx_k. When this is done the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \quad \text{is replaced by} \quad \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

We also need to replace the expression $n \to +\infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression $\max \Delta x_k \to 0$ for this purpose. Based on our intuitive concept of area, we would then expect the net signed area A between the graph of f and the interval $[a, b]$ to satisfy the equation

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Some writers use the symbol $\|\Delta\|$ rather than $\max \Delta x_k$ for the mesh size of the partition, in which case $\max \Delta x_k \to 0$ would be replaced by $\|\Delta\| \to 0$.

(We will see in a moment that this is the case.) The limit that appears in this expression is one of the fundamental concepts of integral calculus and forms the basis for the following definition.

4.5.1 DEFINITION A function f is said to be ***integrable*** on a finite closed interval $[a, b]$ if the limit

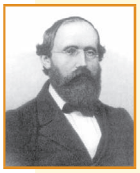
$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

which is called the ***definite integral*** of f from a to b. The numbers a and b are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and $f(x)$ is called the ***integrand***.

The notation used for the definite integral deserves some comment. Historically, the expression "$f(x)\,dx$" was interpreted to be the "infinitesimal area" of a rectangle with height $f(x)$ and "infinitesimal" width dx. By "summing" these infinitesimal areas, the entire area under the curve was obtained. The integral symbol "$\int$" is an "elongated s" that was used to indicate this summation. For us, the integral symbol "$\int$" and the symbol "dx" can serve as reminders that the definite integral is actually a limit of a *summation* as $\Delta x_k \to 0$. The sum that appears in Definition 4.5.1 is called a ***Riemann sum***, and the definite integral is sometimes called the ***Riemann integral*** in honor of the German mathematician Bernhard Riemann who formulated many of the basic concepts of integral calculus. (The reason for the similarity in notation between the definite integral and the indefinite integral will become clear in the next section, where we will establish a link between the two types of "integration.")



Georg Friedrich Bernhard Riemann (1826–1866) German mathematician. Bernhard Riemann, as he is commonly known, was the son of a Protestant minister. He received his elementary education from his father and showed brilliance in arithmetic at an early age. In 1846 he enrolled at Göttingen University to study theology and philology, but he soon transferred to mathematics. He studied physics under W. E. Weber and mathematics under Carl Friedrich Gauss, whom some people consider to be the greatest mathematician who ever lived. In 1851 Riemann received his Ph.D. under Gauss, after which he remained at Göttingen to teach. In 1862, one month after his marriage, Riemann suffered an attack of pleuritis, and for the remainder of his life was an extremely sick man. He finally succumbed to tuberculosis in 1866 at age 39.

An interesting story surrounds Riemann's work in geometry. For his introductory lecture prior to becoming an associate professor, Riemann submitted three possible topics to Gauss. Gauss surprised Riemann by choosing the topic Riemann liked the least, the foundations of geometry. The lecture was like a scene from a movie. The old and failing Gauss, a giant in his day, watching intently as his brilliant and youthful protégé skillfully pieced together portions of the old man's own work into a complete and beautiful system. Gauss is said to have gasped with delight as the lecture neared its end, and on the way home he marveled at his student's brilliance. Gauss died shortly thereafter. The results presented by Riemann that day eventually evolved into a fundamental tool that Einstein used some 50 years later to develop relativity theory.

In addition to his work in geometry, Riemann made major contributions to the theory of complex functions and mathematical physics. The notion of the definite integral, as it is presented in most basic calculus courses, is due to him. Riemann's early death was a great loss to mathematics, for his mathematical work was brilliant and of fundamental importance.

[*Image: http://commons.wikimedia.org/wiki/File:Georg_Friedrich_Bernhard_Riemann.jpeg*]

The limit that appears in Definition 4.5.1 is somewhat different from the kinds of limits discussed in Chapter 1. Loosely phrased, the expression

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = L$$

is intended to convey the idea that we can force the Riemann sums to be as close as we please to L, regardless of how the values of x_k^* are chosen, by making the mesh size of the partition sufficiently small. While it is possible to give a more formal definition of this limit, we will simply rely on intuitive arguments when applying Definition 4.5.1.

Although a function need not be continuous on an interval to be integrable on that interval (Exercise 42), we will be interested primarily in definite integrals of continuous functions. The following theorem, which we will state without proof, says that if a function is continuous on a finite closed interval, then it is integrable on that interval, and its definite integral is the net signed area between the graph of the function and the interval.

4.5.2 THEOREM *If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is*

$$A = \int_a^b f(x)\, dx \tag{1}$$

REMARK Formula (1) follows from the integrability of f, since the integrability allows us to use any partitions to evaluate the integral. In particular, if we use *regular* partitions of $[a, b]$, then

$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

for all values of k. This implies that $\max \Delta x_k = (b-a)/n$, from which it follows that $\max \Delta x_k \to 0$ if and only if $n \to +\infty$. Thus,

$$\int_a^b f(x)\, dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = A$$

In the simplest cases, definite integrals of continuous functions can be calculated using formulas from plane geometry to compute signed areas.

In Example 1, it is understood that the units of area are the squared units of length, even though we have not stated the units of length explicitly.

▶ **Example 1** Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

$$\text{(a)}\ \int_1^4 2\, dx \qquad \text{(b)}\ \int_{-1}^{2} (x+2)\, dx \qquad \text{(c)}\ \int_0^1 \sqrt{1-x^2}\, dx$$

Solution (a). The graph of the integrand is the horizontal line $y = 2$, so the region is a rectangle of height 2 extending over the interval from 1 to 4 (Figure 4.5.4*a*). Thus,

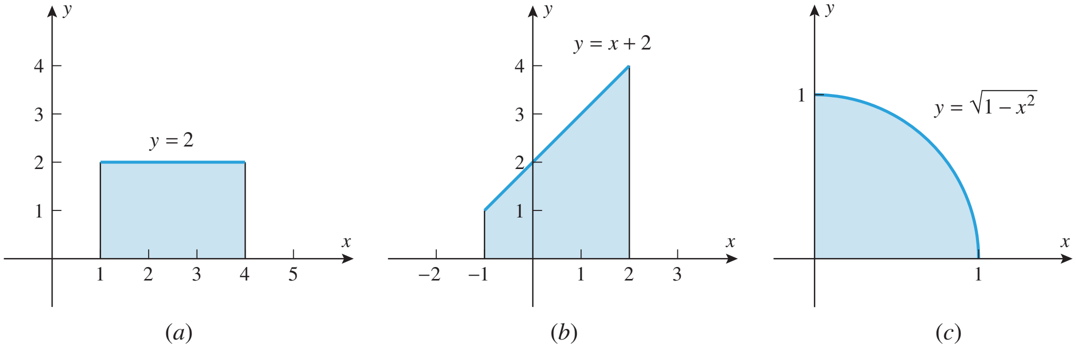
$$\int_1^4 2\, dx = (\text{area of rectangle}) = 2(3) = 6$$

Solution (b). The graph of the integrand is the line $y = x + 2$, so the region is a trapezoid whose base extends from $x = -1$ to $x = 2$ (Figure 4.5.4*b*). Thus,

$$\int_{-1}^{2} (x+2)\, dx = (\text{area of trapezoid}) = \frac{1}{2}(1+4)(3) = \frac{15}{2}$$

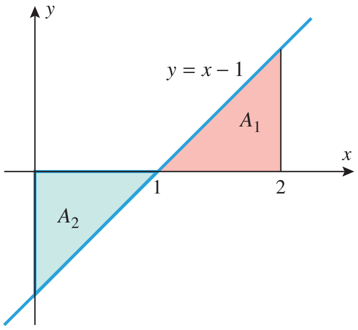
Solution (c). The graph of $y = \sqrt{1 - x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = 0$ to $x = 1$ (Figure 4.5.4c). Thus,

$$\int_0^1 \sqrt{1 - x^2}\,dx = (\text{area of quarter-circle}) = \frac{1}{4}\pi(1^2) = \frac{\pi}{4} \quad ◀$$



(a) (b) (c)

▲ **Figure 4.5.4**



▲ **Figure 4.5.5**

▶ **Example 2** Evaluate

$$\text{(a)} \int_0^2 (x-1)\,dx \qquad \text{(b)} \int_0^1 (x-1)\,dx$$

Solution. The graph of $y = x - 1$ is shown in Figure 4.5.5, and we leave it for you to verify that the shaded triangular regions both have area $\frac{1}{2}$. Over the interval $[0, 2]$ the net signed area is $A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = 0$, and over the interval $[0, 1]$ the net signed area is $-A_2 = -\frac{1}{2}$. Thus,

$$\int_0^2 (x-1)\,dx = 0 \quad \text{and} \quad \int_0^1 (x-1)\,dx = -\tfrac{1}{2}$$

(Recall that in Example 7 of Section 4.4, we used Definition 4.4.5 to show that the net signed area between the graph of $y = x - 1$ and the interval $[0, 2]$ is zero.) ◀

PROPERTIES OF THE DEFINITE INTEGRAL

It is assumed in Definition 4.5.1 that $[a, b]$ is a finite closed interval with $a < b$, and hence the upper limit of integration in the definite integral is greater than the lower limit of integration. However, it will be convenient to extend this definition to allow for cases in which the upper and lower limits of integration are equal or the lower limit of integration is greater than the upper limit of integration. For this purpose we make the following special definitions.

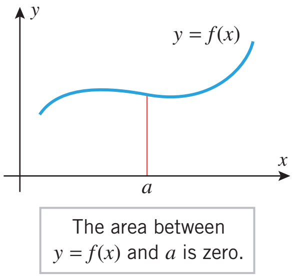
4.5.3 DEFINITION

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\,dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$



The area between $y = f(x)$ and a is zero.

▲ **Figure 4.5.6**

Part (a) of this definition is consistent with the intuitive idea that the area between a point on the x-axis and a curve $y = f(x)$ should be zero (Figure 4.5.6). Part (b) of the definition is simply a useful convention; it states that interchanging the limits of integration reverses the sign of the integral.

▶ Example 3

(a) $$\int_1^1 x^2\,dx = 0$$

(b) $$\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}\,dx = -\frac{\pi}{4} \blacktriangleleft$$

Example 1(c)

Because definite integrals are defined as limits, they inherit many of the properties of limits. For example, we know that constants can be moved through limit signs and that the limit of a sum or difference is the sum or difference of the limits. Thus, you should not be surprised by the following theorem, which we state without formal proof.

4.5.4 THEOREM *If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f + g$, and $f - g$ are integrable on $[a, b]$ and*

(a) $$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

(b) $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(c) $$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

▶ Example 4

Evaluate

$$\int_0^1 (5 - 3\sqrt{1-x^2}\,)\,dx$$

Solution. From parts (a) and (c) of Theorem 4.5.4 we can write

$$\int_0^1 (5 - 3\sqrt{1-x^2}\,)\,dx = \int_0^1 5\,dx - \int_0^1 3\sqrt{1-x^2}\,dx = \int_0^1 5\,dx - 3\int_0^1 \sqrt{1-x^2}\,dx$$

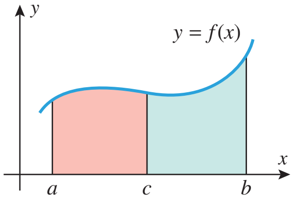
The first integral in this difference can be interpreted as the area of a rectangle of height 5 and base 1, so its value is 5, and from Example 1 the value of the second integral is $\pi/4$. Thus,

$$\int_0^1 (5 - 3\sqrt{1-x^2}\,)\,dx = 5 - 3\left(\frac{\pi}{4}\right) = 5 - \frac{3\pi}{4} \blacktriangleleft$$

Part (b) of Theorem 4.5.4 can be extended to more than two functions. More precisely,

$$\int_a^b [f_1(x) + f_2(x) + \cdots + f_n(x)]\,dx$$
$$= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx + \cdots + \int_a^b f_n(x)\,dx$$

Some properties of definite integrals can be motivated by interpreting the integral as an area. For example, if f is continuous and nonnegative on the interval $[a, b]$, and if c is a point between a and b, then the area under $y = f(x)$ over the interval $[a, b]$ can be split into two parts and expressed as the area under the graph from a to c plus the area under the graph from c to b (Figure 4.5.7), that is,



▲ **Figure 4.5.7**

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This is a special case of the following theorem about definite integrals, which we state without proof.

4.5.5 THEOREM *If f is integrable on a closed interval containing the three points a, b, and c, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the points are ordered.

The following theorem, which we state without formal proof, can also be motivated by interpreting definite integrals as areas.

Part (*b*) of Theorem 4.5.6 states that the direction (sometimes called the *sense*) of the inequality $f(x) \geq g(x)$ is unchanged if one integrates both sides. Moreover, if $b > a$, then both parts of the theorem remain true if $\geq$ is replaced by $\leq$, $>$, or $<$ throughout.

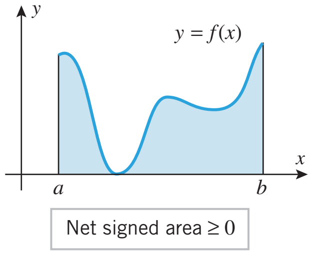
4.5.6 THEOREM

(*a*) *If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then*

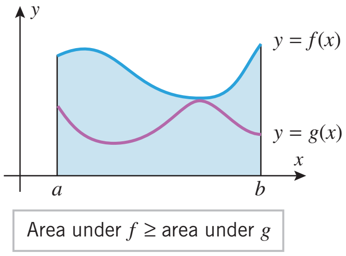
$$\int_a^b f(x)\,dx \geq 0$$

(*b*) *If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$



▲ **Figure 4.5.8**

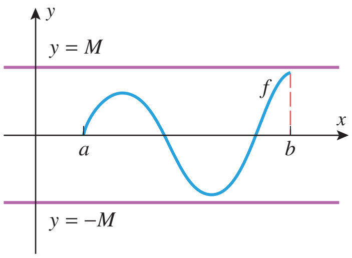


▲ **Figure 4.5.9**

Geometrically, part (*a*) of this theorem states the obvious fact that if f is nonnegative on $[a, b]$, then the net signed area between the graph of f and the interval $[a, b]$ is also nonnegative (Figure 4.5.8). Part (*b*) has its simplest interpretation when f and g are nonnegative on $[a, b]$, in which case the theorem states that if the graph of f does not go below the graph of g, then the area under the graph of f is at least as large as the area under the graph of g (Figure 4.5.9).

DISCONTINUITIES AND INTEGRABILITY

In the late nineteenth and early twentieth centuries, mathematicians began to investigate conditions under which the limit that defines an integral fails to exist, that is, conditions under which a function fails to be integrable. The matter is quite complex and beyond the scope of this text. However, there are a few basic results about integrability that are important to know; we begin with a definition.



f is bounded on $[a, b]$.

▲ **Figure 4.5.10**

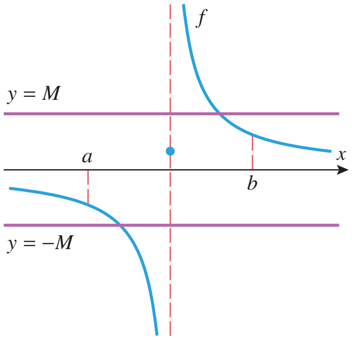
4.5.7 DEFINITION A function f that is defined on an interval is said to be ***bounded*** on the interval if there is a positive number M such that

$$-M \leq f(x) \leq M$$

for all x in the interval. Geometrically, this means that the graph of f over the interval lies between the lines $y = -M$ and $y = M$.

For example, a continuous function f is bounded on *every* finite closed interval because the Extreme-Value Theorem (3.4.2) implies that f has an absolute maximum and an absolute minimum on the interval; hence, its graph will lie between the lines $y = -M$ and $y = M$, provided we make M large enough (Figure 4.5.10). In contrast, a function that has a vertical asymptote inside of an interval is not bounded on that interval because its graph over the interval cannot be made to lie between the lines $y = -M$ and $y = M$, no matter how large we make the value of M (Figure 4.5.11).

The following theorem, which we state without proof, provides some facts about integrability for functions with discontinuities. In the exercises we have included some problems that are concerned with this theorem (Exercises 42, 43, and 44).



f is not bounded on $[a, b]$.

▲ **Figure 4.5.11**

4.5.8 THEOREM *Let f be a function that is defined on the finite closed interval $[a, b]$.*

(*a*) *If f has finitely many discontinuities in $[a, b]$ but is bounded on $[a, b]$, then f is integrable on $[a, b]$.*

(*b*) *If f is not bounded on $[a, b]$, then f is not integrable on $[a, b]$.*

✓ QUICK CHECK EXERCISES 4.5 *(See page 309 for answers.)*

1. In each part, use the partition of [2, 7] in the accompanying figure.

▲ **Figure Ex-1**

(a) What is n, the number of subintervals in this partition?

(b) $x_0 =$ ________; $x_1 =$ ________; $x_2 =$ ________;
$x_3 =$ ________; $x_4 =$ ________

(c) $\Delta x_1 =$ ________; $\Delta x_2 =$ ________; $\Delta x_3 =$ ________;
$\Delta x_4 =$ ________

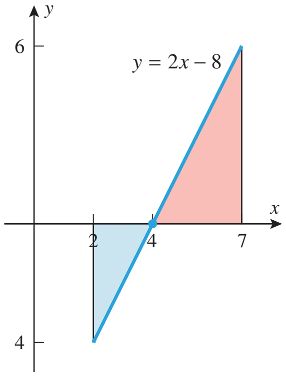
(d) The mesh of this partition is ________.

2. Let $f(x) = 2x - 8$. Use the partition of [2, 7] in Quick Check Exercise 1 and the choices $x_1^* = 2$, $x_2^* = 4$, $x_3^* = 5$, and $x_4^* = 7$ to evaluate the Riemann sum

$$\sum_{k=1}^{4} f(x_k^*)\Delta x_k$$

3. Use the accompanying figure to evaluate

$$\int_2^7 (2x - 8)\, dx$$



◀ **Figure Ex-3**

4. Suppose that $g(x)$ is a function for which

$$\int_{-2}^{1} g(x)\, dx = 5 \quad \text{and} \quad \int_{1}^{2} g(x)\, dx = -2$$

Use this information to evaluate the definite integrals.

(a) $\displaystyle\int_1^2 5g(x)\, dx$

(b) $\displaystyle\int_{-2}^2 g(x)\, dx$

(c) $\displaystyle\int_1^1 [g(x)]^2\, dx$

(d) $\displaystyle\int_2^{-2} 4g(x)\, dx$

EXERCISE SET 4.5

1–4 Find the value of

(a) $\displaystyle\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ (b) $\max \Delta x_k$. ■

1. $f(x) = x + 1;\ a = 0, b = 4;\ n = 3;$
$\Delta x_1 = 1, \Delta x_2 = 1, \Delta x_3 = 2;$
$x_1^* = \frac{1}{3}, x_2^* = \frac{3}{2}, x_3^* = 3$

2. $f(x) = \cos x;\ a = 0, b = 2\pi;\ n = 4;$
$\Delta x_1 = \pi/2, \Delta x_2 = 3\pi/4, \Delta x_3 = \pi/2, \Delta x_4 = \pi/4;$
$x_1^* = \pi/4, x_2^* = \pi, x_3^* = 3\pi/2, x_4^* = 7\pi/4$

3. $f(x) = 4 - x^2;\ a = -3, b = 4;\ n = 4;$
$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 1, \Delta x_4 = 3;$
$x_1^* = -\frac{5}{2}, x_2^* = -1, x_3^* = \frac{1}{4}, x_4^* = 3$

4. $f(x) = x^3;\ a = -3, b = 3;\ n = 4;$
$\Delta x_1 = 2, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = 2;$
$x_1^* = -2, x_2^* = 0, x_3^* = 0, x_4^* = 2$

5–8 Use the given values of a and b to express the following limits as integrals. (Do not evaluate the integrals.) ■

5. $\displaystyle\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k;\ a = -1, b = 2$

6. $\displaystyle\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k;\ a = 1, b = 2$

7. $\displaystyle\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1 - 3x_k^*)\Delta x_k;\ a = -3, b = 3$

8. $\displaystyle\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\sin^2 x_k^*)\Delta x_k;\ a = 0, b = \pi/2$

9–10 Use Definition 4.5.1 to express the integrals as limits of Riemann sums. (Do not evaluate the integrals.) ■

9. (a) $\displaystyle\int_1^2 2x\,dx$ (b) $\displaystyle\int_0^1 \frac{x}{x+1}\,dx$

10. (a) $\displaystyle\int_1^2 \sqrt{x}\,dx$ (b) $\displaystyle\int_{-\pi/2}^{\pi/2} (1 + \cos x)\,dx$

FOCUS ON CONCEPTS

11. Explain informally why Theorem 4.5.4(*a*) follows from Definition 4.5.1.

12. Explain informally why Theorem 4.5.6(*a*) follows from Definition 4.5.1.

13–16 Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed. ■

13. (a) $\displaystyle\int_0^3 x\,dx$ (b) $\displaystyle\int_{-2}^{-1} x\,dx$
(c) $\displaystyle\int_{-1}^{4} x\,dx$ (d) $\displaystyle\int_{-5}^{5} x\,dx$

14. (a) $\displaystyle\int_0^2 \left(1 - \tfrac{1}{2}x\right)dx$ (b) $\displaystyle\int_{-1}^{1} \left(1 - \tfrac{1}{2}x\right)dx$
(c) $\displaystyle\int_2^3 \left(1 - \tfrac{1}{2}x\right)dx$ (d) $\displaystyle\int_0^3 \left(1 - \tfrac{1}{2}x\right)dx$

15. (a) $\displaystyle\int_0^5 2\,dx$ (b) $\displaystyle\int_0^{\pi} \cos x\,dx$
(c) $\displaystyle\int_{-1}^{2} |2x - 3|\,dx$ (d) $\displaystyle\int_{-1}^{1} \sqrt{1 - x^2}\,dx$

16. (a) $\displaystyle\int_{-10}^{-5} 6\,dx$ (b) $\displaystyle\int_{-\pi/3}^{\pi/3} \sin x\,dx$
(c) $\displaystyle\int_0^3 |x - 2|\,dx$ (d) $\displaystyle\int_0^2 \sqrt{4 - x^2}\,dx$

17. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} |x - 2|, & x \ge 0 \\ x + 2, & x < 0 \end{cases}$$

(a) $\displaystyle\int_{-2}^{0} f(x)\,dx$ (b) $\displaystyle\int_{-2}^{2} f(x)\,dx$
(c) $\displaystyle\int_0^6 f(x)\,dx$ (d) $\displaystyle\int_{-4}^{6} f(x)\,dx$

18. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} 2x, & x \le 1 \\ 2, & x > 1 \end{cases}$$

(a) $\displaystyle\int_0^1 f(x)\,dx$ (b) $\displaystyle\int_{-1}^{1} f(x)\,dx$
(c) $\displaystyle\int_1^{10} f(x)\,dx$ (d) $\displaystyle\int_{1/2}^{5} f(x)\,dx$

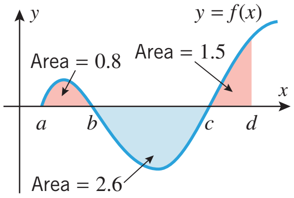
FOCUS ON CONCEPTS

19–20 Use the areas shown in the figure to find

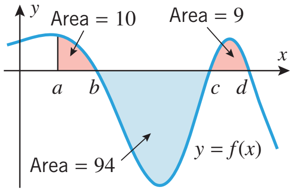
(a) $\displaystyle\int_a^b f(x)\,dx$ (b) $\displaystyle\int_b^c f(x)\,dx$

(c) $\displaystyle\int_a^c f(x)\,dx$ (d) $\displaystyle\int_a^d f(x)\,dx$. ■

19.



20.



21. Find $\int_{-1}^{2} [f(x) + 2g(x)]\,dx$ if

$$\int_{-1}^{2} f(x)\,dx = 5 \quad \text{and} \quad \int_{-1}^{2} g(x)\,dx = -3$$

22. Find $\int_{1}^{4} [3f(x) - g(x)]\,dx$ if

$$\int_{1}^{4} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{4} g(x)\,dx = 10$$

23. Find $\int_{1}^{5} f(x)\,dx$ if

$$\int_{0}^{1} f(x)\,dx = -2 \quad \text{and} \quad \int_{0}^{5} f(x)\,dx = 1$$

24. Find $\int_{3}^{-2} f(x)\,dx$ if

$$\int_{-2}^{1} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{3} f(x)\,dx = -6$$

25–28 Use Theorem 4.5.4 and appropriate formulas from geometry to evaluate the integrals. ■

25. $\int_{-1}^{3} (4 - 5x)\,dx$ **26.** $\int_{-2}^{2} (1 - 3|x|)\,dx$

27. $\int_{0}^{1} (x + 2\sqrt{1 - x^2}\,)\,dx$ **28.** $\int_{-3}^{0} (2 + \sqrt{9 - x^2}\,)\,dx$

29–32 True–False Determine whether the statement is true or false. Explain your answer. ■

29. If $f(x)$ is integrable on $[a, b]$, then $f(x)$ is continuous on $[a, b]$.

30. It is the case that

$$0 < \int_{-1}^{1} \frac{\cos x}{\sqrt{1 + x^2}}\,dx$$

31. If the integral of $f(x)$ over the interval $[a, b]$ is negative, then $f(x) \le 0$ for $a \le x \le b$.

32. The function

$$f(x) = \begin{cases} 0, & x \le 0 \\ x^2, & x > 0 \end{cases}$$

is integrable over every closed interval $[a, b]$.

33–34 Use Theorem 4.5.6 to determine whether the value of the integral is positive or negative. ■

33. (a) $\int_{2}^{3} \frac{\sqrt{x}}{1 - x}\,dx$ (b) $\int_{0}^{4} \frac{x^2}{3 - \cos x}\,dx$

34. (a) $\int_{-3}^{-1} \frac{x^4}{\sqrt{3 - x}}\,dx$ (b) $\int_{-2}^{2} \frac{x^3 - 9}{|x| + 1}\,dx$

35. Prove that if f is continuous and if $m \le f(x) \le M$ on $[a, b]$, then

$$m(b - a) \le \int_{a}^{b} f(x)\,dx \le M(b - a)$$

36. Find the maximum and minimum values of $\sqrt{x^3 + 2}$ for $0 \le x \le 3$. Use these values, and the inequalities in Exercise 35, to find bounds on the value of the integral

$$\int_{0}^{3} \sqrt{x^3 + 2}\,dx$$

37–38 Evaluate the integrals by completing the square and applying appropriate formulas from geometry. ■

37. $\int_{0}^{10} \sqrt{10x - x^2}\,dx$ **38.** $\int_{0}^{3} \sqrt{6x - x^2}\,dx$

39–40 Evaluate the limit by expressing it as a definite integral over the interval $[a, b]$ and applying appropriate formulas from geometry. ■

39. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (3x_k^* + 1)\Delta x_k;\ a = 0, b = 1$

40. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{4 - (x_k^*)^2}\,\Delta x_k;\ a = -2, b = 2$

FOCUS ON CONCEPTS

41. Let $f(x) = C$ be a constant function.

(a) Use a formula from geometry to show that

$$\int_{a}^{b} f(x)\,dx = C(b - a)$$

(b) Show that any Riemann sum for $f(x)$ over $[a, b]$ evaluates to $C(b - a)$. Use Definition 4.5.1 to show that

$$\int_{a}^{b} f(x)\,dx = C(b - a)$$

42. Define a function f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Use Definition 4.5.1 to show that

$$\int_{0}^{1} f(x)\,dx = 1$$

43. It can be shown that every interval contains both rational and irrational numbers. Accepting this to be so, do you believe that the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases}$$

is integrable on a closed interval $[a, b]$? Explain your reasoning.

44. Define the function f by

$$f(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

It follows from Theorem 4.5.8(b) that f is not integrable on the interval [0, 1]. Prove this to be the case by applying Definition 4.5.1. [*Hint:* Argue that no matter how small the mesh size is for a partition of [0, 1], there will always be a choice of x_1^* that will make the Riemann sum in Definition 4.5.1 as large as we like.]

45. In each part, use Theorems 4.5.2 and 4.5.8 to determine whether the function f is integrable on the interval $[-1, 1]$.

(a) $f(x) = \cos x$

(b) $f(x) = \begin{cases} x/|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(d) $f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$

46. **Writing** Write a short paragraph that discusses the similarities and differences between indefinite integrals and definite integrals.

47. **Writing** Write a paragraph that explains informally what it means for a function to be "integrable."

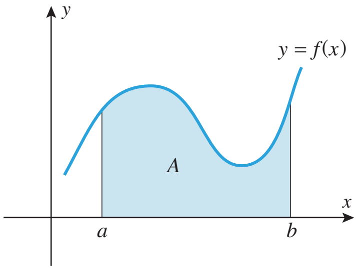
✔QUICK CHECK ANSWERS 4.5

1. (a) $n = 4$ (b) 2, 3, 4.5, 6.5, 7 (c) 1, 1.5, 2, 0.5 (d) 2 **2.** 3 **3.** 5 **4.** (a) -10 (b) 3 (c) 0 (d) -12

4.6 THE FUNDAMENTAL THEOREM OF CALCULUS

In this section we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the "Fundamental Theorem of Calculus." One part of this theorem will relate the rectangle and antiderivative methods for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.

THE FUNDAMENTAL THEOREM OF CALCULUS

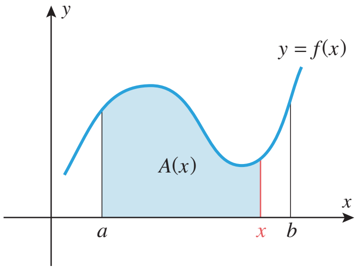


▲ **Figure 4.6.1**

As in earlier sections, let us begin by assuming that f is nonnegative and continuous on an interval $[a, b]$, in which case the area A under the graph of f over the interval $[a, b]$ is represented by the definite integral

$$A = \int_a^b f(x)\,dx \tag{1}$$

(Figure 4.6.1).



▲ **Figure 4.6.2**

Recall that our discussion of the antiderivative method in Section 4.1 suggested that if $A(x)$ is the area under the graph of f from a to x (Figure 4.6.2), then

- $A'(x) = f(x)$
- $A(a) = 0$ The area under the curve from a to a is the area above the single point a, and hence is zero.
- $A(b) = A$ The area under the curve from a to b is A.

The formula $A'(x) = f(x)$ states that $A(x)$ is an antiderivative of $f(x)$, which implies that every other antiderivative of $f(x)$ on $[a, b]$ can be obtained by adding a constant to $A(x)$. Accordingly, let

$$F(x) = A(x) + C$$

be any antiderivative of $f(x)$, and consider what happens when we subtract $F(a)$ from $F(b)$:

$$F(b) - F(a) = [A(b) + C] - [A(a) + C] = A(b) - A(a) = A - 0 = A$$

Hence (1) can be expressed as

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

In words, this equation states:

The definite integral can be evaluated by finding any antiderivative of the integrand and then subtracting the value of this antiderivative at the lower limit of integration from its value at the upper limit of integration.

Although our evidence for this result assumed that f is nonnegative on $[a, b]$, this assumption is not essential.

4.6.1 THEOREM (*The Fundamental Theorem of Calculus, Part 1*) *If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{2}$$

PROOF Let $x_1, x_2, \ldots, x_{n-1}$ be any points in $[a, b]$ such that

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

These values divide $[a, b]$ into n subintervals

$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b] \tag{3}$$

whose lengths, as usual, we denote by

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_n$$

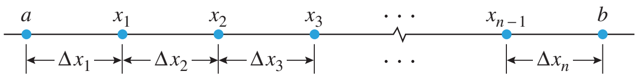
(see Figure 4.6.3). By hypothesis, $F'(x) = f(x)$ for all x in $[a, b]$, so F satisfies the hypotheses of the Mean-Value Theorem (3.8.2) on each subinterval in (3). Hence, we can find points $x_1^*, x_2^*, \ldots, x_n^*$ in the respective subintervals in (3) such that

$$\begin{aligned}
F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) = f(x_1^*)\Delta x_1\\
F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2\\
F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) = f(x_3^*)\Delta x_3\\
&\;\;\vdots\\
F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\Delta x_n
\end{aligned}$$

Adding the preceding equations yields

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{4}$$

Let us now increase n in such a way that $\max \Delta x_k \to 0$. Since f is assumed to be continuous, the right side of (4) approaches $\int_a^b f(x)\,dx$ by Theorem 4.5.2 and Definition 4.5.1. However,



▶ **Figure 4.6.3**

the left side of (4) is independent of n; that is, the left side of (4) remains constant as n increases. Thus,

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx \quad ■$$

It is standard to denote the difference $F(b) - F(a)$ as

$$F(x)\Big]_a^b = F(b) - F(a) \quad \text{or} \quad \Big[F(x)\Big]_a^b = F(b) - F(a)$$

For example, using the first of these notations we can express (2) as

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big]_a^b} \tag{5}$$

We will sometimes write

$$F(x)\Big]_{x=a}^b = F(b) - F(a)$$

when it is important to emphasize that a and b are values for the variable x.

The integral in Example 1 represents the area of a certain trapezoid. Sketch the trapezoid, and find its area using geometry.

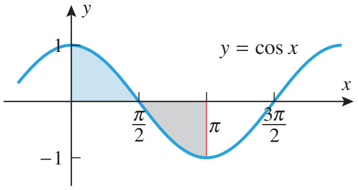
▶ **Example 1** Evaluate $\int_1^2 x\,dx$.

Solution. The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$; thus, from (2)

$$\int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2} \quad ◀$$

▶ **Example 2** In Example 5 of Section 4.4 we used the definition of area to show that the area under the graph of $y = 9 - x^2$ over the interval [0, 3] is 18 (square units). We can now solve that problem much more easily using the Fundamental Theorem of Calculus:

$$A = \int_0^3 (9 - x^2)\,dx = \left[9x - \frac{x^3}{3}\right]_0^3 = \left(27 - \frac{27}{3}\right) - 0 = 18 \quad ◀$$



▲ **Figure 4.6.4**

▶ **Example 3**

(a) Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$ (Figure 4.6.4).

(b) Make a conjecture about the value of the integral

$$\int_0^\pi \cos x\,dx$$

and confirm your conjecture using the Fundamental Theorem of Calculus.

Solution (a). Since $\cos x \geq 0$ over the interval $[0, \pi/2]$, the area A under the curve is

$$A = \int_0^{\pi/2} \cos x\,dx = \sin x\Big]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$$

Solution (b). The given integral can be interpreted as the signed area between the graph of $y = \cos x$ and the interval $[0, \pi]$. The graph in Figure 4.6.4 suggests that over the interval $[0, \pi]$ the portion of area above the x-axis is the same as the portion of area below the x-axis, so we conjecture that the signed area is zero; this implies that the value of the integral is zero. This is confirmed by the computations

$$\int_0^{\pi} \cos x\, dx = \sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0 \blacktriangleleft$$

THE RELATIONSHIP BETWEEN DEFINITE AND INDEFINITE INTEGRALS

Observe that in the preceding examples we did not include a constant of integration in the antiderivatives. In general, when applying the Fundamental Theorem of Calculus there is no need to include a constant of integration because it will drop out anyway. To see that this is so, let F be any antiderivative of the integrand on $[a, b]$, and let C be any constant; then

$$\int_a^b f(x)\, dx = \big[F(x) + C\big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Thus, for purposes of evaluating a definite integral we can omit the constant of integration in

$$\int_a^b f(x)\, dx = \big[F(x) + C\big]_a^b$$

and express (5) as

$$\int_a^b f(x)\, dx = \int f(x)\, dx \Big]_a^b \tag{6}$$

which relates the definite and indefinite integrals.

▶ Example 4

$$\int_1^9 \sqrt{x}\, dx = \int x^{1/2}\, dx \Big]_1^9 = \frac{2}{3}x^{3/2}\Big]_1^9 = \frac{2}{3}(27 - 1) = \frac{52}{3} \blacktriangleleft$$

▶ Example 5 Table 4.2.1 will be helpful for the following computations.

$$\int_4^9 x^2\sqrt{x}\, dx = \int_4^9 x^{5/2}\, dx = \frac{2}{7}x^{7/2}\Big]_4^9 = \frac{2}{7}(2187 - 128) = \frac{4118}{7} = 588\frac{2}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\, dx = -\frac{1}{5}\cos x\Big]_0^{\pi/2} = -\frac{1}{5}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{1}{5}[0 - 1] = \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\, dx = \tan x\Big]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$\int_{-\pi/4}^{\pi/4} \sec x \tan x\, dx = \sec x\Big]_{-\pi/4}^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec\left(-\frac{\pi}{4}\right) = \sqrt{2} - \sqrt{2} = 0 \blacktriangleleft$$

TECHNOLOGY MASTERY

If you have a CAS, read the documentation on evaluating definite integrals and then check the results in Example 5.

WARNING The requirements in the Fundamental Theorem of Calculus that f be continuous on $[a, b]$ and that F be an antiderivative for f over the entire interval $[a, b]$ are important to keep in mind. Disregarding these assumptions will likely lead to incorrect results. For example, the function $f(x) = 1/x^2$ fails on two counts to be continuous at $x = 0$: $f(x)$ is not defined at $x = 0$ and $\lim_{x\to 0} f(x)$ does not exist. Thus, the Fundamental Theorem of Calculus should not be used to integrate f on any interval that contains $x = 0$. However, if we ignore this and mistakenly apply Formula (2) over the interval $[-1, 1]$, we might *incorrectly* compute $\int_{-1}^{1}(1/x^2)\,dx$ by evaluating an antiderivative, $-1/x$, at the endpoints, arriving at the answer

$$-\frac{1}{x}\bigg]_{-1}^{1} = -[1-(-1)] = -2$$

But $f(x) = 1/x^2$ is a nonnegative function, so clearly a negative value for the definite integral is impossible.

The Fundamental Theorem of Calculus can be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

Example 6

$$\int_1^1 x^2\,dx = \frac{x^3}{3}\bigg]_1^1 = \frac{1}{3} - \frac{1}{3} = 0$$

$$\int_4^0 x\,dx = \frac{x^2}{2}\bigg]_4^0 = \frac{0}{2} - \frac{16}{2} = -8$$

The latter result is consistent with the result that would be obtained by first reversing the limits of integration in accordance with Definition 4.5.3(b):

$$\int_4^0 x\,dx = -\int_0^4 x\,dx = -\frac{x^2}{2}\bigg]_0^4 = -\left[\frac{16}{2} - \frac{0}{2}\right] = -8 \blacktriangleleft$$

To integrate a continuous function that is defined piecewise on an interval $[a, b]$, split this interval into subintervals at the breakpoints of the function, and integrate separately over each subinterval in accordance with Theorem 4.5.5.

Example 7 Evaluate $\int_0^3 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Solution. See Figure 4.6.5. From Theorem 4.5.5 we can integrate from 0 to 2 and from 2 to 3 separately and add the results. This yields

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx$$

$$= \frac{x^3}{3}\bigg]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6} \blacktriangleleft$$

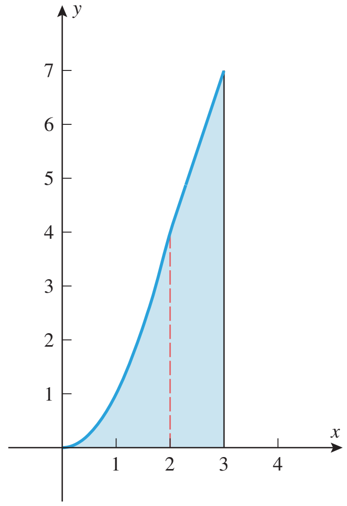
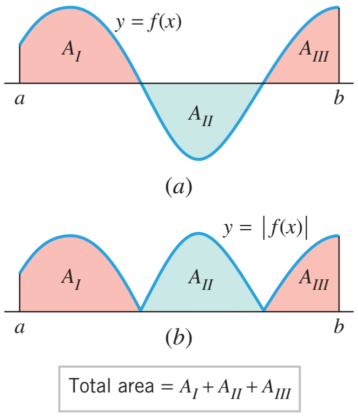


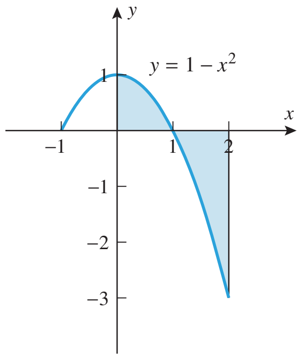
Figure 4.6.5

If f is a continuous function on the interval $[a, b]$, then we define the ***total area*** between the curve $y = f(x)$ and the interval $[a, b]$ to be

$$\text{total area} = \int_a^b |f(x)|\,dx \tag{7}$$



▲ **Figure 4.6.6**



▲ **Figure 4.6.7**

(Figure 4.6.6). To compute total area using Formula (7), begin by dividing the interval of integration into subintervals on which $f(x)$ does not change sign. On the subintervals for which $0 \leq f(x)$ replace $|f(x)|$ by $f(x)$, and on the subintervals for which $f(x) \leq 0$ replace $|f(x)|$ by $-f(x)$. Adding the resulting integrals then yields the total area.

▶ **Example 8** Find the total area between the curve $y = 1 - x^2$ and the x-axis over the interval [0, 2] (Figure 4.6.7).

Solution. The area A is given by

$$A = \int_0^2 |1 - x^2|\, dx = \int_0^1 (1 - x^2)\, dx + \int_1^2 -(1 - x^2)\, dx$$

$$= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2 \blacktriangleleft$$

DUMMY VARIABLES

To evaluate a definite integral using the Fundamental Theorem of Calculus, one needs to be able to find an antiderivative of the integrand; thus, it is important to know what kinds of functions have antiderivatives. It is our next objective to show that all continuous functions have antiderivatives, but to do this we will need some preliminary results.

Formula (6) shows that there is a close relationship between the integrals

$$\int_a^b f(x)\, dx \quad \text{and} \quad \int f(x)\, dx$$

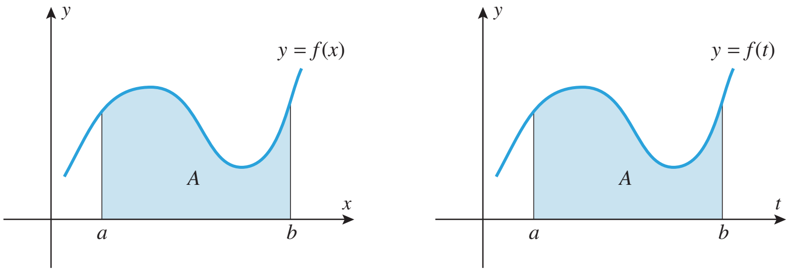
However, the definite and indefinite integrals differ in some important ways. For one thing, the two integrals are different kinds of objects—the definite integral is a *number* (the net signed area between the graph of $y = f(x)$ and the interval $[a, b]$), whereas the indefinite integral is a *function*, or more accurately a family of functions [the antiderivatives of $f(x)$]. However, the two types of integrals also differ in the role played by the variable of integration. In an indefinite integral, the variable of integration is “passed through” to the antiderivative in the sense that integrating a function of x produces a function of x, integrating a function of t produces a function of t, and so forth. For example,

$$\int x^2\, dx = \frac{x^3}{3} + C \quad \text{and} \quad \int t^2\, dt = \frac{t^3}{3} + C$$

In contrast, the variable of integration in a definite integral is not passed through to the end result, since the end result is a number. Thus, integrating a function of x over an interval and integrating the same function of t over the same interval of integration produce the same value for the integral. For example,

$$\int_1^3 x^2\, dx = \frac{x^3}{3}\bigg]_{x=1}^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3} \quad \text{and} \quad \int_1^3 t^2\, dt = \frac{t^3}{3}\bigg]_{t=1}^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$$

However, this latter result should not be surprising, since the area under the graph of the curve $y = f(x)$ over an interval $[a, b]$ on the x-axis is the same as the area under the graph of the curve $y = f(t)$ over the interval $[a, b]$ on the t-axis (Figure 4.6.8).



$$A = \int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Figure 4.6.8

Because the variable of integration in a definite integral plays no role in the end result, it is often referred to as a ***dummy variable***. In summary:

> *Whenever you find it convenient to change the letter used for the variable of integration in a definite integral, you can do so without changing the value of the integral.*

THE MEAN-VALUE THEOREM FOR INTEGRALS

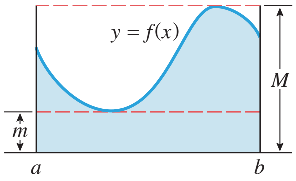


Figure 4.6.9

To reach our goal of showing that continuous functions have antiderivatives, we will need to develop a basic property of definite integrals, known as the *Mean-Value Theorem for Integrals.* In Section 4.8 we will interpret this theorem using the concept of the "average value" of a continuous function over an interval. Here we will need it as a tool for developing other results.

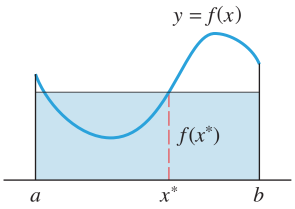
Let f be a continuous nonnegative function on $[a, b]$, and let m and M be the minimum and maximum values of $f(x)$ on this interval. Consider the rectangles of heights m and M over the interval $[a, b]$ (Figure 4.6.9). It is clear geometrically from this figure that the area

$$A = \int_a^b f(x)\,dx$$

under $y = f(x)$ is at least as large as the area of the rectangle of height m and no larger than the area of the rectangle of height M. It seems reasonable, therefore, that there is a rectangle over the interval $[a, b]$ of some appropriate height $f(x^*)$ between m and M whose area is precisely A; that is,

$$\int_a^b f(x)\,dx = f(x^*)(b - a)$$

(Figure 4.6.10). This is a special case of the following result.



The area of the shaded rectangle is equal to the area of the shaded region in Figure 4.6.9.

Figure 4.6.10

4.6.2 THEOREM (*The Mean-Value Theorem for Integrals*) *If f is continuous on a closed interval $[a, b]$, then there is at least one point x^* in $[a, b]$ such that*

$$\int_a^b f(x)\,dx = f(x^*)(b - a) \tag{8}$$

PROOF By the Extreme-Value Theorem (3.4.2), f assumes a maximum value M and a minimum value m on $[a, b]$. Thus, for all x in $[a, b]$,

$$m \le f(x) \le M$$

and from Theorem 4.5.6(*b*)

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

or

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \tag{9}$$

or

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

This implies that

$$\frac{1}{b-a}\int_a^b f(x)\,dx \tag{10}$$

is a number between m and M, and since $f(x)$ assumes the values m and M on $[a, b]$, it follows from the Intermediate-Value Theorem (1.5.8) that $f(x)$ must assume the value (10) at some x^* in $[a, b]$; that is,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(x^*) \quad \text{or} \quad \int_a^b f(x)\,dx = f(x^*)(b-a) \quad ■$$

▶ Example 9 Since $f(x) = x^2$ is continuous on the interval $[1, 4]$, the Mean-Value Theorem for Integrals guarantees that there is a point x^* in $[1, 4]$ such that

$$\int_1^4 x^2\,dx = f(x^*)(4-1) = (x^*)^2(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\bigg]_1^4 = 21$$

so that

$$3(x^*)^2 = 21 \quad \text{or} \quad (x^*)^2 = 7 \quad \text{or} \quad x^* = \pm\sqrt{7}$$

Thus, $x^* = \sqrt{7} \approx 2.65$ is the point in the interval $[1, 4]$ whose existence is guaranteed by the Mean-Value Theorem for Integrals. ◀

■ PART 2 OF THE FUNDAMENTAL THEOREM OF CALCULUS

In Section 4.1 we suggested that if f is continuous and nonnegative on $[a, b]$, and if $A(x)$ is the area under the graph of $y = f(x)$ over the interval $[a, x]$ (Figure 4.6.2), then $A'(x) = f(x)$. But $A(x)$ can be expressed as the definite integral

$$A(x) = \int_a^x f(t)\,dt$$

(where we have used t rather than x as the variable of integration to avoid confusion with the x that appears as the upper limit of integration). Thus, the relationship $A'(x) = f(x)$ can be expressed as

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

This is a special case of the following more general result, which applies even if f has negative values.

4.6.3 THEOREM **(*The Fundamental Theorem of Calculus, Part 2*)** *If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \tag{11}$$

PROOF We will show first that $F(x)$ is defined at each x in the interval. If $x > a$ and x is in the interval, then Theorem 4.5.2 applied to the interval $[a, x]$ and the continuity of f ensure that $F(x)$ is defined; and if x is in the interval and $x \le a$, then Definition 4.5.3 combined with Theorem 4.5.2 ensures that $F(x)$ is defined. Thus, $F(x)$ is defined for all x in the interval.

Next we will show that $F'(x) = f(x)$ for each x in the interval. If x is not an endpoint, then it follows from the definition of a derivative that

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \qquad \boxed{\text{Theorem 4.5.5}} \end{aligned} \tag{12}$$

Applying the Mean-Value Theorem for Integrals (4.6.2) to the integral in (12) we obtain

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = \frac{1}{h}[f(t^*)\cdot h] = f(t^*) \tag{13}$$

where t^* is some number between x and $x+h$. Because t^* is trapped between x and $x+h$, it follows that $t^* \to x$ as $h \to 0$. Thus, the continuity of f at x implies that $f(t^*) \to f(x)$ as $h \to 0$. Therefore, it follows from (12) and (13) that

$$F'(x) = \lim_{h\to 0}\left(\frac{1}{h}\int_x^{x+h} f(t)\,dt\right) = \lim_{h\to 0} f(t^*) = f(x)$$

If x is an endpoint of the interval, then the two-sided limits in the proof must be replaced by the appropriate one-sided limits, but otherwise the arguments are identical. ■

In words, Formula (11) states:

If a definite integral has a variable upper limit of integration, a constant lower limit of integration, and a continuous integrand, then the derivative of the integral with respect to its upper limit is equal to the integrand evaluated at the upper limit.

▶ **Example 10** Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

by applying Part 2 of the Fundamental Theorem of Calculus, and then confirm the result by performing the integration and then differentiating.

Solution. The integrand is a continuous function, so from (11)

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$$

Alternatively, evaluating the integral and then differentiating yields

$$\int_1^x t^3\,dt = \frac{t^4}{4}\bigg]_{t=1}^x = \frac{x^4}{4} - \frac{1}{4}, \qquad \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$$

so the two methods for differentiating the integral agree. ◀

▶ Example 11 Since

$$f(x) = \frac{\sin x}{x}$$

is continuous on any interval that does not contain the origin, it follows from (11) that on the interval $(0, +\infty)$ we have

$$\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t}\,dt\right] = \frac{\sin x}{x}$$

Unlike the preceding example, there is no way to evaluate the integral in terms of familiar functions, so Formula (11) provides the only simple method for finding the derivative. ◀

DIFFERENTIATION AND INTEGRATION ARE INVERSE PROCESSES

The two parts of the Fundamental Theorem of Calculus, when taken together, tell us that differentiation and integration are inverse processes in the sense that each undoes the effect of the other. To see why this is so, note that Part 1 of the Fundamental Theorem of Calculus (4.6.1) implies that

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

which tells us that if the value of $f(a)$ is known, then the function f can be recovered from its derivative f' by integrating. Conversely, Part 2 of the Fundamental Theorem of Calculus (4.6.3) states that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

which tells us that the function f can be recovered from its integral by differentiating. Thus, differentiation and integration can be viewed as inverse processes.

It is common to treat parts 1 and 2 of the Fundamental Theorem of Calculus as a single theorem and refer to it simply as the *Fundamental Theorem of Calculus*. This theorem ranks as one of the greatest discoveries in the history of science, and its formulation by Newton and Leibniz is generally regarded to be the "discovery of calculus."

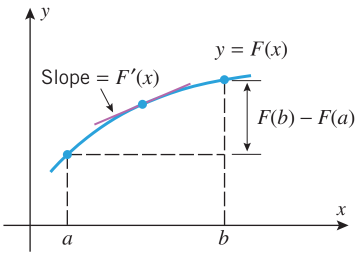
INTEGRATING RATES OF CHANGE

The Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{14}$$

has a useful interpretation that can be seen by rewriting it in a slightly different form. Since F is an antiderivative of f on the interval $[a, b]$, we can use the relationship $F'(x) = f(x)$ to rewrite (14) as

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{15}$$



Integrating the slope of $y = F(x)$ over the interval $[a, b]$ produces the change $F(b) - F(a)$ in the value of $F(x)$.

▲ **Figure 4.6.11**

In this formula we can view $F'(x)$ as the rate of change of $F(x)$ with respect to x, and we can view $F(b) - F(a)$ as the *change* in the value of $F(x)$ as x increases from a to b (Figure 4.6.11). Thus, we have the following useful principle.

4.6.4 INTEGRATING A RATE OF CHANGE Integrating the rate of change of $F(x)$ with respect to x over an interval $[a, b]$ produces the change in the value of $F(x)$ that occurs as x increases from a to b.

Here are some examples of this idea:

- If $s(t)$ is the position of a particle in rectilinear motion, then $s'(t)$ is the instantaneous velocity of the particle at time t, and
$$\int_{t_1}^{t_2} s'(t)\,dt = s(t_2) - s(t_1)$$
is the displacement (or the change in the position) of the particle between the times t_1 and t_2.
- If $P(t)$ is a population (e.g., plants, animals, or people) at time t, then $P'(t)$ is the rate at which the population is changing at time t, and
$$\int_{t_1}^{t_2} P'(t)\,dt = P(t_2) - P(t_1)$$
is the change in the population between times t_1 and t_2.
- If $A(t)$ is the area of an oil spill at time t, then $A'(t)$ is the rate at which the area of the spill is changing at time t, and
$$\int_{t_1}^{t_2} A'(t)\,dt = A(t_2) - A(t_1)$$
is the change in the area of the spill between times t_1 and t_2.
- If $P'(x)$ is the marginal profit that results from producing and selling x units of a product (see Section 3.5), then
$$\int_{x_1}^{x_2} P'(x)\,dx = P(x_2) - P(x_1)$$
is the change in the profit that results when the production level increases from x_1 units to x_2 units.

Mitchell Funk/Getty Images

Mathematical analysis plays an important role in understanding human population growth.

✔QUICK CHECK EXERCISES 4.6 *(See page 322 for answers.)*

1. (a) If $F(x)$ is an antiderivative for $f(x)$, then
$$\int_a^b f(x)\,dx = \underline{\qquad\qquad}$$
(b) $\displaystyle\int_a^b F'(x)\,dx = \underline{\qquad\qquad}$

(c) $\displaystyle\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = \underline{\qquad\qquad}$

2. (a) $\displaystyle\int_0^2 (3x^2 - 2x)\,dx = \underline{\qquad\qquad}$

(b) $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = \underline{\qquad\qquad}$

3. For the function $f(x) = 3x^2 - 2x$ and an interval $[a, b]$, the point x^* guaranteed by the Mean-Value Theorem for Integrals is $x^* = \frac{2}{3}$. It follows that
$$\int_a^b (3x^2 - 2x)\,dx = \underline{\qquad\qquad}$$

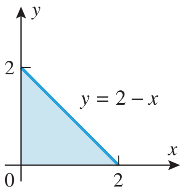
4. The area of an oil spill is increasing at a rate of $25t$ ft^2/s t seconds after the spill. Between times $t = 2$ and $t = 4$ the area of the spill increases by ________ ft^2.

EXERCISE SET 4.6

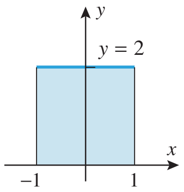
Graphing Utility   C CAS

1. In each part, use a definite integral to find the area of the region, and check your answer using an appropriate formula from geometry.

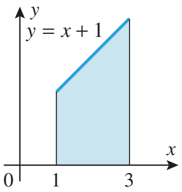
(a)



(b)



(c)



2. In each part, use a definite integral to find the area under the curve $y = f(x)$ over the stated interval, and check your answer using an appropriate formula from geometry.
(a) $f(x) = x$; $[0, 5]$
(b) $f(x) = 5$; $[3, 9]$
(c) $f(x) = x + 3$; $[-1, 2]$

3. In each part, sketch the analogue of Figure 4.6.10 for the specified region. [Let $y = f(x)$ denote the upper boundary of the region. If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
(a) The region in part (a) of Exercise 1.
(b) The region in part (b) of Exercise 1.
(c) The region in part (c) of Exercise 1.

4. In each part, sketch the analogue of Figure 4.6.10 for the function and interval specified. [If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
(a) The function and interval in part (a) of Exercise 2.
(b) The function and interval in part (b) of Exercise 2.
(c) The function and interval in part (c) of Exercise 2.

5–8 Find the area under the curve $y = f(x)$ over the stated interval. ■

5. $f(x) = x^3$; $[2, 3]$
6. $f(x) = x^4$; $[-1, 1]$
7. $f(x) = 3\sqrt{x}$; $[1, 4]$
8. $f(x) = x^{-2/3}$; $[1, 27]$

9–10 Find all values of x^* in the stated interval that satisfy Equation (8) in the Mean-Value Theorem for Integrals (4.6.2), and explain what these numbers represent. ■

9. (a) $f(x) = \sqrt{x}$; $[0, 3]$
(b) $f(x) = x^2 + x$; $[-12, 0]$

10. (a) $f(x) = \sin x$; $[-\pi, \pi]$ (b) $f(x) = 1/x^2$; $[1, 3]$

11–22 Evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus. ■

11. $\int_{-2}^{1} (x^2 - 6x + 12)\,dx$
12. $\int_{-1}^{2} 4x(1 - x^2)\,dx$
13. $\int_{1}^{4} \frac{4}{x^2}\,dx$
14. $\int_{1}^{2} \frac{1}{x^6}\,dx$
15. $\int_{4}^{9} 2x\sqrt{x}\,dx$
16. $\int_{1}^{4} \frac{1}{x\sqrt{x}}\,dx$
17. $\int_{-\pi/2}^{\pi/2} \sin\theta\,d\theta$
18. $\int_{0}^{\pi/4} \sec^2\theta\,d\theta$
19. $\int_{-\pi/4}^{\pi/4} \cos x\,dx$
20. $\int_{0}^{\pi/3} (2x - \sec x \tan x)\,dx$
21. $\int_{1}^{4} \left(\frac{1}{\sqrt{t}} - 3\sqrt{t}\right) dt$
22. $\int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right) dx$

23–24 Use Theorem 4.5.5 to evaluate the given integrals. ■

23. (a) $\int_{-1}^{1} |2x - 1|\,dx$ (b) $\int_{0}^{3\pi/4} |\cos x|\,dx$

24. (a) $\int_{-1}^{2} \sqrt{2 + |x|}\,dx$ (b) $\int_{0}^{\pi/2} \left|\frac{1}{2} - \cos x\right|\,dx$

25–26 A function $f(x)$ is defined piecewise on an interval. In these exercises: (a) Use Theorem 4.5.5 to find the integral of $f(x)$ over the interval. (b) Find an antiderivative of $f(x)$ on the interval. (c) Use parts (a) and (b) to verify Part 1 of the Fundamental Theorem of Calculus. ■

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x^2, & 1 < x \le 2 \end{cases}$

26. $f(x) = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ 1/x^2, & 1 \le x \le 4 \end{cases}$

27–30 True–False Determine whether the statement is true or false. Explain your answer. ■

27. There does not exist a differentiable function $F(x)$ such that $F'(x) = |x|$.

28. If $f(x)$ is continuous on the interval $[a, b]$, and if the definite integral of $f(x)$ over this interval has value 0, then the equation $f(x) = 0$ has at least one solution in the interval $[a, b]$.

29. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

if and only if

$$G(a) + F(b) = F(a) + G(b)$$

30. If $f(x)$ is continuous everywhere and

$$F(x) = \int_0^x f(t)\,dt$$

then the equation $F(x) = 0$ has at least one solution.

31–34 Use a calculating utility to find the midpoint approximation of the integral using $n = 20$ subintervals, and then find the exact value of the integral using Part 1 of the Fundamental Theorem of Calculus. ■

31. $\displaystyle\int_1^3 \frac{1}{x^2}\,dx$ **32.** $\displaystyle\int_0^{\pi/2} \sin x\,dx$

33. $\displaystyle\int_{-1}^1 \sec^2 x\,dx$ **34.** $\displaystyle\int_1^5 \frac{1}{x^3}\,dx$

35–38 Sketch the region described and find its area. ■

35. The region under the curve $y = x^2 + 1$ and over the interval $[0, 3]$.

36. The region below the curve $y = x - x^2$ and above the x-axis.

37. The region under the curve $y = 3\sin x$ and over the interval $[0, 2\pi/3]$.

38. The region below the interval $[-2, -1]$ and above the curve $y = x^3$.

42 Sketch the curve and find the total area between the curve and the given interval on the x-axis. ■

39. $y = x^2 - x;\ [0, 2]$ **40.** $y = \sin x;\ [0, 3\pi/2]$

41. $y = 2\sqrt{x+1} - 3;\ [0, 3]$ **42.** $y = \dfrac{x^2 - 1}{x^2};\ \left[\frac{1}{2}, 2\right]$

43. A student wants to find the area enclosed by the graphs of $y = \cos x$, $y = 0$, $x = 0$, and $x = 0.8$.

(a) Show that the exact area is $\sin 0.8$.

(b) The student uses a calculator to approximate the result in part (a) to three decimal places and obtains an incorrect answer of 0.014. What was the student's error? Find the correct approximation.

FOCUS ON CONCEPTS

44. Explain why the Fundamental Theorem of Calculus may be applied without modification to definite integrals in which the lower limit of integration is greater than or equal to the upper limit of integration.

45. (a) If $h'(t)$ is the rate of change of a child's height measured in inches per year, what does the integral $\int_0^{10} h'(t)\,dt$ represent, and what are its units?

(b) If $r'(t)$ is the rate of change of the radius of a spherical balloon measured in centimeters per second, what does the integral $\int_1^2 r'(t)\,dt$ represent, and what are its units?

(c) If $H(t)$ is the rate of change of the speed of sound with respect to temperature measured in ft/s per °F, what does the integral $\int_{32}^{100} H(t)\,dt$ represent, and what are its units?

(d) If $v(t)$ is the velocity of a particle in rectilinear motion, measured in cm/h, what does the integral $\int_{t_1}^{t_2} v(t)\,dt$ represent, and what are its units?

46. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{1}{100}(x+2)(x+1)(x-3)(x-5)$$

and use the graph to make a conjecture about the sign of the integral

$$\int_{-2}^5 f(x)\,dx$$

(b) Check your conjecture by evaluating the integral.

47. Define $F(x)$ by

$$F(x) = \int_1^x (3t^2 - 3)\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

48. Define $F(x)$ by

$$F(x) = \int_{\pi/4}^x \cos 2t\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

49–52 Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives. ■

49. (a) $\displaystyle\frac{d}{dx}\int_1^x \sin(t^2)\,dt$ (b) $\displaystyle\frac{d}{dx}\int_1^x \sqrt{1-\cos t}\,dt$

50. (a) $\displaystyle\frac{d}{dx}\int_0^x \frac{dt}{1+\sqrt{t}}$ (b) $\displaystyle\frac{d}{dx}\int_2^x \frac{dt}{t^2+3t-4}$

51. $\displaystyle\frac{d}{dx}\int_x^0 t\sec t\,dt$ [*Hint:* Use Definition 4.5.3(b).]

52. $\displaystyle\frac{d}{du}\int_0^u |x|\,dx$

53. Let $F(x) = \displaystyle\int_4^x \sqrt{t^2+9}\,dt$. Find

(a) $F(4)$ (b) $F'(4)$ (c) $F''(4)$.

54. Let $F(x) = \displaystyle\int_0^x \frac{\cos t}{t^2+3t+5}\,dt$. Find

(a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.

55. Let $F(x) = \displaystyle\int_0^x \frac{t-3}{t^2+7}\,dt$ for $-\infty < x < +\infty$.

(a) Find the value of x where F attains its minimum value.

(b) Find intervals over which F is only increasing or only decreasing.

(c) Find open intervals over which F is only concave up or only concave down.

C **56.** Use the plotting and numerical integration commands of a CAS to generate the graph of the function F in Exercise 55 over the interval $-20 \le x \le 20$, and confirm that the graph is consistent with the results obtained in that exercise.

57. (a) Over what open interval does the formula

$$F(x) = \int_1^x \frac{dt}{t}$$

represent an antiderivative of $f(x) = 1/x$?

(b) Find a point where the graph of F crosses the x-axis.

58. (a) Over what open interval does the formula

$$F(x) = \int_1^x \frac{1}{t^2 - 9}\,dt$$

represent an antiderivative of

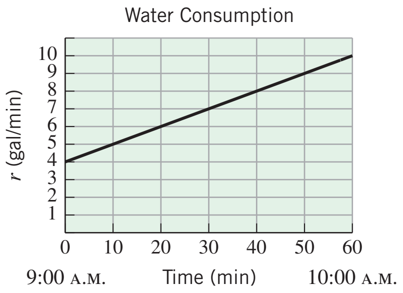
$$f(x) = \frac{1}{x^2 - 9}?$$

(b) Find a point where the graph of F crosses the x-axis.

59. (a) Suppose that a reservoir supplies water to an industrial park at a constant rate of $r = 4$ gallons per minute (gal/min) between 8:30 A.M. and 9:00 A.M. How much water does the reservoir supply during that time period?

(b) Suppose that one of the industrial plants increases its water consumption between 9:00 A.M. and 10:00 A.M. and that the rate at which the reservoir supplies water increases linearly, as shown in the accompanying figure. How much water does the reservoir supply during that 1-hour time period?

(c) Suppose that from 10:00 A.M. to 12 noon the rate at which the reservoir supplies water is given by the formula $r(t) = 10 + \sqrt{t}$ gal/min, where t is the time (in minutes) since 10:00 A.M. How much water does the reservoir supply during that 2-hour time period?



◀ **Figure Ex-59**

60. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M.

(a) When does the peak traffic flow into the highway occur?

(b) Estimate the number of cars that enter the highway during the rush hour.

61–62 Evaluate each limit by interpreting it as a Riemann sum in which the given interval is divided into n subintervals of equal width. ■

61. $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{\pi}{4n} \sec^2\left(\frac{\pi k}{4n}\right);\ \left[0, \frac{\pi}{4}\right]$

62. $\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{n}{(n+k)^2};\ [1, 2]$

63. Prove the Mean-Value Theorem for Integrals (Theorem 4.6.2) by applying the Mean-Value Theorem (3.8.2) to an antiderivative F for f.

64. **Writing** Write a short paragraph that describes the various ways in which integration and differentiation may be viewed as inverse processes. (Be sure to discuss both definite and indefinite integrals.)

65. **Writing** Let f denote a function that is continuous on an interval $[a, b]$, and let x^* denote the point guaranteed by the Mean-Value Theorem for Integrals. Explain geometrically why $f(x^*)$ may be interpreted as a "mean" or average value of $f(x)$ over $[a, b]$. (In Section 4.8 we will discuss the concept of "average value" in more detail.)

✔QUICK CHECK ANSWERS 4.6

1. (a) $F(b) - F(a)$ (b) $F(b) - F(a)$ (c) $f(x)$ **2.** (a) 4 (b) 0 **3.** 0 **4.** 150 ft^2

4.7 RECTILINEAR MOTION REVISITED USING INTEGRATION

In Section 3.6 we used the derivative to define the notions of instantaneous velocity and acceleration for a particle in rectilinear motion. In this section we will resume the study of such motion using the tools of integration.

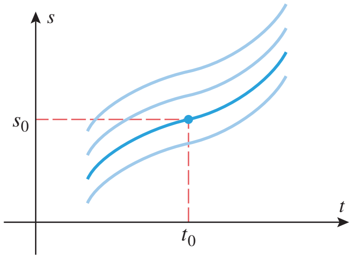
FINDING POSITION AND VELOCITY BY INTEGRATION

Recall from Formulas (1) and (3) of Section 3.6 that if a particle in rectilinear motion has position function $s(t)$, then its instantaneous velocity and acceleration are given by the formulas

$$v(t) = s'(t) \quad \text{and} \quad a(t) = v'(t)$$

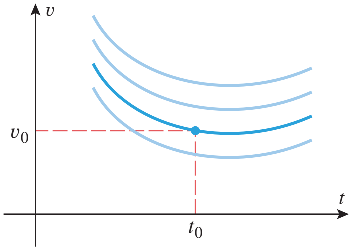
It follows from these formulas that $s(t)$ is an antiderivative of $v(t)$ and $v(t)$ is an antiderivative of $a(t)$; that is,

$$\boxed{s(t) = \int v(t)\,dt} \quad \text{and} \quad \boxed{v(t) = \int a(t)\,dt} \tag{1–2}$$



There is a unique position function such that $s(t_0) = s_0$.

▲ **Figure 4.7.1**



There is a unique velocity function such that $v(t_0) = v_0$.

▲ **Figure 4.7.2**

By Formula (1), if we know the velocity function $v(t)$ of a particle in rectilinear motion, then by integrating $v(t)$ we can produce a family of position functions with that velocity function. If, in addition, we know the position s_0 of the particle at any time t_0, then we have sufficient information to find the constant of integration and determine a unique position function (Figure 4.7.1). Similarly, if we know the acceleration function $a(t)$ of the particle, then by integrating $a(t)$ we can produce a family of velocity functions with that acceleration function. If, in addition, we know the velocity v_0 of the particle at any time t_0, then we have sufficient information to find the constant of integration and determine a unique velocity function (Figure 4.7.2).

▶ **Example 1** Suppose that a particle moves with velocity $v(t) = \cos \pi t$ along a coordinate line. Assuming that the particle has coordinate $s = 4$ at time $t = 0$, find its position function.

Solution. The position function is

$$s(t) = \int v(t)\,dt = \int \cos \pi t\,dt = \frac{1}{\pi}\sin \pi t + C$$

Since $s = 4$ when $t = 0$, it follows that

$$4 = s(0) = \frac{1}{\pi}\sin 0 + C = C$$

Thus,

$$s(t) = \frac{1}{\pi}\sin \pi t + 4 \quad \blacktriangleleft$$

COMPUTING DISPLACEMENT AND DISTANCE TRAVELED BY INTEGRATION

Recall that the displacement over a time interval of a particle in rectilinear motion is its final coordinate minus its initial coordinate. Thus, if the position function of the particle is $s(t)$, then its displacement (or change in position) over the time interval $[t_0, t_1]$ is $s(t_1) - s(t_0)$. This can be written in integral form as

$$\boxed{\begin{bmatrix}\text{displacement}\\ \text{over the time}\\ \text{interval } [t_0, t_1]\end{bmatrix} = \int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} s'(t)\,dt = s(t_1) - s(t_0)} \tag{3}$$

Recall that Formula (3) is a special case of the formula

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

for integrating a rate of change.

In contrast, to find the distance traveled by the particle over the time interval $[t_0, t_1]$ (distance traveled in the positive direction plus the distance traveled in the negative direction), we must integrate the absolute value of the velocity function; that is,

$$\boxed{\begin{bmatrix}\text{distance traveled}\\ \text{during time}\\ \text{interval } [t_0, t_1]\end{bmatrix} = \int_{t_0}^{t_1} |v(t)|\,dt} \tag{4}$$

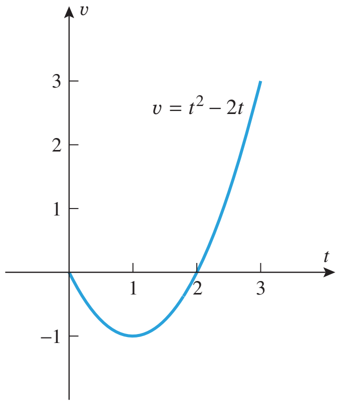
Since the absolute value of velocity is speed, Formulas (3) and (4) can be summarized informally as follows:

> *Integrating velocity over a time interval produces displacement, and integrating speed over a time interval produces distance traveled.*

▶ Example 2 Suppose that a particle moves on a coordinate line so that its velocity at time t is $v(t) = t^2 - 2t$ m/s (Figure 4.7.3).

(a) Find the displacement of the particle during the time interval $0 \le t \le 3$.

(b) Find the distance traveled by the particle during the time interval $0 \le t \le 3$.



▲ **Figure 4.7.3**

Solution (a). From (3) the displacement is

$$\int_0^3 v(t)\,dt = \int_0^3 (t^2 - 2t)\,dt = \left[\frac{t^3}{3} - t^2\right]_0^3 = 0$$

Thus, the particle is at the same position at time $t = 3$ as at $t = 0$.

Solution (b). The velocity can be written as $v(t) = t^2 - 2t = t(t-2)$, from which we see that $v(t) \le 0$ for $0 \le t \le 2$ and $v(t) \ge 0$ for $2 \le t \le 3$. Thus, it follows from (4) that the distance traveled is

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^2 -v(t)\,dt + \int_2^3 v(t)\,dt\\ &= \int_0^2 -(t^2 - 2t)\,dt + \int_2^3 (t^2 - 2t)\,dt\\ &= -\left[\frac{t^3}{3} - t^2\right]_0^2 + \left[\frac{t^3}{3} - t^2\right]_2^3 = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \text{ m} \blacktriangleleft\end{aligned}$$

In physical problems it is important to associate correct units with definite integrals. In general, the units for

$$\int_a^b f(x)\,dx$$

are units of $f(x)$ times units of x, since the integral is the limit of Riemann sums, each of whose terms has these units. For example, if $v(t)$ is in meters per second (m/s) and t is in seconds (s), then

$$\int_a^b v(t)\,dt$$

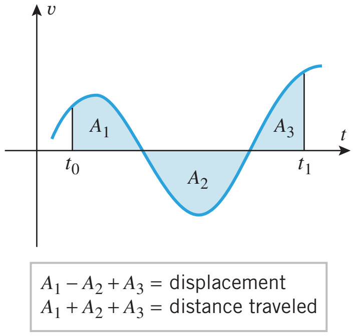
is in meters since

$$(\text{m/s}) \times \text{s} = \text{m}$$

ANALYZING THE VELOCITY VERSUS TIME CURVE

In Section 3.6 we showed how to use the position versus time curve to obtain information about the behavior of a particle in rectilinear motion (Table 3.6.1). Similarly, there is valuable information that can be obtained from the velocity versus time curve. For example, the integral in (3) can be interpreted geometrically as the *net signed area* between the graph of $v(t)$ and the interval $[t_0, t_1]$, and the integral in (4) can be interpreted as the *total area* between the graph of $v(t)$ and the interval $[t_0, t_1]$. Thus we have the following result.

4.7.1 FINDING DISPLACEMENT AND DISTANCE TRAVELED FROM THE VELOCITY VERSUS TIME CURVE For a particle in rectilinear motion, the net signed area between the velocity versus time curve and the interval $[t_0, t_1]$ on the t-axis represents the displacement of the particle over that time interval, and the total area between the velocity versus time curve and the interval $[t_0, t_1]$ on the t-axis represents the distance traveled by the particle over that time interval (Figure 4.7.4).



$A_1 - A_2 + A_3$ = displacement
$A_1 + A_2 + A_3$ = distance traveled

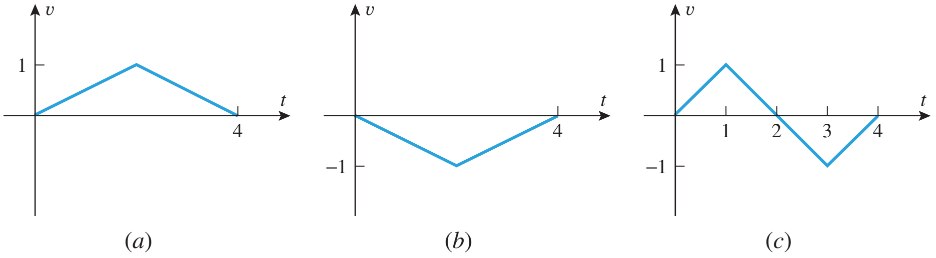
▲ **Figure 4.7.4**

▶ Example 3 Figure 4.7.5 shows three velocity versus time curves for a particle in rectilinear motion along a horizontal line with the positive direction to the right. In each case find the displacement and the distance traveled over the time interval $0 \le t \le 4$, and explain what that information tells you about the motion of the particle.

Solution (a). In part (*a*) of the figure the area and the net signed area over the interval are both 2. Thus, at the end of the time period the particle is 2 units to the right of its starting point and has traveled a distance of 2 units.

Solution (b). In part (*b*) of the figure the net signed area is -2, and the total area is 2. Thus, at the end of the time period the particle is 2 units to the left of its starting point and has traveled a distance of 2 units.

Solution (c). In part (*c*) of the figure the net signed area is 0, and the total area is 2. Thus, at the end of the time period the particle is back at its starting point and has traveled a distance of 2 units. More specifically, it traveled 1 unit to the right over the time interval $0 \le t \le 1$ and then 1 unit to the left over the time interval $1 \le t \le 2$ (why?). ◀



(*a*) (*b*) (*c*)

▲ **Figure 4.7.5**

CONSTANT ACCELERATION

One of the most important cases of rectilinear motion occurs when a particle has ***constant acceleration***. We will show that if a particle moves with constant acceleration along an s-axis, and if the position and velocity of the particle are known at some point in time, say when $t = 0$, then it is possible to derive formulas for the position $s(t)$ and the velocity $v(t)$ at any time t. To see how this can be done, suppose that the particle has constant acceleration

$$a(t) = a \tag{5}$$

and

$$s = s_0 \quad \text{when} \quad t = 0 \tag{6}$$

$$v = v_0 \quad \text{when} \quad t = 0 \tag{7}$$

where s_0 and v_0 are known. We call (6) and (7) the ***initial conditions***.

With (5) as a starting point, we can integrate $a(t)$ to obtain $v(t)$, and we can integrate $v(t)$ to obtain $s(t)$, using an initial condition in each case to determine the constant of integration. The computations are as follows:

$$v(t) = \int a(t)\,dt = \int a\,dt = at + C_1 \tag{8}$$

To determine the constant of integration C_1 we apply initial condition (7) to this equation to obtain

$$v_0 = v(0) = a \cdot 0 + C_1 = C_1$$

Substituting this in (8) and putting the constant term first yields

$$v(t) = v_0 + at$$

Since v_0 is constant, it follows that

$$s(t) = \int v(t)\,dt = \int (v_0 + at)\,dt = v_0 t + \tfrac{1}{2}at^2 + C_2 \tag{9}$$

To determine the constant C_2 we apply initial condition (6) to this equation to obtain

$$s_0 = s(0) = v_0 \cdot 0 + \tfrac{1}{2}a \cdot 0 + C_2 = C_2$$

Substituting this in (9) and putting the constant term first yields

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2$$

In summary, we have the following result.

How can you tell from the graph of the velocity versus time curve whether a particle moving along a line has constant acceleration?

4.7.2 CONSTANT ACCELERATION If a particle moves with constant acceleration a along an s-axis, and if the position and velocity at time $t = 0$ are s_0 and v_0, respectively, then the position and velocity functions of the particle are

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}at^2 \tag{10}$$

$$v(t) = v_0 + at \tag{11}$$

▶ **Example 4** Suppose that an intergalactic spacecraft uses a sail and the "solar wind" to produce a constant acceleration of $0.032\ \text{m/s}^2$. Assuming that the spacecraft has a velocity of 10,000 m/s when the sail is first raised, how far will the spacecraft travel in 1 hour, and what will its velocity be at the end of this hour?

Solution. In this problem the choice of a coordinate axis is at our discretion, so we will choose it to make the computations as simple as possible. Accordingly, let us introduce an s-axis whose positive direction is in the direction of motion, and let us take the origin to coincide with the position of the spacecraft at the time $t = 0$ when the sail is raised. Thus, Formulas (10) and (11) apply with

$$s_0 = s(0) = 0, \quad v_0 = v(0) = 10{,}000, \quad \text{and} \quad a = 0.032$$

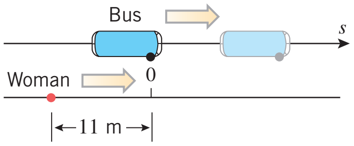
Since 1 hour corresponds to $t = 3600$ s, it follows from (10) that in 1 hour the spacecraft travels a distance of

$$s(3600) = 10{,}000(3600) + \tfrac{1}{2}(0.032)(3600)^2 \approx 36{,}200{,}000\ \text{m}$$

and it follows from (11) that after 1 hour its velocity is

$$v(3600) = 10{,}000 + (0.032)(3600) \approx 10{,}100\ \text{m/s} \blacktriangleleft$$

▶ **Example 5** A bus has stopped to pick up riders, and a woman is running at a constant velocity of 5 m/s to catch it. When she is 11 m behind the front door the bus pulls away with a constant acceleration of $1\ \text{m/s}^2$. From that point in time, how long will it take for the woman to reach the front door of the bus if she keeps running with a velocity of 5 m/s?



▲ **Figure 4.7.6**

Solution. As shown in Figure 4.7.6, choose the s-axis so that the bus and the woman are moving in the positive direction, and the front door of the bus is at the origin at the time $t = 0$ when the bus begins to pull away. To catch the bus at some later time t, the woman will have to cover a distance $s_w(t)$ that is equal to 11 m plus the distance $s_b(t)$ traveled by the bus; that is, the woman will catch the bus when

$$s_w(t) = s_b(t) + 11 \tag{12}$$

Since the woman has a constant velocity of 5 m/s, the distance she travels in t seconds is $s_w(t) = 5t$. Thus, (12) can be written as

$$s_b(t) = 5t - 11 \tag{13}$$

Since the bus has a constant acceleration of $a = 1$ m/s^2, and since $s_0 = v_0 = 0$ at time $t = 0$ (why?), it follows from (10) that

$$s_b(t) = \tfrac{1}{2}t^2$$

Substituting this equation into (13) and reorganizing the terms yields the quadratic equation

$$\tfrac{1}{2}t^2 - 5t + 11 = 0 \quad \text{or} \quad t^2 - 10t + 22 = 0$$

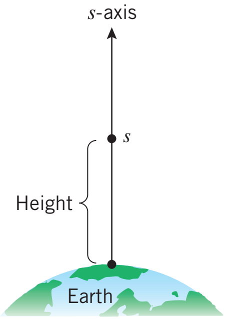
Solving this equation for t using the quadratic formula yields two solutions:

$$t = 5 - \sqrt{3} \approx 3.3 \quad \text{and} \quad t = 5 + \sqrt{3} \approx 6.7$$

(verify). Thus, the woman can reach the door at two different times, $t = 3.3$ s and $t = 6.7$ s. The reason that there are two solutions can be explained as follows: When the woman first reaches the door, she is running faster than the bus and can run past it if the driver does not see her. However, as the bus speeds up, it eventually catches up to her, and she has another chance to flag it down. ◀

FREE-FALL MODEL

Motion that occurs when an object near the Earth is imparted some initial velocity (up or down) and thereafter moves along a vertical line is called ***free-fall motion***. In modeling free-fall motion we assume that the only force acting on the object is the Earth's gravity and that the object stays sufficiently close to the Earth that the gravitational force is constant. In particular, air resistance and the gravitational pull of other celestial bodies are neglected.



▲ **Figure 4.7.7**

In our model we will ignore the physical size of the object by treating it as a particle, and we will assume that it moves along an s-axis whose origin is at the surface of the Earth and whose positive direction is up. With this convention, the s-coordinate of the particle is the height of the particle above the surface of the Earth (Figure 4.7.7).

It is a fact of physics that a particle with free-fall motion has constant acceleration. The magnitude of this constant, denoted by the letter g, is called the ***acceleration due to gravity*** and is approximately 9.8 m/s^2 or 32 ft/s^2, depending on whether distance is measured in meters or feet.[*]

Recall that a particle is speeding up when its velocity and acceleration have the same sign and is slowing down when they have opposite signs. Thus, because we have chosen the positive direction to be up, it follows that the acceleration $a(t)$ of a particle in free fall is negative for all values of t. To see that this is so, observe that an upward-moving particle (positive velocity) is slowing down, so its acceleration must be negative; and a downward-moving particle (negative velocity) is speeding up, so its acceleration must also be negative. Thus, we conclude that

$$a(t) = -g \tag{14}$$

It now follows from this and Formulas (10) and (11) that the position and velocity functions

---

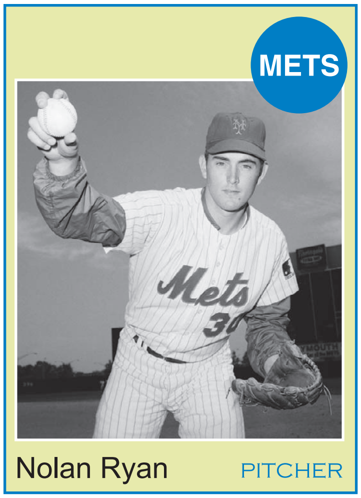
[*] Strictly speaking, the constant g varies with the latitude and the distance from the Earth's center. However, for motion at a fixed latitude and near the surface of the Earth, the assumption of a constant g is satisfactory for many applications.

for a particle in free-fall motion are

$$s(t) = s_0 + v_0 t - \tfrac{1}{2}gt^2 \tag{15}$$

$$v(t) = v_0 - gt \tag{16}$$

How would Formulas (14), (15), and (16) change if we choose the direction of the positive s-axis to be down?



Corbis.Bettmann

Nolan Ryan as a rookie.

Example 6 Nolan Ryan, a member of the Baseball Hall of Fame and one of the fastest baseball pitchers of all time, was able to throw a baseball 150 ft/s (over 102 mi/h). During his career, he had the opportunity to pitch in the Houston Astrodome, home to the Houston Astros Baseball Team from 1965 to 1999. The Astrodome was an indoor stadium with a ceiling 208 ft high. Could Nolan Ryan have hit the ceiling of the Astrodome if he were capable of giving a baseball an upward velocity of 100 ft/s from a height of 7 ft?

Solution. Since distance is in feet, we take $g = 32$ ft/s^2. Initially, we have $s_0 = 7$ ft and $v_0 = 100$ ft/s, so from (15) and (16) we have

$$s(t) = 7 + 100t - 16t^2$$
$$v(t) = 100 - 32t$$

The ball will rise until $v(t) = 0$, that is, until $100 - 32t = 0$. Solving this equation we see that the ball is at its maximum height at time $t = \frac{25}{8}$. To find the height of the ball at this instant we substitute this value of t into the position function to obtain

$$s\left(\tfrac{25}{8}\right) = 7 + 100\left(\tfrac{25}{8}\right) - 16\left(\tfrac{25}{8}\right)^2 = 163.25 \text{ ft}$$

which is roughly 45 ft short of hitting the ceiling. ◀

In Example 6 the ball is moving up when the velocity is positive and is moving down when the velocity is negative, so it makes sense physically that the velocity is zero when the ball reaches its peak.

Example 7 A penny is released from rest near the top of the Empire State Building at a point that is 1250 ft above the ground (Figure 4.7.8). Assuming that the free-fall model applies, how long does it take for the penny to hit the ground, and what is its speed at the time of impact?

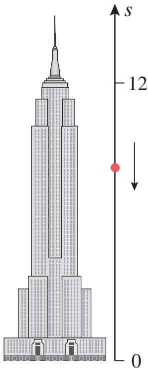


Figure 4.7.8

Solution. Since distance is in feet, we take $g = 32$ ft/s^2. Initially, we have $s_0 = 1250$ and $v_0 = 0$, so from (15)

$$s(t) = 1250 - 16t^2 \tag{17}$$

Impact occurs when $s(t) = 0$. Solving this equation for t, we obtain

$$1250 - 16t^2 = 0$$
$$t^2 = \frac{1250}{16} = \frac{625}{8}$$
$$t = \pm\frac{25}{\sqrt{8}} \approx \pm 8.8 \text{ s}$$

Since $t \geq 0$, we can discard the negative solution and conclude that it takes $25/\sqrt{8} \approx 8.8$ s for the penny to hit the ground. To obtain the velocity at the time of impact, we substitute $t = 25/\sqrt{8}$, $v_0 = 0$, and $g = 32$ in (16) to obtain

$$v\left(\frac{25}{\sqrt{8}}\right) = 0 - 32\left(\frac{25}{\sqrt{8}}\right) = -200\sqrt{2} \approx -282.8 \text{ ft/s}$$

Thus, the speed at the time of impact is

$$\left|v\left(\frac{25}{\sqrt{8}}\right)\right| = 200\sqrt{2} \approx 282.8 \text{ ft/s}$$

which is more than 192 mi/h. ◀

✔QUICK CHECK EXERCISES 4.7 *(See page 331 for answers.)*

1. Suppose that a particle is moving along an s-axis with velocity $v(t) = 2t + 1$. If at time $t = 0$ the particle is at position $s = 2$, the position function of the particle is $s(t) =$ ________.

2. Let $v(t)$ denote the velocity function of a particle that is moving along an s-axis with constant acceleration $a = -2$. If $v(1) = 4$, then $v(t) =$ ________.

3. Let $v(t)$ denote the velocity function of a particle in rectilinear motion. Suppose that $v(0) = -1$, $v(3) = 2$, and the velocity versus time curve is a straight line. The displacement of the particle between times $t = 0$ and $t = 3$ is ________, and the distance traveled by the particle over this period of time is ________.

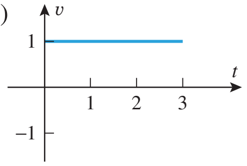
4. Based on the free-fall model, from what height must a coin be dropped so that it strikes the ground with speed 48 ft/s?

EXERCISE SET 4.7 Graphing Utility C CAS

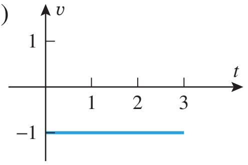
FOCUS ON CONCEPTS

1. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the displacement and the distance traveled by the particle over the time interval $0 \le t \le 3$.

(a)



(b)



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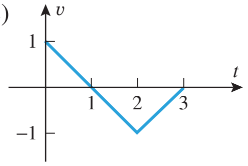

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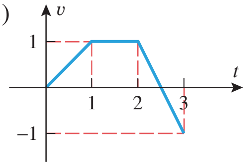
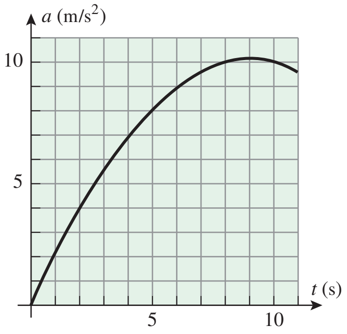


2. Given a push, a ball rolls up an inclined plane and returns to its original position after 4 s. If the total distance traveled by the ball is 16 ft, sketch a velocity versus time curve for the ball.

3. The accompanying figure shows the acceleration versus time curve for a particle moving along a coordinate line. If the initial velocity of the particle is 20 m/s, estimate

(a) the velocity at time $t = 4$ s

(b) the velocity at time $t = 6$ s.



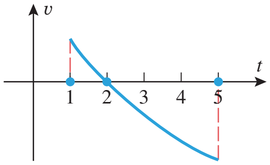
◀ **Figure Ex-3**

4. The accompanying figure shows the velocity versus time curve over the time interval $1 \le t \le 5$ for a particle moving along a horizontal coordinate line.

(a) What can you say about the sign of the acceleration over the time interval?

(b) When is the particle speeding up? Slowing down?

(c) What can you say about the location of the particle at time $t = 5$ relative to its location at time $t = 1$? Explain your reasoning.



◀ **Figure Ex-4**

5–8 A particle moves along an s-axis. Use the given information to find the position function of the particle. ■

5. (a) $v(t) = 3t^2 - 2t;\ s(0) = 1$
(b) $a(t) = 3\sin 3t;\ v(0) = 3;\ s(0) = 3$

6. (a) $v(t) = 1 + \sin t;\ s(0) = -3$
(b) $a(t) = t^2 - 3t + 1;\ v(0) = 0;\ s(0) = 0$

7. (a) $v(t) = 3t + 1;\ s(2) = 4$
(b) $a(t) = 2t^{-3};\ v(1) = 0;\ s(1) = 2$

8. (a) $v(t) = t^{2/3};\ s(8) = 0$
(b) $a(t) = \sqrt{t};\ v(4) = 1;\ s(4) = -5$

9–12 A particle moves with a velocity of $v(t)$ m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval. ■

9. (a) $v(t) = \sin t;\ 0 \le t \le \pi/2$
(b) $v(t) = \cos t;\ \pi/2 \le t \le 2\pi$

10. (a) $v(t) = 3t - 2;\ 0 \le t \le 2$
(b) $v(t) = |1 - 2t|;\ 0 \le t \le 2$

11. (a) $v(t) = t^3 - 3t^2 + 2t;\ 0 \le t \le 3$
(b) $v(t) = \sqrt{t} - 2;\ 0 \le t \le 3$

12. (a) $v(t) = t - \sqrt{t};\ 0 \le t \le 4$

(b) $v(t) = \dfrac{1}{\sqrt{t+1}};\ 0 \le t \le 3$

13–16 A particle moves with acceleration $a(t)$ m/s^2 along an s-axis and has velocity v_0 m/s at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval. ■

13. $a(t) = 3;\ v_0 = -1;\ 0 \le t \le 2$

14. $a(t) = t - 2;\ v_0 = 0;\ 1 \le t \le 5$

15. $a(t) = 1/\sqrt{3t+1};\ v_0 = \frac{4}{3};\ 1 \le t \le 5$

16. $a(t) = \sin t;\ v_0 = 1;\ \pi/4 \le t \le \pi/2$

17. In each part, use the given information to find the position, velocity, speed, and acceleration at time $t = 1$.

(a) $v = \sin \frac{1}{2}\pi t;\ s = 0$ when $t = 0$

(b) $a = -3t;\ s = 1$ and $v = 0$ when $t = 0$

18. In each part, use the given information to find the position, velocity, speed, and acceleration at time $t = 1$.

(a) $v = \cos \frac{1}{3}\pi t;\ s = 0$ when $t = \frac{3}{2}$

(b) $a = 5t - t^3;\ s = 1$ and $v = -\frac{1}{2}$ when $t = 0$

19. The velocity of an ant running along the edge of a shelf is modeled by the function

$$v(t) = \begin{cases} 5t, & 0 \le t < 1 \\ 6\sqrt{t} - \dfrac{1}{t}, & 1 \le t \le 2 \end{cases}$$

where t is in seconds and v is in centimeters per second. Estimate the time at which the ant is 4 cm from its starting position.

20. The velocity of a mouse running alongside the baseboard of a room is modeled by the function

$$v(t) = 3\cos(\pi t/12) - 0.5t, \quad 0 \le t \le 8$$

where t is in seconds and v is in meters per second. (Assume that positive values of v indicate motion to the right.) Estimate the time(s) at which the mouse is 2 m from its starting position.

21. Suppose that the velocity function of a particle moving along an s-axis is $v(t) = 20t^2 - 110t + 120$ ft/s and that the particle is at the origin at time $t = 0$. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 6 s of motion.

22. Suppose that the acceleration function of a particle moving along an s-axis is $a(t) = 4t - 30$ m/s^2 and that the position and velocity at time $t = 0$ are $s_0 = -5$ m and $v_0 = 3$ m/s. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 25 s of motion.

23–26 True–False Determine whether the statement is true or false. Explain your answer. Each question refers to a particle in rectilinear motion. ■

23. If the particle has constant acceleration, the velocity versus time graph will be a straight line.

24. If the particle has constant nonzero acceleration, its position versus time curve will be a parabola.

25. If the total area between the velocity versus time curve and a time interval $[a, b]$ is positive, then the displacement of the particle over this time interval will be nonzero.

26. If $D(t)$ denotes the distance traveled by the particle over the time interval $[0, t]$, then $D(t)$ is an antiderivative for the speed of the particle.

C **27–28** For the given velocity function $v(t)$:

(a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the given time interval.

(b) Use a CAS to find the displacement. ■

27. $v(t) = 0.5 - t\sin t;\ 0 \le t \le 5$

28. $v(t) = 0.5 - t\cos \pi t;\ 0 \le t \le 1$

29. Suppose that at time $t = 0$ a particle is at the origin of an x-axis and has a velocity of $v_0 = 25$ cm/s. For the first 4 s thereafter it has no acceleration, and then it is acted on by a retarding force that produces a constant negative acceleration of $a = -10$ cm/s^2.

(a) Sketch the acceleration versus time curve over the interval $0 \le t \le 12$.

(b) Sketch the velocity versus time curve over the time interval $0 \le t \le 12$.

(c) Find the x-coordinate of the particle at times $t = 8$ s and $t = 12$ s.

(d) What is the maximum x-coordinate of the particle over the time interval $0 \le t \le 12$?

30–36 In these exercises assume that the object is moving with constant acceleration in the positive direction of a coordinate line, and apply Formulas (10) and (11) as appropriate. In some of these problems you will need the fact that 88 ft/s = 60 mi/h. ■

30. A car traveling 60 mi/h along a straight road decelerates at a constant rate of 11 ft/s^2.

(a) How long will it take until the speed is 45 mi/h?

(b) How far will the car travel before coming to a stop?

31. Spotting a police car, you hit the brakes on your new Porsche to reduce your speed from 90 mi/h to 60 mi/h at a constant rate over a distance of 200 ft.

(a) Find the acceleration in ft/s^2.

(b) How long does it take for you to reduce your speed to 55 mi/h?

(c) At the acceleration obtained in part (a), how long would it take for you to bring your Porsche to a complete stop from 90 mi/h?

32. A particle moving along a straight line is accelerating at a constant rate of 5 m/s^2. Find the initial velocity if the particle moves 60 m in the first 4 s.

33. A motorcycle, starting from rest, speeds up with a constant acceleration of 2.6 m/s^2. After it has traveled 120 m, it slows down with a constant acceleration of -1.5 m/s^2 until it attains a speed of 12 m/s. What is the distance traveled by the motorcycle at that point?

34. A sprinter in a 100 m race explodes out of the starting block with an acceleration of 4.0 m/s^2, which she sustains for 2.0 s. Her acceleration then drops to zero for the rest of race.
(a) What is her time for the race?
(b) Make a graph of her distance from the starting block versus time.

35. A car that has stopped at a toll booth leaves the booth with a constant acceleration of 4 ft/s^2. At the time the car leaves the booth it is 2500 ft behind a truck traveling with a constant velocity of 50 ft/s. How long will it take for the car to catch the truck, and how far will the car be from the toll booth at that time?

36. In the final sprint of a rowing race the challenger is rowing at a constant speed of 12 m/s. At the point where the leader is 100 m from the finish line and the challenger is 15 m behind, the leader is rowing at 8 m/s but starts accelerating at a constant 0.5 m/s^2. Who wins?

37–43 Assume that a free-fall model applies. Solve these exercises by applying Formulas (15) and (16). In these exercises take $g = 32\ \text{ft/s}^2$ or $g = 9.8\ \text{m/s}^2$, depending on the units. ■

37. A projectile is launched vertically upward from ground level with an initial velocity of 112 ft/s.
(a) Find the velocity at $t = 3$ s and $t = 5$ s.
(b) How high will the projectile rise?
(c) Find the speed of the projectile when it hits the ground.

38. A rock tossed downward from a height of 112 ft reaches the ground in 2 s. What is its initial velocity?

39. A projectile is fired vertically upward from ground level with an initial velocity of 16 ft/s.
(a) How long will it take for the projectile to hit the ground?
(b) How long will the projectile be moving upward?

40. In 1939, Joe Sprinz of the San Francisco Seals Baseball Club attempted to catch a ball dropped from a blimp at a height of 800 ft (for the purpose of breaking the record for catching a ball dropped from the greatest height set the preceding year by members of the Cleveland Indians).
(a) How long does it take for a ball to drop 800 ft?
(b) What is the velocity of a ball in miles per hour after an 800 ft drop (88 ft/s = 60 mi/h)?
[*Note:* As a practical matter, it is unrealistic to ignore wind resistance in this problem; however, even with the slowing effect of wind resistance, the impact of the ball slammed Sprinz's glove hand into his face, fractured his upper jaw in 12 places, broke five teeth, and knocked him unconscious. He dropped the ball!]

41. A model rocket is launched upward from ground level with an initial speed of 60 m/s.
(a) How long does it take for the rocket to reach its highest point?
(b) How high does the rocket go?
(c) How long does it take for the rocket to drop back to the ground from its highest point?
(d) What is the speed of the rocket when it hits the ground?

42. (a) Use the results in Exercise 41 to make a conjecture about the relationship between the initial and final speeds of a projectile that is launched upward from ground level and returns to ground level.
(b) Prove your conjecture.

43. A model rocket is fired vertically upward with an initial velocity of 49 m/s from a tower 150 m high.
(a) How long will it take for the rocket to reach its maximum height?
(b) What is the maximum height?
(c) How long will it take for the rocket to pass its starting point on the way down?
(d) What is the velocity when it passes the starting point on the way down?
(e) How long will it take for the rocket to hit the ground?
(f) What will be its speed at impact?

44. **Writing** Make a list of important features of a velocity versus time curve, and interpret each feature in terms of the motion.

45. **Writing** Use Riemann sums to argue informally that integrating speed over a time interval produces the distance traveled.

✔QUICK CHECK ANSWERS 4.7

1. $t^2 + t + 2$ **2.** $6 - 2t$ **3.** $\frac{3}{2}; \frac{5}{2}$ **4.** 36 ft

4.8 AVERAGE VALUE OF A FUNCTION AND ITS APPLICATIONS

In this section we will define the notion of the "average value" of a function, and we will give various applications of this idea.

AVERAGE VELOCITY REVISITED

Let $s = s(t)$ denote the position function of a particle in rectilinear motion. In Section 2.1 we defined the average velocity v_{ave} of the particle over the time interval $[t_0, t_1]$ to be

$$v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}$$

Let $v(t) = s'(t)$ denote the velocity function of the particle. We saw in Section 4.7 that integrating $s'(t)$ over a time interval gives the displacement of the particle over that interval. Thus,

$$\int_{t_0}^{t_1} v(t)\,dt = \int_{t_0}^{t_1} s'(t)\,dt = s(t_1) - s(t_0)$$

It follows that

$$v_{\text{ave}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} v(t)\,dt \tag{1}$$

▶ **Example 1** Suppose that a particle moves along a coordinate line so that its velocity at time t is $v(t) = 2 + \cos t$. Find the average velocity of the particle during the time interval $0 \le t \le \pi$.

Solution. From (1) the average velocity is

$$\frac{1}{\pi - 0}\int_0^{\pi} (2 + \cos t)\,dt = \frac{1}{\pi}\big[2t + \sin t\big]_0^{\pi} = \frac{1}{\pi}(2\pi) = 2 \blacktriangleleft$$

We will see that Formula (1) is a special case of a formula for what we will call the *average value* of a continuous function over a given interval.

AVERAGE VALUE OF A CONTINUOUS FUNCTION

In scientific work, numerical information is often summarized by an *average value* or *mean value* of the observed data. There are various kinds of averages, but the most common is the ***arithmetic mean*** or ***arithmetic average***, which is formed by adding the data and dividing by the number of data points. Thus, the arithmetic average $\overline{a}$ of n numbers $a_1, a_2, \ldots, a_n$ is

$$\overline{a} = \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = \frac{1}{n}\sum_{k=1}^{n} a_k$$

In the case where the a_k's are values of a function f, say,

$$a_1 = f(x_1), a_2 = f(x_2), \ldots, a_n = f(x_n)$$

then the arithmetic average $\overline{a}$ of these function values is

$$\overline{a} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

We will now show how to extend this concept so that we can compute not only the arithmetic average of finitely many function values but an average of *all* values of $f(x)$ as

x varies over a closed interval $[a, b]$. For this purpose recall the Mean-Value Theorem for Integrals (4.6.2), which states that if f is continuous on the interval $[a, b]$, then there is at least one point x^* in this interval such that

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

The quantity

$$f(x^*) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

will be our candidate for the average value of f over the interval $[a, b]$. To explain what motivates this, divide the interval $[a, b]$ into n subintervals of equal length

$$\Delta x = \frac{b-a}{n} \tag{2}$$

and choose arbitrary points $x_1^*, x_2^*, \ldots, x_n^*$ in successive subintervals. Then the arithmetic average of the values $f(x_1^*), f(x_2^*), \ldots, f(x_n^*)$ is

$$\text{ave} = \frac{1}{n}[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]$$

or from (2)

$$\text{ave} = \frac{1}{b-a}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

Taking the limit as $n \to +\infty$ yields

$$\lim_{n\to+\infty}\frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Since this equation describes what happens when we compute the average of "more and more" values of $f(x)$, we are led to the following definition.

Note that the Mean-Value Theorem for Integrals, when expressed in form (3), ensures that there is always at least one point x^* in $[a, b]$ at which the value of f is equal to the average value of f over the interval.

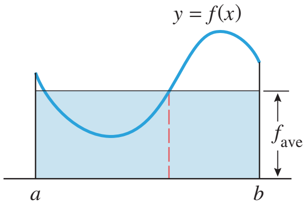
4.8.1 DEFINITION If f is continuous on $[a, b]$, then the ***average value*** (or ***mean value***) of f on $[a, b]$ is defined to be

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{3}$$

REMARK When f is nonnegative on $[a, b]$, the quantity f_{ave} has a simple geometric interpretation, which can be seen by writing (3) as

$$f_{\text{ave}} \cdot (b-a) = \int_a^b f(x)\,dx$$

The left side of this equation is the area of a rectangle with a height of f_{ave} and base of length $b-a$, and the right side is the area under $y = f(x)$ over $[a, b]$. Thus, f_{ave} is the height of a rectangle constructed over the interval $[a, b]$, whose area is the same as the area under the graph of f over that interval (Figure 4.8.1).



▲ **Figure 4.8.1**

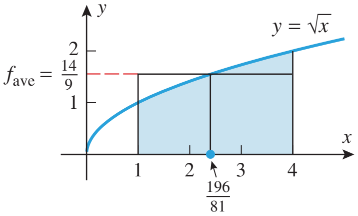
▶ **Example 2** Find the average value of the function $f(x) = \sqrt{x}$ over the interval [1, 4], and find all points in the interval at which the value of f is the same as the average.

Solution.

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\left[\frac{2x^{3/2}}{3}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{16}{3} - \frac{2}{3}\right] = \frac{14}{9} \approx 1.6$$

The x-values at which $f(x) = \sqrt{x}$ is the same as this average satisfy $\sqrt{x} = 14/9$, from which we obtain $x = 196/81 \approx 2.4$ (Figure 4.8.2). ◀



▲ **Figure 4.8.2**

AVERAGE VALUE AND AVERAGE VELOCITY

We now have two ways to calculate the average velocity of a particle in rectilinear motion, since

$$\frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} v(t)\,dt \tag{4}$$

and both of these expressions are equal to the average velocity. The left side of (4) gives the average rate of change of s over $[t_0, t_1]$, while the right side gives the average value of $v = s'$ over the interval $[t_0, t_1]$. That is, *the average velocity of the particle over the time interval* $[t_0, t_1]$ *is the same as the average value of the velocity function over that interval.*

Since velocity functions are generally continuous, it follows from the marginal note associated with Definition 4.8.1 that a particle's average velocity over a time interval matches the particle's velocity at some time in the interval.

▶ **Example 3** Show that if a body released from rest (initial velocity zero) is in free fall, then its average velocity over a time interval $[0, T]$ during its fall is its velocity at time $t = T/2$.

Solution. It follows from Formula (16) of Section 4.7 with $v_0 = 0$ that the velocity function of the body is $v(t) = -gt$. Thus, its average velocity over a time interval $[0, T]$ is

$$\begin{aligned} v_{\text{ave}} &= \frac{1}{T-0}\int_0^T v(t)\,dt \\ &= \frac{1}{T}\int_0^T -gt\,dt \\ &= -\frac{g}{T}\left[\frac{1}{2}t^2\right]_0^T = -g\cdot\frac{T}{2} = v\left(\frac{T}{2}\right) \end{aligned}$$ ◀

The result of Example 3 can be generalized to show that the average velocity of a particle with constant acceleration during a time interval $[a, b]$ is the velocity at time $t = (a+b)/2$. (See Exercise 14.)

✔QUICK CHECK EXERCISES 4.8 *(See page 336 for answers.)*

1. The arithmetic average of n numbers, $a_1, a_2, \ldots, a_n$ is ________.
2. If f is continuous on $[a, b]$, then the average value of f on $[a, b]$ is ________.
3. If f is continuous on $[a, b]$, then the Mean-Value Theorem for Integrals guarantees that for at least one point x^* in $[a, b]$ ________ equals the average value of f on $[a, b]$.
4. The average value of $f(x) = 4x^3$ on [1, 3] is ________.

EXERCISE SET 4.8

$\boxed{\text{C}}$ CAS

1. (a) Find f_{ave} of $f(x) = 2x$ over $[0, 4]$.
(b) Find a point x^* in $[0, 4]$ such that $f(x^*) = f_{\text{ave}}$.
(c) Sketch a graph of $f(x) = 2x$ over $[0, 4]$, and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

2. (a) Find f_{ave} of $f(x) = x^2$ over $[0, 2]$.
(b) Find a point x^* in $[0, 2]$ such that $f(x^*) = f_{\text{ave}}$.
(c) Sketch a graph of $f(x) = x^2$ over $[0, 2]$, and construct a rectangle over the interval whose area is the same as the area under the graph of f over the interval.

3–8 Find the average value of the function over the given interval. ■

3. $f(x) = 3x;\ [1, 3]$

4. $f(x) = \sqrt[3]{x};\ [-1, 8]$

5. $f(x) = \sin x;\ [0, \pi]$

6. $f(x) = \sec x \tan x;\ [0, \pi/3]$

7. $f(x) = \dfrac{x}{(5x^2+1)^2};\ [0, 2]$

8. $f(x) = \sec^2 x;\ [-\pi/4, \pi/4]$

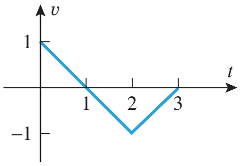
FOCUS ON CONCEPTS

9. Let $f(x) = 3x^2$.
(a) Find the arithmetic average of the values $f(0.4)$, $f(0.8)$, $f(1.2)$, $f(1.6)$, and $f(2.0)$.
(b) Find the arithmetic average of the values $f(0.1)$, $f(0.2)$, $f(0.3), \ldots, f(2.0)$.
(c) Find the average value of f on $[0, 2]$.
(d) Explain why the answer to part (c) is less than the answers to parts (a) and (b).

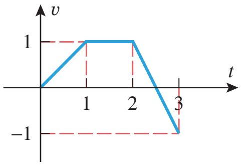
10. In parts (a)–(d), let $f(x) = 1 + \dfrac{1}{x^2}$.
(a) Find the arithmetic average of the values $f\left(\frac{6}{5}\right)$, $f\left(\frac{7}{5}\right)$, $f\left(\frac{8}{5}\right)$, $f\left(\frac{9}{5}\right)$, and $f(2)$.
(b) Find the arithmetic average of the values $f(1.1)$, $f(1.2)$, $f(1.3), \ldots, f(2)$.
(c) Find the average value of f on $[1, 2]$.
(d) Explain why the answer to part (c) is greater than the answers to parts (a) and (b).

11. In each part, the velocity versus time curve is given for a particle moving along a line. Use the curve to find the average velocity of the particle over the time interval $0 \le t \le 3$.

(a)



(b)



12. Suppose that a particle moving along a line starts from rest and has an average velocity of 2 ft/s over the time interval $0 \le t \le 5$. Sketch a velocity versus time curve for the particle assuming that the particle is also at rest at time $t = 5$. Explain how your curve satisfies the required properties.

13. Suppose that f is a linear function. Using the graph of f, explain why the average value of f on $[a, b]$ is

$$f\left(\frac{a+b}{2}\right)$$

14. Suppose that a particle moves along a coordinate line with constant acceleration. Show that the average velocity of the particle during a time interval $[a, b]$ matches the velocity of the particle at the midpoint of the interval.

15–18 True–False Determine whether the statement is true or false. Explain your answer. (Assume that f and g denote continuous functions on an interval $[a, b]$ and that f_{ave} and g_{ave} denote the respective average values of f and g on $[a, b]$.) ■

15. If $g_{\text{ave}} < f_{\text{ave}}$, then $g(x) \le f(x)$ on $[a, b]$.

16. The average value of a constant multiple of f is the same multiple of f_{ave}; that is, if c is any constant,

$$(c \cdot f)_{\text{ave}} = c \cdot f_{\text{ave}}$$

17. The average of the sum of two functions on an interval is the sum of the average values of the two functions on the interval; that is,

$$(f + g)_{\text{ave}} = f_{\text{ave}} + g_{\text{ave}}$$

18. The average of the product of two functions on an interval is the product of the average values of the two functions on the interval; that is,

$$(f \cdot g)_{\text{ave}} = f_{\text{ave}} \cdot g_{\text{ave}}$$

19. (a) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = 3t^3 + 2$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ by integrating.
(b) Suppose that the position function of a particle moving along a coordinate line is $s(t) = 6t^2 + t$. Find the average velocity of the particle over the time interval $1 \le t \le 4$ algebraically.

20. (a) Suppose that the acceleration function of a particle moving along a coordinate line is $a(t) = t + 1$. Find the average acceleration of the particle over the time interval $0 \le t \le 5$ by integrating.
(b) Suppose that the velocity function of a particle moving along a coordinate line is $v(t) = \cos t$. Find the average acceleration of the particle over the time interval $0 \le t \le \pi/4$ algebraically.

21. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3.]

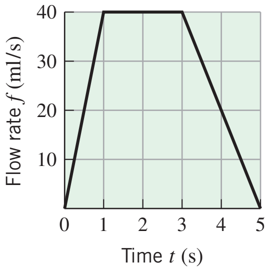
22. (a) The temperature of a 10 m long metal bar is 15°C at one end and 30°C at the other end. Assuming that the temperature increases linearly from the cooler end to the hotter end, what is the average temperature of the bar?

(b) Explain why there must be a point on the bar where the temperature is the same as the average, and find it.

23. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M. Find the average rate, in cars per minute, at which cars enter the highway during the first half hour of rush hour.

24. Suppose that the value of a yacht in dollars after t years of use is $V(t) = 275{,}000\sqrt{\frac{20}{t+20}}$. What is the average value of the yacht over its first 10 years of use?

25. A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second (ml/s). Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.



◀ **Figure Ex-25**

C **26.** The function J_0 defined by

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)\, dt$$

is called the ***Bessel function of order zero***.

(a) Find a function f and an interval $[a, b]$ for which $J_0(1)$ is the average value of f over $[a, b]$.

(b) Estimate $J_0(1)$.

(c) Use a CAS to graph the equation $y = J_0(x)$ over the interval $0 \le x \le 8$.

(d) Estimate the smallest positive zero of J_0.

27. Find a positive value of k such that the average value of $f(x) = \sqrt{3x}$ over the interval $[0, k]$ is 6.

28. Suppose that a tumor grows at the rate of $r(t) = kt$ grams per week for some positive constant k, where t is the number of weeks since the tumor appeared. When, during the second 26 weeks of growth, is the mass of the tumor the same as its average mass during that period?

29. **Writing** Consider the following statement: *The average value of the rate of change of a function over an interval is equal to the average rate of change of the function over that interval.* Write a short paragraph that explains why this statement may be interpreted as a rewording of Part 1 of the Fundamental Theorem of Calculus.

30. **Writing** If an automobile gets an average of 25 miles per gallon of gasoline, then it is also the case that on average the automobile expends 1/25 gallon of gasoline per mile. Interpret this statement using the concept of the average value of a function over an interval.

✔QUICK CHECK ANSWERS 4.8

1. $\frac{1}{n}\sum_{k=1}^{n} a_k$ **2.** $\frac{1}{b-a}\int_a^b f(x)\,dx$ **3.** $f(x^*)$ **4.** 40

4.9 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

In this section we will discuss two methods for evaluating definite integrals in which a substitution is required.

TWO METHODS FOR MAKING SUBSTITUTIONS IN DEFINITE INTEGRALS

Recall from Section 4.3 that indefinite integrals of the form

$$\int f(g(x))g'(x)\,dx$$

can sometimes be evaluated by making the u-substitution

$$u = g(x), \quad du = g'(x)\,dx \tag{1}$$

which converts the integral to the form

$$\int f(u)\,du$$

To apply this method to a definite integral of the form

$$\int_a^b f(g(x))g'(x)\,dx$$

we need to account for the effect that the substitution has on the x-limits of integration. There are two ways of doing this.

Method 1.

First evaluate the indefinite integral

$$\int f(g(x))g'(x)\,dx$$

by substitution, and then use the relationship

$$\int_a^b f(g(x))g'(x)\,dx = \left[\int f(g(x))g'(x)\,dx\right]_a^b$$

to evaluate the definite integral. This procedure does not require any modification of the x-limits of integration.

Method 2.

Make the substitution (1) directly in the definite integral, and then use the relationship $u = g(x)$ to replace the x-limits, $x = a$ and $x = b$, by corresponding u-limits, $u = g(a)$ and $u = g(b)$. This produces a new definite integral

$$\int_{g(a)}^{g(b)} f(u)\,du$$

that is expressed entirely in terms of u.

▶ **Example 1** Use the two methods above to evaluate $\int_0^2 x(x^2+1)^3\,dx$.

Solution by Method 1. If we let

$$u = x^2 + 1 \quad \text{so that} \quad du = 2x\,dx \tag{2}$$

then we obtain

$$\int x(x^2+1)^3\,dx = \frac{1}{2}\int u^3\,du = \frac{u^4}{8} + C = \frac{(x^2+1)^4}{8} + C$$

Thus,

$$\begin{aligned}\int_0^2 x(x^2+1)^3\,dx &= \left[\int x(x^2+1)^3\,dx\right]_{x=0}^2 \\ &= \frac{(x^2+1)^4}{8}\bigg]_{x=0}^2 = \frac{625}{8} - \frac{1}{8} = 78\end{aligned}$$

Solution by Method 2. If we make the substitution $u = x^2+1$ in (2), then

$$\begin{aligned}&\text{if} \quad x = 0, \quad u = 1 \\ &\text{if} \quad x = 2, \quad u = 5\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^2 x(x^2+1)^3\,dx &= \frac{1}{2}\int_1^5 u^3\,du \\ &= \frac{u^4}{8}\bigg]_{u=1}^5 = \frac{625}{8} - \frac{1}{8} = 78\end{aligned}$$

which agrees with the result obtained by Method 1. ◀

The following theorem states precise conditions under which Method 2 can be used.

4.9.1 THEOREM *If g' is continuous on $[a, b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

PROOF Since f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, it follows that f has an antiderivative F on that interval. If we let $u = g(x)$, then the chain rule implies that

$$\frac{d}{dx}F(g(x)) = \frac{d}{dx}F(u) = \frac{dF}{du}\frac{du}{dx} = f(u)\frac{du}{dx} = f(g(x))g'(x)$$

for each x in $[a, b]$. Thus, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ on $[a, b]$. Therefore, by Part 1 of the Fundamental Theorem of Calculus (4.6.1)

$$\int_a^b f(g(x))g'(x)\,dx = F(g(x))\bigg]_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du \quad ■$$

The choice of methods for evaluating definite integrals by substitution is generally a matter of taste, but in the following examples we will use the second method, since the idea is new.

▶ **Example 2** Evaluate

$$\text{(a)} \int_0^{\pi/8} \sin^5 2x \cos 2x\,dx \qquad \text{(b)} \int_2^5 (2x-5)(x-3)^9\,dx$$

Solution (a). Let

$$u = \sin 2x \quad \text{so that} \quad du = 2\cos 2x\, dx \quad \left(\text{or } \tfrac{1}{2}\, du = \cos 2x\, dx\right)$$

With this substitution,

$$\begin{aligned} &\text{if} \quad x = 0, \quad u = \sin(0) = 0 \\ &\text{if} \quad x = \pi/8, \quad u = \sin(\pi/4) = 1/\sqrt{2} \end{aligned}$$

so

$$\begin{aligned} \int_0^{\pi/8} \sin^5 2x \cos 2x\, dx &= \frac{1}{2}\int_0^{1/\sqrt{2}} u^5\, du \\ &= \frac{1}{2}\cdot \frac{u^6}{6}\bigg]_{u=0}^{1/\sqrt{2}} = \frac{1}{2}\left[\frac{1}{6(\sqrt{2})^6} - 0\right] = \frac{1}{96} \end{aligned}$$

Solution (b). Let

$$u = x - 3 \quad \text{so that} \quad du = dx$$

This leaves a factor of $2x - 5$ unresolved in the integrand. However,

$$x = u + 3, \quad \text{so} \quad 2x - 5 = 2(u + 3) - 5 = 2u + 1$$

With this substitution,

$$\begin{aligned} &\text{if} \quad x = 2, \quad u = 2 - 3 = -1 \\ &\text{if} \quad x = 5, \quad u = 5 - 3 = 2 \end{aligned}$$

so

$$\begin{aligned} \int_2^5 (2x - 5)(x - 3)^9\, dx &= \int_{-1}^2 (2u + 1)u^9\, du = \int_{-1}^2 (2u^{10} + u^9)\, du \\ &= \left[\frac{2u^{11}}{11} + \frac{u^{10}}{10}\right]_{u=-1}^2 = \left(\frac{2^{12}}{11} + \frac{2^{10}}{10}\right) - \left(-\frac{2}{11} + \frac{1}{10}\right) \\ &= \frac{52{,}233}{110} \approx 474.8 \; ◀ \end{aligned}$$

▶ **Example 3** Evaluate $\displaystyle\int_1^3 \frac{\cos(\pi/x)}{x^2}\, dx$.

Solution. Let

$$u = \frac{\pi}{x} \quad \text{so that} \quad du = -\frac{\pi}{x^2}\, dx = -\pi \cdot \frac{1}{x^2}\, dx \quad \text{or} \quad -\frac{1}{\pi}\, du = \frac{1}{x^2}\, dx$$

With this substitution,

$$\begin{aligned} &\text{if} \quad x = 1, \quad u = \pi \\ &\text{if} \quad x = 3, \quad u = \pi/3 \end{aligned}$$

Thus,

$$\begin{aligned} \int_1^3 \frac{\cos(\pi/x)}{x^2}\, dx &= -\frac{1}{\pi}\int_\pi^{\pi/3} \cos u\, du \\ &= -\frac{1}{\pi}\sin u\bigg]_{u=\pi}^{\pi/3} = -\frac{1}{\pi}\big(\sin(\pi/3) - \sin \pi\big) \\ &= -\frac{\sqrt{3}}{2\pi} \approx -0.2757 \; ◀ \end{aligned}$$

The u-substitution in Example 3 produces an integral in which the upper u-limit is smaller than the lower u-limit. Use Definition 4.5.3(b) to convert this integral to one whose lower limit is smaller than the upper limit and verify that it produces an integral with the same value as that in the example.

✔ QUICK CHECK EXERCISES 4.9

(See page 342 for answers.)

1. Assume that g' is continuous on $[a, b]$ and that f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$. If F is an antiderivative for f, then

$$\int_a^b f(g(x))g'(x)\,dx = \underline{\hspace{3em}}$$

2. In each part, use the substitution to replace the given integral with an integral involving the variable u. (Do not evaluate the integral.)

(a) $\displaystyle\int_0^2 3x^2(1+x^3)^3\,dx;\; u = 1+x^3$

(b) $\displaystyle\int_0^2 \frac{x}{\sqrt{5-x^2}}\,dx;\; u = 5-x^2$

3. Evaluate the integral by making an appropriate substitution.

(a) $\displaystyle\int_{-\pi}^0 \sin(3x-\pi)\,dx = \underline{\hspace{3em}}$

(b) $\displaystyle\int_0^{\pi/2} \sqrt[3]{\sin x}\cos x\,dx = \underline{\hspace{3em}}$

EXERCISE SET 4.9

Graphing Utility C CAS

1–2 Express the integral in terms of the variable u, but do not evaluate it. ■

1. (a) $\displaystyle\int_1^3 (2x-1)^3\,dx;\; u = 2x-1$

(b) $\displaystyle\int_0^4 3x\sqrt{25-x^2}\,dx;\; u = 25-x^2$

(c) $\displaystyle\int_{-1/2}^{1/2} \cos(\pi\theta)\,d\theta;\; u = \pi\theta$

(d) $\displaystyle\int_0^1 (x+2)(x+1)^5\,dx;\; u = x+1$

2. (a) $\displaystyle\int_{-1}^4 (5-2x)^8\,dx;\; u = 5-2x$

(b) $\displaystyle\int_{-\pi/3}^{2\pi/3} \frac{\sin x}{\sqrt{2+\cos x}}\,dx;\; u = 2+\cos x$

(c) $\displaystyle\int_0^{\pi/4} \tan^2 x\sec^2 x\,dx;\; u = \tan x$

(d) $\displaystyle\int_0^1 x^3\sqrt{x^2+3}\,dx;\; u = x^2+3$

3–12 Evaluate the definite integral two ways: first by a u-substitution in the definite integral and then by a u-substitution in the corresponding indefinite integral. ■

3. $\displaystyle\int_0^1 (2x+1)^3\,dx$

4. $\displaystyle\int_1^2 (4x-2)^3\,dx$

5. $\displaystyle\int_0^1 (2x-1)^3\,dx$

6. $\displaystyle\int_1^2 (4-3x)^8\,dx$

7. $\displaystyle\int_0^8 x\sqrt{1+x}\,dx$

8. $\displaystyle\int_{-3}^0 x\sqrt{1-x}\,dx$

9. $\displaystyle\int_0^{\pi/2} 4\sin(x/2)\,dx$

10. $\displaystyle\int_0^{\pi/6} 2\cos 3x\,dx$

11. $\displaystyle\int_{-2}^{-1} \frac{x}{(x^2+2)^3}\,dx$

12. $\displaystyle\int_{1-\pi}^{1+\pi} \sec^2\left(\tfrac{1}{4}x - \tfrac{1}{4}\right)\,dx$

13–16 Evaluate the definite integral by expressing it in terms of u and evaluating the resulting integral using a formula from geometry. ■

13. $\displaystyle\int_{-5/3}^{5/3} \sqrt{25-9x^2}\,dx;\; u = 3x$

14. $\displaystyle\int_0^2 x\sqrt{16-x^4}\,dx;\; u = x^2$

15. $\displaystyle\int_{\pi/3}^{\pi/2} \sin\theta\sqrt{1-4\cos^2\theta}\,d\theta;\; u = 2\cos\theta$

16. $\displaystyle\int_{-3}^1 \sqrt{3-2x-x^2}\,dx;\; u = x+1$

17. A particle moves with a velocity of $v(t) = \sin \pi t$ m/s along an s-axis. Find the distance traveled by the particle over the time interval $0 \le t \le 1$.

18. A particle moves with a velocity of $v(t) = 3\cos 2t$ m/s along an s-axis. Find the distance traveled by the particle over the time interval $0 \le t \le \pi/8$.

19. Find the area under the curve $y = 9/(x+2)^2$ over the interval $[-1, 1]$.

20. Find the area under the curve $y = 1/(3x+1)^2$ over the interval $[0, 1]$.

21–34 Evaluate the integrals by any method. ■

21. $\displaystyle\int_1^5 \frac{dx}{\sqrt{2x-1}}$

22. $\displaystyle\int_1^2 \sqrt{5x-1}\,dx$

23. $\displaystyle\int_{-1}^1 \frac{x^2\,dx}{\sqrt{x^3+9}}$

24. $\displaystyle\int_{\pi/2}^{\pi} 6\sin x(\cos x+1)^5\,dx$

25. $\displaystyle\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$

26. $\displaystyle\int_1^2 \frac{dx}{x^2-6x+9}$

27. $\displaystyle\int_0^{\pi/4} 4\sin x\cos x\,dx$

28. $\displaystyle\int_0^{\pi/4} \sqrt{\tan x}\sec^2 x\,dx$

29. $\displaystyle\int_0^{\sqrt{\pi}} 5x\cos(x^2)\,dx$

30. $\displaystyle\int_{\pi^2}^{4\pi^2} \frac{1}{\sqrt{x}}\sin\sqrt{x}\,dx$

31. $\displaystyle\int_{\pi/12}^{\pi/9} \sec^2 3\theta\, d\theta$ **32.** $\displaystyle\int_{\pi/6}^{\pi/3} \csc^2 2\theta\, d\theta$

33. $\displaystyle\int_0^1 \frac{y^2\, dy}{\sqrt{4-3y}}$ **34.** $\displaystyle\int_{-1}^4 \frac{x\, dx}{\sqrt{5+x}}$

C **35.** (a) Use a CAS to find the exact value of the integral

$$\int_0^{\pi/6} \sin^4 x \cos^3 x\, dx$$

(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $\cos^2 x = 1 - \sin^2 x$.]

C **36.** (a) Use a CAS to find the exact value of the integral

$$\int_{-\pi/4}^{\pi/4} \tan^4 x\, dx$$

(b) Confirm the exact value by hand calculation. [*Hint:* Use the identity $1 + \tan^2 x = \sec^2 x$.]

37. (a) Find $\displaystyle\int_0^1 f(3x+1)\, dx$ if $\displaystyle\int_1^4 f(x)\, dx = 5$.

(b) Find $\displaystyle\int_0^3 f(3x)\, dx$ if $\displaystyle\int_0^9 f(x)\, dx = 5$.

(c) Find $\displaystyle\int_{-2}^0 xf(x^2)\, dx$ if $\displaystyle\int_0^4 f(x)\, dx = 1$.

38. Given that m and n are positive integers, show that

$$\int_0^1 x^m(1-x)^n\, dx = \int_0^1 x^n(1-x)^m\, dx$$

by making a substitution. Do not attempt to evaluate the integrals.

39. Given that n is a positive integer, show that

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx$$

by using a trigonometric identity and making a substitution. Do not attempt to evaluate the integrals.

40. Given that n is a positive integer, evaluate the integral

$$\int_0^1 x(1-x)^n\, dx$$

41–44 Suppose that an object is launched vertically upwards from the surface of the earth with an initial speed of 3 mi/s, and assume that the only force acting on the object is Earth's gravity. It follows from Newton's Law of Universal Gravitation that the speed v of the object at a distance of b miles above the surface of the earth is approximated by the formula

$$v \approx \left[9 - 191{,}000 \int_a^b \frac{1}{(3963+x)^2}\, dx\right]^{1/2} \quad \text{(mi/s)}$$ ■

41. Estimate the speed of the object at a distance of 200 mi above the surface of the earth.

42. Estimate the speed of the object at a distance of 400 mi above the surface of the earth.

43. Estimate the distance of the object from the surface of the earth when its speed is half its initial speed.

44. Estimate the maximum distance from the surface of the earth achieved by the object.

45. (a) The accompanying table shows the fraction of the Moon that is illuminated (as seen from Earth) at midnight (Eastern Standard Time) for the first week of 2005. Find the average fraction of the Moon illuminated during the first week of 2005.

Source: Data from the U.S Naval Observatory Astronomical Applications Department.

(b) The function $f(x) = 0.5 + 0.5\sin(0.213x + 2.481)$ models data for illumination of the Moon for the first 60 days of 2005. Find the average value of this illumination function over the interval $[0, 7]$.

DAY	1	2	3	4	5	6	7
ILLUMINATION	0.74	0.65	0.56	0.45	0.35	0.25	0.16

▲ **Table Ex-45**

46. Electricity is supplied to homes in the form of ***alternating current***, which means that the voltage has a sinusoidal waveform described by an equation of the form

$$V = V_p \sin(2\pi ft)$$

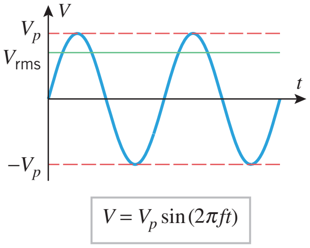
(see the accompanying figure). In this equation, V_p is called the ***peak voltage*** or ***amplitude*** of the current, f is called its ***frequency***, and $1/f$ is called its ***period***. The voltages V and V_p are measured in volts (V), the time t is measured in seconds (s), and the frequency is measured in hertz (Hz). (1 Hz = 1 cycle per second; a ***cycle*** is the electrical term for one period of the waveform.) Most alternating-current voltmeters read what is called the ***rms*** or ***root-mean-square*** value of V. By definition, this is the square root of the average value of V^2 over one period.

(a) Show that

$$V_{\text{rms}} = \frac{V_p}{\sqrt{2}}$$

[*Hint:* Compute the average over the cycle from $t = 0$ to $t = 1/f$, and use the identity $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ to help evaluate the integral.]

(b) In the United States, electrical outlets supply alternating current with an rms voltage of 120 V at a frequency of 60 Hz. What is the peak voltage at such an outlet?



◀ **Figure Ex-46**

C **47.** (a) Find the limit

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{\sin(k\pi/n)}{n}$$

by evaluating an appropriate definite integral over the interval $[0, 1]$.

(b) Check your answer to part (a) by evaluating the limit directly with a CAS.

FOCUS ON CONCEPTS

48. Let

$$I = \int_{-1}^{1} \frac{1}{1+x^2}\, dx$$

(a) Explain why $I > 0$.
(b) Show that the substitution $x = 1/u$ results in

$$I = -\int_{-1}^{1} \frac{1}{1+x^2}\, dx = -I$$

Thus, $2I = 0$, which implies that $I = 0$. But this contradicts part (a). What is the error?

49. (a) Prove that if f is an odd function, then

$$\int_{-a}^{a} f(x)\, dx = 0$$

and give a geometric explanation of this result. [*Hint:* One way to prove that a quantity q is zero is to show that $q = -q$.]
(b) Prove that if f is an even function, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$$

and give a geometric explanation of this result. [*Hint:* Split the interval of integration from $-a$ to a into two parts at 0.]

50. Show that if f and g are continuous functions, then

$$\int_{0}^{t} f(t-x)g(x)\, dx = \int_{0}^{t} f(x)g(t-x)\, dx$$

51. (a) Let

$$I = \int_{0}^{a} \frac{f(x)}{f(x)+f(a-x)}\, dx$$

Show that $I = a/2$.

[*Hint:* Let $u = a - x$, and then note the difference between the resulting integrand and 1.]
(b) Use the result of part (a) to find

$$\int_{0}^{3} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{3-x}}\, dx$$

(c) Use the result of part (a) to find

$$\int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$$

52. Evaluate

(a) $\displaystyle\int_{-1}^{1} x\sqrt{\cos(x^2)}\, dx$

(b) $\displaystyle\int_{0}^{\pi} \sin^8 x \cos^5 x\, dx.$

[*Hint:* Use the substitution $u = x - (\pi/2)$.]

53. **Writing** The two substitution methods discussed in this section yield the same result when used to evaluate a definite integral. Write a short paragraph that carefully explains why this is the case.

54. **Writing** In some cases, the second method for the evaluation of definite integrals has distinct advantages over the first. Provide some illustrations, and write a short paragraph that discusses the advantages of the second method in each case. [*Hint:* To get started, consider the results in Exercises 38–40, 49, and 51.]

✔ QUICK CHECK ANSWERS 4.9

1. $F(g(b)) - F(g(a))$ **2.** (a) $\displaystyle\int_{1}^{9} u^3\, du$ (b) $\displaystyle\int_{1}^{5} \frac{1}{2\sqrt{u}}\, du$ **3.** (a) $\dfrac{2}{3}$ (b) $\dfrac{3}{4}$

CHAPTER 4 REVIEW EXERCISES

Graphing Utility C CAS

1–4 Evaluate the integrals. ■

1. $\displaystyle\int \left[\frac{1}{2x^3} + 4\sqrt{x}\right] dx$ **2.** $\displaystyle\int [u^3 - 2u + 7]\, du$

3. $\displaystyle\int [4\sin x + 2\cos x]\, dx$ **4.** $\displaystyle\int \sec x(\tan x + \cos x)\, dx$

5. Solve the initial-value problems.

(a) $\dfrac{dy}{dx} = \dfrac{1-x}{\sqrt{x}},\ y(1) = 0$

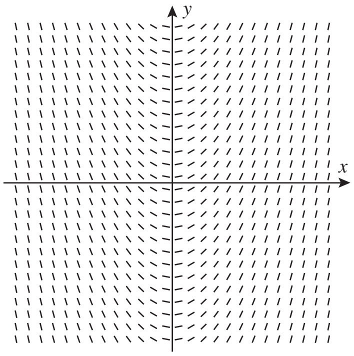
(b) $\dfrac{dy}{dx} = \cos x - 5x,\ y(0) = 1$

(c) $\dfrac{dy}{dx} = \sqrt[3]{x},\ y(1) = 2$

6. The accompanying figure on the next page shows the slope field for a differential equation $dy/dx = f(x)$. Which of the following functions is most likely to be $f(x)$?

$$\sqrt{x}, \quad \sin x, \quad x^4, \quad x$$

Explain your reasoning.



◀ **Figure Ex-6**

7. (a) Show that the substitutions $u = \sec x$ and $u = \tan x$ produce different values for the integral

$$\int \sec^2 x \tan x\, dx$$

(b) Explain why both are correct.

8. Use the two substitutions in Exercise 7 to evaluate the definite integral

$$\int_0^{\pi/4} \sec^2 x \tan x\, dx$$

and confirm that they produce the same result.

9. Evaluate the integral

$$\int \frac{x^7}{\sqrt{x^4+2}}\, dx$$

by making the substitution $u = x^4 + 2$.

10. Evaluate the integral

$$\int \sqrt{1 + x^{-2/3}}\, dx$$

by making the substitution $u = 1 + x^{2/3}$.

C **11–14** Evaluate the integrals by hand, and check your answers with a CAS if you have one. ■

11. $\displaystyle\int \frac{\cos 3x}{\sqrt{5 + 2\sin 3x}}\, dx$

12. $\displaystyle\int \frac{\sqrt{3 + \sqrt{x}}}{\sqrt{x}}\, dx$

13. $\displaystyle\int \frac{x^2}{(ax^3+b)^2}\, dx$

14. $\displaystyle\int x \sec^2(ax^2)\, dx$

15. Express

$$\sum_{k=4}^{18} k(k-3)$$

in sigma notation with

(a) $k = 0$ as the lower limit of summation

(b) $k = 5$ as the lower limit of summation.

16. (a) Fill in the blank:

$$1 + 3 + 5 + \cdots + (2n-1) = \sum_{k=1}^{n} \underline{\qquad\qquad}$$

(b) Use part (a) to prove that the sum of the first n consecutive odd integers is a perfect square.

17. Find the area under the graph of $f(x) = 4x - x^2$ over the interval $[0, 4]$ using Definition 4.4.3 with x_k^* as the *right* endpoint of each subinterval.

18. Find the area under the graph of $f(x) = 5x - x^2$ over the interval $[0, 5]$ using Definition 4.4.3 with x_k^* as the *left* endpoint of each subinterval.

19–20 Use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y = f(x)$ over the stated interval using $n = 10$ subintervals. ■

19. $y = 1/x$; $[1, 2]$

20. $y = \tan x$; $[0, 1]$

21. The *definite integral* of f over the interval $[a, b]$ is defined as the limit

$$\int_a^b f(x)\, dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Explain what the various symbols on the right side of this equation mean.

22. Use a geometric argument to evaluate

$$\int_0^1 |2x - 1|\, dx$$

23. Suppose that

$$\int_0^1 f(x)\, dx = \tfrac{1}{2}, \quad \int_1^2 f(x)\, dx = \tfrac{1}{4},$$

$$\int_0^3 f(x)\, dx = -1, \quad \int_0^1 g(x)\, dx = 2$$

In each part, use this information to evaluate the given integral, if possible. If there is not enough information to evaluate the integral, then say so.

(a) $\displaystyle\int_0^2 f(x)\, dx$ (b) $\displaystyle\int_1^3 f(x)\, dx$ (c) $\displaystyle\int_2^3 5f(x)\, dx$

(d) $\displaystyle\int_1^0 g(x)\, dx$ (e) $\displaystyle\int_0^1 g(2x)\, dx$ (f) $\displaystyle\int_0^1 [g(x)]^2\, dx$

24. In parts (a)–(d), use the information in Exercise 23 to evaluate the given integral. If there is not enough information to evaluate the integral, then say so.

(a) $\displaystyle\int_0^1 [f(x) + g(x)]\, dx$ (b) $\displaystyle\int_0^1 f(x)g(x)\, dx$

(c) $\displaystyle\int_0^1 \frac{f(x)}{g(x)}\, dx$ (d) $\displaystyle\int_0^1 [4g(x) - 3f(x)]\, dx$

25. In each part, evaluate the integral. Where appropriate, you may use a geometric formula.

(a) $\displaystyle\int_{-1}^1 (1 + \sqrt{1 - x^2})\, dx$

(b) $\displaystyle\int_0^3 (x\sqrt{x^2+1} - \sqrt{9 - x^2})\, dx$

(c) $\displaystyle\int_0^1 x\sqrt{1 - x^4}\, dx$

26. In each part, find the limit by interpreting it as a limit of Riemann sums in which the interval [0, 1] is divided into n subintervals of equal length.

(a) $\lim_{n\to+\infty} \dfrac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{3/2}}$

(b) $\lim_{n\to+\infty} \dfrac{1^4+2^4+3^4+\cdots+n^4}{n^5}$

27–34 Evaluate the integrals using the Fundamental Theorem of Calculus and (if necessary) properties of the definite integral.

27. $\displaystyle\int_{-3}^{0} (x^2-4x+7)\,dx$ **28.** $\displaystyle\int_{-1}^{2} x(1+x^3)\,dx$

29. $\displaystyle\int_{1}^{3} \frac{1}{x^2}\,dx$ **30.** $\displaystyle\int_{1}^{8} (5x^{2/3}-4x^{-2})\,dx$

31. $\displaystyle\int_{0}^{1} (x-\sec x\tan x)\,dx$

32. $\displaystyle\int_{1}^{4} \left(\frac{3}{\sqrt{t}}-5\sqrt{t}-t^{-3/2}\right) dt$

33. $\displaystyle\int_{0}^{2} |2x-3|\,dx$ **34.** $\displaystyle\int_{0}^{\pi/2} \left|\tfrac{1}{2}-\sin x\right|\,dx$

35–36 Find the area under the curve $y=f(x)$ over the stated interval.

35. $f(x)=\sqrt{x}$; [1, 9] **36.** $f(x)=x^{-3/5}$; [1, 4]

37. Find the area that is above the x-axis but below the curve $y=(1-x)(x-2)$. Make a sketch of the region.

C **38.** Use a CAS to find the area of the region in the first quadrant that lies below the curve $y=x+x^2-x^3$ and above the x-axis.

39–40 Sketch the curve and find the total area between the curve and the given interval on the x-axis.

39. $y=x^2-1$; [0, 3] **40.** $y=\sqrt{x+1}-1$; [−1, 1]

41. Define $F(x)$ by

$$F(x)=\int_{1}^{x} (t^3+1)\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

42. Define $F(x)$ by

$$F(x)=\int_{4}^{x} \frac{1}{\sqrt{t}}\,dt$$

(a) Use Part 2 of the Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check the result in part (a) by first integrating and then differentiating.

43–46 Use Part 2 of the Fundamental Theorem of Calculus to find the derivatives.

43. $\dfrac{d}{dx}\left[\displaystyle\int_{0}^{x} \frac{1}{t^4+5}\,dt\right]$ **44.** $\dfrac{d}{dx}\left[\displaystyle\int_{0}^{x} \frac{t}{\cos t^2}\,dt\right]$

45. $\dfrac{d}{dx}\left[\displaystyle\int_{0}^{x} |t-1|\,dt\right]$ **46.** $\dfrac{d}{dx}\left[\displaystyle\int_{\pi}^{x} \cos\sqrt{t}\,dt\right]$

47. State the two parts of the Fundamental Theorem of Calculus, and explain what is meant by the statement "Differentiation and integration are inverse processes."

C **48.** Let $F(x)=\displaystyle\int_{0}^{x} \frac{t^2-3}{t^4+7}\,dt$.

(a) Find the intervals on which F is increasing and those on which F is decreasing.

(b) Find the open intervals on which F is concave up and those on which F is concave down.

(c) Find the x-values, if any, at which the function F has absolute extrema.

(d) Use a CAS to graph F, and confirm that the results in parts (a), (b), and (c) are consistent with the graph.

49. Use differentiation to prove that the function

$$F(x)=\int_{0}^{x} \frac{1}{1+t^2}\,dt+\int_{0}^{1/x} \frac{1}{1+t^2}\,dt$$

is constant on the interval $(0,+\infty)$.

50. What is the natural domain of the function

$$F(x)=\int_{1}^{x} \frac{1}{t^2-9}\,dt?$$

Explain your reasoning.

51. In each part, determine the values of x for which $F(x)$ is positive, negative, or zero without performing the integration; explain your reasoning.

(a) $F(x)=\displaystyle\int_{1}^{x} \frac{t^4}{t^2+3}\,dt$ (b) $F(x)=\displaystyle\int_{-1}^{x} \sqrt{4-t^2}\,dt$

C **52.** Use a CAS to approximate the largest and smallest values of the integral

$$\int_{-1}^{x} \frac{t}{\sqrt{2+t^3}}\,dt$$

for $1\le x\le 3$.

53. Find all values of x^* in the stated interval that are guaranteed to exist by the Mean-Value Theorem for Integrals, and explain what these numbers represent.

(a) $f(x)=\sqrt{x}$; [0, 3] (b) $f(x)=2x-x^2$; [0, 2]

54. A 10-gram tumor is discovered in a laboratory rat on March 1. The tumor is growing at a rate of $r(t)=t/7$ grams per week, where t denotes the number of weeks since March 1. What will be the mass of the tumor on June 7?

55. Use the graph of f shown in the accompanying figure on the next page to find the average value of f on the interval [0, 10].

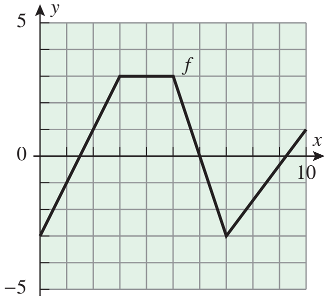


Figure Ex-55

56. Find the average value of $f(x) = x^2 + \dfrac{1}{x^2}$ over the interval $\left[\frac{1}{2}, 2\right]$.

57. Derive the formulas for the position and velocity functions of a particle that moves with constant acceleration along a coordinate line.

58. The velocity of a particle moving along an s-axis is measured at 5 s intervals for 40 s, and the velocity function is modeled by a smooth curve. (The curve and the data points are shown in the accompanying figure.) Use this model in each part.

(a) Does the particle have constant acceleration? Explain your reasoning.

(b) Is there any 15 s time interval during which the acceleration is constant? Explain your reasoning.

(c) Estimate the distance traveled by the particle from time $t = 0$ to time $t = 40$.

(d) Estimate the average velocity of the particle over the 40 s time period.

(e) Is the particle ever slowing down during the 40 s time period? Explain your reasoning.

(f) Is there sufficient information for you to determine the s-coordinate of the particle at time $t = 10$? If so, find it. If not, explain what additional information you need.

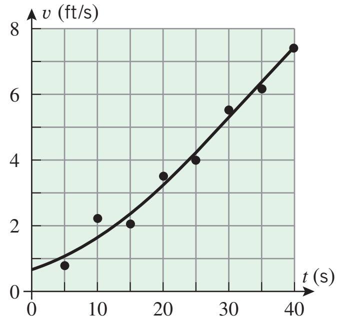


Figure Ex-58

59–62 A particle moves along an s-axis. Use the given information to find the position function of the particle.

59. $v(t) = t^3 - 2t^2 + 1;\ s(0) = 1$

60. $a(t) = 4\cos 2t;\ v(0) = -1, s(0) = -3$

61. $v(t) = 2t - 3;\ s(1) = 5$

62. $a(t) = \cos t - 2t;\ v(0) = 0, s(0) = 0$

63–66 A particle moves with a velocity of $v(t)$ m/s along an s-axis. Find the displacement and the distance traveled by the particle during the given time interval.

63. $v(t) = 2t - 4;\ 0 \le t \le 6$

64. $v(t) = |t - 3|;\ 0 \le t \le 5$

65. $v(t) = \dfrac{1}{2} - \dfrac{1}{t^2};\ 1 \le t \le 3$

66. $v(t) = \dfrac{3}{\sqrt{t}};\ 4 \le t \le 9$

67–68 A particle moves with acceleration $a(t)$ m/s^2 along an s-axis and has velocity v_0 m/s at time $t = 0$. Find the displacement and the distance traveled by the particle during the given time interval.

67. $a(t) = -2;\ v_0 = 3;\ 1 \le t \le 4$

68. $a(t) = \dfrac{1}{\sqrt{5t+1}};\ v_0 = 2;\ 0 \le t \le 3$

69. A car traveling 60 mi/h (= 88 ft/s) along a straight road decelerates at a constant rate of 10 ft/s^2.

(a) How long will it take until the speed is 45 mi/h?

(b) How far will the car travel before coming to a stop?

70. Suppose that the velocity function of a particle moving along an s-axis is $v(t) = 20t^2 - 100t + 50$ ft/s and that the particle is at the origin at time $t = 0$. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for the first 6 s of motion.

71. A ball is thrown vertically upward from a height of s_0 ft with an initial velocity of v_0 ft/s. If the ball is caught at height s_0, determine its average speed through the air using the free-fall model.

72. A rock, dropped from an unknown height, strikes the ground with a speed of 24 m/s. Find the height from which the rock was dropped.

73–77 Evaluate the integrals by making an appropriate substitution.

73. $\displaystyle\int_0^1 (2x+1)^4\,dx$

74. $\displaystyle\int_{-5}^0 x\sqrt{4-x}\,dx$

75. $\displaystyle\int_0^1 \frac{dx}{\sqrt{3x+1}}$

76. $\displaystyle\int_0^{\sqrt{\pi}} x\sin x^2\,dx$

77. $\displaystyle\int_0^1 \sin^2(\pi x)\cos(\pi x)\,dx$

78. Find a function f and a number a such that

$$2 + \int_a^x f(t)\,dt = \frac{8}{x+3}$$

CHAPTER 4 MAKING CONNECTIONS

1. Consider a Riemann sum

$$\sum_{k=1}^{n} 2x_k^* \Delta x_k$$

for the integral of $f(x) = 2x$ over an interval $[a, b]$.

(a) Show that if x_k^* is the midpoint of the kth subinterval, the Riemann sum is a telescoping sum. (See Exercises 57–60 of Section 4.4 for other examples of telescoping sums.)

(b) Use part (a), Definition 4.5.1, and Theorem 4.5.2 to evaluate the definite integral of $f(x) = 2x$ over $[a, b]$.

2. The function $f(x) = \sqrt{x}$ is continuous on $[0, 4]$ and therefore integrable on this interval. Evaluate

$$\int_0^4 \sqrt{x}\, dx$$

by using Definition 4.5.1. Use subintervals of unequal length given by the partition

$$0 < 4(1)^2/n^2 < 4(2)^2/n^2 < \cdots < 4(n-1)^2/n^2 < 4$$

and let x_k^* be the right endpoint of the kth subinterval.

3. Make appropriate modifications and repeat Exercise 2 for

$$\int_0^8 \sqrt[3]{x}\, dx$$

4. Given a continuous function f and a positive real number m, let g denote the function defined by the composition $g(x) = f(mx)$.

(a) Suppose that

$$\sum_{k=1}^{n} g(x_k^*) \Delta x_k$$

is any Riemann sum for the integral of g over $[0, 1]$. Use the correspondence $u_k = mx_k$, $u_k^* = mx_k^*$ to create a Riemann sum for the integral of f over $[0, m]$. How are the values of the two Riemann sums related?

(b) Use part (a), Definition 4.5.1, and Theorem 4.5.2 to find an equation that relates the integral of g over $[0, 1]$ with the integral of f over $[0, m]$.

(c) How is your answer to part (b) related to Theorem 4.9.1?

5. Given a continuous function f, let g denote the function defined by $g(x) = 2xf(x^2)$.

(a) Suppose that

$$\sum_{k=1}^{n} g(x_k^*) \Delta x_k$$

is any Riemann sum for the integral of g over $[2, 3]$, with $x_k^* = (x_k + x_{k-1})/2$ the midpoint of the kth subinterval. Use the correspondence $u_k = x_k^2$, $u_k^* = (x_k^*)^2$ to create a Riemann sum for the integral of f over $[4, 9]$. How are the values of the two Riemann sums related?

(b) Use part (a), Definition 4.5.1, and Theorem 4.5.2 to find an equation that relates the integral of g over $[2, 3]$ with the integral of f over $[4, 9]$.

(c) How is your answer to part (b) related to Theorem 4.9.1?

Chapter V

EXPONENTIAL,LOGARITHMIC,ANDINVERSETRIGONOMETRIC FUNCTIONS

Craig Lovell/Corbis Images

6

EXPONENTIAL, LOGARITHMIC, AND INVERSE TRIGONOMETRIC FUNCTIONS

The growth and decline of animal populations and natural resources can be modeled using basic functions studied in this chapter.

We begin this chapter with a review of exponential and logarithmic functions. These functions have important applications, from modeling population growth and the spread of disease, to the measurement of the magnitude of an earthquake or the perceived loudness of a sound. Logarithmic and exponential functions are best understood within the context of inverse functions and we will derive an important relationship between the derivative of a function and the derivative of its inverse. This connection will allow us to compute derivative formulas for logarithmic and exponential functions, along with their associated integration formulas. Later in the chapter we will exploit this connection again, to find the derivatives of inverse trigonometric functions, together with some related integration formulas. Along the way, we will discuss L'Hôpital's rule, a powerful tool for evaluating limits. We conclude the chapter with a study of some important combinations of exponential functions known as "hyperbolic functions."

6.1 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

When logarithms were introduced in the seventeenth century as a computational tool, they provided scientists of that period computing power that was previously unimaginable. Although computers and calculators have replaced logarithm tables for numerical calculations, the logarithmic functions have wide-ranging applications in mathematics and science. In this section we will review some properties of exponents and logarithms and then develop results about exponential and logarithmic functions.

IRRATIONAL EXPONENTS

Recall from algebra that if b is a nonzero real number, then nonzero *integer* powers of b are defined by

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n\text{ factors}} \quad\text{and}\quad b^{-n} = \frac{1}{b^n}$$

and if $n = 0$, then $b^0 = 1$. Also, if p/q is a positive *rational* number expressed in lowest terms, then

$$b^{p/q} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p \quad\text{and}\quad b^{-p/q} = \frac{1}{b^{p/q}}$$

If b is negative, then some fractional powers of b will have imaginary values—the quantity $(-2)^{1/2} = \sqrt{-2}$, for example. To avoid this complication, we will assume throughout this section that $b > 0$, even if it is not stated explicitly.

There are various methods for defining *irrational* powers such as

$$2^{\pi}, \quad 3^{\sqrt{2}}, \quad \pi^{-\sqrt{7}}$$

One approach is to define irrational powers of b via successive approximations using rational powers of b. For example, to define 2^{π} consider the decimal representation of π:

$$3.1415926\ldots$$

Table 6.1.1

x	2^x
3	8.000000
3.1	8.574188
3.14	8.815241
3.141	8.821353
3.1415	8.824411
3.14159	8.824962
3.141592	8.824974
3.1415926	8.824977

From this decimal we can form a sequence of rational numbers that gets closer and closer to π, namely,

$$3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159$$

and from these we can form a sequence of *rational* powers of 2:

$$2^{3.1}, \quad 2^{3.14}, \quad 2^{3.141}, \quad 2^{3.1415}, \quad 2^{3.14159}$$

Since the exponents of the terms in this sequence get successively closer to π, it seems plausible that the terms themselves will get successively closer to some number. It is that number that we *define* to be 2^{π}. This is illustrated in Table 6.1.1, which we generated using a calculator. The table suggests that to four decimal places the value of 2^{π} is

$$2^{\pi} \approx 8.8250 \tag{1}$$

TECHNOLOGY MASTERY

Use a calculating utility to verify the results in Table 6.1.1, and then verify (1) by using the utility to compute 2^{π} directly.

With this notion for irrational powers, we remark without proof that the following familiar laws of exponents hold for all real values of p and q:

$$b^p b^q = b^{p+q}, \qquad \frac{b^p}{b^q} = b^{p-q}, \qquad \left(b^p\right)^q = b^{pq}$$

THE FAMILY OF EXPONENTIAL FUNCTIONS

A function of the form $f(x) = b^x$, where $b > 0$, is called an ***exponential function with base b***. Some examples are

$$f(x) = 2^x, \qquad f(x) = \left(\tfrac{1}{2}\right)^x, \qquad f(x) = \pi^x$$

Note that an exponential function has a constant base and variable exponent. Thus, functions such as $f(x) = x^2$ and $f(x) = x^{\pi}$ would *not* be classified as exponential functions, since they have a variable base and a constant exponent.

Figure 6.1.1 illustrates that the graph of $y = b^x$ has one of three general forms, depending on the value of b. The graph of $y = b^x$ has the following properties:

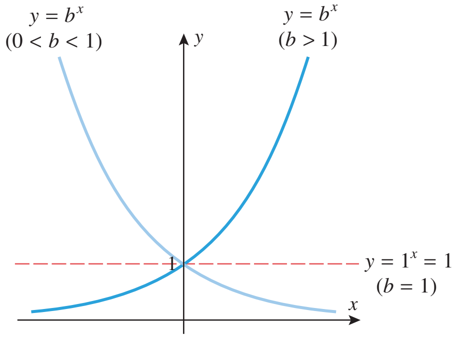
- The graph passes through (0, 1) because $b^0 = 1$.
- If $b > 1$, the value of b^x increases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x increase indefinitely. If you traverse the graph from right to left, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x.
- If $0 < b < 1$, the value of b^x decreases as x increases. As you traverse the graph of $y = b^x$ from left to right, the values of b^x decrease toward zero but never reach zero. Thus, the x-axis is a horizontal asymptote of the graph of b^x. If you traverse the graph from right to left, the values of b^x increase indefinitely.
- If $b = 1$, then the value of b^x is constant.

Some typical members of the family of exponential functions are graphed in Figure 6.1.2. This figure illustrates that the graph of $y = (1/b)^x$ is the reflection of the graph of

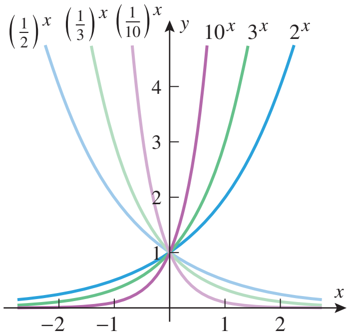
$y = b^x$ about the y-axis. This is because replacing x by $-x$ in the equation $y = b^x$ yields

$$y = b^{-x} = (1/b)^x$$

The figure also conveys that for $b > 1$, the larger the base b, the more rapidly the function $f(x) = b^x$ increases for $x > 0$.



▲ **Figure 6.1.1**



▲ **Figure 6.1.2** The family $y = b^x \, (b > 0)$

Since it is not our objective in this section to develop the properties of exponential functions in rigorous mathematical detail, we will simply observe without proof that the following properties of exponential functions are consistent with the graphs shown in Figures 6.1.1 and 6.1.2.

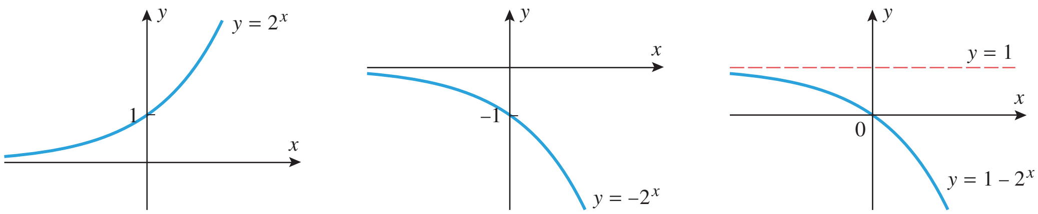
6.1.1 THEOREM *If $b > 0$ and $b \neq 1$, then:*

(*a*) *The function $f(x) = b^x$ is defined for all real values of x, so its natural domain is* $(-\infty, +\infty)$.

(*b*) *The function $f(x) = b^x$ is continuous on the interval* $(-\infty, +\infty)$, *and its range is* $(0, +\infty)$.

▶ **Example 1** Sketch the graph of the function $f(x) = 1 - 2^x$ and find its domain and range.

Solution. Start with a graph of $y = 2^x$. Reflect this graph across the x-axis to obtain the graph of $y = -2^x$, then translate that graph upward by 1 unit to obtain the graph of $y = 1 - 2^x$ (Figure 6.1.3). The dashed line in the third part of Figure 6.1.3 is a horizontal asymptote for the graph. You should be able to see from the graph that the domain of f is $(-\infty, +\infty)$ and the range is $(-\infty, 1)$. ◀



▲ **Figure 6.1.3**

THE NATURAL EXPONENTIAL FUNCTION

Among all possible bases for exponential functions there is one particular base that plays a special role in calculus. That base, denoted by the letter e, is a certain irrational number whose value to six decimal places is

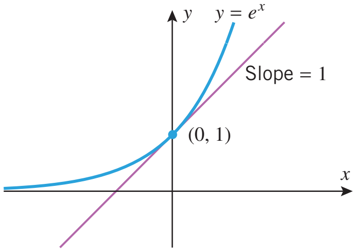
$$e \approx 2.718282 \tag{2}$$

This base is important in calculus because, as we will prove later, $b = e$ is the only base for which the slope of the tangent line to the curve $y = b^x$ at any point P on the curve is equal to the y-coordinate at P. Thus, for example, the tangent line to $y = e^x$ at (0, 1) has slope 1 (Figure 6.1.4).

The function $f(x) = e^x$ is called the ***natural exponential function***. To simplify typography, the natural exponential function is sometimes written as $\exp(x)$, in which case the relationship $e^{x_1+x_2} = e^{x_1}e^{x_2}$ would be expressed as

$$\exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$

The use of the letter e is in honor of the Swiss mathematician Leonhard Euler (biography on p. 3) who is credited with recognizing the mathematical importance of this constant.



▲ **Figure 6.1.4** The tangent line to the graph of $y = e^x$ at (0, 1) has slope 1.

TECHNOLOGY MASTERY

Your technology utility should have keys or commands for approximating e and for graphing the natural exponential function. Read your documentation on how to do this and use your utility to confirm (2) and to generate the graphs in Figures 6.1.2 and 6.1.4.

The constant e also arises in the context of the graph of the equation

$$y = \left(1 + \frac{1}{x}\right)^x \tag{3}$$

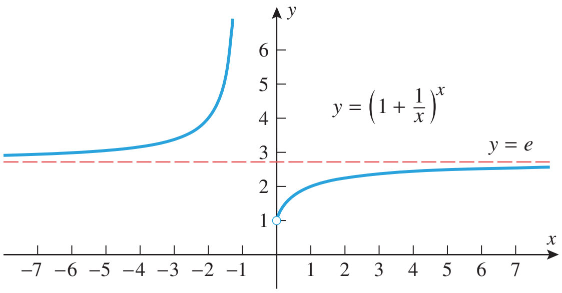
As suggested by Figure 6.1.5 and Table 6.1.2, $y = e$ is a horizontal asymptote of this graph, and the limits

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e \quad \text{and} \quad \lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e \tag{4–5}$$

are satisfied. These limits can be derived from the limit

$$\lim_{x \to 0} (1 + x)^{1/x} = e \tag{6}$$

which is sometimes taken as the definition of e.



▲ **Figure 6.1.5**

Table 6.1.2

THE VALUES OF $(1 + 1/x)^x$ APPROACH e AS $x \to +\infty$

x	$1 + \frac{1}{x}$	$\left(1 + \frac{1}{x}\right)^x$
1	2	≈ 2.000000
10	1.1	2.593742
100	1.01	2.704814
1000	1.001	2.716924
10,000	1.0001	2.718146
100,000	1.00001	2.718268
1,000,000	1.000001	2.718280

LOGARITHMIC FUNCTIONS

Recall from algebra that a logarithm is an exponent. More precisely, if $b > 0$ and $b \neq 1$, then for a positive value of x the expression

$$\log_b x$$

(read "the logarithm to the base b of x") denotes that exponent to which b must be raised to produce x. Thus, for example,

$$\underset{\boxed{10^2 = 100}}{\log_{10} 100 = 2}, \quad \underset{\boxed{10^{-3} = 1/1000}}{\log_{10}(1/1000) = -3}, \quad \underset{\boxed{2^4 = 16}}{\log_2 16 = 4}, \quad \underset{\boxed{b^0 = 1}}{\log_b 1 = 0}, \quad \underset{\boxed{b^1 = b}}{\log_b b = 1}$$

Logarithms with base 10 are called ***common logarithms*** and are often written without explicit reference to the base. Thus, the symbol $\log x$ generally denotes $\log_{10} x$.

We call the function $f(x) = \log_b x$ the ***logarithmic function with base b***.

Logarithmic functions can also be viewed as inverses of exponential functions. To see why this is so, observe from Figure 6.1.1 that if $b > 0$ and $b \neq 1$, then the graph of $f(x) = b^x$ passes the horizontal line test, so b^x has an inverse. We can find a formula for this inverse with x as the independent variable by solving the equation

$$x = b^y$$

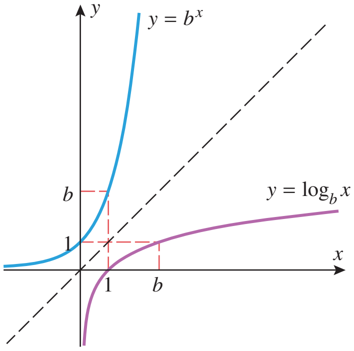
for y as a function of x. But this equation states that y is the logarithm to the base b of x, so it can be rewritten as

$$y = \log_b x$$

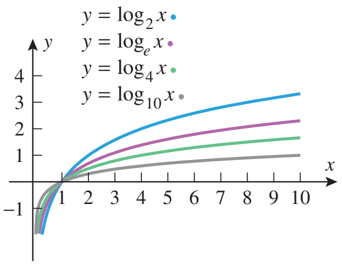
Thus, we have established the following result.

6.1.2 THEOREM *If $b > 0$ and $b \neq 1$, then b^x and $\log_b x$ are inverse functions.*

It follows from this theorem that the graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$ (see Figure 6.1.6 for the case where $b > 1$). Figure 6.1.7 shows the graphs of $y = \log_b x$ for various values of b. Observe that they all pass through the point $(1, 0)$.



▲ **Figure 6.1.6**



▲ **Figure 6.1.7** The family $y = \log_b x$ $(b > 1)$

The most important logarithm function in applications is the one with base e. This is called the ***natural logarithm function*** because the function $\log_e x$ is the inverse of the natural exponential function e^x. It is standard to denote the natural logarithm of x by $\ln x$ (read "ell en of x"), rather than $\log_e x$. For example,

$$\underset{\boxed{\text{Since } e^0 = 1}}{\ln 1 = 0}, \quad \underset{\boxed{\text{Since } e^1 = e}}{\ln e = 1}, \quad \underset{\boxed{\text{Since } e^{-1} = 1/e}}{\ln 1/e = -1}, \quad \underset{\boxed{\text{Since } e^2 = e^2}}{\ln(e^2) = 2}$$

In general,

$$y = \ln x \quad \text{if and only if} \quad x = e^y$$

As shown in Table 6.1.3, the inverse relationship between b^x and $\log_b x$ produces a correspondence between some basic properties of those functions.

It also follows from the cancellation properties of inverse functions [see (3) in Section 0.4] that

$$\begin{aligned} \log_b(b^x) &= x \quad \text{for all real values of } x \\ b^{\log_b x} &= x \quad \text{for } x > 0 \end{aligned} \tag{7}$$

In the special case where $b = e$, these equations become

$$\begin{aligned} \ln(e^x) &= x \quad \text{for all real values of } x \\ e^{\ln x} &= x \quad \text{for } x > 0 \end{aligned} \tag{8}$$

In words, the functions b^x and $\log_b x$ cancel out the effect of one another when composed

TECHNOLOGY MASTERY

Use your graphing utility to generate the graphs of $y = \ln x$ and $y = \log x$.

Table 6.1.3

CORRESPONDENCE BETWEEN PROPERTIES OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

PROPERTY OF b^x	PROPERTY OF $\log_b x$
$b^0 = 1$	$\log_b 1 = 0$
$b^1 = b$	$\log_b b = 1$
Range is $(0, +\infty)$	Domain is $(0, +\infty)$
Domain is $(-\infty, +\infty)$	Range is $(-\infty, +\infty)$
x-axis is a horizontal asymptote	y-axis is a vertical asymptote

in either order; for example,

$$\log 10^x = x, \quad 10^{\log x} = x, \quad \ln e^x = x, \quad e^{\ln x} = x, \quad \ln e^5 = 5, \quad e^{\ln \pi} = \pi$$

SOLVING EQUATIONS INVOLVING EXPONENTIALS AND LOGARITHMS

You should be familiar with the following properties of logarithms from your earlier studies.

6.1.3 THEOREM (*Algebraic Properties of Logarithms*) *If $b > 0$, $b \neq 1$, $a > 0$, $c > 0$, and r is any real number, then:*

(*a*) $\log_b(ac) = \log_b a + \log_b c$ Product property

(*b*) $\log_b(a/c) = \log_b a - \log_b c$ Quotient property

(*c*) $\log_b(a^r) = r \log_b a$ Power property

(*d*) $\log_b(1/c) = -\log_b c$ Reciprocal property

WARNING

Expressions of the form $\log_b(u + v)$ and $\log_b(u - v)$ have no useful simplifications. In particular,

$$\log_b(u + v) \neq \log_b(u) + \log_b(v)$$

$$\log_b(u - v) \neq \log_b(u) - \log_b(v)$$

These properties are often used to expand a single logarithm into sums, differences, and multiples of other logarithms and, conversely, to condense sums, differences, and multiples of logarithms into a single logarithm. For example,

$$\log \frac{xy^5}{\sqrt{z}} = \log xy^5 - \log \sqrt{z} = \log x + \log y^5 - \log z^{1/2} = \log x + 5 \log y - \tfrac{1}{2} \log z$$

$$5 \log 2 + \log 3 - \log 8 = \log 32 + \log 3 - \log 8 = \log \frac{32 \cdot 3}{8} = \log 12$$

$$\tfrac{1}{3} \ln x - \ln(x^2 - 1) + 2 \ln(x + 3) = \ln x^{1/3} - \ln(x^2 - 1) + \ln(x + 3)^2 = \ln \frac{\sqrt[3]{x}(x + 3)^2}{x^2 - 1}$$

An equation of the form $\log_b x = k$ can be solved for x by rewriting it in the exponential form $x = b^k$, and an equation of the form $b^x = k$ can be solved by rewriting it in the logarithm form $x = \log_b k$. Alternatively, the equation $b^x = k$ can be solved by taking *any* logarithm of both sides (but usually log or ln) and applying part (*c*) of Theorem 6.1.3. These ideas are illustrated in the following example.

▶ **Example 2** Find x such that

(a) $\log x = \sqrt{2}$ (b) $\ln(x + 1) = 5$ (c) $5^x = 7$

Solution (a). Converting the equation to exponential form yields

$$x = 10^{\sqrt{2}} \approx 25.95$$

Solution (b). Converting the equation to exponential form yields

$$x + 1 = e^5 \quad \text{or} \quad x = e^5 - 1 \approx 147.41$$

Solution (c). Converting the equation to logarithmic form yields

$$x = \log_5 7 \approx 1.21$$

Alternatively, taking the natural logarithm of both sides and using the power property of logarithms yields

$$x \ln 5 = \ln 7 \quad \text{or} \quad x = \frac{\ln 7}{\ln 5} \approx 1.21 \blacktriangleleft$$

Erik Simonsen/Getty Images

Power to satellites can be supplied by batteries, fuel cells, solar cells, or radioisotope devices.

▶ **Example 3** A satellite that requires 7 watts of power to operate at full capacity is equipped with a radioisotope power supply whose power output P in watts is given by the equation

$$P = 75e^{-t/125}$$

where t is the time in days that the supply is used. How long can the satellite operate at full capacity?

Solution. The power P will fall to 7 watts when

$$7 = 75e^{-t/125}$$

The solution for t is as follows:

$$\begin{aligned} 7/75 &= e^{-t/125} \\ \ln(7/75) &= \ln(e^{-t/125}) \\ \ln(7/75) &= -t/125 \\ t &= -125\ln(7/75) \approx 296.4 \end{aligned}$$

so the satellite can operate at full capacity for about 296 days. ◀

Here is a more complicated example.

▶ **Example 4** Solve $\dfrac{e^x - e^{-x}}{2} = 1$ for x.

Solution. Multiplying both sides of the given equation by 2 yields

$$e^x - e^{-x} = 2$$

or equivalently,

$$e^x - \frac{1}{e^x} = 2$$

Multiplying through by e^x yields

$$e^{2x} - 1 = 2e^x \quad \text{or} \quad e^{2x} - 2e^x - 1 = 0$$

This is really a quadratic equation in disguise, as can be seen by rewriting it in the form

$$\left(e^x\right)^2 - 2e^x - 1 = 0$$

and letting $u = e^x$ to obtain

$$u^2 - 2u - 1 = 0$$

Solving for u by the quadratic formula yields

$$u = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

or, since $u = e^x$,

$$e^x = 1 \pm \sqrt{2}$$

But e^x cannot be negative, so we discard the negative value $1 - \sqrt{2}$; thus,

$$\begin{aligned} e^x &= 1 + \sqrt{2} \\ \ln e^x &= \ln(1 + \sqrt{2}\,) \\ x &= \ln(1 + \sqrt{2}\,) \approx 0.881 \end{aligned}$$ ◄

CHANGE OF BASE FORMULA FOR LOGARITHMS

Scientific calculators generally have no keys for evaluating logarithms with bases other than 10 or e. However, this is not a serious deficiency because it is possible to express a logarithm with any base in terms of logarithms with any other base (see Exercise 42). For example, the following formula expresses a logarithm with base b in terms of natural logarithms:

$$\boxed{\log_b x = \frac{\ln x}{\ln b}} \tag{9}$$

We can derive this result by letting $y = \log_b x$, from which it follows that $b^y = x$. Taking the natural logarithm of both sides of this equation we obtain $y \ln b = \ln x$, from which (9) follows.

▶ **Example 5** Use a calculating utility to evaluate $\log_2 5$ by expressing this logarithm in terms of natural logarithms.

Solution. From (9) we obtain

$$\log_2 5 = \frac{\ln 5}{\ln 2} \approx 2.321928$$ ◄

LOGARITHMIC SCALES IN SCIENCE AND ENGINEERING

Logarithms are used in science and engineering to deal with quantities whose units vary over an excessively wide range of values. For example, the "loudness" of a sound can be measured by its ***intensity*** I (in watts per square meter), which is related to the energy transmitted by the sound wave—the greater the intensity, the greater the transmitted energy, and the louder the sound is perceived by the human ear. However, intensity units are unwieldy because they vary over an enormous range. For example, a sound at the threshold of human hearing has an intensity of about $10^{-12}\ \mathrm{W/m^2}$, a close whisper has an intensity that is about 100 times the hearing threshold, and a jet engine at 50 meters has an intensity that is about $10{,}000{,}000{,}000{,}000 = 10^{13}$ times the hearing threshold. To see how logarithms can be used to reduce this wide spread, observe that if

$$y = \log x$$

then increasing x by a *factor* of 10 *adds* 1 unit to y since

$$\log 10x = \log 10 + \log x = 1 + y$$

Table 6.1.4

β (dB)	I/I_0
0	$10^0 = 1$
10	$10^1 = 10$
20	$10^2 = 100$
30	$10^3 = 1000$
40	$10^4 = 10{,}000$
50	$10^5 = 100{,}000$
⋮	⋮
120	$10^{12} = 1{,}000{,}000{,}000{,}000$

Physicists and engineers take advantage of this property by measuring loudness in terms of the ***sound level*** β, which is defined by

$$\beta = 10\log(I/I_0)$$

where $I_0 = 10^{-12}\ \mathrm{W/m^2}$ is a reference intensity close to the threshold of human hearing. The units of β are ***decibels*** (dB), named in honor of the telephone inventor Alexander Graham Bell. With this scale of measurement, *multiplying* the intensity I by a factor of 10 *adds* 10 dB to the sound level β (verify). This results in a more tractable scale than intensity for measuring sound loudness (Table 6.1.4). Some other familiar logarithmic scales are the ***Richter scale*** used to measure earthquake intensity and the **pH *scale*** used to measure acidity in chemistry, both of which are discussed in the exercises.

▶ **Example 6** A space shuttle taking off generates a sound level of 150 dB near the launch-pad. A person exposed to this level of sound would experience severe physical injury. By comparison, a car horn at one meter has a sound level of 110 dB, near the threshold of pain for many people. What is the ratio of sound intensity of a space shuttle takeoff to that of a car horn?

Solution. Let I_1 and β_1 $(= 150\text{ dB})$ denote the sound intensity and sound level of the space shuttle taking off, and let I_2 and β_2 $(= 110\text{ dB})$ denote the sound intensity and sound level of a car horn. Then

$$\begin{aligned}
I_1/I_2 &= (I_1/I_0)/(I_2/I_0)\\
\log(I_1/I_2) &= \log(I_1/I_0) - \log(I_2/I_0)\\
10\log(I_1/I_2) &= 10\log(I_1/I_0) - 10\log(I_2/I_0) = \beta_1 - \beta_2\\
10\log(I_1/I_2) &= 150 - 100 = 40\\
\log(I_1/I_2) &= 4
\end{aligned}$$

Thus, $I_1/I_2 = 10^4$, which tells us that the sound intensity of the space shuttle taking off is 10,000 times greater than a car horn! ◀

Regina Mitchell-Ryall, Tony Gray/NASA/Getty Images

The roar of a space shuttle near the launchpad would damage your hearing without ear protection.

EXPONENTIAL AND LOGARITHMIC GROWTH

Table 6.1.5

x	e^x	$\ln x$
1	2.72	0.00
2	7.39	0.69
3	20.09	1.10
4	54.60	1.39
5	148.41	1.61
6	403.43	1.79
7	1096.63	1.95
8	2980.96	2.08
9	8103.08	2.20
10	22026.47	2.30
100	2.69×10^{43}	4.61
1000	1.97×10^{434}	6.91

The growth patterns of e^x and $\ln x$ illustrated in Table 6.1.5 are worth noting. Both functions increase as x increases, but they increase in dramatically different ways—the value of e^x increases extremely rapidly and that of $\ln x$ increases extremely slowly. For example, the value of e^x at $x = 10$ is over 22,000, but at $x = 1000$ the value of $\ln x$ has not even reached 7.

A function f is said to ***increase without bound*** as x increases if the values of $f(x)$ eventually exceed any specified positive number M (no matter how large) as x increases indefinitely. Table 6.1.5 strongly suggests that $f(x) = e^x$ increases without bound, which is consistent with the fact that the range of this function is $(0, +\infty)$. Indeed, if we choose any positive number M, then we will have $e^x = M$ when $x = \ln M$, and since the values of e^x increase as x increases, we will have

$$e^x > M \quad \text{if} \quad x > \ln M$$

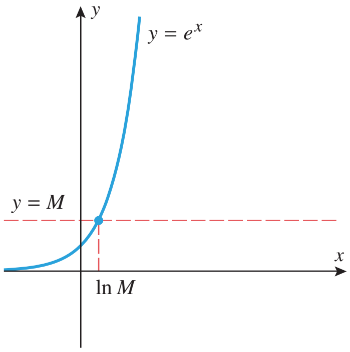
(Figure 6.1.8). It is not clear from Table 6.1.5 whether $\ln x$ increases without bound as x increases because the values grow so slowly, but we know this to be so since the range of this function is $(-\infty, +\infty)$. To see this algebraically, let M be any positive number. We will have $\ln x = M$ when $x = e^M$, and since the values of $\ln x$ increase as x increases, we will have

$$\ln x > M \quad \text{if} \quad x > e^M$$

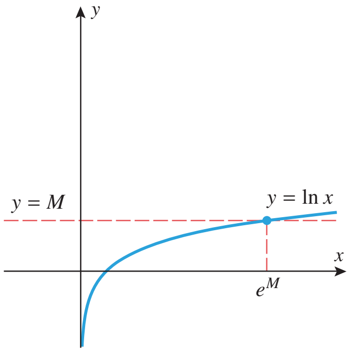
(Figure 6.1.9).

In summary,

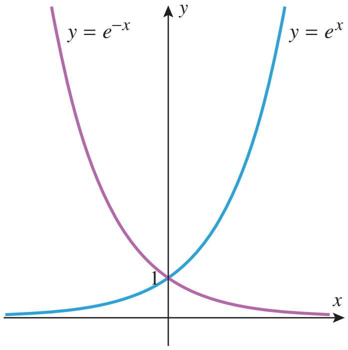
$$\lim_{x\to+\infty} e^x = +\infty \qquad \lim_{x\to+\infty} \ln x = +\infty \tag{10–11}$$



▲ **Figure 6.1.8** The value of $y = e^x$ will exceed an arbitrary positive value of M when $x > \ln M$.



▲ **Figure 6.1.9** The value of $y = \ln x$ will exceed an arbitrary positive value of M when $x > e^M$.



▲ **Figure 6.1.10**

The following limits can be deduced numerically by constructing appropriate tables of values (verify):

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to 0^+} \ln x = -\infty \tag{12–13}$$

The following limits can be deduced numerically, but they can be seen more readily by noting that the graph of $y = e^{-x}$ is the reflection about the y-axis of the graph of $y = e^x$ (Figure 6.1.10):

$$\lim_{x\to+\infty} e^{-x} = 0 \qquad \lim_{x\to-\infty} e^{-x} = +\infty \tag{14–15}$$

✔QUICK CHECK EXERCISES 6.1 *(See page 420 for answers.)*

1. The function $y = \left(\frac{1}{2}\right)^x$ has domain ________ and range ________.

2. The function $y = \ln(1 - x)$ has domain ________ and range ________.

3. Express as a power of 4:
(a) 1 (b) 2 (c) $\frac{1}{16}$ (d) $\sqrt{8}$ (e) 5.

4. Solve each equation for x.
(a) $e^x = \frac{1}{2}$ (b) $10^{3x} = 1{,}000{,}000$
(c) $7e^{3x} = 56$

5. Solve each equation for x.
(a) $\ln x = 3$ (b) $\log(x - 1) = 2$
(c) $2\log x - \log(x + 1) = \log 4 - \log 3$

EXERCISE SET 6.1 Graphing Utility

1–2 Simplify the expression without using a calculating utility. ■

1. (a) $-8^{2/3}$ (b) $(-8)^{2/3}$ (c) $8^{-2/3}$

2. (a) 2^{-4} (b) $4^{1.5}$ (c) $9^{-0.5}$

3–4 Use a calculating utility to approximate the expression. Round your answer to four decimal places. ■

3. (a) $2^{1.57}$ (b) $5^{-2.1}$

4. (a) $\sqrt[5]{24}$ (b) $\sqrt[8]{0.6}$

5–6 Find the exact value of the expression without using a calculating utility. ■

5. (a) $\log_2 16$ (b) $\log_2\left(\frac{1}{32}\right)$
(c) $\log_4 4$ (d) $\log_9 3$

6. (a) $\log_{10}(0.001)$ (b) $\log_{10}(10^4)$
(c) $\ln(e^3)$ (d) $\ln(\sqrt{e}\,)$

7–8 Use a calculating utility to approximate the expression. Round your answer to four decimal places. ■

7. (a) $\log 23.2$ (b) $\ln 0.74$

8. (a) $\log 0.3$ (b) $\ln \pi$

9–10 Use the logarithm properties in Theorem 6.1.3 to rewrite the expression in terms of r, s, and t, where $r = \ln a$, $s = \ln b$, and $t = \ln c$. ■

9. (a) $\ln a^2\sqrt{bc}$ (b) $\ln \dfrac{b}{a^3c}$

10. (a) $\ln \dfrac{\sqrt[3]{c}}{ab}$ (b) $\ln \sqrt{\dfrac{ab^3}{c^2}}$

11–12 Expand the logarithm in terms of sums, differences, and multiples of simpler logarithms.

11. (a) $\log(10x\sqrt{x-3})$ (b) $\ln \dfrac{x^2 \sin^3 x}{\sqrt{x^2+1}}$

12. (a) $\log \dfrac{\sqrt[3]{x+2}}{\cos 5x}$ (b) $\ln \sqrt{\dfrac{x^2+1}{x^3+5}}$

13–15 Rewrite the expression as a single logarithm.

13. $4 \log 2 - \log 3 + \log 16$

14. $\frac{1}{2} \log x - 3 \log(\sin 2x) + 2$

15. $2 \ln(x+1) + \frac{1}{3} \ln x - \ln(\cos x)$

16–23 Solve for x without using a calculating utility.

16. $\log_{10}(1+x) = 3$

17. $\log_{10}(\sqrt{x}) = -1$

18. $\ln(x^2) = 4$

19. $\ln(1/x) = -2$

20. $\log_3(3^x) = 7$

21. $\log_5(5^{2x}) = 8$

22. $\ln 4x - 3 \ln(x^2) = \ln 2$

23. $\ln(1/x) + \ln(2x^3) = \ln 3$

24–29 Solve for x without using a calculating utility. Use the natural logarithm anywhere that logarithms are needed.

24. $3^x = 2$

25. $5^{-2x} = 3$

26. $3e^{-2x} = 5$

27. $2e^{3x} = 7$

28. $e^x - 2xe^x = 0$

29. $xe^{-x} + 2e^{-x} = 0$

30. Solve $e^{-2x} - 3e^{-x} = -2$ for x without using a calculating utility. [*Hint:* Rewrite the equation as a quadratic equation in $u = e^{-x}$.]

FOCUS ON CONCEPTS

31–34 In each part, identify the domain and range of the function, and then sketch the graph of the function without using a graphing utility.

31. (a) $f(x) = \left(\frac{1}{2}\right)^{x-1} - 1$ (b) $g(x) = \ln|x|$

32. (a) $f(x) = 1 + \ln(x-2)$ (b) $g(x) = 3 + e^{x-2}$

33. (a) $f(x) = \ln(x^2)$ (b) $g(x) = e^{-x^2}$

34. (a) $f(x) = 1 - e^{-x+1}$ (b) $g(x) = 3 \ln \sqrt[3]{x-1}$

35–38 True–False Determine whether the statement is true or false. Explain your answer.

35. The function $y = x^3$ is an exponential function.

36. The graph of the exponential function with base b passes through the point $(0, 1)$.

37. The natural logarithm function is the logarithmic function with base e.

38. The domain of a logarithmic function is the interval $x > 1$.

39. Use a calculating utility and the change of base formula (9) to find the values of $\log_2 7.35$ and $\log_5 0.6$, rounded to four decimal places.

40–41 Graph the functions on the same screen of a graphing utility. [Use the change of base formula (9), where needed.]

40. $\ln x,\ e^x,\ \log x,\ 10^x$

41. $\log_2 x,\ \ln x,\ \log_5 x,\ \log x$

42. (a) Derive the general change of base formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

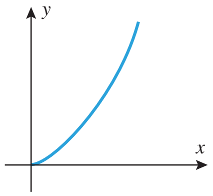
(b) Use the result in part (a) to find the exact value of $(\log_2 81)(\log_3 32)$ without using a calculating utility.

43. Use a graphing utility to estimate the two points of intersection of the graphs of $y = 1.3^x$ and $y = \log_{1.3} x$.

44. Use a graphing utility to estimate the two points of intersection of the graphs of $y = 0.6^{(x^2)}$ and $y = \log_{0.6}(x^2)$.

FOCUS ON CONCEPTS

45. (a) Is the curve in the accompanying figure the graph of an exponential function? Explain your reasoning.
(b) Find the equation of an exponential function that passes through the point $(4, 2)$.
(c) Find the equation of an exponential function that passes through the point $\left(2, \frac{1}{4}\right)$.
(d) Use a graphing utility to generate the graph of an exponential function that passes through the point $(2, 5)$.



◀ **Figure Ex-45**

46. (a) Make a conjecture about the general shape of the graph of $y = \log(\log x)$, and sketch the graph of this equation and $y = \log x$ in the same coordinate system.
(b) Check your work in part (a) with a graphing utility.

47. Find the fallacy in the following "proof" that $\frac{1}{8} > \frac{1}{4}$. Multiply both sides of the inequality $3 > 2$ by $\log \frac{1}{2}$ to get

$$3 \log \tfrac{1}{2} > 2 \log \tfrac{1}{2}$$
$$\log \left(\tfrac{1}{2}\right)^3 > \log \left(\tfrac{1}{2}\right)^2$$
$$\log \tfrac{1}{8} > \log \tfrac{1}{4}$$
$$\tfrac{1}{8} > \tfrac{1}{4}$$

48. Prove the four algebraic properties of logarithms in Theorem 6.1.3.

49. If equipment in the satellite of Example 3 requires 15 watts to operate correctly, what is the operational lifetime of the power supply?

50. The equation $Q = 12e^{-0.055t}$ gives the mass Q in grams of radioactive potassium-42 that will remain from some initial quantity after t hours of radioactive decay.
(a) How many grams were there initially?
(b) How many grams remain after 4 hours?
(c) How long will it take to reduce the amount of radioactive potassium-42 to half of the initial amount?

51. The acidity of a substance is measured by its pH value, which is defined by the formula

$$\text{pH} = -\log[H^+]$$

where the symbol $[H^+]$ denotes the concentration of hydrogen ions measured in moles per liter. Distilled water has a pH of 7; a substance is called *acidic* if it has pH < 7 and *basic* if it has pH > 7. Find the pH of each of the following substances and state whether it is acidic or basic.

	SUBSTANCE	$[H^+]$
(a)	Arterial blood	3.9×10^{-8} mol/L
(b)	Tomatoes	6.3×10^{-5} mol/L
(c)	Milk	4.0×10^{-7} mol/L
(d)	Coffee	1.2×10^{-6} mol/L

52. Use the definition of pH in Exercise 51 to find $[H^+]$ in a solution having a pH equal to
(a) 2.44 (b) 8.06.

53. The perceived loudness β of a sound in decibels (dB) is related to its intensity I in watts per square meter (W/m^2) by the equation

$$\beta = 10\log(I/I_0)$$

where $I_0 = 10^{-12}$ W/m^2. Damage to the average ear occurs at 90 dB or greater. Find the decibel level of each of the following sounds and state whether it will cause ear damage.

	SOUND	I
(a)	Jet aircraft (from 50 ft)	1.0×10^{2} W/m^2
(b)	Amplified rock music	1.0 W/m^2
(c)	Garbage disposal	1.0×10^{-4} W/m^2
(d)	TV (mid volume from 10 ft)	3.2×10^{-5} W/m^2

54–56 Use the definition of the decibel level of a sound (see Exercise 51). ■

54. If one sound is three times as intense as another, how much greater is its decibel level?

55. According to one source, the noise inside a moving automobile is about 70 dB, whereas an electric blender generates 93 dB. Find the ratio of the intensity of the noise of the blender to that of the automobile.

56. Suppose that the intensity level of an echo is $\frac{2}{3}$ the intensity level of the original sound. If each echo results in another echo, how many echoes will be heard from a 120 dB sound given that the average human ear can hear a sound as low as 10 dB?

57. On the ***Richter scale***, the magnitude M of an earthquake is related to the released energy E in joules (J) by the equation

$$\log E = 4.4 + 1.5M$$

(a) Find the energy E of the 1906 San Francisco earthquake that registered $M = 8.2$ on the Richter scale.
(b) If the released energy of one earthquake is 10 times that of another, how much greater is its magnitude on the Richter scale?

58. Suppose that the magnitudes of two earthquakes differ by 1 on the Richter scale. Find the ratio of the released energy of the larger earthquake to that of the smaller earthquake. [*Note:* See Exercise 57 for terminology.]

QUICK CHECK ANSWERS 6.1

1. $(-\infty, +\infty)$; $(0, +\infty)$ **2.** $(-\infty, 1)$; $(-\infty, +\infty)$ **3.** (a) 4^0 (b) $4^{1/2}$ (c) 4^{-2} (d) $4^{3/4}$ (e) $4^{\log_4 5}$ **4.** (a) $\ln\frac{1}{2} = -\ln 2$ (b) 2 (c) $\ln 2$ **5.** (a) e^3 (b) 101 (c) 2

6.2 DERIVATIVES AND INTEGRALS INVOLVING LOGARITHMIC FUNCTIONS

In this section we will obtain derivative formulas for logarithmic functions, and we will explain why the natural logarithm function is preferred in calculus over logarithms with other bases. The derivative formulas that we derive will allow us to find and use corresponding integral formulas.

DERIVATIVES OF LOGARITHMIC FUNCTIONS

We begin by establishing that $f(x) = \ln x$ is differentiable for $x > 0$ by using the derivative definition to find its derivative. To obtain this derivative, we need the fact that $\ln x$ is continuous for $x > 0$. Since e^x is continuous by Theorem 6.1.1(b), we know that $\ln x$ is continuous for $x > 0$ by Theorem 1.5.7. We will also need the limit

$$\lim_{v\to 0}(1+v)^{1/v} = e \tag{1}$$

that was given in Formula (6) of Section 6.1 (with x rather than v as the variable). Using the definition of the derivative, we obtain

$$\begin{aligned}
\frac{d}{dx}[\ln x] &= \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) && \boxed{\text{The quotient property of logarithms in Theorem 6.1.3}} \\
&= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right) \\
&= \lim_{v\to 0}\frac{1}{vx}\ln(1+v) && \boxed{\text{Let } v = h/x \text{ and note that } v\to 0 \text{ if and only if } h\to 0.} \\
&= \frac{1}{x}\lim_{v\to 0}\frac{1}{v}\ln(1+v) && \boxed{x \text{ is fixed in this limit computation, so } 1/x \text{ can be moved through the limit sign.}} \\
&= \frac{1}{x}\lim_{v\to 0}\ln(1+v)^{1/v} && \boxed{\text{The power property of logarithms in Theorem 6.1.3}} \\
&= \frac{1}{x}\ln\left[\lim_{v\to 0}(1+v)^{1/v}\right] && \boxed{\ln x \text{ is continuous on } (0,+\infty) \text{ so we can move the limit through the function symbol.}} \\
&= \frac{1}{x}\ln e \\
&= \frac{1}{x} && \boxed{\text{Since } \ln e = 1}
\end{aligned}$$

Thus,

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0 \tag{2}$$

A derivative formula for the general logarithmic function $\log_b x$ can be obtained from (2) by using Formula (9) of Section 6.1 to write

$$\frac{d}{dx}[\log_b x] = \frac{d}{dx}\left[\frac{\ln x}{\ln b}\right] = \frac{1}{\ln b}\frac{d}{dx}[\ln x]$$

It follows from this that

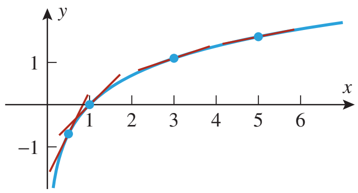
$$\frac{d}{dx}[\log_b x] = \frac{1}{x\ln b}, \quad x > 0 \tag{3}$$

Note that, among all possible bases, the base $b = e$ produces the simplest formula for the derivative of $\log_b x$. This is one of the reasons why the natural logarithm function is preferred over other logarithms in calculus.

▶ Example 1

(a) Figure 6.2.1 shows the graph of $y = \ln x$ and its tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5. Find the slopes of those tangent lines.

(b) Does the graph of $y = \ln x$ have any horizontal tangent lines? Use the derivative of $\ln x$ to justify your answer.



$y = \ln x$ with tangent lines

▲ **Figure 6.2.1**

Solution (a). From (2), the slopes of the tangent lines at the points $x = \frac{1}{2}$, 1, 3, and 5 are $1/x = 2, 1, \frac{1}{3}$, and $\frac{1}{5}$, respectively, which is consistent with Figure 6.2.1.

Solution (b). It does not appear from the graph of $y = \ln x$ that there are any horizontal tangent lines. This is confirmed by the fact that $dy/dx = 1/x$ is not equal to zero for any real value of x. ◀

If u is a differentiable function of x, and if $u(x) > 0$, then applying the chain rule to (2) and (3) produces the following generalized derivative formulas:

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[\log_b u] = \frac{1}{u \ln b} \cdot \frac{du}{dx} \tag{4–5}$$

▶ **Example 2** Find $\frac{d}{dx}[\ln(x^2 + 1)]$.

Solution. Using (4) with $u = x^2 + 1$ we obtain

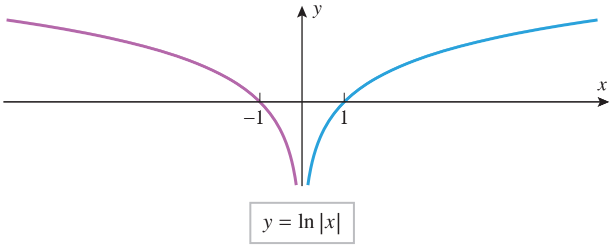
$$\frac{d}{dx}[\ln(x^2 + 1)] = \frac{1}{x^2 + 1} \cdot \frac{d}{dx}[x^2 + 1] = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1} \blacktriangleleft$$

When possible, the properties of logarithms in Theorem 6.1.3 should be used to convert products, quotients, and exponents into sums, differences, and constant multiples *before* differentiating a function involving logarithms.

▶ **Example 3**

$$\begin{aligned} \frac{d}{dx}\left[\ln\left(\frac{x^2 \sin x}{\sqrt{1 + x}}\right)\right] &= \frac{d}{dx}\left[2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1 + x)\right] \\ &= \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1 + x)} \\ &= \frac{2}{x} + \cot x - \frac{1}{2 + 2x} \blacktriangleleft \end{aligned}$$

Figure 6.2.2 shows the graph of $f(x) = \ln|x|$. This function is important because it "extends" the domain of the natural logarithm function in the sense that the values of $\ln|x|$ and $\ln x$ are the same for $x > 0$, but $\ln|x|$ is defined for all nonzero values of x, and $\ln x$ is only defined for positive values of x.



▶ **Figure 6.2.2**

The derivative of $\ln|x|$ for $x \neq 0$ can be obtained by considering the cases $x > 0$ and $x < 0$ separately:

Case $x > 0$. In this case $|x| = x$, so

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln x] = \frac{1}{x}$$

Case $x < 0$. In this case $|x| = -x$, so it follows from (4) that

$$\frac{d}{dx}[\ln|x|] = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)} \cdot \frac{d}{dx}[-x] = \frac{1}{x}$$

Since the same formula results in both cases, we have shown that

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{if } x \neq 0 \tag{6}$$

▶ **Example 4** From (6) and the chain rule,

$$\frac{d}{dx}[\ln|\sin x|] = \frac{1}{\sin x} \cdot \frac{d}{dx}[\sin x] = \frac{\cos x}{\sin x} = \cot x$$ ◀

LOGARITHMIC DIFFERENTIATION

We now consider a technique called ***logarithmic differentiation*** that is useful for differentiating functions that are composed of products, quotients, and powers.

▶ **Example 5** The derivative of

$$y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4} \tag{7}$$

is messy to calculate directly. However, if we first take the natural logarithm of both sides and then use its properties, we can write

$$\ln y = 2\ln x + \tfrac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Differentiating both sides with respect to x yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7/3}{7x-14} - \frac{8x}{1+x^2}$$

Thus, on solving for dy/dx and using (7) we obtain

$$\frac{dy}{dx} = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\left[\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right]$$ ◀

REMARK Since $\ln y$ is only defined for $y > 0$, the computations in Example 5 are only valid for $x > 2$ (verify). However, because the derivative of $\ln y$ is the same as the derivative of $\ln|y|$, and because $\ln|y|$ is defined for $y < 0$ as well as $y > 0$, it follows that the formula obtained for dy/dx is valid for $x < 2$ as well as $x > 2$. In general, whenever a derivative dy/dx is obtained by logarithmic differentiation, the resulting derivative formula will be valid for all values of x for which $y \neq 0$. It may be valid at those points as well, but it is not guaranteed.

INTEGRALS INVOLVING ln *x*

Formula (2) states that the function $\ln x$ is an antiderivative of $1/x$ on the interval $(0, +\infty)$, whereas Formula (6) states that the function $\ln|x|$ is an antiderivative of $1/x$ on each of the intervals $(-\infty, 0)$ and $(0, +\infty)$. Thus we have the companion integration formula to (6),

$$\int \frac{1}{u}\,du = \ln|u| + C \tag{8}$$

with the implicit understanding that this formula is applicable only across an interval that does not contain 0.

▶ Example 6 Applying Formula (8),

$$\int_1^e \frac{1}{x}\,dx = \ln|x|\Big]_1^e = \ln|e| - \ln|1| = 1 - 0 = 1$$

$$\int_{-e}^{-1} \frac{1}{x}\,dx = \ln|x|\Big]_{-e}^{-1} = \ln|-1| - \ln|-e| = 0 - 1 = -1 \quad ◀$$

▶ Example 7 Evaluate $\displaystyle\int \frac{3x^2}{x^3+5}\,dx$.

Solution. Make the substitution

$$u = x^3 + 5, \qquad du = 3x^2\,dx$$

so that

$$\int \frac{3x^2}{x^3+5}\,dx = \int \frac{1}{u}\,du \underset{\boxed{\text{Formula (8)}}}{=} \ln|u| + C = \ln|x^3+5| + C \quad ◀$$

▶ Example 8 Evaluate $\displaystyle\int \tan x\,dx$.

Solution.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \underset{\boxed{\substack{u\,=\,\cos x \\ du\,=\,-\sin x\,dx}}}{=} -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C \quad ◀$$

REMARK The last two examples illustrate an important point: any integral of the form

$$\int \frac{g'(x)}{g(x)}\,dx$$

(where the numerator of the integrand is the derivative of the denominator) can be evaluated by the u-substitution $u = g(x)$, $du = g'(x)\,dx$, since this substitution yields

$$\int \frac{g'(x)}{g(x)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|g(x)| + C$$

DERIVATIVES OF REAL POWERS OF *x*

We know from Theorem 2.3.2 and Exercise 82 in Section 2.3 that the differentiation formula

$$\frac{d}{dx}[x^r] = rx^{r-1} \tag{9}$$

holds for constant integer values of r. We will now use logarithmic differentiation to show that this formula holds if r is *any* real number (rational or irrational). In our computations we will assume that x^r is a differentiable function and that the familiar laws of exponents hold for real exponents.

In the next section we will discuss differentiating functions that have exponents which are not constant.

Let $y = x^r$, where r is a real number. The derivative dy/dx can be obtained by logarithmic differentiation as follows:

$$\ln|y| = \ln|x^r| = r\ln|x|$$

$$\frac{d}{dx}[\ln|y|] = \frac{d}{dx}[r\ln|x|]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x}y = \frac{r}{x}x^r = rx^{r-1}$$

QUICK CHECK EXERCISES 6.2 (See page 427 for answers.)

1. The equation of the tangent line to the graph of $y = \ln x$ at $x = e^2$ is ________.
2. Find dy/dx.
   (a) $y = \ln 3x$
   (b) $y = \ln\sqrt{x}$
   (c) $y = \log(1/|x|)$
3. Use logarithmic differentiation to find the derivative of
$$f(x) = \frac{\sqrt{x+1}}{\sqrt[3]{x-1}}$$
4. $\lim_{h\to 0}\frac{\ln(1+h)}{h} =$ ________
5. $\int_2^5 \frac{1}{t}\,dt =$ ________

EXERCISE SET 6.2

1–26 Find dy/dx.

1. $y = \ln 5x$
2. $y = \ln\frac{x}{3}$
3. $y = \ln|1+x|$
4. $y = \ln(2+\sqrt{x})$
5. $y = \ln|x^2-1|$
6. $y = \ln|x^3-7x^2-3|$
7. $y = \ln\left(\frac{x}{1+x^2}\right)$
8. $y = \ln\left|\frac{1+x}{1-x}\right|$
9. $y = \ln x^2$
10. $y = (\ln x)^3$
11. $y = \sqrt{\ln x}$
12. $y = \ln\sqrt{x}$
13. $y = x\ln x$
14. $y = x^3\ln x$
15. $y = x^2\log_2(3-2x)$
16. $y = x[\log_2(x^2-2x)]^3$
17. $y = \frac{x^2}{1+\log x}$
18. $y = \frac{\log x}{1+\log x}$
19. $y = \ln(\ln x)$
20. $y = \ln(\ln(\ln x))$
21. $y = \ln(\tan x)$
22. $y = \ln(\cos x)$
23. $y = \cos(\ln x)$
24. $y = \sin^2(\ln x)$
25. $y = \log(\sin^2 x)$
26. $y = \log(1-\sin^2 x)$

27–30 Use the method of Example 3 to help perform the indicated differentiation.

27. $\frac{d}{dx}[\ln((x-1)^3(x^2+1)^4)]$
28. $\frac{d}{dx}[\ln((\cos^2 x)\sqrt{1+x^4})]$
29. $\frac{d}{dx}\left[\ln\frac{\cos x}{\sqrt{4-3x^2}}\right]$
30. $\frac{d}{dx}\left[\ln\sqrt{\frac{x-1}{x+1}}\right]$

31–34 True–False Determine whether the statement is true or false. Explain your answer.

31. The slope of the tangent line to the graph of $y = \ln x$ at $x = a$ approaches infinity as $a \to 0^+$.
32. If $\lim_{x\to+\infty} f'(x) = 0$, then the graph of $y = f(x)$ has a horizontal asymptote.
33. The derivative of $\ln|x|$ is an odd function.
34. We have
$$\frac{d}{dx}((\ln x)^2) = \frac{d}{dx}(2(\ln x)) = \frac{2}{x}$$

35–38 Find dy/dx using logarithmic differentiation.

35. $y = x\sqrt[3]{1+x^2}$
36. $y = \sqrt[5]{\frac{x-1}{x+1}}$
37. $y = \frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}$
38. $y = \frac{\sin x\cos x\tan^3 x}{\sqrt{x}}$
39. Find
   (a) $\frac{d}{dx}[\log_x e]$
   (b) $\frac{d}{dx}[\log_x 2]$.
40. Find
   (a) $\frac{d}{dx}[\log_{(1/x)} e]$
   (b) $\frac{d}{dx}[\log_{(\ln x)} e]$.

41–44 Find the equation of the tangent line to the graph of $y = f(x)$ at $x = x_0$.

41. $f(x) = \ln x;\ x_0 = e^{-1}$ **42.** $f(x) = \log x;\ x_0 = 10$

43. $f(x) = \ln(-x);\ x_0 = -e$ **44.** $f(x) = \ln|x|;\ x_0 = -2$

FOCUS ON CONCEPTS

45. (a) Find the equation of a line through the origin that is tangent to the graph of $y = \ln x$.

(b) Explain why the y-intercept of a tangent line to the curve $y = \ln x$ must be 1 unit less than the y-coordinate of the point of tangency.

46. Use logarithmic differentiation to verify the product and quotient rules. Explain what properties of $\ln x$ are important for this verification.

47. Find a formula for the area $A(w)$ of the triangle bounded by the tangent line to the graph of $y = \ln x$ at $P(w, \ln w)$, the horizontal line through P, and the y-axis.

48. Find a formula for the area $A(w)$ of the triangle bounded by the tangent line to the graph of $y = \ln x^2$ at $P(w, \ln w^2)$, the horizontal line through P, and the y-axis.

49. Verify that $y = \ln(x + e)$ satisfies $dy/dx = e^{-y}$, with $y = 1$ when $x = 0$.

50. Verify that $y = -\ln(e^2 - x)$ satisfies $dy/dx = e^y$, with $y = -2$ when $x = 0$.

51. Find a function f such that $y = f(x)$ satisfies $dy/dx = e^{-y}$, with $y = 0$ when $x = 0$.

52. Find a function f such that $y = f(x)$ satisfies $dy/dx = e^y$, with $y = -\ln 2$ when $x = 0$.

53. Let p denote the number of paramecia in a nutrient solution t days after the start of an experiment, and assume that p is defined implicitly as a function of t by the equation

$$0 = \ln p + 0.83 - \ln(2.3 - 0.0046p) - 2.3t$$

Use implicit differentiation to show that the rate of change of p with respect to t satisfies the equation

$$\frac{dp}{dt} = 0.0046p(500 - p)$$

54. Let p denote the population of the United States in the year t, and assume that p is defined implicitly as a function of t by the equation

$$0 = \ln p + 45.817 - \ln(2225 - 4.2381p) - 0.02225t$$

Use implicit differentiation to show that the rate of change of p with respect to t satisfies the equation

$$\frac{dp}{dt} = 10^{-5}p(2225 - 4.2381p)$$

55–57 Find the limit by interpreting the expression as an appropriate derivative. ■

55. (a) $\displaystyle\lim_{x\to 0} \frac{\ln(1+3x)}{x}$ (b) $\displaystyle\lim_{x\to 0} \frac{\ln(1-5x)}{x}$

56. (a) $\displaystyle\lim_{\Delta x\to 0} \frac{\ln(e^2+\Delta x) - 2}{\Delta x}$ (b) $\displaystyle\lim_{w\to 1} \frac{\ln w}{w-1}$

57. (a) $\displaystyle\lim_{x\to 0} \frac{\ln(\cos x)}{x}$ (b) $\displaystyle\lim_{h\to 0} \frac{(1+h)^{\sqrt{2}} - 1}{h}$

58. Modify the derivation of Equation (2) to give another proof of Equation (3).

59–60 Evaluate the integral and check your answer by differentiating. ■

59. $\displaystyle\int \left[\frac{2}{x} + 3\sin x\right] dx$ **60.** $\displaystyle\int \left[\frac{1}{2t} + 2t\right] dt$

61–62 Evaluate the integrals using the indicated substitutions. ■

61. $\displaystyle\int \frac{dx}{x\ln x};\ u = \ln x$

62. $\displaystyle\int \frac{\sin 3\theta}{1+\cos 3\theta}\, d\theta;\ u = 1 + \cos 3\theta$

63–64 Evaluate the integrals using appropriate substitutions. ■

63. $\displaystyle\int \frac{x^4}{1+x^5}\, dx$ **64.** $\displaystyle\int \frac{dx}{2x}$

65–66 Evaluate each integral by first modifying the form of the integrand and then making an appropriate substitution, if needed. ■

65. $\displaystyle\int \frac{t+1}{t}\, dt$ **66.** $\displaystyle\int \cot x\, dx$

67–68 Evaluate the integrals. ■

67. $\displaystyle\int_0^2 \frac{3x}{1+x^2}\, dx$ **68.** $\displaystyle\int_{1/2}^1 \frac{1}{2x}\, dx$

69. Evaluate the definite integral by making the indicated u-substitution.

$$\int_e^{e^2} \frac{\ln x}{x}\, dx; \quad u = \ln x$$

70. Evaluate the definite integral by expressing it in terms of u and evaluating the resulting integral using a formula from geometry.

$$\int_{e^{-3}}^{e^3} \frac{\sqrt{9-(\ln x)^2}}{x}\, dx; \quad u = \ln x$$

71–72 Evaluate the integrals by any method. ■

71. $\displaystyle\int_0^e \frac{dx}{2x+e}$ **72.** $\displaystyle\int_0^{\pi/3} \frac{\sin x}{1+\cos x}\, dx$

73. Solve the initial-value problem.

$$\frac{dy}{dt} = \frac{1}{t}, \quad y(-1) = 5$$

74. **Writing** Review the derivation of the formula

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

and then write a paragraph that discusses all the ingredients (theorems, limit properties, etc.) that are needed for this derivation.

75. **Writing** Write a paragraph that explains how logarithmic differentiation can replace a difficult differentiation computation with a simpler computation.

✔QUICK CHECK ANSWERS 6.2

1. $y = \dfrac{x}{e^2} + 1$ **2.** (a) $\dfrac{dy}{dx} = \dfrac{1}{x}$ (b) $\dfrac{dy}{dx} = \dfrac{1}{2x}$ (c) $\dfrac{dy}{dx} = -\dfrac{1}{x \ln 10}$ **3.** $\dfrac{\sqrt{x+1}}{\sqrt[3]{x-1}}\left[\dfrac{1}{2(x+1)} - \dfrac{1}{3(x-1)}\right]$ **4.** 1 **5.** $\ln\left(\dfrac{5}{2}\right)$

6.3 DERIVATIVES OF INVERSE FUNCTIONS; DERIVATIVES AND INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

In this section we will show how the derivative of a one-to-one function can be used to obtain the derivative of its inverse function. This will provide the tools we need to obtain derivative formulas for exponential functions from the derivative formulas for logarithmic functions. We will then find the corresponding integration formulas for exponential functions.

See Section 0.4 for a review of one-to-one functions and inverse functions.

DIFFERENTIABILITY OF INVERSE FUNCTIONS

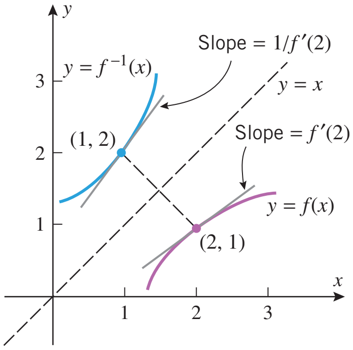
Our first goal in this section is to obtain a formula that relates the derivative of an invertible function f to the derivative of its inverse function.

Example 1 Suppose that f is a one-to-one differentiable function such that $f(2) = 1$ and $f'(2) = \frac{3}{4}$. Then the tangent line to $y = f(x)$ at the point (2, 1) has equation

$$y - 1 = \tfrac{3}{4}(x - 2)$$

Since the graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$, the tangent line to $y = f^{-1}(x)$ at the point $(1, 2)$ is the reflection about the line $y = x$ of the tangent line to $y = f(x)$ at the point (2, 1) (Figure 6.3.1). Its equation can be obtained from that of the tangent line to the graph of f by interchanging x and y:

$$x - 1 = \tfrac{3}{4}(y - 2) \quad \text{or} \quad y - 2 = \tfrac{4}{3}(x - 1)$$



▲ **Figure 6.3.1**

Notice that the slope of the tangent line to $y = f^{-1}(x)$ at $x = 1$ is the reciprocal of the slope of the tangent line to $y = f(x)$ at $x = 2$. That is,

$$(f^{-1})'(1) = \frac{1}{f'(2)} = \frac{4}{3} \quad ◀ \tag{1}$$

Since $2 = f^{-1}(1)$ for the function f in Example 1, it follows that $f'(2) = f'(f^{-1}(1))$. Thus, Formula (1) can also be expressed as

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

In general, if f is a differentiable and one-to-one function, then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \tag{2}$$

provided $f'(f^{-1}(x)) \neq 0$.

Formula (2) can be confirmed using implicit differentiation. The equation $y = f^{-1}(x)$ is equivalent to $x = f(y)$. Differentiating with respect to x we obtain

$$1 = \frac{d}{dx}[x] = \frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx}$$

so that

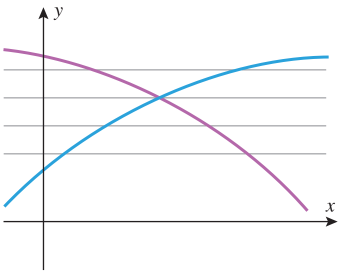
$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Also from $x = f(y)$ we have $dx/dy = f'(y)$, which gives the following alternative version of Formula (2):

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{3}$$

INCREASING OR DECREASING FUNCTIONS ARE ONE-TO-ONE

If the graph of a function f is always increasing or always decreasing over the domain of f, then a horizontal line will cut the graph of f in at most one point (Figure 6.3.2), so f must have an inverse function (see Section 0.4). In Theorem 3.1.2 we saw that f must be increasing on any interval on which $f'(x) > 0$ and must be decreasing on any interval on which $f'(x) < 0$. These observations, together with Formula (2), suggest the following theorem, which we state without formal proof.



▲ **Figure 6.3.2** The graph of an increasing function (blue) or a decreasing function (purple) is cut at most once by any horizontal line.

6.3.1 THEOREM *Suppose that the domain of a function f is an open interval on which $f'(x) > 0$ or on which $f'(x) < 0$. Then f is one-to-one, $f^{-1}(x)$ is differentiable at all values of x in the range of f, and the derivative of $f^{-1}(x)$ is given by Formula (2).*

In general, once it is established that f^{-1} is differentiable, one has the option of calculating the derivative of f^{-1} using Formula (2) or (3), or by differentiating implicitly.

▶ **Example 2** Consider the function $f(x) = x^5 + x + 1$.

(a) Show that f has a differentiable inverse function.

(b) Use implicit differentiation to find a formula for the derivative of f^{-1}.

(c) Compute $(f^{-1})'(1)$.

Solution (a). Since

$$f'(x) = 5x^4 + 1 > 0$$

for all real values of x, it follows from Theorem 6.3.1 that f is one-to-one on the interval $(-\infty, +\infty)$ and has a differentiable inverse function.

Solution (b). Let $y = f^{-1}(x)$. Differentiating $x = f(y) = y^5 + y + 1$ implicitly with respect to x yields

$$\frac{d}{dx}[x] = \frac{d}{dx}[y^5 + y + 1]$$

$$1 = (5y^4 + 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{5y^4 + 1} \tag{4}$$

We cannot solve $x = y^5 + y + 1$ for y in terms of x, so we leave the expression for dy/dx in Equation (4) in terms of y.

Solution (c). From Equation (4),

$$(f^{-1})'(1) = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{5y^4 + 1}\right|_{x=1}$$

Thus, we need to know the value of $y = f^{-1}(x)$ at $x = 1$, which we can obtain by solving the equation $f(y) = 1$ for y. This equation is $y^5 + y + 1 = 1$, which, by inspection, is satisfied by $y = 0$. Thus,

$$(f^{-1})'(1) = \left.\frac{1}{5y^4 + 1}\right|_{y=0} = 1 \blacktriangleleft$$

DERIVATIVES OF EXPONENTIAL FUNCTIONS

Our next objective is to show that the general exponential function b^x $(b > 0, b \neq 1)$ is differentiable everywhere and to find its derivative. To do this, we will use the fact that b^x is the inverse of the function $f(x) = \log_b x$. We will assume that $b > 1$. With this assumption we have $\ln b > 0$, so

$$f'(x) = \frac{d}{dx}[\log_b x] = \frac{1}{x \ln b} > 0 \quad \text{for all } x \text{ in the interval } (0, +\infty)$$

It now follows from Theorem 6.3.1 that $f^{-1}(x) = b^x$ is differentiable for all x in the range of $f(x) = \log_b x$. But we know from Table 6.1.3 that the range of $\log_b x$ is $(-\infty, +\infty)$, so we have established that b^x is differentiable everywhere.

To obtain a derivative formula for b^x we rewrite $y = b^x$ as

$$x = \log_b y$$

and differentiate implicitly using Formula (5) of Section 6.2 to obtain

$$1 = \frac{1}{y \ln b} \cdot \frac{dy}{dx}$$

Solving for dy/dx and replacing y by b^x we have

$$\frac{dy}{dx} = y \ln b = b^x \ln b$$

How does the derivation of Formula (5) change if $0 < b < 1$?

Thus, we have shown that

$$\frac{d}{dx}[b^x] = b^x \ln b \tag{5}$$

In the special case where $b = e$ we have $\ln e = 1$, so that (5) becomes

$$\frac{d}{dx}[e^x] = e^x \tag{6}$$

In Section 6.1 we stated that $b = e$ is the only base for which the slope of the tangent line to the curve $y = b^x$ at any point P on the curve is the y-coordinate at P (see page 412). Verify this statement.

Moreover, if u is a differentiable function of x, then it follows from (5) and (6) that

$$\frac{d}{dx}[b^u] = b^u \ln b \cdot \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx}[e^u] = e^u \cdot \frac{du}{dx} \tag{7–8}$$

It is important to distinguish between differentiating an exponential function b^x (variable exponent and constant base) and a power function x^b (variable base and constant exponent). For example, compare the derivative

$$\frac{d}{dx}[x^2] = 2x$$

to the derivative of 2^x in Example 3.

▶ Example 3 The following computations use Formulas (5), (7) and (8).

$$\frac{d}{dx}[2^x] = 2^x \ln 2$$

$$\frac{d}{dx}[e^{-2x}] = e^{-2x} \cdot \frac{d}{dx}[-2x] = -2e^{-2x}$$

$$\frac{d}{dx}[e^{x^3}] = e^{x^3} \cdot \frac{d}{dx}[x^3] = 3x^2 e^{x^3}$$

$$\frac{d}{dx}[e^{\cos x}] = e^{\cos x} \cdot \frac{d}{dx}[\cos x] = -(\sin x)e^{\cos x} \blacktriangleleft$$

Functions of the form $f(x) = u^v$ in which u and v are *nonconstant* functions of x are neither exponential functions nor power functions. Functions of this form can be differentiated using logarithmic differentiation.

▶ **Example 4** Use logarithmic differentiation to find $\dfrac{d}{dx}[(x^2+1)^{\sin x}]$.

Solution. Setting $y=(x^2+1)^{\sin x}$ we have

$$\ln y = \ln[(x^2+1)^{\sin x}] = (\sin x)\ln(x^2+1)$$

Differentiating both sides with respect to x yields

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}[(\sin x)\ln(x^2+1)]\\ &= (\sin x)\frac{1}{x^2+1}(2x)+(\cos x)\ln(x^2+1)\end{aligned}$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= y\left[\frac{2x\sin x}{x^2+1}+(\cos x)\ln(x^2+1)\right]\\ &= (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1}+(\cos x)\ln(x^2+1)\right] \quad \blacktriangleleft\end{aligned}$$

INTEGRALS INVOLVING EXPONENTIAL FUNCTIONS

Associated with derivatives (7) and (8) are the companion integration formulas

$$\int b^u\,du = \frac{b^u}{\ln b}+C \quad \text{and} \quad \int e^u\,du = e^u + C \tag{9–10}$$

▶ **Example 5**

$$\int 2^x\,dx = \frac{2^x}{\ln 2}+C \quad \blacktriangleleft$$

▶ **Example 6** Evaluate $\displaystyle\int e^{5x}\,dx$.

Solution. Let $u=5x$ so that $du=5\,dx$ or $dx=\frac{1}{5}\,du$, which yields

$$\int e^{5x}\,dx = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x}+C \quad \blacktriangleleft$$

▶ **Example 7** The following computations use Formula (10).

$$\int e^{-x}\,dx \underset{\boxed{\begin{array}{l}u=-x\\ du=-dx\end{array}}}{=} -\int e^u\,du = -e^u + C = -e^{-x}+C$$

$$\int x^2e^{x^3}\,dx \underset{\boxed{\begin{array}{l}u=x^3\\ du=3x^2\,dx\end{array}}}{=} \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3}+C$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx \underset{\boxed{\begin{array}{l}u=\sqrt{x}\\ du=\frac{1}{2\sqrt{x}}\,dx\end{array}}}{=} 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}}+C \quad \blacktriangleleft$$

▶ **Example 8** Evaluate $\displaystyle\int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx$.

Solution. Make the u-substitution

$$u = 1+e^x, \quad du = e^x\,dx$$

and change the x-limits of integration ($x=0$, $x=\ln 3$) to the u-limits

$$u = 1+e^0 = 2, \quad u = 1+e^{\ln 3} = 1+3 = 4$$

This yields

$$\int_0^{\ln 3} e^x(1+e^x)^{1/2}\,dx = \int_2^4 u^{1/2}\,du = \frac{2}{3}u^{3/2}\bigg]_2^4 = \frac{2}{3}[4^{3/2}-2^{3/2}] = \frac{16-4\sqrt{2}}{3}$$ ◀

✔ QUICK CHECK EXERCISES 6.3 *(See page 434 for answers.)*

1. Suppose that a one-to-one function f has tangent line $y=5x+3$ at the point $(1, 8)$. Evaluate $(f^{-1})'(8)$.

2. In each case, from the given derivative, determine whether the function f is invertible.
(a) $f'(x) = x^2+1$ (b) $f'(x) = x^2-1$
(c) $f'(x) = \sin x$ (d) $f'(x) = \dfrac{1}{2} - e^{x^2}$

3. Evaluate the derivative.
(a) $\dfrac{d}{dx}[e^x]$ (b) $\dfrac{d}{dx}[7^x]$
(c) $\dfrac{d}{dx}[\cos(e^x+1)]$ (d) $\dfrac{d}{dx}[e^{3x-2}]$

4. Let $f(x) = e^{x^3+x}$. Use $f'(x)$ to verify that f is one-to-one.

5. $\displaystyle\int_0^{\frac{1}{2}\ln 5} e^x\,dx =$ ________

EXERCISE SET 6.3 Graphing Utility

FOCUS ON CONCEPTS

1. Let $f(x) = x^5+x^3+x$.
(a) Show that f is one-to-one and confirm that $f(1)=3$.
(b) Find $(f^{-1})'(3)$.

2. Let $f(x) = x^3+2e^x$.
(a) Show that f is one-to-one and confirm that $f(0)=2$.
(b) Find $(f^{-1})'(2)$.

3–4 Find $(f^{-1})'(x)$ using Formula (2), and check your answer by differentiating f^{-1} directly. ■

3. $f(x) = 2/(x+3)$ **4.** $f(x) = \ln(2x+1)$

5–6 Determine whether the function f is one-to-one by examining the sign of $f'(x)$. ■

5. (a) $f(x) = x^2+8x+1$
(b) $f(x) = 2x^5+x^3+3x+2$
(c) $f(x) = 2x+\sin x$
(d) $f(x) = \left(\frac{1}{2}\right)^x$

6. (a) $f(x) = x^3+3x^2-8$
(b) $f(x) = x^5+8x^3+2x-1$
(c) $f(x) = \dfrac{x}{x+1}$
(d) $f(x) = \log_b x, \quad 0<b<1$

7–10 Find the derivative of f^{-1} by using Formula (3), and check your result by differentiating implicitly. ■

7. $f(x) = 5x^3+x-7$ **8.** $f(x) = 1/x^2, \quad x>0$

9. $f(x) = 2x^5+x^3+1$

10. $f(x) = 5x-\sin 2x, \quad -\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$

FOCUS ON CONCEPTS

11. Figure 0.4.8 is a "proof by picture" that the reflection of a point $P(a, b)$ about the line $y=x$ is the point $Q(b, a)$. Establish this result rigorously by completing each part.
(a) Prove that if P is not on the line $y=x$, then P and Q are distinct, and the line $\overleftrightarrow{PQ}$ is perpendicular to the line $y=x$.
(b) Prove that if P is not on the line $y=x$, the midpoint of segment PQ is on the line $y=x$.
(c) Carefully explain what it means geometrically to reflect P about the line $y=x$.
(d) Use the results of parts (a)–(c) to prove that Q is the reflection of P about the line $y=x$.

12. Prove that the reflection about the line $y=x$ of a line with slope m, $m\neq 0$, is a line with slope $1/m$. [*Hint:* Apply the result of the previous exercise to a pair of points on the line of slope m and to a corresponding pair of points on the reflection of this line about the line $y = x$.]

13. Suppose that f and g are increasing functions. Determine which of the functions $f(x) + g(x)$, $f(x)g(x)$, and $f(g(x))$ must also be increasing.

14. Suppose that f and g are one-to-one functions. Determine which of the functions $f(x) + g(x)$, $f(x)g(x)$, and $f(g(x))$ must also be one-to-one.

15–26 Find dy/dx.

15. $y = e^{7x}$
16. $y = e^{-5x^2}$
17. $y = x^3 e^x$
18. $y = e^{1/x}$
19. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
20. $y = \sin(e^x)$
21. $y = e^{x \tan x}$
22. $y = \dfrac{e^x}{\ln x}$
23. $y = e^{(x - e^{3x})}$
24. $y = \exp(\sqrt{1 + 5x^3})$
25. $y = \ln(1 - xe^{-x})$
26. $y = \ln(\cos e^x)$

27–30 Find $f'(x)$ by Formula (7) and then by logarithmic differentiation.

27. $f(x) = 2^{x^2}$
28. $f(x) = 3^{-x}$
29. $f(x) = \pi^{\sin x}$
30. $f(x) = \pi^{x \tan x}$

31–35 Find dy/dx using the method of logarithmic differentiation.

31. $y = (x^3 - 2x)^{\ln x}$
32. $y = x^{\sin x}$
33. $y = (\ln x)^{\tan x}$
34. $y = (x^2 + 3)^{\ln x}$
35. $y = (\ln x)^{\ln x}$

36. (a) Explain why Formula (5) cannot be used to find $(d/dx)[x^x]$.
(b) Find this derivative by logarithmic differentiation.

37–42 Find dy/dx using any method.

37. $y = (x^3 - 2x^2 + 1)e^x$
38. $y = (2x^2 - 2x + 1)e^{2x}$
39. $y = (x^2 + \sqrt{x})3^x$
40. $y = (x^3 + \sqrt[3]{x})5^x$
41. $y = 4^{3 \sin x - e^x}$
42. $y = 2^{\cos x + \ln x}$

43–46 True–False Determine whether the statement is true or false. Explain your answer.

43. If a function $y = f(x)$ satisfies $dy/dx = y$, then $y = e^x$.

44. If $y = f(x)$ is a function such that dy/dx is a rational function, then $f(x)$ is also a rational function.

45. $\dfrac{d}{dx}(\log_b |x|) = \dfrac{1}{x \ln b}$

46. If the tangent line to the graph of $f(x) = b^x$ has slope 1 at the point $(0, 1)$, then f is the natural exponential function.

47. (a) Show that $f(x) = x^3 - 3x^2 + 2x$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the largest value of k such that f is one-to-one on the interval $(-k, k)$.

48. (a) Show that the function $f(x) = x^4 - 2x^3$ is not one-to-one on $(-\infty, +\infty)$.
(b) Find the smallest value of k such that f is one-to-one on the interval $[k, +\infty)$.

49. Let $f(x) = x^4 + x^3 + 1$, $0 \le x \le 2$.
(a) Show that f is one-to-one.
(b) Let $g(x) = f^{-1}(x)$ and define $F(x) = f(2g(x))$. Find an equation for the tangent line to $y = F(x)$ at $x = 3$.

50. Let $f(x) = \dfrac{\exp(4 - x^2)}{x}$, $x > 0$.
(a) Show that f is one-to-one.
(b) Let $g(x) = f^{-1}(x)$ and define $F(x) = f([g(x)]^2)$. Find $F'\left(\frac{1}{2}\right)$.

51. Show that for any constants A and k, the function $y = Ae^{kt}$ satisfies the equation $dy/dt = ky$.

52. Show that for any constants A and B, the function

$$y = Ae^{2x} + Be^{-4x}$$

satisfies the equation

$$y'' + 2y' - 8y = 0$$

53. Show that
(a) $y = xe^{-x}$ satisfies the equation $xy' = (1 - x)y$
(b) $y = xe^{-x^2/2}$ satisfies the equation $xy' = (1 - x^2)y$.

54. Show that the rate of change of $y = 100e^{-0.2x}$ with respect to x is proportional to y.

55. Suppose that the percentage of U.S. households with broadband Internet access is modeled by the equation

$$P(t) = \frac{5300}{53 + 47e^{-0.182t}}$$

where $P(t)$ is the percentage t years after an initial survey result made in the year 2007.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the percentage over time. Check your conclusion by finding $\lim_{t \to +\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

56. Suppose that the population of oxygen-dependent bacteria in a pond is modeled by the equation

$$P(t) = \frac{60}{5 + 7e^{-t}}$$

where $P(t)$ is the population (in billions) t days after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t \to +\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

57–59 Find the limit by interpreting the expression as an appropriate derivative.

57. $\displaystyle\lim_{x \to 0} \frac{e^{3x} - 1}{x}$
58. $\displaystyle\lim_{x \to 0} \frac{\exp(x^2) - 1}{x}$

59. $\lim\limits_{h \to 0} \dfrac{10^h - 1}{h}$

60. Suppose that a steel ball bearing is released within a vat of fluid and begins to sink. According to one model, the speed $v(t)$ (in m/s) of the ball bearing t seconds after its release is given by the formula

$$v(t) = \frac{9.8(1 - e^{-kt})}{k}$$

where k is a positive constant that corresponds to the resistance the fluid offers against the motion of the bearing. (The smaller the value of k, the weaker will be the resistance.) For t fixed, determine the limiting value of the speed as $k \to 0^+$, and give a physical interpretation of the limit. [*Hint:* Interpret the limit as an appropriate derivative.]

61–62 Evaluate the integral and check your answer by differentiating.

61. $\displaystyle\int \left[\frac{2}{x} + 3e^x\right] dx$ **62.** $\displaystyle\int \left[\frac{1}{2t} - \sqrt{2}e^t\right] dt$

63–64 Evaluate the integrals using the indicated substitutions.

63. $\displaystyle\int e^{-5x}\,dx;\ u = -5x$ **64.** $\displaystyle\int \frac{e^x}{1 + e^x}\,dx;\ u = 1 + e^x$

65–72 Evaluate the integrals using appropriate substitutions.

65. $\displaystyle\int e^{2x}\,dx$ **66.** $\displaystyle\int e^{-x/2}\,dx$ **67.** $\displaystyle\int e^{\sin x}\cos x\,dx$

68. $\displaystyle\int x^3 e^{x^4}\,dx$ **69.** $\displaystyle\int x^2 e^{-2x^3}\,dx$ **70.** $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$

71. $\displaystyle\int \frac{dx}{e^x}$ **72.** $\displaystyle\int \sqrt{e^x}\,dx$

73–74 Evaluate each integral by first modifying the form of the integrand.

73. $\displaystyle\int [\ln(e^x) + \ln(e^{-x})]\,dx$ **74.** $\displaystyle\int e^{2\ln x}\,dx$

75–76 Evaluate the integrals.

75. $\displaystyle\int_{\ln 2}^{3} 5e^x\,dx$ **76.** $\displaystyle\int_0^1 (e^x - x)\,dx$

77. Evaluate the definite integral by making the indicated u-substitution.

$$\int_0^1 e^{2x-1}\,dx; \qquad u = 2x - 1$$

78–80 Evaluate the integrals by any method.

78. $\displaystyle\int_0^{\ln 5} e^x(3 - 4e^x)\,dx$ **79.** $\displaystyle\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x + 4}\,dx$

80. $\displaystyle\int_1^{\sqrt{2}} xe^{-x^2}\,dx$

81–84 Medication can be administered to a patient in different ways. For a given method, let $c(t)$ denote the concentration of medication in the patient's bloodstream (measured in mg/L) t hours after the dose is given. Over the time interval $0 \le t \le b$, the area between the graph of $c = c(t)$ and the interval $[0, b]$ indicates the "availability" of the medication for the patient's body over that time period. Determine which method provides the greater availability over the given interval.

81. Method 1: $c(t) = 5(e^{-0.2t} - e^{-t})$,
Method 2: $c(t) = 4(e^{-0.2t} - e^{-3t})$; $[0, 4]$

82. Method 1: $c(t) = 5(e^{-0.2t} - e^{-t})$,
Method 2: $c(t) = 4(e^{-0.2t} - e^{-3t})$; $[0, 24]$

83. Method 1: $c(t) = 5.78(e^{-0.4t} - e^{-1.3t})$,
Method 2: $c(t) = 4.15(e^{-0.4t} - e^{-3t})$; $[0, 4]$

84. Method 1: $c(t) = 5.78(e^{-0.4t} - e^{-1.3t})$,
Method 2: $c(t) = 4.15(e^{-0.4t} - e^{-3t})$; $[0, 24]$

85. Suppose that at time $t = 0$ there are 750 bacteria in a growth medium and the bacteria population $y(t)$ grows at the rate $y'(t) = 802.137e^{1.528t}$ bacteria per hour. How many bacteria will there be in 12 hours?

86. Suppose that a particle moving along a coordinate line has velocity $v(t) = 25 + 10e^{-0.05t}$ ft/s.

(a) What is the distance traveled by the particle from time $t = 0$ to time $t = 10$?

(b) Does the term $10e^{-0.05t}$ have much effect on the distance traveled by the particle over that time interval? Explain your reasoning.

87. Find a positive value of k such that the area under the graph of $y = e^{2x}$ over the interval $[0, k]$ is 3 square units.

88. Solve the initial-value problem.

$$\frac{dy}{dt} = -e^{2t}, \quad y(0) = 6$$

89. Let $y(t)$ denote the number of *E. coli* cells in a container of nutrient solution t minutes after the start of an experiment. Assume that $y(t)$ is modeled by the initial-value problem

$$\frac{dy}{dt} = (\ln 2)2^{t/20}, \qquad y(0) = 20$$

Use this model to estimate the number of *E. coli* cells in the container 2 hours after the start of the experiment.

90. **Writing** Let G denote the graph of an invertible function f and consider G as a fixed set of points in the plane. Suppose we relabel the coordinate axes so that the x-axis becomes the y-axis and vice versa. Carefully explain why now the same set of points G becomes the graph of f^{-1} (with the coordinate axes in a nonstandard position). Use this result to explain Formula (2).

91. **Writing** Suppose that f has an inverse function. Carefully explain the connection between Formula (2) and implicit differentiation of the equation $x = f(y)$.

✔QUICK CHECK ANSWERS 6.3

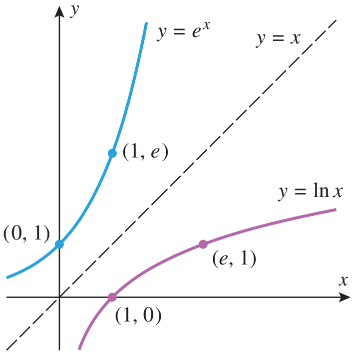
1. $\frac{1}{5}$ **2.** (a) yes (b) no (c) no (d) yes **3.** (a) e^x (b) $7^x \ln 7$ (c) $-e^x \sin(e^x + 1)$ (d) $3e^{3x-2}$
4. $f'(x) = e^{x^3+x} \cdot (3x^2 + 1) > 0$ for all x **5.** $\sqrt{5} - 1$

6.4 GRAPHS AND APPLICATIONS INVOLVING LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this section we will apply the techniques developed in Chapter 3 to graphing functions involving logarithmic or exponential functions. We will also look at applications of differentiation and integration in some contexts that involve logarithmic or exponential functions.

SOME PROPERTIES OF e^x AND $\ln x$

In Figure 6.4.1 we present computer-generated graphs of $y = e^x$ and $y = \ln x$. Since $f(x) = e^x$ and $g(x) = \ln x$ are inverses, their graphs are reflections of each other about the line $y = x$. Table 6.4.1 summarizes some important properties of e^x and $\ln x$.



▲ **Figure 6.4.1** The functions e^x and $\ln x$ are inverses.

Table 6.4.1

PROPERTIES OF e^x	PROPERTIES OF $\ln x$
$e^x > 0$ for all x	$\ln x > 0$ if $x > 1$ $\ln x < 0$ if $0 < x < 1$ $\ln x = 0$ if $x = 1$
e^x is increasing on $(-\infty, +\infty)$	$\ln x$ is increasing on $(0, +\infty)$
The graph of e^x is concave up on $(-\infty, +\infty)$	The graph of $\ln x$ is concave down on $(0, +\infty)$

We can verify that $y = e^x$ is increasing and its graph is concave up from its first and second derivatives. For all x in $(-\infty, +\infty)$ we have

$$\frac{d}{dx}[e^x] = e^x > 0 \quad \text{and} \quad \frac{d^2}{dx^2}[e^x] = \frac{d}{dx}[e^x] = e^x > 0$$

The first of these inequalities demonstrates that e^x is increasing on $(-\infty, +\infty)$, and the second inequality shows that the graph of $y = e^x$ is concave up on $(-\infty, +\infty)$.

Similarly, for all x in $(0, +\infty)$ we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x} > 0 \quad \text{and} \quad \frac{d^2}{dx^2}[\ln x] = \frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2} < 0$$

The first of these inequalities demonstrates that $\ln x$ is increasing on $(0, +\infty)$, and the second inequality shows that the graph of $y = \ln x$ is concave down on $(0, +\infty)$.

GRAPHING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The calculus tools developed in Sections 1–3 of Chapter 3 can be used to graph functions that involve exponential and logarithmic functions.

▶ **Example 1** Sketch the graph of $y = e^{-x^2/2}$ and identify the locations of all relative extrema and inflection points.

Solution.

- *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.
- *x- and y-intercepts:* Setting $y = 0$ leads to the equation $e^{-x^2/2} = 0$, which has no solutions since all powers of e have positive values. Thus, there are no x-intercepts. Setting $x = 0$ yields the y-intercept $y = 1$.
- *Vertical asymptotes:* There are no vertical asymptotes since $e^{-x^2/2}$ is continuous on $(-\infty, +\infty)$.
- *End behavior:* The x-axis ($y = 0$) is a horizontal asymptote since

$$\lim_{x\to-\infty} e^{-x^2/2} = \lim_{x\to+\infty} e^{-x^2/2} = 0$$

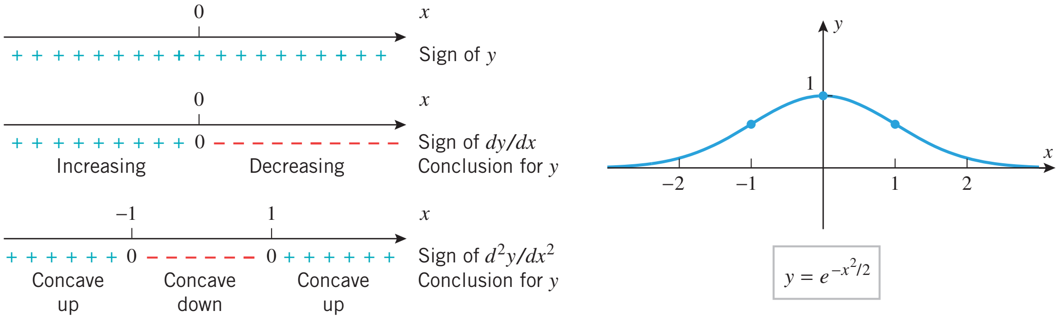
- *Derivatives:*

$$\begin{aligned}\frac{dy}{dx} &= e^{-x^2/2}\frac{d}{dx}\left[-\frac{x^2}{2}\right] = -xe^{-x^2/2}\\ \frac{d^2y}{dx^2} &= -x\frac{d}{dx}[e^{-x^2/2}] + e^{-x^2/2}\frac{d}{dx}[-x]\\ &= x^2e^{-x^2/2} - e^{-x^2/2} = (x^2-1)e^{-x^2/2}\end{aligned}$$

Conclusions and graph:

- The sign analysis of y in Figure 6.4.2*a* is based on the fact that $e^{-x^2/2} > 0$ for all x. This shows that the graph is always above the x-axis.
- The sign analysis of dy/dx in Figure 6.4.2*a* is based on the fact that $dy/dx = -xe^{-x^2/2}$ has the same sign as $-x$. This analysis and the first derivative test show that there is a stationary point at $x = 0$ at which there is a relative maximum. The value of y at the relative maximum is $y = e^0 = 1$.
- The sign analysis of d^2y/dx^2 in Figure 6.4.2*a* is based on the fact that $d^2y/dx^2 = (x^2-1)e^{-x^2/2}$ has the same sign as $x^2 - 1$. This analysis shows that there are inflection points at $x = -1$ and $x = 1$. The graph changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = 1$. The coordinates of the inflection points are $(-1, e^{-1/2}) \approx (-1, 0.61)$ and $(1, e^{-1/2}) \approx (1, 0.61)$.

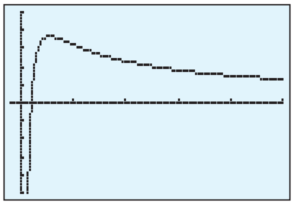
The graph is shown in Figure 6.4.2*b*. ◀



(*a*) (*b*)

▲ **Figure 6.4.2**

▶ **Example 2** Use a graphing utility to generate the graph of $f(x) = (\ln x)/x$, and discuss what it tells you about relative extrema, inflection points, asymptotes, and end behavior. Use calculus to find the locations of all key features of the graph.



▲ **Figure 6.4.3**

Solution. Figure 6.4.3 shows a graph of f produced by a graphing utility. The graph suggests that there is an x-intercept near $x = 1$, a relative maximum somewhere between $x = 0$ and $x = 5$, an inflection point near $x = 5$, a vertical asymptote at $x = 0$, and possibly a horizontal asymptote $y = 0$. For a more precise analysis of this information we need to consider the derivatives

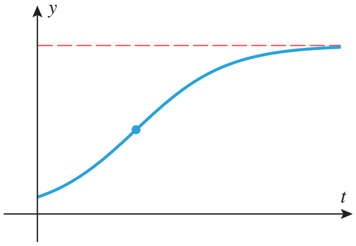
$$f'(x) = \frac{x\left(\dfrac{1}{x}\right) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f''(x) = \frac{x^2\left(-\dfrac{1}{x}\right) - (1 - \ln x)(2x)}{x^4} = \frac{2x \ln x - 3x}{x^4} = \frac{2 \ln x - 3}{x^3}$$

- *Relative extrema:* Solving $f'(x) = 0$ yields the stationary point $x = e$ (verify). Since
$$f''(e) = \frac{2-3}{e^3} = -\frac{1}{e^3} < 0$$
there is a relative maximum at $x = e \approx 2.7$ by the second derivative test.
- *Inflection points:* Since $f(x) = (\ln x)/x$ is only defined for positive values of x, the second derivative $f''(x)$ has the same sign as $2 \ln x - 3$. We leave it for you to use the inequalities $(2 \ln x - 3) < 0$ and $(2 \ln x - 3) > 0$ to show that $f''(x) < 0$ if $x < e^{3/2}$ and $f''(x) > 0$ if $x > e^{3/2}$. Thus, there is an inflection point at $x = e^{3/2} \approx 4.5$.
- *Asymptotes:* We will develop a technique in Section 6.5 that will allow us to conclude that
$$\lim_{x \to +\infty} \frac{\ln x}{x} = 0$$
so that $y = 0$ is a horizontal asymptote. Also, there is a vertical asymptote at $x = 0$ since
$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty$$
(why?).
- *Intercepts:* Setting $f(x) = 0$ yields $(\ln x)/x = 0$. The only real solution of this equation is $x = 1$, so there is an x-intercept at this point. ◀

LOGISTIC CURVES

When a population grows in an environment in which space or food is limited, the graph of population versus time is typically an S-shaped curve of the form shown in Figure 6.4.4. The scenario described by this curve is a population that grows slowly at first and then more and more rapidly as the number of individuals producing offspring increases. However, at a certain point in time (where the inflection point occurs) the environmental factors begin to show their effect, and the growth rate begins a steady decline. Over an extended period of time the population approaches a limiting value that represents the upper limit on the number of individuals that the available space or food can sustain. Population growth curves of this type are called ***logistic growth curves***.

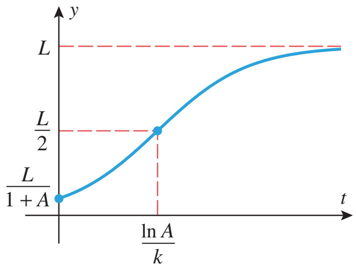


▲ **Figure 6.4.4**

▶ **Example 3** We will see in a later chapter that logistic growth curves arise from equations of the form

$$y = \frac{L}{1 + Ae^{-kt}} \tag{1}$$

where y is the population at time t ($t \geq 0$) and A, k, and L are positive constants. Show that Figure 6.4.5 correctly describes the graph of this equation when $A > 1$.



▲ **Figure 6.4.5**

Solution. It follows from (1) that at time $t = 0$ the value of y is

$$y = \frac{L}{1+A}$$

and it follows from (1) and the fact that $0 < e^{-kt} \leq 1$ for $t \geq 0$ that

$$\frac{L}{1+A} \leq y < L \tag{2}$$

(verify). This is consistent with the graph in Figure 6.4.5. The horizontal asymptote at $y = L$ is confirmed by the limit

$$\lim_{t \to +\infty} y = \lim_{t \to +\infty} \frac{L}{1 + Ae^{-kt}} = \frac{L}{1+0} = L \tag{3}$$

Physically, Formulas (2) and (3) tell us that L is an upper limit on the population and that the population approaches this limit over time. Again, this is consistent with the graph in Figure 6.4.5.

To investigate intervals of increase and decrease, concavity, and inflection points, we need the first and second derivatives of y with respect to t. By multiplying both sides of Equation (1) by $e^{kt}(1 + Ae^{-kt})$, we can rewrite (1) as

$$ye^{kt} + Ay = Le^{kt}$$

Using implicit differentiation, we can derive that

$$\frac{dy}{dt} = \frac{k}{L}y(L-y) \tag{4}$$

$$\frac{d^2y}{dt^2} = \frac{k^2}{L^2}y(L-y)(L-2y) \tag{5}$$

(Exercise 36). Since $k > 0$, $y > 0$, and $L - y > 0$, it follows from (4) that $dy/dt > 0$ for all t. Thus, y is always increasing, which is consistent with Figure 6.4.5.

Since $y > 0$ and $L - y > 0$, it follows from (5) that

$$\frac{d^2y}{dt^2} > 0 \quad \text{if} \quad L - 2y > 0$$

$$\frac{d^2y}{dt^2} < 0 \quad \text{if} \quad L - 2y < 0$$

Thus, the graph of y versus t is concave up if $y < L/2$, concave down if $y > L/2$, and has an inflection point where $y = L/2$, all of which is consistent with Figure 6.4.5.

Finally, we leave it for you to solve the equation

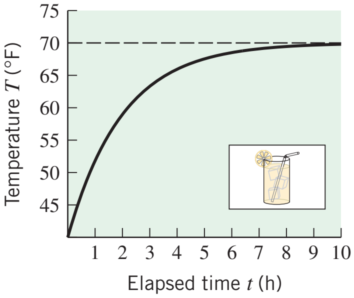
$$\frac{L}{2} = \frac{L}{1 + Ae^{-kt}}$$

for t to show that the inflection point occurs at

$$t = \frac{1}{k}\ln A = \frac{\ln A}{k} \quad ◀ \tag{6}$$

NEWTON'S LAW OF COOLING

▶ **Example 4** A glass of lemonade with a temperature of 40°F is left to sit in a room whose temperature is a constant 70°F. Using a principle of physics called ***Newton's Law of Cooling***, one can show that if the temperature of the lemonade reaches 52°F in 1 hour,



▲ **Figure 6.4.6**

then the temperature T of the lemonade as a function of the elapsed time t is modeled by the equation

$$T = 70 - 30e^{-0.5t}$$

where T is in degrees Fahrenheit and t is in hours. The graph of this equation, shown in Figure 6.4.6, conforms to our everyday experience that the temperature of the lemonade gradually approaches the temperature of the room. Find the average temperature T_{ave} of the lemonade over the first 5 hours.

Solution. From Definition 4.8.1 the average value of T over the time interval $[0, 5]$ is

$$T_{\text{ave}} = \frac{1}{5}\int_0^5 (70 - 30e^{-0.5t})\,dt \tag{7}$$

To evaluate this integral, we make the substitution

$$u = -0.5t \quad \text{so that} \quad du = -0.5\,dt \quad (\text{or } dt = -2\,du)$$

With this substitution, if

$$\begin{aligned} t = 0, &\quad u = 0 \\ t = 5, &\quad u = (-0.5)5 = -2.5 \end{aligned}$$

Thus, (7) can be expressed as

$$\begin{aligned} \frac{1}{5}\int_0^{-2.5} (70 - 30e^u)(-2)\,du &= -\frac{2}{5}\int_0^{-2.5} (70 - 30e^u)\,du \\ &= -\frac{2}{5}\Big[70u - 30e^u\Big]_{u=0}^{-2.5} \\ &= -\frac{2}{5}[(-175 - 30e^{-2.5}) - (-30)] \\ &= 58 + 12e^{-2.5} \approx 59°\text{F} \ ◂ \end{aligned}$$

In Example 4, the temperature T of the lemonade rises from an initial temperature of 40°F toward the room temperature of 70°F. Explain why the formula

$$T = 70 - 30e^{-0.5t}$$

is a good model for this situation.

✔QUICK CHECK EXERCISES 6.4 *(See page 441 for answers.)*

1. Suppose that $f(x)$ has derivative $f'(x) = (x-4)^2e^{-x/2}$. Then $f''(x) = -\frac{1}{2}(x-4)(x-8)e^{-x/2}$.
  (a) The function f is increasing on the interval(s) ________.
  (b) The function f is concave up on the interval(s) ________.
  (c) The function f is concave down on the interval(s) ________.

2. Let $f(x) = x^2(2\ln x - 3)$. Given that

$$f'(x) = 4x(\ln x - 1), \qquad f''(x) = 4\ln x$$

determine the following properties of the graph of f.
  (a) The graph is increasing on the interval ________.
  (b) The graph is concave down on the interval ________.

3. Let $f(x) = (x-2)^2e^{x/2}$. Given that

$$f'(x) = \tfrac{1}{2}(x^2-4)e^{x/2}, \qquad f''(x) = \tfrac{1}{4}(x^2+4x-4)e^{x/2}$$

determine the following properties of the graph of f.
  (a) The graph is above the x-axis on the intervals ________.
  (b) The graph is increasing on the intervals ________.
  (c) The graph is concave up on the intervals ________.
  (d) The relative minimum point on the graph is ________.
  (e) The relative maximum point on the graph is ________.
  (f) Inflection points occur at $x =$ ________.

EXERCISE SET 6.4 Graphing Utility C CAS

1–4 Use the given derivative to find all critical points of f, and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere. ■

1. $f'(x) = xe^{1-x^2}$

2. $f'(x) = x^4(e^x - 3)$

3. $f'(x) = \ln\left(\dfrac{2}{1+x^2}\right)$

4. $f'(x) = e^{2x} - 5e^x + 6$

5–8 Use a graphing utility to estimate the absolute maximum and minimum values of f, if any, on the stated interval, and then use calculus methods to find the exact values. ■

5. $f(x) = x^3e^{-2x}$; $[1, 4]$ **6.** $f(x) = \dfrac{\ln(2x)}{x}$; $[1, e]$

7. $f(x) = 5\ln(x^2 + 1) - 3x$; $[0, 4]$

8. $f(x) = (x^2 - 1)e^x$; $[-2, 2]$

9–18 We will develop techniques in Section 6.5 to verify that

$$\lim_{x\to+\infty} \frac{e^x}{x} = +\infty, \quad \lim_{x\to+\infty} \frac{x}{e^x} = 0, \quad \lim_{x\to-\infty} xe^x = 0$$

In these exercises: (a) Use these results, as necessary, to find the limits of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. (b) Sketch a graph of $f(x)$ and identify all relative extrema, inflection points, and asymptotes (as appropriate). Check your work with a graphing utility. ■

9. $f(x) = xe^x$ **10.** $f(x) = xe^{-x}$

11. $f(x) = x^2e^{-2x}$ **12.** $f(x) = x^2e^{2x}$

13. $f(x) = x^2e^{-x^2}$ **14.** $f(x) = e^{-1/x^2}$

15. $f(x) = \dfrac{e^x}{1 - x}$ **16.** $f(x) = x^{2/3}e^x$

17. $f(x) = x^2e^{1-x}$ **18.** $f(x) = x^3e^{x-1}$

19–24 We will develop techniques in Section 6.5 to verify that

$$\lim_{x\to+\infty} \frac{\ln x}{x^r} = 0, \quad \lim_{x\to+\infty} \frac{x^r}{\ln x} = +\infty, \quad \lim_{x\to0^+} x^r \ln x = 0$$

for any positive real number r. In these exercises: (a) Use these results, as necessary, to find the limits of $f(x)$ as $x \to +\infty$ and as $x \to 0^+$. (b) Sketch a graph of $f(x)$ and identify all relative extrema, inflection points, and asymptotes (as appropriate). Check your work with a graphing utility. ■

19. $f(x) = x\ln x$ **20.** $f(x) = x^2\ln x$

21. $f(x) = x^2\ln(2x)$ **22.** $f(x) = \ln(x^2 + 1)$

23. $f(x) = x^{2/3}\ln x$ **24.** $f(x) = x^{-1/3}\ln x$

FOCUS ON CONCEPTS

25. Consider the family of curves $y = xe^{-bx}$ $(b > 0)$.

(a) Use a graphing utility to generate some members of this family.

(b) Discuss the effect of varying b on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

26. Consider the family of curves $y = e^{-bx^2}$ $(b > 0)$.

(a) Use a graphing utility to generate some members of this family.

(b) Discuss the effect of varying b on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

27. (a) Determine whether the following limits exist, and if so, find them:

$$\lim_{x\to+\infty} e^x \cos x, \quad \lim_{x\to-\infty} e^x \cos x$$

(b) Sketch the graphs of the equations $y = e^x$, $y = -e^x$, and $y = e^x \cos x$ in the same coordinate system, and label any points of intersection.

(c) Use a graphing utility to generate some members of the family $y = e^{ax}\cos bx$ $(a > 0$ and $b > 0)$, and discuss the effect of varying a and b on the shape of the curve.

28. Consider the family of curves $y = x^n e^{-x^2/n}$, where n is a positive integer.

(a) Use a graphing utility to generate some members of this family.

(b) Discuss the effect of varying n on the shape of the graph, and discuss the locations of the relative extrema and inflection points.

29–32 True–False Determine whether the statement is true or false. Explain your answer. ■

29. The graph of $y = e^x$ is the reflection of the graph of $y = \ln x$ across the y-axis.

30. If f is a function with derivative $f'(x) = e^{(x-1)^2}$, then f has a relative minimum at $x = 1$.

31. The average value of $f(x) = \ln x$ over the interval $[1, e^2]$ is greater than 1.

32. Assume that A, k, and L are positive constants. The graph of the logistic curve $y = L/(1 + Ae^{-kt})$, $t \geq 0$, is increasing with horizontal asymptote $y = L$.

33. Suppose that a population y grows according to the logistic model given by Formula (1).

(a) At what rate is y increasing at time $t = 0$?

(b) In words, describe how the rate of growth of y varies with time.

(c) At what time is the population growing most rapidly?

34. Suppose that the number of individuals at time t in a certain wildlife population is given by

$$N(t) = \frac{340}{1 + 9(0.77)^t}, \quad t \geq 0$$

where t is in years. Use a graphing utility to estimate the time at which the size of the population is increasing most rapidly.

35. Suppose that the spread of a flu virus on a college campus is modeled by the function

$$y(t) = \frac{1000}{1 + 999e^{-0.9t}}$$

where $y(t)$ is the number of infected students at time t (in days, starting with $t = 0$). Use a graphing utility to estimate the day on which the virus is spreading most rapidly.

36. The logistic growth model given in Formula (1) is equivalent to

$$ye^{kt} + Ay = Le^{kt}$$

where y is the population at time t $(t \geq 0)$ and A, k, and L are positive constants. Use implicit differentiation to verify that

$$\frac{dy}{dt} = \frac{k}{L}y(L - y)$$

$$\frac{d^2y}{dt^2} = \frac{k^2}{L^2}y(L - y)(L - 2y)$$

37. Assuming that A, k, and L are positive constants, verify that the graph of $y = L/(1 + Ae^{-kt})$ has an inflection point at $\left(\frac{1}{k}\ln A, \frac{1}{2}L\right)$.

38. Suppose that the number of bacteria in a culture at time t is given by $N = 5000(25 + te^{-t/20})$.
- (a) Find the largest and smallest number of bacteria in the culture during the time interval $0 \le t \le 100$.
- (b) At what time during the time interval in part (a) is the number of bacteria decreasing most rapidly?

39. The concentration $C(t)$ of a drug in the bloodstream t hours after it has been injected is commonly modeled by an equation of the form

$$C(t) = \frac{K(e^{-bt} - e^{-at})}{a - b}$$

where $K > 0$ and $a > b > 0$.
- (a) At what time does the maximum concentration occur?
- (b) Let $K = 1$ for simplicity, and use a graphing utility to check your result in part (a) by graphing $C(t)$ for various values of a and b.

40. Let $s(t) = te^{-t}$ be the position function of a particle moving along a coordinate line, where s is in meters and t is in seconds. Use a graphing utility to generate the graphs of $s(t)$, $v(t)$, and $a(t)$ for $t \ge 0$, and use those graphs where needed.
- (a) Use the appropriate graph to make a rough estimate of the time at which the particle first reverses the direction of its motion; and then find the time exactly.
- (b) Find the exact position of the particle when it first reverses the direction of its motion.
- (c) Use the appropriate graphs to make a rough estimate of the time intervals on which the particle is speeding up and on which it is slowing down; and then find those time intervals exactly.

41–42 Find the area under the curve $y = f(x)$ over the stated interval.

41. $f(x) = e^{2x}$; $[0, \ln 2]$ **42.** $f(x) = \frac{1}{x}$; $[1, 5]$

43–44 Sketch the area enclosed by the curves and find its area.

43. $y = e^x$, $y = e^{2x}$, $x = 0$, $x = \ln 2$

44. $x = 1/y$, $x = 0$ $y = 1$, $y = e$

45–46 Sketch the curve and find the total area between the curve and the given interval on the x-axis.

45. $y = e^x - 1$; $[-1, 1]$ **46.** $y = \frac{x-2}{x}$; $[1, 3]$

47–49 Find the average value of the function over the given interval.

47. $f(x) = 1/x$; $[1, e]$ **48.** $f(x) = e^x$; $[-1, \ln 5]$

49. $f(x) = e^{-2x}$; $[0, 4]$

50. Suppose that the value of a yacht in dollars after t years of use is $V(t) = 275{,}000e^{-0.17t}$. What is the average value of the yacht over its first 10 years of use?

C **51–52** For the given velocity function $v(t)$:
- (a) Generate the velocity versus time curve, and use it to make a conjecture about the sign of the displacement over the given time interval.
- (b) Use a CAS to find the displacement.

51. $v(t) = 0.5 - te^{-t}$; $0 \le t \le 5$

52. $v(t) = t\ln(t + 0.1)$; $0 \le t \le 1$

53–54 Use a graphing utility to determine the number of times the curves intersect and then apply Newton's Method, where needed, to approximate the x-coordinates of all intersections.

53. $y = 1$ and $y = e^x \sin x$; $0 < x < \pi$

54. $y = e^{-x}$ and $y = \ln x$

55. For the function

$$f(x) = \frac{e^{-x}}{1 + x^2}$$

use Newton's Method to approximate the x-coordinates of the inflection points to two decimal places.

56. (a) Show that $e^x \ge 1 + x$ if $x \ge 0$.
- (b) Show that $e^x \ge 1 + x + \frac{1}{2}x^2$ if $x \ge 0$.
- (c) Confirm the inequalities in parts (a) and (b) with a graphing utility.

57–58 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

57. $y = e^x$, $y = 0$, $x = 0$, $x = \ln 3$

58. $y = e^{-2x}$, $y = 0$, $x = 0$, $x = 1$

59–60 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

59. $y = \frac{1}{x^2 + 1}$, $x = 0$, $x = 1$, $y = 0$

60. $y = e^{x^2}$, $x = 1$, $x = \sqrt{3}$, $y = 0$

61–62 Use the arc length formula from Exercise 24 of Section 5.4 to find the arc length of the curve.

61. $x = e^t \cos t$, $y = e^t \sin t$ $(0 \le t \le \pi/2)$

62. $x = e^t(\sin t + \cos t)$, $y = e^t(\cos t - \sin t)$ $(1 \le t \le 4)$

C **63–64** Express the exact arc length of the curve over the given interval as an integral that has been simplified to eliminate the radical, and then evaluate the integral using a CAS.

63. $y = \ln(\sec x)$ from $x = 0$ to $x = \pi/4$

64. $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$

C **65–66** Use a CAS or a calculating utility with numerical integration capabilities to approximate the area of the surface generated by revolving the curve about the stated axis. Round your answer to two decimal places.

65. $y = e^x$, $0 \le x \le 1$; x-axis

66. $y = e^x$, $1 \le y \le e$; y-axis

✔QUICK CHECK ANSWERS 6.4

1. (a) $(-\infty, +\infty)$ (b) $(4, 8)$ (c) $(-\infty, 4), (8, +\infty)$ **2.** (a) $(e, +\infty)$ (b) $(0, 1)$ **3.** (a) $(-\infty, 2)$ and $(2, +\infty)$ (b) $(-\infty, -2]$ and $[2, +\infty)$ (c) $(-\infty, -2-2\sqrt{2})$ and $(-2+2\sqrt{2}, +\infty)$ (d) $(2, 0)$ (e) $(-2, 16e^{-1}) \approx (-2, 5.89)$ (f) $-2 \pm 2\sqrt{2}$

6.5 L'HÔPITAL'S RULE; INDETERMINATE FORMS

In this section we will discuss a general method for using derivatives to find limits. This method will enable us to establish limits with certainty that earlier in the text we were only able to conjecture using numerical or graphical evidence. The method that we will discuss in this section is an extremely powerful tool that is used internally by many computer programs to calculate limits of various types.

INDETERMINATE FORMS OF TYPE 0/0

Recall that a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)} \tag{1}$$

in which $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$ is called an ***indeterminate form of type* 0/0**. Some examples encountered earlier in the text are

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2, \quad \lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

The first limit was obtained algebraically by factoring the numerator and canceling the common factor of $x-1$, and the second two limits were obtained using geometric methods. However, there are many indeterminate forms for which neither algebraic nor geometric methods will produce the limit, so we need to develop a more general method.

To motivate such a method, suppose that (1) is an indeterminate form of type $0/0$ in which f' and g' are continuous at $x=a$ and $g'(a)\neq 0$. Since f and g can be closely approximated by their local linear approximations near a, it is reasonable to expect that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)} \tag{2}$$

Since we are assuming that f' and g' are continuous at $x=a$, we have

$$\lim_{x\to a} f'(x) = f'(a) \quad \text{and} \quad \lim_{x\to a} g'(x) = g'(a)$$

and since the differentiability of f and g at $x=a$ implies the continuity of f and g at $x=a$, we have

$$f(a) = \lim_{x\to a} f(x) = 0 \quad \text{and} \quad g(a) = \lim_{x\to a} g(x) = 0$$

Thus, we can rewrite (2) as

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \lim_{x\to a} \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{3}$$

This result, called ***L'Hôpital's rule***, converts the given indeterminate form into a limit involving derivatives that is often easier to evaluate.

Although we motivated (3) by assuming that f and g have continuous derivatives at $x=a$ and that $g'(a)\neq 0$, the result is true under less stringent conditions and is also valid for one-sided limits and limits at $+\infty$ and $-\infty$. The proof of the following precise statement of L'Hôpital's rule is omitted.

6.5.1 THEOREM (*L'Hôpital's Rule for Form* 0/0) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

If $\lim_{x \to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a^-$, $x \to a^+$, $x \to -\infty$, or as $x \to +\infty$.

WARNING

Note that in L'Hôpital's rule the numerator and denominator are differentiated individually. This is *not* the same as differentiating $f(x)/g(x)$.

In the examples that follow we will apply L'Hôpital's rule using the following three-step process:

Applying L'Hôpital's Rule

Step 1. Check that the limit of $f(x)/g(x)$ is an indeterminate form of type $0/0$.

Step 2. Differentiate f and g separately.

Step 3. Find the limit of $f'(x)/g'(x)$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $f(x)/g(x)$.

Example 1 Find the limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

using L'Hôpital's rule, and check the result by factoring.

Solution. The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type $0/0$. Applying L'Hôpital's rule yields

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\dfrac{d}{dx}[x^2 - 4]}{\dfrac{d}{dx}[x - 2]} = \lim_{x \to 2} \frac{2x}{1} = 4$$

This agrees with the computation

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4 \blacktriangleleft$$

The limit in Example 1 can be interpreted as the limit form of a certain derivative. Use that derivative to evaluate the limit.

Guillaume François Antoine de L'Hôpital (1661–1704) French mathematician. L'Hôpital, born to parents of the French high nobility, held the title of Marquis de Sainte-Mesme Comte d'Autrement. He showed mathematical talent quite early and at age 15 solved a difficult problem about cycloids posed by Pascal. As a young man he served briefly as a cavalry officer, but resigned because of nearsightedness. In his own time he gained fame as the author of the first textbook ever published on differential calculus, *L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (1696). L'Hôpital's rule appeared for the first time in that book. Actually, L'Hôpital's rule and most of the material in the calculus text were due to John Bernoulli, who was L'Hôpital's teacher. L'Hôpital dropped his plans for a book on integral calculus when Leibniz informed him that he intended to write such a text. L'Hôpital was apparently generous and personable, and his many contacts with major mathematicians provided the vehicle for disseminating major discoveries in calculus throughout Europe.

[*Image: http://en.wikipedia.org/wiki/File:Guillaume_de_l%27H%C3%B4pital.jpg*]

▶ **Example 2** In each part confirm that the limit is an indeterminate form of type 0/0, and evaluate it using L'Hôpital's rule.

(a) $\lim_{x\to 0}\frac{\sin 2x}{x}$ (b) $\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}$ (c) $\lim_{x\to 0}\frac{e^x-1}{x^3}$

(d) $\lim_{x\to 0^-}\frac{\tan x}{x^2}$ (e) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (f) $\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$

Solution (a). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]}=\lim_{x\to 0}\frac{2\cos 2x}{1}=2$$

Observe that this result agrees with that obtained by substitution in Example 2(b) of Section 1.6.

WARNING

Applying L'Hôpital's rule to limits that are not indeterminate forms can produce incorrect results. For example, the computation

$$\lim_{x\to 0}\frac{x+6}{x+2}=\lim_{x\to 0}\frac{\frac{d}{dx}[x+6]}{\frac{d}{dx}[x+2]}$$
$$=\lim_{x\to 0}\frac{1}{1}=1$$

is ***not valid***, since the limit is not an indeterminate form. The correct result is

$$\lim_{x\to 0}\frac{x+6}{x+2}=\frac{0+6}{0+2}=3$$

Solution (b). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to \pi/2}\frac{1-\sin x}{\cos x}=\lim_{x\to \pi/2}\frac{\frac{d}{dx}[1-\sin x]}{\frac{d}{dx}[\cos x]}=\lim_{x\to \pi/2}\frac{-\cos x}{-\sin x}=\frac{0}{-1}=0$$

Solution (c). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{e^x-1}{x^3}=\lim_{x\to 0}\frac{\frac{d}{dx}[e^x-1]}{\frac{d}{dx}[x^3]}=\lim_{x\to 0}\frac{e^x}{3x^2}=+\infty$$

Solution (d). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}=\lim_{x\to 0^-}\frac{\sec^2 x}{2x}=-\infty$$

Solution (e). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}$$

Since the new limit is another indeterminate form of type 0/0, we apply L'Hôpital's rule again:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

Solution (f). The numerator and denominator have a limit of 0, so the limit is an indeterminate form of type 0/0. Applying L'Hôpital's rule yields

$$\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}=\lim_{x\to +\infty}\frac{-\frac{4}{3}x^{-7/3}}{(-1/x^2)\cos(1/x)}=\lim_{x\to +\infty}\frac{\frac{4}{3}x^{-1/3}}{\cos(1/x)}=\frac{0}{1}=0$$ ◀

INDETERMINATE FORMS OF TYPE ∞/∞

When we want to indicate that the limit (or a one-sided limit) of a function is $+\infty$ or $-\infty$ without being specific about the sign, we will say that the limit is ∞. For example,

$$\lim_{x\to a^+} f(x) = \infty \quad \text{means} \quad \lim_{x\to a^+} f(x) = +\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty$$

$$\lim_{x\to +\infty} f(x) = \infty \quad \text{means} \quad \lim_{x\to +\infty} f(x) = +\infty \quad \text{or} \quad \lim_{x\to +\infty} f(x) = -\infty$$

$$\lim_{x\to a} f(x) = \infty \quad \text{means} \quad \lim_{x\to a^+} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

The limit of a ratio, $f(x)/g(x)$, in which the numerator has limit ∞ and the denominator has limit ∞ is called an ***indeterminate form of type*** ∞/∞. The following version of L'Hôpital's rule, which we state without proof, can often be used to evaluate limits of this type.

6.5.2 THEOREM (*L'Hôpital's Rule for Form* ∞/∞) *Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that*

$$\lim_{x\to a} f(x) = \infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \infty$$

If $\lim\limits_{x\to a} [f'(x)/g'(x)]$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x\to a^-$, $x\to a^+$, $x\to -\infty$, or as $x\to +\infty$.

▶ **Example 3** In each part confirm that the limit is an indeterminate form of type ∞/∞ and apply L'Hôpital's rule.

$$\text{(a)}\quad \lim_{x\to +\infty} \frac{x}{e^x} \qquad \text{(b)}\quad \lim_{x\to 0^+} \frac{\ln x}{\csc x}$$

Solution (a). The numerator and denominator both have a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to +\infty} \frac{x}{e^x} = \lim_{x\to +\infty} \frac{1}{e^x} = 0$$

Solution (b). The numerator has a limit of $-\infty$ and the denominator has a limit of $+\infty$, so we have an indeterminate form of type ∞/∞. Applying L'Hôpital's rule yields

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} \tag{4}$$

This last limit is again an indeterminate form of type ∞/∞. Moreover, any additional applications of L'Hôpital's rule will yield powers of $1/x$ in the numerator and expressions involving $\csc x$ and $\cot x$ in the denominator; thus, repeated application of L'Hôpital's rule simply produces new indeterminate forms. We must try something else. The last limit in (4) can be rewritten as

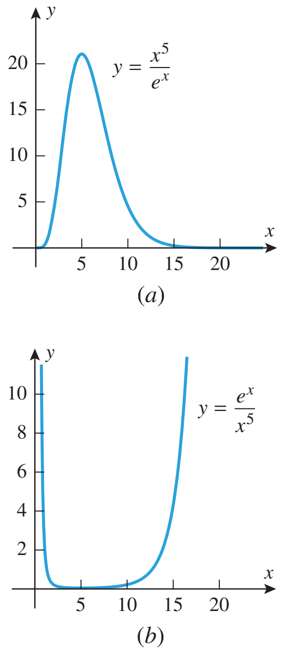
$$\lim_{x\to 0^+} \left(-\frac{\sin x}{x} \tan x\right) = -\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \tan x = -(1)(0) = 0$$

Thus,

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = 0 \;◀$$

ANALYZING THE GROWTH OF EXPONENTIAL FUNCTIONS USING L'HÔPITAL'S RULE

If n is any positive integer, then $x^n \to +\infty$ as $x \to +\infty$. Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example, we know that $e^x \to +\infty$ as $x \to +\infty$ and that the growth of e^x is very rapid (Table 6.1.5); however, the growth of x^n is also rapid when n is a high power, so it is reasonable to ask whether high powers of x grow more or less rapidly than e^x. One way to investigate this is to examine the behavior of the ratio x^n/e^x as $x \to +\infty$. For example, Figure 6.5.1*a* shows the graph of $y = x^5/e^x$. This graph suggests that $x^5/e^x \to 0$ as $x \to +\infty$, and this implies that the growth of the function e^x is sufficiently rapid that its values eventually overtake those of x^5 and force the ratio toward zero. Stated informally, "e^x eventually grows more rapidly than x^5." The same conclusion could have been reached by putting e^x on top and examining the behavior of e^x/x^5 as $x \to +\infty$ (Figure 6.5.1*b*). In this case the values of e^x eventually overtake those of x^5 and force the ratio toward $+\infty$. More generally, we can use L'Hôpital's rule to show that e^x *eventually grows more rapidly than any positive integer power of* x, that is,



(*a*) (*b*)

▲ **Figure 6.5.1**

$$\boxed{\lim_{x \to +\infty} \frac{x^n}{e^x} = 0} \quad \text{and} \quad \boxed{\lim_{x \to +\infty} \frac{e^x}{x^n} = +\infty} \tag{5–6}$$

Both limits are indeterminate forms of type ∞/∞ that can be evaluated using L'Hôpital's rule. For example, to establish (5), we will need to apply L'Hôpital's rule n times. For this purpose, observe that successive differentiations of x^n reduce the exponent by 1 each time, thus producing a constant for the nth derivative. For example, the successive derivatives of x^3 are $3x^2$, $6x$, and 6. In general, the nth derivative of x^n is $n(n-1)(n-2)\cdots 1 = n!$ (verify).[*] Thus, applying L'Hôpital's rule n times to (5) yields

$$\lim_{x \to +\infty} \frac{x^n}{e^x} = \lim_{x \to +\infty} \frac{n!}{e^x} = 0$$

Limit (6) can be established similarly.

INDETERMINATE FORMS OF TYPE $0 \cdot \infty$

Thus far we have discussed indeterminate forms of type $0/0$ and ∞/∞. However, these are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \quad f(x) \cdot g(x), \quad f(x)^{g(x)}, \quad f(x) - g(x), \quad f(x) + g(x)$$

is called an *indeterminate form* if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example, the limit

$$\lim_{x \to 0^+} x \ln x$$

is an ***indeterminate form of type* $0 \cdot \infty$** because the limit of the first factor is 0, the limit of the second factor is $-\infty$, and these two limits exert conflicting influences on the product. On the other hand, the limit

$$\lim_{x \to +\infty} [\sqrt{x}(1 - x^2)]$$

is not an indeterminate form because the first factor has a limit of $+\infty$, the second factor has a limit of $-\infty$, and these influences work together to produce a limit of $-\infty$ for the product.

Indeterminate forms of type $0 \cdot \infty$ can sometimes be evaluated by rewriting the product as a ratio, and then applying L'Hôpital's rule for indeterminate forms of type $0/0$ or ∞/∞.

---

[*] Recall that for $n \geq 1$ the expression $n!$, read ***n-factorial***, denotes the product of the first n positive integers.

WARNING

It is tempting to argue that an indeterminate form of type $0 \cdot \infty$ has value 0 since "zero times anything is zero." However, this is fallacious since $0 \cdot \infty$ is not a product of numbers, but rather a statement about limits. For example, here are two indeterminate forms of type $0 \cdot \infty$ whose limits are *not* zero:

$$\lim_{x\to 0}\left(x \cdot \frac{1}{x}\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{x\to 0^+}\left(\sqrt{x} \cdot \frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}}\right)$$

$$= +\infty$$

▶ Example 4 Evaluate

$$\text{(a)} \quad \lim_{x\to 0^+} x \ln x \qquad \text{(b)} \quad \lim_{x\to \pi/4} (1 - \tan x) \sec 2x$$

Solution (a). The factor x has a limit of 0 and the factor $\ln x$ has a limit of $-\infty$, so the stated problem is an indeterminate form of type $0 \cdot \infty$. There are two possible approaches: we can rewrite the limit as

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \text{or} \quad \lim_{x\to 0^+} \frac{x}{1/\ln x}$$

the first being an indeterminate form of type ∞/∞ and the second an indeterminate form of type $0/0$. However, the first form is the preferred initial choice because the derivative of $1/x$ is less complicated than the derivative of $1/\ln x$. That choice yields

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

Solution (b). The stated problem is an indeterminate form of type $0 \cdot \infty$. We will convert it to an indeterminate form of type $0/0$:

$$\lim_{x\to \pi/4} (1 - \tan x) \sec 2x = \lim_{x\to \pi/4} \frac{1 - \tan x}{1/\sec 2x} = \lim_{x\to \pi/4} \frac{1 - \tan x}{\cos 2x}$$

$$= \lim_{x\to \pi/4} \frac{-\sec^2 x}{-2 \sin 2x} = \frac{-2}{-2} = 1 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty) - (+\infty), \quad (-\infty) - (-\infty),$$
$$(+\infty) + (-\infty), \quad (-\infty) + (+\infty)$$

is called an ***indeterminate form of type*** $\boldsymbol{\infty - \infty}$. Such limits are indeterminate because the two terms exert conflicting influences on the expression: one pushes it in the positive direction and the other pushes it in the negative direction. However, limit problems that lead to one of the expressions

$$(+\infty) + (+\infty), \quad (+\infty) - (-\infty),$$
$$(-\infty) + (-\infty), \quad (-\infty) - (+\infty)$$

are not indeterminate, since the two terms work together (those on the top produce a limit of $+\infty$ and those on the bottom produce a limit of $-\infty$).

Indeterminate forms of type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $0/0$ or ∞/∞.

▶ Example 5 Evaluate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Solution. Both terms have a limit of $+\infty$, so the stated problem is an indeterminate form of type $\infty - \infty$. Combining the two terms yields

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$$

which is an indeterminate form of type $0/0$. Applying L'Hôpital's rule twice yields

$$\lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{2} = 0 \blacktriangleleft$$

INDETERMINATE FORMS OF TYPE 0^0, ∞^0, 1^∞

Limits of the form

$$\lim f(x)^{g(x)}$$

can give rise to ***indeterminate forms of the types*** $\mathbf{0^0}$***,*** $\boldsymbol{\infty^0}$***, and*** $\mathbf{1^\infty}$. (The interpretations of these symbols should be clear.) For example, the limit

$$\lim_{x\to 0^+}(1+x)^{1/x}$$

whose value we know to be e [see Formula (1) of Section 6.2] is an indeterminate form of type 1^∞. It is indeterminate because the expressions $1+x$ and $1/x$ exert two conflicting influences: the first approaches 1, which drives the expression toward 1, and the second approaches $+\infty$, which drives the expression toward $+\infty$.

Indeterminate forms of types 0^0, ∞^0, and 1^∞ can sometimes be evaluated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then computing the limit of $\ln y$. Since

$$\ln y = \ln[f(x)^{g(x)}] = g(x)\cdot\ln[f(x)]$$

the limit of $\ln y$ will be an indeterminate form of type $0\cdot\infty$ (verify), which can be evaluated by methods we have already studied. Once the limit of $\ln y$ is known, it is a straightforward matter to determine the limit of $y = f(x)^{g(x)}$, as we will illustrate in the next example.

▶ **Example 6** Find $\lim\limits_{x\to 0}(1+\sin x)^{1/x}$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1+\sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1+\sin x)^{1/x} = \frac{1}{x}\ln(1+\sin x) = \frac{\ln(1+\sin x)}{x}$$

Thus,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$$

which is an indeterminate form of type $0/0$, so by L'Hôpital's rule

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x} = \lim_{x\to 0}\frac{(\cos x)/(1+\sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x\to 0$, the continuity of the exponential function implies that $e^{\ln y}\to e^1$ as $x\to 0$, and this implies that $y\to e$ as $x\to 0$. Thus,

$$\lim_{x\to 0}(1+\sin x)^{1/x} = e \blacktriangleleft$$

✔QUICK CHECK EXERCISES 6.5 *(See page 450 for answers.)*

1. In each part, does L'Hôpital's rule apply to the given limit?

(a) $\lim\limits_{x\to 1}\dfrac{2x-2}{x^3+x-2}$ (b) $\lim\limits_{x\to 0}\dfrac{\cos x}{x}$

(c) $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{\tan x}$

2. Evaluate each of the limits in Quick Check Exercise 1.

3. Using L'Hôpital's rule, $\lim\limits_{x\to+\infty}\dfrac{e^x}{500x^2} =$ ________.

EXERCISE SET 6.5

Graphing Utility C CAS

1–2 Evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule. ■

1. (a) $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x^2+2x-8}$ (b) $\displaystyle\lim_{x\to +\infty} \frac{2x-5}{3x+7}$

2. (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{\tan x}$ (b) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^3-1}$

3–6 True–False Determine whether the statement is true or false. Explain your answer. ■

3. L'Hôpital's rule does not apply to $\displaystyle\lim_{x\to -\infty} \frac{\ln x}{x}$.

4. For any polynomial $p(x)$, $\displaystyle\lim_{x\to +\infty} \frac{p(x)}{e^x} = 0$.

5. If n is chosen sufficiently large, then $\displaystyle\lim_{x\to +\infty} \frac{(\ln x)^n}{x} = +\infty$.

6. $\displaystyle\lim_{x\to 0^+} (\sin x)^{1/x} = 0$

7–43 Find the limits. ■

7. $\displaystyle\lim_{x\to 0} \frac{e^x-1}{\sin x}$

8. $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{\sin 5x}$

9. $\displaystyle\lim_{\theta\to 0} \frac{\tan\theta}{\theta}$

10. $\displaystyle\lim_{t\to 0} \frac{te^t}{1-e^t}$

11. $\displaystyle\lim_{x\to \pi^+} \frac{\sin x}{x-\pi}$

12. $\displaystyle\lim_{x\to 0^+} \frac{\sin x}{x^2}$

13. $\displaystyle\lim_{x\to +\infty} \frac{\ln x}{x}$

14. $\displaystyle\lim_{x\to +\infty} \frac{e^{3x}}{x^2}$

15. $\displaystyle\lim_{x\to 0^+} \frac{\cot x}{\ln x}$

16. $\displaystyle\lim_{x\to 0^+} \frac{1-\ln x}{e^{1/x}}$

17. $\displaystyle\lim_{x\to +\infty} \frac{x^{100}}{e^x}$

18. $\displaystyle\lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$

19. $\displaystyle\lim_{x\to +\infty} xe^{-x}$

20. $\displaystyle\lim_{x\to \pi^-} (x-\pi)\tan\tfrac{1}{2}x$

21. $\displaystyle\lim_{x\to +\infty} x\sin\frac{\pi}{x}$

22. $\displaystyle\lim_{x\to 0^+} \tan x \ln x$

23. $\displaystyle\lim_{x\to \pi/2^-} \sec 3x \cos 5x$

24. $\displaystyle\lim_{x\to \pi} (x-\pi)\cot x$

25. $\displaystyle\lim_{x\to +\infty} (1-3/x)^x$

26. $\displaystyle\lim_{x\to 0} (1+2x)^{-3/x}$

27. $\displaystyle\lim_{x\to 0} (e^x+x)^{1/x}$

28. $\displaystyle\lim_{x\to +\infty} (1+a/x)^{bx}$

29. $\displaystyle\lim_{x\to 1} (2-x)^{\tan[(\pi/2)x]}$

30. $\displaystyle\lim_{x\to +\infty} [\cos(2/x)]^{x^2}$

31. $\displaystyle\lim_{x\to 0} (\csc x - 1/x)$

32. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x^2}-\frac{\cos 3x}{x^2}\right)$

33. $\displaystyle\lim_{x\to +\infty} (\sqrt{x^2+x}-x)$

34. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

35. $\displaystyle\lim_{x\to +\infty} [x-\ln(x^2+1)]$

36. $\displaystyle\lim_{x\to +\infty} [\ln x - \ln(1+x)]$

37. $\displaystyle\lim_{x\to 0^+} x^{\sin x}$

38. $\displaystyle\lim_{x\to 0^+} (e^{2x}-1)^x$

39. $\displaystyle\lim_{x\to 0^+} \left[-\frac{1}{\ln x}\right]^x$

40. $\displaystyle\lim_{x\to +\infty} x^{1/x}$

41. $\displaystyle\lim_{x\to +\infty} (\ln x)^{1/x}$

42. $\displaystyle\lim_{x\to 0^+} (-\ln x)^x$

43. $\displaystyle\lim_{x\to \pi/2^-} (\tan x)^{(\pi/2)-x}$

44. Show that for any positive integer n

(a) $\displaystyle\lim_{x\to +\infty} \frac{\ln x}{x^n} = 0$ (b) $\displaystyle\lim_{x\to +\infty} \frac{x^n}{\ln x} = +\infty$.

FOCUS ON CONCEPTS

45. (a) Find the error in the following calculation:

$$\lim_{x\to 1} \frac{x^3-x^2+x-1}{x^3-x^2} = \lim_{x\to 1} \frac{3x^2-2x+1}{3x^2-2x} = \lim_{x\to 1} \frac{6x-2}{6x-2} = 1$$

(b) Find the correct limit.

46. (a) Find the error in the following calculation:

$$\lim_{x\to 2} \frac{e^{3x^2-12x+12}}{x^4-16} = \lim_{x\to 2} \frac{(6x-12)e^{3x^2-12x+12}}{4x^3} = 0$$

(b) Find the correct limit.

47–50 Make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule. ■

47. $\displaystyle\lim_{x\to +\infty} \frac{\ln(\ln x)}{\sqrt{x}}$

48. $\displaystyle\lim_{x\to 0^+} x^x$

49. $\displaystyle\lim_{x\to 0^+} (\sin x)^{3/\ln x}$

50. $\displaystyle\lim_{x\to (\pi/2)^-} \frac{4\tan x}{1+\sec x}$

51–54 Make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule. ■

51. $y = \ln x - e^x$

52. $y = x - \ln(1+2e^x)$

53. $y = (\ln x)^{1/x}$

54. $y = \left(\dfrac{x+1}{x+2}\right)^x$

55. Limits of the type

$$0/\infty,\quad \infty/0,\quad 0^\infty,\quad \infty\cdot\infty,\quad +\infty+(+\infty),$$
$$+\infty-(-\infty),\quad -\infty+(-\infty),\quad -\infty-(+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\displaystyle\lim_{x\to 0^+} \frac{x}{\ln x}$ (b) $\displaystyle\lim_{x\to +\infty} \frac{x^3}{e^{-x}}$

(c) $\displaystyle\lim_{x\to (\pi/2)^-} (\cos x)^{\tan x}$ (d) $\displaystyle\lim_{x\to 0^+} (\ln x)\cot x$

(e) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x}-\ln x\right)$ (f) $\displaystyle\lim_{x\to -\infty} (x+x^3)$

56. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types 0^0, ∞^0, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^0, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples, which were suggested by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:

(a) $\lim\limits_{x\to 0^+} [x^{(\ln a)/(1+\ln x)}] = a \quad$ (form 0^0)

(b) $\lim\limits_{x\to +\infty} [x^{(\ln a)/(1+\ln x)}] = a \quad$ (form ∞^0)

(c) $\lim\limits_{x\to 0} [(x+1)^{(\ln a)/x}] = a \quad$ (form 1^∞).

Verify these results.

57–60 Verify that L'Hôpital's rule is of no help in finding the limit; then find the limit, if it exists, by some other method.

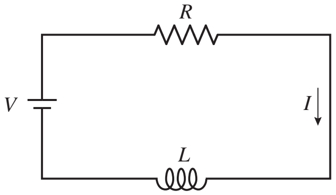
57. $\lim\limits_{x\to +\infty} \dfrac{x+\sin 2x}{x}$ **58.** $\lim\limits_{x\to +\infty} \dfrac{2x-\sin x}{3x+\sin x}$

59. $\lim\limits_{x\to +\infty} \dfrac{x(2+\sin 2x)}{x+1}$ **60.** $\lim\limits_{x\to +\infty} \dfrac{x(2+\sin x)}{x^2+1}$

61. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t=0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e., $R \to 0^+$)?



◀ **Figure Ex-61**

62. (a) Show that $\lim\limits_{x\to \pi/2} (\pi/2 - x)\tan x = 1$.

(b) Show that

$$\lim_{x\to \pi/2}\left(\frac{1}{\pi/2 - x} - \tan x\right) = 0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2 - x}$$

should be good for values of x near $\pi/2$. Use a calculator to find $\tan x$ and $1/(\pi/2 - x)$ for $x = 1.57$; compare the results.

C **63.** (a) Use a CAS to show that if k is a positive constant, then

$$\lim_{x\to +\infty} x(k^{1/x} - 1) = \ln k$$

(b) Confirm this result using L'Hôpital's rule. [*Hint:* Express the limit in terms of $t = 1/x$.]

(c) If n is a positive integer, then it follows from part (a) with $x = n$ that the approximation

$$n(\sqrt[n]{k} - 1) \approx \ln k$$

should be good when n is large. Use this result and the square root key on a calculator to approximate the values of $\ln 0.3$ and $\ln 2$ with $n = 1024$, then compare the values obtained with values of the logarithms generated directly from the calculator. [*Hint:* The nth roots for which n is a power of 2 can be obtained as successive square roots.]

64. Find all values of k and l such that

$$\lim_{x\to 0} \frac{k + \cos lx}{x^2} = -4$$

FOCUS ON CONCEPTS

65. Let $f(x) = x^2 \sin(1/x)$.

(a) Are the limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ indeterminate forms?

(b) Use a graphing utility to generate the graph of f, and use the graph to make conjectures about the limits in part (a).

(c) Use the Squeezing Theorem (1.6.2) to confirm that your conjectures in part (b) are correct.

66. (a) Explain why L'Hôpital's rule does not apply to the problem

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}$$

(b) Find the limit.

67. Find $\lim\limits_{x\to 0^+} \dfrac{x\sin(1/x)}{\sin x}$ if it exists.

68. Suppose that functions f and g are differentiable at $x = a$ and that $f(a) = g(a) = 0$. If $g'(a) \neq 0$, show that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

without using L'Hôpital's rule. [*Hint:* Divide the numerator and denominator of $f(x)/g(x)$ by $x - a$ and use the definitions for $f'(a)$ and $g'(a)$.]

69. **Writing** Were we to use L'Hôpital's rule to evaluate either

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{or} \quad \lim_{x\to +\infty} \left(1 + \frac{1}{x}\right)^x$$

we could be accused of circular reasoning. Explain why.

70. **Writing** Exercise 56 shows that the indeterminate forms 0^0 and ∞^0 can assume any positive real value. However, it is often the case that these indeterminate forms have value 1. Read the article "Indeterminate Forms of Exponential Type" by John Baxley and Elmer Hayashi in the June–July 1978 issue of *The American Mathematical Monthly*, and write a short report on why this is the case.

✔QUICK CHECK ANSWERS 6.5

1. (a) yes (b) no (c) yes **2.** (a) $\frac{1}{2}$ (b) does not exist (c) 2 **3.** $+\infty$

6.6 LOGARITHMIC AND OTHER FUNCTIONS DEFINED BY INTEGRALS

In Section 6.1 we defined the natural logarithm function $\ln x$ *to be the inverse of* e^x*. Although this was convenient and enabled us to deduce many properties of* $\ln x$*, the mathematical foundation was shaky in that we accepted the continuity of* e^x *and of all exponential functions without proof. In this section we will show that* $\ln x$ *can be defined as a certain integral, and we will use this new definition to prove that exponential functions are continuous. This integral definition is also important in applications because it provides a way of recognizing when integrals that appear in solutions of problems can be expressed as natural logarithms.*

THE CONNECTION BETWEEN NATURAL LOGARITHMS AND INTEGRALS

The connection between natural logarithms and integrals was made in the middle of the seventeenth century in the course of investigating areas under the curve $y = 1/t$. The problem being considered was to find values of $t_1, t_2, t_3, \ldots, t_n, \ldots$ for which the areas $A_1, A_2, A_3, \ldots, A_n, \ldots$ in Figure 6.6.1a would be equal. Through the combined work of Isaac Newton, the Belgian Jesuit priest Gregory of St. Vincent (1584–1667), and Gregory's student Alfons A. de Sarasa (1618–1667), it was shown that by taking the points to be

$$t_1 = e, \quad t_2 = e^2, \quad t_3 = e^3, \ldots, \quad t_n = e^n, \ldots$$

each of the areas would be 1 (Figure 6.6.1b). Thus, in modern integral notation

$$\int_1^{e^n} \frac{1}{t}\,dt = n$$

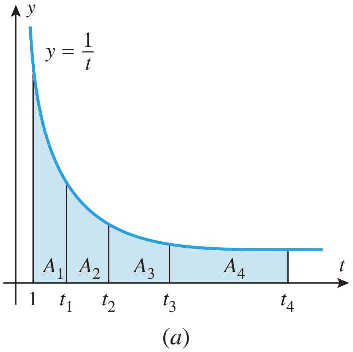
which can be expressed as

$$\int_1^{e^n} \frac{1}{t}\,dt = \ln(e^n)$$

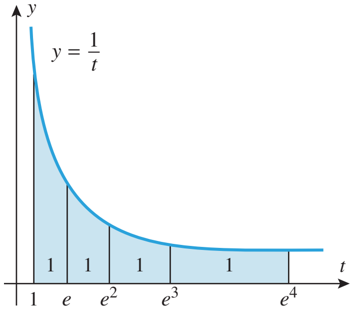
By comparing the upper limit of the integral and the expression inside the logarithm, it is a natural leap to the more general result

$$\int_1^{x} \frac{1}{t}\,dt = \ln x$$

which today we take as the formal definition of the natural logarithm.



(*a*)



(*b*)

Not drawn to scale

▲ **Figure 6.6.1**

6.6.1 DEFINITION The ***natural logarithm*** of x is denoted by $\ln x$ and is defined by the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0 \tag{1}$$

Review Theorem 4.5.8 and then explain why x is required to be positive in Definition 6.6.1.

Our strategy for putting the study of logarithmic and exponential functions on a sound mathematical footing is to use (1) as a starting point and then define e^x as the inverse of $\ln x$. This is the exact opposite of our previous approach in which we defined $\ln x$ to be the inverse of e^x. However, whereas previously we had to *assume* that e^x is continuous, the continuity of e^x will now follow from our definitions as a *theorem*. Our first challenge is to demonstrate that the properties of $\ln x$ resulting from Definition 6.6.1 are consistent with those obtained earlier. To start, observe that Part 2 of the Fundamental Theorem of Calculus (4.6.3) implies that $\ln x$ is differentiable and

None of the properties of $\ln x$ obtained in this section should be new, but now, for the first time, we give them a sound mathematical footing.

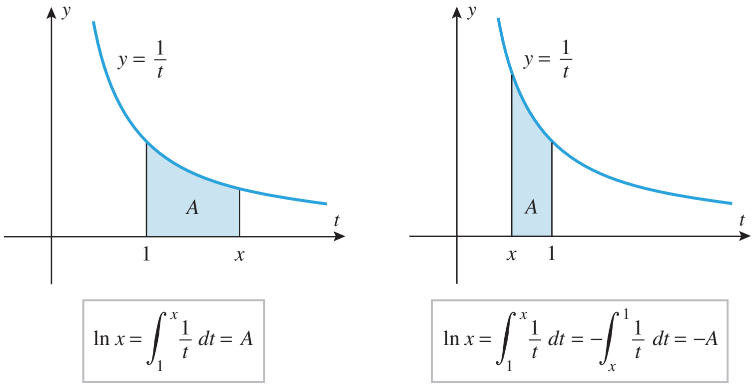
$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x} \quad (x > 0) \tag{2}$$

This is consistent with the derivative formula for $\ln x$ that we obtained previously. Moreover, because differentiability implies continuity, it follows that $\ln x$ is a continuous function on the interval $(0, +\infty)$.

Other properties of $\ln x$ can be obtained by interpreting the integral in (1) geometrically: In the case where $x > 1$, this integral represents the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 6.6.2*a*); in the case where $0 < x < 1$, the integral represents the negative of the area under the curve $y = 1/t$ from $t = x$ to $t = 1$ (Figure 6.6.2*b*); and in the case where $x = 1$, the integral has value 0 because its upper and lower limits of integration are the same. These geometric observations imply that

$$\begin{aligned} \ln x &> 0 \quad \text{if} \quad x > 1 \\ \ln x &< 0 \quad \text{if} \quad 0 < x < 1 \\ \ln x &= 0 \quad \text{if} \quad x = 1 \end{aligned}$$

Also, since $1/x$ is positive for $x > 0$, it follows from (2) that $\ln x$ is an increasing function on the interval $(0, +\infty)$. This is all consistent with the graph of $\ln x$ in Figure 6.6.3.

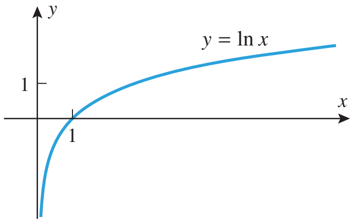


$$\ln x = \int_1^x \frac{1}{t}\,dt = A$$

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt = -A$$

(*a*) (*b*)

▶ **Figure 6.6.2**



▲ **Figure 6.6.3**

ALGEBRAIC PROPERTIES OF ln *x*

We can use (1) to show that Definition 6.6.1 produces the standard algebraic properties of logarithms.

6.6.2 THEOREM *For any positive numbers a and c and any rational number r:*

(*a*) $\ln ac = \ln a + \ln c$ (*b*) $\ln \frac{1}{c} = -\ln c$

(*c*) $\ln \frac{a}{c} = \ln a - \ln c$ (*d*) $\ln a^r = r \ln a$

PROOF (*a*) Treating a as a constant, consider the function $f(x) = \ln(ax)$. Then

$$f'(x) = \frac{1}{ax} \cdot \frac{d}{dx}(ax) = \frac{1}{ax} \cdot a = \frac{1}{x}$$

Thus, $\ln ax$ and $\ln x$ have the same derivative on $(0, +\infty)$, so these functions must differ by a constant on this interval. That is, there is a constant k such that

$$\ln ax - \ln x = k \tag{3}$$

on $(0, +\infty)$. Substituting $x = 1$ into this equation we conclude that $\ln a = k$ (verify). Thus, (3) can be written as

$$\ln ax - \ln x = \ln a$$

Setting $x = c$ establishes that

$$\ln ac - \ln c = \ln a \quad \text{or} \quad \ln ac = \ln a + \ln c$$

PROOFS (*b*) AND (*c*) Part (*b*) follows immediately from part (*a*) by substituting $1/c$ for a (verify). Then

$$\ln \frac{a}{c} = \ln\left(a \cdot \frac{1}{c}\right) = \ln a + \ln \frac{1}{c} = \ln a - \ln c$$

PROOF (*d*) First, we will argue that part (*d*) is satisfied if r is any nonnegative integer. If $r = 1$, then (*d*) is clearly satisfied; if $r = 0$, then (*d*) follows from the fact that $\ln 1 = 0$. Suppose that we know (*d*) is satisfied for r equal to some integer n. It then follows from part (*a*) that

$$\ln a^{n+1} = \ln[a \cdot a^n] = \ln a + \ln a^n = \ln a + n \ln a = (n+1)\ln a$$

That is, if (*d*) is valid for r equal to some integer n, then it is also valid for $r = n + 1$. However, since we know (*d*) is satisfied if $r = 1$, it follows that (*d*) is valid for $r = 2$. But this implies that (*d*) is satisfied for $r = 3$, which in turn implies that (*d*) is valid for $r = 4$, and so forth. We conclude that (*d*) is satisfied if r is any nonnegative integer.

How is the proof of Theorem 6.6.2(*d*) for the case where r is a nonnegative integer analogous to a row of falling dominos? (This "domino" argument uses an informal version of a property of the integers known as the *principle of mathematical induction.*)

Next, suppose that $r = -m$ is a negative integer. Then

$$\begin{aligned} \ln a^r = \ln a^{-m} = \ln \frac{1}{a^m} &= -\ln a^m && \boxed{\text{By part }(b)} \\ &= -m \ln a && \boxed{\text{Part }(d)\text{ is valid for positive powers.}} \\ &= r \ln a \end{aligned}$$

which shows that (*d*) is valid for any negative integer r. Combining this result with our previous conclusion that (*d*) is satisfied for a nonnegative integer r shows that (*d*) is valid if r is *any* integer.

Finally, suppose that $r = m/n$ is any rational number, where $m \neq 0$ and $n \neq 0$ are integers. Then

$$\begin{aligned} \ln a^r = \frac{n \ln a^r}{n} &= \frac{\ln[(a^r)^n]}{n} && \boxed{\text{Part }(d)\text{ is valid for integer powers.}} \\ &= \frac{\ln a^{rn}}{n} && \boxed{\text{Property of exponents}} \\ &= \frac{\ln a^m}{n} && \boxed{\text{Definition of }r} \\ &= \frac{m \ln a}{n} && \boxed{\text{Part }(d)\text{ is valid for integer powers.}} \\ &= \frac{m}{n} \ln a = r \ln a \end{aligned}$$

which shows that (*d*) is valid for any rational number r. ■

APPROXIMATING ln x NUMERICALLY

For specific values of x, the value of $\ln x$ can be approximated numerically by approximating the definite integral in (1), say by using the midpoint approximation that was discussed in Section 4.4.

Example 1 Approximate $\ln 2$ using the midpoint approximation with $n = 10$.

Solution. From (1), the exact value of $\ln 2$ is represented by the integral

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt$$

The midpoint rule is given in Formulas (5) and (6) of Section 4.4. Expressed in terms of t, the latter formula is

$$\int_a^b f(t)\,dt \approx \Delta t \sum_{k=1}^{n} f(t_k^*)$$

where Δt is the common width of the subintervals and $t_1^*, t_2^*, \dots, t_n^*$ are the midpoints. In this case we have 10 subintervals, so $\Delta t = (2-1)/10 = 0.1$. The computations to six decimal places are shown in Table 6.6.1. By comparison, a calculator set to display six decimal places gives $\ln 2 \approx 0.693147$, so the magnitude of the error in the midpoint approximation is about 0.000311. Greater accuracy in the midpoint approximation can be obtained by increasing n. For example, the midpoint approximation with $n = 100$ yields $\ln 2 \approx 0.693144$, which is correct to five decimal places. ◀

Table 6.6.1

$n = 10$
$\Delta t = (b - a)/n = (2 - 1)/10 = 0.1$

k	t_k^*	$1/t_k^*$
1	1.05	0.952381
2	1.15	0.869565
3	1.25	0.800000
4	1.35	0.740741
5	1.45	0.689655
6	1.55	0.645161
7	1.65	0.606061
8	1.75	0.571429
9	1.85	0.540541
10	1.95	0.512821
		6.928355

$$\Delta t \sum_{k=1}^{n} f(t_k^*) \approx (0.1)(6.928355) \approx 0.692836$$

DOMAIN, RANGE, AND END BEHAVIOR OF ln x

6.6.3 THEOREM

(*a*) *The domain of* $\ln x$ *is* $(0, +\infty)$.

(*b*) $\lim_{x \to 0^+} \ln x = -\infty$ *and* $\lim_{x \to +\infty} \ln x = +\infty$

(*c*) *The range of* $\ln x$ *is* $(-\infty, +\infty)$.

PROOFS (*a*) AND (*b*) We have already shown that $\ln x$ is defined and increasing on the interval $(0, +\infty)$. To prove that $\ln x \to +\infty$ as $x \to +\infty$, we must show that given any number $M > 0$, the value of $\ln x$ exceeds M for sufficiently large values of x. To do this, let N be any integer. If $x > 2^N$, then

$$\ln x > \ln 2^N = N \ln 2 \tag{4}$$

by Theorem 6.6.2(*d*). Since

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt > 0$$

it follows that $N \ln 2$ can be made arbitrarily large by choosing N sufficiently large. In particular, we can choose N so that $N \ln 2 > M$. It now follows from (4) that if $x > 2^N$, then $\ln x > M$, and this proves that

$$\lim_{x \to +\infty} \ln x = +\infty$$

Furthermore, by observing that $v = 1/x \to +\infty$ as $x \to 0^+$, we can use the preceding limit and Theorem 6.6.2(*b*) to conclude that

$$\lim_{x \to 0^+} \ln x = \lim_{v \to +\infty} \ln \frac{1}{v} = \lim_{v \to +\infty} (-\ln v) = -\infty$$

PROOF (*c*) It follows from part (*a*), the continuity of $\ln x$, and the Intermediate-Value Theorem (1.5.8) that $\ln x$ assumes every real value as x varies over the interval $(0, +\infty)$ (why?). ■

DEFINITION OF e^x

In Section 6.1 we defined $\ln x$ to be the inverse of the natural exponential function e^x. Now that we have a formal definition of $\ln x$ in terms of an integral, we will define the natural exponential function to be the inverse of $\ln x$.

Since $\ln x$ is increasing and continuous on $(0, +\infty)$ with range $(-\infty, +\infty)$, there is exactly one (positive) solution to the equation $\ln x = 1$. We *define* e to be the unique solution to $\ln x = 1$, so

$$\ln e = 1 \tag{5}$$

Furthermore, if x is any real number, there is a unique positive solution y to $\ln y = x$, so for irrational values of x we *define* e^x to be this solution. That is, when x is irrational, e^x is defined by

$$\ln e^x = x \tag{6}$$

Note that for rational values of x, we also have $\ln e^x = x \ln e = x$ from Theorem 6.6.2(*d*). Moreover, it follows immediately that $e^{\ln x} = x$ for any $x > 0$. Thus, (6) defines the exponential function for all real values of x as the inverse of the natural logarithm function.

6.6.4 DEFINITION The inverse of the natural logarithm function $\ln x$ is denoted by e^x and is called the ***natural exponential function***.

We can now establish the differentiability of e^x and confirm that

$$\frac{d}{dx}[e^x] = e^x$$

6.6.5 THEOREM *The natural exponential function e^x is differentiable, and hence continuous, on $(-\infty, +\infty)$, and its derivative is*

$$\frac{d}{dx}[e^x] = e^x$$

PROOF Because $\ln x$ is differentiable and

$$\frac{d}{dx}[\ln x] = \frac{1}{x} > 0$$

for all x in $(0, +\infty)$, it follows from Theorem 6.3.1, with $f(x) = \ln x$ and $f^{-1}(x) = e^x$, that e^x is differentiable on $(-\infty, +\infty)$ and its derivative is

$$\frac{d}{dx}\underbrace{[e^x]}_{f^{-1}(x)} = \underbrace{\frac{1}{1/e^x}}_{f'(f^{-1}(x))} = e^x \quad ■$$

IRRATIONAL EXPONENTS

Recall from Theorem 6.6.2(*d*) that if $a > 0$ and r is a rational number, then $\ln a^r = r \ln a$. Then $a^r = e^{\ln a^r} = e^{r \ln a}$ for any positive value of a and any rational number r. But the expression $e^{r \ln a}$ makes sense for *any* real number r, whether rational or irrational, so it is a good candidate to give meaning to a^r for any real number r.

Use Definition 6.6.6 to prove that if $a > 0$ and r is a real number, then $\ln a^r = r \ln a$.

6.6.6 DEFINITION If $a > 0$ and r is a real number, a^r is defined by

$$a^r = e^{r \ln a} \tag{7}$$

With this definition it can be shown that the standard algebraic properties of exponents, such as

$$a^p a^q = a^{p+q}, \quad \frac{a^p}{a^q} = a^{p-q}, \quad (a^p)^q = a^{pq}, \quad (a^p)(b^p) = (ab)^p$$

hold for any real values of a, b, p, and q, where a and b are positive. In addition, using (7) for a real exponent r, we can define the power function x^r whose domain consists of all positive real numbers, and for a positive base b we can define the ***base b exponential function*** $\boldsymbol{b^x}$ whose domain consists of all real numbers.

6.6.7 THEOREM

(*a*) *For any real number r, the power function x^r is differentiable on $(0, +\infty)$ and its derivative is*

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

(*b*) *For $b > 0$ and $b \neq 1$, the base b exponential function b^x is differentiable on $(-\infty, +\infty)$ and its derivative is*

$$\frac{d}{dx}[b^x] = b^x \ln b$$

PROOF The differentiability of $x^r = e^{r \ln x}$ and $b^x = e^{x \ln b}$ on their domains follows from the differentiability of $\ln x$ on $(0, +\infty)$ and of e^x on $(-\infty, +\infty)$:

$$\frac{d}{dx}[x^r] = \frac{d}{dx}[e^{r \ln x}] = e^{r \ln x} \cdot \frac{d}{dx}[r \ln x] = x^r \cdot \frac{r}{x} = rx^{r-1}$$

$$\frac{d}{dx}[b^x] = \frac{d}{dx}[e^{x \ln b}] = e^{x \ln b} \cdot \frac{d}{dx}[x \ln b] = b^x \ln b \quad ■$$

We expressed e as the value of a limit in Formulas (4) and (5) of Section 6.1 and in Formula (1) of Section 6.2. We now have the mathematical tools necessary to prove the existence of these limits.

6.6.8 THEOREM

(*a*) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (*b*) $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$ (*c*) $\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$

PROOF We will prove part (*a*); the proofs of parts (*b*) and (*c*) follow from this limit and are left as exercises. We first observe that

$$\frac{d}{dx}[\ln(x+1)]\bigg|_{x=0} = \frac{1}{x+1} \cdot 1 \bigg|_{x=0} = 1$$

However, using the definition of the derivative, we obtain

$$1 = \frac{d}{dx}[\ln(x+1)]\bigg|_{x=0} = \lim_{h\to 0}\frac{\ln(0+h+1)-\ln(0+1)}{h}$$
$$= \lim_{h\to 0}\left[\frac{1}{h}\cdot\ln(1+h)\right]$$

or, equivalently,

$$\lim_{x\to 0}\frac{1}{x}\cdot\ln(1+x) = 1 \tag{8}$$

Now

$$\begin{aligned}\lim_{x\to 0}(1+x)^{1/x} &= \lim_{x\to 0} e^{[\ln(1+x)]/x} && \boxed{\text{Definition 6.6.6}}\\ &= e^{\lim_{x\to 0}([\ln(1+x)]/x)} && \boxed{\text{Theorem 1.5.5}}\\ &= e^1 && \boxed{\text{Equation (8)}}\\ &= e \ \blacksquare\end{aligned}$$

GENERAL LOGARITHMS

We note that for $b > 0$ and $b \neq 1$, the function b^x is one-to-one and so has an inverse function. Using the definition of b^x, we can solve $y = b^x$ for x as a function of y:

$$\begin{aligned}&y = b^x = e^{x\ln b}\\ &\ln y = \ln(e^{x\ln b}) = x\ln b\\ &\frac{\ln y}{\ln b} = x\end{aligned}$$

Thus, the inverse function for b^x is $(\ln x)/(\ln b)$.

6.6.9 DEFINITION For $b > 0$ and $b \neq 1$, the ***base b logarithm*** function, denoted $\log_b x$, is defined by

$$\log_b x = \frac{\ln x}{\ln b} \tag{9}$$

It follows immediately from this definition that $\log_b x$ is the inverse function for b^x and satisfies the properties in Table 6.1.3. Furthermore, $\log_b x$ is differentiable, and hence continuous, on $(0, +\infty)$, and its derivative is

$$\frac{d}{dx}[\log_b x] = \frac{1}{x\ln b}$$

As a final note of consistency, we observe that $\log_e x = \ln x$.

FUNCTIONS DEFINED BY INTEGRALS

The functions we have dealt with thus far in this text are called ***elementary functions***; they include polynomial, rational, power, exponential, logarithmic, and trigonometric functions, and all other functions that can be obtained from these by addition, subtraction, multiplication, division, root extraction, and composition.

However, there are many important functions that do not fall into this category. Such functions occur in many ways, but they commonly arise in the course of solving initial-value problems of the form

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0 \tag{10}$$

Recall from Example 6 of Section 4.2 and the discussion preceding it that the basic method for solving (10) is to integrate $f(x)$, and then use the initial condition to determine the constant of integration. It can be proved that if f is continuous, then (10) has a unique solution and that this procedure produces it. However, there is another approach: Instead of solving each initial-value problem individually, we can find a general formula for the solution of (10), and then apply that formula to solve specific problems. We will now show that

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt \tag{11}$$

is a formula for the solution of (10). To confirm this we must show that $dy/dx = f(x)$ and that $y(x_0) = y_0$. The computations are as follows:

$$\frac{dy}{dx} = \frac{d}{dx}\left[y_0 + \int_{x_0}^{x} f(t)\,dt\right] = 0 + f(x) = f(x)$$

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(t)\,dt = y_0 + 0 = y_0$$

▶ **Example 2** In Example 6 of Section 4.2 we showed that the solution of the initial-value problem

$$\frac{dy}{dx} = \cos x, \quad y(0) = 1$$

is $y(x) = 1 + \sin x$. This initial-value problem can also be solved by applying Formula (11) with $f(x) = \cos x$, $x_0 = 0$, and $y_0 = 1$. This yields

$$y(x) = 1 + \int_{0}^{x} \cos t\,dt = 1 + \big[\sin t\big]_{t=0}^{x} = 1 + \sin x \quad ◀$$

In the last example we were able to perform the integration in Formula (11) and express the solution of the initial-value problem as an elementary function. However, sometimes this will not be possible, in which case the solution of the initial-value problem must be left in terms of an "unevaluated" integral. For example, from (11), the solution of the initial-value problem

$$\frac{dy}{dx} = e^{-x^2}, \quad y(0) = 1$$

is

$$y(x) = 1 + \int_{0}^{x} e^{-t^2}\,dt$$

However, it can be shown that there is no way to express the integral in this solution as an elementary function. Thus, we have encountered a *new* function, which we regard to be *defined* by the integral. A close relative of this function, known as the ***error function***, plays an important role in probability and statistics; it is denoted by erf(x) and is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\,dt \tag{12}$$

Indeed, many of the most important functions in science and engineering are defined as integrals that have special names and notations associated with them. For example, the functions defined by

$$S(x) = \int_{0}^{x} \sin\left(\frac{\pi t^2}{2}\right) dt \quad \text{and} \quad C(x) = \int_{0}^{x} \cos\left(\frac{\pi t^2}{2}\right) dt \tag{13–14}$$

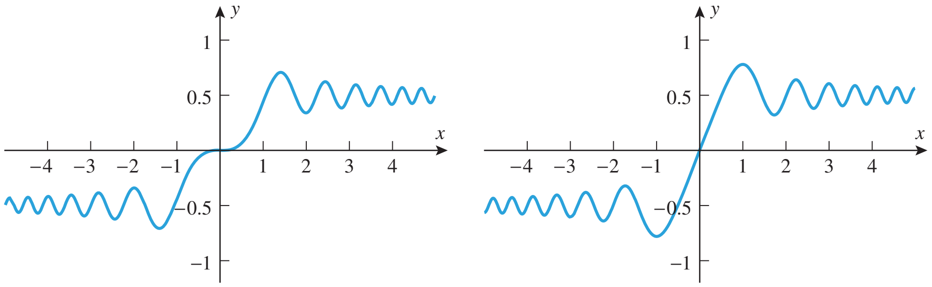
are called the ***Fresnel sine and cosine functions***, respectively, in honor of the French physicist Augustin Fresnel (1788–1827), who first encountered them in his study of diffraction of light waves.

EVALUATING AND GRAPHING FUNCTIONS DEFINED BY INTEGRALS

The following values of $S(1)$ and $C(1)$ were produced by a CAS that has a built-in algorithm for approximating definite integrals:

$$S(1) = \int_0^1 \sin\left(\frac{\pi t^2}{2}\right) dt \approx 0.438259, \qquad C(1) = \int_0^1 \cos\left(\frac{\pi t^2}{2}\right) dt \approx 0.779893$$

To generate graphs of functions defined by integrals, computer programs choose a set of x-values in the domain, approximate the integral for each of those values, and then plot the resulting points. Thus, there is a lot of computation involved in generating such graphs, since each plotted point requires the approximation of an integral. The graphs of the Fresnel functions in Figure 6.6.4 were generated in this way using a CAS.



Fresnel sine function

Fresnel cosine function

▲ **Figure 6.6.4**

REMARK Although it required a considerable amount of computation to generate the graphs of the Fresnel functions, the derivatives of $S(x)$ and $C(x)$ are easy to obtain using Part 2 of the Fundamental Theorem of Calculus (4.6.3); they are

$$S'(x) = \sin\left(\frac{\pi x^2}{2}\right) \quad \text{and} \quad C'(x) = \cos\left(\frac{\pi x^2}{2}\right) \tag{15–16}$$

These derivatives can be used to determine the locations of the relative extrema and inflection points and to investigate other properties of $S(x)$ and $C(x)$.

INTEGRALS WITH FUNCTIONS AS LIMITS OF INTEGRATION

Various applications can lead to integrals in which at least one of the limits of integration is a function of x. Some examples are

$$\int_x^1 \sqrt{\sin t}\, dt, \qquad \int_{x^2}^{\sin x} \sqrt{t^3+1}\, dt, \qquad \int_{\ln x}^{\pi} \frac{dt}{t^7 - 8}$$

We will complete this section by showing how to differentiate integrals of the form

$$\int_a^{g(x)} f(t)\, dt \tag{17}$$

where a is constant. Derivatives of other kinds of integrals with functions as limits of integration will be discussed in the exercises.

To differentiate (17) we can view the integral as a composition $F(g(x))$, where

$$F(x) = \int_a^x f(t)\, dt$$

If we now apply the chain rule, we obtain

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\, dt\right] = \frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x)$$

Theorem 4.6.3

Thus,

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f(g(x))g'(x) \tag{18}$$

In words:

> *To differentiate an integral with a constant lower limit and a function as the upper limit, substitute the upper limit into the integrand, and multiply by the derivative of the upper limit.*

▶ Example 3

$$\frac{d}{dx}\left[\int_1^{\sin x} (1-t^2)\,dt\right] = (1-\sin^2 x)\cos x = \cos^3 x \quad ◀$$

✔QUICK CHECK EXERCISES 6.6 *(See page 462 for answers.)*

1. $\int_1^{1/e} \frac{1}{t}\,dt =$ ________

2. Estimate ln 2 using Definition 6.6.1 and
  (a) a left endpoint approximation with $n = 2$
  (b) a right endpoint approximation with $n = 2$.

3. $\pi^{1/(\ln \pi)} =$ ________

4. A solution to the initial-value problem

$$\frac{dy}{dx} = \cos x^3, \quad y(0) = 2$$

that is defined by an integral is $y =$ ________.

5. $\frac{d}{dx}\left[\int_0^{e^{-x}} \frac{1}{1+t^4}\,dt\right] =$ ________

EXERCISE SET 6.6 Graphing Utility C CAS

1. Sketch the curve $y = 1/t$, and shade a region under the curve whose area is
  (a) ln 2 (b) $-\ln 0.5$ (c) 2.

2. Sketch the curve $y = 1/t$, and shade two different regions under the curve whose areas are ln 1.5.

3. Given that $\ln a = 2$ and $\ln c = 5$, find
  (a) $\int_1^{ac} \frac{1}{t}\,dt$ (b) $\int_1^{1/c} \frac{1}{t}\,dt$
  (c) $\int_1^{a/c} \frac{1}{t}\,dt$ (d) $\int_1^{a^3} \frac{1}{t}\,dt$.

4. Given that $\ln a = 9$, find
  (a) $\int_1^{\sqrt{a}} \frac{1}{t}\,dt$ (b) $\int_1^{2a} \frac{1}{t}\,dt$
  (c) $\int_1^{2/a} \frac{1}{t}\,dt$ (d) $\int_2^{a} \frac{1}{t}\,dt$.

5. Approximate ln 5 using the midpoint rule with $n = 10$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.

6. Approximate ln 3 using the midpoint rule with $n = 20$, and estimate the magnitude of the error by comparing your answer to that produced directly by a calculating utility.

7. Simplify the expression and state the values of x for which your simplification is valid.
  (a) $e^{-\ln x}$ (b) $e^{\ln x^2}$
  (c) $\ln\left(e^{-x^2}\right)$ (d) $\ln(1/e^x)$
  (e) $\exp(3\ln x)$ (f) $\ln(xe^x)$
  (g) $\ln\left(e^{x-\sqrt[3]{x}}\right)$ (h) $e^{x-\ln x}$

8. (a) Let $f(x) = e^{-2x}$. Find the simplest exact value of the function $f(\ln 3)$.
  (b) Let $f(x) = e^x + 3e^{-x}$. Find the simplest exact value of the function $f(\ln 2)$.

9–10 Express the given quantity as a power of e.

9. (a) 3^π (b) $2^{\sqrt{2}}$

10. (a) π^{-x} (b) $x^{2x}, \quad x > 0$

11–12 Find the limits by making appropriate substitutions in the limits given in Theorem 6.6.8.

11. (a) $\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^x$ (b) $\lim_{x\to 0}(1+2x)^{1/x}$

12. (a) $\lim_{x\to+\infty}\left(1+\frac{3}{x}\right)^x$ (b) $\lim_{x\to 0}(1+x)^{1/(3x)}$

13–14 Find $g'(x)$ using Formula (18) and check your answer by evaluating the integral and then differentiating. ■

13. $g(x)=\int_1^{x^3}(t^2-t)\,dt$ **14.** $g(x)=\int_\pi^{1/x}(1-\cos t)\,dt$

15–16 Find the derivative using Formula (18), and check your answer by evaluating the integral and then differentiating the result. ■

15. (a) $\frac{d}{dx}\int_1^{x^3}\frac{1}{t}\,dt$ (b) $\frac{d}{dx}\int_1^{\ln x}e^t\,dt$

16. (a) $\frac{d}{dx}\int_{-1}^{x^2}\sqrt{t+1}\,dt$ (b) $\frac{d}{dx}\int_\pi^{1/x}\sin t\,dt$

17. Let $F(x)=\int_0^x\frac{\sin t}{t^2+1}\,dt$. Find
(a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.

18. Let $F(x)=\int_2^x\sqrt{3t^2+1}\,dt$. Find
(a) $F(2)$ (b) $F'(2)$ (c) $F''(2)$.

19–22 True–False Determine whether the equation is true or false. Explain your answer. ■

19. $\int_1^{1/a}\frac{1}{t}\,dt=-\int_1^a\frac{1}{t}\,dt,\quad \text{for } 0<a$

20. $\int_1^{\sqrt{a}}\frac{1}{t}\,dt=\frac{1}{2}\int_1^a\frac{1}{t}\,dt,\quad \text{for } 0<a$

21. $\int_{-1}^{e}\frac{1}{t}\,dt=1$

22. $\int\frac{2x}{1+x^2}\,dx=\int_1^{1+x^2}\frac{1}{t}\,dt+C$

C **23.** (a) Use Formula (18) to find

$$\frac{d}{dx}\int_1^{x^2}t\sqrt{1+t}\,dt$$

(b) Use a CAS to evaluate the integral and differentiate the resulting function.
(c) Use the simplification command of the CAS, if necessary, to confirm that the answers in parts (a) and (b) are the same.

24. Show that

(a) $\frac{d}{dx}\left[\int_x^a f(t)\,dt\right]=-f(x)$

(b) $\frac{d}{dx}\left[\int_{g(x)}^a f(t)\,dt\right]=-f(g(x))g'(x)$.

25–26 Use the results in Exercise 24 to find the derivative. ■

25. (a) $\frac{d}{dx}\int_x^\pi\cos(t^3)\,dt$ (b) $\frac{d}{dx}\int_{\tan x}^3\frac{t^2}{1+t^2}\,dt$

26. (a) $\frac{d}{dx}\int_x^0\frac{1}{(t^2+1)^2}\,dt$ (b) $\frac{d}{dx}\int_{1/x}^\pi\cos^3 t\,dt$

27. Find

$$\frac{d}{dx}\left[\int_{3x}^{x^2}\frac{t-1}{t^2+1}\,dt\right]$$

by writing

$$\int_{3x}^{x^2}\frac{t-1}{t^2+1}\,dt=\int_{3x}^{0}\frac{t-1}{t^2+1}\,dt+\int_{0}^{x^2}\frac{t-1}{t^2+1}\,dt$$

28. Use Exercise 24(b) and the idea in Exercise 27 to show that

$$\frac{d}{dx}\int_{h(x)}^{g(x)}f(t)\,dt=f(g(x))g'(x)-f(h(x))h'(x)$$

29. Use the result obtained in Exercise 28 to perform the following differentiations:

(a) $\frac{d}{dx}\int_{x^2}^{x^3}\sin^2 t\,dt$ (b) $\frac{d}{dx}\int_{-x}^{x}\frac{1}{1+t}\,dt$.

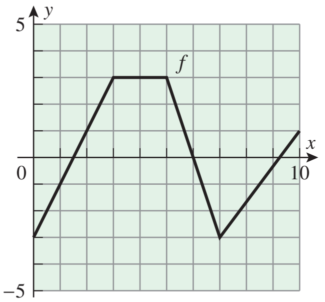
30. Prove that the function

$$F(x)=\int_x^{5x}\frac{1}{t}\,dt$$

is constant on the interval $(0,+\infty)$ by using Exercise 28 to find $F'(x)$. What is that constant?

FOCUS ON CONCEPTS

31. Let $F(x)=\int_0^x f(t)\,dt$, where f is the function whose graph is shown in the accompanying figure.
(a) Find $F(0)$, $F(3)$, $F(5)$, $F(7)$, and $F(10)$.
(b) On what subintervals of the interval $[0, 10]$ is F increasing? Decreasing?
(c) Where does F have its maximum value? Its minimum value?
(d) Sketch the graph of F.



◀ **Figure Ex-31**

32. Determine the inflection point(s) for the graph of F in Exercise 31.

33–34 Express $F(x)$ in a piecewise form that does not involve an integral. ■

33. $F(x)=\int_{-1}^x|t|\,dt$

34. $F(x)=\int_0^x f(t)\,dt$, where $f(x)=\begin{cases}x, & 0\le x\le 2\\ 2, & x>2\end{cases}$

35–38 Use Formula (11) to solve the initial-value problem. ■

35. $\frac{dy}{dx}=\frac{2x^2+1}{x},\ y(1)=2$ **36.** $\frac{dy}{dx}=\frac{x+1}{\sqrt{x}},\ y(1)=0$

37. $\dfrac{dy}{dx} = \sec^2 x - \sin x, \quad y(\pi/4) = 1$

38. $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}, \quad y(e) = 1$

39. Suppose that at time $t = 0$ there are P_0 individuals who have disease X, and suppose that a certain model for the spread of the disease predicts that the disease will spread at the rate of $r(t)$ individuals per day. Write a formula for the number of individuals who will have disease X after x days.

40. Suppose that $v(t)$ is the velocity function of a particle moving along an s-axis. Write a formula for the coordinate of the particle at time T if the particle is at s_1 at time $t = 1$.

FOCUS ON CONCEPTS

41. The accompanying figure shows the graphs of $y = f(x)$ and $y = \int_0^x f(t)\,dt$. Determine which graph is which, and explain your reasoning.

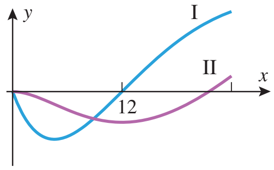


Figure Ex-41

42. (a) Make a conjecture about the value of the limit

$$\lim_{k\to 0} \int_1^b t^{k-1}\,dt \quad (b > 0)$$

(b) Check your conjecture by evaluating the integral and finding the limit. [*Hint:* Interpret the limit as the definition of the derivative of an exponential function.]

43. Let $F(x) = \int_0^x f(t)\,dt$, where f is the function graphed in the accompanying figure.

(a) Where do the relative minima of F occur?

(b) Where do the relative maxima of F occur?

(c) Where does the absolute maximum of F on the interval [0, 5] occur?

(d) Where does the absolute minimum of F on the interval [0, 5] occur?

(e) Where is F concave up? Concave down?

(f) Sketch the graph of F.

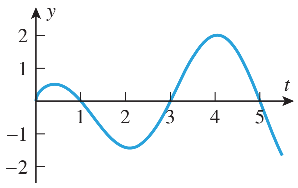


Figure Ex-43

C **44.** CAS programs have commands for working with most of the important nonelementary functions. Check your CAS documentation for information about the error function $\operatorname{erf}(x)$ [see Formula (12)], and then complete the following.

(a) Generate the graph of $\operatorname{erf}(x)$.

(b) Use the graph to make a conjecture about the existence and location of any relative maxima and minima of $\operatorname{erf}(x)$.

(c) Check your conjecture in part (b) using the derivative of $\operatorname{erf}(x)$.

(d) Use the graph to make a conjecture about the existence and location of any inflection points of $\operatorname{erf}(x)$.

(e) Check your conjecture in part (d) using the second derivative of $\operatorname{erf}(x)$.

(f) Use the graph to make a conjecture about the existence of horizontal asymptotes of $\operatorname{erf}(x)$.

(g) Check your conjecture in part (f) by using the CAS to find the limits of $\operatorname{erf}(x)$ as $x \to \pm\infty$.

45. The Fresnel sine and cosine functions $S(x)$ and $C(x)$ were defined in Formulas (13) and (14) and graphed in Figure 6.6.4. Their derivatives were given in Formulas (15) and (16).

(a) At what points does $C(x)$ have relative minima? Relative maxima?

(b) Where do the inflection points of $C(x)$ occur?

(c) Confirm that your answers in parts (a) and (b) are consistent with the graph of $C(x)$.

46. Find the limit

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \ln t\,dt$$

47. Find a function f and a number a such that

$$4 + \int_a^x f(t)\,dt = e^{2x}$$

48. (a) Give a geometric argument to show that

$$\frac{1}{x+1} < \int_x^{x+1} \frac{1}{t}\,dt < \frac{1}{x}, \qquad x > 0$$

(b) Use the result in part (a) to prove that

$$\frac{1}{x+1} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}, \qquad x > 0$$

(c) Use the result in part (b) to prove that

$$e^{x/(x+1)} < \left(1 + \frac{1}{x}\right)^x < e, \qquad x > 0$$

and hence that

$$\lim_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e$$

(d) Use the result in part (b) to prove that

$$\left(1 + \frac{1}{x}\right)^x < e < \left(1 + \frac{1}{x}\right)^{x+1}, \qquad x > 0$$

49. Use a graphing utility to generate the graph of

$$y = \left(1 + \frac{1}{x}\right)^{x+1} - \left(1 + \frac{1}{x}\right)^x$$

in the window [0, 100] × [0, 0.2], and use that graph and part (d) of Exercise 48 to make a rough estimate of the error in the approximation

$$e \approx \left(1 + \frac{1}{50}\right)^{50}$$

50. (a) Divide the interval $[1, 2]$ into 5 subintervals of equal length, and use approximate Riemann sums to show that

$$0.2\left[\tfrac{1}{1.2}+\tfrac{1}{1.4}+\tfrac{1}{1.6}+\tfrac{1}{1.8}+\tfrac{1}{2.0}\right] < \ln 2$$
$$< 0.2\left[\tfrac{1}{1.0}+\tfrac{1}{1.2}+\tfrac{1}{1.4}+\tfrac{1}{1.6}+\tfrac{1}{1.8}\right]$$

(b) Show that if the interval $[1, 2]$ is divided into n subintervals of equal length, then

$$\sum_{k=1}^{n} \frac{1}{n+k} < \ln 2 < \sum_{k=0}^{n-1} \frac{1}{n+k}$$

(c) Show that the difference between the two sums in part (b) is $1/(2n)$, and use this result to show that the sums in part (a) approximate $\ln 2$ with an error of at most 0.1.

(d) How large must n be to ensure that the sums in part (b) approximate $\ln 2$ to three decimal places?

51. Prove: If f is continuous on an open interval and a is any point in that interval, then

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on the interval.

52. **Writing** A student objects that it is circular reasoning to make the definition

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

since to evaluate the integral we need to know the value of $\ln x$. Write a short paragraph that answers this student's objection.

53. **Writing** Write a short paragraph that compares Definition 6.6.1 with the definition of the natural logarithm function given in Section 6.1. Be sure to discuss the issues surrounding continuity and differentiability.

✓QUICK CHECK ANSWERS 6.6

1. -1 **2.** (a) $\frac{5}{6}$ (b) $\frac{7}{12}$ **3.** e **4.** $y = 2 + \int_0^x \cos t^3\,dt$ **5.** $-\dfrac{e^{-x}}{1+e^{-4x}}$

6.7 DERIVATIVES AND INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle x using a known value of $\sin x$, $\cos x$, or some other trigonometric function. Problems of this type involve the computation of inverse trigonometric functions. In this section we will study these functions from the viewpoint of general inverse functions, with the goal of developing derivative formulas for the inverse trigonometric functions. We will also derive some related integration formulas that involve inverse trigometric functions.

INVERSE TRIGONOMETRIC FUNCTIONS

The six basic trigonometric functions do not have inverses because their graphs repeat periodically and hence do not pass the horizontal line test. To circumvent this problem we will restrict the domains of the trigonometric functions to produce one-to-one functions and then define the "inverse trigonometric functions" to be the inverses of these restricted functions. The top part of Figure 6.7.1 shows geometrically how these restrictions are made for $\sin x$, $\cos x$, $\tan x$, and $\sec x$, and the bottom part of the figure shows the graphs of the corresponding inverse functions

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \sec^{-1} x$$

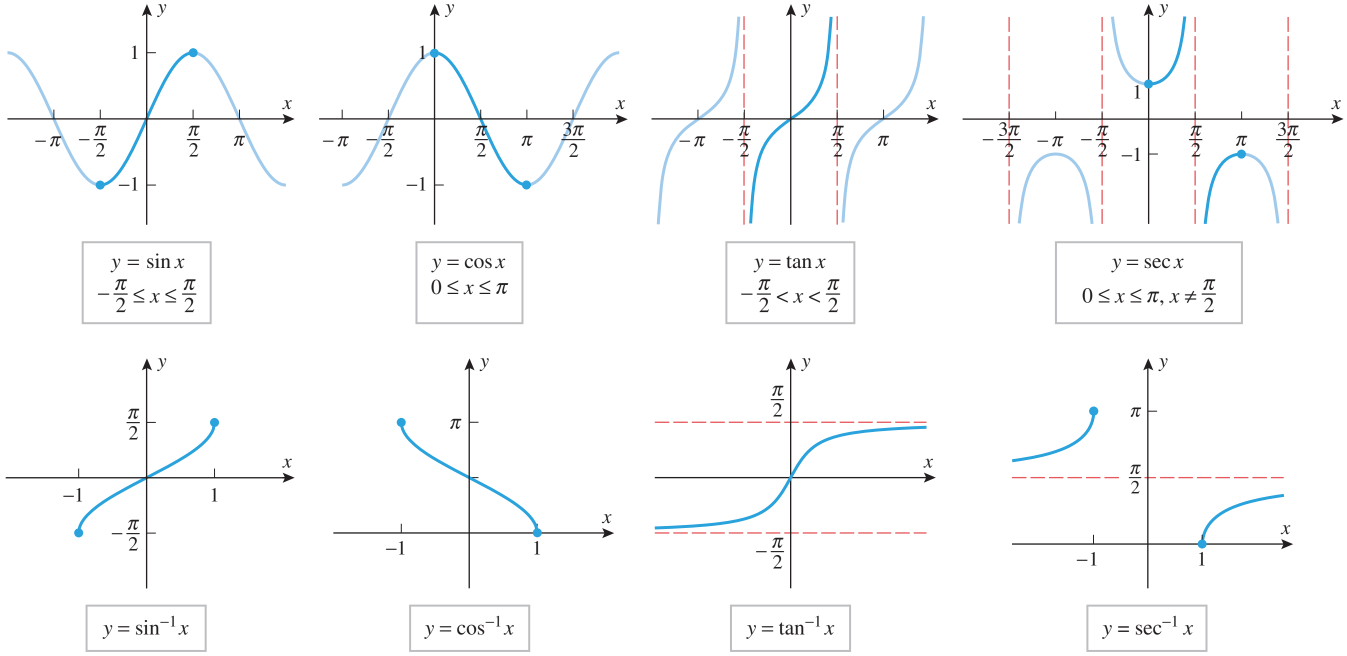
(also denoted by arcsin x, arccos x, arctan x, and arcsec x). Inverses of $\cot x$ and $\csc x$ are of lesser importance and will be considered in the exercises.

The following formal definitions summarize the preceding discussion.

If you have trouble visualizing the correspondence between the top and bottom parts of Figure 6.7.1, keep in mind that a reflection about $y = x$ converts vertical lines into horizontal lines, and vice versa; and it converts x-intercepts into y-intercepts, and vice versa.

6.7.1 DEFINITION The ***inverse sine function***, denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\pi/2 \le x \le \pi/2$$



$y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$y = \cos x$, $0 \le x \le \pi$

$y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$y = \sec x$, $0 \le x \le \pi,\ x \ne \frac{\pi}{2}$

$y = \sin^{-1} x$

$y = \cos^{-1} x$

$y = \tan^{-1} x$

$y = \sec^{-1} x$

▲ **Figure 6.7.1**

6.7.2 DEFINITION The ***inverse cosine function***, denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$

6.7.3 DEFINITION The ***inverse tangent function***, denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\pi/2 < x < \pi/2$$

6.7.4 DEFINITION* The ***inverse secant function***, denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \text{ with } x \ne \pi/2$$

Table 6.7.1 summarizes the basic properties of the inverse trigonometric functions we have considered. You should confirm that the domains and ranges listed in this table are consistent with the graphs shown in Figure 6.7.1.

WARNING

The notations $\sin^{-1} x, \cos^{-1} x, \ldots$ are reserved exclusively for the inverse trigonometric functions and are not used for reciprocals of the trigonometric functions. If we want to express the reciprocal $1/\sin x$ using an exponent, we would write $(\sin x)^{-1}$ and *never* $\sin^{-1} x$.

---

*There is no universal agreement on the definition of $\sec^{-1} x$, and some mathematicians prefer to restrict the domain of $\sec x$ so that $0 \le x < \pi/2$ or $\pi \le x < 3\pi/2$, which was the definition used in some earlier editions of this text. Each definition has advantages and disadvantages, but we will use the current definition to conform with the conventions used by the CAS programs *Mathematica*, *Maple*, and *Sage*.

Table 6.7.1

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$ $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\cos^{-1}(\cos x) = x$ if $0 \le x \le \pi$ $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$
$\tan^{-1}$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\tan^{-1}(\tan x) = x$ if $-\pi/2 < x < \pi/2$ $\tan(\tan^{-1} x) = x$ if $-\infty < x < +\infty$
$\sec^{-1}$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$	$\sec^{-1}(\sec x) = x$ if $0 \le x \le \pi, x \ne \pi/2$ $\sec(\sec^{-1} x) = x$ if $\lvert x\rvert \ge 1$

EVALUATING INVERSE TRIGONOMETRIC FUNCTIONS

A common problem in trigonometry is to find an angle whose sine is known. For example, you might want to find an angle x in radian measure such that

$$\sin x = \tfrac{1}{2} \tag{1}$$

and, more generally, for a given value of y in the interval $-1 \le y \le 1$ you might want to solve the equation

$$\sin x = y \tag{2}$$

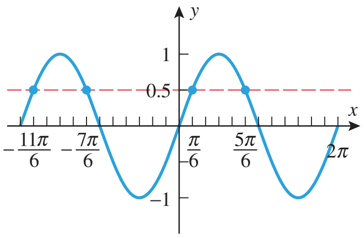
Because $\sin x$ repeats periodically, this equation has infinitely many solutions for x; however, if we solve this equation as

$$x = \sin^{-1} y$$

then we isolate the specific solution that lies in the interval $[-\pi/2, \pi/2]$, since this is the range of the inverse sine. For example, Figure 6.7.2 shows four solutions of Equation (1), namely, $-11\pi/6$, $-7\pi/6$, $\pi/6$, and $5\pi/6$. Of these, $\pi/6$ is the solution in the interval $[-\pi/2, \pi/2]$, so

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \pi/6 \tag{3}$$

In general, if we view $x = \sin^{-1} y$ as an angle in radian measure whose sine is y, then the restriction $-\pi/2 \le x \le \pi/2$ imposes the geometric requirement that the angle x in standard position terminate in either the first or fourth quadrant or on an axis adjacent to those quadrants.



▲ Figure 6.7.2

TECHNOLOGY MASTERY

Refer to the documentation for your calculating utility to determine how to calculate inverse sines, inverse cosines, and inverse tangents; and then confirm Equation (3) numerically by showing that

$$\sin^{-1}(0.5) \approx 0.523598775598\ldots \approx \pi/6$$

▶ **Example 1** Find exact values of

$$\text{(a)}\ \sin^{-1}(1/\sqrt{2}) \qquad \text{(b)}\ \sin^{-1}(-1)$$

by inspection, and confirm your results numerically using a calculating utility.

Solution (a). Because $\sin^{-1}(1/\sqrt{2}) > 0$, we can view $x = \sin^{-1}(1/\sqrt{2})$ as that angle in the first quadrant such that $\sin\theta = 1/\sqrt{2}$. Thus, $\sin^{-1}(1/\sqrt{2}) = \pi/4$. You can confirm this with your calculating utility by showing that $\sin^{-1}(1/\sqrt{2}) \approx 0.785 \approx \pi/4$.

Solution (b). Because $\sin^{-1}(-1) < 0$, we can view $x = \sin^{-1}(-1)$ as an angle in the fourth quadrant (or an adjacent axis) such that $\sin x = -1$. Thus, $\sin^{-1}(-1) = -\pi/2$. You can confirm this with your calculating utility by showing that $\sin^{-1}(-1) \approx -1.57 \approx -\pi/2$. ◀

If $x = \cos^{-1} y$ is viewed as an angle in radian measure whose cosine is y, in what possible quadrants can x lie? Answer the same question for

$$x = \tan^{-1} y \quad \text{and} \quad x = \sec^{-1} y$$

TECHNOLOGY MASTERY Most calculators do not provide a direct method for calculating inverse secants. In such situations the identity

$$\sec^{-1} x = \cos^{-1}(1/x) \tag{4}$$

is useful (Exercise 56). Use this formula to show that

$$\sec^{-1}(2.25) \approx 1.11 \quad \text{and} \quad \sec^{-1}(-2.25) \approx 2.03$$

If you have a calculating utility (such as a CAS) that can find $\sec^{-1} x$ directly, use it to check these values.

IDENTITIES FOR INVERSE TRIGONOMETRIC FUNCTIONS

If we interpret $\sin^{-1} x$ as an angle in radian measure whose sine is x, and if that angle is *nonnegative*, then we can represent $\sin^{-1} x$ geometrically as an angle in a right triangle in which the hypotenuse has length 1 and the side opposite to the angle $\sin^{-1} x$ has length x (Figure 6.7.3*a*). Moreover, the unlabeled acute angle in Figure 6.7.3*a* is $\cos^{-1} x$, since the cosine of that angle is x, and the unlabeled side in that figure has length $\sqrt{1-x^2}$ by the Theorem of Pythagoras (Figure 6.7.3*b*). This triangle motivates a number of useful identities involving inverse trigonometric functions that are valid for $-1 \le x \le 1$; for example,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \tag{5}$$

$$\cos(\sin^{-1} x) = \sqrt{1-x^2} \tag{6}$$

$$\sin(\cos^{-1} x) = \sqrt{1-x^2} \tag{7}$$

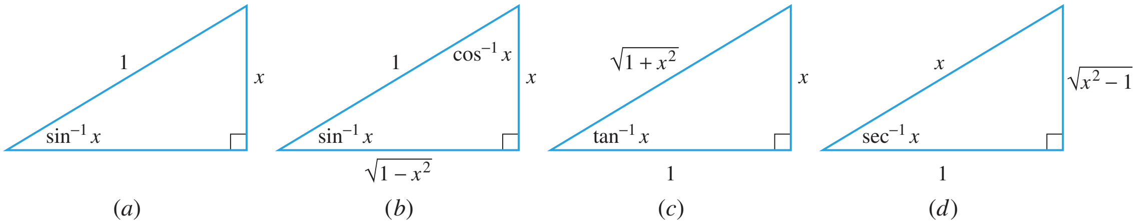
$$\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}} \tag{8}$$

There is little to be gained by memorizing these identities. What is important is the mastery of the *method* used to obtain them.

In a similar manner, $\tan^{-1} x$ and $\sec^{-1} x$ can be represented as angles in the right triangles shown in Figures 6.7.3*c* and 6.7.3*d* (verify). Those triangles reveal additional useful identities; for example,

$$\sec(\tan^{-1} x) = \sqrt{1+x^2} \tag{9}$$

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{x} \quad (x \ge 1) \tag{10}$$



▲ **Figure 6.7.3**

REMARK The triangle technique does not always produce the most general form of an identity. For example, in Exercise 91 we will ask you to derive the following extension of Formula (10) that is valid for $x \le -1$ as well as $x \ge 1$:

$$\sin(\sec^{-1} x) = \frac{\sqrt{x^2-1}}{|x|} \quad (|x| \ge 1) \tag{11}$$

Referring to Figure 6.7.1, observe that the inverse sine and inverse tangent are odd functions; that is,

$$\sin^{-1}(-x) = -\sin^{-1}(x) \quad \text{and} \quad \tan^{-1}(-x) = -\tan^{-1}(x) \tag{12–13}$$

▶ **Example 2** Figure 6.7.4 shows a computer-generated graph of $y = \sin^{-1}(\sin x)$. One might think that this graph should be the line $y = x$, since $\sin^{-1}(\sin x) = x$. Why isn't it?

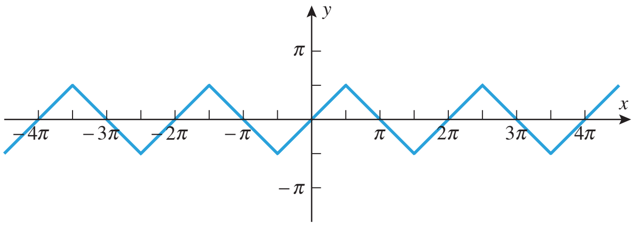
Solution. The relationship $\sin^{-1}(\sin x) = x$ is valid on the interval $-\pi/2 \le x \le \pi/2$, so we can say with certainty that the graphs of $y = \sin^{-1}(\sin x)$ and $y = x$ coincide on this interval (which is confirmed by Figure 6.7.4). However, outside of this interval the relationship $\sin^{-1}(\sin x) = x$ does not hold. For example, if the quantity x lies in the interval $\pi/2 \le x \le 3\pi/2$, then the quantity $x - \pi$ lies in the interval $-\pi/2 \le x \le \pi/2$, so

$$\sin^{-1}[\sin(x - \pi)] = x - \pi$$

Thus, by using the identity $\sin(x - \pi) = -\sin x$ and the fact that $\sin^{-1}$ is an odd function, we can express $\sin^{-1}(\sin x)$ as

$$\sin^{-1}(\sin x) = \sin^{-1}[-\sin(x - \pi)] = -\sin^{-1}[\sin(x - \pi)] = -(x - \pi)$$

This shows that on the interval $\pi/2 \le x \le 3\pi/2$ the graph of $y = \sin^{-1}(\sin x)$ coincides with the line $y = -(x - \pi)$, which has slope -1 and an x-intercept at $x = \pi$. This agrees with Figure 6.7.4. ◀



▶ **Figure 6.7.4**

DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

To begin, consider the function $\sin^{-1} x$. If we let $f(x) = \sin x$ $(-\pi/2 \le x \le \pi/2)$, then it follows from Formula (2) in Section 6.3 that $f^{-1}(x) = \sin^{-1} x$ will be differentiable at any point x where $\cos(\sin^{-1} x) \ne 0$. This is equivalent to the condition

$$\sin^{-1} x \ne -\frac{\pi}{2} \quad \text{and} \quad \sin^{-1} x \ne \frac{\pi}{2}$$

so it follows that $\sin^{-1} x$ is differentiable on the interval $(-1, 1)$.

Observe that $\sin^{-1} x$ is only differentiable on the interval $(-1, 1)$, even though its domain is $[-1, 1]$. This is because the graph of $y = \sin x$ has horizontal tangent lines at the points $(\pi/2, 1)$ and $(-\pi/2, -1)$, so the graph of $y = \sin^{-1} x$ has vertical tangent lines at $x = \pm 1$.

A derivative formula for $\sin^{-1} x$ on $(-1, 1)$ can be obtained by using Formula (2) or (3) in Section 6.3 or by differentiating implicitly. We will use the latter method. Rewriting the equation $y = \sin^{-1} x$ as $x = \sin y$ and differentiating implicitly with respect to x, we obtain

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin y]$$

$$1 = \cos y \cdot \frac{dy}{dx}$$

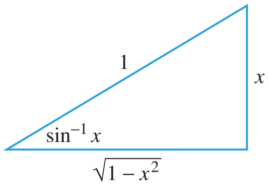
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

At this point we have succeeded in obtaining the derivative; however, this derivative formula can be simplified using the identity indicated in Figure 6.7.5. This yields

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Thus, we have shown that

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}} \qquad (-1 < x < 1)$$



$\cos(\sin^{-1} x) = \sqrt{1 - x^2}$

▲ **Figure 6.7.5**

More generally, if u is a differentiable function of x, then the chain rule produces the following generalized version of this formula:

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (-1 < u < 1)$$

The method used to derive this formula can be used to obtain generalized derivative formulas for the remaining inverse trigonometric functions. The following is a complete list of these formulas, each of which is valid on the natural domain of the function that multiplies du/dx.

The appearance of $|u|$ in (18) and (19) will be explained in Exercise 59.

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}[\cos^{-1} u] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad (14\text{–}15)$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx} \qquad (16\text{–}17)$$

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad \frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad (18\text{–}19)$$

▶ **Example 3** Find dy/dx if

(a) $y = \sin^{-1}(x^3)$ (b) $y = \sec^{-1}(e^x)$

Solution (a). From (14)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^3)^2}}(3x^2) = \frac{3x^2}{\sqrt{1-x^6}}$$

Solution (b). From (18)

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{(e^x)^2-1}}(e^x) = \frac{1}{\sqrt{e^{2x}-1}} \blacktriangleleft$$

INTEGRATION FORMULAS

Differentiation formulas (14)–(19) yield useful integration formulas. Those most commonly needed are

See Exercise 62 for a justification of Formula (22).

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C \qquad (20)$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C \qquad (21)$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} |u| + C \qquad (22)$$

▶ **Example 4** Evaluate $\displaystyle\int \frac{dx}{1+3x^2}$.

Solution. Substituting

$$u = \sqrt{3}x, \quad du = \sqrt{3}\,dx$$

yields

$$\int \frac{dx}{1+3x^2} = \frac{1}{\sqrt{3}}\int \frac{du}{1+u^2} = \frac{1}{\sqrt{3}}\tan^{-1} u + C = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + C \blacktriangleleft$$

▶ **Example 5** Evaluate $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.

Solution. Substituting

$$u = e^x, \quad du = e^x\,dx$$

yields

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(e^x) + C \quad ◀$$

▶ **Example 6** Evaluate $\displaystyle\int \frac{dx}{a^2+x^2}\,dx$, where $a \neq 0$ is a constant.

Solution. Some simple algebra and an appropriate u-substitution will allow us to use (21).

$$\int \frac{dx}{a^2+x^2} = \int \frac{a(dx/a)}{a^2(1+(x/a)^2)} = \frac{1}{a}\int \frac{dx/a}{1+(x/a)^2} \qquad \boxed{\begin{array}{c} u = x/a \\ du = dx/a \end{array}}$$

$$= \frac{1}{a}\int \frac{du}{1+u^2} = \frac{1}{a}\tan^{-1} u + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad ◀$$

The method of Example 6 leads to the following generalizations of (20), (21), and (22) for $a > 0$:

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C \tag{23}$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C \tag{24}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \tag{25}$$

▶ **Example 7** Evaluate $\displaystyle\int \frac{dx}{\sqrt{2-x^2}}$.

Solution. Applying (24) with $u = x$ and $a = \sqrt{2}$ yields

$$\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + C \quad ◀$$

✔ QUICK CHECK EXERCISES 6.7 *(See page 472 for answers.)*

1. In each part, determine the exact value without using a calculating utility.
 (a) $\sin^{-1}(-1) = $ ________
 (b) $\tan^{-1}(1) = $ ________
 (c) $\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right) = $ ________
 (d) $\cos^{-1}\left(\frac{1}{2}\right) = $ ________
 (e) $\sec^{-1}(-2) = $ ________

2. In each part, determine the exact value without using a calculating utility.
 (a) $\sin^{-1}(\sin \pi/7) = $ ________
 (b) $\sin^{-1}(\sin 5\pi/7) = $ ________
 (c) $\tan^{-1}(\tan 13\pi/6) = $ ________
 (d) $\cos^{-1}(\cos 12\pi/7) = $ ________

3. $\dfrac{d}{dx}[\sin^{-1}(2x)] = $ ________

4. $\displaystyle\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = $ ________

EXERCISE SET 6.7

Graphing Utility C CAS

1. Given that $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, find the exact values of $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

2. Given that $\theta = \sec^{-1} 2.6$, find the exact values of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, and $\csc\theta$.

3. For which values of x is it true that
(a) $\cos^{-1}(\cos x) = x$ (b) $\cos(\cos^{-1} x) = x$
(c) $\tan^{-1}(\tan x) = x$ (d) $\tan(\tan^{-1} x) = x$?

4–5 Find the exact value of the given quantity. ■

4. $\sec\left[\sin^{-1}\left(-\frac{3}{4}\right)\right]$ **5.** $\sin\left[2\cos^{-1}\left(\frac{3}{5}\right)\right]$

6–7 Complete the identities using the triangle method (Figure 6.7.3). ■

6. (a) $\sin(\cos^{-1} x) = ?$ (b) $\tan(\cos^{-1} x) = ?$
(c) $\csc(\tan^{-1} x) = ?$ (d) $\sin(\tan^{-1} x) = ?$

7. (a) $\cos(\tan^{-1} x) = ?$ (b) $\tan(\cos^{-1} x) = ?$
(c) $\sin(\sec^{-1} x) = ?$ (d) $\cot(\sec^{-1} x) = ?$

8. (a) Use a calculating utility set to radian measure to make tables of values of $y = \sin^{-1} x$ and $y = \cos^{-1} x$ for $x = -1, -0.8, -0.6, \ldots, 0, 0.2, \ldots, 1$. Round your answers to two decimal places.
(b) Plot the points obtained in part (a), and use the points to sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$. Confirm that your sketches agree with those in Figure 6.7.1.
(c) Use your graphing utility to graph $y = \sin^{-1} x$ and $y = \cos^{-1} x$; confirm that the graphs agree with those in Figure 6.7.1.

9. In each part, sketch the graph and check your work with a graphing utility.
(a) $y = \sin^{-1} 2x$ (b) $y = \tan^{-1} \frac{1}{2}x$

10. The ***law of cosines*** states that

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where a, b, and c are the lengths of the sides of a triangle and θ is the angle formed by sides a and b. Find θ, to the nearest degree, for the triangle with $a = 2$, $b = 3$, and $c = 4$.

11–12 Use a calculating utility to approximate the solution of each equation. Where radians are used, express your answer to four decimal places, and where degrees are used, express it to the nearest tenth of a degree. [*Note:* In each part, the solution is not in the range of the relevant inverse trigonometric function.] ■

11. (a) $\sin x = 0.37$, $\pi/2 < x < \pi$
(b) $\sin\theta = -0.61$, $180° < \theta < 270°$

12. (a) $\cos x = -0.85$, $\pi < x < 3\pi/2$
(b) $\cos\theta = 0.23$, $-90° < \theta < 0°$

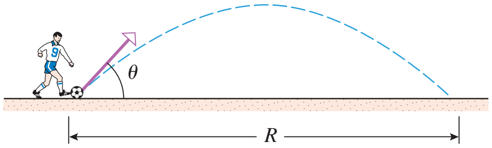
FOCUS ON CONCEPTS

13. (a) Use a calculating utility to evaluate the expressions $\sin^{-1}(\sin^{-1} 0.25)$ and $\sin^{-1}(\sin^{-1} 0.9)$, and explain what you think is happening in the second calculation.
(b) For what values of x in the interval $-1 \le x \le 1$ will your calculating utility produce a real value for the function $\sin^{-1}(\sin^{-1} x)$?

14. A soccer player kicks a ball with an initial speed of 14 m/s at an angle θ with the horizontal (see the accompanying figure). The ball lands 18 m down the field. If air resistance is neglected, then the ball will have a parabolic trajectory and the horizontal range R will be given by

$$R = \frac{v^2}{g}\sin 2\theta$$

where v is the initial speed of the ball and g is the acceleration due to gravity. Using $g = 9.8\ \text{m/s}^2$, approximate two values of θ, to the nearest degree, at which the ball could have been kicked. Which angle results in the shorter time of flight? Why?



▲ **Figure Ex-14**

15–26 Find dy/dx. ■

15. $y = \sin^{-1}(3x)$ **16.** $y = \cos^{-1}\left(\dfrac{x+1}{2}\right)$

17. $y = \sin^{-1}(1/x)$ **18.** $y = \cos^{-1}(\cos x)$

19. $y = \tan^{-1}(x^3)$ **20.** $y = \sec^{-1}(x^5)$

21. $y = (\tan x)^{-1}$ **22.** $y = \dfrac{1}{\tan^{-1} x}$

23. $y = e^x \sec^{-1} x$ **24.** $y = \ln(\cos^{-1} x)$

25. $y = \sin^{-1} x + \cos^{-1} x$ **26.** $y = x^2(\sin^{-1} x)^3$

27–28 Find dy/dx by implicit differentiation. ■

27. $x^3 + x\tan^{-1} y = e^y$ **28.** $\sin^{-1}(xy) = \cos^{-1}(x - y)$

29–30 Evaluate the integral and check your answer by differentiating. ■

29. $\displaystyle\int \left[\frac{1}{2\sqrt{1-x^2}} - \frac{3}{1+x^2}\right] dx$

30. $\displaystyle\int \left[\frac{4}{x\sqrt{x^2-1}} + \frac{1+x+x^3}{1+x^2}\right] dx$

31–48 Evaluate the integral. ■

31. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$ **32.** $\displaystyle\int \frac{dx}{1+16x^2}$

33. $\displaystyle\int \frac{e^x}{1+e^{2x}}\, dx$ **34.** $\displaystyle\int \frac{t}{t^4+1}\, dt$

35. $\displaystyle\int \frac{\sec^2 x\, dx}{\sqrt{1-\tan^2 x}}$ **36.** $\displaystyle\int \frac{\sin\theta}{\cos^2\theta + 1}\, d\theta$

37. $\displaystyle\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$

38. $\displaystyle\int_{-1}^{1} \frac{dx}{1+x^2}$

39. $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2-1}}$

40. $\displaystyle\int_{-\sqrt{2}}^{-2/\sqrt{3}} \frac{dx}{x\sqrt{x^2-1}}$

41. $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{\tan^{-1} x}}{1+x^2}\,dx$

42. $\displaystyle\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(\ln x)^2}}$

43. $\displaystyle\int_1^{3} \frac{dx}{\sqrt{x}(x+1)}$

44. $\displaystyle\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}\,dx}{\sqrt{1-e^{-2x}}}$

45. $\displaystyle\int_0^{1} \frac{x}{\sqrt{4-3x^4}}\,dx$

46. $\displaystyle\int_1^{2} \frac{1}{\sqrt{x}\sqrt{4-x}}\,dx$

47. $\displaystyle\int_0^{1/\sqrt{3}} \frac{1}{1+9x^2}\,dx$

48. $\displaystyle\int_1^{\sqrt{2}} \frac{x}{3+x^4}\,dx$

49–50 Evaluate the integrals with the aid of Formulas (23), (24), and (25). ■

49. (a) $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$ (b) $\displaystyle\int \frac{dx}{5+x^2}$ (c) $\displaystyle\int \frac{dx}{x\sqrt{x^2-\pi}}$

50. (a) $\displaystyle\int \frac{e^x}{4+e^{2x}}\,dx$ (b) $\displaystyle\int \frac{dx}{\sqrt{9-4x^2}}$ (c) $\displaystyle\int \frac{dy}{y\sqrt{5y^2-3}}$

51–54 True–False Determine whether the statement is true or false. Explain your answer. ■

51. By definition, $\sin^{-1}(\sin x) = x$ for all real numbers x.

52. The range of the inverse tangent function is the interval $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$.

53. The graph of $y = \sec^{-1} x$ has a horizontal asymptote.

54. We can conclude from the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ that $\sin^{-1} x + \cos^{-1} x$ is constant.

FOCUS ON CONCEPTS

55–56 The function $\cot^{-1} x$ is defined to be the inverse of the restricted cotangent function

$$\cot x, \quad 0 < x < \pi$$

and the function $\csc^{-1} x$ is defined to be the inverse of the restricted cosecant function

$$\csc x, \quad -\pi/2 \le x \le \pi/2, \quad x \ne 0$$

Use these definitions in these and in all subsequent exercises that involve these functions. ■

55. (a) Sketch the graphs of $\cot^{-1} x$ and $\csc^{-1} x$.
(b) Find the domain and range of $\cot^{-1} x$ and $\csc^{-1} x$.

56. Show that

(a) $\cot^{-1} x = \begin{cases} \tan^{-1}(1/x), & \text{if } x > 0 \\ \pi + \tan^{-1}(1/x), & \text{if } x < 0 \end{cases}$

(b) $\sec^{-1} x = \cos^{-1} \dfrac{1}{x}$, if $|x| \ge 1$

(c) $\csc^{-1} x = \sin^{-1} \dfrac{1}{x}$, if $|x| \ge 1$.

57. Most scientific calculators have keys for the values of only $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$. The formulas in Exercise 54 show how a calculator can be used to obtain values of $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ for positive values of x. Use these formulas and a calculator to find numerical values for each of the following inverse trigonometric functions. Express your answers in degrees, rounded to the nearest tenth of a degree.
(a) $\cot^{-1} 0.7$ (b) $\sec^{-1} 1.2$ (c) $\csc^{-1} 2.3$

58. A camera is positioned x feet from the base of a missile launching pad (see the accompanying figure). If a missile of length a feet is launched vertically, show that when the base of the missile is b feet above the camera lens, the angle θ subtended at the lens by the missile is

$$\theta = \cot^{-1}\frac{x}{a+b} - \cot^{-1}\frac{x}{b}$$

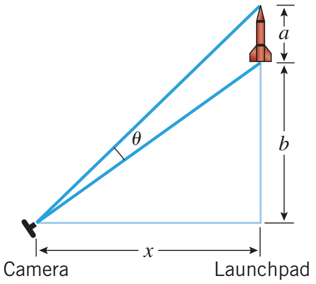


Figure Ex-58

59. Use identity (5) and Formula (14) to obtain the derivative of $y = \cos^{-1} x$.

60. (a) Use Formula (2) in Section 6.3 to prove that

$$\left.\frac{d}{dx}[\cot^{-1} x]\right|_{x=0} = -1$$

(b) Use part (a) above, part (a) of Exercise 56 and the chain rule to show that

$$\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{1+x^2}$$

for $-\infty < x < +\infty$.

(c) Conclude from part (b) that

$$\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{1+u^2}\frac{du}{dx}$$

for $-\infty < u < +\infty$.

61. (a) Use part (c) of Exercise 56 and the chain rule to show that

$$\frac{d}{dx}[\csc^{-1} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$

for $1 < |x|$.

(b) Conclude from part (a) that

$$\frac{d}{dx}[\csc^{-1} u] = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

for $1 < |u|$.

(c) Use Equation (5) and parts (b) and (c) of Exercise 56 to show that if $|x| \ge 1$ then, $\sec^{-1} x + \csc^{-1} x = \pi/2$. Conclude from part (a) that

$$\frac{d}{dx}[\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

(d) Conclude from part (c) that

$$\frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

62. Use the derivative formula from part (d) of Exercise 61 to verify Formula (22).

63–66 Find dy/dx. ■

63. $y = \sec^{-1} x + \csc^{-1} x$ **64.** $y = \csc^{-1}(e^x)$

65. $y = \cot^{-1}(\sqrt{x})$ **66.** $y = \sqrt{\cot^{-1} x}$

67. The number of hours of daylight on a given day at a given point on the Earth's surface depends on the latitude λ of the point, the angle γ through which the Earth has moved in its orbital plane during the time period from the vernal equinox (March 21), and the angle of inclination ϕ of the Earth's axis of rotation measured from ecliptic north ($\phi \approx 23.45°$). The number of hours of daylight h can be approximated by the formula

$$h = \begin{cases} 24, & D \geq 1 \\ 12 + \frac{2}{15}\sin^{-1} D, & |D| < 1 \\ 0, & D \leq -1 \end{cases}$$

where

$$D = \frac{\sin\phi \sin\gamma \tan\lambda}{\sqrt{1-\sin^2\phi \sin^2\gamma}}$$

and $\sin^{-1} D$ is in degree measure. Given that Fairbanks, Alaska, is located at a latitude of $\lambda = 65°$ N and also that $\gamma = 90°$ on June 20 and $\gamma = 270°$ on December 20, approximate

(a) the maximum number of daylight hours at Fairbanks to one decimal place

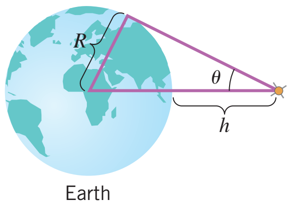
(b) the minimum number of daylight hours at Fairbanks to one decimal place.

Source: This problem was adapted from *TEAM, A Path to Applied Mathematics*, The Mathematical Association of America, Washington, D.C., 1985.

68. An Earth-observing satellite has horizon sensors that can measure the angle θ shown in the accompanying figure. Let R be the radius of the Earth (assumed spherical) and h the distance between the satellite and the Earth's surface.

(a) Show that $\sin\theta = \dfrac{R}{R+h}$.

(b) Find θ, to the nearest degree, for a satellite that is 10,000 km from the Earth's surface (use $R = 6378$ km).

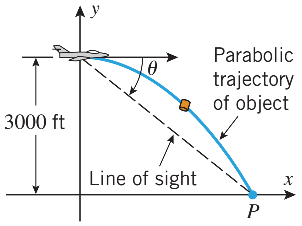


◀ **Figure Ex-68**

69. An airplane is flying at a constant height of 3000 ft above water at a speed of 400 ft/s. The pilot is to release a survival package so that it lands in the water at a sighted point P. If air resistance is neglected, then the package will follow a parabolic trajectory whose equation relative to the coordinate system in the accompanying figure is

$$y = 3000 - \frac{g}{2v^2}x^2$$

where g is the acceleration due to gravity and v is the speed of the airplane. Using $g = 32$ ft/s^2, find the "line of sight" angle θ, to the nearest degree, that will result in the package hitting the target point.



◀ **Figure Ex-69**

70. Sketch the region enclosed by the curves

$$y = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad y = 2$$

and find the area of this region.

71. Estimate the value of k $(0 < k < 1)$ so that the region enclosed by $y = 1/\sqrt{1-x^2}$, $y = x$, $x = 0$, and $x = k$ has an area of 1 square unit.

72. Estimate the area of the region in the first quadrant enclosed by $y = \sin 2x$ and $y = \sin^{-1} x$.

73. Find the volume of the solid that results when the region enclosed by the curves $y = 1/\sqrt{4+x^2}$, $x = -2$, $x = 2$, and $y = 0$ is revolved about the x-axis.

C **74.** Use a CAS to estimate the volume of the solid that results when the region enclosed by the curves $y = x\sqrt{\tan^{-1} x}$ and $y = x$ is revolved about the x-axis.

C **75.** Consider the region enclosed by $y = \sin^{-1} x$, $y = 0$, and $x = 1$. Find the volume of the solid generated by revolving the region about the x-axis using

(a) disks (b) cylindrical shells.

76. (a) Find the volume V of the solid generated when the region bounded by $y = 1/(1+x^4)$, $y = 0$, $x = 1$, and $x = b$ $(b > 1)$ is revolved about the y-axis.

(b) Find $\lim\limits_{b\to+\infty} V$.

77–79 Find the average value of the function over the given interval. ■

77. $f(x) = \dfrac{1}{1+x^2}$; $[1, \sqrt{3}]$ **78.** $f(x) = \dfrac{1}{\sqrt{1-x^2}}$; $[-\frac{1}{2}, 0]$

79. $f(x) = \dfrac{e^{3x}}{1+e^{6x}}$; $\left[-\dfrac{\ln 3}{6}, 0\right]$

80. Find a positive value of k such that the average value of $f(x) = 1/(k^2+x^2)$ over the interval $[-k, k]$ is π.

81–83 Solve the initial-value problems. ■

81. $\dfrac{dy}{dt} = \dfrac{3}{\sqrt{1-t^2}}$, $y\left(\dfrac{\sqrt{3}}{2}\right) = 0$

82. $\dfrac{dy}{dx} = \dfrac{x^2-1}{x^2+1}$, $y(1) = \dfrac{\pi}{2}$

83. $\dfrac{dy}{dt} = \dfrac{1}{25 + 9t^2}, \quad y\left(-\dfrac{5}{3}\right) = \dfrac{\pi}{30}$

84. Evaluate the limit by interpreting it as a Riemann sum in which the interval [0, 1] is divided into n subintervals of equal width:

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$$

85. Prove:
(a) $\sin^{-1}(-x) = -\sin^{-1} x$
(b) $\tan^{-1}(-x) = -\tan^{-1} x$.

86. Prove:
(a) $\cos^{-1}(-x) = \pi - \cos^{-1} x$
(b) $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

87. Use the Mean-Value Theorem to prove that

$$\frac{x}{1 + x^2} < \tan^{-1} x < x \quad (x > 0)$$

88. Prove:

(a) $\sin^{-1} x = \tan^{-1} \dfrac{x}{\sqrt{1 - x^2}} \quad (|x| < 1)$

(b) $\cos^{-1} x = \dfrac{\pi}{2} - \tan^{-1} \dfrac{x}{\sqrt{1 - x^2}} \quad (|x| < 1)$.

89. Prove:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

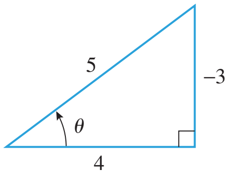
provided $-\pi/2 < \tan^{-1} x + \tan^{-1} y < \pi/2$. [*Hint:* Use an identity for $\tan(\alpha + \beta)$.]

90. Use the result in Exercise 86 to show that
(a) $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \pi/4$
(b) $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \pi/4$.

91. Use identities (4) and (7) to obtain identity (11).

92. **Writing** Suppose that f is a nonconstant function that is twice-differentiable everywhere. Is it always possible to restrict the domain of f to an open interval so that the restricted function has an inverse? Justify your answer by appealing to appropriate theorems.

93. **Writing** Let $\theta = \tan^{-1}(-3/4)$ and explain why the triangle in the accompanying figure may be used to evaluate $\sin\theta$ and $\cos\theta$. More generally, suppose that q denotes a rational number and that $\theta = \tan^{-1} q$ or that $\theta = \sin^{-1} q$. Find right triangles, with at least two sides labeled by integers, that may be used to evaluate $\sin\theta$ and $\cos\theta$.



◀ **Figure Ex-93**

QUICK CHECK ANSWERS 6.7

1. (a) $-\pi/2$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/3$ (e) $2\pi/3$ **2.** (a) $\pi/7$ (b) $2\pi/7$ (c) $\pi/6$ (d) $2\pi/7$ **3.** $\dfrac{2}{\sqrt{1 - 4x^2}}$ **4.** $\pi/3$

6.8 HYPERBOLIC FUNCTIONS AND HANGING CABLES

In this section we will study certain combinations of e^x and e^{-x}, called "hyperbolic functions." These functions, which arise in various engineering applications, have many properties in common with the trigonometric functions. This similarity is somewhat surprising, since there is little on the surface to suggest that there should be any relationship between exponential and trigonometric functions. This is because the relationship occurs within the context of complex numbers, a topic which we will leave for more advanced courses.

DEFINITIONS OF HYPERBOLIC FUNCTIONS

To introduce the hyperbolic functions, observe from Exercise 65 in Section 0.2 that the function e^x can be expressed in the following way as the sum of an even function and an odd function:

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{Even}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{Odd}}$$

These functions are sufficiently important that there are names and notation associated with them: the odd function is called the *hyperbolic sine* of x and the even function is called the *hyperbolic cosine* of x. They are denoted by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

where sinh is pronounced “cinch” and cosh rhymes with “gosh.” From these two building blocks we can create four more functions to produce the following set of six ***hyperbolic functions***.

The terms “tanh,” “sech,” and “csch” are pronounced “tanch,” “seech,” and “coseech,” respectively.

6.8.1 DEFINITION

Hyperbolic sine $\sinh x = \dfrac{e^x - e^{-x}}{2}$

Hyperbolic cosine $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Hyperbolic tangent $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

Hyperbolic cosecant $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

TECHNOLOGY MASTERY

Computer algebra systems have built-in capabilities for evaluating hyperbolic functions directly, but some calculators do not. However, if you need to evaluate a hyperbolic function on a calculator, you can do so by expressing it in terms of exponential functions, as in Example 1.

▶ Example 1

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$\sinh 2 = \frac{e^2 - e^{-2}}{2} \approx 3.6269 \blacktriangleleft$$

GRAPHS OF THE HYPERBOLIC FUNCTIONS

The graphs of the hyperbolic functions, which are shown in Figure 6.8.1, can be generated with a graphing utility, but it is worthwhile to observe that the general shape of the graph of $y = \cosh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ separately and adding the corresponding y-coordinates [see part (a) of the figure]. Similarly, the general shape of the graph of $y = \sinh x$ can be obtained by sketching the graphs of $y = \frac{1}{2}e^x$ and $y = -\frac{1}{2}e^{-x}$ separately and adding corresponding y-coordinates [see part (b) of the figure].

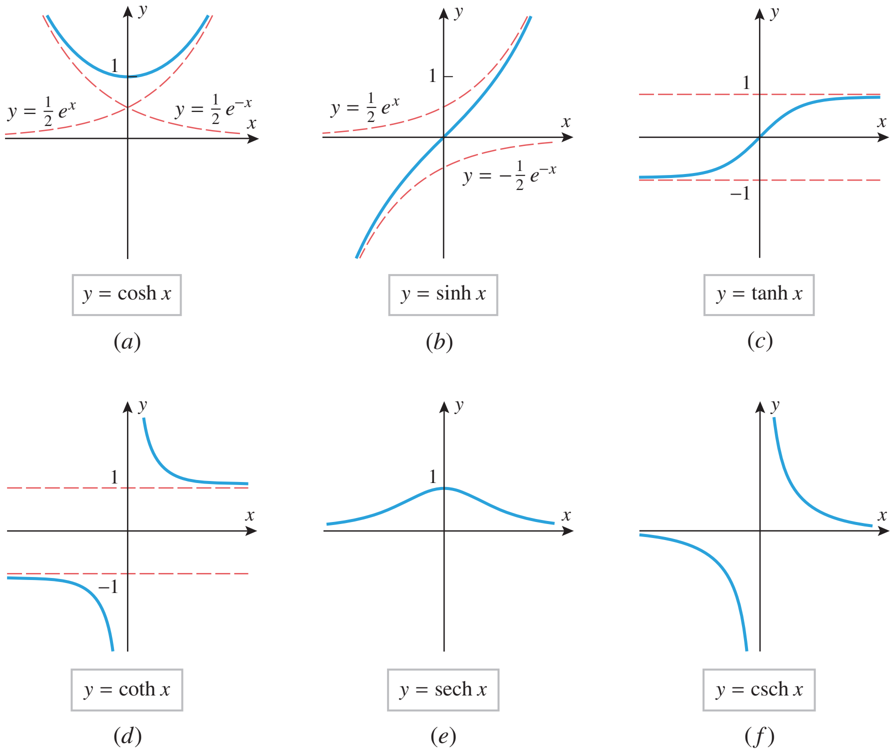
Observe that $\sinh x$ has a domain of $(-\infty, +\infty)$ and a range of $(-\infty, +\infty)$, whereas $\cosh x$ has a domain of $(-\infty, +\infty)$ and a range of $[1, +\infty)$. Observe also that $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ are ***curvilinear asymptotes*** for $y = \cosh x$ in the sense that the graph of $y = \cosh x$ gets closer and closer to the graph of $y = \frac{1}{2}e^x$ as $x \to +\infty$ and gets closer and closer to the graph of $y = \frac{1}{2}e^{-x}$ as $x \to -\infty$. (See Section 3.3.) Similarly, $y = \frac{1}{2}e^x$ is a curvilinear asymptote for $y = \sinh x$ as $x \to +\infty$ and $y = -\frac{1}{2}e^{-x}$ is a curvilinear asymptote as $x \to -\infty$. Other properties of the hyperbolic functions are explored in the exercises.

Glen Allison/Stone/Getty Images

The design of the Gateway Arch in St. Louis is based on an inverted hyperbolic cosine curve (Exercise 73).

HANGING CABLES AND OTHER APPLICATIONS

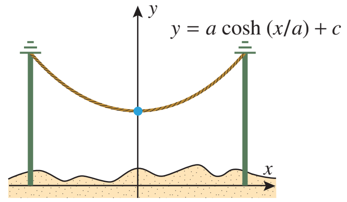
Hyperbolic functions arise in vibratory motions inside elastic solids and more generally in many problems where mechanical energy is gradually absorbed by a surrounding medium. They also occur when a homogeneous, flexible cable is suspended between two points, as with a telephone line hanging between two poles. Such a cable forms a curve, called a ***catenary*** (from the Latin *catena*, meaning “chain”). If, as in Figure 6.8.2, a coordinate



$y = \cosh x$ (*a*) $y = \sinh x$ (*b*) $y = \tanh x$ (*c*)

$y = \coth x$ (*d*) $y = \operatorname{sech} x$ (*e*) $y = \operatorname{csch} x$ (*f*)

▲ **Figure 6.8.1**



▲ **Figure 6.8.2**

system is introduced so that the low point of the cable lies on the y-axis, then it can be shown using principles of physics that the cable has an equation of the form

$$y = a \cosh\left(\frac{x}{a}\right) + c$$

where the parameters a and c are determined by the distance between the poles and the composition of the cable.

HYPERBOLIC IDENTITIES

The hyperbolic functions satisfy various identities that are similar to identities for trigonometric functions. The most fundamental of these is

$$\cosh^2 x - \sinh^2 x = 1 \qquad (1)$$

which can be proved by writing

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= (\cosh x + \sinh x)(\cosh x - \sinh x)\\ &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)\\ &= e^x \cdot e^{-x} = 1\end{aligned}$$

Other hyperbolic identities can be derived in a similar manner or, alternatively, by performing algebraic operations on known identities. For example, if we divide (1) by $\cosh^2 x$, we obtain

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

and if we divide (1) by $\sinh^2 x$, we obtain

$$\coth^2 x - 1 = \operatorname{csch}^2 x$$

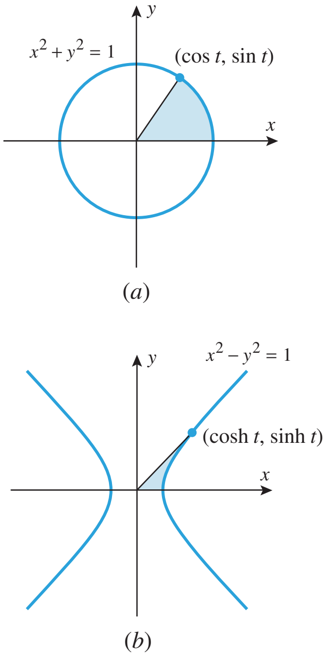
© Lorenz Britt/Alamy

A flexible cable suspended between two poles forms a catenary.

The following theorem summarizes some of the more useful hyperbolic identities. The proofs of those not already obtained are left as exercises.

6.8.2 THEOREM

$$\begin{array}{ll}
\cosh x + \sinh x = e^x & \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \\
\cosh x - \sinh x = e^{-x} & \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \\
\cosh^2 x - \sinh^2 x = 1 & \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y \\
1 - \tanh^2 x = \operatorname{sech}^2 x & \cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y \\
\coth^2 x - 1 = \operatorname{csch}^2 x & \sinh 2x = 2 \sinh x \cosh x \\
\cosh(-x) = \cosh x & \cosh 2x = \cosh^2 x + \sinh^2 x \\
\sinh(-x) = -\sinh x & \cosh 2x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1
\end{array}$$



▲ **Figure 6.8.3**

WHY THEY ARE CALLED HYPERBOLIC FUNCTIONS

Recall that the parametric equations

$$x = \cos t, \quad y = \sin t \qquad (0 \le t \le 2\pi)$$

represent the unit circle $x^2 + y^2 = 1$ (Figure 6.8.3*a*), as may be seen by writing

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

If $0 \le t \le 2\pi$, then the parameter t can be interpreted as the angle in radians from the positive x-axis to the point $(\cos t, \sin t)$ or, alternatively, as twice the shaded area of the sector in Figure 6.8.3*a* (verify). Analogously, the parametric equations

$$x = \cosh t, \quad y = \sinh t \qquad (-\infty < t < +\infty)$$

represent a portion of the curve $x^2 - y^2 = 1$, as may be seen by writing

$$x^2 - y^2 = \cosh^2 t - \sinh^2 t = 1$$

and observing that $x = \cosh t > 0$. This curve, which is shown in Figure 6.8.3*b*, is the right half of a larger curve called the ***unit hyperbola***; this is the reason why the functions in this section are called *hyperbolic* functions. It can be shown that if $t \ge 0$, then the parameter t can be interpreted as twice the shaded area in Figure 6.8.3*b*. (We omit the details.)

DERIVATIVE AND INTEGRAL FORMULAS

Derivative formulas for $\sinh x$ and $\cosh x$ can be obtained by expressing these functions in terms of e^x and e^{-x}:

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{e^x + e^{-x}}{2}\right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

Derivatives of the remaining hyperbolic functions can be obtained by expressing them in terms of sinh and cosh and applying appropriate identities. For example,

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \dfrac{d}{dx}[\sinh x] - \sinh x \dfrac{d}{dx}[\cosh x]}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

The following theorem provides a complete list of the generalized derivative formulas and corresponding integration formulas for the hyperbolic functions.

6.8.3 THEOREM

$$\frac{d}{dx}[\sinh u] = \cosh u \frac{du}{dx} \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = \sinh u \frac{du}{dx} \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = \operatorname{sech}^2 u \frac{du}{dx} \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -\operatorname{csch}^2 u \frac{du}{dx} \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -\operatorname{sech} u \tanh u \frac{du}{dx} \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -\operatorname{csch} u \coth u \frac{du}{dx} \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

▶ Example 2

$$\frac{d}{dx}[\cosh(x^3)] = \sinh(x^3) \cdot \frac{d}{dx}[x^3] = 3x^2 \sinh(x^3)$$

$$\frac{d}{dx}[\ln(\tanh x)] = \frac{1}{\tanh x} \cdot \frac{d}{dx}[\tanh x] = \frac{\operatorname{sech}^2 x}{\tanh x}$$ ◀

▶ Example 3

$$\int \sinh^5 x \cosh x\, dx = \tfrac{1}{6}\sinh^6 x + C \qquad \boxed{\begin{aligned} u &= \sinh x \\ du &= \cosh x\, dx \end{aligned}}$$

$$\begin{aligned} \int \tanh x\, dx &= \int \frac{\sinh x}{\cosh x}\, dx \\ &= \ln|\cosh x| + C \qquad \boxed{\begin{aligned} u &= \cosh x \\ du &= \sinh x\, dx \end{aligned}} \\ &= \ln(\cosh x) + C \end{aligned}$$

We were justified in dropping the absolute value signs since $\cosh x > 0$ for all x. ◀

▶ Example 4 A 100 ft wire is attached at its ends to the tops of two 50 ft poles that are positioned 90 ft apart. How high above the ground is the middle of the wire?

Solution. From above, the wire forms a catenary curve with equation

$$y = a \cosh\left(\frac{x}{a}\right) + c$$

where the origin is on the ground midway between the poles. Using Formula (4) of Section 5.4 for the length of the catenary, we have

$$\begin{aligned}
100 &= \int_{-45}^{45} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= 2\int_{0}^{45} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx && \boxed{\text{By symmetry about the } y\text{-axis}} \\
&= 2\int_{0}^{45} \sqrt{1+\sinh^2\left(\frac{x}{a}\right)}\,dx \\
&= 2\int_{0}^{45} \cosh\left(\frac{x}{a}\right)dx && \boxed{\text{By (1) and the fact that } \cosh x > 0} \\
&= 2a\sinh\left(\frac{x}{a}\right)\Bigg]_0^{45} = 2a\sinh\left(\frac{45}{a}\right)
\end{aligned}$$

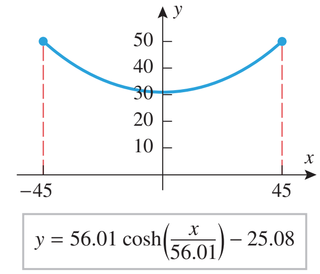
Using a calculating utility's numeric solver to solve

$$100 = 2a\sinh\left(\frac{45}{a}\right)$$

for a gives $a \approx 56.01$. Then

$$50 = y(45) = 56.01\cosh\left(\frac{45}{56.01}\right) + c \approx 75.08 + c$$

so $c \approx -25.08$. Thus, the middle of the wire is $y(0) \approx 56.01 - 25.08 = 30.93$ ft above the ground (Figure 6.8.4). ◀

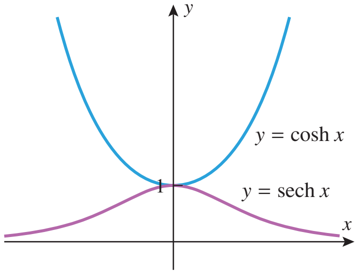


▲ **Figure 6.8.4**

INVERSES OF HYPERBOLIC FUNCTIONS

Referring to Figure 6.8.1, it is evident that the graphs of $\sinh x$, $\tanh x$, $\coth x$, and $\operatorname{csch} x$ pass the horizontal line test, but the graphs of $\cosh x$ and $\operatorname{sech} x$ do not. In the latter case, restricting x to be nonnegative makes the functions invertible (Figure 6.8.5). The graphs of the six inverse hyperbolic functions in Figure 6.8.6 were obtained by reflecting the graphs of the hyperbolic functions (with the appropriate restrictions) about the line $y = x$.

Table 6.8.1 summarizes the basic properties of the inverse hyperbolic functions. You should confirm that the domains and ranges listed in this table agree with the graphs in Figure 6.8.6.



With the restriction that $x \geq 0$, the curves $y = \cosh x$ and $y = \operatorname{sech} x$ pass the horizontal line test.

▲ **Figure 6.8.5**

LOGARITHMIC FORMS OF INVERSE HYPERBOLIC FUNCTIONS

Because the hyperbolic functions are expressible in terms of e^x, it should not be surprising that the inverse hyperbolic functions are expressible in terms of natural logarithms; the next theorem shows that this is so.

6.8.4 THEOREM *The following relationships hold for all x in the domains of the stated inverse hyperbolic functions:*

$$\begin{aligned}
\sinh^{-1} x &= \ln(x+\sqrt{x^2+1}\,) & \cosh^{-1} x &= \ln(x+\sqrt{x^2-1}\,) \\
\tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) & \coth^{-1} x &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \\
\operatorname{sech}^{-1} x &= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) & \operatorname{csch}^{-1} x &= \ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right)
\end{aligned}$$

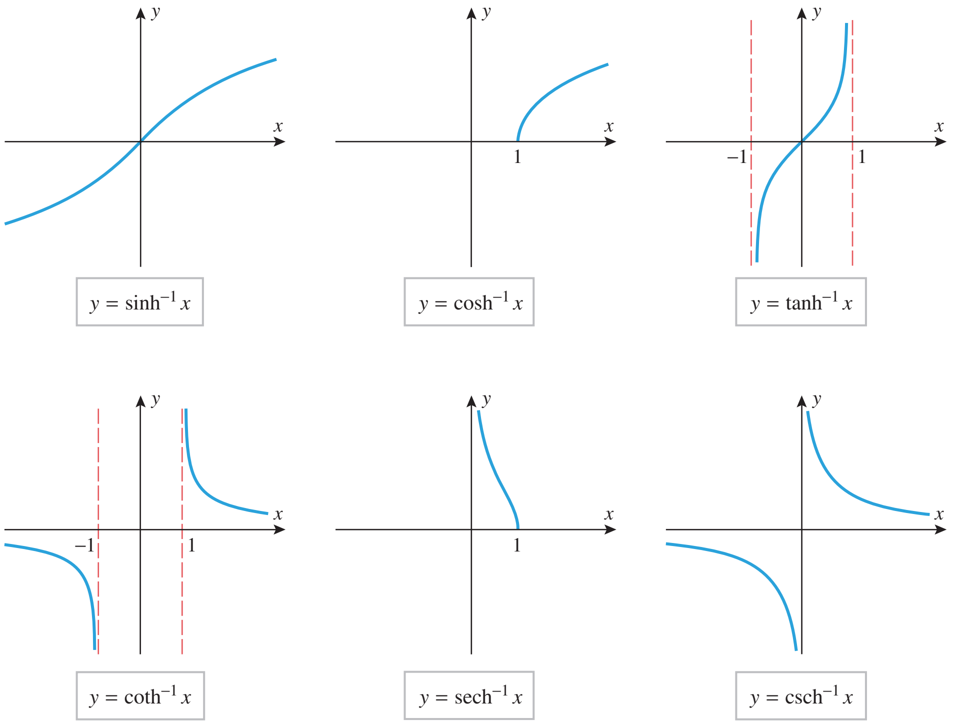


Figure 6.8.6

Table 6.8.1

PROPERTIES OF INVERSE HYPERBOLIC FUNCTIONS

FUNCTION	DOMAIN	RANGE	BASIC RELATIONSHIPS
$\sinh^{-1} x$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$\sinh^{-1}(\sinh x) = x$ if $-\infty < x < +\infty$ $\sinh(\sinh^{-1} x) = x$ if $-\infty < x < +\infty$
$\cosh^{-1} x$	$[1, +\infty)$	$[0, +\infty)$	$\cosh^{-1}(\cosh x) = x$ if $x \geq 0$ $\cosh(\cosh^{-1} x) = x$ if $x \geq 1$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, +\infty)$	$\tanh^{-1}(\tanh x) = x$ if $-\infty < x < +\infty$ $\tanh(\tanh^{-1} x) = x$ if $-1 < x < 1$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\coth^{-1}(\coth x) = x$ if $x < 0$ or $x > 0$ $\coth(\coth^{-1} x) = x$ if $x < -1$ or $x > 1$
$\operatorname{sech}^{-1} x$	$(0, 1]$	$[0, +\infty)$	$\operatorname{sech}^{-1}(\operatorname{sech} x) = x$ if $x \geq 0$ $\operatorname{sech}(\operatorname{sech}^{-1} x) = x$ if $0 < x \leq 1$
$\operatorname{csch}^{-1} x$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$\operatorname{csch}^{-1}(\operatorname{csch} x) = x$ if $x < 0$ or $x > 0$ $\operatorname{csch}(\operatorname{csch}^{-1} x) = x$ if $x < 0$ or $x > 0$

We will show how to derive the first formula in this theorem and leave the rest as exercises. The basic idea is to write the equation $x = \sinh y$ in terms of exponential functions and solve this equation for y as a function of x. This will produce the equation $y = \sinh^{-1} x$ with $\sinh^{-1} x$ expressed in terms of natural logarithms. Expressing $x = \sinh y$ in terms of exponentials yields

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

which can be rewritten as

$$e^y - 2x - e^{-y} = 0$$

Multiplying this equation through by e^y we obtain

$$e^{2y} - 2xe^y - 1 = 0$$

and applying the quadratic formula yields

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $e^y > 0$, the solution involving the minus sign is extraneous and must be discarded. Thus,

$$e^y = x + \sqrt{x^2+1}$$

Taking natural logarithms yields

$$y = \ln(x + \sqrt{x^2+1}\,) \quad \text{or} \quad \sinh^{-1} x = \ln(x + \sqrt{x^2+1}\,)$$

▶ Example 5

$$\sinh^{-1} 1 = \ln(1 + \sqrt{1^2+1}\,) = \ln(1 + \sqrt{2}\,) \approx 0.8814$$

$$\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \frac{1}{2}\ln 3 \approx 0.5493 \blacktriangleleft$$

DERIVATIVES AND INTEGRALS INVOLVING INVERSE HYPERBOLIC FUNCTIONS

Formulas for the derivatives of the inverse hyperbolic functions can be obtained from Theorem 6.8.4. For example,

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{d}{dx}[\ln(x + \sqrt{x^2+1}\,)] = \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1} + x}{(x + \sqrt{x^2+1}\,)(\sqrt{x^2+1}\,)} = \frac{1}{\sqrt{x^2+1}}$$

Show that the derivative of the function $\sinh^{-1} x$ can also be obtained by letting $y = \sinh^{-1} x$ and then differentiating $x = \sinh y$ implicitly.

This computation leads to two integral formulas, a formula that involves $\sinh^{-1} x$ and an equivalent formula that involves logarithms:

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C = \ln(x + \sqrt{x^2+1}\,) + C$$

The following two theorems list the generalized derivative formulas and corresponding integration formulas for the inverse hyperbolic functions. Some of the proofs appear as exercises.

6.8.5 THEOREM

$$\frac{d}{dx}(\sinh^{-1} u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} \qquad \frac{d}{dx}(\coth^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1 \qquad \frac{d}{dx}(\operatorname{sech}^{-1} u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}(\tanh^{-1} u) = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1 \qquad \frac{d}{dx}(\operatorname{csch}^{-1} u) = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

6.8.6 THEOREM *If* $a > 0$, *then*

$$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \quad \textit{or} \quad \ln(u + \sqrt{u^2+a^2}\,) + C$$

$$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \quad \textit{or} \quad \ln(u + \sqrt{u^2-a^2}\,) + C, \quad u > a$$

$$\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & |u| < a \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & |u| > a \end{cases} \quad \textit{or} \quad \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C, \quad |u| \neq a$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left|\frac{u}{a}\right| + C \quad \textit{or} \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right) + C, \quad 0 < |u| < a$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C \quad \textit{or} \quad -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right) + C, \quad u \neq 0$$

▶ **Example 6** Evaluate $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}}$, $x > \dfrac{3}{2}$.

Solution. Let $u = 2x$. Thus, $du = 2\,dx$ and

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-3^2}}$$

$$= \frac{1}{2}\cosh^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{2}\cosh^{-1}\left(\frac{2x}{3}\right) + C$$

Alternatively, we can use the logarithmic equivalent of $\cosh^{-1}(2x/3)$,

$$\cosh^{-1}\left(\frac{2x}{3}\right) = \ln(2x + \sqrt{4x^2-9}\,) - \ln 3$$

(verify), and express the answer as

$$\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\ln(2x + \sqrt{4x^2-9}\,) + C \quad ◀$$

✔QUICK CHECK EXERCISES 6.8 *(See page 483 for answers.)*

1. $\cosh x =$ ________ $\sinh x =$ ________
$\tanh x =$ ________

2. Complete the table.

	cosh x	sinh x	tanh x	coth x	sech x	csch x
DOMAIN						
RANGE						

3. The parametric equations

$$x = \cosh t, \quad y = \sinh t \quad (-\infty < t < +\infty)$$

represent the right half of the curve called a ________. Eliminating the parameter, the equation of this curve is ________.

4. $\dfrac{d}{dx}[\cosh x] =$ ________ $\dfrac{d}{dx}[\sinh x] =$ ________

$\dfrac{d}{dx}[\tanh x] =$ ________

5. $\int \cosh x\,dx = \_\_\_\_\_\_\_\_$ $\quad \int \sinh x\,dx = \_\_\_\_\_\_\_\_$

$\int \tanh x\,dx = \_\_\_\_\_\_\_\_$

6. $\frac{d}{dx}[\cosh^{-1} x] = \_\_\_\_\_\_\_\_$ $\quad \frac{d}{dx}[\sinh^{-1} x] = \_\_\_\_\_\_\_\_$

$\frac{d}{dx}[\tanh^{-1} x] = \_\_\_\_\_\_\_\_$

EXERCISE SET 6.8

Graphing Utility

1–2 Approximate the expression to four decimal places.

1. (a) $\sinh 3$ (b) $\cosh(-2)$ (c) $\tanh(\ln 4)$
(d) $\sinh^{-1}(-2)$ (e) $\cosh^{-1} 3$ (f) $\tanh^{-1} \frac{3}{4}$

2. (a) $\operatorname{csch}(-1)$ (b) $\operatorname{sech}(\ln 2)$ (c) $\coth 1$
(d) $\operatorname{sech}^{-1} \frac{1}{2}$ (e) $\coth^{-1} 3$ (f) $\operatorname{csch}^{-1}(-\sqrt{3})$

3. Find the exact numerical value of each expression.
(a) $\sinh(\ln 3)$ (b) $\cosh(-\ln 2)$
(c) $\tanh(2\ln 5)$ (d) $\sinh(-3\ln 2)$

4. In each part, rewrite the expression as a ratio of polynomials.
(a) $\cosh(\ln x)$ (b) $\sinh(\ln x)$
(c) $\tanh(2\ln x)$ (d) $\cosh(-\ln x)$

5. In each part, a value for one of the hyperbolic functions is given at an unspecified positive number x_0. Use appropriate identities to find the exact values of the remaining five hyperbolic functions at x_0.
(a) $\sinh x_0 = 2$ (b) $\cosh x_0 = \frac{5}{4}$ (c) $\tanh x_0 = \frac{4}{5}$

6. Obtain the derivative formulas for $\operatorname{csch} x$, $\operatorname{sech} x$, and $\coth x$ from the derivative formulas for $\sinh x$, $\cosh x$, and $\tanh x$.

7. Find the derivatives of $\cosh^{-1} x$ and $\tanh^{-1} x$ by differentiating the formulas in Theorem 6.8.4.

8. Find the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$ by differentiating the equations $x = \sinh y$, $x = \cosh y$, and $x = \tanh y$ implicitly.

9–28 Find dy/dx.

9. $y = \sinh(4x - 8)$ **10.** $y = \cosh(x^4)$

11. $y = \coth(\ln x)$ **12.** $y = \ln(\tanh 2x)$

13. $y = \operatorname{csch}(1/x)$ **14.** $y = \operatorname{sech}(e^{2x})$

15. $y = \sqrt{4x + \cosh^2(5x)}$ **16.** $y = \sinh^3(2x)$

17. $y = x^3 \tanh^2(\sqrt{x})$ **18.** $y = \sinh(\cos 3x)$

19. $y = \sinh^{-1}\left(\frac{1}{3}x\right)$ **20.** $y = \sinh^{-1}(1/x)$

21. $y = \ln(\cosh^{-1} x)$ **22.** $y = \cosh^{-1}(\sinh^{-1} x)$

23. $y = \dfrac{1}{\tanh^{-1} x}$ **24.** $y = (\coth^{-1} x)^2$

25. $y = \cosh^{-1}(\cosh x)$ **26.** $y = \sinh^{-1}(\tanh x)$

27. $y = e^x \operatorname{sech}^{-1}\sqrt{x}$ **28.** $y = (1 + x\operatorname{csch}^{-1} x)^{10}$

29–44 Evaluate the integrals.

29. $\int \sinh^6 x \cosh x\,dx$ **30.** $\int \cosh(2x - 3)\,dx$

31. $\int \sqrt{\tanh x}\,\operatorname{sech}^2 x\,dx$ **32.** $\int \operatorname{csch}^2(3x)\,dx$

33. $\int \tanh 2x\,dx$ **34.** $\int \coth^2 x \operatorname{csch}^2 x\,dx$

35. $\int_{\ln 2}^{\ln 3} \tanh x \operatorname{sech}^3 x\,dx$ **36.** $\int_0^{\ln 3} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

37. $\int \dfrac{dx}{\sqrt{1 + 9x^2}}$ **38.** $\int \dfrac{dx}{\sqrt{x^2 - 2}}$ $(x > \sqrt{2})$

39. $\int \dfrac{dx}{\sqrt{1 - e^{2x}}}$ $(x < 0)$ **40.** $\int \dfrac{\sin\theta\,d\theta}{\sqrt{1 + \cos^2\theta}}$

41. $\int \dfrac{dx}{x\sqrt{1 + 4x^2}}$ **42.** $\int \dfrac{dx}{\sqrt{9x^2 - 25}}$ $(x > 5/3)$

43. $\int_0^{1/2} \dfrac{dx}{1 - x^2}$ **44.** $\int_0^{\sqrt{3}} \dfrac{dt}{\sqrt{t^2 + 1}}$

45–48 True–False Determine whether the statement is true or false. Explain your answer.

45. The equation $\cosh x = \sinh x$ has no solutions.

46. Exactly two of the hyperbolic functions are bounded.

47. There is exactly one hyperbolic function $f(x)$ such that for all real numbers a, the equation $f(x) = a$ has a unique solution x.

48. The identities in Theorem 6.8.2 may be obtained from the corresponding trigonometric identities by replacing each trigonometric function with its hyperbolic analogue.

49. Find the area enclosed by $y = \sinh 2x$, $y = 0$, and $x = \ln 3$.

50. Find the volume of the solid that is generated when the region enclosed by $y = \operatorname{sech} x$, $y = 0$, $x = 0$, and $x = \ln 2$ is revolved about the x-axis.

51. Find the volume of the solid that is generated when the region enclosed by $y = \cosh 2x$, $y = \sinh 2x$, $x = 0$, and $x = 5$ is revolved about the x-axis.

52. Approximate the positive value of the constant a such that the area enclosed by $y = \cosh ax$, $y = 0$, $x = 0$, and $x = 1$ is 2 square units. Express your answer to at least five decimal places.

53. Find the arc length of the catenary $y = \cosh x$ between $x = 0$ and $x = \ln 2$.

54. Find the arc length of the catenary $y = a\cosh(x/a)$ between $x = 0$ and $x = x_1$ $(x_1 > 0)$.

55. In parts (a)–(f) find the limits, and confirm that they are consistent with the graphs in Figures 6.8.1 and 6.8.6.

(a) $\lim_{x \to +\infty} \sinh x$ (b) $\lim_{x \to -\infty} \sinh x$

(c) $\lim_{x \to +\infty} \tanh x$ (d) $\lim_{x \to -\infty} \tanh x$

(e) $\lim_{x \to +\infty} \sinh^{-1} x$ (f) $\lim_{x \to 1^-} \tanh^{-1} x$

FOCUS ON CONCEPTS

56. Explain how to obtain the asymptotes for $y = \tanh x$ from the curvilinear asymptotes for $y = \cosh x$ and $y = \sinh x$.

57. Prove that $\sinh x$ is an odd function of x and that $\cosh x$ is an even function of x, and check that this is consistent with the graphs in Figure 6.8.1.

58–59 Prove the identities.

58. (a) $\cosh x + \sinh x = e^x$

(b) $\cosh x - \sinh x = e^{-x}$

(c) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

(d) $\sinh 2x = 2 \sinh x \cosh x$

(e) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(f) $\cosh 2x = \cosh^2 x + \sinh^2 x$

(g) $\cosh 2x = 2 \sinh^2 x + 1$

(h) $\cosh 2x = 2 \cosh^2 x - 1$

59. (a) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(b) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

(c) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

60. Prove:

(a) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$

(b) $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1 + x}{1 - x}\right), \quad -1 < x < 1.$

61. Use Exercise 60 to obtain the derivative formulas for $\cosh^{-1} x$ and $\tanh^{-1} x$.

62. Prove:

$$\operatorname{sech}^{-1} x = \cosh^{-1}(1/x), \quad 0 < x \leq 1$$
$$\coth^{-1} x = \tanh^{-1}(1/x), \quad |x| > 1$$
$$\operatorname{csch}^{-1} x = \sinh^{-1}(1/x), \quad x \neq 0$$

63. Use Exercise 62 to express the integral

$$\int \frac{du}{1 - u^2}$$

entirely in terms of $\tanh^{-1}$.

64. Show that

(a) $\dfrac{d}{dx}[\operatorname{sech}^{-1}|x|] = -\dfrac{1}{x\sqrt{1 - x^2}}$

(b) $\dfrac{d}{dx}[\operatorname{csch}^{-1}|x|] = -\dfrac{1}{x\sqrt{1 + x^2}}.$

65. In each part, find the limit.

(a) $\lim_{x \to +\infty} (\cosh^{-1} x - \ln x)$ (b) $\lim_{x \to +\infty} \dfrac{\cosh x}{e^x}$

66. Use the first and second derivatives to show that the graph of $y = \tanh^{-1} x$ is always increasing and has an inflection point at the origin.

67. The integration formulas for $1/\sqrt{u^2 - a^2}$ in Theorem 6.8.6 are valid for $u > a$. Show that the following formula is valid for $u < -a$:

$$\int \frac{du}{\sqrt{u^2 - a^2}} = -\cosh^{-1}\left(-\frac{u}{a}\right) + C \quad \text{or} \quad \ln\left|u + \sqrt{u^2 - a^2}\right| + C$$

68. Show that $(\sinh x + \cosh x)^n = \sinh nx + \cosh nx$.

69. Show that

$$\int_{-a}^{a} e^{tx}\, dx = \frac{2 \sinh at}{t}$$

70. A cable is suspended between two poles as shown in Figure 6.8.2. Assume that the equation of the curve formed by the cable is $y = a \cosh(x/a)$, where a is a positive constant. Suppose that the x-coordinates of the points of support are $x = -b$ and $x = b$, where $b > 0$.

(a) Show that the length L of the cable is given by

$$L = 2a \sinh \frac{b}{a}$$

(b) Show that the sag S (the vertical distance between the highest and lowest points on the cable) is given by

$$S = a \cosh \frac{b}{a} - a$$

71–72 These exercises refer to the hanging cable described in Exercise 70.

71. Assuming that the poles are 400 ft apart and the sag in the cable is 30 ft, approximate the length of the cable by approximating a. Express your final answer to the nearest tenth of a foot. [*Hint:* First let $u = 200/a$.]

72. Assuming that the cable is 120 ft long and the poles are 100 ft apart, approximate the sag in the cable by approximating a. Express your final answer to the nearest tenth of a foot. [*Hint:* First let $u = 50/a$.]

73. The design of the Gateway Arch in St. Louis, Missouri, by architect Eero Saarinan was implemented using equations provided by Dr. Hannskarl Badel. The equation used for the centerline of the arch was

$$y = 693.8597 - 68.7672 \cosh(0.0100333x) \text{ ft}$$

for x between -299.2239 and 299.2239.

(a) Use a graphing utility to graph the centerline of the arch.

(b) Find the length of the centerline to four decimal places.

(c) For what values of x is the height of the arch 100 ft? Round your answers to four decimal places.

(d) Approximate, to the nearest degree, the acute angle that the tangent line to the centerline makes with the ground at the ends of the arch.

74. Suppose that a hollow tube rotates with a constant angular velocity of ω rad/s about a horizontal axis at one end of the tube, as shown in the accompanying figure (on the next page). Assume that an object is free to slide without friction in the tube while the tube is rotating. Let r be the distance from the object to the pivot point at time $t \geq 0$, and assume that the object is at rest and $r = 0$ when $t = 0$. It can be shown that if the tube is horizontal at time $t = 0$ and rotating as shown in the figure, then

$$r = \frac{g}{2\omega^2}[\sinh(\omega t) - \sin(\omega t)]$$

during the period that the object is in the tube. Assume that t is in seconds and r is in meters, and use $g = 9.8 \text{ m/s}^2$ and $\omega = 2$ rad/s.

(a) Graph r versus t for $0 \leq t \leq 1$.

(b) Assuming that the tube has a length of 1 m, approximately how long does it take for the object to reach the end of the tube?

(c) Use the result of part (b) to approximate dr/dt at the instant that the object reaches the end of the tube.

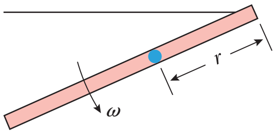


Figure Ex-74

75. The accompanying figure shows a person pulling a boat by holding a rope of length a attached to the bow and walking along the edge of a dock. If we assume that the rope is always tangent to the curve traced by the bow of the boat, then this curve, which is called a ***tractrix***, has the property that the segment of the tangent line between the curve and the y-axis has a constant length a. It can be proved that the equation of this tractrix is

$$y = a \operatorname{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}$$

(a) Show that to move the bow of the boat to a point (x, y), the person must walk a distance

$$D = a \operatorname{sech}^{-1}\frac{x}{a}$$

from the origin.

(b) If the rope has a length of 15 m, how far must the person walk from the origin to bring the boat 10 m from the dock? Round your answer to two decimal places.

(c) Find the distance traveled by the bow along the tractrix as it moves from its initial position to the point where it is 5 m from the dock.

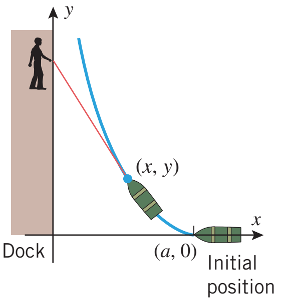


Figure Ex-75

76. Writing Suppose that, by analogy with the trigonometric functions, we *define* $\cosh t$ and $\sinh t$ geometrically using Figure 6.8.3*b*:

"For any real number t, define $x = \cosh t$ and $y = \sinh t$ to be the unique values of x and y such that

(i) $P(x, y)$ is on the right branch of the unit hyperbola $x^2 - y^2 = 1$;

(ii) t and y have the same sign (or are both 0);

(iii) the area of the region bounded by the x-axis, the right branch of the unit hyperbola, and the segment from the origin to P is $|t|/2$."

Discuss what properties would first need to be verified in order for this to be a legitimate definition.

77. Writing Investigate what properties of $\cosh t$ and $\sinh t$ can be proved directly from the geometric definition in Exercise 76. Write a short description of the results of your investigation.

QUICK CHECK ANSWERS 6.8

1. $\dfrac{e^x + e^{-x}}{2}$; $\dfrac{e^x - e^{-x}}{2}$; $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

2.

	$\cosh x$	$\sinh x$	$\tanh x$	$\coth x$	$\operatorname{sech} x$	$\operatorname{csch} x$
DOMAIN	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$	$(-\infty, +\infty)$	$(-\infty, 0) \cup (0, +\infty)$
RANGE	$[1, +\infty)$	$(-\infty, +\infty)$	$(-1, 1)$	$(-\infty, -1) \cup (1, +\infty)$	$(0, 1]$	$(-\infty, 0) \cup (0, +\infty)$

3. unit hyperbola; $x^2 - y^2 = 1$ **4.** $\sinh x$; $\cosh x$; $\operatorname{sech}^2 x$ **5.** $\sinh x + C$; $\cosh x + C$; $\ln(\cosh x) + C$

6. $\dfrac{1}{\sqrt{x^2 - 1}}$; $\dfrac{1}{\sqrt{1 + x^2}}$; $\dfrac{1}{1 - x^2}$

CHAPTER 6 REVIEW EXERCISES

Graphing Utility

1. In each part, find $f^{-1}(x)$ if the inverse exists.

(a) $f(x) = (e^x)^2 + 1$

(b) $f(x) = \sin\left(\dfrac{1-2x}{x}\right), \quad \dfrac{2}{4+\pi} \le x \le \dfrac{2}{4-\pi}$

(c) $f(x) = \dfrac{1}{1 + 3\tan^{-1} x}$

2. (a) State the restrictions on the domains of $\sin x$, $\cos x$, $\tan x$, and $\sec x$ that are imposed to make those functions one-to-one in the definitions of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} x$.

(b) Sketch the graphs of the restricted trigonometric functions in part (a) and their inverses.

3. In each part, find the exact numerical value of the given expression.

(a) $\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)]$

(b) $\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)]$

4. In each part, sketch the graph, and check your work with a graphing utility.

(a) $f(x) = 3\sin^{-1}(x/2)$

(b) $f(x) = \cos^{-1} x - \pi/2$

(c) $f(x) = 2\tan^{-1}(-3x)$

(d) $f(x) = \cos^{-1} x + \sin^{-1} x$

5. Suppose that the graph of $y = \log x$ is drawn with equal scales of 1 inch per unit in both the x- and y-directions. If a bug wants to walk along the graph until it reaches a height of 5 ft above the x-axis, how many miles to the right of the origin will it have to travel?

6. Find the largest value of a such that the function $f(x) = xe^{-x}$ has an inverse on the interval $(-\infty, a]$.

7. Express the following function as a rational function of x:

$$3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1)$$

8. Suppose that $y = Ce^{kt}$, where C and k are constants, and let $Y = \ln y$. Show that the graph of Y versus t is a line, and state its slope and Y-intercept.

9. (a) Sketch the curves $y = \pm e^{-x/2}$ and $y = e^{-x/2}\sin 2x$ for $-\pi/2 \le x \le 3\pi/2$ in the same coordinate system, and check your work using a graphing utility.

(b) Find all x-intercepts of the curve $y = e^{-x/2}\sin 2x$ in the stated interval, and find the x-coordinates of all points where this curve intersects the curves $y = \pm e^{-x/2}$.

10. Suppose that a package of medical supplies is dropped from a helicopter straight down by parachute into a remote area. The velocity v (in feet per second) of the package t seconds after it is released is given by $v = 24.61(1 - e^{-1.3t})$.

(a) Graph v versus t.

(b) Show that the graph has a horizontal asymptote $v = c$.

(c) The constant c is called the ***terminal velocity***. Explain what the terminal velocity means in practical terms.

(d) Can the package actually reach its terminal velocity? Explain.

(e) How long does it take for the package to reach 98% of its terminal velocity?

11. A breeding group of 20 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, N, after t years will be given by the formula

$$N = \frac{220}{1 + 10(0.83^t)}$$

and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 80.

(a) Graph N versus t.

(b) How many years must the state of Colorado maintain a program to care for the sheep?

(c) How many bighorn sheep can the environment in the protected area support? [*Hint:* Examine the graph of N versus t for large values of t.]

12. An oven is preheated and then remains at a constant temperature. A potato is placed in the oven to bake. Suppose that the temperature T (in °F) of the potato t minutes later is given by $T = 400 - 325(0.97^t)$. The potato will be considered done when its temperature is anywhere between 260°F and 280°F.

(a) During what interval of time would the potato be considered done?

(b) How long does it take for the difference between the potato and oven temperatures to be cut in half?

13. (a) Show that the graphs of $y = \ln x$ and $y = x^{0.2}$ intersect.

(b) Approximate the solution(s) of the equation $\ln x = x^{0.2}$ to three decimal places.

14. (a) Show that for $x > 0$ and $k \ne 0$ the equations

$$x^k = e^x \quad \text{and} \quad \frac{\ln x}{x} = \frac{1}{k}$$

have the same solutions.

(b) Use the graph of $y = (\ln x)/x$ to determine the values of k for which the equation $x^k = e^x$ has two distinct positive solutions.

(c) Estimate the positive solution(s) of $x^8 = e^x$.

15–18 Find the limits. ■

15. $\displaystyle\lim_{t\to\pi/2^+} e^{\tan t}$

16. $\displaystyle\lim_{\theta\to 0^+} \ln(\sin 2\theta) - \ln(\tan\theta)$

17. $\displaystyle\lim_{x\to+\infty} \left(1 + \frac{3}{x}\right)^{-x}$

18. $\displaystyle\lim_{x\to+\infty} \left(1 + \frac{a}{x}\right)^{bx}, \quad a, b > 0$

19–20 Find dy/dx by first using algebraic properties of the natural logarithm function. ■

19. $y = \ln\left(\dfrac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4}\right)$

20. $y = \ln\left(\dfrac{\sqrt{x}\,\sqrt[3]{x+1}}{\sin x \sec x}\right)$

21–38 Find dy/dx. ■

21. $y = \ln 2x$

22. $y = (\ln x)^2$

23. $y = \sqrt[3]{\ln x + 1}$

24. $y = \ln(\sqrt[3]{x} + 1)$

25. $y = \log(\ln x)$

26. $y = \dfrac{1 + \log x}{1 - \log x}$

27. $y = \ln(x^{3/2}\sqrt{1 + x^4})$

28. $y = \ln\left(\dfrac{\sqrt{x}\cos x}{1 + x^2}\right)$

29. $y = e^{\ln(x^2+1)}$

30. $y = \ln\left(\dfrac{1 + e^x + e^{2x}}{1 - e^{3x}}\right)$

31. $y = 2xe^{\sqrt{x}}$

32. $y = \dfrac{a}{1 + be^{-x}}$

33. $y = \dfrac{1}{\pi}\tan^{-1} 2x$

34. $y = 2^{\sin^{-1} x}$

35. $y = x^{(e^x)}$

36. $y = (1 + x)^{1/x}$

37. $y = \sec^{-1}(2x + 1)$

38. $y = \sqrt{\cos^{-1} x^2}$

39–40 Find dy/dx using logarithmic differentiation. ■

39. $y = \dfrac{x^3}{\sqrt{x^2 + 1}}$

40. $y = \sqrt[3]{\dfrac{x^2 - 1}{x^2 + 1}}$

41. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
(b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
(c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
(d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain.
(e) Find the exact x-coordinates of all horizontal tangent lines to the curve.

42. Recall from Section 6.1 that the loudness β of a sound in decibels (dB) is given by $\beta = 10\log(I/I_0)$, where I is the intensity of the sound in watts per square meter (W/m^2) and I_0 is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of β with respect to I at the point where
(a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$.

43. A particle is moving along the curve $y = x\ln x$. Find all values of x at which the rate of change of y with respect to time is three times that of x. [Assume that dx/dt is never zero.]

44. Find the equation of the tangent line to the graph of $y = \ln(5 - x^2)$ at $x = 2$.

45. Find the value of b so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

46. In each part, find the value of k for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
(a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

47. If f and g are inverse functions and f is differentiable on its domain, must g be differentiable on its domain? Give a reasonable informal argument to support your answer.

48. In each part, find $(f^{-1})'(x)$ using Formula (2) of Section 6.3, and check your answer by differentiating f^{-1} directly.
(a) $f(x) = 3/(x + 1)$ (b) $f(x) = \sqrt{e^x}$

49. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

50. Show that the rate of change of $y = 5000e^{1.07x}$ is proportional to y.

51. Show that the function $y = e^{ax}\sin bx$ satisfies

$$y'' - 2ay' + (a^2 + b^2)y = 0$$

for any real constants a and b.

52. Show that the function $y = \tan^{-1} x$ satisfies

$$y'' = -2\sin y\cos^3 y$$

53. Suppose that the population of deer on an island is modeled by the equation

$$P(t) = \frac{95}{5 - 4e^{-t/4}}$$

where $P(t)$ is the number of deer t weeks after an initial observation at time $t = 0$.
(a) Use a graphing utility to graph the function $P(t)$.
(b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
(c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

54. The equilibrium constant k of a balanced chemical reaction changes with the absolute temperature T according to the law

$$k = k_0\exp\left(-\frac{q(T - T_0)}{2T_0T}\right)$$

where k_0, q, and T_0 are constants. Find the rate of change of k with respect to T.

55–56 Find the limit by interpreting the expression as an appropriate derivative. ■

55. $\displaystyle\lim_{h\to 0}\frac{(1 + h)^\pi - 1}{h}$

56. $\displaystyle\lim_{x\to e}\frac{1 - \ln x}{(x - e)\ln x}$

57. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim[f(x) - g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

58. (a) Under what conditions will a limit of the form

$$\lim_{x\to a}[f(x)/g(x)]$$

be an indeterminate form?
(b) If $\lim_{x\to a} g(x) = 0$, must $\lim_{x\to a}[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your answer.

59–62 Evaluate the given limit. ■

59. $\lim_{x \to +\infty} (e^x - x^2)$

60. $\lim_{x \to 1} \sqrt{\dfrac{\ln x}{x^4 - 1}}$

61. $\lim_{x \to 0} \dfrac{x^2 e^x}{\sin^2 3x}$

62. $\lim_{x \to 0} \dfrac{a^x - 1}{x}, \quad a > 0$

63–64 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points. ■

63. $f(x) = 1/e^{x^2}$

64. $f(x) = \tan^{-1} x^2$

65–66 Use any method to find the relative extrema of the function f. ■

65. $f(x) = \ln(1 + x^2)$

66. $f(x) = x^2 e^x$

67–68 In each part, find the absolute minimum m and the absolute maximum M of f on the given interval (if they exist), and state where the absolute extrema occur. ■

67. $f(x) = e^x/x^2;\ (0, +\infty)$

68. $f(x) = x^x;\ (0, +\infty)$

69. Use a graphing utility to estimate the absolute maximum and minimum values of $f(x) = x/2 + \ln(x^2 + 1)$, if any, on the interval $[-4, 0]$, and then use calculus methods to find the exact values.

70. Prove that $x \le \sin^{-1} x$ for all x in $[0, 1]$.

71–74 Evaluate the integrals. ■

71. $\int [x^{-2/3} - 5e^x]\,dx$

72. $\int \left[\dfrac{3}{4x} - \sec^2 x\right] dx$

73. $\int \left[\dfrac{1}{1 + x^2} + \dfrac{2}{\sqrt{1 - x^2}}\right] dx$

74. $\int \left[\dfrac{12}{x\sqrt{x^2 - 1}} + \dfrac{1 - x^4}{1 + x^2}\right] dx$

75–76 Use a calculating utility to find the left endpoint, right endpoint, and midpoint approximations to the area under the curve $y = f(x)$ over the stated interval using $n = 10$ subintervals. ■

75. $y = \ln x;\ [1, 2]$

76. $y = e^x;\ [0, 1]$

77. Interpret the expression as a definite integral over $[0, 1]$, and then evaluate the limit by evaluating the integral.

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} e^{x_k^*} \Delta x_k$$

78. Find the limit

$$\lim_{n \to +\infty} \frac{e^{1/n} + e^{2/n} + e^{3/n} + \cdots + e^{n/n}}{n}$$

by interpreting it as a limit of Riemann sums in which the interval $[0, 1]$ is divided into n subintervals of equal length,

79–80 Find the area under the curve $y = f(x)$ over the stated interval. ■

79. $f(x) = e^x;\ [1, 3]$

80. $f(x) = \dfrac{1}{x};\ [1, e^3]$

81. Solve the initial-value problems.

(a) $\dfrac{dy}{dx} = \cos x - 5e^x,\ y(0) = 0$

(b) $\dfrac{dy}{dx} = xe^{x^2},\ y(0) = 0$

82–84 Evaluate the integrals by making an appropriate substitution. ■

82. $\displaystyle\int_e^{e^2} \frac{dx}{x \ln x}$

83. $\displaystyle\int_0^1 \frac{dx}{\sqrt{e^x}}$

84. $\displaystyle\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2}\,dx$

85. Find the volume of the solid whose base is the region bounded between the curves $y = \sqrt{x}$ and $y = 1/\sqrt{x}$ for $1 \le x \le 4$ and whose cross sections taken perpendicular to the x-axis are squares.

86. Find the average value of $f(x) = e^x + e^{-x}$ over the interval $\left[\ln \frac{1}{2}, \ln 2\right]$.

87. In each part, prove the identity.

(a) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

(b) $\cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)}$

(c) $\sinh \frac{1}{2}x = \pm\sqrt{\frac{1}{2}(\cosh x - 1)}$

88. Show that for any constant a, the function $y = \sinh(ax)$ satisfies the equation $y'' = a^2 y$.

CHAPTER 6 MAKING CONNECTIONS

1. Consider a simple model of radioactive decay. We assume that given any quantity of a radioactive element, the fraction of the quantity that decays over a period of time will be a constant that depends on only the particular element and the length of the time period. We choose a time parameter $-\infty < t < +\infty$ and let $A = A(t)$ denote the amount of the element remaining at time t. We also choose units of measure such that the initial amount of the element is $A(0) = 1$, and we let $b = A(1)$ denote the amount at time $t = 1$. Prove that the function $A(t)$ has the following properties.

(a) $A(-t) = \dfrac{1}{A(t)}$ [*Hint:* For $t > 0$, you can interpret $A(t)$ as the fraction of any given amount that remains after a time period of length t.]

(b) $A(s+t) = A(s) \cdot A(t)$ [*Hint:* First consider positive s and t. For the other cases use the property in part (a).]

(c) If n is any nonzero integer, then

$$A\left(\frac{1}{n}\right) = (A(1))^{1/n} = b^{1/n}$$

(d) If m and n are integers with $n \neq 0$, then

$$A\left(\frac{m}{n}\right) = (A(1))^{m/n} = b^{m/n}$$

(e) Assuming that $A(t)$ is a continuous function of t, then $A(t) = b^t$. [*Hint:* Prove that if two continuous functions agree on the set of rational numbers, then they are equal.]

(f) If we replace the assumption that $A(0) = 1$ by the condition $A(0) = A_0$, prove that $A = A_0 b^t$.

2. Refer to Figure 6.1.5.

(a) Make the substitution $h = 1/x$ and conclude that

$$(1+h)^{1/h} < e < (1-h)^{-1/h} \quad \text{for } h > 0$$

and

$$(1-h)^{-1/h} < e < (1+h)^{1/h} \quad \text{for } h < 0$$

(b) Use the inequalities in part (a) and the Squeezing Theorem to prove that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(c) Explain why the limit in part (b) confirms Figure 6.1.4.

(d) Use the limit in part (b) to prove that

$$\frac{d}{dx}(e^x) = e^x$$

3. Give a convincing geometric argument to show that

$$\int_1^e \ln x \, dx + \int_0^1 e^x \, dx = e$$

RATIONAL FUNCTIONS CONTAINING POWERS OF $a + bu$ IN THE DENOMINATOR

60. $\int \frac{u\,du}{a+bu} = \frac{1}{b^2}[bu - a\ln|a+bu|] + C$

61. $\int \frac{u^2\,du}{a+bu} = \frac{1}{b^3}\left[\frac{1}{2}(a+bu)^2 - 2a(a+bu) + a^2\ln|a+bu|\right] + C$

62. $\int \frac{u\,du}{(a+bu)^2} = \frac{1}{b^2}\left[\frac{a}{a+bu} + \ln|a+bu|\right] + C$

63. $\int \frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\left[bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right] + C$

64. $\int \frac{u\,du}{(a+bu)^3} = \frac{1}{b^2}\left[\frac{a}{2(a+bu)^2} - \frac{1}{a+bu}\right] + C$

65. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

66. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

67. $\int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} + \frac{1}{a^2}\ln\left|\frac{u}{a+bu}\right| + C$

RATIONAL FUNCTIONS CONTAINING $a^2 \pm u^2$ IN THE DENOMINATOR ($a > 0$)

68. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

69. $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C$

70. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$

71. $\int \frac{bu+c}{a^2+u^2}\,du = \frac{b}{2}\ln(a^2+u^2) + \frac{c}{a}\tan^{-1}\frac{u}{a} + C$

INTEGRALS OF $\sqrt{a^2+u^2}$, $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$ AND THEIR RECIPROCALS ($a > 0$)

72. $\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$

73. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

74. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

75. $\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u+\sqrt{u^2+a^2}) + C$

76. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln|u+\sqrt{u^2-a^2}| + C$

77. $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a^2-u^2}$ OR ITS RECIPROCAL

78. $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\sin^{-1}\frac{u}{a} + C$

79. $\int \frac{\sqrt{a^2-u^2}\,du}{u} = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

80. $\int \frac{\sqrt{a^2-u^2}\,du}{u^2} = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$

81. $\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

82. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

83. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u} + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{u^2 \pm a^2}$ OR THEIR RECIPROCALS

84. $\int u\sqrt{u^2+a^2}\,du = \frac{1}{3}(u^2+a^2)^{3/2} + C$

85. $\int u\sqrt{u^2-a^2}\,du = \frac{1}{3}(u^2-a^2)^{3/2} + C$

86. $\int \frac{du}{u\sqrt{u^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

87. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

88. $\int \frac{\sqrt{u^2-a^2}\,du}{u} = \sqrt{u^2-a^2} - a\sec^{-1}\left|\frac{u}{a}\right| + C$

89. $\int \frac{\sqrt{u^2+a^2}\,du}{u} = \sqrt{u^2+a^2} - a\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

90. $\int \frac{du}{u^2\sqrt{u^2\pm a^2}} = \mp\frac{\sqrt{u^2\pm a^2}}{a^2u} + C$

91. $\int u^2\sqrt{u^2+a^2}\,du = \frac{u}{8}(2u^2+a^2)\sqrt{u^2+a^2} - \frac{a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$

92. $\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$

93. $\int \frac{\sqrt{u^2+a^2}}{u^2}\,du = -\frac{\sqrt{u^2+a^2}}{u} + \ln(u+\sqrt{u^2+a^2}) + C$

94. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln|u+\sqrt{u^2-a^2}| + C$

95. $\int \frac{u^2}{\sqrt{u^2+a^2}}\,du = \frac{u}{2}\sqrt{u^2+a^2} - \frac{a^2}{2}\ln(u+\sqrt{u^2+a^2}) + C$

96. $\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

INTEGRALS CONTAINING $(a^2+u^2)^{3/2}$, $(a^2-u^2)^{3/2}$, $(u^2-a^2)^{3/2}$ ($a > 0$)

97. $\int \frac{du}{(a^2-u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$

98. $\int \frac{du}{(u^2\pm a^2)^{3/2}} = \pm\frac{u}{a^2\sqrt{u^2\pm a^2}} + C$

99. $\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}(2u^2-5a^2)\sqrt{a^2-u^2} + \frac{3a^4}{8}\sin^{-1}\frac{u}{a} + C$

100. $\int (u^2+a^2)^{3/2}\,du = \frac{u}{8}(2u^2+5a^2)\sqrt{u^2+a^2} + \frac{3a^4}{8}\ln(u+\sqrt{u^2+a^2}) + C$

101. $\int (u^2-a^2)^{3/2}\,du = \frac{u}{8}(2u^2-5a^2)\sqrt{u^2-a^2} + \frac{3a^4}{8}\ln|u+\sqrt{u^2-a^2}| + C$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{a+bu}$ OR ITS RECIPROCAL

102. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2}+C$

103. $\int u^2\sqrt{a+bu}\,du = \frac{2}{105b^3}(15b^2u^2-12abu+8a^2)(a+bu)^{3/2}+C$

104. $\int u^n\sqrt{a+bu}\,du = \frac{2u^n(a+bu)^{3/2}}{b(2n+3)} - \frac{2an}{b(2n+3)}\int u^{n-1}\sqrt{a+bu}\,du$

105. $\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C$

106. $\int \frac{u^2\,du}{\sqrt{a+bu}} = \frac{2}{15b^3}(3b^2u^2-4abu+8a^2)\sqrt{a+bu}+C$

107. $\int \frac{u^n\,du}{\sqrt{a+bu}} = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2an}{b(2n+1)}\int \frac{u^{n-1}\,du}{\sqrt{a+bu}}$

108. $\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right|+C & (a>0) \\ \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bu}{-a}}+C & (a<0) \end{cases}$

109. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

110. $\int \frac{\sqrt{a+bu}\,du}{u} = 2\sqrt{a+bu}+a\int \frac{du}{u\sqrt{a+bu}}$

111. $\int \frac{\sqrt{a+bu}\,du}{u^n} = -\frac{(a+bu)^{3/2}}{a(n-1)u^{n-1}} - \frac{b(2n-5)}{2a(n-1)}\int \frac{\sqrt{a+bu}\,du}{u^{n-1}}$

POWERS OF u MULTIPLYING OR DIVIDING $\sqrt{2au-u^2}$ OR ITS RECIPROCAL

112. $\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

113. $\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2}+\frac{a^3}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

114. $\int \frac{\sqrt{2au-u^2}\,du}{u} = \sqrt{2au-u^2}+a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

115. $\int \frac{\sqrt{2au-u^2}\,du}{u^2} = -\frac{2\sqrt{2au-u^2}}{u} - \sin^{-1}\left(\frac{u-a}{a}\right)+C$

116. $\int \frac{du}{\sqrt{2au-u^2}} = \sin^{-1}\left(\frac{u-a}{a}\right)+C$

117. $\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au}+C$

118. $\int \frac{u\,du}{\sqrt{2au-u^2}} = -\sqrt{2au-u^2}+a\sin^{-1}\left(\frac{u-a}{a}\right)+C$

119. $\int \frac{u^2\,du}{\sqrt{2au-u^2}} = -\frac{(u+3a)}{2}\sqrt{2au-u^2}+\frac{3a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right)+C$

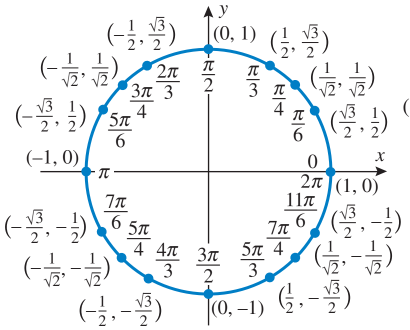
INTEGRALS CONTAINING $(2au-u^2)^{3/2}$

120. $\int \frac{du}{(2au-u^2)^{3/2}} = \frac{u-a}{a^2\sqrt{2au-u^2}}+C$

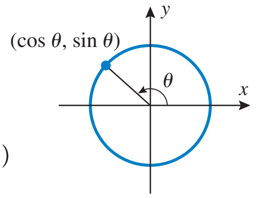
121. $\int \frac{u\,du}{(2au-u^2)^{3/2}} = \frac{u}{a\sqrt{2au-u^2}}+C$

THE WALLIS FORMULA

122. $\int_0^{\pi/2}\sin^n u\,du = \int_0^{\pi/2}\cos^n u\,du = \frac{1\cdot3\cdot5\cdot\cdots\cdot(n-1)}{2\cdot4\cdot6\cdot\cdots\cdot n}\cdot\frac{\pi}{2}\left(\begin{array}{l} n \text{ an even} \\ \text{integer and} \\ n\geq 2 \end{array}\right)$ or $\frac{2\cdot4\cdot6\cdot\cdots\cdot(n-1)}{3\cdot5\cdot7\cdot\cdots\cdot n}\left(\begin{array}{l} n \text{ an odd} \\ \text{integer and} \\ n\geq 3 \end{array}\right)$



TRIGONOMETRY REVIEW



PYTHAGOREAN IDENTITIES

$\sin^2\theta+\cos^2\theta=1$ $\qquad$ $\tan^2\theta+1=\sec^2\theta$ $\qquad$ $1+\cot^2\theta=\csc^2\theta$

SIGN IDENTITIES

$\sin(-\theta)=-\sin\theta$ $\qquad$ $\cos(-\theta)=\cos\theta$ $\qquad$ $\tan(-\theta)=-\tan\theta$

$\csc(-\theta)=-\csc\theta$ $\qquad$ $\sec(-\theta)=\sec\theta$ $\qquad$ $\cot(-\theta)=-\cot\theta$

COMPLEMENT IDENTITIES

$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ $\qquad$ $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ $\qquad$ $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$

$\csc\left(\frac{\pi}{2}-\theta\right)=\sec\theta$ $\qquad$ $\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$ $\qquad$ $\cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$

SUPPLEMENT IDENTITIES

$\sin(\pi-\theta)=\sin\theta$ $\qquad$ $\cos(\pi-\theta)=-\cos\theta$ $\qquad$ $\tan(\pi-\theta)=-\tan\theta$

$\csc(\pi-\theta)=\csc\theta$ $\qquad$ $\sec(\pi-\theta)=-\sec\theta$ $\qquad$ $\cot(\pi-\theta)=-\cot\theta$

$\sin(\pi+\theta)=-\sin\theta$ $\qquad$ $\cos(\pi+\theta)=-\cos\theta$ $\qquad$ $\tan(\pi+\theta)=\tan\theta$

$\csc(\pi+\theta)=-\csc\theta$ $\qquad$ $\sec(\pi+\theta)=-\sec\theta$ $\qquad$ $\cot(\pi+\theta)=\cot\theta$

ADDITION FORMULAS

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

DOUBLE-ANGLE FORMULAS

$\sin 2\alpha=2\sin\alpha\cos\alpha$ $\qquad$ $\cos 2\alpha=2\cos^2\alpha-1$

$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$ $\qquad$ $\cos 2\alpha=1-2\sin^2\alpha$

HALF-ANGLE FORMULAS

$\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}$ $\qquad$ $\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}$