South Valley University. Faculty Of Commerce.

Dep. Of Quantitative Techniques.

"Business Mathematics "

"An Introduction"

For the 1st Year Students

English Section

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Contents

Chapter (1)

Linear Programming Models

(1-1): Introduction:

The main objective of this chapter is to provide an overall view of operations research (abbreviated OR) and its origin as well as the fundamental phases in an OR study. Although it is not possible to detail all the phases of OR here, the objective is to present a unified treatment of the subject that can be used as a general guideline for solving OR problems. Therefore, this chapter provides an overview of OR applied to problem formulation and its solution.

(1-2): Development of Operations Research:

During the world war II, the military management in U.K. (or England) called on a teamwork of scientists to study the strategic and tactical problems associated with air and land defense of the country. Their objective was to determine the most effective utilization of limited military resources. The applications included among others, studies of the way to use the newly invented radar and of effectiveness of new types of bombers. The establishment of this scientific teamwork marked the first formal operations research activity.

The name of operation research was apparently coined because the teamwork was dealing with research on military operations. Since its birth, this new decision-making field has been characterized by the use of scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.

The encouraging results achieved by the British operations research teams motivated the United States military management to start similar activities. Many successful application of the U.S. teamwork's included the study of complex logistical problems, the invention of new flight patterns, the planning of sea mining, and the effective utilization of electronic equipment.

Following the war, the success of this military teamworks attracted the attention of industrial managers who were seeking solutions to their problems, which were becoming more acute because of the introduction of functional specialization into business organizations. Despite the fact that specialized functions are established primarily to serve the overall objective of the organization, the individual objectives of these functions may not always be consistent with the goals of the organization. This has resulted in complex decision problems that ultimately have

forced business organizations to seek the utilization of the effective tools of OR.

Although Great Britain is credited with the initiation of OR as a new discipline, the leadership in the rapidly growing field was soon taken over by the United States. The first widely accepted mathematical technique in the field, called the simplex method of linear programming, was developed in 1947 by the American mathematician George B. Dantzig. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in both academic institutions and industry.

Today, the impact of OR can be felt in many areas. This is indicated by the number of academic institutions offering courses in this subject at all degree levels.

Many management consulting firms are currently engaged in OR activities. These activities have gone beyond military and business applications to include hospitals, financial institutions, libraries, city planning, transportation systems and even crime investigation studies.

(1.3): The Concept of OR:

An OR study consists in building a model of the physical situation. An OR model is defined as simplified

representation of a real-life system. This system may already be in existence or may still be an idea awaiting execution. In the first case, the model's objective is to analyze the behavior of the system in order to improve its performance. While, in the second case, the objective is to identify the best structure of the future system. The complexity of a real system results from the very large number of elements or variables controlling the behavior of the system.

A teamwork of scientists in the branch of OR defined OR as that scientific branch which concerned with the model construction, its formulation, how it can be found its optimum technique for its solution and how can use these techniques in order to answer for the quarry about what will happen if there is an specific condition (or a set of conditions) will be hold (or not), which is called the sensitivity analysis. Others defined that OR is the branch which concerned with the cases studies or the applied statistics.

From the preceding definitions, the branch of OR depends on a set of basic characteristics shown as follows:

1- **Applied the scientific methods for treating the different problems.**

2- Construct a mathematical model for solving the problems and take the optimal decision from the feasible solutions.

3- Determine the model which can be used in the sensitivity analysis.

(1.3.1): Types of OR Models:

Although it is not possible to present fixed rules about how a model is constructed, it may be helpful to present ideas about possible OR model types, their general structures, and their general characteristics.

The most important type of OR model is the symbolic or mathematical model. In formulating this type, one assumes that all relevant variables are quantifiable. Thus mathematical symbols are used to represent variables, which are then related by the appropriate mathematical functions to describe the behavior of the system. The solution of the model is then achieved by appropriate mathematical manipulation as will be shown in the succeeding chapters.

Most of OR analysts identify the name operations research primarily with mathematical models. The reason may be that such models are amenable to mathematical analysis, which usually makes it possible to find the "best"

solution by means of convenient mathematical tools. It is not surprising then that most of the attention in OR has been directed toward the development of mathematical models.

In addition to mathematical models, simulation and heuristic models are used. Simulation models "imitate" the behavior of a specific system over a period of time. This is achieved by specifying a number of events which are point in time whose occurrence signifies that important information pertaining to the behavior of the system can be gathered. Once such events are defined, it is necessary to pay attention to the system only when an event occurs. The information yielding measures of performance for the system is accumulated as statistical observations, which are updated as each event takes place. Naturally, the simulation model is not as convenient as the (successful) mathematical models which yield a general solution to the problem.

While mathematical seek the determination of the best or the optimum solution, sometimes the mathematical formulation may be too complex to allow an exact solution. Even, if the optimum solution can be attained eventually, the required computation may be impractically long. In this case, heuristics can be rules that given a current solution to the model, allow the determination of an improved solution.

(1.3.2.): Structure of Mathematical Models:

Mathematical model includes three basic sets of elements stated as follows:

1- **Decision variables and parameters: The decision variables are the unknowns to be determined from the solution of the model. The parameters represent the controlled variables of the system. For example, the production level represents a decision variable. Example of the parameters in this case include the production and consumption rates of the stocked item. In general, the parameters of the model may be deterministic or probabilistic.**

2- **Constraints or restrictions: To account for the physical limitations of the system, the model must include constraints which limit the decision variables to their feasible (or permissible) values. This is usually expressed in the form of constraining mathematical functions. For example, let x_1 and x_2 be the number of units to be produced of two products (decision variables) and let a_1 and a_2 be their respective per unit requirements of raw material (parameters). If the total amount available of this raw material is A, the corresponding constraint function is given by:**

$a_1x_1 + a_2x_2 \leq A$.

3- **<u>Objective Function:</u> This defines the measure of effective-ness of the system as a mathematical function of its decision variables. For example, if the objective of the system is to maximize the total profit, the objective function must specify the profit in terms of the decision variables. In general, the *optimum* solution to the model is obtained when the corresponding values of the decision variables yield the best value of the objective function while satisfying all the constraints. This means that the objective function acts as an indicator for the achievement of the optimum solution.**

Mathematical models in operations research may be viewed generally as determining the values of the decision variables x_J for $j = 1, 2, \ldots, n$, which will

$$\text{Optimize} \quad x_0 = f(x_1, \ldots, x_n)$$

Subject to

$$g_i(x_1, \ldots, x_n) \leq b_i \;, \quad i = 1,2,\ldots,m$$

$$x_J \geq 0, \quad j = 1,2, \ldots, n$$

The function f is the objective function, while $g_i \leq b_i$ represents the i^{th} constraint, where b_i is a known constant.

The constraints $x_i \geq 0$ are called the non-negativity constraints, which restrict the variables to zero or positive values only. In most real-life systems, the non-negativity constraints appear to be a natural requirement.

***Model Optimization:**

The above discussion indicates that a mathematical model seeks to "optimize" a given objective function subject to a set of constraints. Optimization is generally taken to signify the "maximization" or "minimization" of the objective function. But this is about the extent to which the word "optimization" goes in unifying mathematical models. By this it is meant that two analysts working on the same problem independently may yield two different models with different objective criteria. For example, analyst A may prefer to maximize profit, while Analyst B may rightly prefer to minimize cost. The two criteria are not equivalent in the sense that with the same constraints the two models may not produce the same optimum solution. This can be made clear by realizing that, although cost may be under the immediate control of the organization in which the study is made, profit could be effected by uncontrollable factors such as the market situation dictated by competitors.

The main conclusion at this point is that "the" optimum solution of a model is the best only relative to that model. In other words, one must not think that this optimum is *the* best for the problem under consideration. Rather, it is the *best* only if the specified criterion can be justified as a true representation of the goals of the entire organization in which the problem exists.

(1.4): Phases of Operations Research Study:

An OR study cannot be conducted and controlled by the OR analyst alone. Although he may be the expert on modeling and model solution techniques, he cannot possibly be an expert in al the areas where OR problems arise. Consequently, an OR team should include members of the organization directly responsible for the functions in which the problem exists as well as for the execution and implementation of the recommended solution. In other words, an OR analyst commits a grave mistake by assuming that he can solve problems without the cooperation of those who will implement his recommendations.

The major phases through which the OR team would proceed in order to effect an OR study include

1-Definition of the problem.

2-Construction of the model.

3-Solution of the model.

4-Validation of the model.

5-Implementation of the final results.

Although the above sequence is by no means standard, it seems generally acceptable. Except for the "model solution" phase, which is based generally on well-developed techniques, the remaining phases do not seem to follow fixed rules. This stems from the fact that the procedures for these phases depend on the type of problem under investigation, and the operating enviro-nment in which it exists. In this respect, an operations research team would be guided in the study principally by the different professional experiences of its members rather than by fixed rules.

(1.5): Linear Programming Models and its Applications:

Linear programming is a class of mathematical programming models concerned with the efficient allocation of limited resources to known activities with the objective of meeting a desired goal (such as maximizing profit or minimizing cost). The distinct characteristic of linear programming models is that the functions representing the objective and the constraints are linear.

This chapter introduces the reader to some of the applications of linear programming. The examples are taken from actual applications in different fields in order to illustrate the diverse uses of this type of model. As stated in Chapter 1, a presentation of the procedure for gathering data for the model will take the discussion far a field. Instead, the analysis will concentrate on how the assumption of linearity can be justified.

The linearity of some models can be justified based on the physical properties of the problem; other models, which in the direct sense are nonlinear, can be linearlized by the proper use of mathematical transformations. Examples of these types will be presented in the next section. Formulation of the theoretical problem in a linear programming models means that we have to determine the following basics:

1-Decision variables and parameters.

2-Objective function.

3-Constraints and nonnegativity constraints.

These basic components may be stated in matrix form as follows: Determine the decision variable x_1, x_2, …., x_n in which will optimize

$f(x) = f(x_1, x_2, \ldots, x_n)$ **subject to:** $\underline{A}\ \underline{x} \ \underline{\le}\ \underline{b}\ ,\ \underline{x}\ \underline{\ge}\ 0$

Examples of Linear Programming Applications:

The applications in the this chapter are excerpted from the following areas:

1-Production planning.

2-Feed mix.

3-Stock cutting or slitting.

4-Water-quality management.

5-Oil drilling and production.

6-Assembly balancing.

7-Inventory.

***Example (1): (Production Planning):**

Three products are processed through three different operations. The times (in minutes) required per unit of each product, the daily capacity of the operations (in minutes per day) and the profit per unit sold of each product (in dollars) are as follows:

Operation	Time per unit (minutes)			Operation capacity (minutes/day)
	Product 1	Product 2	Product 3	
1	1	2	1	430
2	3	0	2	460
3	1	4	0	420
Profit/unit ($)	3	2	5	

The zero times indicate that the product does not require the given operation. It is assumed that all units produced are sold. Moreover, the given profits per unit are net values that result after all pertinent expenses are deducted. The goal of the model is to determine the optimum daily production for the three products that maximizes profit.

As mentioned in Chapter 1, the main elements of a mathematical model are (1) the variables or unknowns, (2) the objective function, and (3) the constraints. The variables are immediately identified as the daily number of units to be manufactured of each product. Let x_1, x_2, and x_3 be the

number of daily units produced of products 1, 2, and 3. Because of the assumption that all units produced are sold, the total profit x_0 (in dollars) for the three products is:

$$x_0 = 3x_1 + 2x_2 + 5x_3.$$

The constraints of the problem must ensure that the total processing time required by all produced units does not exceed the daily capacity of each operation. These are expressed as:

Operation 1: $1x_1 + 2x_2 + 1x_3 \leq 430$

Operation 2: $3x_1 + 0x_2 + 2x_3 \leq 460$

Operation 3: $1x_1 + 4x_2 + 0x_3 \leq 420$

Because it is nonsensical to produce negative quantities, the additional *non-negativity constraints* $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$ must be added.

Some operations research users have a tendency to replace the inequality ($\leq$) in the "operations constraints" by a strict equation (=). The justification is that it is better to use all available resources than to "waste" part of it. This reasoning does not hold since the use of ($\leq$) automatically implies (=). Thus, if the *optimum* solution requires that all constraints be satisfied exactly, the inequalities ($\leq$) still

allow this. In other words, the strict equalities should not be imposed unless the problem requires that all operations must work to full capacity. This is completely different from simply stipulating that the capacity of each operation should not be exceeded.

The linear programming model is now summarized as follows.

$$\text{Maximize } x_0 = 3x_1 + 2x_2 + 5x_3$$

Subject to

$$x_1 + 2x_2 + x_3 \leq 430$$

$$2x_1 + 2x_3 \leq 460$$

$$x_1 + 4x_2 \leq 420$$

$$x_1, \quad x_2, \quad x_3 \geq 0$$

What makes the above problem fit a linear programming model? Several implicit assumptions allow (1) imposing constant proportionality between the number of units of a product and its total contribution to the objective function (or its usage of each operation's time), and (2) adding directly the profit contributions (or the time requirements) of each product to obtain the total profit of the system (or the total usage of a given operation's time).

Suppose, for example, a price break is allowed so that if the size of an order exceeds a certain quantity, the sale price (and hence profit) per unit decreases by a fixed amount. In this case, the constant proportionality assumption built in the objective function is invalid. Another example is that if defective pieces are reworked on the same operation, it is no longer true that the time requirement per unit is constant for each operation. A third example is that the volume of sales for the three products may be interdependent. Unless the relationships between volumes of sales are linear, the direct addition of the individual profit contributions as given in the above objective function will be unacceptable. A specific illustration is as follows. Let y and z be the sales volumes of two competing products where an increase in the sales volume of one product adversely affects the sales volume of the other. Mathematically, this means that y is proportional to $1/z$. If b is the proportionality constant, then $y = b/z$, or $yz = b$, which is not a linear constraint.

The above discussion suggests situations where the linearity assumption is not justified. Some nonlinearities, however, may be "approximated" by linear functions. For example, the nonlinearities created by the quantity discount may be approximated by a linear function.

<u>Example (2):</u>

A company produce three products 1, 2 and 3 by using three different raw materials A, B and C. The following table represents the required per unit of each product, the daily capacity of each material (A and B in kilograms, C in hours) and price per unit sold of each product (in dollars) are as follows:

Product / Raw material	K.G and hours per unit produced			Available capacity
	1	2	3	
A	5	3	7	2100 K.G
B	4	2	5	1600 K.G
C	3	2	4	1700 Hour
Sold price per unit produced	51	32	52	

<u>In addition you have the following data:</u>

- **The unit cost for each of the raw materials A and B are 4 and 3$ respectively, and the wage rate for each hour**

in the C operation is one dollar. Beside that, each unit produced needs 5, 6 and 3$ respectively as a tips.

- **The demand units for marketing the three products are 100, 150 and 200 units respectively.**
- **The number of units produced from the product A must be twice of B.**
- **The fixed cost of this company is 2500$.**

Required: Formulate the problem as a linear programming problem.

Solution:

In order to formulate this theoretical problem as a LPM, we have to firstly suppose a set of decision variables as follows:

***Decision variable and parameters:**

Assume that x_1 , x_2 and x_3 are the three decision variables that denote the number of unit must be produced from the three different products respectively A, B and C, then the following table represents the different parameters:

Product / Raw mat.	1	2	3	Cost per unit of raw mat.	Capacity
A B C	5 4 3	3 2 2	7 5 4	4 $ / 1K.G 3 $ / 1K.G 1 $ / 1 H	2100 1600 1700
Var. cost (tips)	5	6	3		
Var. cost (raw mat.)	4(5)+3(4) +1(3) =35	4(3)+3(2) +1(2) =20	4(7)+3(5) +1(4) =47		
Total var. cost	35+5= 40	26	50		
Price unite	51	32	52		
Profit unite	11	6	2	Maximization	

Then, we have to find the values for x_1 , x_2 and x_3 by which make the total net profit function:

$$f(x) = 11x_1 + 6x_2 + 2x_3 - 2500 \qquad \text{Maximization}$$

Subject to:

1) $5x_1 + 3x_2 + 7x_3 \leq 2100$
2) $4x_1 + 2x_2 + 5x_3 \leq 1600$
3) $3x_1 + 2x_2 + 4x_3 \leq 1700$
4) $x_1 \geq 100$
   $x_2 \geq 150$
   $x_3 \geq 200$
5) $x_1 - 2x_2 = 0$
6) $x_1 \geq 0$
   $x_2 \geq 0$
   $x_3 \geq 0$

(1.6): Definitions of linear programming forms:

The real-life examples in the preceding section show that a linear program may be of the maximization or minimization type. The constraints may be of the type ($\leq$) , (=), or ($\geq$) and the variables may be nonnegative or unrestricted in sign. A general linear programming model thus is usually defined as follows.

Maximize or minimize $x_0 = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Subject to*:

$$a_{11}x + a_{12}x_2 + \dots + a_{1n}x_n \ (\leq, =, \text{or} \geq)\ b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \ (\leq, =, \text{or} \geq)\ b_2$$

:

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \ (\leq, =, \text{or} \geq)\ b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

Where c_j, b_i , and a_{ij} (i= 1,2, …, m; j = 1,2, …, n) are constants determined from the technology or the problem, and x_j are the decision variables. Only one sign ($\leq$, = , or $\geq$) holds for each constraint. Although all variables are declared nonnegative, the preceding section shows that every unrestricted variable can be converted equivalently to nonnegative variables. The non-negative restriction is essential for the development of the solution method for linear programming.

Linear programming models often represent "allocation" problems in which limited resources are allocated to a number of activities. In terms of the above formulation, the coefficients c_i, a_{ij}, and b_i are interpreted physically as follows.

If b_i is the available amount of resource *i*, then a_{ij} is the amount of resource *i* that must be allocated to each unit of activity *j*. The "worth" per unit of activity *j* is equal to c_j.

<u>Note that:</u>

If there is a constraint in the absolute value form, i.e., we have an constraint as the following form:

$$| ax_1 \pm bx_2 | < C$$

Then we have the two forms:

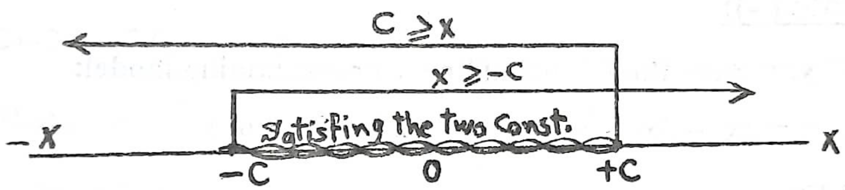
1- If $| ax_1 \pm bx_2 | \leq c$

i.e., $-c \leq ax_1 \pm bx_2 \leq c$

put $y = a_1x \pm bx_2$

Then: $-c \leq y \leq c$

Graphically (As we will show in the following chapter) we can imagine the follow sketch:



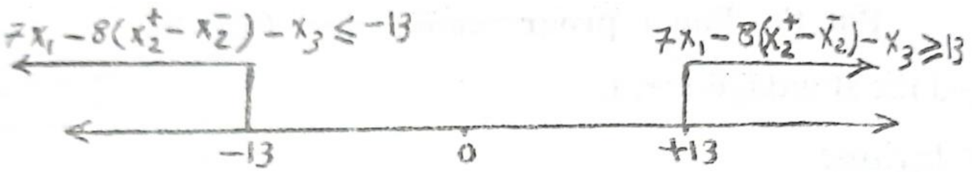
The intersection between the two arrows which are represent the area by which satisfied the two sub constraints $y = ax_1 \pm bx_2 \geq - c$ and $y = ax_1 \pm bx_2 \leq c$ means that we must take the two constraints resulted from the constraint of the absolute value in the solution.

2- If $| ax_1 \pm bx_2 | \geq c$ meaning that

$-c \geq ax_1 \pm bx_2 \geq c$

i.e., $-c \geq y \geq c$, where $y = ax_1 \pm bx_2$

Graphically:



Since there is no intersection between the two arrows, (i.e., there is no feasible solution between the two arrows), then, we have to take only one of the two sub constraints $y = ax_1 \pm bx_2 \geq c$ or $y = ax_1 \pm bx_2 \geq c$ in the solution.

Exercises Or Problems

1-Four products are processed successively on two machines. The manufacturing times in hours per unit of each product are tabulated below for the two machines.

Machine	Time per unit (hours)			
	Product 1	Product 2	Product 3	Product 4
1	2	3	4	2
2	3	2	1	2

The total cost of producing a unit of each product is based directly on the machine time. Assume the cost per hour for machines 1 and 2 are \$10 and \$15. The total hours budgeted for all the products on machines 1 and 2 are 500 and 380. If the sales price per unit for products 1,2,3, and 4 are \$65, \$70, \$55 and \$45, formulate the problem as a linear programming model to maximize total net profit.

2-A company produces two types of cowboy hats. Each hat of the first type requires twice as much labor time as the

second type. If all hats are of the second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second types to 150 and 250 hats. Assume the profits per hat are $8 for type 1 and $5 for type 2. Determine the number of hats to be produced of each type in order to maximize profit.

3-A manufacturer produces three models. (I, II, and III) of a certain product. He uses two types of raw material (A and B) of which 2000 and 3000 units are available, respectively. The raw material requirements per unit of the three models are given below:

Raw material	Requirements per unit of given model		
A	2	3	5
B	4	2	7

The labor time for each unit of Model I is twice that of Model II and three times that of Model III. The entire labor force of the factory can produce the equivalent of 700 units of Model 1. A market survey indicates that the minimum demand of the three models are 200, 200, and 150 units,

respectively. However, the ratios of the number of units produced must be equal to 3: 2: 5. Assume that the profit per unit of Models I, II, and III are 30, 20, and 50 dollars. Formulate the problem as a linear programming model in order to determine the number of units of each product which will maximize profit.

Solving the Linear Programming Models

The purpose of this chapter is to present the procedure for solving the linear programming models. The presentation starts with a graphical solution of two-decision variables problem, which is subsequently used to develop an understanding of the algebraic procedure for solving linear programs which called the simplex procedure.

(2-1): Graphical Solution of Two-Variable Linear Programs:

The purpose of the graphical solution is not to provide a practical method for solving the linear programming models, since practical problems usually include a large number of decision variables. Instead, the graphical method demonstrates the basic concepts for developing the general algebraic technique (simplex methods) for linear programs with more than two variables.

The graphical solution is based on how can we graph of either linear equations or inequalities, and determine the solution space for each of them. In summary, the graphical solution passes through the following steps:-

1-Skech graphically the coordinates of any two points for each constraints, then plot the feasible solution space by which enclosed by all constraints in the (x_1, x_2) plane. The

non-negativity constraints specify that the feasible solutions must lie in the first quadrant defined by $x_1 \geq 0$ and $x_2 \geq 0$. Note that each of the constraints which will be plotted with ($\leq$ or $\geq$) replaced by (=), thus yielding simple straight-line equations. The region in which each constraint holds is indicated by an arrow on its associated straight line. After, determining the resulting feasible solution space (area), if any constraint can be deleted without effecting the solution space, then it is called a redundant constraint. Every point within or on the boundaries of the solution space satisfies all the constraints is called a feasible solution. And every corner in the solution space is called a basic feasible solution. (extreme point).

2-Skech graphically the coordinates of any two point for the linear function of the objective function f(x) or $x_0 = 0$, then the optimum solution is that point in the solution space which yields the largest value (in case of maximum) or the lowest value (in case of minimum) of *f*(*x*) or x_0. The optimum solution can be determined by moving the line of *f*(*x*) or $x_0 = 0$ parallel to itself in the direction of the solution space, then it will be tangent the feasible solution space through the first or the latest point in the solution space which determined the minimum or maximum value of *f*(*x*) or x_0 since the two coefficients for the two decision variables are

positive coefficients, then we have to determine the coordinates of these points. Substituting these values into the objective function gives the optimum solution.

<u>*Revision of graphical representation for the linear equations and inequalities:</u>

In this section we will represent how can we graph the linear equations and inequalities:

<u>a)Linear equation:</u>

A linear equation in two unknowns x_1 and x_2 has the standard form:

$Ax_1 + bx_2 = C$

Where a, b and c are real numbers.

Linear equations are first-degree equations. It is better to discuss first how can we represent graphically the linear equations. Graphing straight line of a linear equation of the form:

$ax_1 + bx_2 = c$ where a, b and $C \neq 0$ intersects the x_1 axis in (c/a) units from the origin, and similarly intersects the x_2 axis in (c/b) units from the origin. When the linear equation is in the form $ax_1 + bx_2 = 0$, then the straight line for this

equation passes through the origin point (0,0) and has no intercepts at all x_1 and x_2 axis.

<u>Example(1):</u>

If you have the linear equation:

$2x_1 + 3x_3 = 6$

then it can be put in the intersects by dividing both sides by 6 then we have the following form:

$(2x_1) / 6 + (3x_2) / 6 = 1$ or $x_1 / (6/2) + x_2 / (6/3) = 1$

Then $x_1 /3 + x_2/2 = 1$

Which mean that the linear equation $2x_1 + 3x_2 = 6$ is intersects the two axis x_1 and x_2 in 3 and 2 units from the origin point as it be shown in figure (1)

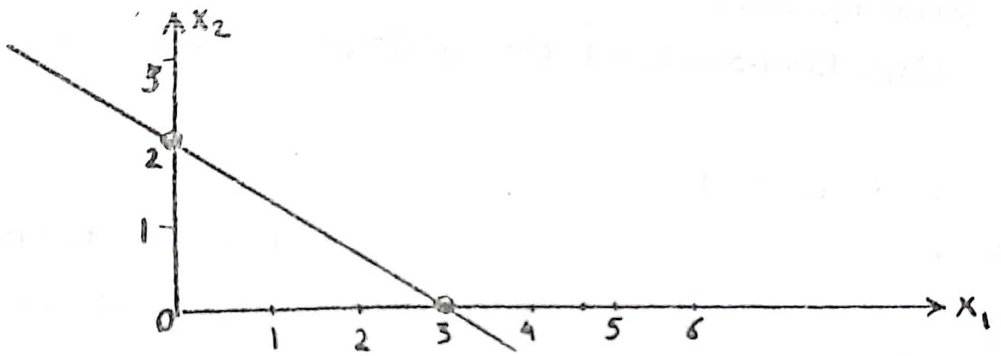


Figure (1)

Figure (1)

The preceding method for graphing the straight line for the linear equation can be achieved in another way by determining the intercepts of an equation simply, we first setting x_1 = 0 and solve the equation for x_2 and second setting x_2 = 0 and solve the equation for x_1.

In the preceding example, the following table summarizes the two intercepts from the origin point.

X_1	0	3
X_2	2	0

Note that , any point lies in the strait line passes through the two points (0 , 2) , (3 , 0) will be satisfied the linear equation:

$2x_1 + 3x_3 = 6$.

<u>Example (2):</u>

Graph a straight line using the two distinct intercepts for the following linear equation:

$3x_1 – 5x = 15$ by using two different methods.

<u>Solution:</u>

We can graph the straight line by using the two methods of intercepts as follows:

1-Since $3x_1 - 5x_2 = 15$, then dividing the two sides of this equation by (15) then we have:

$$(3x_1) / 15 + (-5x_2)/15 = 1 \quad \text{Or} \quad x_1/ (15/3) + x_2\ (15/(-5)) = 1$$

i.e.,

$$x_1 / 5 + x_2 /-3 = 1$$

which means that the equation $3x_1 - 5x_2 = 15$ intercepts the two axis x_1 and x_2 in 5 and (-3) units respectively as it be shown in figure (2).

2-The two intercepts can be achieved in another way by setting $x_1 = 0$ and solving for x_2, the: $-5x_2 = 15$

i.e., $x_2 = -3$

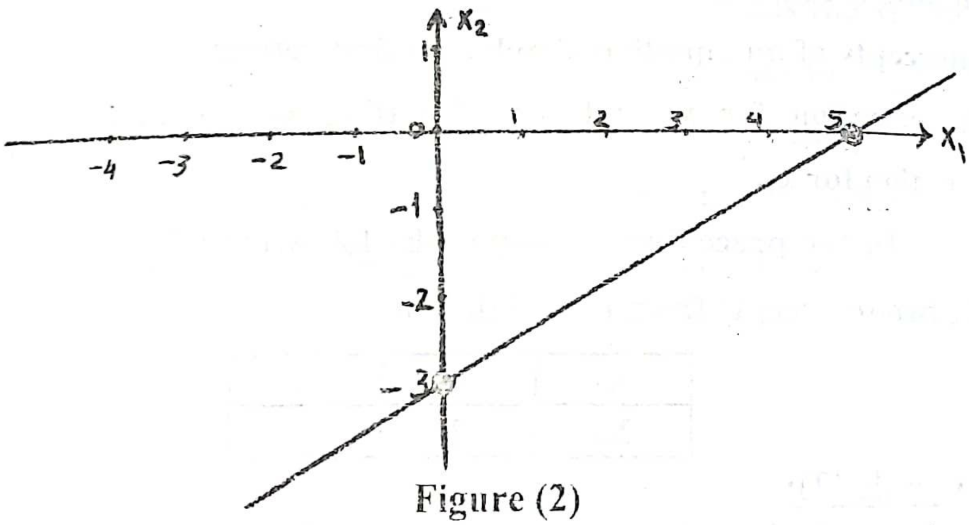


Figure (2)

Figure (2)

Also, by setting $x_2 = 0$ in the equation and solving for x_1 , then we have:

$3x_1 = 15$, i.e., $x_1 = 5$

The following table summarizes the preceding results:

X_1	0	5
X_2	-3	0

Finally, the linear equation (=) means that each point on the graph line only will satisfy the equation.

<u>Example (3):</u>

Use the two intercepts point to graph $2x_1 + 5x_2 = 10$

<u>Solution:</u>

We find the two intercepts by first setting $x_1 = 0$ and solving for x_2, then setting $x_2 = 0$ and solving for x_1. when $x_1 = 0$, we get $x_2 = 2$ and when $x_2 = 0$ we get $x_1 = 5$, the following table summarizes these two points:

X_1	0	5
X_2	2	0

Or,

Since $2x_1 + 5x_2 = 10$

Then $(2x_1) / 10 + (5x_2)/ 10 = 1$

i.e.,

$$x_1 / (10/2) + x_2 (10/5) = 1$$

i.e.,

$$(x_1) / 5 + (x_2) / 2 = 1$$

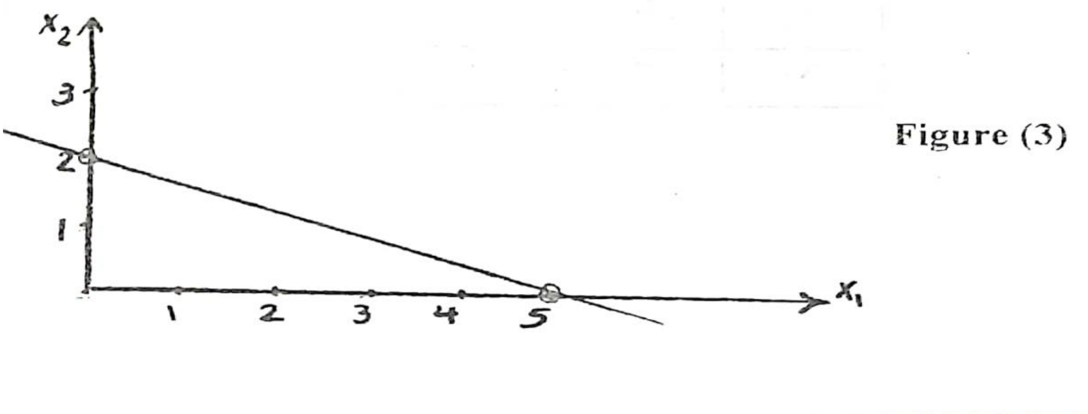


Figure (3)

i.e., the graph line for this equation intercepts the two axis x_1 and x_2 in 2 and 5 units from the origin respectively as it be shown in figure (3)

Note that each point on the graph line for the equation only satisfies the equation.

<u>Remark:</u>

If you have the following equation $ax_1 + bx_2 = 0$, then the graph line will passes through the origin point (0 , 0).

Also this line passes through the coefficients for the two variable with simple different, i.e., if you set that x_1 will equal to (b) the coefficient of the second unknown variable (x_2) then, the value of x_2 will be equal the same coefficient of x_1 but with opposite sign. i.e., if you set that $x_1 = b$ then the value of x_2 must be equal to (-a).

<u>Example (4):</u>

Graph the line for the equation: $2x_1 + 3x_2 = 0$

<u>Solution:</u>

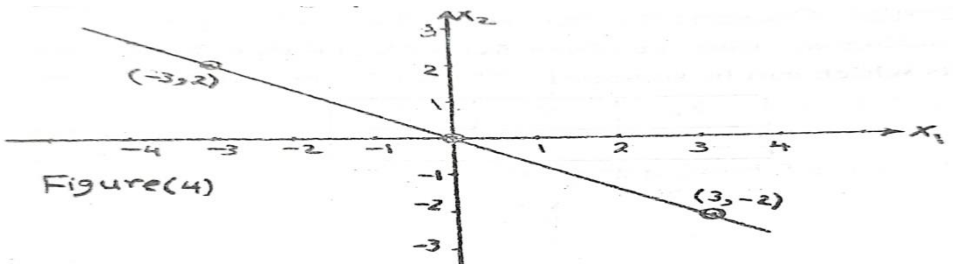
Since the equation in the form ax + bx = 0, then it will passes through the origin point (0 , 0) and (3 , -2) i.e., we have the following table

X_1	0	3
X_2	0	-2

Or

X_1	0	-3
X_2	0	2

Then we have the graph line as it be shown in figure (4)



Figure(4)

(b): Inequalities:

Graphing the linear inequalities in the form:

$Ax_1 + bx_2 \leq$ or $\geq$ C , where a, b and $C \neq 0$ are real numbers by converting the inequality into a linear equation in the form $ax_1 + 6x_2 = C$ and then we have to determine the two intercepts from the origin as it be shown in the preceding examples.

After plotting the graph line for the inequality, then we can determine the direction which will satisfy this inequality by using the any point , and for simplicity, we will use the origin point for determining its direction. The following examples will represent this procedure.

<u>Example (5):</u>

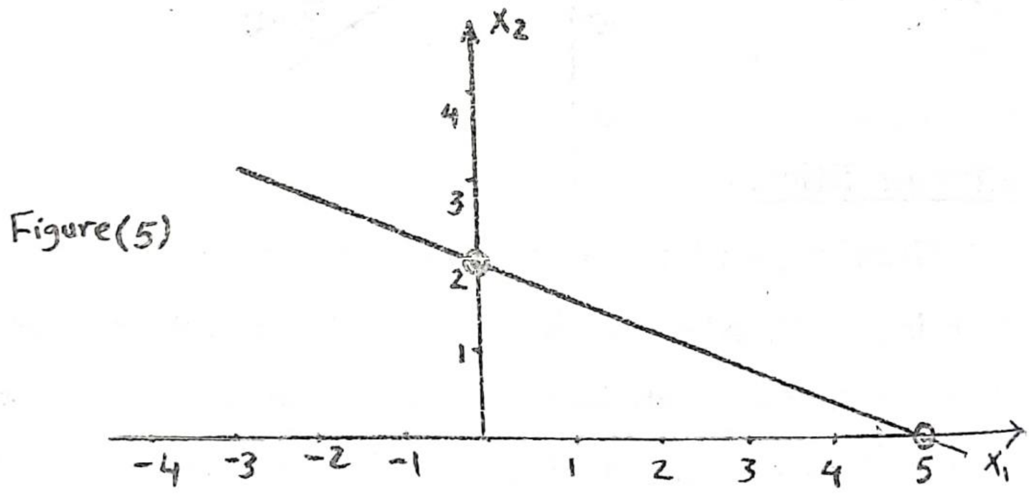
Determine graphically the region in which satisfy the following inequality: $2x_1 + 5x_2 \leq 10$

<u>Solution:</u>

<u>Firstly</u>: Convert the inequality $2x_1 + 5x_2 \leq 10$ to its corresponding equation. i.e., $2x_1 + 5x_2 = 10$, and then find the two intercepts which can be summarized by the following table:

X_1	0	5
X_2	2	0

Then we have the following graph (figure 5):



<u>Secondly</u>: in order to determine the area by which satisfy the inequality $2x_1 + 5x_2 \le 10$, we can use any point in the two dimension (x_1 , x_2) ~ plane. For simplicity, we use the origin point (0 , 0) for achieving this task. Then, we have to put

$x_1=0$ and $x_2 = 0$ in both sides for this inequality.

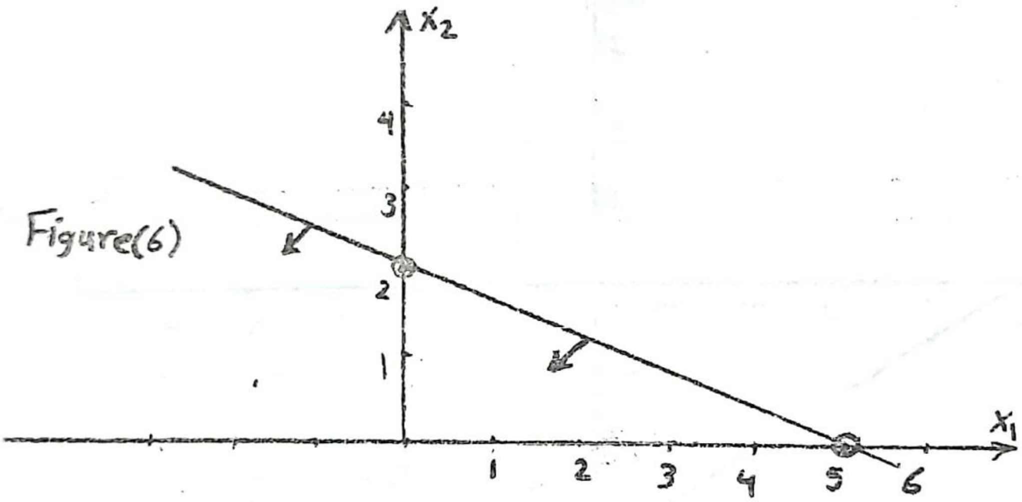
i.e.,

2 (0) + 5(0) [ ] 10

i.e., 0 [ ] 10

then put the suitable relation between the two sides. Then, we have $0 \le 10$, i.e., the origin point lies in the area which satisfy the relation less than (<), i.e., any point in the area which the origin point exist will satisfy this inequalities.

Then we can determine the region in which this inequality holds is indicated by an arrow on its associated straight line as it be shown in Figure (6)



Remark:

If the relation of the inequality ($\leq$ or $\geq$) have the same relation for substituting by the origin point (0 , 0) in this inequality, then the origin point in the same region for the inequality. Conversely; if the relation is opposite, then the origin point is in the converse region for the region which satisfy the inequality.

Example (6):

Determine graphically the region in which satisfy the following inequality: $3x_1 + 5x_2 \leq -15$

Solution:

Firstly convert the inequality $3x_1 + 5x_2 \leq -15$ to an equation $3x_1 + 5x_2 = -15$, then we have the following two points (intercepts)

X_1	0	-5
X_2	3	0

Graphing the two points, then we have the following

Figure (7):

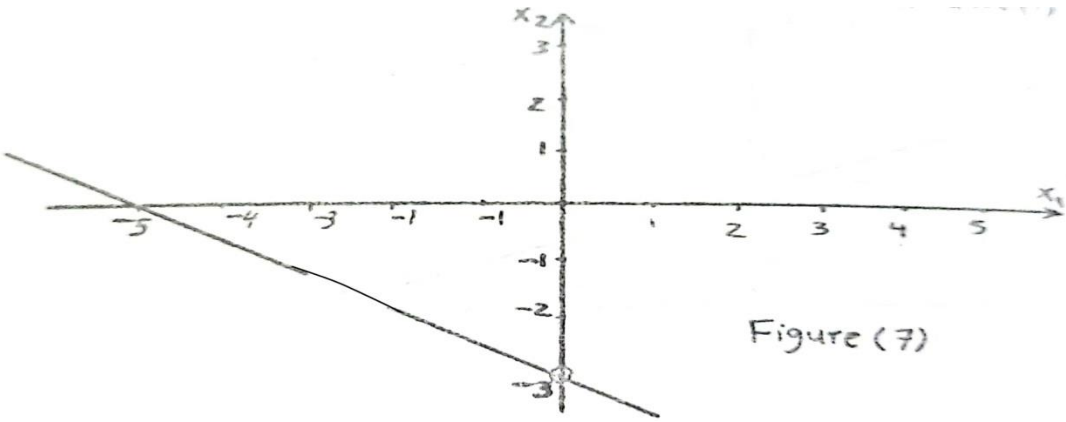


Figure (7)

Now, in order to determine the region by which the inequality $3x_1 + 5x_2 \leq -15$, then we have to substitute the origin point

(0 , 0) in the inequality, then we have :

$3 (0) + 5 (0)$ [ ] -15

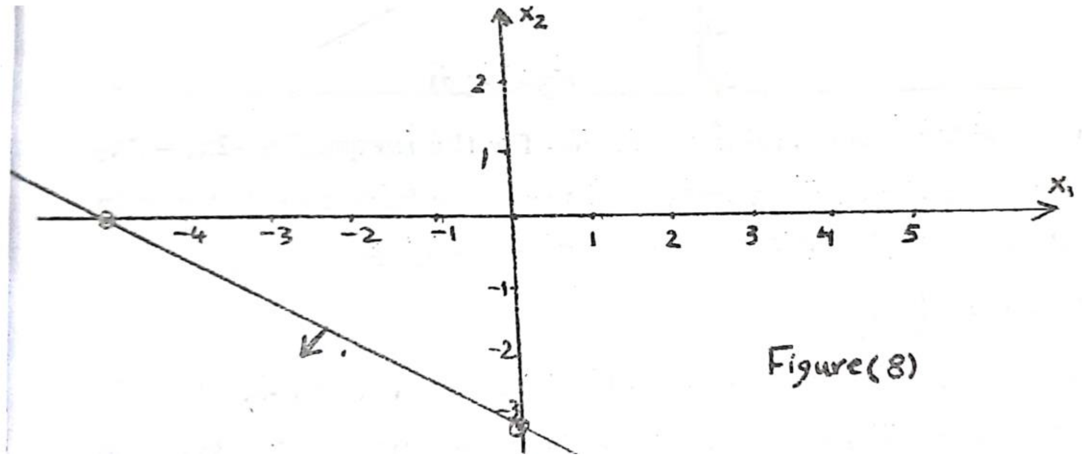
i.e., 0 [ ] -15

Then the correct relation between the two sides of the last relation is (>), i.e., the origin point (0 , 0) lies in the opposite region of this inequality.

In other meaning, since each of the two inequalities:

$3x_1 + 5x_2 \leq - 15$ and 0 (>) -15 are conversely, then any point in the line graph in Figure (7) or in the opposite region for the origin point exist is satisfied the inequality $3x_1 + 5x_2 \leq -15$.

Therefore, the region in which holds the inequality is indicated by an arrow on its associated straight line as it be shown in figure (8)



Figure(8)

<u>Example (7):</u>

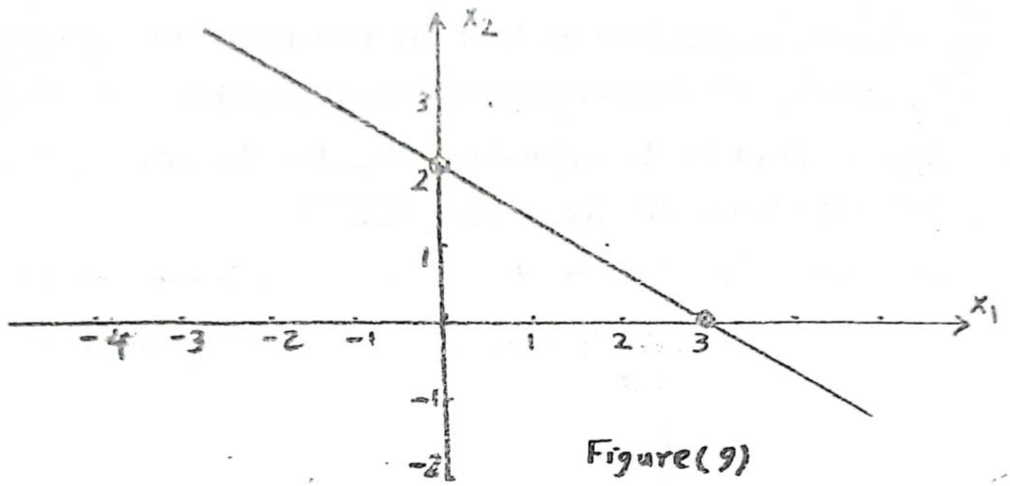
Determine graphically the region in which satisfy the following inequality: $-2x_1 - 3x_2 \leq -6$

Solution:

Firstly, convert the inequality into the following equation then, we have: $-2x_1 - 3x_2 = -6$, and the following table represents the two intercepts:

X_1	0	3
X_2	2	0

Graphing the two points then we have the following figure (9)



Figure(9)

Now, in order to determine the region for the inequality $-2x_1 - 3x_2 \leq -6$ by using the origin point (0 , 0) then we have to substitute in both the two sides of this inequality,

then we have:

-2 (0) – 3 (0) [ ] -6

0 [ ] -6

Then the correct relation between the two sides is 0 (>) -6.

Now, since the relation for each of the inequality :

$-2x_1 - 3x_2 \leq -6$ and the relation resulted from substituting with the origin point 0 (>) are conversely, then the region by which hold the inequality is conversely with the origin point.

i.e., any point in the graph line $-2x_1 - 3x_2 = -6$ or in the opposite region by which the origin point exist is the region which satisfy this inequality.

Then the region in which holds the inequality:

$-2x_1 - 3x_2 \leq -6$ is indicated by an arrow on its associated straight line as in figure (10).

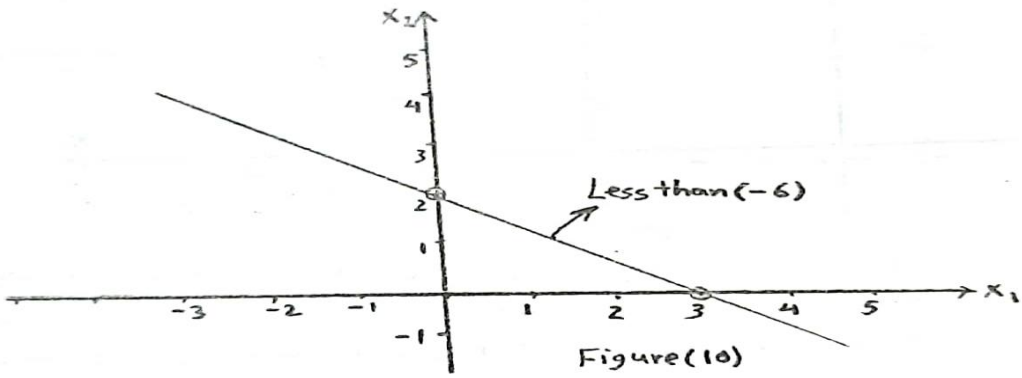


Figure (10)

<u>Example (8):</u>

Determine graphically the direction which satisfy each of the following

(a) $2x_1 \leq -3$

(b) $3x_1 \geq -9$

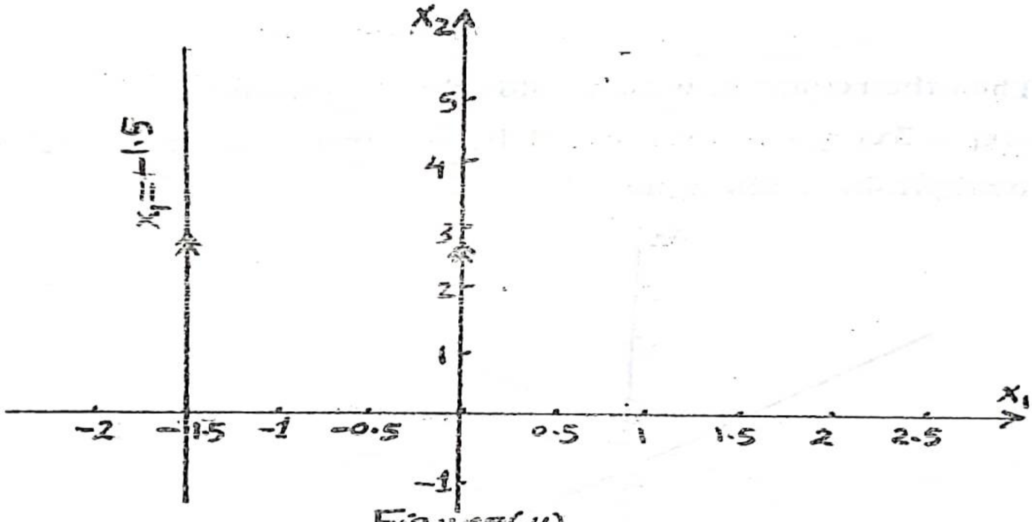
Solution :

(a)In order to find the direction which hold the inequality :

$2x_1 \leq -3$ convert the inequality into an equation : $2x_1 = -3$,

i.e., $x_1 = -1.5$

Then: $x_1 = -1.5$ can be represented graphically through a line by which intercept x_1 axis in (-1.5) unit and parallel with the (x_2) axis as it be shown in figure (11).



Figure(11)

Now in order to determine the direction by which satisfy the inequality $2x_1 \leq -3$ by using the origin point, then 2 (0) [ ] -3 i.e., 0 [ ] -3 the right relation between the two sides is:

0 (>) -3 i.e., the region for the inequality $2x_1 \leq -6$ is conversely with the region which the origin point is exist. i.e., the direction of the inequality $2x_1 \leq -3$ can be represented by an arrow on its straight line as it be shown in figure (12) .

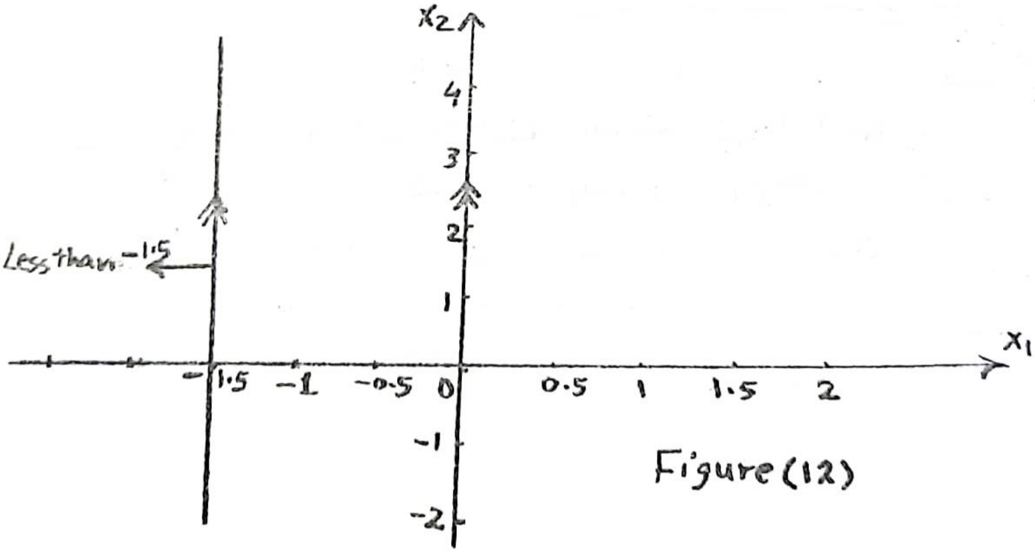


Figure (12)

(b)Similarly, it can be show that figure (13) represents the direction by which the inequality $3x_1 \geq -9$ is satisfied or hold.

Graphical solution steps for the linear programming models:

We can summarize the steps for solving the Linear programming models graphically as follows:

1- Each constraint will be plotted first with ($\leq$ or $\geq$) replaced by (=), thus yielding simple straight line equation, the region in which each constraint holds is indicated by an arrow on its associated straight line, then the resulting feasible space solution is given [feasible solution].

2- The optimum solution is the point in the feasible solution which yields the largest (maximum) value or the lowest (minimum) value for the equation of the objective function x_0 or $f(x) = 0$. This optimum point can be determined from moving the graph line for the objective function $f(x)$ or $x_0 = 0$ parallel to itself in the direction of the feasible solution even so it passes or tangent the first or the latest point in the feasible solution, then, we can determine the optimum point in the feasible solution.

Remarks:

1- If the linear programming model have one constraint in equation form (=) from the set of constraints, then the feasible solution will be a part of the graph line for the constraint in which it be found in the equation form, subject to it must be lie in the feasible solution for the remaining set of constraints for the linear programming model.

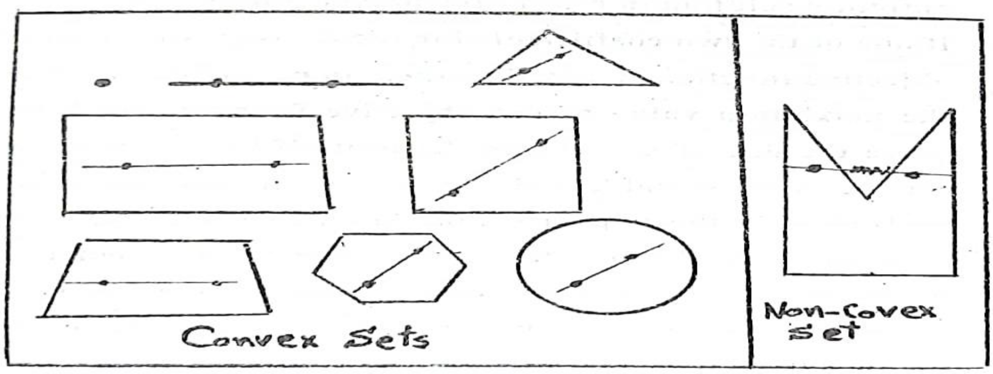
2- If the linear programming model have at least two constraints in equations form (=) from the set of constraints, then the feasible solution will be at most one point. This point is the intersection between the two constraints in which they are found in the equation form (=) subject to this intersection lies in the feasible solution for the remaining set of constraints for the linear programming model. Furthermore, this intersection point is considered the optimum solution in either determination the maximum or the minimum value of the objective function in this case .

3- If the two coefficients for the two decision variables x_1 and x_2 are positive numbers, then the optimum solution for the LPM is the first point that the line of the objective function $x_0 = 0$ passes or tangent the feasible solution in case of minimization for x_0. Conversely the latest point that the line $x_0 = 0$ passes or tangent the feasible solution is considered the optimum solution in case of the maximization of x_0.

4- If one of the two coefficients for the decision variable in the objective function is negative, then, in case of determination the maximum value of the objective function, we have to move the line of $x_0 = 0$ parallel to itself in the direction of the decision variable axis by which it has the positive coefficient in the objective function x_0. conversely, we will have to move the line for $x_0 = 0$ parallel to itself in the direction of the decision variable axis which have the negative coefficient in the objective function in case of minimization the value of x_0.

5- If the two coefficient for the decision variables x_1 and x_2 are negative values, then we have to move the line x_0 parallel to itself in the direction for the decision variable which have the lowest negative coefficient in case of maximization x_0, and conversely in case of the minimization value of x_0. Or, we have to move the line x_0 parallel to itself in the direction for the feasible solution space , and then, the 1st point that the line $X_0 = 0$ is considered the optimum solution in case of minimizing X_0 , and vice versa .

6- The feasible space solution resulted from the graphical solution for the linear programming model is a convex set, where the convex set is a solution space by which the line passes between any two arbitrary points must be lies in the solution space.



7- After we have determine the feasible solution space, if there is line graph for any constraint can be deleted without affecting the solution space, then this constraint for this line is considered a redundant constraint.

8- The types of solutions resulted from the graphical solution for any linear programming model: there are four types of solutions:

(a)Feasible Solutions:

After determining the feasible solution space, then any point in this space is considered a feasible solution.

(b)Basic Solutions:

Any intersection between two arbitrary constraint lines is considered a basic solution.

(c)Basic Feasible Solution: (or the Extreme Points):

The corners points for the feasible solution is considered the basic feasible solutions.

(d)Optimal Solution:

If the feasible solution is exist, then the optimal solution is at least one point of these basic feasible solutions by which it makes the value of the objective function in its maximum

value in case of determination the value of x_i's that make x_0 maximization, and vice versa.

Example (9):

Determine the feasible solution space for the following constraints and determine the redundant constraints if there are exist:

(1) $x_1 + x_2 \leq 4$

(2) $4x_1 + 3x_2 \leq 12$

(3) $-x_1 + x_2 \leq 1$

(4) $x_1 + x_2 \leq 6$

(5) $x_1 \geq 0$

(6) $x_2 \geq 0$

Solution:

In order to determine the feasible solution space, we have to graph the line for each constraint and determine the region by which satisfied this constraint as follows:

(1)The nonnegativity constraints:

$x_1 \geq 0$, $x_2 \geq 0$ specify that the feasible solution must lie in the first quadrant.

<u>*The 1st constraint:</u> $x_1 + x_2 \leq 4$

Convert into an equation: $x_1 + x_2 = 4$**, then the following table represents the two intercepts.**

X_1	0	4
X_2	4	0

and since all the coefficients of x_1 , x_2 and the right hand side of the constraint is positive, then the region in witch satisfies the 1st constraint lie down its line graph.

<u>*The 2nd constraint:</u> $4x_1 + 3x_2 \leq 12$

Convert into an equation: $4x_1 + 3x_2 = 12$**,**

Then the following table represents the two intercepts:

X_1	0	3
X_2	4	0

and since, all the coefficient of x_1 , x_2 and the R.H.S of the constraint is positive, then the region in witch satisfies the 2nd constraint lie down its line graph.

<u>*The 3rd constraint:</u> $-x_1 + x_2 \leq 1$

Convert into an equation: $-x_1 + x_2 = 1$**,**

then the following table represents the two intercepts.

X_1	0	-1
X_2	1	0

Now, in order to determine the direction for this constraint, we will use the origin point as follow:

-(0) + 0 [ ] 1

0 [<] 1

Since, we have the same relation for the inequality resulted from substituting the origin point and the constraint, then the region for the line constraint can be indicated with an arrow at the same direction of the origin point.

<u>*The 4th constraint:</u> $x_1 + x_2 \leq 6$

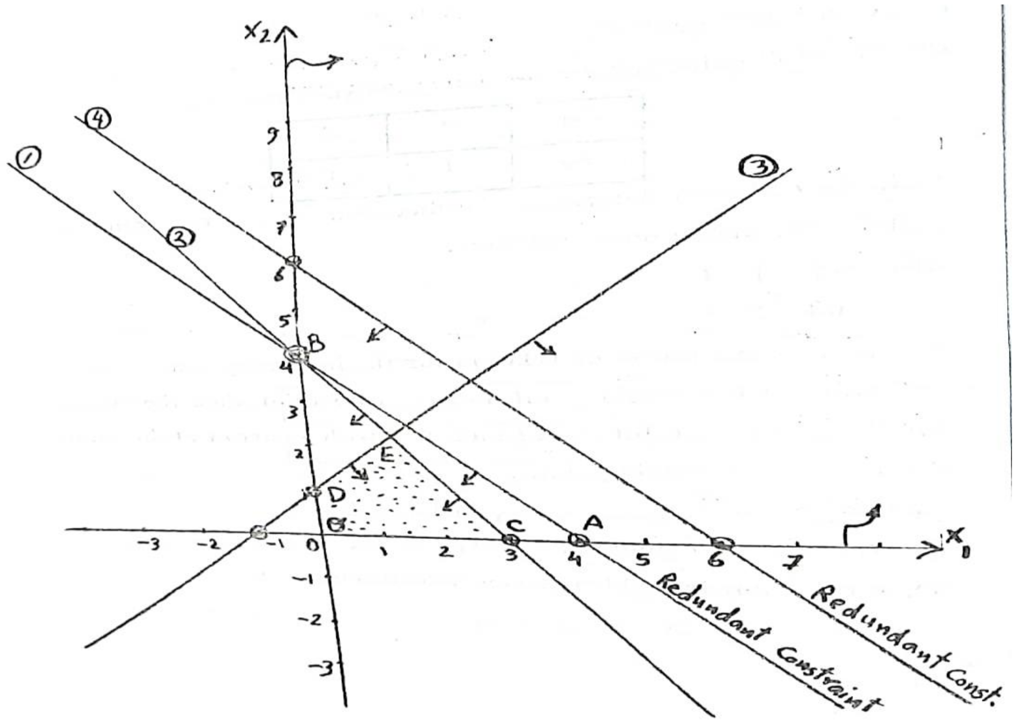
Convert into an equation : $x_1 + x_2 = 6$,

Then the following table represents the two intercepts.

X_1	0	6
X_2	6	0

and the region by which satisfy this constraint is in the direction of the origin point.

(2)Determination the feasible solution space graphically:



Reduction the feasible solution

-Nonnegativity constraints : $x_1 Ox_2$

-1st constraint : AOB

-2nd constraint : COB

-3rd constraint : CODE

-4th constraint : CODE (Feasible Solution)

Therefore the feasible solution space is CODE. And the two constraints enumerated with (1) and (4) are redundants,

since their lines can be deleted without affecting in the solution space CODE.

Example (10):

If you have the following model:

$x_0 = x_1 + 2x_2$

Subject to:

(1) $-3x_1 + 3x_2 \leq 9$

(2) $x_1 + x_2 \leq 2$

(3) $x_1 + x_2 \leq 6$

(4) $x_1 + 3x_2 \geq 6$

(5) $x_1 \geq 0$

$x_2 \geq 0$

Required:

1- Determine the optimum solution in either maximization or minimization for x_0.
2- Determine the different types of solution.
3- Determine the redundant constraints if there are exist.

Solution:

1-The idea of the graphical solution is to plot the feasible solution space, which is defined as the space enclosed by constraints (1) through (5). The optimum solution is the point in the solution space which maximize or minimize the value of the objective function x_0.

***Non-negativity restrictions: $x_1 \geq 0$ and $x_2 \geq 0$ implies that the feasible solution must be lie in the 1st quadrant.**

-1st constraint: $-3x_1 + 3x_2 \leq 9$

Convert into an equation $-3x_1 + x_2 = 9$,

Then the following table represent the two intercepts:

X_1	0	-3
X_2	3	0

And by using the origin point (0 , 0) for determining the direction for this inequality

-3 (0)+ 3 (0) [ ] 9

0 [<] 9

i.e., each of the direction of the origin point and the direction of this inequality are the same.

-2nd constraint: $x_1 - x_2 \leq 2$

Convert into an equation, then we have $x_1 - x_2 = 2$, and the following table represents the two intercepts:

X_1	0	2
X_2	-2	0

And by using the origin point,

Then: $0 - 0$ [ ] 2

0 [<] 2

i.e., both the direction for the inequality and the origin point are the same.

3rd constraint: $x_1 + x_2 \leq 6$

Convert into an equation, then $x_1 + x_2 = 6$, and the following table represents the two intercepts:

X_1	0	6
X_2	6	0

since the two coefficients for the decision variables x_1 , x_2 and the R.H.S for the constraint is positive number, then the direction for this inequality and the origin point are the same direction.

<u>4th constraint:</u> $x_1 + 3x_2 \geq 6$

Convert into an equation $x_1 + 3x_2 = 6$,

Then, the following table represents the two intercepts:

X_1	0	6
X_2	2	0

Since the two coefficients for x_1 , x_2 and the R.H.S are positive numbers, then the direction for this inequality ($\geq$) is in the opposite direction which implies the origin point.

<u>*Graphing the objective function line:</u>

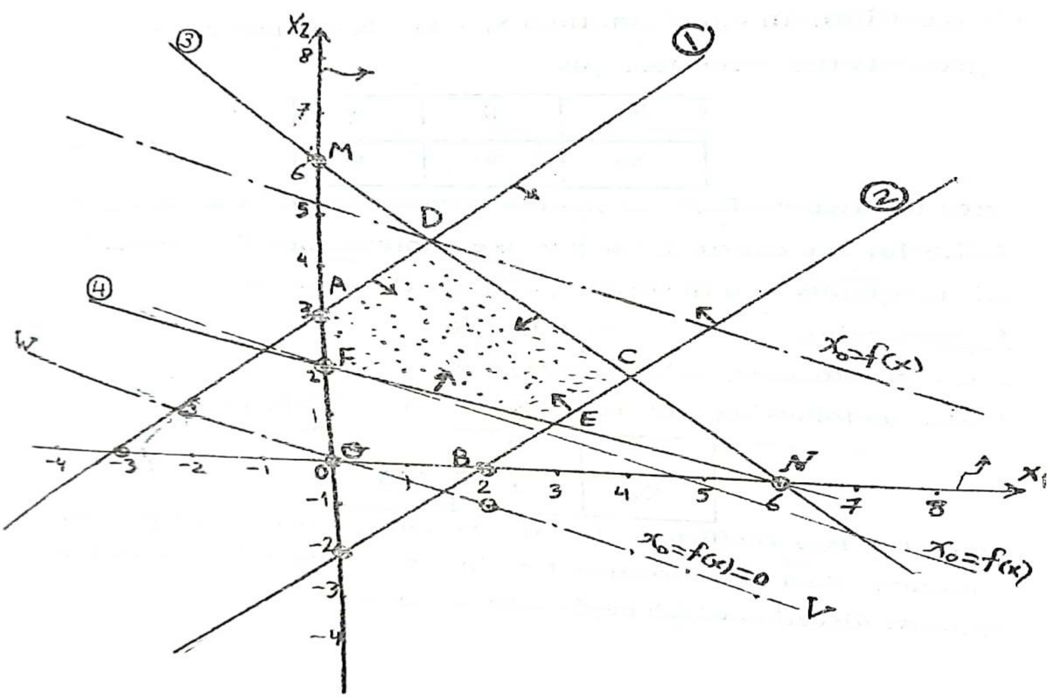
Suppose that $x_0 = 0$, then we have the following equation:

$$x_1 + 2x_2 = 0$$

The following table represents the points that passes through that line:

X_1	0	2	-2
X_2	0	-1	1

then, we have the following graph:



Reduction for the feasible solution:

***Nonnegativity constraints: x_1 OX$_2$**

***1st constraints = x_1 OA (1)**

***2nd constraints = (2) BOA (1)**

***3rd constraints = OBCDA**

***4th constraints = ECDAF (Feasible solution space)**

-The line for the objective function $x_0 = 0$ is VOW

Now, for determining the optimal solution in either the minimization or the maximization cases, we have to move

the line graph for the objective function $x_1 + 2x_2 = 0$ (VOW) parallel to itself in the direction of the feasible solution ECDAF even it passes through the first point (since the two coefficients are positive value) in the feasible solution space in case of minimization for x_0 or even it passes through the latest point in the feasible solution in case of the maximization for x_0. one can see that the minimum value of x_0 occurs where the line of the objective function passes through point F whose coordinates are $x_1 = 0$ and $x_2 = 2$ units. Substituting these values into the objective function gives $x_0 = x_1 + 2x_2 = 0 + 2(2) = 4$ unit. Also, one can see that the maximum value of x_0 occurs where the line of the objective function passes through the point D whose coordinates are $x_1 = {}^3/_2$ and $x_2 = {}^9/_2$. Substituting these values in the objective function gives $x_0 = x_1 + 2x_2 = ({}^3/_2) + 2({}^9/_2) = ({}^{21}/_2) = 10.5$ unit.

An interesting observation is that either the minimum value or the maximum value of the objective function x_0 always occurs at one of the corner points (Extreme points) E, C, D, A and F of the solution space. The choice of a specific corner point as the optimum depends on the slope of the objective function. As an illustration, the reader can verify graphically that the changes in the objective function given

in the table blow produce the optimum solution in the two cases:

Coordinates for the corner points	$f(x) = x_0 = x_1 + 2x_2$	Remarks
F (0 , 2)	$f_F(x) = 0 + 2(2) = 4$	Minimum value
E (3 , 1)	$f_E(x) = 3 + 2(1) = 5$	
C (4 , 2)	$f_C(x) = 4 + 2(2) = 8$	
D ($^3/_2$, $^9/_2$)	$f_D(x) = 3/2 + 2(^9/_2) = 10.5$	Maximum value
A (0 , 3)	$f_A(x) = 0 + 2(3) = 6$	

2-The Different Types of Solutions are:

(a)Feasible solutions:

Any point lie in the feasible solution space ECDAF is considered a feasible solution. Henceforth, there are unfinite numbers of solutions in this problem.

(b)Basic solutions: The set of Basic solutions were the set of points O, B, N, F, E, C, D, A, and M. therefore, any point

resulted from the intersection between the lines of two constraint in the 1st quadrant is defined as a basic solution.

(c)Basic feasible solutions (Extreme points)

The Basic Feasible solutions in this problem were E, C, D, A and F.

(d)The optimal solution:

The point F(0 , 2) is considered the optimum solution in case of minimization the value of x_0. And the point D ($^3/_2$, $^9/_2$) is considered the optimum solution in case of maximization the value of x_0.

3-Since all the lines graph for the set of all constraints implies the feasible solution, then all constraints for the problem considered basically (non-redundant), i.e., there is no redundant constraint in this problem.

Example (11):

Suppose that you have the following (LPM):

$f(y) = 5y_1 + 2y_2$ **Max (Min)**

subject to:

$y_1 + y_2 \leq 10$

$y_1 \quad = \quad 5$

$y_1 \,,\, y_2 \quad \geq \quad 0$

find the optimum solution and the types of solutions.

Solution:

In order to determine the optimal solution for the LPM in either maximization or minimization form graphically we have to determine the feasible solution space as the following:

***The non-negativity constraints: $x_1 \geq 0$ and $x_2 \geq 0$ implies that the feasible solution space must be lie in the 1st quadrant.**

-1st constraint: $y_1 + y_2 \leq 10$

Convert it into an equation yields to $y_1 + y_2 = 10$,

The following table represents the two intercepts:

X_1	0	-10
X_2	10	0

The region by which it is satisfy the inequality for this constraint is the same region that the origin point is exist.

-2nd constraint: $y_1 = 5$

The line graph for this equation is vertical on y_1 axis and intercept the y_1 axis in 5 units vertically and parallel to the y_2 axis. Each point in the line graph for this constraint is only satisfy this equation.

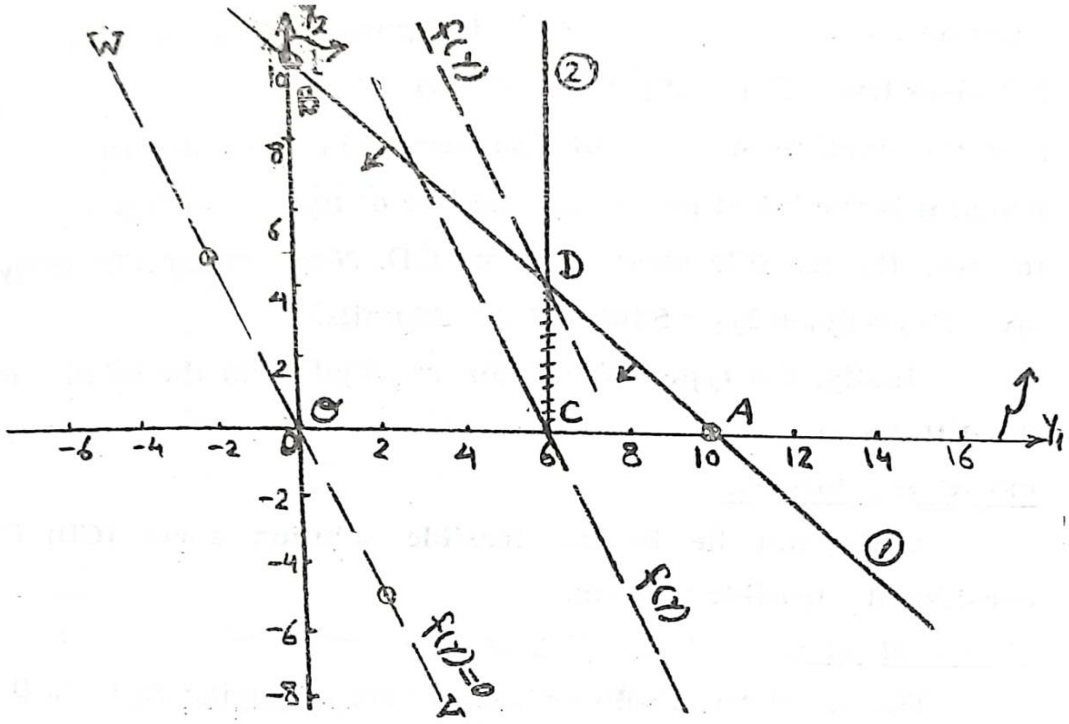
-Graphing the objective function:

Suppose that $f(y) = 0$, then we have the following;

$5y_1 + 2y_2 = 0$,

y_1	0	2	-2
y_2	0	-5	5

The following table represents the point that passes through the line for this equation then, we have the following graph:



Reduction of the feasible solution space:

***Nonnegativity constraints : x_1 O x_2**

***1st constraint : AOB**

***2nd constraint : CD (feasible solution)**

-The line graph for the objective function *f*(*y*) = 0 is VOW.

Then, in order to determine the optimal solution, we have to move the line VOW parallel to itself in the direction of the feasible solution space (CD) even it passes through the first point in the line CD (since the two coefficients of y_1 and y_2 are positive numbers) in case of minimization of *f*(*y*). One

can see that the minimum value of f(y) occurs in the point C(6,0). Substituting in f(y) gives f(y) = $5y_1 + 2y_2$ = 5 (6) + 2(0) = 30 units. Also, one can see that the maximum value of f(y) occurs in the point D(6, 4) by which it is the latest point that the line of f(y) = 0 or VOW passes through the feasible solution space CD. Now substituting in f(y) gives:

f(y) = $5y_1 + 2y_2$ = 5 (6) + 2(4) = 38 units.

Finally, the types of solution resulted from the solution of this LPM are:

<u>*Feasible solutions:</u>

Any point lie in the feasible solution space (CD) is considered a feasible solution.

<u>*Basic solution:</u>

The set of basic solution are the set of points: O, C, A, D and B

<u>*Basic feasible solutions:</u>

The to corners of the feasible solution space (CD) are the basic feasible solution space, i.e., the two points C and D are the basic feasible solution (Extreme Points).

<u>*Optimal solutions:</u>

The point C (6, 0) is considered the optimal solution in case of minimization the value of the objective function f(y), and the point D (6 , 4) is considered the optimal solution in case of maximization the value of the objective function f(y).

Example (12):

Determine the value of x_1 and x_2 by which:

$f(x) = -x_2$ **Max (Min)**

Subject to:

(1) $x_1 + x_2 \geq 1$

(2) $x_1 + x_2 \leq 2$

(3) $x_1 - x_2 \leq 1$

(4) $x_1 - x_2 \geq -1$

(5) $x_1 , x_2 \geq 0$

Solution:

***Nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$ implies that the feasible solution space must lie in the 1^{st} quadrant.**

-1^{st} constraint:

$x_1 + x_2 \geq 1$, **convert into an equation,** $x_1 + x_2 = 1$,

then the following table represent the two intercepts:

X_1	0	1
X_2	1	0

And the direction of the inequality of this constraint is opposite to the region in which the origin point is exist.

-2nd constraint:

$x_1 + x_2 \leq 2$, convert into an equation: $x_1 + x_2 = 2$,

Then the following table represents the two intercepts:

X_1	0	2
X_2	2	0

And the direction of the inequality of this constraint is the same direction by which the origin point is exist.

-3rd constraint: **$x_1 – x_1 \leq 1$, convert into an equation:**

$x_1 – x_2 = 1$,

X_1	0	1
X_2	-1	0

then the following table represents the two intercepts. Now, in order to determine the region by which satisfies the

inequality of this constraint, let us used the origin point for this determination:

0 – 0 [ ] 1

0 [<] 1

i.e., the direction of this inequality is the same region by which the origin point is exist, since the two relation have the same (<).

-4th constraint:

$x_1 - x_2 \geq -1$, convert into an equation: $x_1 - x_2 = -1$,

The following table represents the two intercepts:

X_1	0	1
X_2	1	0

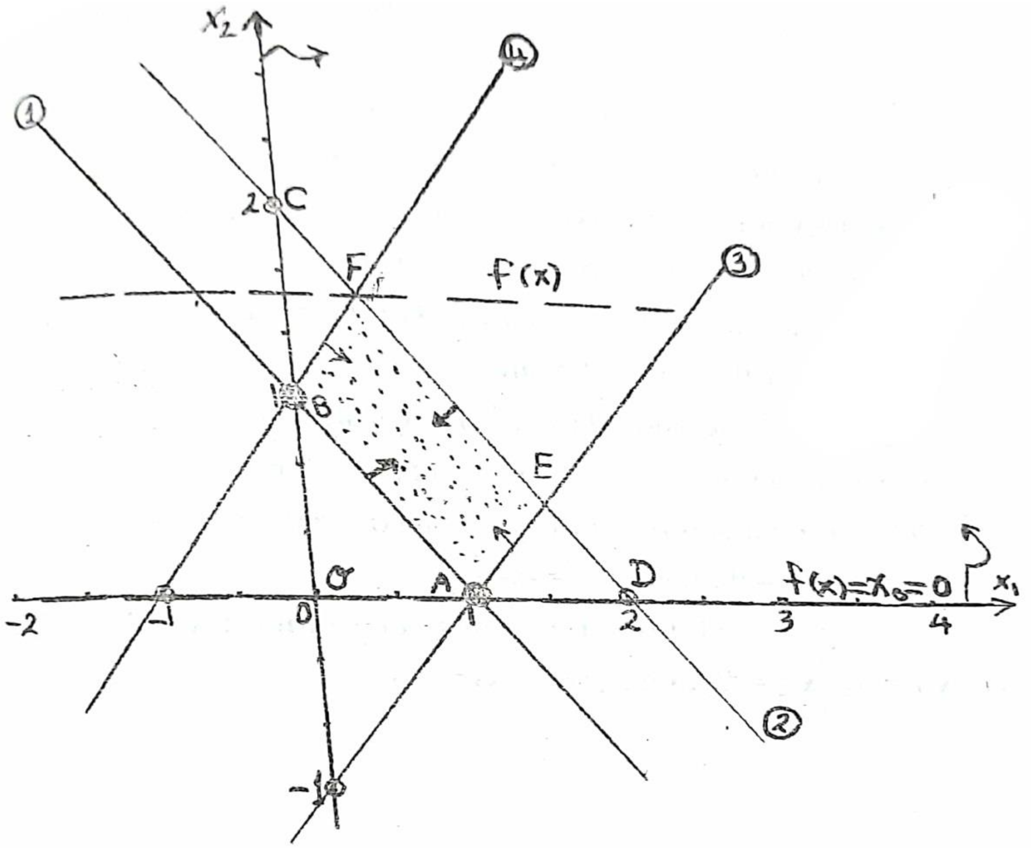
And: 0 – 0 [ ] -1

0 [>] -1

i.e., Each of the region satisfying this inequality and the region by which the origin point is exist are the same.

***Graphing the objective function: Suppose that *f*(*x*) = 0, then we have the following: $-x_2 = 0$, this equation implies that the**

line for $f(x) = -x_2 = 0$ is the same line for the x_1 axis. Then, we have the following graph:



<u>Reduction of the feasible solution</u>:

-Nonnegativity constraints : $x_1 Ox_2$

-1st constraint : $x_1 ABX_2$

-2nd constraint : ABCD

-3rd constraint : ABCE

-4th constraint : ABFE (Feasible Solution)

<u>*The line graph for the objective function is (-x_1 O x_1).</u>

Since the coefficient of the decision variable x_2 is negative, therefore the decision is conversely, i.e., in order to determine the optimal solution, we have to move the line graph for the objective function f(x) = 0 or the line (-x_1 O x_1) parallel to itself even it passes through the first point in the feasible solution space (ABFE) in case of maximization the value of *f*(*x*), or even it passes through the latest point in the feasible solution in case of minimization the value of *f*(*x*). Therefore, the point A (1 , 0) is considered the optimal solution in case of maximization the value of *f*(*x*), and the point *F*($^1/_2$, $^3/_2$) is considered the optimal solution in case of minimization the value of *f*(*x*). Hence:

-The optimal solution in case of maximization the value of *f*(*x*) is $x_1^* = 1$, $x^*_2 = 0$, then *f*(*x*) = $-x_2 = 0$

-The optimal solution in case of minimization the value of *f*(*x*) is $x^*_1 = {}^1/_2$, $x^*_2 = {}^3/_2$, then *f*(*x*) = $-x_2 = -{}^3/_2$.

Note that, one can see that the solution is correct from substitution in the objective function by the corners coordinates (extreme points) for the feasible solution, as it be shown in the following table:

Extreme points	***f*(x) = -x_2**	**Remarks**
A (1 , 0)	**f_A (x) = 0**	**Maximum value**
E ($^3/_2$, $^1/_2$)	**f_E (x) = -$^1/_2$**	
F ($^1/_2$, $^3/_2$)	**f_F (x) = -$^3/_2$**	**Minimum value**
B (0 , 1)	**f_B (x) = -1**	

Note that if *f*(*x*) = -x_1 , then the optimal solution is the point B (0 , 1) in case of maximization the value of *f*(*x*), and the point E ($^3/_2$, $^1/_2$) in case of minimization the value of *f*(*x*), since the line graph for the objective function *f*(*x*) = -x_1 = 0 will be the same line for the x_2 axis in this case.

Example (13):

If you have the following LPM:

F(x) = 2x_1 + x_2 (minimize)

Subject to:

$$3x_1 + x_2 \geq 3$$
$$4x_1 + 3x_2 \geq 6$$
$$X_1 + 2x_2 \leq 3$$
$$X_1 , x_2 \geq 0$$

Required : Solve the LPM graphically. Determine the redundant constraint if there is exist?

<u>Solution:</u>

1- To solve the primal problem graphically, we have to determine the feasible solution space as follows:

- **<u>The non-negativity constraint:</u> $x_1 \geq 0$ and $x_2 \geq 0$, implies that the feasible solution spec lies in the 1st quadrant.**
- **<u>The 1st constraint:</u> $3x_1 + x_2 \geq 3$**

Convert the inequality to an equation as: $3x_1 + x_2 = 3$. Then, the following table represents the two intercepts

X_1	0	1
X_2	3	0

And the region by which the origin point is not exist satisfied the constraint (up the line graph).

- **<u>The 2nd constraint</u>: $4x_1 + 3x_2 \geq 6$, convert into an equation, then, we have $4x_1 + 3x_2 = 6$, the following table represents the two intercepts**

X_1	0	1.5
X_2	2	0

And the region up to the tine graph satisfies the constraint, since all the coefficients and the R.H.S are positive numbers.

- **<u>The 3rd constraint</u>: $x_1 + 2x_2 \leq 3$, convert into an equation, then, we have $x_1 + 2x_2 = 3$, the following table represents the two intercepts,**

X_1	0	3
X_2	1.5	0

And the region by which the origin point is exist (down the line graph) satisfies the constraint.

- **<u>Graph the line for the objective function</u>: suppose that f(x) = 0, then, we have $2x_1 + x_2 = 0$, the following table represents three point passes through the line f(x) = 0.**

X_1	0	1	-1
X_2	0	-2	2

- **Then, we can determine the feasible solution space and the optimal solution graphically as follows:**

(<u>Each student must solve this example</u>)

You can used the following Reductions:

<u>Reduction the feasible solution space</u>:

- **Non-negativity constraint = x_1 O x_2**
- **1st constraint = x_1 AB x_2**
- **2nd constraint = x_1 CDB x_2**
- **3rd constraint = ECD (feasible solution)**
- **The line graph for objective function f(x) = 0 is VOW. Now for determining the optimal solution for the primal LPM graphically, we have to move the line graph VOW parallel to itself in the direction of the feasible solution**

ECD even so it passes through the 1st point in the feasible solution since this case is to minimize f(x). one can see that the minimum value of f(x) occurs in the point (D) whose coordinates are $x_1 = 3/5$ and $x_2 = 6/5$

- **Substituting these values into the objective function gives:**

$F(x^* = 2x^*_1 + x^*_2 = 2(3/5) + 6/5 = 12/5$, i.e., the optimal solution graphically for the primal LPM is:

$x^*_1 = 3/5$, $x^*_2 = 6/5$ and $f(x^*) = 12/5$.

- **The 1st constraint is considered a redundant constraint since it can be deleted without effect on the feasible solution space.**

Exercises Or Problems

1) -Two product A and, B passes through two machines (1) and (2). The unit produced from the product A needs four hours in the 1st machine and three hours in the 2nd machine, and the unit produced from the product B needs two hours in the 1st machine and only one hour in the 2nd machine. If the available capacity for the two machines are 18 and 12 hours respectively, and the unit profit for each product are 4 (L.E) and 2 (L.E) respectively.

Required:

- **(A)Formulate the problem in a Linear programming model.**
- **(B)Determine the optimal solution for the LPM.**

2) -Determine the solution space graphically for the following inequalities:

$$x_1 + x_2 \leq 4$$

$$4x_1 + 3x_2 \leq 12$$

$$-x_1 + x_2 \geq 1$$

$$x_1 + x_2 \leq 6$$

$$x_1 , x_2 \geq 0$$

which constraints implies by others? Reduce the system to the smallest number of constraints which will define the same solution space.

3) -Solve the following problem graphically:

$$f(x) = 5x_1 + 2x_2$$

<u>Subject to:</u>

$$x_1 + x_2 \leq 10$$
$$x_1 = 5$$
$$x_1 , x_2 \geq 0$$

4) -Consider the graphical representation of the following LPM:

$F(x) = 5x_1 + 3x_2$ **Maximize (or Minimize)**

$$x_1 + x_2 \leq 6$$
$$x_1 \geq 3$$
$$x_2 \geq 3$$
$$2x_1 + 3x_2 \geq 3$$
$$x_1 , x_2 \geq 0$$

A-In each of the following cases indicate if the solution space has one point, infinite number of points, or no points:

(i)The constraints are as given above.

(ii)The constraint $x_1 + x_2 \leq 6$ is changed to $x_1 + x_2 \leq 5$

(iii)The constraint $x_1 + x_2 \leq 6$ is changed to $x_1 + x_2 \leq 7$.

B-For all cases in (A), determine the number of feasible extreme points if any.

C- For the cases in (A), in which a feasible solution space exists, determine the maximum and minimum value of *f(x)* and their associated extreme points.

5) -Consider the following LPM:

$$f(x) = 6x_1 - 2x_2 \qquad \text{(Maximize)}$$

<u>Subject to:</u>

$$x_1 - x_2 \leq 1$$
$$3x_1 - x_2 \leq 6$$
$$x_1 , x_2 \geq 0$$

Show graphically that at the optimal solution, the two decision variable x_1 and x_2 can be increased indefinitely while the value objective functions remains constant.

6) -Consider the following LPM:

$$F(x) = 3x_1 + 2x_2 \qquad \text{(Maximize)}$$

<u>Subject to:</u>

$$2x_1 + x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$X_1 , x_2 \geq 0$$

Show graphically that the problem has no feasible extreme point. What can conclude concerning the solution to the problem?

7) - Solve the following linear programming models graphically:

(A): $f(x) = x_1 + 2x_2$ (Maximize)

Subject to:

$$-3x_1 + 3x_2 \leq 9$$
$$x_1 - x_2 \leq 2$$
$$x_1 + x_2 \leq 6$$
$$x_1 + 3x_2 \leq 6$$
$$x_1 , x_2 \geq 0$$

(B): $f(x) = -x_1 + x_2$ (Maximize)

Subject t to:

$$x_1 + x_2 \leq 2$$
$$x_1 \leq 2$$
$$x_1 , x_2 \geq 0$$

(C): $f(x) = -x_1 + x_2$ (Maximize)

Subject to:

$$x_1 + x_2 \leq 4$$
$$2x_1 + 5x_2 \leq 10$$

$x_1 \; , \; x_2 \quad \geq \quad 0$

8) - Solve the following LPM graphically and determine the different types of solutions:

$f(x) = -x_2$ (Minimize)

Subject to:

$x_1 + x_2 \quad \geq \quad 1$

$x_1 + x_2 \quad \leq \quad 2$

$x_1 - x_2 \quad \leq \quad 1$

$x_1 - x_2 \quad \geq \quad -1$

$x_1 \; , \; x_2 \quad \geq \quad 0$

and solve the same LPM considering that $f(x) = -x_1$ and in maximization form.

9) -Consider the following LPM:

$f(x) = 2x_1 + x_2$ (Maximize)

Subject to:

$x_1 + 2x_2 \quad \leq \quad 10$

$x_1 + x_2 \quad \leq \quad 6$

$x_1 - x_2 \quad \leq \quad 2$

$x_1 - x_2 \quad \geq \quad 0$

Required:

Solve the LPM graphically and determine the types of solutions.

10)-Solve the following LPM graphically:

$f(x) = -2x_1 - x_2$ **(Minimize)**

<u>Subject to:</u>

$$x_1 + x_2 \leq 2$$
$$x_1 + 3x_2 \geq 3$$
$$x_1 , x_2 \geq 0$$

11)-Solve the following LPM graphically:

$f(x) = 4x_1 + 8x_2$ **(Minimize)**

<u>Subject to:</u>

$$x_1 - x_2 \geq 2$$
$$2x_1 + x_2 \geq 5$$
$$x_1 , x_2 \geq 0$$

12)-Consider the following LPM:

$f(x) = -x_2$ **(Minimize)**

<u>Subject to:</u>

$$x_1 + 2x_2 \geq 6$$
$$x_1 - x_2 \leq 2$$
$$x_2 \leq 2$$
$$x_1 , x_2 \geq 0$$

<u>Required:</u>

Determine the optimal solution for the LPM.

(14)-Consider the following LPM:

$f(x) = -x_1$ **(Minimize)**

Subject to:

$$x_1 + x_2 \leq 2$$

$$| x_1 - x_2 | \leq 1$$

$$x_1 \ , \ x_2 \geq 0$$

Required:

Solve the LPM graphically.

Chapter (2)

"Mathematical Relations"

- **Introduction:**

Mathematical quantities are divided into two types:

1-*Constant quantities*: These quantities do not change during the mathematical calculations. The constant quantities are expressed by symbols *a*, *b*, *c*,…… . For example: the sum of the three angles for any triangle is a constant that is always equal to 180^o.

2-*Variable quantities*: It takes different values, and it is usually expressed by variables and denoted by the symbols: *x* , *y* , *z* ,……..and so on . The value of the variables depends on the values that other variables take according to a specific mathematical relation. For example : weights and ages of students are variables, because the weight or the age is changed from one student to another.

3-*Mathematical Relation:* It is a mathematical formula that contains one (or a set of) variable (s) , and probably with addition a constant (or a set of) constant(s) , by which the value of any variable can be determined if it is known the values of the other variables value. Mathematical relations take the following forms:

(1) Equations. (2) : Inequalities. (3) :Functions.

(2-1): Equations

The equation is a statement that expresses the equality of two algebra expressions. The algebra expression may be stated in terms of one or more *variables* and the equality symbol " = ". The following are examples of equations:

$$5x - 8 = 6 - 2x \qquad (1)$$

$$, \; 2x + 3y = 17 \qquad (2)$$

$$, \; y^2 - 2y + 5 = 8 \qquad (3)$$

Equation (1) is called the 1st order equation in one variable *x*, while the equation (2) is called the 1st order or degree equation in two variables (*x*, *y*) or it is called a linear equation because it can be represented graphically as a straight line. Finally, Equation (3) is a 2nd order equation in one variable (*y*). In general, the degree of the equation is equal to the largest power of the variables included in the equation. The mathematical expressions in any equation is separated by the equality symbol (=) which are called the two sides of the equation; individually they are called the left hand side and the right hand side.

The value of the variable that makes an equation a true statement is called the root or solution of the given equation. We say that the equation is satisfied by such the value of the variable. we are often interested in finding the roots e of some given equation-that is , in

determining all the values of the variable that make the equation a true statement.

<u>For example</u>: consider the equation:

$$2x - 3 = x + 2$$

If *x* takes the value 5, this equation becomes

$$2(5) - 3 = (5) + 2$$

$$7 = 7$$

Which is a true statement? On the other hand, if *x* takes the value 4, then we have:

$$2(4) - 3 = (4) + 2$$

$$5 \neq 6$$

This is a false statement. Henceforth, the value x = 4 is not a root for the equation, while the value x = 5 is considered a root for the equation.

<u>We can summarize three types of equations:</u>

- **An *identity* equation is an equation which is true for all values of the variables. Examples of an identity is the following equations:**

$$2x + 5 = \frac{10x + 25}{5}$$

$$3(x + y) = 3x + 3y$$

for each of the preceding equations, any value for the variable x will make both sides of the equation equal.

- **A *conditional* equation is true for only a limited number of values of the variables. For example, the equation:**

$$x + 5 = 9$$

is true only when $x = 4$

- **A *false* statement, or *contradiction*, is an equation which is never true. That is, there are no values of the variables which make the two sides of the equation equal. An example is the equation:**

$$x = x + 8$$

We indicate that the two sides are not equal by using the symbol "≠", for this example,

$$x \neq x + 8$$

(2-1-1): Solving 1st order Equations in One Variable:

The standard form for the linear equation in one variable(x)is:

$$ax + b = 0 \qquad ,where \quad (a \neq 0)$$

Where a and b are constant.

The solution of this equation is : $ax = -b$

$$i.e., \quad x = \frac{-b}{a}$$

For example, if we have the following equation:

$$3x + 8 = 23 \;\;,$$

Then, $3x = 23 - 8 = 15 \quad i.e., \quad x = \frac{15}{3} = 5$

Note that by substituting in the main equation for *x* = 5, we find that the equation is satisfied, i.e.

$$the\ left\ hand\ side\ = 3x + 8 = 3(5) + 8 = 23$$
$$= right\ hand\ side$$

<u>Example (2-1):</u>

Solve the following equation:

$$3x - 7 = \frac{x + 5}{4}$$

<u>Solution:</u>

Multiplying both sides of the equation by 4, then we have:

$$12x - 28 = x + 5$$

$$12x - x = 5 + 28$$

$$11x = 33$$

$$i.e., \quad x = \frac{33}{11} = 3$$

By substituting in both sides of the main equation for *x* = 3, we find that :

$$left\ hand\ side = 3x - 7 = 3(3) - 7 = 2$$

$$right\ hand\ side = \frac{x+5}{4} = \frac{3+5}{4} = \frac{8}{4} = 2$$

$$i.e.,\quad the\ left\ hand\ side = the\ right\ hand\ side = 2$$

So that, *x* = 3 is the only root for the equation.

(2-1-2): Solving 2nd order Equations in One Variable:

The 2nd order (degree) equation or the quadratic formula in one variable *x* has the following general form:

$$ax^2 + bx + c = 0 \qquad ,where \quad (a \neq 0)$$

Where *a* , *b* and *c* are constants . The 2nd order equations are usually called *quadratic equation* or *quadratic form*.

There are several methods to solve the quadratic equations. We will summarize two methods for determining the two roots for the 2nd order equation. These two methods are:

- **Factorization Method.**
- **Quadratic Formula law.**

Factorization Method:

If the left hand side of the quadratic equation can be factorized, then the roots can be identified very easily. The factorized form of the equation suggests that the product of two terms equals *zero*. The product will equal *to zero* if either of the two factors is equal to *zero*.

For example, consider the following quadratic equation:

$$x^2 - 7x = 0$$

The left hand side of the equation can be factorized, as the following form:

$$x(x - 7) = 0$$

Now, in order to make the multiplication for the two factors (x) and (x -7) is equal to zero then ,either the 1st factor is equal to *zero* : i.e., when $x = 0$ make the two sides for the equation are equal , or also the 2nd factor $(x - 7)$ *is equal to zero* , then we conclude that $x = 7$. Therefore , the two roots for this equation are (0) and (7).

<u>* Quadratic formula</u>:

When the quadratic equation cannot be factorized, or if you are unable to identify the factors, you can apply the quadratic formula. The quadratic formula will allow you to identify the two roots for the equation of the general following form :

$$ax^2 + bx + c = 0 \qquad ,where \quad (a \neq 0)$$

Then, the two roots for the quadratic formula are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

<u>For example</u>: if we have the following 2nd order equation:

$$x^2 + 4x - 21 = 0$$

Then, the coefficients are: $a = 1$, $b = 4$ and $c = -21$,then by Substituting these coefficients in the quadratic formula , the two roots for this equation are calculated as follows:

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-21)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 + 84}}{2} = \frac{-4 \pm \sqrt{100}}{2} = \frac{-4 \pm 10}{2}$$

Using the plus sign, we have the 1st root as follows :

$$x = \frac{-4 + 10}{2} = 3$$

Using the minus sign, we have the 2nd root as follows :

$$x = \frac{-4 - 10}{2} = -7$$

Therefore, the two real values or the two roots for (x) by which satisfy the quadratic equation are : $x = 3$ and $x = -7$.

<u>Example (2-2):</u>

Solve the following equation (Determine the roots for the following equations) :

(a) $x^2 - 3x - 10 = 0$

(b) $x^2 - 6x + 9 = 0$

(c) $x^2 + 9x - 322 = 0$

(d) $3x^2 - 14x + 8 = 0$

$(e)\ 4x^2 - 25 = 0$

Solution:

(a): $x^2 - 3x - 10 = 0$

The equation can be factorized as the following:

$$(x+2)(x-5) = 0$$

Then, by setting each factor equal to *zero*, we find that:

If : $x + 2 = 0$ $\therefore x = -2$

Or : $x - 5 = 0$ $\therefore x = 5$

Therefore, the two roots for this equation are $x = -2\ and\ x = 5$.

(b): $x^2 - 6x + 9 = 0$

Note that, this quadratic form has a square term in the first term its square root is equal to (*x*) , also , the square root for the last term is equal to (3) , and the product for the double of the square root of the 1st term by the square root of the last term is equal to (6*x*) . This formula is called *perfect square* . Then, this equation can be factorized as the following form :

$$(x-3)^2 = 0 \quad or \quad (x-3)(x-3) = 0$$

Therefore, this equation has a unique root , which is equal to $x = 3$ **.**

(c): $x^2 + 9x - 322 = 0$

In this equation, note that ,the left hand side cannot be factorized. Therefore, we apply the quadratic formula as follows:

The coefficients for the quadratic formula are $a = 1$, $b = 9$ and $c = -322$. Then , Substituting these coefficients in the quadratic formula, then, we have the following two roots for the equation which are computed as follows:

$$x = \frac{-9 \pm \sqrt{9^2 - 4(1)(-322)}}{2(1)}$$

$$x = \frac{-9 \pm \sqrt{81 + 1288}}{2} = \frac{-9 \pm \sqrt{1369}}{2} = \frac{-9 \pm 37}{2}$$

Using the plus sign, we get :

$$x = \frac{-9 + 37}{2} = 14$$

Using the minus sign, we get:

$$x = \frac{-9 - 37}{2} = -23$$

Therefore, the two roots for this equation are 14 , -23 which satisfied this equation.

(d): $3x^2 - 14x + 8 = 0$

The coefficients for the quadratic formula are $a = 3$, $b = -14$ and $c = 8$. Substituting these coefficients in the quadratic

formula, then the two roots for this equation are calculated as follows:

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(3)(8)}}{2(3)}$$

$$x = \frac{14 \pm \sqrt{196 - 96}}{6} = \frac{14 \pm \sqrt{100}}{6} = \frac{14 \pm 10}{6}$$

So,

$$x = \frac{14 + 10}{6} = 4 \qquad and \qquad x = \frac{14 - 10}{6} = \frac{2}{3}$$

Therefore, the two roots for this equation are: 4 , $\frac{2}{3}$ which are satisfied this equation.

(e): $4x^2 - 25 = 0$

This equation can be analyzed as a difference between two square terms as follows:

$$(2x - 5)(2x + 5) = 0$$

$$2x - 5 = 0 \qquad \therefore x = \frac{5}{2}$$

$$2x + 5 = 0 \qquad \therefore x = \frac{-5}{2}$$

Therefore, the two roots for this equation are : $\frac{5}{2}$ and $\frac{-5}{2}$.

Example (2-3):

Determine the two roots for the following equation:

$$x^2 = x - 10$$

Solution:

Before starting the solution, the equation should be written on the general form for the 2nd order equation as follows:

$$ax^2 + bx + c = 0$$

So that ; the equation is arranged to become as follows:

$$x^2 - x + 10 = 0$$

Henceforth, the coefficients are $a = 1$, $b = -1$ and $c = 10$. Then; Substituting these coefficients in the quadratic formula as follows:

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(10)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1 - 40}}{2} = \frac{1 \pm \sqrt{-39}}{6}$$

Now, because there is no real square root for the value (-39), then we conclude that there are no real values of (x) by which satisfy the quadratic equation. In other words, the equation has no real roots.

Note that the expression under the radical of quadratic formula $(b^2 - 4ac)$ is called the *discriminant*. The value of the discriminant helps us to determine the number of roots for the quadratic equation. For a quadratic equation of the form:

$ax^2 + bx + c = 0$:

- **If $b^2 - 4ac > 0$,** ***then there are two real roots.***
- **If $b^2 - 4ac = 0$,** ***then there is one real root.***
- **If $b^2 - 4ac < 0$,** ***then there are no real roots.***

(2-1-3): Finding the Linear Equations in Two Variables:

In general, *the linear equation* (or the 1st order equation) in two variables (*x*)and (*y*) is an equation of the following general form:

$$ax + by + c = 0 \quad , where \quad a \neq 0, b \neq 0$$

Where *a* , *b* and *c* are constants.

Determining the equation of a straight line (linear equation):

(I): By using the coordinates of two points.

The equation of a straight line passing through two points of known coordinates$(x_1, y_1), (x_2, y_2)$, is as follows:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

<u>Example (2-4):</u>

Determine the mathematical formula for the equation of a straight line passing through the two points $(3, 8), (-2, 5)$

<u>Solution:</u>

$$Since, \qquad \frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$Then, \quad \frac{y-8}{x-3} = \frac{5-8}{-2-3} = \frac{-3}{-5}$$

$$i.e., \quad -5(y-8) = -3(x-3)$$

$$-5y + 40 = -3x + 9$$

Therefore, the required equation is: $3x - 5y = -31$

Now to verify the validity of the conclusion of the previous mathematical formula, it is possible to substitute by the value of the *x*-coordinate for one of the two points into the resulted mathematical formula, then the resulting value must be the *y*-coordinate of the same point.

Let's substitute in the equation for the value of $x = x_1 = 3$, **as follows:**

$$3(3) - 5y = -31$$

$$-5y = -31 - 9 = -40$$

$$\therefore y = \frac{-40}{-5} = 8 = y_1$$

Therefore, the resulted mathematical formula is mathematically correct.

(II): ***By using the coordinate of a point and the slope.***

The equation of a straight line in terms of a point (x_1, y_1) and having a slope *(m)*, is as follows:

$$y - y_1 = m(x - x_1)$$

Example (2-5):

Determine the mathematical formula for the equation of a straight line passing through the point $(-3, 7)$ and its slope is equal to 2.

Solution:

Since, we have the point $(-3, 7)$by which the line passing through it, and having 2 slope, then we have the following data:

$$x_1 = -3 \quad , \quad y_1 = 7 \quad , \quad m = 2$$

$$y - y_1 = m(x - x_1)$$

$$y - 7 = 2(x - (-3))$$

$$y - 7 = 2x + 6$$

Therefore, the required linear equation is : $-2x + y = 13$.

To verify the validity of the conclusion of the previous mathematical formula, substitute by $x = -3$,then, if the resulting value is $y = 7$. Then, the equation is correct.

$$-2(-3) + y = 13$$

$$y = 13 - 6 = 7$$

So, the mathematical formula of equation is correct.

Remark (1):

If we have a linear equation on the form: $ax + by + c = 0$. Then, the slope for the straight line is:

$$the\ slope(m) = \frac{-\ coefficent\ of\ x}{coefficent\ of\ y}$$

Applying on to the preceding equation in the previous example, the slope of equation: $-2x + y = 13$ is :

$$the\ slope(m) = \frac{-\ (-2)}{1} = 2$$

Remark (2):

The slope m of the straight line connecting two points $(x_1, y_1), (x_2, y_2)$ is:

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, the slope of the straight line passing through the two points $(1, 5), (4, -4)$ is:

$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{\Delta y}{\Delta x} = \frac{-4 - 5}{4 - 1} = \frac{-9}{3} = -3$$

(III): By using the X-Intercept and Y-Intercept:

The equation of a straight line in terms of the two intercepts from the(*x)* and(*y*) axis is on the form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

This equation is for a straight line intercepts (*a*) units from the *x*-axis and (*b*) units from the *y*-axis.

Example (2-6):

Determine the mathematical formula for the equation of a straight line cutting (-3) units from the *x*-axis, and $(\frac{5}{2})$ units from the *y*-axis.

Solution:

Since, the formula for the equation of a straight line by which it is cutting (*a*) unit from the *x*-axis and (*b*) unit from the *y*-axis is as follows:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Then, substituting that: $a = -3 \; and \; b = \frac{5}{2}$ in the previous equation; then we have the following:

$$\frac{x}{-3} + \frac{y}{5/2} = 1$$

$$\therefore \quad \frac{-x}{3} + \frac{2y}{5} = 1$$

By multiplying both sides of the equation by 15, $therefore$:

$$-5x + 6y = 15 \qquad \textit{(Required Equation)}$$

Example (2-7):

Write the equation $-2x + 5y = -20$ in terms of the two intercepts for the(*x*)and (*y*)axis.

Solution:

Since, the equation of a straight line by which intercepts (*a*) unit from the *x*-axis and (*b*) unit from the *y*-axis is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

By dividing both sides of the equation $(-2x + 5y = -20)$ by (-20), then we have the following:

$$\frac{-2x}{-20} + \frac{5y}{-20} = \frac{-20}{-20}$$

$$i.e., \qquad \frac{x}{-20/_{-2}} + \frac{y}{-20/_{5}} = 1$$

$$\therefore \qquad \frac{x}{10} + \frac{y}{-4} = 1$$

Which indicates that this equation is an equation of a straight line cutting (*10*) unit from the *x*-axis and (-*4*) unit from the *y*-axis.

(2-1-4): Economic Applications for the Equations:

**(Mix Production):

The following example indicates what is the meaning of the Mix-Production:

Example (2-8): A company produces two types of products. The weekly available working hours are 250 hours. If each unit of the 1st product requires 5 working hours, and each unit of the 2nd product requires 2 working hours. If the company's management wants to use the available capacity for the working hours full used.

Required:

- ***a)*** **Find the equation by which shows the use of all the available working hours in producing *(x*) units from the 1st product and *(y*)units from the 2nd product.**
- ***b)*** **Determine both the two intercepts from the (*x*)-axis and the (*y*)-axis. Explain the economic significance for each intercept.**

c) **Find the slope for the straight line that represents the equation computed in (*a*). Then interpret its economic significance.**

d) **How many units can be produced from the 1st product if 60 units of the 2nd product are produced?**

e) **If the company decided to produce only one product, what is the maximum quantity that it can produce from the first product?**

Solution:

a): **Since,** ***the number of work hours required for*** **producing(*x*) units of the 1st product** is: $5(x) = 5x$

And, ***the number of work hours required for*** **producing(*y*)units of the 2nd product** is: $2(y) = 2y$

And since the company wants to use all the available working hours, i.e.; full used capacity therefore the required equation is:

$$5x + 2y = 250$$

b)**: Since, the *x*-intercept from the *x*-axis can be obtained by substituting** $y = 0$ **in the preceding equation computed in (*a*), as follows:**

$$5x + 2(0) = 250$$

$$\therefore\ x = \frac{250}{5} = 50$$

Therefore, the *x*-intercept from the *x*-axis is (50,0) , and it indicates that if the company wanted to use all the available work hours in producing the 1st product only, it can produced 50 units of this product, at a time when production from the second product is completely absent.

Also, the *y*-intercept from the *y*-axis can be obtained by substituting $x = 0$ in the equation computed in (*a*), as follows:

$$5(0) + 2y = 250$$

$$\therefore\ y = \frac{250}{2} = 125$$

Henceforth, the *y*-intercept of the *y*-axis is (0,125) ,and it indicates that if the company wanted to use all the available work hours in producing the 2nd product only, it can produce 125 units of this product, at a time when production from the 1st product is completely absent.

***c)*:Where the equation for the full used capacity for producing (*x*) and (*y*) units from the two products respectively is:**

$$5x + 2y = 250$$

$$the\ slope(m) = \frac{-\ coefficent\ of\ x}{coefficent\ of\ y} = \frac{-5}{2} = -2.5$$

It indicates that if all the working hours are full used, then increasing the 1st product by one unit must be decrease in the 2nd product by 2.5 units.

d) **: In the case of producing (60) units of the 2nd product, then the number of units that can be produced from the 1st product in case of using all the available working hours is determined as follows:**

$$5x + 2(60) = 250$$

$$5x = 250 - 120 = 130$$

$$\therefore \quad x = \frac{130}{5} = 26\, units$$

Therefore, the quantity that can be produced from the 1st product in case of producing 60 units of the 2nd product is 26 units from the 1st product.

e): **The maximum quantity that it can produced from the 1st product if the company decided to produce only one product is determined by substituting $y = 0$ in the previous equation computed in (*a*), as follows:**

$$5x + 2(0) = 250$$

$$\therefore \; x = \frac{250}{5} = 50$$

***i.e.*, the maximum units from the 1st product is 50 units from the 1st product with the absence of producing the 2nd product.**

<u>(2-2): Functions:</u>

The *function* is a mathematical relation between at least two variables, one of them is related to another variable (or a set of variables), so if a variable (*y*)is related to another variable *(x)* by a specific relationship where the value of (*y*) is determined after knowing the value of (*x*), then it is said that *(y)* is a function of (*x*), and it is written as follows:

$$y = f(x)$$

It is a functional relation between two variables *(x)* and *(y)*. when we say that *y* is a function of(*x*), we mean that the value of the variable(*y*) depends on and is uniquely determined by the value of the variable *(x)* . The respective roles of the two variables result in the variable (*x)* is being called the *independent* variable and the variable (*y*) is being called the *dependent* variable.

For example, it is known in economic that the quantity of demand for a commodity (or the quantity of supply) is considered a dependent variable, as it is a function of the commodity price, which is considered an independent variable. This may be multiple independent variables (two or more) , for example, the monthly family spending is a dependent variable, which is a function of a set

of the independent variables, including, for example, the monthly family income, family size, number of children in the education stages, and other independent variables.

In general, the *simple* function is a relation between two variables; (*y*) as an dependent variable , and *(x)* as an independent variable, and it is written as: $y = f(x)$

Also, for the multivariate function, it is a relationship between a dependent variable (*y*) and a set of independent variables $(x_1, x_2, x_3, \ldots \ldots \ldots .. x_n)$ **and it is written as:**

$$y = f(x_1, x_2, x_3, \ldots \ldots \ldots .. x_n)$$

(2-2-1): Types of Functions

Functions can be classified into several types according to their structural characteristics. And we will discuss some of these types as follows:

(2-2-1-1): Constant Functions:

A *constant function* has the following general form:

$$y = f(x) = a$$

Where (*a*) is a real number or constant. The value of *(y)* is constant for any value for (*x*) , for example, the equation:

$$y = f(x) = 22$$

is a constant function for any value for *x* . Hence, the value of (y) is equal to 22 always.

An important concept in economics is the *marginal revenue*. Marginal revenue is the additional revenue resulted from selling an additional one unit of a product. If each unit of a product sells at the same price, the marginal revenue is always equal to the price. For example, if a product is sold by $7.5 per unit, the marginal revenue function can be stated as the constant function:

$$\boldsymbol{MR = f(x) = 7.5}$$

Where *MR* denotes the marginal revenue and *(x*) equal to the number of units sold from the product.

(2-2-1-2): Linear Functions:

A *linear function* has the following general form (slope-intercept):

$$\boldsymbol{y = f(x) = ax + b}$$

Where (*a*) and (*b*) are constants. This function is a 1st degree(order) function which is represented by a straight line having slope *(a*) and y-intercept (*b*).

Example (2-9):

if we have the following function:

$$\boldsymbol{f(x) = 4x - 7}$$

Find the value of the function when $\boldsymbol{x = 2\,, x = 3\,, x = -1}$

Solution:

To find the values of the function at the different values of *(x*), we substitute in the function for each value of (*x*), as follow:

$\boldsymbol{f(2) = 4(2) - 7 = 1}$

$\boldsymbol{f(3) = 4(3) - 7 = 5}$ **, and** $\boldsymbol{f(-1) = 4(-1) - 7 = -11}$

(2-2-1-3): Quadratic Functions:

A *quadratic function* has the following general form:

$$\boldsymbol{y = f(x) = ax^2 + bx + c \qquad ; \quad a \neq 0}$$

Where *a* , *b* and *c* are constants. This function is a 2nd degree (order) function which is one of the most used functions in practical life. For example, the function:

$$\boldsymbol{y = f(x) = 8x^2 - 3x + 1}$$

is a 2nd degree (order) function (quadratic function).

(2-2-1-4): Cubic Functions:

A *cubic function* has the following general form:

$$\boldsymbol{y = f(x) = ax^3 + bx^2 + cx + d \qquad ; \quad a \neq 0}$$

Where *a* , *b* , *c* and *d* are constants. This function is a 3rd degree(order) function. For example, the function:

$$\boldsymbol{y = f(x) = 5x^3 + 2x^2 - x + 7}$$

is a 3nd order function (cubic function).

(2-2-1-5): Polynomial Functions:

Each of the previous functions are a special case of the polynomial functions. A *polynomial function of degree (n)* has the following general form:

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a \qquad ; \qquad a_n \neq 0$$

Where a_n , a_{n-1} , … … , a_1 , a are constants . The power for each of (x) must be nonnegative integer and the degree (order) for the polynomial is the highest power for the independent variable (x) in the function.

For example, the function:

$$y = f(x) = 5x^6$$

is a polynomial function of (6)th order or degree.

There are other famous functions such as logarithmic functions and exponential functions, which each have its uses in the field of economics.

(2-2-2): Economic Applications for Functions:

There are many economic applications for the functions, we will cover the following topics:

<u>(2-2-2-1): Demand Function:</u>

A *demand function* is a mathematical relationship expressing the way in which the demanded quantity for any commodity depends on its price . The relationship between these two variables (*demanded quantity* and the *price per unit*) is usually have an inverse relationship; i.e., a *decrease* in price leads to an *increase* in the demand units.

Although most of the demand functions are nonlinear, there are some cases by which the demand function may be approximated may be a linear function . The demand function takes the following form:

$$\boldsymbol{Demand\ quantity\ = f(price\ per\ unit)}$$

$$\boldsymbol{i.e.,\qquad q = f(p)}$$

<u>Example (2-10):</u>

From the previous experience of one of the producers, the level of demand for his product reached to 6000 units at a price 20 pounds per unit. And the level of demand reached to 4500 units at a price 30 pounds per unit.

<u>Required:</u>

i. **Determine the linear demand function which takes the form $\boldsymbol{q = f(p)}$, where (*q*) is the demand quantity and (*p*) is the unit price of product.**

ii. **If the demand quantity reached to 1500 units, what is the unit selling price?**

iii. **What is the expected demand quantity if the unit selling price reached to 40 pounds?**

iv. **Economically explain the meaning of the slope of the demand function determined in (*i*).**

Solution:

***i)*: Assuming that the demand function is linear on the form: $q = f(p)$; and to deduce the linear formula(function), it passes through the two points: $(p_1, q_1), (p_2, q_2)$ then, the two points are $(20, 6000), (30, 4500)$, and since the equation of the straight line passing through the two points is:**

$$\frac{q - q_1}{p - p_1} = \frac{q_2 - q_1}{p_2 - p_1}$$

Hence; $p_1 = 20, q_1 = 6000 , p_2 = 30, q_2 = 4500$

By substitution in the previous equation, then we have the following:

$$\frac{q - 6000}{p - 20} = \frac{4500 - 6000}{30 - 20} = \frac{-1500}{10} = \frac{-150}{1}$$

$$q - 6000 = -150(p - 20) = -150p + 3000$$

$$\therefore q = -150p + 9000 \qquad or \qquad 150p + q = 9000$$

ii): **If the demand quantity reached to 2000 units, then to determine the selling price unit , we substitute by $q = 1500$ in the demand function as:**

$$1500 = -150p + 9000$$

$$150p = 9000 - 1500 = 7500$$

$$p = \frac{7500}{150} = 50$$

Therefore, the price by which the demand quantity reached to 1500 units is 50 pounds.

iii) **: To determine the demand quantity at price 40, we substitute by $p = 40$ in the demand function as follows:**

$$q = -150(40) + 9000$$

$$q = 3000$$

So, at the selling price unit 40 pounds, the demand quantity is 3000 units.

***iv)*:** **Since, the demand function is $150p + q = 9000$**

$$\textit{The slope of demand function}:$$

$$= \frac{-\textit{the coefficient of the independent variable}}{\textit{the coefficient of the dependent variable}}$$

$$= \frac{-\textit{ the coefficient of price}}{\textit{the coefficient of quantity}} = \frac{-\textit{ the coefficient of p}}{\textit{the coefficient of q}}$$

$$i.e., \quad the\ slope\ of\ the\ demand\ function = \frac{-150}{1} = -150$$

The economic interpretation of the slope of the demand function means that if the selling price of the unit increases by one pound, it leads to decreasing the demand quantity by 150 units.

(2-2-2-2): Supply Function:

A *supply function* relates market price to the quantities that suppliers are willing to produce and sell. The implication of supply functions is that what is brought to the market depends upon the price people are willing to pay. As opposed to the inverse nature of price and quantity demanded, the quantity which suppliers are willing to supply usually varies directly with the market price.

As with demand functions, supply functions can be approximated sometimes using linear functions.

The supply function takes the following form:

$$supplied\ Quantity = f(market\ price)$$

$$q = f(p)$$

Example (2-11):

If the supply function for a commodity is:

$$q = 0.5p + 35$$

Where *(q)*: is the supplied quantity in units of a commodity,

(p): is the unit price in pounds.

<u>Required:</u>

i. **Finding the supplied quantity of this commodity when the unit price of the commodity is 20 pounds.**

ii. **Determine the slope of the supply function with an explanation of its economic significance.**

<u>Solution:</u>

***i)*: To find the supplied quantity at price 20, we substitute by $p = 40$ in the supply function as follows:**

$$q = 0.5(20) + 35$$

$$= 10 + 35 = 45 \qquad units$$

***ii)*: To determine the slope for the supply function, the function is reformulated on the form: $ax + by = c$ *i.e.*,**

$-0.5p + q = 35$ **,then :**

$$The\ slope\ of\ the\ supply\ function \text{ is}$$

$$= \frac{-\,the\ coefficient\ of\ price}{the\ coefficient\ of\ quantity} = \frac{-\,the\ coefficient\ of\ p}{the\ coefficient\ of\ q}$$

$$\therefore The\ slope\ of supply\ function = \frac{-(-0.5)}{1} = 0.5$$

Which means that for every one pound increasing in the price of the commodity, the supplied quantity of it increases by 0.5 unit, and in

another meaning for every two pounds increasing in the commodity price, the supplied quantity of it increases by one unit.

(2-2-2-3): Market Equilibrium:

One of the most important economic applications for the demand and supply functions is to determine the point of equilibrium (or *market equilibrium*). If we have the demand and supply functions for a specific product, then the market *equilibrium* exists if there is a price at which the demand quantity equal to the supplied quantity. Therefore, to determine the market equilibrium point, the demand and supply functions are solved simultaneously to obtain the equilibrium price and the equilibrium quantity.

Example (2-12):

If the demand function for a specific product is:

$$p^2 + q^2 = 169$$

Where q: is the demand quantity.

p: is the unit price.

And the supply function for the same product is:

$$p = q + 7$$

Required:

Determine the equilibrium price and quantity for this product.

<u>Solution:</u>

In the equilibrium condition (or point) the demand quantity is equal to the supplied quantity. Therefore; in order to find the equilibrium point we must make the demand quantity (*q*) from the demand function is equal to the supplied quantity *(q*) from the supply function , i.e.;

From the demand function: $\boldsymbol{p^2 + q^2 = 169}$

$$\boldsymbol{then, \qquad q^2 = 169 - p^2}$$

$$\boldsymbol{i.e., \qquad q = \pm\sqrt{169 - p^2} = \sqrt{169 - p^2}}$$

From the supply function: $\boldsymbol{p = q + 7}$

$$\boldsymbol{\therefore q = p - 7}$$

And in the equilibrium case, the demand quantity is equal to the supplied quantity, *i.e.*, the demand and supply equations are equal.

Therefore:

$$\boldsymbol{\sqrt{169 + p^2} = p - 7}$$

And by squaring both sides, then we get:

$$\boldsymbol{169 - p^2 = p^2 - 14p + 49}$$

$$\boldsymbol{2p^2 - 14p - 120 = 0}$$

And by dividing both sides by 2, then we have the following:

$$\boldsymbol{p^2 - 7p - 60 = 0}$$

And by factorization, then we have the following :

$$(p-12)(p+5)=0$$

$$so, either \quad (p-12)=0 \qquad \therefore p=12$$

$$or \quad (p+5)=0 \qquad \therefore p=-5 \qquad \textbf{(Rejected)}$$

Therefore, the equilibrium price is: $p = 12\, pound$, and to determine the equilibrium quantity, we substitute by $p = 12$ in the supply function:

$$\therefore q = 12 - 7 = 5 \qquad units$$

Henceforth, in the equilibrium case:

$$\text{the demand quantity} = \text{the supplied quantity} = 5\, units$$

<u>(2-2-2-4): Total Revenue Function:</u>

The money by which flows into an organization from either selling the products or providing services is often referred to as *revenue.* The most fundamental way of computing total revenue from selling a product is as follows:

$$Total\ revenue = (unit\ selling\ price)(sold\ quantity\ )$$

So, the total revenue(R) is a function of two independent variables the selling price (p) and the sold quantity(q) of it, as follows:

$$R = p \times q$$

And the total revenue function can be written as a function only of either the quantity or of price, since the demand function:

$$q = f(p)$$

Then the total revenue function can be determined using the demand function as follows:

$$Total\ revenue\ R(p) = pq = p.f(p)$$

Therefore, the total revenue can be expressed as a function of 2nd order in price.

Also, we can express the total revenue as a function of the sold quantity as follows: since; From the demand function, the price is expressed as a function of quantity as follows:

$$p = f(q)$$

$$\therefore\ Total\ revenue\ R(q) = pq = q.f(q)$$

In conclusion, the total revenue function *R(q)* is either a function of *(p)* and *(q)*, or a function of 2nd order in either (*p*) or (*q)* only.

<u>Example (2-13):</u>

If the demand function for a specific product is:

$$q = 1500 - 50p$$

Where *(q*): is the demand quantity in thousands of units,

And, (*p)*: is the unit price.

Required:

i. **Find the total revenue function.**

ii. **What is the total revenue if the unit price reached to 20 pounds?**

iii. **What is the total revenue if the amount of sales reached to 400 units?**

Solution:

***i)* :** **Since,** $The\ Total\ Revenue\ R = pq$

And from the demand function: $q = 1500 - 50p$

Hence, $The\ total\ Revenue\ R(p) = p(1500 - 50p)$

$$\therefore R(p) = 1500p - 50p^2$$

***ii)*: The total revenue if the unit price reached to 20 pounds is:**

$$R(20) = 1500(20) - 50(20)^2$$

$$\therefore R(20) = 30000 - 20000 = 10000 \quad pounds$$

***iii)*: The total revenue if the amount of sales reached to 400 units, *i.e.*, at , *q* = 400 , then:**

Substituting by $q = 400$ **in the demand function, then we get:**

$$400 = 1500 - 50p$$

$$50p = 1500 - 400 = 1100$$

$$\therefore p = \frac{1100}{50} = 22 \quad pounds$$

Therefore, the amount of revenue when the quantity of sales reached to 400 units is the same as the amount of revenue when the unit price reached to 22 pounds.

$$\therefore R(22) = 1500(22) - 50(22)^2 = 8800$$

$$or \qquad R = pq = 22 \times 400 = 8800$$

<u>(2-2-2-5): Total Cost Function:</u>

The *Total cost* is the total cost for producing the total production. It is defined in terms of two components: *total variable cost* and *fixed cost* .Therefore the total cost has the f0llowing form:

$$Total\ cost\ C(q) = total\ variable\ cost + fixed\ cost$$

= *a q* + *b*   where : (a) and (b) are real numbers , And (a) is the variable cost for each unit produced , (b) is the fixed cost .

<u>(2-2-2-6): Total Profit Function:</u>

***Profit* is the difference between the total revenue and the total cost. It can be derived as in the following equation:**

The $Total\ Profit = theTotal\ revenue - the\ Total\ cost$

If $\boldsymbol{the\,Total\,revenue = R(q)}$ **and** $\boldsymbol{the\,Total\,cost = C(q)}$

Where *(q)*is the produced and sold units from the commodity. Hence the total profit function is defined as follows:

$$\boldsymbol{P(q) = R(q) - C(q)}$$

When the total revenue exceeds the total cost, profit is positive. In such cases the profit may be referred to as a *net profit.* When the total cost exceeds the total revenue, the profit is negative. In such cases the negative profit may be referred to as a *net loss.*

Example (2-14): A company can sell each one of its product at price 55 pounds per unit. And the variable cost is equal to 23 pounds per unit, while the fixed costs is 400,000 pounds, so if we assume that *(q)* is the number of unit produced and sold during the year.

Required:

- *i.* **Find the total cost function.**
- *ii.* **Find the total revenue function.**
- *iii.* **Find the total profit function.**
- *iv.* **What is the amount of profit if 15,000 units were produced and sold during the year?**
- *v.* **What is the amount of production required for the company to achieve profits equal to 64,000 pounds?**

Solution:

i) **:** $\boldsymbol{Since,\ the\ total\ cost = total\ variable\ cost + fixed\ cost}$

$$\therefore \quad \boldsymbol{C(q) = 23q + 400000}$$

ii) : $\boldsymbol{Since,\ the\ total\ revenue}$

$$\boldsymbol{= (unit\ selling\ price)(quantity\ sold)}$$

$$\boldsymbol{R(q) = pq}$$

$$\therefore \boldsymbol{R(q) = 55q}$$

iii) $\boldsymbol{Since,\ the\ total\ Profit = Total\ revenue - Total\ cost}$

$$\boldsymbol{P(q) = R(q) - C(q)}$$

$$\boldsymbol{= 55q - (23q + 400000)}$$

$$\boldsymbol{= 55q - 23q - 400000}$$

$$\therefore \boldsymbol{P(q) = 32q - 400000}$$

***iv)* :The amount of profit if 15,000 units are produced and sold is:**

$$\boldsymbol{P(15000) = 32(15000) - 400000}$$

$$\boldsymbol{= 480000 - 400000 = 80000\ \ pounds}$$

***v):* In order to make the company achieve profits of 64,000 pounds, the amount of production can be determined as follows:**

$$\boldsymbol{P(q) = 32q - 400000}$$

$$64000 = 32q - 400000$$

$$32q = 400000 + 64000 = 464000$$

$$\therefore q = \frac{464000}{32} = 14500 \qquad units$$

(2-2-2-7): Break - Even Models:

***Break - Even models* are a set of planning tools which can be useful in managing organization. These models enable decision-makers to determine the amount of production at which the organization covers the cost of its production only without achieving any profits (*zero profit*) . This level of production is called the *break – even point.* This point represents the level of production at which the total revenue equal to the total cost. Any changes from this level of production will result in either a profit or loss.**

Some Assumptions of Break - Even Models:

- **The total cost function is linear, which implies that the variable cost per unit is either a constant or can be assumed to be a constant. The linear cost function assumes that the total variable costs depend on the level of production.**
- **The total revenue function is linear; this assumes that the selling price per unit is constant.**
- **The price per unit is greater than the unit variable cost. If price per unit is less than variable cost per unit, a firm will**

lose money on every unit produced and sold. A break – even condition could never exist.

In break – even analysis, the primary objective is to determine the break-even point by the following steps:

1) **Formulate the total cost as a function of the level of production (*q*).**
2) **Formulate the total revenue as a function of the level of production (*q*).**
3) **Since break-even conditions exist when the total revenue is equal to the total cost and set *C(q)* equal to *R(q)* and find the value of (q) . The resulted value of *(q)* is the break-even level for production.**

An alternative to step (3) is to construct the profit function $P(q) = R(q) - C(q)$ and set *P(q)* is equal to zero and solve for(*q*).

Example (2-15): **A company sell each one of its product at price 45 pounds per unit. If the variable cost is equal to 33 pounds per unit, and the fixed costs is 450,000 pounds.**

Required:

What is the number of units that must be produced and sold in order to the company to cover the cost of its production without achieving any profits?

Solution:

Assuming that the number of unit produced is *(q)*. Then :

The Total revenue function: $R(q) = 45q$ **and;**

The Total cost function: $C(q) = 33q + 450000$

Hence to determine the break-even point we have to find the value for (q) by which satisfied the following equation:

$$R(q) = C(q)$$

$$45q = 33q + 450000$$

$$12q = 450000$$

$$\therefore q = \frac{450000}{12} = 37500 \ units$$

Therefore; we conclude that the company must produce and sell 37500 units in order to break-even.

Exercises for Chapter (2)

1- Solve the following equations:

$(a)\ \frac{3x+7}{2}=\frac{1+x}{3}$

$(b)\ (x-4)^2=(x-2)^2$

$(c)\ x^2-2x-3=0$

$(d)\ 4x^2-5x=0$

$(e)\ 6x^2+2.5x+0.25=0$

$(f)\ 3x^2+6x-2=0$

$(g)\ x+3=\sqrt{5x+11}$

2- Find the equation of the straight line that satisfies the conditions for each of the following:

(a) **Passing through the point** $(2,5)$ and **its slope** $=0.$

(b) **Passing through the point** $(1,-2)$ and **its slope** $=-3.$

(c) **The slope** $=0$ and **the part cutting from the** y**-axis** $=5$

3- Find the slope and the the(y)-intercept for each of the following linear equations:

$(a)\ 4x+5y=20$

$(b)\ \frac{x}{5}+\frac{y}{7}=1$

$(c)\ y-2x+3=0$

$(d)\ 3x + 4y = 0$

4- Find the equation of the straight line by which it passing through the following two points:

(a): $(3, 1)$ and $(2, 7)$

(b): $(2, 1)$ and $(3, -1)$

(c): $(-2, 3)$ and $(-3, 4)$

5- A company manufactures two types of products (A, B), and the available working hours per week are 120 working hours used in the manufacture of these two products. The company wants to use the available working hours in its full used capacity. If you know that the unit production from the 1st product (A) needs 3 working hours, while the unit from the 2nd product needs 2.5 working hours.

Required:

- ***a)*** **Find the equation by which show the use of all the available working hours.**
- ***b)*** **Through your results in (a), determine both the two intercepts from the *x*-axis and the *y*-axis, then explain the economic significance for each one.**
- ***c)*** **Find the slope of the straight line that represents the equation computed in (*a*). Then interpret its economic significance.**
- ***d)*** **If the company decided to produce only one product, what are the maximum units that it can produce from the 1st product.**

6- If the number of units required of a particular product is indicated by the following equation:

$$q = 50000 - 12.5p$$

$Where \quad q: is\ the\ quantity\ of\ demand\ per\ units$

$and \quad p: is\ the\ unit\ selling\ price.$

Required:

- ***a)*** **Find both the two intercepts from the *q*-axis and the *p*-axis, then explain the economic significance for each one.**
- ***b)*** **Find the slope of the equation. Then interpret its economic significance.**
- ***c)*** **What is the expected quantity of demand if the unit selling price reached to 200 pounds?**
- ***d)*** **If the level of demand reached to 25000 units. What is the expected unit selling price?**

7- A phone dealer can sell one device at a price 150 pounds, if the fixed cost is 15,000 pounds per month, in addition to the cost of the raw materials and labor for each device amounts to 100 pounds.

Required: **Determine the number of devices that must be produced and sold in order for the trader to achieve a monthly profit of at least 1000 pounds.**

8- A factory owner sells a product at a price 25 pounds per unit, and the total cost of producing (*q*) units is determined by the following mathematical formula:

$$C(q) = 300 + 20q - q^2$$

<u>Required:</u>

What is the number of units that must be produced and sold weekly in order for the factory owner to achieve profits?

9- From the previous experience of one of the producers indicates that he can sell 8,000 units from his product at a price 20 pounds per unit, also he can sell 62,500 units at a price 30 pounds per unit.

<u>Required:</u>

- ***a)*** **Determine the linear demand function which takes the form $q = f(p)$, where (*q*) is the demand quantity and (*p*) is the unit price from the product.**
- ***b)*** **If the demand quantity reached to 70,000 units, what is the unit selling price of this product?**
- ***c)*** **What is the expected demand quantity if the unit selling price reached to 25 pounds?**
- ***d)*** **Find the slope of the demand function. Then interpret its economic significance.**

10- If the demand and supply functions for one of the products are:

$$q = p^2 - 40p + 400 \qquad \textit{Demand function}$$

$$q = p^2 - 100 \qquad\qquad \textit{Supply function}$$

Required:

Determine the market equilibrium price and quantity of this product.

11- A company sells its product at a price of 65 pounds per unit, if you know that the variable cost for the unit is 20 pounds from the raw materials, 27.5 pounds from work hours, and the fixed cost is 100,000 pounds.

Required:

- ***a)*** **Express the profit as a function of the number of produced and sold units (*q*).**
- ***b)*** **Find the slope of the profit function and interpret its economic significance.**
- ***c)*** **How much the profit did the company get as a result of selling 20,000 units?**

12- If the fixed costs in a factory for producing the television devices amount to 10,000 pounds, while the cost of the labor and the raw materials is 500 pounds per device.

Required:

- ***a)*** **Find the total cost function as a function of the number of sold devices.**

b) **Assuming that each device is sold at a price 2,000 pounds, determine the total revenue function and the total profit function.**

c) **What is the amount of profit if the factory produced and sold 200 devices?**

13- If the total cost function for a specific product is:

$$C(q) = 100q^2 + 1300q + 1000$$

Where (q) is the number of units produced in thousands, and $C(q)$ is the total cost in thousands of pounds. If each unit of the product is sold at a price 2,000 .

Required:

a) **Find the total revenue function.**

b) **What is the amount of production that covers the production cost without achieving any profits?**

c) **Find the total profit function, and then calculate the total profit when producing 4,000 units.**

Chapter (3)

Determinants

Determinants are considered from the important and common mathematical methods used in many practical applications of mathematics. The determinants were not known until the middle of the last century. Once it appeared it was widely used in various branches of mathematics.

The determinants are used as a tool to simplify mathematical operations that seem to be complexity. In the following we will clarify the concept of determinants, as we show the most important characteristics and the uses of the determinants, supported by examples that facilitate the reader to understand this topic.

(2-1): The Definition of Determinant:

The determinant is a set of numbers or symbols(or elements) organized in a set of rows and a set of columns between two vertical lines subject to the number of rows must be equal to the number of columns by which is called the order of the determinant. The symbol Δ (delta) is often used to denote the value of a given determinant.

(2-1-1): The Value for the 2^{nd} order determinants:

From the preceding concept for the determinant, the 2^{nd} order determinant is a square system of elements consisting of two rows (horizontal) and two columns (vertical). It is defined by the following expression:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

In other words, the determinant is given by the product of elements a_1 and b_2 on the main diagonal ($\searrow$) minus the product of elements a_2 and b_1 on the cross-diagonal ($\swarrow$). We can indicate these two diagonals by means of arrows as follows:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b2 - a_2 b_1$$

Example (2-1):

Determine the value of the following determinants:

a- $\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$ b- $\begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix}$ c- $\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix}$

Solution:

a- $\Delta = \begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$ **= 2(5) – 4(-3) = 10 + 12 = 22**

b- $\Delta = \begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix}$ **= 3(4) – 0(2) = 12 – 0 = 12**

c- $\Delta = \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix}$ **= 3(1) – 2(5) = 3 – 10 = -7**

(2-1-2): The value for the 3rd order determinants:

The determinant is of the 3rd order if it consists of three rows and three columns. It is easy to show the 3rd order determinant as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Where:

a_{11} **: The element that is in the 1st row and the 1st column.**

a_{12} **: The element that is in the 1st row and the 2nd column.**

a_{13} **: The element that is in the 1st row and the 3rd column.**

In general, the element (a_{ij} **)is the element that is in the** i^{th} **row and the** j^{th} **column.**

Also, the determinant of 3rd order can be expressed as follows:

$$\Lambda \quad - \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

When we want to determine the value for the a 3rd order determinant, we use what are called *cofactor*. Each element in the determinant has a cofactor, which is denoted by the corresponding capital letter. For example, A_2 denotes the cofactor of a_2, B_3 denotes the cofactor of b_3, and so on. They are defined as follows.

The minor of an element in a determinant Δ is equal to the determinant obtained by deleting the row and column in Δ that contain the given element. If the given element lies in the i^{th} row and the j^{th} column of Δ, then its cofactor is equal to $(-1)^{i+j}$ times its minor.

For example, in the above determinant Δ, a_2 lies in the second row and the first column ($i = 2$ and $j = 1$), so its cofactor is:

$$A_2 = (-1)^{2+1} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

$$= - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = -(b_1 c_3 - b_3 c_1)$$

Also, since c_3 lies in the 3rd row and the 3rd column (i.e.; i = 3 and j = 3), then its cofactor is:

$$C_3 = (-1)^{3+3} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^6 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The value of a determinant can be found by multiplying the elements in any row (or column) by their cofactors and adding the products for all elements in the this row (or column).

For example, in the previous determinant the value of this determinant ($\triangle$), and by using the first row, then the value of this determinant ($\triangle$) is as follows:

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1$$

Note that, the value of the determinant can be found by using the elements of any row or column. Here it should be noted that when calculating the value of any determinant, it is more appropriate to use the row or column that contains the largest number of zeros elements (if it exists), because we will not need to calculate the values of its cofactors for the elements that are zeros, which will facilitate the execution of the computational operations.

It is easy to find the sign factors{ $(-1)^{i+j}$ }of cofactors for any determinant, as the sign change from positive to negative than positive, and so on in the horizontal or vertical direction as shown in the following:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} , \qquad \begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$$

And so on according to the order of the determinant.

<u>Example (2-2):</u>

Determine the value of the following determinant:

$$\begin{vmatrix} 2 & 1 & 3 \\ 5 & 3 & 2 \\ 1 & 0 & 4 \end{vmatrix}$$

Solution:

The value of this determinant can be found by using the elements of any row or column. However, it is noted in this determinant that the third row or the second column have zero as one of its elements. Therefore, it is easier to find the value of this determinant by using the elements for the third row or the elements for the second column, as follows:

By using the third row:

$$\Delta = 1 \times \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} - 0 + 4 \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix}$$

= 1 (1 x 2 – 3 x 3) + 4 (2 x 3 – 5 x 1)

= (2 – 9) + 4 (6 – 5) = –7 + 4(1) = – 3

By using the second column:

$$\Delta = -1 \times \begin{vmatrix} 5 & 2 \\ 1 & 4 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} - 0$$

= –1 (5 x 4 – 1 x 2) + 3 (2 x 4 – 1 x 3)

= –1 (20 – 2) + 3 (8 – 3) = –18 + 3(5) = –3

Note that; It is the same as the previous result.

To verify that the value of the determinant does not differ according to the row or column used, we will find the value of this determinant by using any other row or column (let it be the first row):

$$\Delta = 2 \times \begin{vmatrix} 3 & 2 \\ 0 & 4 \end{vmatrix} - 1 \times \begin{vmatrix} 5 & 2 \\ 1 & 4 \end{vmatrix} + 3 \times \begin{vmatrix} 5 & 3 \\ 1 & 0 \end{vmatrix}$$

= 2 (3 x 4 – 0 x 2) – 1 (5 x 4 – 1 x 2) + 3 (5 x 0 – 1 x 3)

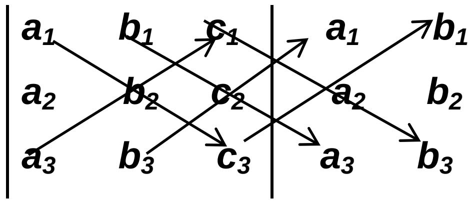
= 2 (12 – 0) – 1 (20 – 2) + 3 (0 – 3)

= 24 – 18 – 9 = –3

It is the same value that we obtained before.

Other method to find the value of the 3rd order determinant is the :(<u>Sarrus Diagram</u>):

This method is summarized in writing the elements of the three columns and then rewriting the first and second columns in the same order in the right of the determinant as follows:



The three diagonals heading to downwards represent the positive terms in the determinant expansion. The three diagonals heading to up represent the negative terms in the determinant expansion.

That is, the positive terms are:

$$a_1b_2c_3\ ,\ \ b_1c_2a_3\ ,\ \ c_1a_2b_3$$

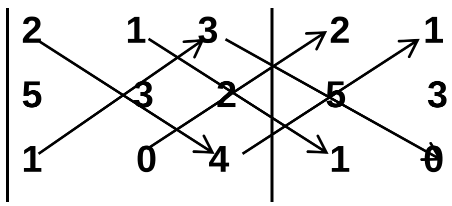
And the negative terms are:

$$a_3b_2c_1\ ,\ \ b_3c_2a_1\ ,\ \ c_3a_2b_1$$

So, the value of determinant is:

$$\Delta = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

For example, by applying Sarrus Diagram to find the value of the determinant given in Example 2:



$$\Delta = (2 \times 3 \times 4) + (1 \times 2 \times 1) + (3 \times 5 \times 0) - (1 \times 3 \times 3) - (0 \times 2 \times 2)$$

– (4 x 5 x 1)

= 24 + 2 + 0 – 9 – 0 – 20

= 26 – 29 = – 3 , which is the same preceding result.

<u>Example (2-3):</u>

Find the value of the following determinant.

$$\begin{vmatrix} 2 & 3 & -1 \\ 5 & 2 & 4 \\ 3 & -2 & 6 \end{vmatrix}$$

<u>Solution:</u>

Finding the value of determinant using the elements of the first row:

$$\Delta = 2 \times \begin{vmatrix} 2 & 4 \\ -2 & 6 \end{vmatrix} - 3 \times \begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix} + (-1) \times \begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix}$$

= 2 [2 x 6 – (-2) x 4] – 3 (5 x 6 – 3 x 4) + (-1) [5 x (-2) – 3 x 2]

= 2 (12 + 8) – 3 (30 – 12) – 1 (–10 – 6)

= 40 – 54 + 16 = 2

And by Using Sarrus Diagram; then we have :

$$\left|\begin{array}{ccc|cc} 2 & 3 & -1 & 2 & 3 \\ 5 & 2 & 4 & 5 & 2 \\ 3 & -2 & 6 & 3 & -2 \end{array}\right.$$

$\Delta =$ (2 x 2 x 6) + (3 x 4 x 3) + (-1 x 5 x -2) – (3 x 2 x -1) – (-2x 4 x 2) – (6 x 5 x 3)

= 24 + 36 + 10 – (-6) – (-16) – 90

= 70 – 68 = 2

The same result is obtained by using two different methods.

(2-1-3): The value for the determinant of at least 4th order :

The determinant of 4th order consists of four rows and four columns. Any determinant can be calculate its value by expansion by using any row or column. The value of the determinant is obtained by finding the summation of multiplying each element in the row (or column) by its cofactor. For example, consider the following determinant of 4th order:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

Its expansion by the first row is given by

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}$$

$$+ c_1 \begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1 \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

The value of each minor (of the 3^{rd} order) can be obtained by the two previous methods covered when we study how can we determined the value of the 3^{rd} order determinant.

As for the determinants of the fifth order, we need to find the values of the minors of the fourth order, and so on for the determinants of the higher order.

<u>Example (2-4):</u>

Determine the value of the following determinant:

$$\begin{vmatrix} 5 & 3 & 4 & 2 \\ 1 & 2 & 5 & 3 \\ 2 & 5 & 6 & 5 \\ 3 & 1 & 2 & 4 \end{vmatrix}$$

Solution:

The value of determinant using the elements of the first row:

$$\Delta = 5\begin{vmatrix} 2 & 5 & 3 \\ 5 & 6 & 5 \\ 1 & 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 & 3 \\ 2 & 6 & 5 \\ 3 & 2 & 4 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 6 \\ 3 & 1 & 2 \end{vmatrix}$$

After that, it can find the value of each third-order determinant.

(The student must continue and complete the solution…… , and then , you have the following result)

Result : Therefore, the value of the determinant is:

$$\Delta = 5(-35) - 3(7) + 4(-10) - 2(-33)$$
$$= -175 - 21 - 40 + 66 \qquad = -170$$

(2-2): The Properties of Determinants

Determinants have some properties that are useful as it permits us to get the same results with different and simple configurations of entries (elements). Some of these properties will have been covered below in a detailed way along with solved examples:

Property (1):

The determinant remains unchanged if its rows are changed into columns and the columns into rows.

If we denote the value of the determinant by Δ_1. And for the new value for the determinant after changing the rows into columns or changing columns into rows by Δ_2 as follow:

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

According to this property, we find that $\Delta_1 = \Delta_2$

<u>Example (2-5):</u>

Prove that:

$$\Delta_1 = \begin{vmatrix} 2 & 3 & -1 \\ 5 & 2 & 4 \\ 3 & -2 & 6 \end{vmatrix} = \Delta_2 = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 2 & -2 \\ -1 & 4 & 6 \end{vmatrix}$$

<u>Solution:</u>

The value of Δ_1 using the elements of the first row is:

$$\Delta_1 = 2 \begin{vmatrix} 2 & 4 \\ -2 & 6 \end{vmatrix} - 3 \begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix} + (-1) \begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix}$$

$$= 2(12 - (-8)) - 3(30 - 12) - 1(-10 - 6)$$

$$= 40 - 54 + 16 = 2$$

The value of Δ_2 using the elements of the first row is:

$$\Delta_2 = 2 \begin{vmatrix} 2 & -2 \\ 4 & 6 \end{vmatrix} - 5 \begin{vmatrix} 3 & -2 \\ -1 & 6 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}$$

= 2(12 – (-8)) – 5(18 – 2) + 3(12 – (-2))

= 40 – 80 + 42 = 2

$$\therefore \Delta_1 = \Delta_2 = 2$$

Property (2):

If any two rows (or columns) of the determinant are exchanged, the value of determinant doesn't change but the sign for the value the determinant is changed. It can be shown as follow:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}$$

Example (2-6):

Determine the value of the following determinant.

$$\begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix}$$

And find its value after exchanging the 1st row with the 2nd row.

Solution:

$$\Delta_1 = \begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix} = 4 - (-12) = 16$$

After exchanging the 1st row with the 2nd row:

$$\Delta_2 = \begin{vmatrix} 4 & 2 \\ 2 & -3 \end{vmatrix} = -12 - 4 = -16$$

$$\therefore \Delta_1 = -\Delta_2$$

Property (3):

If we move a row (or a column) in a determinant, then if the row (or column) skip a certain number of rows (columns) = *m*, the numerical value of the determinant does not change if *m* is an even number, while its sign changes only if *m* is an odd number. For example ;If we have the following determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

And ; If we moved the first column to skip the second and third columns and settled in place of the third column. In this case, the first column will have skipped two columns (i.e., *m* =2, which is an even number), then the value of the new determinant will be equal the same value of original determinant as follow:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 & d_1 \\ b_2 & c_2 & a_2 & d_2 \\ b_3 & c_3 & a_3 & d_3 \\ b_4 & c_4 & a_4 & d_4 \end{vmatrix}$$

But if the first column will have skip three columns (i.e., *m* =3, which is an odd number) to settled in place of the fourth column, then the sign of the new determinant will change as follow:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = - \begin{vmatrix} b_1 & c_1 & d_1 & a_1 \\ b_2 & c_2 & d_2 & a_2 \\ b_3 & c_3 & d_3 & a_3 \\ b_4 & c_4 & d_4 & a_4 \end{vmatrix}$$

Example (2-7):

Prove that:

$$\Delta_1 = \begin{vmatrix} 2 & 4 & 3 \\ -3 & 1 & 5 \\ 0 & 2 & 6 \end{vmatrix} = \Delta_2 = \begin{vmatrix} 4 & 3 & 2 \\ 1 & 5 & -3 \\ 2 & 6 & 0 \end{vmatrix}$$

Solution:

Note that, in this example, the 1st column and the 3rd column have exchanged their positions, in other meaning the first column skip two columns (*m* = 2), therefore the value of the determinant does not change, and we can verify that as follows:

The value of(Δ_1)by using the elements of the first column:

$$= 2 \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} - (-)3 \begin{vmatrix} 4 & 3 \\ 2 & 6 \end{vmatrix} + 0$$

$$= 2(6 - 10) + 3(24 - 6)$$

$$= -8 + 54 = 46$$

The value of (Δ_2)by using the elements of the 3rd row is:

$$\Delta_2 = 2 \begin{vmatrix} 3 & 2 \\ 5 & -3 \end{vmatrix} - 6 \begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix} + 0$$

$$= 2(-9-10) - 6(-12-2)$$

$$= -38 + 84 = 46$$

i.e.;

$$\therefore \Delta_1 = \Delta_2 = 46$$

<u>Property (4):</u>

If all the elements of a row (or column) are identical (or proportional) to the elements of other row (or column), then the value of the determinant is equal to zero.

<u>For example:</u>

$$\begin{vmatrix} 3 & 1 & 3 \\ 4 & -3 & 4 \\ 2 & 5 & 2 \end{vmatrix} = 0$$

This is because the 1st and 3rd columns are identical.

Also,

$$\begin{vmatrix} 6 & 4 & 2 \\ -2 & 5 & 6 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

This is because : the 1st row = 2 × the 3rd row.

<u>Property (5):</u>

If we find in one of the rows (or columns) in the determinant a common factor(*k*), and divide each element of this row or (column) by this common factor, then the value of the original determinant is equal to *k* multiplied by the value for the new value of the resulted determinant.

In other words, in any determinant Δ, if all elements in any row or (column) are multiplied by a constant *k,* then the resulting determinant equal to $k\Delta$.

<u>For example:</u>

$$3\begin{vmatrix} 2 & 3 & 6 \\ 4 & 5 & 1 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 6 & 3 & 6 \\ 12 & 5 & 1 \\ -3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 9 & 6 \\ 4 & 15 & 1 \\ -1 & 6 & 3 \end{vmatrix}$$

Or

$$3\begin{vmatrix} 2 & 3 & 6 \\ 4 & 5 & 1 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 6 & 9 & 18 \\ 4 & 5 & 1 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 6 \\ 12 & 15 & 3 \\ -1 & 2 & 3 \end{vmatrix}$$

This can be verified by considering only one case, and let it be:

$$3\begin{vmatrix} 2 & 3 & 6 \\ 4 & 5 & 1 \\ -1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 6 & 3 & 6 \\ 12 & 5 & 1 \\ -3 & 2 & 3 \end{vmatrix}$$

Where we find that:

$$\text{The right hand side} = 3\begin{vmatrix} 2 & 3 & 6 \\ 4 & 5 & 1 \\ -1 & 2 & 3 \end{vmatrix}$$

$$= 3\left(2\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ -1 & 3 \end{vmatrix} + 6\begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix}\right)$$

$$= 3[2(15 - 2) - 3(12 - (-1)) + 6(8 - (-5))]$$

$$= 3[2(13) - 3(13) + 6(13)]$$

$$= 3[26 - 39 + 78] = 3[65] = 195$$

$$\text{And, The left hand side} = \begin{vmatrix} 6 & 3 & 6 \\ 12 & 5 & 1 \\ -3 & 2 & 3 \end{vmatrix}$$

$$= 6\begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 12 & 1 \\ -3 & 3 \end{vmatrix} + 6\begin{vmatrix} 12 & 5 \\ -3 & 2 \end{vmatrix}$$

$$= 6(15 - 2) - 3(36 - (-3)) + 6(24 - (-15))$$

$$= 6(13) - 3(39) + 6(39)$$

$$= 78 - 117 + 234 = 195$$

Therefore, both sides are equal.

Property (6):

If each element of any row (or column) of a determinant is the sum of *m* terms, then this determinant can be expressed by the sum of the determinants of number *m*. it can be shown as follow:

$$\begin{vmatrix} a_1 & b_1+c_1 & d_1 \\ a_2 & b_2+c_2 & d_2 \\ a_3 & b_3+c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}$$

Example (2-8):

Prove that:

$$\begin{vmatrix} x & x+x^2 & x^2 \\ y & y+y^2 & y^2 \\ z & z+z^2 & y^2 \end{vmatrix} = 0$$

Solution:

According to property (6):

$$\begin{vmatrix} x & x+x^2 & x^2 \\ y & y+y^2 & y^2 \\ z & z+z^2 & y^2 \end{vmatrix} = \begin{vmatrix} x & x & x^2 \\ y & y & y^2 \\ z & z & y^2 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^2 \\ y & y^2 & y^2 \\ z & z^2 & y^2 \end{vmatrix}$$

$= 0 + 0 = 0$

This is because the 1st and 2nd columns in the first determinant on the right hand side are identical, and the 2nd and 3rd columns in the second determinant on the right hand side are also identical. Therefore, the numerical value of each of them is equal to zero, according to the property (4).

Property (7):

The sum of the products of the elements of any row (or column) of a determinant in the cofactors of the elements corresponding to another row (or column) is equal to zero.
For example, if we consider the following determinant:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If we consider A_1, A_2, A_3 are the cofactors for the elements of the 1st column, then:

$$b_1A_1 + b_2A_2 + b_3A_3 = 0$$

Also, $$c_1A_1 + c_2A_2 + c_3A_3 = 0$$

Property (8):

If the multiples of the elements of any row (or column) are added to the elements of another row (or column), then the value of the determinant is not change.

For example, if we have the following determinant:

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then, if we multiply each of the elements in the 1st row by a constant *k*, and we add the results to the corresponding elements in the 2nd row, then we will get:

$$\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+ka_1 & b_2+kb_1 & c_2+kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

According to this property, we find that:

$$\Delta_1 = \Delta_2$$

This property can be proven as follows:

$$\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+ka_1 & b_2+kb_1 & c_2+kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

According to property (6):

$$\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Taking *k* as a common factor between the elements of the 2^{nd} row in the determinant of 2^{nd} term in the right hand side (Applying to property 5). Then, we have the following:

$$\therefore \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and according to property (4), we have:

$$k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \times 0 = 0$$

Then, we have:

$$\Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + 0 = \Delta_1$$

As a special case, if *k* = 1, this means that if we added the elements of one row (or column) to the elements of another row (or column), Also if *k* = -1, then that means if we subtracted the elements of a row (or column) from the elements of another row (or column); In both cases, the value of the determinant does not change.

This property is considered one of the most important properties for determinants as it enables us easily to make many of the determinant elements equal to zero, and thus it facilitates the operations of finding the numerical values of the determinants.

Example (2-9):

Find the value of the following determinant (Δ) by using the properties of the determinants:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 5 \\ -2 & 4 & 6 \end{vmatrix}$$

Solution: Multiplying the elements of the 1st row by 2 and adding to the elements of the third row $(R_3 \to R_3 + 2\,R_1)$,then we have the following:

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 5 \\ 0 & 8 & 12 \end{vmatrix}$$

Also, by multiplying the elements of the 1st row by (-3) and adding the results to the elements of the 2nd row $(R_2 \rightarrow R_2 - 3R_1)$**, then we get the following:**

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -4 \\ 0 & 8 & 12 \end{vmatrix}$$

Then, the value of Δ ***by using the elements of the 1st column becomes:***

$$\Delta = 1 \begin{vmatrix} -5 & -4 \\ 8 & 12 \end{vmatrix} - 0 + 0$$

$$= (-5\times12) - (8\times(-4))$$

$$= -60 - (-32) = -28$$

<u>Example (2-10):</u>

By Using the properties of the determinants, Find the value of the following determinant (Δ**):**

$$\Delta = \begin{vmatrix} 2 & 4 & 6 \\ -1 & 5 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

<u>Solution:</u>

Firstly, by taking 2 as a common factor between the elements of the 1st row (property 5). Then we have:

$$\Delta = 2\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 3 \\ 3 & 1 & 2 \end{vmatrix}$$

Secondly, by adding the elements of the 1st row to the elements of the 2nd row i.e.,$(R_2 \to R_2 + R_1)$. And if, we multiply the elements of the 1st row by (-3) and adding the results to the elements of the 3rd row i.e., $(R_3 \to R_3 - 3R_1)$, then we have the following:

$$\Delta = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 7 & 6 \\ 0 & -5 & -7 \end{vmatrix}$$

Hence , the value of (Δ) by using the elements of the 1st column becomes:

$$\Delta = 2\times 1\begin{vmatrix} 7 & 6 \\ -5 & -7 \end{vmatrix} - 0 + 0$$

$$= 2[(7\times(-7)) - (-5\times 6)]$$

= 2 [(– 49) – (-30)]

= 2 [– 19] = –38

Note that when we use this property to find the numerical value for the determinant, we assumed that R_i , C_i, denotes the (i)th row ,the (*j*)th column respectively to facilitate the operations of the calculations that we perform in this property.

These properties may be useful in computing the value of the determinant. For example, the magnitude of the numbers being manipulated can be reduced if all elements in a row or column have a common factor. This can be achieved also by adding (or subtracting) multiples of one row (column) to another. Significant efficiencies can be introduced if, prior to using the method of cofactors, multiples of rows (columns) are combined to create a row (column) containing mostly zeros as occurred in Examples (9, 10).

(2-3): Some Types of Determinants

(2-3-1): Diagonal Determinant.

The diagonal determinant is the one of the determinants whose elements are all zeros except for the diagonal elements.

If the non-zero elements are located on the main diagonal then the value of the determinant is equal to the

product of multiplying the elements of the diagonal. And if the non-zero elements are located on the cross diagonal, then the value of the determinant is equal to the product of multiplying the elements of the cross diagonal, but with a negative sign. This can be expressed as follows:

If we have the following determinant:

$$\Delta = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

Since, we have diagonal determinant with its main diagonal (a , b , c)different from zero, then we have:

$$\Delta = a\ b\ c$$

But if we have the following determinant:

$$\Delta = \begin{vmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{vmatrix}$$

Since, we have diagonal determinant with its cross diagonal (a , b , c)different from zero, then we have:

$$\Delta = -a\ b\ c$$

<u>Example (2-11):</u>

Find the value of the following determinant (Δ):

$$\Delta = \begin{vmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{vmatrix}$$

Solution:

Since, we have diagonal determinant with its main diagonal (-2 , 3 , 5)different from zero, then we have:

$$\Delta = -2 \times 3 \times 5 = -30$$

Example (2-12):

Find the value of the following determinant (Δ):

$$\Delta = \begin{vmatrix} 0 & 0 & 8 \\ 0 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix}$$

Solution:

Since, we have diagonal determinant with its cross diagonal (8 , 1 , - 4)different from zero, then we have:

$$\Delta = -(8 \times 1 \times (-4)) = 32$$

(2-3-2): Triangular Determinant.

There are two types of triangular determinants:

- **Upper Triangular Determinant:**

It is the determinant by which all the elements below the main diagonal are zeros.

- **Lower Triangular Determinant:**

  It is the determinant by which all the elements above the main diagonal are zeros.

The value of each of both determinants is equal to the product of multiplying the elements of its main diagonal.

For example, if we have the following determinant:

$$\Delta = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 1 \end{vmatrix}$$

Then,

$$\Delta = 2 \times 5 \times 1 = 10$$

Since, it is an upper triangular determinant.

Also, if we have the following determinant:

$$\Delta = \begin{vmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 1 & 3 \end{vmatrix}$$

Then,

$$\Delta = 3 \times 2 \times 3 = 18$$

Since, it is a lower triangular determinant.

The properties of the determinants can be used to turn them into triangular or diagonal determinants, which it is

easy to find the value of these determinants, regardless of its higher order. Therefore, the value of the determinant, with the higher order, can be easily found without expanding it.

(2-4): Solving the linear equations by using the determinants (Cramer's Rule)

One of the most important applications of the determinants is to using it in the solution of linear equations when the number of equations is equal the number of unknowns variable. In fact, the concept of determinants originated from the study of such systems of equations. The main result, known as *Cramer's Rule*, is stated as the following:

Consider the following system of three equations in three unknowns *x* , *y* , and *z*.

$$a_1x + b_1y + c_1z = k_1$$
$$a_2x + b_2y + c_2z = k_2$$
$$a_3x + b_3y + c_3z = k_3$$

Let ($\triangle$) denotes the value for the determinant coefficients as follows:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

And let Δ_x , Δ_y , and Δ_z be the corresponding determinant obtained by replacing the 1st , 2nd , and 3rd columns in Δ respectively, by the column vector for the constant terms. i.e.,

$$\Delta_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$$

,

$$\Delta_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$$

,

$$\Delta_z = \begin{vmatrix} a_1 & c_1 & k_1 \\ a_2 & c_2 & k_2 \\ a_3 & c_3 & k_3 \end{vmatrix}$$

Then ;If $\Delta \neq 0$, the given system has *a unique solution* given by:

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}, \qquad z = \frac{\Delta_z}{\Delta}$$

Note that:

- **If $\Delta = 0$ and either $\Delta_x \neq 0$, $\Delta_y \neq 0$ or $\Delta_z \neq 0$, then the system has *no solution.***

- **If $\Delta = 0$ and $\Delta_x = \Delta_y = \Delta_z = 0$, then the system has *an infinite number of solutions.* And this case is represented in the existence of a number of independent equations less than the number of unknowns, and perhaps the next group of equations express this case:**

$$\begin{aligned} x - 2y + z &= 1 \\ 3x - y - 2z &= 8 \\ 2x + y - 3z &= 7 \end{aligned}$$

The reader can verify that:

$$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$$

This is due to the fact that the 2nd row in the determinant coefficient is a sum of the 1st and 3rd rows. This means that the number of independent equations is only 2, while there are three unknowns.

Example (2-13):

By using *Cramer's Rule,* solve the following linear equations:

$$2x - y + 2z = 6$$
$$x - y + 3z = 8$$
$$x + y + z = 6$$

Solution :

$$\Delta = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$(R_1 \to R_1 + R_3)$ **and** $(R_2 \to R_2 + R_3)$

$$\Delta = \begin{vmatrix} 3 & 0 & 3 \\ 2 & 0 & 4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix}$$

$$= -1(12 - 6) = -6$$

$$\Delta_x = \begin{vmatrix} 6 & -1 & 2 \\ 8 & -1 & 3 \\ 6 & 1 & 1 \end{vmatrix}$$

$(R_1 \to R_1 + R_3)$ **and** $(R_2 \to R_2 + R_3)$

$$\Delta_x = \begin{vmatrix} 12 & 0 & 3 \\ 14 & 0 & 4 \\ 6 & 1 & 1 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 12 & 3 \\ 14 & 4 \end{vmatrix} - 6$$

= – 1(48 – 42) = - 6 ;

$$\Delta_y = \begin{vmatrix} 2 & 6 & 2 \\ 1 & 8 & 3 \\ 1 & 6 & 1 \end{vmatrix}$$

$(R_1 \rightarrow R_1 - 2R_3)$ **and** $(R_2 \rightarrow R_2 - R_3)$

$$\Delta_y = \begin{vmatrix} 0 & -6 & 0 \\ 0 & 2 & 2 \\ 1 & 6 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -6 & 0 \\ 2 & 2 \end{vmatrix}$$

= 1(–12– 0) = – 12

And;

$$\Delta_z = \begin{vmatrix} 2 & -1 & 6 \\ 1 & -1 & 8 \\ 1 & 1 & 6 \end{vmatrix}$$

$(R_1 \rightarrow R_1 + R_3)$ **and** $(R_2 \rightarrow R_2 + R_3)$

$$\Delta_z = \begin{vmatrix} 3 & 0 & 12 \\ 2 & 0 & 14 \\ 1 & 1 & 6 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 3 & 12 \\ 2 & 14 \end{vmatrix}$$

$$= -1(42 - 24) = -18$$

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{-6}{-6} = 1 \, ,$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-12}{-6} = 2 \, ,$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-18}{-6} = 3$$

This result can be verified by substituting the values of *x, y, z* in any of the three equations (let it be the 3rd equation), where we find that:

$$x + y + z = 1 + 2 + 3 = 6$$

So that, the result is correct.

Example (2-14):

Solve the following linear equations Using the determinants (*Cramer's Rule*):

$$2x + 2y - z = 2$$
$$x + 3z = 7$$
$$x - y + 5z = 10$$

Solution :

$$\Delta = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 0 & 3 \\ 1 & -1 & 5 \end{vmatrix}$$

$(R_1 \rightarrow R_1 + 2R_3)$

$$\Delta = \begin{vmatrix} 4 & 0 & 9 \\ 1 & 0 & 3 \\ 1 & -1 & 5 \end{vmatrix}$$

$$= -(-1) \begin{vmatrix} 4 & 9 \\ 1 & 3 \end{vmatrix}$$

$$= (12 - 9) = 3$$

$$\Delta_x = \begin{vmatrix} 2 & 2 & -1 \\ 7 & 0 & 3 \\ 10 & -1 & 5 \end{vmatrix}$$

$(R_1 \to R_1 + 2R_3)$

$$\Delta_x = \begin{vmatrix} 22 & 0 & 9 \\ 7 & 0 & 3 \\ 10 & -1 & 5 \end{vmatrix}$$

$$= -(-1) \begin{vmatrix} 22 & 9 \\ 7 & 3 \end{vmatrix}$$

= (66 – 63) = 3

$$\Delta_y = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 7 & 3 \\ 1 & 10 & 5 \end{vmatrix}$$

$(C_1 \to C_1 + 2C_3)$ **and** $(C_2 \to C_2 + 2C_3)$

$$\Delta_y = \begin{vmatrix} 0 & 0 & -1 \\ 7 & 13 & 3 \\ 11 & 20 & 5 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 7 & 13 \\ 11 & 20 \end{vmatrix}$$

= –1(140 – 143) = 3

$$\Delta_z = \begin{vmatrix} 2 & 2 & 2 \\ 1 & 0 & 7 \\ 1 & -1 & 10 \end{vmatrix}$$

$(R_1 \rightarrow R_1 + 2 R_3)$

$$\Delta_z = \begin{vmatrix} 4 & 0 & 22 \\ 1 & 0 & 7 \\ 1 & -1 & 10 \end{vmatrix}$$

$$= -(-1) \begin{vmatrix} 4 & 22 \\ 1 & 7 \end{vmatrix}$$

Then ; we can determine the unknown variable as follows:

$$\therefore x = \frac{\Delta_x}{\Delta} = \frac{3}{3} = 1 , \; y = \frac{\Delta_y}{\Delta} = \frac{3}{3} = 1 , \; z = \frac{\Delta_z}{\Delta} = \frac{6}{3} = 2$$

The reader can verify from the result by substituting in one of the given equations.

(2-5): Economic Applications for the determinants:

(2-5-1): Market Equilibrium:

Market equilibrium occurs at the price level by which the quantity of demand equals the quantity of supply. That is, the equilibrium price is achieved at the point of intersection of the two functions of demand and supply. In other words, the equilibrium price and quantity can be determined by solving the demand and supply functions together. In the

following, we will use determinants in determining the point of market equilibrium.

Example (2-15):

Determine the price of the commodity and the quantity of demand for it by which achieve the market equilibrium, if you know that the two functions of demand and supply are:

***Demand function*: $3p + 5q = 22$**

***Supply function:* $2p - 3q = 2$**

Where p: is The price of the commodity unit (L.E.)

q: is The quantity of demand or supply of a commodity (per units).

Solution:

The market equilibrium price is the price at which the quantity of demand for a commodity equals the quantity of its supply, which can be obtained by finding a value of (p) that satisfies both the demand and supply equations. This is achieved by solving the demand and supply equations either by using determinants, which is what we will see here, or by using matrices, which we will discuss in a later topic in this book, or by any other method (deletion and substitution, for example).

Now we explain how to solve the demand and supply equations by using the determinants method as follows:

$$\Delta = \begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} = (-9 - 10) = -19 \,,$$

$$\Delta_p = \begin{vmatrix} 22 & 5 \\ 2 & -3 \end{vmatrix} = (-66 - 10) = -76$$

$$\Delta_q = \begin{vmatrix} 3 & 22 \\ 2 & 2 \end{vmatrix} = (6 - 44) = -38$$

Therefore,

$$p = \frac{\Delta_p}{\Delta} = \frac{-76}{-19} = 4 \,, \qquad q = \frac{\Delta_q}{\Delta} = \frac{-38}{-19} = 2$$

This means that the market equilibrium is achieved when the commodity unit price reached to 4 (L.E.). And at this price, the quantity of demand for this commodity, and the quantity of supply from this commodity is equal to two units only.

(2-5-2): Mix Production:

We know from the economic theory that the factors of production are limited. If the producer wants to determine the quantity of production from the different types of

products that achieve the full operation of production factors(available capacity), then the determinants method can be used to determine the number of units that must be produced from each type of different products that achieve the full operation of production factors.

In addition it is possible to determine the total or net profit as well as the total cost of the production process, if there is available information about the profit (or cost of production) of each unit of the different products in the case of the full operation of the available energies or resources). The following example illustrates the idea of the mix production.

Example (2-16):

A company produces three products A, B, and C. The three products pass through three stages of production. The following table shows the number of work hours required to produce one unit of each product at each stage of production, as well as the weekly available capacity for the production in each stage, expressed in the number of available working hours:

The product / Stages of production	A	B	C	The available capacities
The first stage	2	1	3	1300 *hours*
The second stage	4	3	2	1900 *hours*
The third stage	1	4	2	1100 *hours*

Required:

1) **Formulating the problem as equations that illustrate the full used of the available capacities.**
2) **From your results in (1) and by using the determinants, determine the production levels of each product that the company must produce to achieve the full used for the available capacities.**

3) If the profit of the produced unit from the three products A, B, and C is 50, 100 and 150 pounds, respectively. What is the total profit from the full used of the available capacities?

Solution:

1) Suppose that:

- The number of units to be produced of type (A) is *x* unit.

- The number of units to be produced of type (B) is *y* unit.

- The number of units to be produced of type (C) is *z* unit.

Therefore, the mathematical formulation of equations that determine production levels, *i.e.* the values of *x, y,* and *z* that achieve full used capacity for the operation of the available resources are as follows:

$$\mathbf{2x + 1y + 3z = 1300}$$

$$\mathbf{4x + 3y + 2z = 1900}$$

$$\mathbf{x + 4y + 2z = 1100}$$

2) Through the equations deduced in (1) that explain the full used capacity for operation of the available resources and by using the determinants, the production levels of the three products can be determined, *i.e.,* the values of x, y, and z that achieve the three equations, by solving these equations using determinants, as follows:

- Finding the value of the determinant of the coefficients(Δ) **as follows:**

$$\Delta = \left|\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 4 & 3 & 2 & 4 & 3 \\ 1 & 4 & 2 & 1 & 4 \end{array}\right.$$

$$\Delta = [\mathbf{2 \times 3 \times 2 + 1 \times 2 \times 1 + 3 \times 4 \times 4}]$$

$$-[\mathbf{1 \times 3 \times 3 + 4 \times 2 \times 2 + 2 \times 4 \times 1}]$$

$$= [\mathbf{12 + 2 + 48}] - [\mathbf{9 + 16 + 8}]$$

$$= \mathbf{62 - 33 = 29 \neq 0}$$

There is a unique solution for these linear equations.

- Finding the value of the determinants corresponding to the unknowns as follows:

$$\Delta_x = \left|\begin{array}{ccc|cc} \mathbf{1300} & \mathbf{1} & \mathbf{3} & \mathbf{1300} & \mathbf{1} \\ \mathbf{1900} & \mathbf{3} & \mathbf{2} & \mathbf{1900} & \mathbf{3} \\ \mathbf{1100} & \mathbf{4} & \mathbf{2} & \mathbf{1100} & \mathbf{4} \end{array}\right.$$

$$\Delta_x = [\mathbf{1300 \times 3 \times 2 + 1 \times 2 \times 1100 + 3 \times 1900 \times 4}]$$

$$-[\mathbf{1100 \times 3 \times 3 + 4 \times 2 \times 1300 + 2 \times 1900 \times 1}]$$

$$= [\mathbf{7800 + 2200 + 22800}] - [\mathbf{9900 + 10400 + 3800}]$$

$$= \mathbf{32800 - 24100 = 8700}$$

$$\Delta_y = \left|\begin{array}{ccc|cc} \mathbf{2} & \mathbf{1300} & \mathbf{3} & \mathbf{2} & \mathbf{1300} \\ \mathbf{4} & \mathbf{1900} & \mathbf{2} & \mathbf{4} & \mathbf{1900} \\ \mathbf{1} & \mathbf{1100} & \mathbf{2} & \mathbf{1} & \mathbf{1100} \end{array}\right.$$

$$\Delta_y = [\mathbf{2 \times 1900 \times 2 + 1300 \times 2 \times 1 + 3 \times 4 \times 1100}]$$

$$-[\mathbf{1 \times 1900 \times 3 + 1100 \times 2 \times 2 + 2 \times 4 \times 1300}]$$

$$= [\mathbf{7600 + 2600 + 13200}] - [\mathbf{5700 + 4400 + 10400}]$$

$$= \mathbf{23400 - 20500 = 2900}$$

$$\Delta_z = \left|\begin{array}{ccc|cc} \mathbf{2} & \mathbf{1} & \mathbf{1300} & \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{3} & \mathbf{1900} & \mathbf{4} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{1100} & \mathbf{1} & \mathbf{4} \end{array}\right.$$

$$\Delta_z = [2 \times 3 \times 1100 + 1 \times 1900 \times 1 + 1300 \times 4 \times 4]$$

$$-[1 \times 3 \times 1300 + 4 \times 1900 \times 2 + 1100 \times 4 \times 1]$$

$$= [6600 + 1900 + 20800] - [3900 + 15200 + 4400]$$

$$= 29300 - 23500 = 5800$$

Then , the number of units to be produced are:

$$The\ first\ type(A) = \frac{\Delta_x}{\Delta} = \frac{8700}{29} = 300 \quad units$$

$$The\ second\ type(B) = \frac{\Delta_y}{\Delta} = \frac{2900}{29} = 100 \quad units$$

$$The\ third\ type(C) = \frac{\Delta_z}{\Delta} = \frac{5800}{29} = 200 \quad units$$

Therefore, the production levels that achieve full used capacity for the operations of the production are 300, 100 and 200 units of the three products, respectively.

3) To determine the total profit by which achieved from the full used capacity for the operations of the available capacities, then:

$$\text{The total profit} =$$

$$\text{The sum of the products of multiplying the unit profit for}$$

$$\text{each type of production}$$

$$\times \text{ the number of units produced of each type.}$$

$\boldsymbol{i.e.,}$ $\quad$ $\textbf{The total profit} = \mathbf{50}\boldsymbol{x} + \mathbf{100}\boldsymbol{y} + \mathbf{150}\boldsymbol{z}$

$$= \mathbf{50(300) + 100(100) + 150(200)}$$

$$= \mathbf{15000 + 10000 + 30000}$$

$$= \mathbf{55000} \quad \boldsymbol{pounds}$$

i.e., **the total profit resulted from the operation of the full used available capacities is $\mathbf{55000}$ $\boldsymbol{pounds}$.**

Exercises

Chapter (3)

1- Find the value of the following determinants:

a – $\begin{vmatrix} 3 & -1 \\ 4 & 7 \end{vmatrix}$

b – $\begin{vmatrix} -6 & -7 \\ -8 & -3 \end{vmatrix}$

c – $\begin{vmatrix} 2 & 3 & 1 \\ 5 & 4 & 3 \\ -1 & 2 & 6 \end{vmatrix}$

d – $\begin{vmatrix} 4 & -1 & 3 \\ 5 & 4 & 2 \\ 3 & 0 & 1 \end{vmatrix}$

e – $\begin{vmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 0 & -3 \\ 0 & -2 & 1 & 1 \\ 1 & 2 & 0 & -1 \end{vmatrix}$

2- By using the Properties of determinants, find the value of the following determinant:

$$\begin{vmatrix} 2 & 5 & 4 & -1 \\ 3 & 6 & 2 & 2 \\ -1 & 1 & 3 & 0 \\ 4 & 3 & 1 & -4 \end{vmatrix}$$

3- Without the expansion of determinants, find the value of the following determinants:

a – $\begin{vmatrix} 3 & -1 & 1 \\ 4 & 2 & 3 \\ 6 & 3 & 6 \end{vmatrix}$ **b –** $\begin{vmatrix} -2 & 4 & 1 \\ 0 & 1 & -2 \\ 5 & 3 & 4 \end{vmatrix}$

4- By using determinants, solve the following equations:

(a)-

$$x + y + 2z = 6$$
$$2x - 3y + z = 1$$
$$4x + 2y = 6$$

(b)-

$$2x - y + 4z = 5$$
$$x + y + z = 3$$
$$3x - 2y + 5z = 6$$

(c)-

$$3x - y - z = 5$$
$$4x + 2y + 3z = 11$$
$$x - y + z = 3$$

5- Find the value of *k* that makes the following equations to have a solution:

$$x + y + z = 1 \quad ,$$

$$x - 2y + 4z = k \quad ,$$
$$x + 4y + 10z = k^2$$

6- Determine the equilibrium price and the demand or supply quantity for the following demand and supply functions:

(a) ***Demand function*:** $3p + 7q = 200$

Supply function: $7p - 3q = 56$

(b) ***Demand function*:** $5p + 8q = 80$

Supply function: $3p - 2q = -1$

(c) ***Demand function*:** $4p + q = 50$

Supply function: $6p - 5q = 10$

(d) ***Demand function*:** $3p + 6q = 9$

Supply function: $2p - 3q = 8$

7- A trader can sell 200 units per day of a specific commodity at price 30 pounds per unit, and 250 units per day at price 27 pounds per unit. If the supply function for the same commodity is:

$$6p = q + 48$$

Required:

(a) Find the demand function for this commodity assuming

It's a linear function.

(b) Determine the equilibrium price and quantity.

8- If you have a company that produces three types of products A, B and C, by passing through three production stages (1), (2) and (3).

The requirements for each unit of the three products in these production stages, as well as the unit profit and the available production capacity, are shown in the following table:

The product / The department	A	B	C	The available capacity
(1)	1	2	1	430
(2)	3	0	2	460
(3)	1	4	0	420
The unit profit	3	2	5	

The presence of zero in the previous table indicates that this product does not need this item from the production stages to produce it.

<u>Required:</u>

1) Formulating the problem as equations that illustrate the full operation of the production stages.

2) From your results in (1) and by using the determinants, determine the optimal production quantities that achieve the full used for the available capacities.

3) What is the total profit from the full operation of the available capacities?

Chapter (4)

Matrices

Introduction

Whenever one is dealing with data organization, it should be concerned with organizing the data in such a way that they are meaningful and can be readily identified. Summarizing data in a tabular form can serve this function. The *matrix* is a common tool for summarizing and displaying numbers or data.

In many economic analyses, variables are assumed to be related by sets of linear equations. Matrix algebra provides a clear and concise notation for the formulation and solution of such problems, many of which would be complicated in conventional algebraic notation.

(4-1): The Definition of Matrix:

The *matrix* is a rectangular array of real numbers which is enclosed in large brackets. The real numbers which form the array are called the *entries* or *elements* of the matrix. The elements in any horizontal line form is called a *row* and those in any vertical line form is called a *column* of the matrix. Matrices are generally denoted by capital letters such as A, B, C…. and so on.

The matrix differs from the determinant in which it is not necessary that the number of rows must be equals to the number

of columns. In general, the dimension or the degree (order) of the matrix is determined by the number of rows and the number of columns. If a matrix A has (*m*) rows and *n* columns, then it is said to be with dimension or degree or order of ($m \times n$) and (read *m* by *n*) and it is denoted by (A$_{m\times n}$), and its elements can be expressed as follows:

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & & a_{mn} \end{bmatrix}$$

The matrix A is sometimes denoted by simplified form as $[a_{ij}]_{m\times n}$ $i.e., A = [a_{ij}]_{m\times n}$, the element a_{ij} is the element which is located at the intersection of the i^{th} row and j^{th} column of the matrix. For example, the element a_{23} is located in the 2^{nd} row and the 3^{rd} column, and the element a_{41} is located in the 4^{th} row and the 1^{st} column….. and so on.

(4-2): Types of Matrices:

This section discusses the different types of matrices:

(4-2-1): Row Vector:

The *row vector* is a rectangular matrix consisting of a single row and more than one column($i.e., A_{1\times n}$). For example, the vector

$(A_{1\times 4})$ is a matrix consisting of a single row and four columns as follow:

$$A_{1\times 4} = [8 \quad 3 \quad -4 \quad 2 \quad 1]$$

(4-2-2): Column Vector:

The *column vector* is a rectangular matrix having only one column and more than one row$(i.e., A_{m\times 1})$. For example, the vector $(A_{3\times 1})$ is a column vector consisting of a single column and three rows as follow:

$$A_{3\times 1} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$$

(4-2-3): Square Matrix:

The *square matrix* is a matrix having the same number of rows and columns. It is denoted by $A_{m\times m}$. For example, the matrix $(A_{3\times 3})$ is a square matrix of order 3×3 as follow:

$$A_{3\times 3} = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 3 & 7 \\ 5 & 8 & 0 \end{bmatrix}$$

$$The\ number\ of\ rows(m) = the\ number\ of\ colmns(n) = 3$$

A square matrix of order (or having the dimension) $m \times m$ is simply written as A_m $(i.e., a\ matrix\ of\ order\ m)$.

(4-2-4): Diagonal Matrix:

The *diagonal matrix* is a square matrix in which all elements are zero except for those elements in the main diagonal. Some elements of the main diagonal may be zero but not all.

For example to the diagonal matrices:

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, and \ B = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are diagonal matrices.

In general, the following matrix

$$A_{m\times m} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2m} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & & & a_{mm} \end{bmatrix}$$

is a diagonal matrix if and only if

$$a_{ij} = 0 \qquad for\ i \neq j$$

$$a_{ij} \neq 0 \qquad for\ at\ least\ one\ i = j$$

(4-2-5): Identity Matrix:

The *identity matrix* (*I*), sometimes called the *unit matrix*, is a square diagonal matrix for which all elements along the main diagonal equal to *one*s and all other elements equal to *zero*s.

For example, the matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is an identity matrix of order 2×2 ($i.e., 2^{nd} order\ unit\ matrix$).

Also, the matrix :

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix of order 3×3 ($i.e., 3^{rd} order\ unit\ matrix$).

Note that the identity matrix is a special case of the diagonal matrix. And the role of the unit matrix is as the role of the one in the mathematical algebra.

(4-2-6): Null Matrix:

The *null matrix* is a matrix in which all the elements are zero, it is called *zero matrix*. Zero matrices are generally denoted by the symbol (0). The following matrix is a zero matrix of order 2×3.

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(3-2-7): Transpose Matrix:

If the matrix is rotated so that the rows become columns and the columns become rows, then we get what is called the *transpose* of the matrix. Given that the $(m \times n)$ matrix A with its elements a_{ij}, then the transpose of A (denoted by A^T or A') is an $(n \times m)$ matrix which contains elements ${a_{ij}}^t$ where ${a_{ij}}^t = a_{ji}$.

For example, if we have the following matrix of order 3×2.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}$$

Then the transpose of A is a matrix of order 2×3 as the following:

$$\mathbf{A}^T = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 5 & 1 \end{bmatrix}$$

(4-2-8): Symmetric Matrix:

The *symmetric matrix* is a square matrix that is equal to its transpose.

$$\text{i.e.}, \quad \text{if} \quad \mathbf{A}\ is\ a\ symmetric\ matrix, \quad then, \quad \mathbf{A} = \mathbf{A}^T$$

The entries of a symmetric matrix are symmetric with respect to the main diagonal. So, if a_{ij} denotes the entry in the i^{th} row and j^{th} column, then

$$\mathbf{A}\ is\ symmetric\ matrix \quad \leftrightarrow \quad for\ every\ i, j, \quad a_{ji} = a_{ij}$$

For example, if we have the following matrix A:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 9 \\ -1 & 8 & 0 \\ 9 & 0 & -5 \end{bmatrix}$$

$$and \qquad \mathbf{A}^T = \begin{bmatrix} 2 & -1 & 9 \\ -1 & 8 & 0 \\ 9 & 0 & -5 \end{bmatrix} = \mathbf{A}$$

So, it is said that the matrix A is a symmetric matrix.

(4-2-9): Orthogonal Matrix:

The orthogonal matrix is a square matrix that is satisfied one of the following conditions:

$$AA^T = I \quad \text{or} \quad A^T A = I \quad \text{or} \quad A^T = A^{-1}$$

Where:

A^T: is the transpose of the matrix.

I : is the identity matrix.

A^{-1}: is the inverse of the matrix that we will discuss later.

(4-3): Matrix Algebra

Matrix Algebra means the set of mathematical operations by which it is desired to be known as the set of mathematical operations on numbers(like addition ,subtraction, multiplication and division). But matrix algebra sometimes requires some require conditions. In this section we will discuss some of operations of the matrix algebra.

(4-3-1): Matrix Addition and Subtraction

Two matrices can be added (or subtracted) by adding (or subtracting) their corresponding elements if and only if they have the same dimension. In other words, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, then$(A + B) = [a_{ij} + b_{ij}]$ and $(A - B) = [a_{ij} - b_{ij}]$ $for\ all\ i\ and\ j$.

Example (4-1):

If we have the following two matrices A and B as follow:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -1 \\ 5 & -2 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 9 & 3 & -8 \\ -3 & 2 & 3 \end{bmatrix}$$

Required:

Find $\mathbf{A + B}$, $\mathbf{A - B}$, $\mathbf{B - A}$

Solution:

Note that since the two matrices A and B have the same dimension. So that, they can be added or subtracted as follow:

$$\mathbf{A + B} = \begin{bmatrix} 2 & 4 & -1 \\ 5 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 9 & 3 & -8 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 7 & -9 \\ 2 & 0 & 6 \end{bmatrix}$$

$$\mathbf{A - B} = \begin{bmatrix} 2 & 4 & -1 \\ 5 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 3 & -8 \\ -3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 1 & 7 \\ 8 & -4 & 0 \end{bmatrix}$$

$$\mathbf{B - A} = \begin{bmatrix} 9 & 3 & -8 \\ -3 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 4 & -1 \\ 5 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -1 & -7 \\ -8 & 4 & 0 \end{bmatrix}$$

(4-3-2): Matrix Multiplication:

There are two types of matrix multiplication:

a) ***Multiplication of a Matrix by a Scalar:***

Scalar multiplication of a matrix is the multiplication of the scalar (real number) by each element of the matrix. The product is found by multiplying each element in the matrix by this scalar. For example, if we have the following matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 9 & -2 \\ 8 & 5 \end{bmatrix}$$

And we want to find the matrix B where B = 4A, then:

$$\mathbf{B} = 4\mathbf{A} = 4\begin{bmatrix} 3 & 2 \\ 9 & -2 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 36 & -8 \\ 32 & 20 \end{bmatrix}$$

Note that there is a main difference between determinants and matrices in this process of multiplying by a scalar (constant).where in the case of determinants, multiplying a constant by a determinant means multiplying the constant in the elements of only one row or only one column, but in the case of matrices, multiplying a constant by a matrix means multiplying the constant in all the elements of the matrix.

b) <u>Multiplication of a Matrix by a Matrix:</u>

Two matrices (A) and (B) are said to be conformable for the product (AB) if the number of columns of the matrix (A) is equal to the number of rows for the matrix (B). Then the result from the multiplication or the product matrix process (AB) has the same number of rows as the 1st matrix (A) and the same number of columns as the 2nd matrix (B). In other words, if A = $[a_{ij}]$is an $m \times n$ matrix and B = $[b_{ij}]$ is an $n \times p$ matrix, then the product $\mathrm{C} = [c_{ij}]$=AB is an $m \times p$ matrix:

$$i.e., \quad \mathbf{A}_{m\times n} \times \mathbf{B}_{n\times p} = \mathbf{C}_{m\times p}$$

For simplicity, it can be found the product matrix (C) by multiplying each row of the 1st matrix by each column of the 2nd matrix. In general, if C = AB, then the ij^{th} element of the product matrix (C) is obtained by finding the sum of the product of the i^{th} row of the matrix (A) by the j^{th} column of the matrix (B) as follow:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots\cdots\cdots + a_{in}b_{nj}$$

Example (4-2):

If we have the following two matrices (A) and (B) as follow:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 \\ 5 & 9 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -2 & 1 \\ 3 & 2 \\ 1 & -5 \end{bmatrix}$$

Required:

Find (1): $\mathbf{A} \times \mathbf{B}$, (2): $\mathbf{B} \times \mathbf{A}$

Solution:

(1): Since the matrix (A) is a (2×3) matrix and the matrix (B) is a (3×2) matrix, i.e., the number of columns of (A) is equal to the number of rows of (B), so that, they are conformable for multiplication as follows:

$$\mathbf{A \times B} = \begin{bmatrix} 1 & 3 & -1 \\ 5 & 9 & 3 \end{bmatrix} \times \begin{bmatrix} -2 & 1 \\ 3 & 2 \\ 1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times (-2) + 3 \times 3 + (-1) \times 1 & 1 \times 1 + 3 \times 2 + (-1) \times (-5) \\ 5 \times (-2) + 9 \times 3 + 3 \times 1 & 5 \times 1 + 9 \times 2 + 3 \times (-5) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 12 \\ 20 & 8 \end{bmatrix}$$

(2): Since B is a (3×2) matrix and A is a (2×3) matrix, i.e., the number of columns of B is equal to the number of rows of A, so that , they are conformable for multiplication as follows:

$$\mathbf{B \times A} = \begin{bmatrix} -2 & 1 \\ 3 & 2 \\ 1 & -5 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & -1 \\ 5 & 9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (-2) \times 1 + 1 \times 5 & (-2) \times 3 + 1 \times 9 & (-2) \times (-1) + 1 \times 3 \\ 3 \times 1 + 2 \times 5 & 3 \times 3 + 2 \times 9 & 3 \times (-1) + 2 \times 3 \\ 1 \times 1 + (-5) \times 5 & 1 \times 3 + (-5) \times 9 & 1 \times (-1) + (-5) \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 & 5 \\ 13 & 27 & 3 \\ -24 & -42 & -16 \end{bmatrix}$$

Remark: **If we have the matrices A, B, C, D, and the conditions of addition (or subtraction), or multiplication are achieved, then:**

- $\mathbf{A + B = B + A}$
- $\mathbf{A + (B + C) = (A + B) + C}$
- $\mathbf{A - B = A + (-B)}$
- $\mathbf{AB \neq BA}$
- $\mathbf{A(B + C) = AB + AC}$
- $\mathbf{(B + C)A = BA + CA}$
- $\mathbf{A(BC) = (AB)C}$

Example (4-3):

If we have the following matrices:

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{5} \\ \mathbf{4} & \mathbf{7} \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{3} & \mathbf{2} & \mathbf{5} \end{bmatrix}$$

Prove that:

$(\mathbf{1})$ $\mathbf{AB \neq BA}$

$(\mathbf{2})$ $\mathbf{A(B + C) = AB + AC}$

$(\mathbf{3})$ $\mathbf{A(BD) = (AB)D}$

Solution:

$(\mathbf{1})$ $\boldsymbol{proving\ that}$: $\mathbf{AB \neq BA}$

$$\boldsymbol{The\ left\ hand\ side} = \ \mathbf{A_{2\times3}B_{3\times2} = (AB)_{2\times2}}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + (-2) \times 3 + 3 \times 4 & 1 \times (-1) + (-2) \times 5 + 3 \times 7 \\ 2 \times 2 + 4 \times 3 + (-1) \times 4 & 2 \times (-1) + 4 \times 5 + (-1) \times 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 10 \\ 12 & 11 \end{bmatrix} \qquad (*)$$

$\textit{The right hand side} = \mathbf{B}_{3\times2}\mathbf{A}_{2\times3} = (\mathbf{BA})_{3\times3}$

$$\mathbf{BA} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 1 + (-1) \times 2 & 2 \times (-2) + (-1) \times 4 & 2 \times 3 + (-1) \times (-1) \\ 3 \times 1 + 5 \times 2 & 3 \times (-2) + 5 \times 4 & 3 \times 3 + 5 \times (-1) \\ 4 \times 1 + 7 \times 2 & 4 \times (-2) + 7 \times 4 & 4 \times 3 + 7 \times (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -8 & 7 \\ 13 & 14 & 4 \\ 18 & 20 & 5 \end{bmatrix} \qquad (**)$$

Therefore: from (*) and (**) then BA≠ AB where:

$\textit{The left hand side} \neq \textit{The right hand side}$ **because the dimensions of the products of the two multiplication matrices are different, in addition to their different elements, i.e., BA≠ AB**

<u>(2) $\textit{proving that}$: $\mathbf{A(B + C) = AB + AC}$:</u>

$\boldsymbol{The\ left\ hand\ side} = \mathbf{A(B + C)}$

$$\mathbf{A(B+C)} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \left\{ \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 3 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 12 & 24 \end{bmatrix}$$

$\boldsymbol{The\ right\ hand\ side} = \mathbf{AB + AC}$

$$\mathbf{AB + AC} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 10 \\ 12 & 11 \end{bmatrix} + \begin{bmatrix} 7 & 2 \\ 0 & 13 \end{bmatrix} = \begin{bmatrix} 15 & 12 \\ 12 & 24 \end{bmatrix}$$

Therefore, $\boldsymbol{The\ right\ hand\ side} = \boldsymbol{The\ left\ hand\ side}$

$\boldsymbol{i.e.}$, $\mathbf{A(B + C) = AB + AC}$

<u>**(3)** $\boldsymbol{proving\ that}$: $\mathbf{A(BD) = (AB)D}$:</u>

$\boldsymbol{The\ left\ hand\ side} = \mathbf{A(BD)}$

$$\mathbf{A(BD)} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \left\{ \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & -4 & -5 \\ 21 & 7 & 25 \\ 29 & 10 & 35 \end{bmatrix} = \begin{bmatrix} 46 & 12 & 50 \\ 57 & 10 & 55 \end{bmatrix}$$

$\textit{The Right hand side} = (\mathbf{AB})\mathbf{D}$

$$(\mathbf{AB})\mathbf{D} = \left\{ \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 5 \\ 4 & 7 \end{bmatrix} \right\} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 10 \\ 12 & 11 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 46 & 12 & 50 \\ 57 & 10 & 55 \end{bmatrix}$$

Therefore, $\textit{The right hand side} = \textit{The left hand side}$

$i.e.,\quad \mathbf{A}(\mathbf{BD}) = (\mathbf{AB})\mathbf{D}$

(4-3-3): Matrix Inverse:

The concept of the inverse for the matrix is similar to the division operation in the mathematical operation. some matrices can be identified another matrix called the *inverse* of the matrix. The relationship between the matrix (A)and its inverse which is denoted by(A^{-1}), is that the product of (A)and (A^{-1}) results the identity matrix.

$$i.e.\quad \mathbf{AA^{-1}} = \mathbf{A^{-1}A} = \mathbf{I}$$

The inverse of the matrix is similar to the division in the algebra of real numbers.

Remark:

For a matrix (A), to find the inverse of(A), it must be *square* and its determinant is not equal to *zero*, it said to be a *nonsingular* or *invertible* matrix. The inverse of (A) will also be square and of the

same dimension as (A). But if the determinant of the matrix equal to zero, then there is no inverse for the matrix and it said to be a *singular*. So, not every square matrix has an inverse.

Determining the inverse of a matrix:

There are two main techniques to find the inverse of a matrix, they are:

- **Cofactors technique.**
- **Gaussian reduction procedure.**

And we will restrict our study to the method of the cofactors.

The steps of determining the inverse of a matrix by using the cofactors technique:

1) **Determining the value of the determinant of the matrix ($\Delta = |\mathbf{A}|$).**
2) **Determining the matrix of the cofactors, which is the matrix by which the matrix elements are replaced by their cofactors of the matrix, and it is denoted by(A_c).**
3) **Determining the transpose of the cofactors matrix($\mathbf{A}_c^T$), it is said to be the *adjoint matrix*.**
4) **The inverse of A is found by multiplying the adjoint matrix by the inverse of the determinant of(A), *i.e.*,**

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathbf{A}_c^T$$

Example (4-4): **Determine the inverse of the following matrix:**

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 2 & 4 \\ 3 & -2 & 6 \end{bmatrix}$$

Solution:

To determine the inverse of a matrix, the following steps are taken:

1) Determining the value of the determinant of the matrix as follow:

$$|\mathbf{A}| = \left| \begin{array}{ccc|cc} 2 & 3 & -1 & 2 & 3 \\ 5 & 2 & 4 & 5 & 2 \\ 3 & -2 & 6 & 3 & -2 \end{array} \right.$$

$$|\mathbf{A}| = [\mathbf{2} \times \mathbf{2} \times \mathbf{6} + \mathbf{3} \times \mathbf{4} \times \mathbf{3} + (-\mathbf{1}) \times \mathbf{5} \times (-\mathbf{2})]$$

$$-[\mathbf{3} \times \mathbf{2} \times (-\mathbf{1}) + (-\mathbf{2}) \times \mathbf{4} \times \mathbf{2} + \mathbf{6} \times \mathbf{5} \times \mathbf{3}]$$

$$= [\mathbf{24} + \mathbf{36} + \mathbf{10}] - [-\mathbf{6} - \mathbf{16} + \mathbf{90}]$$

$$= \mathbf{70} - \mathbf{68} = \mathbf{2}$$

$$\neq \mathbf{0} \qquad (\textit{so, there is an inverse for this matrix})$$

2) Determining the matrix of the cofactors $\mathbf{A_c}$ as follow:

$$\mathbf{A_c} = \begin{bmatrix} \begin{vmatrix} 2 & 4 \\ -2 & 6 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -2 & 6 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 20 & -18 & -16 \\ -16 & 15 & 13 \\ 14 & -13 & -11 \end{bmatrix}$$

3) Determining the transpose of the cofactors matrix (A_c^T) as follow:

$$A_c^T = \begin{bmatrix} 20 & -16 & 14 \\ -18 & 15 & -13 \\ -16 & 13 & -11 \end{bmatrix}$$

4) The inverse of the matrix (A) is found as follow:

$$A^{-1} = \frac{1}{|A|} A_c^T = \frac{1}{2}\begin{bmatrix} 20 & -16 & 14 \\ -18 & 15 & -13 \\ -16 & 13 & -11 \end{bmatrix}$$

We can verify from the validity of the previous result by verifying that: $AA^{-1} = I$, as follow:

$$AA^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 3 & -1 \\ 5 & 2 & 4 \\ 3 & -2 & 6 \end{bmatrix}\begin{bmatrix} 20 & -16 & 14 \\ -18 & 15 & -13 \\ -16 & 13 & -11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, the previous result is correct.

(4-4): Some Properties of Matrices:

1) $(A^T)^T = A$
2) $(A + B)^T = A^T + B^T$
3) $(AB)^T = B^T A^T$
4) $(AB)^{-1} = B^{-1} A^{-1}$
5) $AA^{-1} = A^{-1}A = I$
6) $AI = IA = A$

7) **If A and B are symmetric matrices of the same degree, then**
   $(\mathbf{A}+\mathbf{B})$ ***is a symmetric matrix.***
   $(\mathbf{AB})$ ***is not necessary to be a symmetric matrix.***
8) **If we have a square matrix of degree (2×2), then when we find the inverse of the matrix, it is possible to find the transpose of the matrix of the cofactors in one step by exchange the positions of the main diagonal elements of the matrix and reversing the signs of the cross diagonal elements.**

For example, if we have the following matrix:

$$\mathbf{A}=\begin{bmatrix}3 & 5\\ -7 & 2\end{bmatrix}$$

Then,

$$\mathbf{A}_c^T=\begin{bmatrix}2 & -5\\ 7 & 3\end{bmatrix}$$

This result can be verified by performing calculations in two steps as follow:

$$\mathbf{A}_c=\begin{bmatrix}2 & 7\\ -5 & 3\end{bmatrix}$$

Therefore,

$$\mathbf{A}_c^T=\begin{bmatrix}2 & -5\\ 7 & 3\end{bmatrix}$$

It is the same previous result.

<u>Example (4-5):</u>

If we have the following matrices:

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{7} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix}$$

Prove that:

(1) $(\mathbf{A} + \mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}}$

(2) $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$

(3) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

(4) $\mathbf{AB}$ ***is not a symmetric matrix.***

Solution :

(1) ***proving that:*** $(\mathbf{A} + \mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}}$**:**

The left hand side $= (\mathbf{A} + \mathbf{B})^{\mathbf{T}} = \left(\begin{bmatrix} \mathbf{2} & \mathbf{7} \\ \mathbf{1} & \mathbf{4} \end{bmatrix} + \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix}\right)^{\mathbf{T}}$

$$= \begin{bmatrix} \mathbf{5} & \mathbf{9} \\ \mathbf{6} & \mathbf{3} \end{bmatrix}^{\mathbf{T}} = \begin{bmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{9} & \mathbf{3} \end{bmatrix}$$

The right hand side $= \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}} = \begin{bmatrix} \mathbf{2} & \mathbf{7} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}^{\mathbf{T}} + \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix}^{\mathbf{T}}$

$$= \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{7} & \mathbf{4} \end{bmatrix} + \begin{bmatrix} \mathbf{3} & \mathbf{5} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{5} & \mathbf{6} \\ \mathbf{9} & \mathbf{3} \end{bmatrix}$$

Therefore, ***The right hand side = The left hand side***

i.e., $(\mathbf{A} + \mathbf{B})^{\mathbf{T}} = \mathbf{A}^{\mathbf{T}} + \mathbf{B}^{\mathbf{T}}$

(2) ***proving that:*** $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$**:**

The left hand side $= (\mathbf{AB})^{\mathbf{T}} = \left(\begin{bmatrix} \mathbf{2} & \mathbf{7} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}\begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{5} & \mathbf{-1} \end{bmatrix}\right)^{\mathbf{T}}$

$$= \begin{bmatrix} 41 & -3 \\ 23 & -2 \end{bmatrix}^{T} = \begin{bmatrix} 41 & 23 \\ -3 & -2 \end{bmatrix}$$

$$\textit{The right hand side} = \ B^{T}A^{T} = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}^{T} \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}^{T}$$

$$= \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 41 & 23 \\ -3 & -2 \end{bmatrix}$$

Therefore, $\quad$ $\textit{The right hand side} = \textit{The left hand side}$

$i.e.,$ $\quad (AB)^{T} = B^{T}A^{T}$

<u>(3) ***proving that***: $(AB)^{-1} = B^{-1}A^{-1}$:</u>

$$\textit{The left hand side} = \ (AB)^{-1} = \left(\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 41 & -3 \\ 23 & -2 \end{bmatrix}^{-1} = \frac{1}{\Delta} \times \text{(the transpose of the cofactor matrix)}$$

$$= \frac{1}{[41 \times (-2)] - [23 \times (-3)]} \begin{bmatrix} -2 & 3 \\ -23 & 41 \end{bmatrix}$$

$$= \frac{1}{-13} \begin{bmatrix} -2 & 3 \\ -23 & 41 \end{bmatrix}$$

$$\textit{The right hand side} = \ B^{-1}A^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}^{-1}$$

$$= \frac{1}{-13} \begin{bmatrix} -1 & -2 \\ -5 & 3 \end{bmatrix} \times \frac{1}{1} \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$$

$$= \frac{1}{-13} \begin{bmatrix} -2 & 3 \\ -23 & 41 \end{bmatrix}$$

Therefore, $\quad \boldsymbol{The\ right\ hand\ side = The\ left\ hand\ side}$

$\boldsymbol{i.e.},\quad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

<u>$(4)\ \boldsymbol{proving\ that}$: $\mathbf{AB}$ $\boldsymbol{is\ not\ \ a\ symmetric\ matrix}$:</u>

$$\mathbf{AB} = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 41 & -3 \\ 23 & -2 \end{bmatrix}$$

$$(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix} 41 & 23 \\ -3 & -2 \end{bmatrix} \neq \mathbf{AB}$$

Since AB $\neq (\mathbf{AB})^{\mathrm{T}}$**, then AB is not a symmetric matrix.**

<u>(4-5): Solving the linear equations by using the Matrices:</u>

If we have a number of (*n*) linear equations in a number of (*n*) variables (unknowns), then these linear equations can be solved by using the matrices as follows:

Consider the following system of three equations in three unknowns $\boldsymbol{x, y, z}$.

$$\boldsymbol{a_1x + b_1y + c_1z = b_1}$$
$$\boldsymbol{a_2x + b_2y + c_2z = b_2}$$
$$\boldsymbol{a_3x + b_3y + c_3z = b_3}$$

It can be written in a matrix form as follows:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or $$\mathbf{AX = B}$$

Where $\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ **is called the *coefficient matrix*,**

$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **is called the *column vector of unknown variables*,**

$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ **is called the *column vector of constant terms*.**

Since, $$\mathbf{AX = B}$$

Then, by multiplying both sides by ($\mathbf{A^{-1}}$) from the left side (pre-multiplying), we get

$$\mathbf{A^{-1}AX = A^{-1}B}$$

$$\mathbf{IX = A^{-1}B}$$

$$\therefore \mathbf{X = A^{-1}B}$$

Therefore, the values of the variables x, y, z can be obtained by multiplying the inverse of the coefficient matrix by the vector of constant terms.

<u>Example (4-6):</u>

By using the matrices, solve the following equations:

$$x - 2y + z = -1$$

$$y = 1 + z$$

$$3x + y - 2z - 4 = 0$$

<u>Solution:</u>

Firstly, we must rearrange the equations, by making the constant terms are on the right hand side and the variables x, y, z are aligned in columns on the left hand side as follow:

$$x - 2y + z = -1$$

$$0x + y - z = 1$$

$$3x + y - 2z = 4$$

Note that the missing term is written as $0x$ in the second equation.

Then, we can define:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

Now, on order to solve the set of these linear equations, we have to determine the inverse of the matrix(A). then we have the following steps:

- **The value of the determinant of the matrix using the elements of the first column is:**

$$|\mathbf{A}| = \mathbf{1} \times \begin{vmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-2} \end{vmatrix} - \mathbf{0} \times \begin{vmatrix} \mathbf{-2} & \mathbf{1} \\ \mathbf{1} & \mathbf{-2} \end{vmatrix} + \mathbf{3} \times \begin{bmatrix} \mathbf{-2} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$$

$$= \mathbf{1}[\mathbf{1} \times (\mathbf{-2}) - \mathbf{1} \times (\mathbf{-1})] - \mathbf{0} + \mathbf{3}[(\mathbf{-2}) \times (\mathbf{-1}) - \mathbf{1} \times \mathbf{1}]$$

$$= [\mathbf{-1}] + \mathbf{3}[\mathbf{1}] = \mathbf{2} \neq \mathbf{0}$$

- **The cofactors matrix A_c is:**

$$\mathbf{A_c} = \begin{bmatrix} \begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -3 & -3 \\ -3 & -5 & -7 \\ 1 & 1 & 1 \end{bmatrix}$$

- **The transpose of the cofactors matrix (A_c^T) is:**

$$\mathbf{A_c^T} = \begin{bmatrix} -1 & -3 & 1 \\ -3 & -5 & 1 \\ -3 & -7 & 1 \end{bmatrix}$$

- **Then, the inverse of A_c is found as follows:**

$$\mathbf{A^{-1}} = \frac{\mathbf{1}}{|\mathbf{A}|}\mathbf{A_c^T} = \frac{\mathbf{1}}{\mathbf{2}}\begin{bmatrix} -1 & -3 & 1 \\ -3 & -5 & 1 \\ -3 & -7 & 1 \end{bmatrix}$$

$$\textbf{And, Since} \qquad \mathbf{X = A^{-1}B}$$

$$\textit{Then,} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -3 & 1 \\ -3 & -5 & 1 \\ -3 & -7 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\textbf{Therefore,} \quad x = 1 \quad , \quad y = 1 \quad , \quad z = 0$$

(4-6): Economic Applications for the Matrices:

In this section we will represent some economic applications of matrices as discussed in determinants, the most important of these applications are the following:

(4-6-1): Market Equilibrium:

If we have the demand and supply equations on the following linear form:

$$ap \pm bq = c$$

$$\textit{Where:} \quad p \quad \textit{is the price unit} \ ,$$

$$q \quad \textit{is the quantity of demand or supply,}$$

$$a, b, c \quad \textit{are constants.}$$

Then, the matrices can be used in determining the equilibrium price and quantity, as shown in the following example.

<u>Example (4-7):</u>

By using the matrices, determine the price of the commodity and the quantity of demand for it, at which the market equilibrium is achieved, if you know that the demand and supply functions for this commodity are:

$\boldsymbol{Demand\ function}$: $\quad \boldsymbol{3p + 5q = 22}$

$\boldsymbol{Supply\ function}$: $\quad \boldsymbol{p = 1 + 1.5q}$

$\boldsymbol{Where}$: $\quad \boldsymbol{p} \quad \boldsymbol{is\ the\ price\ of\ the\ commodity\ unit(L.E.)}$,

$\boldsymbol{q} \quad \boldsymbol{is\ the\ demanded\ or\ supplied\ quantity\ of\ the\ commodity\ (per\ thousand\ units)}$.

<u>Solution:</u>

To determine the equilibrium price and quantity for this commodity, we first write the demand and supply functions on the following form:

$$\boldsymbol{ap \pm bq = c}$$

Therefore, we get:

$\boldsymbol{Demand\ function}$: $\quad \boldsymbol{3p + 5q = 22}$

$\boldsymbol{Supply\ function}$: $\quad \boldsymbol{p - 1.5q = 1}$

The two linear equations are solved by using matrices as follows:

- **Writing the demand and supply functions in the matrix form as follows:**

$$\begin{bmatrix} \boldsymbol{3} & \boldsymbol{5} \\ \boldsymbol{1} & \boldsymbol{-1.5} \end{bmatrix} \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = \begin{bmatrix} \boldsymbol{22} \\ \boldsymbol{1} \end{bmatrix}$$

This matrix form for these equations can be written simply as

$$\mathbf{AX = B}$$

Where **A is the** *coefficient matrix*,

X is the *column vector of unknown variables* $(\boldsymbol{p}, \boldsymbol{q})$,

B is the *column vector of constant terms*.

Therefore, $$\mathbf{X} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B}$$

- **Then we have to Determine the inverse of the matrix A as follows:**

$$\mathbf{A}^{-1} = \frac{\mathbf{1}}{|\mathbf{A}|}\mathbf{A}_c^T = \frac{\mathbf{1}}{[\mathbf{3} \times (-\mathbf{1.5})] - [\mathbf{1} \times \mathbf{5}]}\begin{bmatrix} -\mathbf{1.5} & -\mathbf{5} \\ -\mathbf{1} & \mathbf{3} \end{bmatrix}$$

$$= \frac{-\mathbf{1}}{\mathbf{9.5}}\begin{bmatrix} -\mathbf{1.5} & -\mathbf{5} \\ -\mathbf{1} & \mathbf{3} \end{bmatrix}$$

Then, $$\mathbf{X} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{bmatrix} = \frac{-\mathbf{1}}{\mathbf{9.5}}\begin{bmatrix} -\mathbf{1.5} & -\mathbf{5} \\ -\mathbf{1} & \mathbf{3} \end{bmatrix}\begin{bmatrix} \mathbf{22} \\ \mathbf{1} \end{bmatrix}$$

$$= \frac{-\mathbf{1}}{\mathbf{9.5}}\begin{bmatrix} -\mathbf{38} \\ -\mathbf{19} \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{2} \end{bmatrix}$$

Therefore, the market equilibrium is achieved when the selling price for the commodity unit is 4 pounds, and at this price the demanded quantity is equal to the supplied quantity which reached to 2 thousand of commodity units, *i.e.*

The equilibrium price $p = 4 \quad pound, \quad and;$**The equilibrium quantity** $q = 2000 \quad units\ from\ this$ **commodity**.

(4-6-2): Mix Production:

Previously we discussed how to use determinants in determining the optimum combination of different products within the limits of the available capacities of production factors, as this optimum combination of products by which achieved the full use of the available capacities (full use for the operation of production factors). And we will represent in this section the same topic, but using matrices instead of determinants in achieving the same objective. We will illustrate the use of matrices in the field of mix production through the following example:

Example (4-8):

A company produces three types of products, A, B, and C, through three production factors: L, N, and M. The produced unit from type (A) needs 5 *kg* of the raw material (L), 4 *kg* of the raw material (N) and 3 working hours in the operation center (M). Also, the produced unit from type (B) needs 3 *kg* of raw material (L), 2 *kg* of raw material (N) and 2 working hours in the operation center (M). While the produced unit from type (C) needs 7 *kg* of raw material (L), 5 *kg* of raw material (N) and 4 working hours in the operation center (M).

The available quantities of the raw material (L) are 2000 *kg* , of the raw material (N) are 1500 *kg*, and the available working hours at the operation center (M) are 1200 working hours.

<u>Required:</u>

1) Formulating the problem as equations that illustrate the full used of the available resources.

2) From your results in (1) and by using the matrices, determine the optimal production quantities that achieve the full used of the available capacities.

3) Assuming that the unit profit for the produced unit from the three products A, B, and C are 100, 50 and 150 pounds respectively. What is the total profit from the full used of the available capacities?

<u>Solution:</u>

1) Suppose that:

- The number of units to be produced of type (A) is x unit.

- The number of units to be produced of type (B) is y unit.

- The number of units to be produced of type (C) is z unit.

The following table summarizes the technical parameters for the unit's production from the three products, as well as the available capacities of production factors and the unit profit for the product unit as follow:

The product / Stages of production	A	B	C	The available capacities
L	**5**	**3**	**7**	**2000 *kg***
N	**4**	**2**	**5**	**1500 *kg***
M	**3**	**2**	**4**	**1200 *hours***
The unit profit	**100**	**50**	**150**	

Therefore, the mathematical formulation for the equations that determine the production levels, *i.e.* the values of *x*, *y*, and *z* that achieve the full used capacity for the operation of the available resources are as follows:

$$5x + 3y + 7z = 2000$$
$$4x + 2y + 5z = 1500$$
$$3x + 2y + 4z = 1200$$

2) Through the equations deduced in (1) that explain the full used capacity for operation of the available resources and by using the matrices, the production levels of the three products can be determined, *i.e.*, the values of x, y, and z that achieve the three equations, by solving these equations using matrices, as follows:

- Writing the linear equations in the matrix form as follows:

$$\begin{bmatrix} 5 & 3 & 7 \\ 4 & 2 & 5 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 1500 \\ 1200 \end{bmatrix}$$

or $$\mathbf{AX = B}$$

Where **A** **is the technical coefficient matrix,**

X **is the vector of unknown variables,**

B **is the vector of constant terms.**

Therefore, $$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A^{-1}B}$$

- Finding the inverse of A, as follow:

- **Determining the value of the determinant of the matrix as follow:**

$$|\mathbf{A}| = \left| \begin{array}{ccc|cc} 5 & 3 & 7 & 5 & 3 \\ 4 & 2 & 5 & 4 & 2 \\ 3 & 2 & 4 & 3 & 2 \end{array} \right.$$

$$|\mathbf{A}| = [5 \times 2 \times 4 + 3 \times 5 \times 3 + 7 \times 4 \times 2]$$

$$-[3 \times 2 \times 7 + 2 \times 5 \times 5 + 4 \times 4 \times 3]$$

$$= [40 + 45 + 56] - [42 + 50 + 48]$$

$$= 141 - 140 = 1$$

$\neq 0$ (*so that, there is an inverse for this matrix*)

- **The matrix of the cofactors A_c is:**

$$\mathbf{A_c} = \begin{bmatrix} \begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 7 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 5 & 7 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 7 \\ 4 & 5 \end{vmatrix} & \begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & 2 \\ 2 & -1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

- **The transpose of the matrix of the cofactors $(\mathbf{A}_c^T)$ is:**

$$\mathbf{A}_c^T = \begin{bmatrix} -2 & 2 & 1 \\ -1 & -1 & 3 \\ 2 & -1 & -2 \end{bmatrix}$$

- **The inverse of A is found as follow:**

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{A}_c^T = \frac{1}{1}\begin{bmatrix} -2 & 2 & 1 \\ -1 & -1 & 3 \\ 2 & -1 & -2 \end{bmatrix}$$

Now, since, $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

Then,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ -1 & -1 & 3 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} 2000 \\ 1500 \\ 1200 \end{bmatrix}$$

$$= \begin{bmatrix} -2(2000) + 2(1500) + 1(1200) \\ -1(2000) - 1(1500) + 3(1200) \\ 2(2000) - 1(1500) - 2(1200) \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \\ 100 \end{bmatrix}$$

Therefore, the production levels that achieve full used capacities for the operations of the production are:

$x =$ **the number of produced units from type (A)** $= 200$ **units,**

$y =$ **the number of produced units from type (B)** $= 100$ **units,**

$z =$ **the number of produced units from type (C)** $= 100$ **units.**

To be making sure the solution is correct, let us substitute in any of the three linear equations, let it be the 1st equation:

$$5x + 3y + 7z = 2000$$

$$\textit{The left hand side} = 5(200) + 3(100) + 7(100)$$

$$= 1000 + 300 + 700$$

$$= 2000 = \textit{The right hand side}$$

Therefore, the solution is correct.

3) To determine the total profit by which achieved the full used capacities for the operations of the available capacities, then

The total profit =

The sum of the products of multiplying the unit profit for each type of production $\times$ the number of units produced of each type.

$$i.e., \quad \text{The total profit} = 100x + 50y + 150z$$

$$= 100(200) + 50(100) + 150(100)$$

$$= 20000 + 5000 + 15000$$

$$= 40000 \quad \textit{pounds}$$

Exercises for Chapter (4)

1- If we have the following matrices:

$$A = \begin{bmatrix} 5 & -1 & -3 \\ -3 & 2 & 4 \\ 2 & 4 & 7 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -3 & 1 \\ -2 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$$

Required:

First: **Prove that:**

(1) $AB \neq BA$ (2) $(AB)^T = B^T A^T$

(3) $(AB)^{-1} = B^{-1}A^{-1}$

Second: **Find the following matrices:**

(1) $3A - 2B$ (2) $B + \frac{1}{2} A$ (3) $2A + B$

2- If we have the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 1 & 2 \\ 3 & -2 & 1 \\ 2 & 4 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 5 & 4 \\ 1 & 3 & -1 \\ 2 & 2 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$$

Required:

First: **Find the inverse, if there are exist for the preceding matrices.**

Second: **Prove that:**

(1) $B(C - D) = BC - BD$

(2) $(C + D)B = CB + DB$

(3) $B(CD) = (BC)D$

***Third*:** **Find:**

$(\mathbf{1})$ $\boldsymbol{The\ matrix}$ **M** $\boldsymbol{that\ make}$ $\mathbf{BM = C}$

$(\mathbf{2})$ $\boldsymbol{The\ matrix}$ **H** $\boldsymbol{that\ make}$ $\mathbf{HM = D}$

3- Solve the following linear equations by using the matrices:

$(\mathbf{1})$

$$\begin{aligned} 2x - 3y &= -1 \\ x + 4y &= 5 \end{aligned}$$

$(\mathbf{2})$

$$\begin{aligned} -4x + 2y - 9z &= 2 \\ 3x + 4y + z &= 5 \\ x - 3y + 2z &= 8 \end{aligned}$$

$(\mathbf{3})$

$$\begin{aligned} 2x + 3y + z &= 9 \\ x + 2y + 3z &= 6 \\ 3x + y + 2z &= 8 \end{aligned}$$

$(\mathbf{4})$

$$\begin{aligned} x + y - 2z &= 3 \\ 3x - y + z &= 0 \\ 3x + 3y - 6z &= 8 \end{aligned}$$

$(\mathbf{5})$

$$\begin{aligned} x + y &= 2 \\ 2x - z &= 1 \\ 2y - 3z &= -1 \end{aligned}$$

4- Determine the equilibrium price and quantity for the following demand and supply functions by using the matrices:

(1) $\quad$ $\boldsymbol{Demand\ function}$: $\quad$ $\boldsymbol{3p + 5q = 200}$

$\boldsymbol{Supply\ function}$: $\quad$ $\boldsymbol{7p - 3q = 56}$

(2) $\quad$ $\boldsymbol{Demand\ function}$: $\quad$ $\boldsymbol{q = 10 - p}$

$\boldsymbol{Supply\ function}$: $\quad$ $\boldsymbol{q = 2 + 3p}$

(3) $\quad$ $\boldsymbol{Demand\ function}$: $\quad$ $\boldsymbol{4p + q = 50}$

$\boldsymbol{Supply\ function}$: $\quad$ $\boldsymbol{6p = 10 + q}$

$\boldsymbol{Where}$: $\quad$ $\boldsymbol{p}$ $\quad$ $\boldsymbol{is\ the\ selling\ price\ of\ the\ commodity\ unit (L.E.)}$,

$\boldsymbol{q}$ $\quad$ $\boldsymbol{is\ the\ demanded\ or\ supplied\ quantity\ of}$

$\boldsymbol{the\ commodity}$

5- A trader can sell 200 units per day of a specific commodity at price 30 pounds per unit, and 250 units per day at price 27 pounds per unit. If the supply function for the same commodity is:

$$6p = q + 48$$

<u>Required:</u>

(a) find the demand function for this commodity assuming it's a linear function.

(b) Determine the equilibrium price and quantity by using matrices.

6- A company produced three products A, B and C which are processed during manufacture in three departments. The following

table shows the number of hours required to produce one unit of each product in each department, and the maximum weekly capacity for production in each department, expressed in the number of available working hours, as well as the profit of the produced unit.

The product / The department	A	B	C	The number of available weekly working hours
(1)	2	1	3	1300
(2)	4	3	2	1900
(3)	1	4	2	1100
The unit profit	3	2	5	

Required:

1) Formulating the problem as equations by which are showed the full used for the available capacities.

2) By using the matrices, determine the production levels of each product that the company must produce to achieve the full used for the available capacities.

3) What is the total profit from the full used for the available capacities?

Chapter (5)

Differentiation

Introduction

In many times in our practical life, we need to measure the changes that occur in the value of the function (the dependent variable) as a result of the changes in the value of the independent variable (or variables), so this chapter will discuss differential calculus and its economic applications.

Differential calculus **is concerned with the study of the changes that occur in a quantity when changes occur in other quantities on which the original quantity depends. The following are examples of such situations:**

- **The change in the total cost of a manufacturing plant that results from each additional unit produced.**
- **The change in the demand for a specific product that resulted from increasing of one unit in the price of this product.**
- **The change in the supply for a specific product that resulted from increasing of one unit in the price of this product.**

- **The change in the gross national product of a country with each additional year.**

In addition to other examples that illustrate the importance of measuring the changes that occur in the dependent variable as a result of changing in the independent variable.

<u>(5-1): The definition of Differentiation:</u>

The process of finding the derivative of a function is called *Differentiation.* Fortunately, for us, this process does not have to be as it may seem when we use the limit approach (it is not discussed in this book). The derivative is also giving the name of the *first differential coefficient* (first differential derivative). Let us assume that $y = f(x)$ be a given function, then the first derivative of (y) with respect to (x) is denoted by: $\frac{dy}{dx}$ or $f'(x)$ $i.e.,$ $\frac{dy}{dx} = f'(x)$

<u>Remark:</u> **-The derivative represents *the rate of change* in the dependent variable given a change in the independent variable.**

- **The derivative is *a general expression for the slope* of the graph of $f(x)$ at any point x in the domain.**
- **If the derivative of a function $f(x)$ exist at a particular value of x, then we say that $f(x)$ is *differentiable* at that point.**

<u>(5-2): Rules of Differentiation</u>

The rules of differentiation which is presented in this section have been developed using the limit approach. The mathematics involved in proving these rules can be very complicated, for this reason we will represent these rules without proof.

The rules of differentiation apply to functions which have specific structural characteristics. The rule will state that if a function has specific characteristics, then the derivative of the function will have a resulting form. As you study these rules, remember that each function can be graphed and that the first derivative is a general expression for the slope of the function.

A set of rules of differentiation exists for finding the derivatives of many common functions. Although there are a lot of functions for which the derivative does not exist, our concern will be with functions which are *differentiable.* In

presenting the rules, let $f'(x)$ (read "$f\ prime\ of\ x$") represent the derivative of the function $f(x)$ with respect to x.

Rule 1: Constant Function

$$\text{if}\quad f(x) = c \qquad ,\text{where } c \text{ is any constant, then:}$$

$$f'(x) = 0$$

This rule states that the derivative of the constant quantity in the function equals to zero. This is logical, since the differential derivative of the function means finding the rate of change of the function and the constant does not change, then it is normal for its derivative to be equal to zero. For example, the fixed costs in any production institution do not change with the change of production level , and therefore do not affect the total cost of production level.

Example (5-1):

Find the 1st derivative of the following function:

$$f(x) = 7$$

Solution:

By applying rule (1), then:

$$f'(x) = 0$$

This result seems reasonable if we consider what this function looks like graphically. The function $f(x) = 7$ graphs as a horizontal line intercept the(y) axis at the point (0 , 7). Therefore, the slope of this line at all points along such a function equals to zero.

Rule 2: Power Rule :

If we have: $f(x) = x^n$

, where n is any real number , then

$$f'(x) = nx^{n-1}$$

Example (5-2):

Find the 1st derivative of the following functions:

(1) $f(x) = x$ (2) $f(x) = x^5$

(3) $f(x) = x^{-4}$ (4) $f(x) = \frac{1}{x^2}$

(5) $f(x) = \sqrt{x}$ (6) $f(x) = \sqrt{x^3}$

Solution:

(1) *Since*, $f(x) = x = x^1$

Hence, by applying rule (2), then:

$$f'(x) = (1)x^{1-1} = x^0 = 1$$

This implies that for the function $f(x) = x$, the slope equals to 1 at all points. We should recognize that this function is a linear function with slope equals to 1.

(2) $\boldsymbol{Since,}$ $\quad f(x) = x^5$

Hence, by applying rule (2), then:

$$f'(x) = (5)x^{5-1} = 5x^4$$

(3) $\boldsymbol{Since,}$ $\quad f(x) = x^{-4}$

Hence, by applying rule (2), then:

$$f'(x) = (-4)x^{-4-1} = -4x^{-5}$$

(4) $\boldsymbol{Since,}$ $\quad f(x) = \frac{1}{x^2} = x^{-2}$ $\quad$; **Note that**: $\frac{1}{x^n} = x^{-n}$

Hence, by applying rule (2), then:

$$f'(x) = (-2)x^{-2-1} = -2x^{-3}$$

(5) $\boldsymbol{Since,}$ $\quad f(x) = \sqrt{x} = x^{\frac{1}{2}}$

Hence, by applying rule (2), then:

$$f'(x) = \left(\frac{1}{2}\right)x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$$

(6) $\boldsymbol{Since,}$ $\quad f(x) = \sqrt{x^3} = x^{\frac{3}{2}}$

; **Note that**: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$

Hence, by applying rule (2), then:

$$f'(x) = \left(\frac{3}{2}\right) x^{\frac{3}{2}-1} = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

<u>Remark:</u>

Note that the 1st rule is a special case from the 2nd rule when $n = 0$.

<u>Example (5-3):</u>

Find the 1st derivative of the following function by using the 2nd rule :

$$f(x) = 8$$

<u>Solution:</u>

$Since, \quad f(x) = 8 = 8x^0$

Hence, by applying rule (2), then:

$$f'(x) = (8)(0)x^{0-1} = 0x^{-1} = \frac{0}{x} = 0$$

It is the same result that we will get when we use the 1st rule.

Rule 3: Constant Times A Function

if $f(x) = c * g(x)$, where c is a constant, then:

$$f'(x) = c * g'(x)$$

Example (5-4):

Find the 1st derivative of the following function:

$$\boldsymbol{f(x) = 5x^4}$$

Solution:

By applying 3rd rule to find the 1st derivative, then:

$$\boldsymbol{c = 5 \quad , \quad g(x) = x^4 \quad , \quad g'(x) = 4x^3}$$

$\boldsymbol{Therefore}$: $\boldsymbol{f'(x) = c^* g'(x) = 5(4x^3) = 20x^3}$

Rule 4: Sum or Difference of Functions

if $f(x) = g(x) \pm u(x)$, then:

$$f'(x) = g'(x) \pm u'(x)$$

This rule implies that the derivative of a function formed by the sum (or difference) of two or more component functions is the sum (or difference) of the derivatives of the component functions.

Example (5-5):

Find the 1st derivative of the following function:

$$f(x) = 3x^3 - 5x^2 + 9x - 14$$

Solution:

By applying rule (4), then:

$$f'(x) = 3(3x^2) - 5(2x) + 9(1) - 0$$

$$f'(x) = 9x^2 - 10x + 9$$

Rule 5: Product Rule

if $f(x) = g(x)^* u(x)$, then:

$$f'(x) = g(x)^* u'(x) + u(x)^* g'(x)$$

Rule (5) implies that the derivative of a product is the 1st function times the derivative of the 2nd function *plus* the 2nd function times the derivative of the 1st function.

Example (5-6):

Find the 1st derivative for the following function:

$$f(x) = (x^2 - 6)(4x^3 - 2x - 1)$$

Solution:

Suppose that:

$$g(x) = (x^2 - 6) \quad , \quad u(x) = (4x^3 - 2x - 1)$$

Hence, by applying 5th rule , then we have:

$$f'(x) = g(x)^{*}u'(x) + u(x)^{*}g'(x)$$

$$= (x^2 - 6)(12x^2 - 2) + \ (4x^3 - 2x - 1)(2x)$$

$$= 12x^4 - 2x^2 - 72x^2 + 12 + 8x^4 - 4x^2 - 2x$$

$$= 20x^4 - 78x^2 - 2x + 12$$

<u>Another solution:</u>

This example can be solved in another way, which is to find the product of multiplying the two functions before performing the differential process, as follows:

$$f(x) = (x^2 - 6)(4x^3 - 2x - 1)$$

$$= 4x^5 - 2x^3 - x^2 - 24x^3 + 12x + 6$$

$$= 4x^5 - 26x^3 - x^2 + 12x + 6$$

Finding the 1st derivative for this form of this function does not require the use of rule (5).

$Therefore$: $f'(x) = 20x^4 - 78x^2 - 2x + 12$

This is the same preceding result obtained before..

Rule 6: Quotient Rule:

If $f(x) = \dfrac{g(x)}{u(x)}$ $\quad ,where\ u(x) \neq 0,$ $\quad$ then:

$$f'(x) = \frac{u(x)^{*}g'(x) - g(x)u'(x)}{[u(x)]^2}$$

Rule (6) implies that the 1st derivative of a quotient is the denominator function times the derivative of the numerator function *minus* the numerator function times the derivative of the denominator function. All are divided by the square of the denominator function.

Example (5-7):

Find the 1st derivative of the following function:

$$\boldsymbol{f(x) = \frac{2x^3 - 7}{4x^2 + 5}}$$

Solution:

Suppose that:

$$\boldsymbol{g(x) = 2x^3 - 7 \qquad , \qquad u(x) = 4x^2 + 5}$$

Hence, by applying rule (6), then:

$$\boldsymbol{f'(x) = \frac{(4x^2 + 5)(6x^2) - (2x^3 - 7)(8x)}{[4x^2 + 5]^2}}$$

$$= \frac{24x^4 + 30x^2 - 16x^4 + 56x}{[4x^2 + 5]^2}$$

$$= \frac{8x^4 + 30x^2 + 56x}{[4x^2 + 5]^2}$$

Rule 7: Power of A Function

If $f(x) = [g(x)]^n$

$, where\ n\ is\ a\ real\ number\ ,$ then

$$f'(x) = n[g(x)]^{n-1}.g'(x)$$

This rule is a very similar to the 2nd rule (power rule). In fact, the power rule is a special case of this rule when $g(x) = x$.

When $g(x) = x \quad , then \quad g'(x) = 1$

Hence, by applying rule (7), we get:

$$f'(x) = n(x)^{n-1}.(1) = nx^{n-1}$$

<u>Example (5-8):</u>

Find the 1st derivative of the following functions:

(1) $f(x) = (2x^3 + 5x^2 - 7)^4$

(2) $f(x) = \sqrt{3x^2 + 8x}$

(3) $f(x) = \left(\frac{2x}{x^2 + 1}\right)^3$

Solution:

(1) $\boldsymbol{Since,} \quad \boldsymbol{g(x) = 2x^3 + 5x^2 - 7} \quad , \quad \boldsymbol{n = 4}$

Hence, by applying rule (7), then:

$$\boldsymbol{f'(x) = n[g(x)]^{n-1}.\, g'(x)}$$

$$\boldsymbol{= 4(2x^3 + 5x^2 - 7)^{4-1}.\,(6x^2 + 10x)}$$

$$\boldsymbol{= 4(2x^3 + 5x^2 - 7)^{3}.\,(6x^2 + 10x)}$$

(2) $\boldsymbol{Since,} \quad \boldsymbol{f(x) = \sqrt{3x^2 + 8x} = (3x^2 + 8x)^{\frac{1}{2}}}$

$\boldsymbol{Therefore}$: $\boldsymbol{g(x) = 3x^2 + 8x} \quad , \quad \boldsymbol{n = \frac{1}{2}}$

Hence, by applying rule (7), then:

$$\boldsymbol{f'(x) = n[g(x)]^{n-1}.\, g'(x)}$$

$$\boldsymbol{= \frac{1}{2}(3x^2 + 8x)^{\frac{1}{2}-1}.\,(6x + 8)}$$

$$\boldsymbol{= \frac{1}{2}(3x^2 + 8x)^{-\frac{1}{2}}.\,(6x + 8)}$$

$$\boldsymbol{= \frac{6x + 8}{2\sqrt{3x^2 + 8x}}}$$

(3) $\boldsymbol{Since,} \quad \boldsymbol{g(x) = \frac{2x}{x^2 + 1}} \quad , \quad \boldsymbol{n = 3}$

Hence, by applying rule (7), then:

$$f'(x) = n[g(x)]^{n-1} . g'(x)$$

$$= 3\left(\frac{2x}{x^2+1}\right)^{3-1} . \left[\frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2}\right]$$

$$= 3\left(\frac{2x}{x^2+1}\right)^{2} . \left[\frac{2x^2+2-4x^2}{(x^2+1)^2}\right]$$

$$= 12\left(\frac{x}{x^2+1}\right)^{2} . \left[\frac{-2x^2+2}{(x^2+1)^2}\right]$$

$$= 24\left(\frac{x}{x^2+1}\right)^{2} . \left[\frac{-x^2+1}{(x^2+1)^2}\right]$$

$$= \frac{24x^2(-x^2+1)}{(x^2+1)^4}$$

Rule 8: Base-*e* Exponential Functions:

If $f(x) = e^{g(x)}$, then:

$$f'(x) = g'(x) . e^{g(x)}$$

Where : $e \approx 2.71828$

<u>Example (5-9):</u>

Find the 1st derivative of the following functions:

(1) $f(x) = e^x$

$(2) \quad f(x) = e^{5x^2+x}$

Solution:

$(1) \quad Since, \qquad f(x) = e^{x} \qquad , then \quad g(x) = x$

Hence, by applying rule (8), then:

$$f'(x) = g'(x).e^{g(x)}$$

$$= (1).e^{x} = e^{x}$$

This result is unique in that the function $f(x) = e^{x}$ and its derivative are identical.

$$i.e., \qquad f(x) = f'(x) = e^{x}$$

Graphically, the interpretation is that for any value of x, the slope of the graph of $f(x) = e^{x}$ is exactly equal to the value of the function.

$(2) \quad Since, \qquad f(x) = e^{5x^2+x} \qquad , then:$

$g(x) = 5x^2 + x$

Hence, by applying rule (8), then:

$$f'(x) = g'(x).e^{g(x)}$$

$$= (10x + 1).e^{5x^2+x}$$

Rule 9: Natural Logarithm Functions:

If $f(x) = \log_e g(x) = \ln g(x)$, then:

$$f'(x) = \frac{g'(x)}{g(x)}$$

Example (5-10):

Find the 1st derivative of the following functions:

(1) $\boldsymbol{f(x) = \ln x}$

(2) $\boldsymbol{f(x) = \ln\ (4x^2 - 3x + 1)}$

Solution:

(1) $\boldsymbol{Since,\quad f(x) = \ln x \quad, then:\ g(x) = x}$

Hence, by applying rule (9), then:

$$\boldsymbol{f'(x) = \frac{g'(x)}{g(x)} = \frac{1}{x}}$$

(2) $\boldsymbol{Since,\quad f(x) = \ln\ (4x^2 - 3x + 1)}$,

$\boldsymbol{then:\ g(x) = 4x^2 - 3x + 1}$

Hence, by applying rule (9), then:

$$\boldsymbol{f'(x) = \frac{g'(x)}{g(x)} = \frac{8x-3}{4x^2-3x+1}}$$

Rule 10: Chain Rule:

If $y = f(u)$ $\quad and \quad$ $u = g(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

In fact, rule 7 (power function) is a special case of the more general *chain rule*. The chain rule specifically applies to the composite functions whose values depend upon other functions.

<u>Example (5-11):</u>

If we have the following two functions:

$$\boldsymbol{y = f(u) = 2u^2 - u + 1}$$

$$\boldsymbol{and \quad u = f(x) = x^3 + 2x^2}$$

Find the 1stderivative: $\boldsymbol{\frac{dy}{dx}}$.

<u>Solution:</u>

$$\boldsymbol{Since, \quad y = 2u^2 - u + 1 \quad and \quad u = x^3 + 2x^2}$$

Therefore, by applying rule (10), then we have the following:

$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

$$= (4u - 1)(3x^2 + 4x)$$

We can express $\frac{dy}{dx}$ strictly in terms of x by substituting $u = x^3 + 2x^2$ as follow:

$$\frac{dy}{dx} = [4(x^3 + 2x^2) - 1](3x^2 + 4x)$$

$$= (4x^3 + 8x^2 - 1)(3x^2 + 4x)$$

$$= 12x^5 + 16x^4 + 24x^4 + 32x^3 - 3x^2 - 4x$$

$$= 12x^5 + 40x^4 + 32x^3 - 3x^2 - 4x$$

<u>Another solution:</u>

The 1st derivative ($\frac{dy}{dx}$) can be found in another way by substituting in $f(u)$ in terms of x from the beginning by substituting $u = x^3 + 2x^2$ into the function $y = 2u^2 - u + 1$, as follow:

$$y = 2u^2 - u + 1$$

$$= 2(x^3 + 2x^2)^2 - (x^3 + 2x^2) + 1$$

$$= 2\left(x^6 + 4x^5 + 4x^4\right) - x^3 - 2x^2 + 1$$

$$= 2x^6 + 8x^5 + 8x^4 - x^3 - 2x^2 + 1$$

Then, we find directly the 1st derivative ($\frac{dy}{dx}$) as follow:

$$\frac{dy}{dx} = 12x^5 + 40x^4 + 32x^3 - 3x^2 - 4x$$

This is the same preceding result.

(5-3): Higher-Order Derivatives:

The *higher-order derivatives* are the 2nd order derivatives or more. So, there are other derivatives which can be defined.

The derivative $f'(x)$ of the function $f(x)$ refers to the *1st derivative* of the function. The adjective "first or 1st " is used to distinguish this derivative from other derivatives (higher-order derivatives) associated with this function. The order of the 1st derivative is 1.

This section discussed these higher-order derivatives and their interpretation.

(5-3-1): The Second Derivative

The *second (2nd)derivative* of a function is the derivative of the 1st derivative. Therefore, if we have the function $y = f(x)$, then the 2nd derivative is obtained by performing the differentiation of the 1st derivative, *i.e.* by differentiating the function twice with respect to *x.*

The 2nd derivative is denoted by either $\frac{d^2y}{dx^2}$ $or\ f''(x)$. It is found by applying the same rules of the differentiation which were used in finding the 1st derivative. The following table

illustrates the computation of the 1st and 2nd derivatives for several functions:

$f(x)$	$f'(x)$	$f''(x)$
$3x^2 - 5x + 4$	$6x - 5$	6
$x^4 + 2x$	$4x^3 + 2$	$12x^2$
e^x	e^x	e^x
$\ln x$	$\frac{1}{x}$	$\frac{-1}{x^2}$
$4x + 1$	4	0

Just as the 1st derivative is a measure of the rate of change in the value of (y) with respect to a change in (x). The 2nd derivative is a measure of the rate of change in the value of the first derivative with respect to the change in (x). In other words, the 2nd derivative is a measure of the rate of change in the slope with respect to a change in (x).

(5-3-2): The 3rd and Higher-Order Derivatives:

The 3rd and the higher-order derivatives become easy to understand as the 3rd derivative can be found by performing the differentiation of the 2nd derivative, and so on for the higher-order derivatives.

In general, the n^{th}-order derivative is obtained by performing the differentiation of the function (n) times with respect to (x). It is denoted by either $\frac{d^n y}{dx^n}$ $or\, f^{(n)}(x)$. In other words, the n^{th}-order derivative is obtained by differentiating the derivative of order (n) with respect to $(x)\, on\, the\, (n-1)\, th\;\; derivitive$. *i.e.,*

$$f^{(n)}(x) = \frac{d}{dx}[f^{(n-1)}(x)]$$

<u>Example (5-12):</u>

If we have the following function:

$$y = f(x) = x^5 - 6x^4 - 2x^3 + 7x + 12$$

<u>Required:</u>

Find the following derivatives:

(1) $f'(x)$ (2) $f''(x)$ (3) $f^{(3)}(x)$

<u>Solution:</u>

Since, $y = f(x) = x^5 - 6x^4 - 2x^3 + 7x + 12$

Then,

(1) $f'(x) = \frac{dy}{dx} = 5x^4 - 24x^3 - 6x^2 + 7$

(2) $f''(x) = \frac{d^2y}{dx^2} = 20x^3 - 72x^2 - 12x$

$$(3)\ \ f^{(3)}(x) = \frac{d^{(3)}y}{dx^{(3)}} = 60x^2 - 144x - 12$$

Example (5-13):

Find all possible derivatives of the following function:

$$y = f(x) = x^3 - 6x^2 + 9x + 16$$

Solution:

$$f'(x) = \frac{dy}{dx} = 3x^2 - 12x + 9$$

$$f''(x) = \frac{d^2y}{dx^2} = 6x - 12$$

$$f^{(3)}(x) = \frac{d^{(3)}y}{dx^{(3)}} = 6$$

$$f^{(4)}(x) = \frac{d^{(4)}y}{dx^{(4)}} = 0$$

All additional higher-order derivatives will also equal to zero.

(5-4): Economic Applications for Differentiation (Marginal Analysis)

The derivatives have a number of applications in business and economics that helps the economist to make decisions in many economic fields. The first derivative is used in many

economic applications in constructing what are called *marginal rates*, that is, a rate of change.

Assuming that the commercial foundation is producing a specific number of units each year, and then the *marginal analysis* provides us the effect of each additional unit produced and sold on the total cost, revenue and profits.

The economic applications will include the following topics:

- **Marginal cost**
- **Marginal Revenue**
- **Marginal Profit**
- **Marginal Price and Marginal Demand**
- **Demand Elasticity**

And we will deal with these topics in some detail as follows:

(5-4-1): Marginal Cost

The *marginal cost* indicates the extra cost of producing and selling an additional unit of the product. Therefore, the marginal cost measures the rate at which the total cost is increasing with respect to increases in the quantity produced.

The linear cost function assumes that the variable cost per unit is constant; therefore, the marginal cost is the same at

all production levels. For example, if we have the following cost function:

$$C(q) = 6000 + 1.5q$$

Where (q) is the production level (or the number of produced units) and $C(q)$ is the total cost for the production level (q).

Then, the variable cost per unit is constant at $1.5\,pounds$, and the marginal cost is always $1.5\,pounds$ at all production levels.

But the nonlinear cost functions are characterized by the change in the marginal cost from one additional level to another of the production levels. For example, if we consider the following cost function:

$$C(q) = 20000 + 10q + 0.003q^2$$

The calculation of the marginal cost of the different levels of production is illustrated in the following table:

Production Level (q)	**Total Cost C(q)**	**Marginal Cost $\Delta C = C(q) - C(q-1)$**
200	**22120.000**	—
201	**22131.203**	**11.203**
202	**22142.412**	**11.209**
203	**22153.627**	**11.215**
204	**22164.848**	**11.221**

It is clear from the third column that the marginal cost differs according to the production level, as we find that, the marginal cost of producing the unit 201 is equal to 11.203 pounds, and it differs from the marginal cost of producing and selling any of the additional units.

The 1st derivative of the total cost function represents the rate of change in the total cost due to the change in the number of units produced. *i.e.,* the marginal cost function $C'(q)$ is the 1st derivative of the total cost function $C(q)$ with respect to the quantity produced. Also, this derivative expressed in general as the slope of the graph of the total cost function.

Therefore, $$\text{The Marginal Cost Function } C'(q) = \frac{d}{dq} C(q)$$

The marginal cost gives an approximate estimate of the cost of producing an additional unit of product. And by finding

the marginal cost function $C'(q)$ of the previously function, we find that:

$$C'(q) = 10 + 0.006q$$

The approximate estimate of the marginal cost as a result of producing the unit 201 can be obtained as follows:

$$C'(200) = 10 + 0.006(200) = 11.2 \quad pounds$$

It is an estimate of the marginal cost of producing the unit 201, which is close to the real value of this cost mentioned in the table, which is 11.203 pounds.

It is necessary to note that the marginal cost in the third column increases each time by 0.006 pounds as a result of producing an additional unit of the product, and this amount represents the slope of the marginal cost function ($note\ that$: $C''(q) = 0.006$).

Note that there is an important difference between the marginal cost and the average cost. If $C(q)$ is the total cost function, then the average cost of producing *(q)* units is the total cost $C(q)$ divided by the number of unit produced, as follows:

$$\text{The Average Cost Function } \bar{C}(q) = \frac{C(q)}{q}$$

Example (5-14):

If the total cost in pounds for producing (*q*) unit of a specific product is given by the following function:

$$\mathbf{C}(q) = 350000 + 7000q + 0.25q^2$$

Required:

a) **Determine the marginal cost as a function in the number of units produced.**

b) **Find the marginal cost at the following production levels:**

$i.\quad q = 100. \qquad\qquad ii.\quad q = 200.$

Then, interpret your results economically in each case.

c) **Find the cost of producing:**

$i.\quad 20\ units. \qquad\qquad ii.\quad the\ 7^{th}\ unit.$

d) **Find the average cost of producing 50 units.**

Solution :

$a)$ **The Marginal Cost Function** $\mathbf{C}'(q) = \frac{d}{dq}\mathbf{C}(q)$

$$= \frac{d}{dq}(350000 + 7000q + 0.25q^2)$$

$$i.e.,\quad \mathbf{C}'(q) = 7000 + 0.5q$$

This function is the marginal cost; it represents the average cost per *additional unit* of a small increase in the production given that (*q*) units are already being produced.

$b)$ $i.$ $When \quad q = 100$

$$C'(100) = 7000 + 0.5(100) = 7050$$

Therefore, at the production level $q = 100$ units, then any small increase in production will costs 7050 pounds on average per unit.

$ii.$ $When \quad q = 200$

$$C'(200) = 7000 + 0.5(200) = 7100$$

This indicates that if the production increased slightly over the quantity $q = 200$ units, then the production of the additional unit will costs 7100 pounds on average per unit.

Generally, in both cases, we can say that the production of 101^th^ unit will cost 7050 pounds, and the production of 201^th^ unit will cost 7100 pounds. Such statement as these is not quite accurate, since the derivative gives the rate for a small increment in production, not for a unit increment.

$c)$ $i.$ **The cost of producing 20 units is:**

$$C(20) = 350000 + 7000(20) + 0.25(20)^2 = 490100$$

$ii.$ **The cost of producing the 7^{th} unit is:**

$$C'(6) = 7000 + 0.5(6) = 7003$$

$d)$ **The Average Cost Function** $\bar{C}(q) = \frac{C(q)}{q}$

$$\bar{C}(q) = \frac{350000 + 7000q + 0.25q^2}{q}$$

$$= \frac{350000}{q} + \frac{7000q}{q} + \frac{0.25q^2}{q}$$

$$i.e., \quad \bar{C}(q) = \frac{350000}{q} + 7000 + 0.25q$$

Therefore, the Average Cost of producing 50 units is:

$$\bar{C}(50) = \frac{350000}{50} + 7000 + 0.25(50) = 14012.5$$

(5-4-2): Marginal Revenue

The *marginal revenue* indicates the extra revenue resulted from selling an additional unit of the product. If the revenue function is linear and the price is constant, then the marginal revenue is always equal to the unit price of the product. For example, if we have the following total revenue function:

$$R(q) = 25q$$

Then, this function indicates that the unit price of the product is 25 pounds, also the marginal revenue from selling an additional unit is 25 pounds at any level of production.

But, in the case of the nonlinear revenue function, then the marginal revenue is not constant as it changes with the change in the level of production.

For example, if we consider the following revenue function:

$$R(q) = 20q - 0.002q^2$$

Then, we can find the marginal revenue of the different levels of production as follows in the following table:

Production Level (q)	Total Revenue $R(q)$	Marginal Revenue $\Delta R = R(q) - R(q-1)$
200	2920.000	———
201	3939.198	19.198
202	3958.392	19.194
203	3977.582	19.190
204	3996.768	19.186

The third column in the table shows the marginal revenue amount at different levels of the production. It is noted that although the differences between production levels are small (one unit at a time), the revenue changes for the additional unit in the level of production.

If $R(q)$ denotes the total revenue function of selling (q) units, then the marginal revenue function is defined to be the 1st derivative $R'(q)$ as follows:

$$\textbf{The Marginal Revenue Function } R'(q) = \frac{d}{dq}R(q)$$

Therefore, we can use the 1st derivative$R'(q)$, to represents the rate of change in the total revenue due to the change in the number of selling units. To determine the approximate marginal revenue of selling an additional unit of the product, then we can find the marginal revenue function $R'(q)$ of the previously mentioned function as follows:

$$R'(q) = \frac{d}{dq}R(q)$$

$$= \frac{d}{dq}(20q - 0.002q^2)$$

$$i.e., \qquad R'(q) = 20 - 0.004q$$

To obtain an approximate value of the marginal revenue of selling the 201th unit, we find the value of $R'(q)$ at $q = 200$, *i.e.,*

$$R'(200) = 20 - 0.004(200) = 19.2$$

This approximation is very close to the real value shown in the table, which is 19.198 pounds.

It is useful for the reader that we indicate that the marginal revenue in the third column decreases each time by 0.004 pounds as a result of selling an additional unit of the product, and this amount represents the slope of the marginal revenue function ($\boldsymbol{note\ that}: \mathbf{R}''(\boldsymbol{q}) = -\mathbf{0.004}$).

<u>Example (5-15):</u>

If the demand function of a specific commodity is given by:

$$\mathbf{q} = \mathbf{1000} - \mathbf{20}\boldsymbol{p}$$

where(*q*) is the demanded quantity (or sales), and (*p*) is the unit price (in pounds).

And if the total cost function is:

$$\boldsymbol{C}(\boldsymbol{q}) = \mathbf{1000} + \mathbf{10}\boldsymbol{q}$$

<u>Required:</u>

a) **Determine the marginal revenue as a function in the quantity of production.**
b) **Find the marginal revenue at the following production levels:**

$\boldsymbol{i}.\quad \boldsymbol{q} = \mathbf{100}.$ $\qquad\qquad$ $\boldsymbol{ii}.\quad \boldsymbol{q} = \mathbf{150}.$

<u>Solution :</u>

$a)$ **Since,**

The total revenue

$$= \text{The number of sold units} \times \text{Unit selling price}$$

$$R(q) = q \times p = qp$$

And, from the demand function:

$$q = 1000 - 20p$$

$$20p = 1000 - q$$

$$\therefore \quad p = 50 - 0.05q$$

Therefore, $\quad$ **The total revenue** $= q(50 - 0.05q)$

$$i.e., \qquad R(q) = 50q - 0.05q^2$$

Therefore, we can determine the marginal revenue function as follows:

The Marginal revenue Function $R'(q) = \dfrac{d}{dq}R(q)$

$$= \frac{d}{dq}(50q - 0.05q^2)$$

$$i.e., \quad R'(q) = 50 - 0.1q$$

$b)$ $\;$ $i.$ $\quad$ $When \quad q = 100$

$$R'(100) = 50 - 0.1(100) = 40$$

Therefore, at the production level $q = 100$ units, the total revenue increased by 40 pounds as a result of the increase in sales by one unit, *i.e.,* as a result of selling the 101th unit.

$ii. \quad When \quad q = 150$

$$R'(150) = 50 - 0.1(150) = 35$$

which indicates that, if the production level is 150 units, then the total revenue increased by 35 pounds as a result of the increase in sales by one unit, *i.e.,* as a result of selling the 151th unit.

(5-4-3): Marginal Profit

The *marginal profit* represents the amount of change in the total profit as a result of producing and selling an additional unit of the product. The marginal profit, as the marginal revenue and the marginal cost, is a constant for all production levels in the case of the total profit function being linear, and in the case of the nonlinear function, then the profit various,

Therefore the total profit function for any level of production is given by the difference between its total revenue and its total costs. If the total revenue function is $R(q)$ when (*q*) units are sold and if the total cost function is $C(q)$ when (*q*)

units are produced, then the total profit function $P(q)$ obtained by producing and selling *(q)* units is given by:

$$P(q) = R(q) - C(q)$$

For example, we can obtain the total profit function by using the linear function of total cost: $C(q) = 6000 + 1.5q$, and the nonlinear function of total revenue: $R(q) = 20q - 0.002q^2$, which were previously mentioned when we were dealing with total cost and total revenue, as follows:

$$P(q) = R(q) - C(q)$$

$$= 20q - 0.002q^2 - (6000 + 1.5q)$$

***i.e.,* the total profit function is:**

$$P(q) = 18.5q - 0.002q^2 - 6000$$

Therefore, we can calculate the marginal profit for the different levels of production as follows in the following table:

Production Level (q)	**Total Profit [*] $P(q)$**	**Marginal Revenue $\Delta P = P(q) - P(q-1)$**
200	-2380.000	——
201	-2362.302	**17.698**
202	-2344.608	**17.694**
203	-2326.918	**17.690**
204	-2309.232	**17.686**

[*] The negative amount of profit means a loss.

The third column in the table show the marginal profit amount at different levels of the production. The first derivative $P'(q)$ is called the *marginal profit* which represents the additional profit of producing and selling an additional unit of the product. *i.e.,* the marginal profit function is defined as follows:

$$\textbf{The Marginal Profit Function } P'(q) = \frac{d}{dq}P(q)$$

For the preceding example, we can find the marginal profit function $P'(q)$ of the previously mentioned function as follows:

$$P'(q) = \frac{d}{dq}P(q)$$

$$= \frac{d}{dq}(18.5q - 0.002q^2 - 6000)$$

$$i.e., \quad P'(q) = 18.5 - 0.004q$$

Then, to obtain an approximate value of the marginal profit of selling the 201th unit, we find the value of $R'(q)$ at $q = 200$, *i.e.,*

$$P'(200) = 18.5 - 0.004(200) = 17.7$$

This approximation is nearest to the real value shown in the table, which is 17.698 pounds.

Here, we can conclude that the marginal profit in the third column decreases each time by 0.004 pounds as a result of producing and selling an additional unit of the product, and this amount represents the slope of the marginal profit function ($note\ that$: $\mathrm{R}''(q) = -0.004$).

Example (5-16):

By using the data in example (5-15), determine the marginal profit as a function of the sales quantity. Then calculate the marginal profit when the sales quantity reaches 200 units.

<u>Solution :</u>

Since,

$$\boldsymbol{The\ total\ profit = The\ total\ revenue - The\ total\ costs}$$

$$\boldsymbol{i.e.,\qquad P(q) = R(q) - C(q)}$$

$$\boldsymbol{= 50q - 0.05q^2 - (1000 + 10q)}$$

Therefore, the total profit function is:

$$\boldsymbol{P(q) = 40q - 0.05q^2 - 1000}$$

Then, the marginal profit function is obtained as follows:

$$\boldsymbol{P'(q) = \frac{d}{dq}P(q)}$$

$$\boldsymbol{= \frac{d}{dq}(40q - 0.05q^2 - 1000)}$$

$$\boldsymbol{i.e.,\qquad P'(q) = 40 - 0.1q}$$

Hence, the marginal profit at the sales quantity $q = 200$ is:

$$\boldsymbol{P'(200) = 40 - 0.1(200) = 20}$$

Which means that the profit increases by 20 pounds as a result of increasing the sales by one unit? *i.e.,* the sale of the 201<u>th</u> unit yields a profit of 20 pounds.

(5-4-4): Marginal Price and Marginal Demand:

If the demand function for a specific commodity is on the form:

$$p = f(q)$$

Where (*p*) is the unit price, and (*q*) is the demanded quantity.

Then, ($\frac{dp}{dq}$) is called the *marginal price*. It represents the amount of change in the unit price as a result of changing in the quantity of demand by a small increment.

But if the demand function is on the form:

$$q = f(p)$$

Then,($\frac{dq}{dp}$) is called the *marginal demand*, which indicates the amount of change in the quantity of demand as a result of changing the unit price by a small increment.

Example (5-17):

If the demand function of a product is:

$$p = 2024 - 2q - q^2$$

Where (*p*) is the unit price, and (*q*) is the demand quantity.

***Required*:**

a) **Determine the marginal price as a function in the demand quantity.**

b) **Find the marginal price at the demand level $q = 30$ units. Then interpret your result economically.**

Solution:

$a)$ **Since,** $$p = 2024 - 2q - q^2$$

Then, The marginal price $= \frac{dp}{dq} = -2 - 2q$

$b)$ **The marginal price at the demand level $q = 30$ is:**

$$\text{The marginal price} = -2 - 2(30) = -62$$

This means that with an increase in the quantity of demand by one unit at the demand level of 30 units, then the price of the commodity will decreased by 62 pounds, *i.e.,* selling the 31th unit leads to decrease in the price by 62 pounds.

Example (5-18):

If the demand quantity for a specific commodity is determined according to the price of this commodity and according to the following function: $p^2 + q = 20$

Where: (*p*) is the unit price, an (*q*) is the demanded quantity.

***Required*:**

Determine the marginal demand when the commodity's price is 2 pounds.

Solution:

Since, $p^2 + q = 200$

Then, $q = 200 - p^2$

Therefore, $\text{The marginal demand} = \frac{dq}{dp} = -2p$

Then, at the unit price is $= 2$, then the marginal demand can be determined as follows:

$$\text{The marginal demand} = -2(2) = -4$$

Therefore, the demand quantity for this commodity is decreased by 4 units if the unit price of the commodity increased by one pound. In other words, if the commodity is offered at price 3 pounds, then the demand for the commodity will decreased by 4 units.

(5-4-5): Elasticity of Demand

The *elasticity of demand* is one of the most widely used concepts in economics. If we have the demand function for a specific product: $q = f(p)$, then the elasticity of demand is defined as :

The elasticity of demand

$$= \frac{\textit{The percentage change in demand}}{\textit{The percentage change in price}}$$

This ratio measures the relative response of demand to the changes in the price. It can be expressed in symbols as follows:

$$\textbf{The elasticity of demand} = \frac{100\left(\frac{\Delta q}{q}\right)}{100\left(\frac{\Delta p}{p}\right)} = \frac{p}{q}\left(\frac{\Delta q}{\Delta p}\right)$$

The elasticity, in its current form, measures the average rate of change in the demand with respect to the change in the price; therefore, the numerical value of elasticity is usually negative due to the inverse relationship between the price and the demanded quantity.

Usually we need to measure the elasticity of demand at a certain price, so we are going to measure the instantaneous rate of change in the demand with respect to the change in the price. In other words, we measure the change in the demand when there is a slight change in the price ($\Delta p \to 0$), therefore the ratio ($\frac{\Delta q}{\Delta p}$) is roughly equal to the derivative ($\frac{dq}{dp}$).

i.e.,

$$\frac{p}{q}\left(\frac{\Delta q}{\Delta p}\right) \approx \frac{p}{q}\left(\frac{dq}{dp}\right) = \textbf{The elasticity } (E)$$

$$i.e., E$$

$$= \frac{price}{quantity}$$

$$\times \text{ the } 1^{st} \text{ derivative of quantity with respect to price}$$

Therefore, we can say that if the change in the price was slight, then:

$$\text{The percentage of change in the demand}$$
$$\approx E \times (\text{The percentage of change in price})$$

For example, if an increase in price by 4% leads to a decrease in demand by 5%, then the elasticity of demand is:

$$E = \frac{-5}{4} = -1.25$$

Also, if the elasticity of demand is equal to -0.8 , then the increase in price by 6% leads to decrease in demand by:

$$(-0.8)(6\%) = -4.8\%$$

<u>Example (5-19):</u>

If the demand function for a specific commodity is given by:

$$q = 250 - 30p + p^2$$

Where (*p*) is the unit price (L.E.), and (*q*) is the demanded quantity.

***Required*:**

a) **Find the elasticity of demand at $p = 12$. Comment on your result.**
b) **If the price increased by 8.5%, what is the approximate percentage of demand decrease?**

Solution:

a) Since, $\quad$ $$\text{The elasticity of demand } (E) = \frac{p}{q}\left(\frac{dq}{dp}\right)$$

And, $$q = 250 - 30p + p^2$$

Then, $$\frac{dq}{dp} = -30 + 2p$$

Therefore, $$\text{The elasticity of demand } (E) = \frac{p}{q}(-30 + 2p)$$

Then, at $p = 12$:

Hence, $$q = 250 - 30(12) + (12)^2 = 34$$

Therefore, the elasticity of demand at $p = 12$ *and* $q = 34$ is determined as follows:

$$\text{The elasticity of demand } (E) = \frac{12}{34}[-30 + 2(12)]$$

$$= \frac{12}{34}[-6] = -2.12$$

This means that when the unit price is 12 pounds, then the increase in the price by 1% leads to decrease in demand by 2.12%. Therefore, it is expected that the percentage change in demand is more than double the percentage change in price.

$\mathbf{b)\ Since,}$

$$\boldsymbol{The\ percentage\ change\ in\ demand} \\ \approx \boldsymbol{E} \times (\boldsymbol{The\ percentage\ change\ in\ price})$$

Then,

$$\boldsymbol{The\ percentage\ change\ in\ demand} \approx (\mathbf{-2.12})\ (\mathbf{8.5}\%) \\ \approx \mathbf{-18}\%$$

Therefore, an 8.5% increase in price of the commodity, it leads to decreasing in the demand quantity by 18%.

Economists have divided the values that the elasticity of demand takes into *three cases* as follows:

- ***<u>Case (1)</u>:*** $(|\boldsymbol{E}| > \mathbf{1})$, $\boldsymbol{i.e.}$ $\boldsymbol{E} < \mathbf{-1}$

In this case, the percentage change in the demand quantity is greater than the percentage change in price (for example: 1% change in price leads to a change in demand by more than 1%). in this case, the demand is said to be *elastic*. Therefore, the increase in the price leads to a decrease in

demand by a greater rate, this leads to a *decreasing* in the total revenue. *i.e.,* the relationship is *negative* between price and the total revenue.

- ***Case (2):*** $(|E| < \mathbf{1})$, $\boldsymbol{i.e.}$ $-\mathbf{1} < \boldsymbol{E} < \mathbf{0}$

In this case, the percentage change in the demand quantity is less than the percentage change in price, then the demand is said to be *inelastic*. in this case, the increase in the price leads to a decrease in demand by a less rate, this leads to an *increase* in the total revenue. *i.e.,* the relationship is *positive* between price and total revenue.

- ***Case (3):*** $(|E| = \mathbf{1})$, $\boldsymbol{i.e.}$ $\boldsymbol{E} = -\mathbf{1}$

a) **In this case, the percentage change in the demand quantity is equal to the percentage change in price. Then, in this case, it is said that demand has *unit elasticity*, and that elasticity is an equivalent. Therefore, the total revenue remains constant for the price increase or decrease change.**

Example (5-20):

If the demand function for a specific commodity is given by:

$$q = 1000 - 50p$$

Where (*p*) is the unit price(L.E.), and (*q*) is the demanded quantity.

***Required*:**

b) **Find the elasticity of demand and Find elasticity of demand at the following prices for the commodity**

$\boldsymbol{i.}\quad \boldsymbol{p = 5}$ $\qquad\qquad$ $\boldsymbol{ii.}\quad \boldsymbol{p = 12}$

Interpret your results from the economical point of view.

c) **Determine the demand level at which total revenue is not affected by price increase or decrease.**

Solution:

a) $\textbf{Since,}\quad \textbf{The elasticity of demand } (\boldsymbol{E}) = \dfrac{\boldsymbol{p}}{\boldsymbol{q}}\left(\dfrac{\boldsymbol{dq}}{\boldsymbol{dp}}\right)$

and the demand function is:

$$\boldsymbol{q = 1000 - 50p}$$

$\boldsymbol{Then,}$

$$\frac{\boldsymbol{dq}}{\boldsymbol{dp}} = \boldsymbol{-50}$$

$\textbf{And, therefore,}\qquad \textbf{The elasticity of demand } (\boldsymbol{E}) = \dfrac{\boldsymbol{p}}{\boldsymbol{q}}(\boldsymbol{-50})$

$\boldsymbol{i.}\quad \boldsymbol{at}\quad \boldsymbol{p = 5:}$

$$\boldsymbol{q = 1000 - 50(5) = 750}$$

Therefore,

$$\textbf{The elasticity of demand } (\boldsymbol{E}) = \frac{\boldsymbol{5}}{\boldsymbol{750}}[\boldsymbol{-50}] = -\frac{\boldsymbol{1}}{\boldsymbol{3}}$$

The explanation for this is that when the price is 5 pounds per unit, then an increase in the price by 1% will lead to decrease in the demanded quantity by 0.33%. Therefore, it is expected that the percentage change in demand is one third of the percentage change in price. This is because the demand is inelastic $\left(|E| = \frac{1}{3} < 1\right)$ and as a result for this, the increase in price leads to an increase in the total revenue.

$\boldsymbol{i.}$ $\quad \boldsymbol{at} \quad \boldsymbol{p = 12}$**:**

$$q = 1000 - 50(12) = 400$$

And, therefore,

$$\text{The elasticity of demand } (E) = \frac{12}{400}[-50] = -1.5$$

Now, since $|E| = 1.5 > 1$, then the demand at this price ($p = 12$) is elastic. This means that the percentage increase in the price leads to decrease in demand by a greater percentage, and this leads to decrease in the total revenue by increasing the price. In fact, the increase in price by 1% leads to a decrease in demand by 1.5%.

b) In order to make the total revenue not to be affected by any increasing or decreasing in the price at a particular

demand level, then the elasticity of demand at this level must be equal to (-1):

($i.e.,\ |E| = 1$).

$$\boldsymbol{Now, since,} \qquad \textbf{The elasticity of demand } (E) = \frac{p}{q}\left(\frac{dq}{dp}\right)$$

Then, by substituting (*p*) in terms of (*q*) in elasticity of demand, as follows:

$$\boldsymbol{Since,} \qquad q = 1000 - 50p$$

$$\boldsymbol{Then,} \qquad p = \frac{1000 - q}{50} = -50$$

$$\textbf{Hence, the elasticity of demand } (E) = \left(\frac{\frac{1000 - q}{50}}{q}\right)(-50)$$

$$= \frac{q - 1000}{q}$$

Then, by substituting the elasticity of demand with the value (-1), we get:

$$\frac{q - 1000}{q} = -1$$

$$q - 1000 = -q$$

$$2q = 1000$$

$$\therefore \boldsymbol{q = 500}$$

Therefore, at the demand level: $q = 500$ units, then the total revenue is not affected by price change, as the demand has unit elasticity.

Exercises For Chapter (5)

1- Find the 1st derivative $f'(x)$ for each of the following functions:

$a.\quad f(x) = \frac{x^4}{2} - 3x^2 + 10$

$b.\quad f(x) = (10 - x)(x^2 + 2)$

$c.\quad f(x) = \frac{2x + 3}{3x^2 - 2x}$

$d.\quad f(x) = (x + \frac{3}{x})(x^2 - 5)$

$e.\quad f(x) = \sqrt{x^2 + 1}$

$f.\quad f(x) = \ln(x^2 - 2x + 5)$

$g.\quad f(x) = e^{3x^2 - x}$

$h.\quad f(x) = \frac{x^4}{4} - \frac{x^3}{3}$

$i.\quad f(x) = (e^x \ln x)^4$

2- Find $\frac{dy}{dx}$ if:

$a.\quad y = u^2 + 2u - 1 \quad , \quad u = x^2 + x$

$b.\quad y = u^3 - 4 \quad , \quad u = x^3 + 3$

$c.\quad y = (2u + 3)^2 \quad , \quad u = x^2 - 3x$

3- Find all the possible derivative for each of the following functions:

$$a.\quad f(x) = \frac{5x^4}{4} - \frac{3x^3}{3} + 6x^2 + 10x$$

$$b.\quad f(x) = 2x^3 - 8x^2 + 5x - 1$$

$$c.\quad f(x) = 3x^2 + 7x + 10$$

$$d.\quad f(x) = 2x^4 - 3x$$

4- If the factory's owner finds that the total cost of produced (*q*) unit of a specific product is determined by the following function:

$$C(q) = 0.001q^3 + 0.3q^2 + 40q + 1000$$

Required:

a) **What is the increase in total cost if the production increases from 50 units to 60 units?**
b) **Calculate the average cost for each additional unit of the production in increasing the production from 50 units to 60 units.**

5- If the total cost of producing (*q*) unit of a product is given by the function:

$$C(q) = 3000 + 10q + 0.1q^2$$

Required:

a) Find the increase in cost if the production increases from 100 units to 200 units.

b) Calculate the average cost of each additional unit of the production in increasing the production from 100 units to 200 units.

c) Determine the marginal cost as a function in the number of units produced.

d) Find the marginal cost at the following production levels:

$i.\quad q = 300. \qquad\qquad ii.\quad q = 400.$

Then, interpret your results economically in each case.

e) Find the cost of producing:

$i.\quad 500\, units. \qquad\qquad ii.\quad the\ 20^{th}\ unit.$

6- If the demand function of a good is given by:

$$q^2 + 400p = 10000$$

where (*q*) is the demand quantity (or sales), and (*p*) is the unit price (in pounds).

<u>Required:</u>

a) Determine the marginal revenue as a function in the level of produced quantity.

b) **Find the marginal revenue at the following production levels:**

$i.\quad q = 20\ units.$ $\qquad ii.\quad q = 100\ \ units.$

c) **If you know that the total cost function takes the form:**

$$C(q) = 1000 + 10q$$

Determine the marginal profit as a function of the sales quantity. Then calculate the marginal profit at the production level $q = 50\ \ units$**.**

7- If the demand function of a product is:

$$2q + 3\ln(p+1) = 60$$

Where (*p*) is the unit price, and (*q*) is the demanded quantity.

***<u>Required</u>*:**

a) **Determine the marginal demand as a function in the unit price.**

b) **Find the marginal demand at the price level** $p = 2$ **pounds. Then interpret your result economically.**

8- If the demand function of a commodity is:

$$p = \frac{300}{q^2 + 1}$$

Where (*p*) is the unit price, and (*q*) is the demanded quantity.

***<u>Required</u>*:**

a) **Determine the marginal price as a function in the demand quantity.**

b) **Find the marginal price at the demand level $q = 3$ units.**

9- A product is manufactured and sold with a profit equal to 10 pounds per unit. If the owner of the factory spent *p* pounds on advertising his product, then the number of units which he can sell equals $[1000(1 - e^{-ap})]$, where $a = 0.001$, If *(P)* denotes the net profit for the sale process.

***<u>Required</u>*:**

a) **Find $\frac{dP}{dp}$. Interpret the mean of this derivative from the economical point of view.**

b) **Calculate the value of $\frac{dP}{dp}$ at:**

$i.\ \ p = 1000.$ $ii.\ \ p = 3000.$

Comment on your results economically in each case.

Chapter (6)
Functions Description

Introduction

Differential calculus presents the behavior of mathematical functions. It is particularly useful in estimating the graphical representation of a function in two dimensions. In this chapter, the tools, which are developed in chapter five, will be extended. We will further our understanding of the 1^{st} and 2^{nd} derivatives. We will see how these derivatives can be useful in describing the behavior of mathematical functions.

The main objective of this chapter is to develop a method for determining the maxima and minima of the functions (determining where a function achieves maximum or minimum values) and how can we use it in many commercial applications.

Now, we will present new dimensions for the use of the 1^{st} and 2^{nd} derivatives in describing functions which enables us to use these derivatives in economic analysis operations.

(6-1): Increasing and Decreasing Functions:

In this section, we will continue to expand our understanding of the 1st derivative. As mentioned in the previous chapter, the 1st derivative represents the rate of change in $f(x)$ with respect to the change in (x).

(6-1-1): Increasing Functions:

The function $f(x)$ is said to be an *increasing function* on an interval *I* if $f(x_1) < f(x_2)$ for any values (x) and (x_2) within the interval, where $x_1 < x_2$.

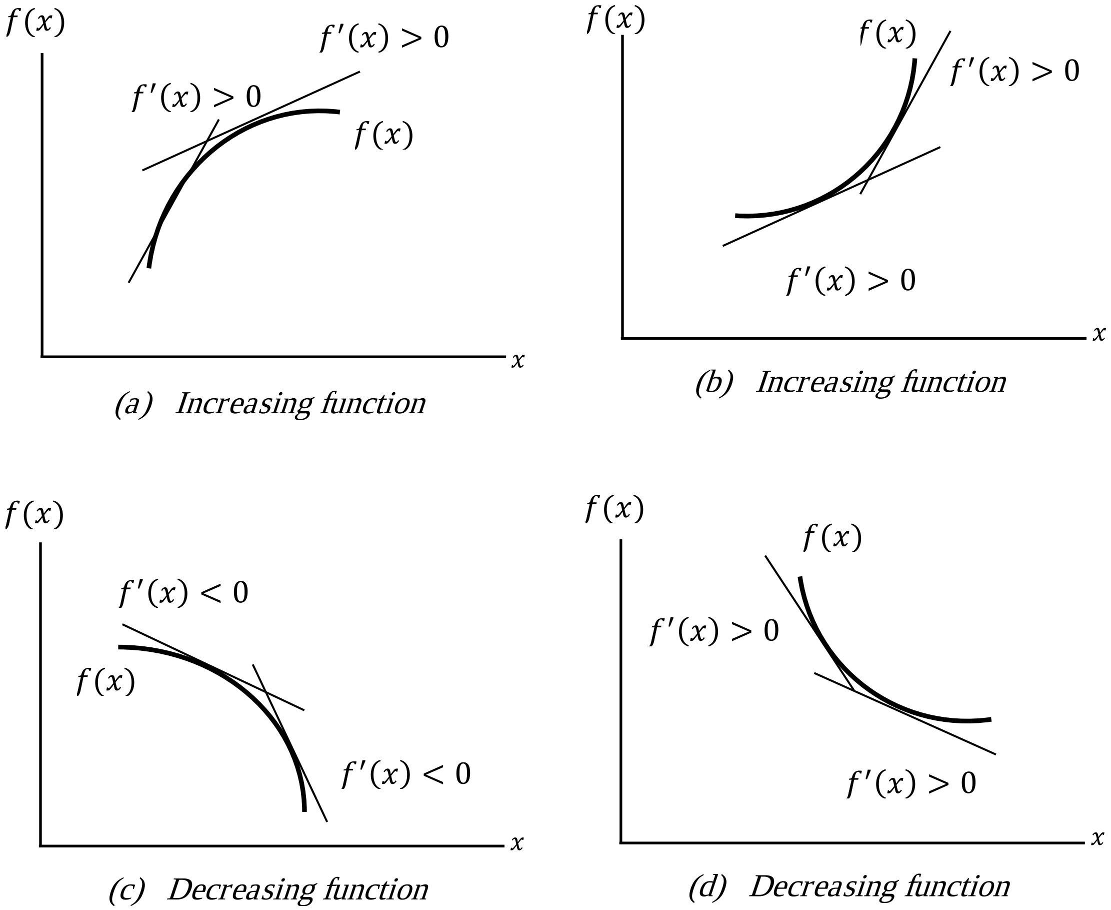
Increasing functions can also be identified by the slope conditions (the 1st derivative). *If the first derivative of $f(x)$ is positive throughout an interval, then the slope is positive and $f(x)$ is an increasing function on this interval*. Therefore, at any point within the interval, a slight increase in the value of x will be accompanied by an increase in the value of $f(x)$.

(6-1-2): Decreasing Functions

The function $f(x)$ is said to be a *decreasing function* on an interval *I* if $f(x_1) > f(x_2)$ for any values of (x_1) and (x_2) within the interval, where$(x_1 < x_2)$.

Also, decreasing functions can also be identified by the slope conditions (the 1st derivative). ***If the 1st derivative of $f(x)$ is negative throughout an interval, then the slope is negative and $f(x)$ is a decreasing function on this interval*****. Therefore, at any point within the interval, a slight increase in the value of (x) will be accompanied by a decrease in the value of $f(x)$.**

The curves in Fig.(6-1*a*) and (6-1*b*) are the graphs of increasing functions of x because the tangent slope at any point is positive $(f'(x) > 0)$. But the curves in Fig. (6-1*c*) and (6-1*d*) are the graphs of decreasing functions of (x) because the tangent slope at any point is negative$(f'(x) < 0)$.



decreasing. /**Figure (6-1):** The relationship between $f'(x)$ and increasing

From the previous figure, it is clear that the function is increasing if $f'(x) > 0$, while the function is decreasing if $f'(x) < 0$.

This can be summarized as follows:

- **If the function is increasing (or decreasing) on an interval, then it is increasing (or decreasing) at every point within this interval.**

- **The increase or decrease of a specific function can easily be determined by using the 1st derivative (slope) as follows:**

$$f'(x)\begin{cases} f'(x) > 0 & \text{the function is increasing} \\ f'(x) < 0 & \text{the function is decreasing} \\ f'(x) = 0 & \text{It is neither increasing nor decreasing} \end{cases}$$

Example (6-1):

If we have the following function:

$$f(x) = x^2 - 3x + 20$$

Required:

a) **Determine whether the function is increasing or decreasing at $x = 1$.**

b) **Determine the values of x over which $f(x)$ can be described as:**

   i. an increasing function.

   ii. an decreasing function.

   iii. neither increasing nor decreasing.

Solution:

a) **To determine whether $f(x)$ is increasing or decreasing at $x = 1$, firstly, we have to find $f'(x)$:**

Since, $f(x) = x^2 - 3x + 20$

Then: $f'(x) = 2x - 3$

at $x = 1$ $f'(1) = 2(1) - 3 = -1$

Since$f'(1) < 0$, then we can say that this function is decreasing at $x = 1$.

b) Since, we have $f'(x) = 2x - 3$, then:

i. $f(x)$ Will be increasing function when $f'(x) > 0$, *i.e.,* or When:

$$2x - 3 > 0$$

$$2x > 3$$

$$i.e., \quad x > 1.5$$

ii. $f(x)$ Will be decreasing function when$f'(x) < 0$, *i.e.,* When:

$$2x - 3 < 0$$

$$2x < 3$$

$$i.e., \quad x < 1.5$$

iii. $f(x)$ Will be neither increasing nor decreasing function when$f'(x) = 0$, *i.e.,* when:

$$2x - 3 = 0$$

$$2x = 3$$

$$i.e., \quad x = 1.5$$

Then we conclude that, $f(x)$ is a decreasing function when $x < 1.5$, neither increasing nor decreasing at $x = 1.5$, and an increasing function when $x > 1.5$.

When the function is neither increasing nor decreasing at a specific point, then it has another characteristic at this point, which will be discussed later in this chapter.

<u>Example (6-2):</u>

If we have the following function:

$$f(x) = x^3 - 3x$$

<u>Required:</u>

a) **Determine whether the function is increasing or decreasing at $x = 2$.**
b) **Find the values of x for which $f(x)$ is increasing or decreasing.**

<u>Solution:</u>

a) Since, $f(x) = x^3 - 3x$

Then, $f'(x) = 3x^2 - 3$ **, and then:**

at $x = 2$ $\qquad f'(2) = 3(2)^2 - 3 = 9$

Since $f'(2) > 0$, then we can say that this function is increasing at $x = 2$.

b) Since, we have $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

$$= 3(x-1)(x+1)$$

To find the interval in which $f(x)$ is increasing, we set $f'(x) > 0$ as follows:

$$3(x-1)(x+1) > 0$$

$$i.e., \qquad (x-1)(x+1) > 0$$

To determine the sign of $f'(x)$, we must specify the sign of $(x-1)$ *and* $(x+1)$ as follows:

$(x-1)$ **is positive *when* $x > 1$, and is negative *when* $x < 1$.**

$(x+1)$ **is positive *when* $x > -1$,and is negative *when* $x < -1$.**

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$(x-1)$	$-\ -\ -$	$0 \quad -$	$- \quad 0$	$+\ +\ +$	
$(x+1)$	$-\ -\ -$	$0 \quad +$	$+ \quad 0$	$+\ +\ +$	
	$f'(x) > 0$	$f'(x) = 0$	$f'(x) < 0$	$f'(x) = 0$	$f'(x) > 0$

We see that $f'(x) > 0$ when $x < -1$ and when$x > 1$, so $f(x)$ is an increasing function of x in each of those intervals. And when$-1 < x < 1$, then$f'(x) < 0$, so $f(x)$ is a decreasing function in this interval. The graph of $f(x)$ is shown in Figure (6-2).

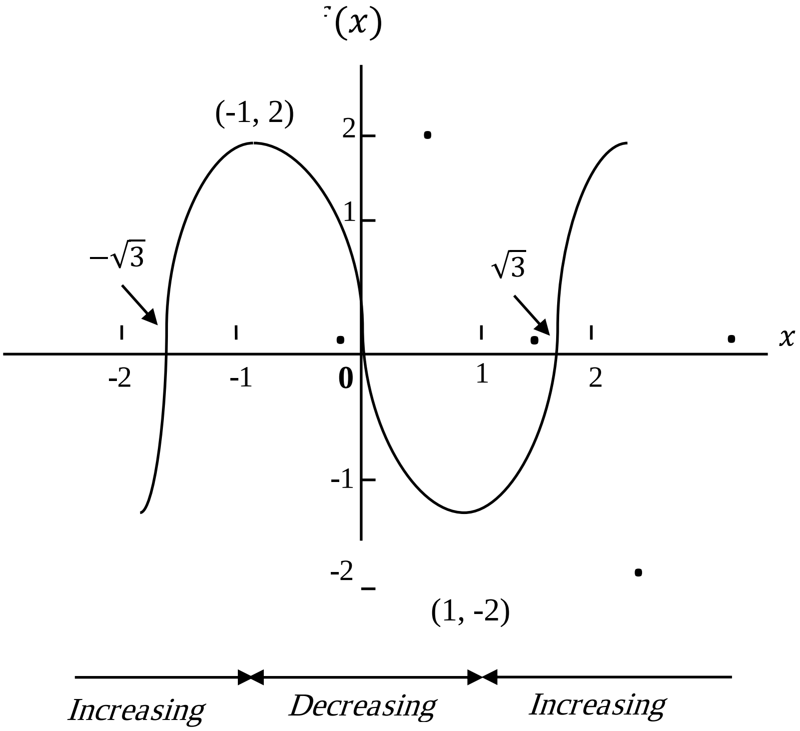


Figure (6-2): Increasing and Decreasing function: $f(x) = x^3 - 3x$

(6-2): Concavity and Inflection points:

This section discussed the concept of the concavity of a function graph and its relationship with the 2nd derivative. Locating inflection point will also be introduced.

(6-2-1): Concavity:

If the graph of the function is open up, and then this graph is *concave up*, but if the graph of the function is opens down, then this graph is *concave down*.

In other word, the graph of a function $f(x)$ is *concave up* on an interval if the 1st derivative (the slope), $f'(x)$, increases over the entire interval. In this case, for any point within such an interval the curve representing $f(x)$ will lie above the tangent line drawn at the point. But if the first derivative (the slope), $f'(x)$, decreases on a specific interval, then the graph of the function $f(x)$ is *concave down* on this interval. In this case, for any point within such an interval the curve representing $f(x)$ will lie below the tangent line drawn at the point.

In Figure (6-3), the graph of $f(x)$ is concave down between *A* and *B*, and it is concave up between *B* and *C.* Note that between *A* and *B* the curve lies below its tangent lines, and between *B* and *C* the curve lies above its tangent lines. Point *B* is where the concavity changes from concave down to concave up. The point by which the concavity is changed called an *inflection point*. Thus, point *B* is an inflection point.

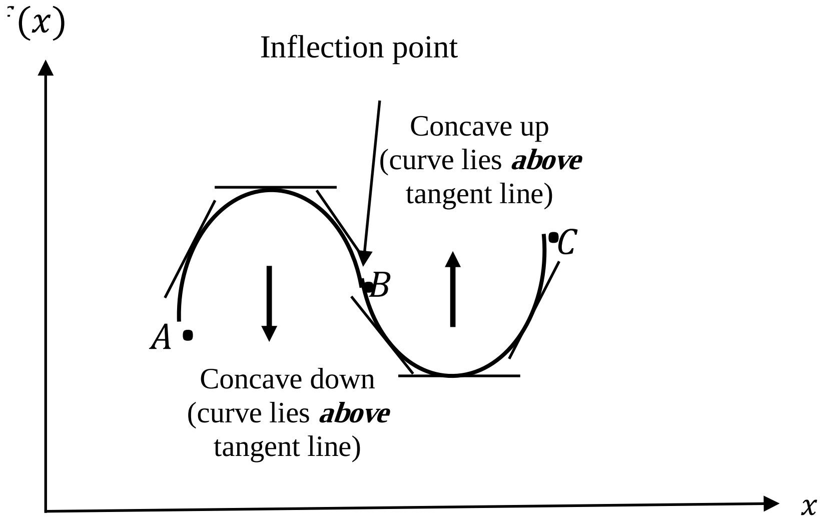


Figure (6-3): Representation of concavity conditions

(6-2-2): Relationship between The 2nd Derivative and the Concavity:

Since the graph of the function is concave up in the interval by which the first derivative of the function (slope) is increases, *i.e.,* when the first derivative is an increasing function. And since the 2nd derivative is in the rule of the derivative of the 1st derivative, then the first derivative $f'(x)$ is an increasing function if $f''(x) > 0$.

This means that the graph of the function is concave up if $f''(x) > 0$. On the contrary, by the same logic, it can be concluded that the graph of the function is concave down if $f''(x) < 0$. Therefore, the relationship between the 2nd derivative and the concavity can be summarized as follows:

i. If $f''(x) > 0$ on the interval$a \le x \ge b$, then the graph of $f(x)$ is concave up over this interval. For any point $x = c$ within this interval, then $f(x)$ is said to be concave up at the point $[c, f(c)]$.

ii. If $f''(x) < 0$ on the interval$a \le x \ge b$, then the graph of $f(x)$ is concave down over this interval. For any point $x = c$ within this interval, then $f(x)$ is said to be concave down at the point $[c, f(c)]$.

iii. If $f''(x) = 0$ at any point $x = c$ in the domain of $f(x)$, no conclusion can be drawn about the concavity at the point $[c, f(c)]$.

We must be careful not to reverse the logic of these relationships. Where we cannot make statement about the sign of the 2nd derivative knowing the concavity of the graph of the function because the relationship (*iii*). For example, if the graph of the function is concave up at $x = a$, then we cannot state that $f''(x) > 0$. In other word, it is possible to find that the graph of the function is concave up, however we find that $f''(x) = 0$ (it is not greater than zero).

Example (6-3):

If we have the following functions:

1) $f(x) = 5x^3 - 4x^2 + 10x$

$\mathbf{2)}\quad \boldsymbol{f(x) = x^2 - 4x + 9}$

Required:

Determine the nature of concavity for each function at:

$\boldsymbol{i.\ \ x = -2}$, $\boldsymbol{ii.\ \ x = 1}$

Solution:

1) Since, $\boldsymbol{f(x) = 5x^3 - 4x^2 + 10x}$

Then, $\boldsymbol{f'(x) = 15x^2 - 8x + 10}$

Therefore, $\boldsymbol{f''(x) = 30x - 8}$

At $\boldsymbol{x = -2}$

$$\boldsymbol{f''(-2) = 30(-2) - 8 = -68}$$

$$\boldsymbol{i.e.,\qquad f''(-2) < 0}$$

Which means that the graph of the function is concave down at $\boldsymbol{x = -2}$**.**

Also, if $\boldsymbol{x = 1}$ **, then:**

$$\boldsymbol{f''(1) = 30(1) - 8 = 22}$$

$$\boldsymbol{i.e.,\qquad f''(1) > 0}$$

Therefore, the graph of the function is concave up at $\boldsymbol{x = 1}$

2) Since, $\boldsymbol{f(x) = x^2 - 4x + 9}$

Then, $f'(x) = 2x - 4$

Therefore, $f''(x) = 2$

$i.e.,\ f''(x) > 0$, regardless of the value of x

Therefore, the graph of the function is concave up at all values of x.

***Note that*: $f''(-2) = f''(1) = 2 > 0$**

Example (6-4):

Determine the concavity of the graph of the function:

$f(x) = x^4$ at $x = 0$.

Solution:

1) Since, $f(x) = x^4$

Then, $f'(x) = 4x^3$

And, $f''(x) = 12x^2$

At $x = 0$

$$f''(0) = 12(0)^2 = 0$$

According to relationship (*iii*), we cannot make statement about the concavity at $x = 0$. However, by substituting a sufficient number of values for x into $f(x)$ and plotting this function, we see that $f(x)$ has the shape shown In Figure

(6-4). From this Figure, we find that the graph is concave up at $x = 0$.

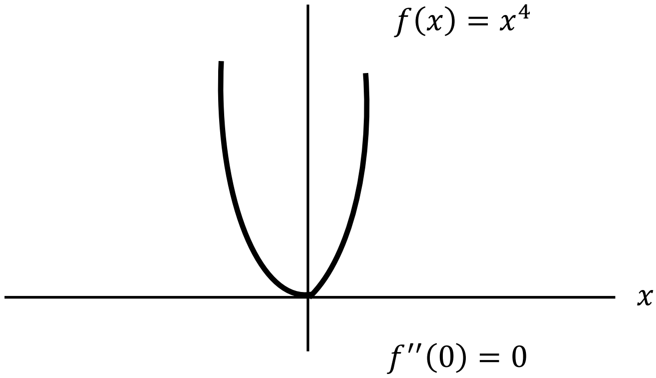


Figure (6-4): No conclusions regarding concavity

Example (6-5):

If we have the following functions:

1) $f(x) = 5x^6 - 6x^5 + 1$

2) $f(x) = x^2 - 6x + 7$

Required:

Find the values of x at which the graph of each function be:

$i.$ *concave up* , $ii.$ *concave down*

Solution:

1) Since, $f(x) = 5x^6 - 6x^5 + 1$

Then, $f'(x) = 30x^5 - 30x^4$

And, $f''(x) = 150x^4 - 120x^3$

i. **In order to the graph of the function be concave up, it must verify that:** $f''(x) > 0$**.**

i.e., $150x^4 - 120x^3 > 0$

$$5x^4 - 4x^3 > 0$$

$$x^3\,(5x - 4) > 0$$

The solution of this inequality is:

Either:

$$x^3 > 0, \qquad i.e., \qquad x > 0$$

$$\text{and}: 5x - 4 > 0, \qquad i.e., \qquad x > \frac{4}{5}$$

Therefore, the values of x **that satisfy these two conditions are:**

$$x > \frac{4}{5}$$

Or:

$$x^3 < 0, \qquad i.e., \qquad x < 0$$

$$\text{and}: 5x - 4 < 0, \qquad i.e., \qquad x < \frac{4}{5}$$

Therefore, the values of x **that satisfy these two conditions are:**

$$x < 0$$

Henceforth, the solution of the inequality is represented in the two intervals:

$$x < 0 \qquad and \qquad x > \frac{4}{5}$$

Therefore, we find that the graph of the function is concave up at $x < 0\ \ or\ x > \frac{4}{5}$.

ii. **In order to the graph of the function be concave down, it must verify that: $f''(x) < 0$.**

i.e., $$150x^4 - 120x^3 < 0$$

$$5x^4 - 4x^3 < 0$$

$$x^3\ (5x - 4) < 0$$

And then, the solution of this inequality is:

<u>Either:</u>

$$x^3 < 0, \qquad i.e., \qquad x < 0$$

$$\text{and} : 5x - 4 > 0, \qquad i.e., \qquad x > \frac{4}{5}$$

Note that: there is no value of x that satisfy these two conditions.

<u>Or:</u>

$$x^3 > 0, \qquad i.e., \qquad x > 0$$

$$\text{and} : 5x - 4 < 0, \qquad i.e., \qquad x < \frac{4}{5}$$

Therefore, the values of x that satisfy these two conditions are in the interval: $0 < x < \frac{4}{5}$.

Henceforth, the solution of the inequality is represented in the interval:

$$0 < x < \frac{4}{5}$$

Therefore, we find that the graph of the function is concave down in the interval: $0 < x < \frac{4}{5}$.

2) Since, $f(x) = x^2 - 6x + 7$

Then, $f'(x) = 2x - 6$

And, $f''(x) = 2 > 0$

***i.e.,* the graph of the function is concave up at all values of x. And it is never concave down.**

(6-2-3): Locating Inflection Points

We mentioned before that the *inflection point* is the point by which the concavity changed. It can be determined as follows:

- ***i.*** **Finding all points *a*, where$f''(a) = 0$.**
- ***ii.*** **If the sign of $f''(x)$ is changed when passing through$x = a$, then there is an inflection point at $x = a$.**

A necessary condition for the existence of an inflection point at $x = a$ is that $f''(a) = 0$. Therefore, by finding all values of x by which $f''(x) = 0$, the candidate locations for inflection points are identified. The condition $f''(a) = 0$ does not ensure that there is an inflection point at $x = a$ [see example (6-4)]. Step *ii* confirms whether the candidate location is an inflection point.

The existence of an inflection point at $x = a$ can be tested by choosing two points, one of them is directly less than the value (a) and the other directly greater than the value (a). In other words, the test is to choose points slightly to the left and right of $x = a$ and determine if the concavity is different on each side. If the value of $f''(a)$ is positive in the left and negative in the right, or vice versa, then there has been a change in concavity when passing through $x = a$. Therefore, an inflection point exists at $x = a$.

<u>Example (6-6):</u>

Determine the inflection points, if there are exist, for the following two functions:

1) $f(x) = x^3 + 6x^2 - 18$

2) $f(x) = -10x^4 + 100$

Solution:

1) Since, $f(x) = x^3 + 6x^2 - 18$

Then: $f'(x) = 3x^2 + 12x$,

And: $f''(x) = 6x + 12$

By setting $f''(x)$ equal to zero in order to find the candidate locations for inflection points, then we have the following:

$$6x + 12 = 0$$

$\therefore x = -2$ ***is considered a possible inflection point***

To verify that the point $x = -2$ is an inflection point, we find the value of $f''(x)$ at the two points $x = -3$ which is located before at $x = -2$, and at the point $x = -1$ which is located after $x = -2$ as follows:

$f''(-3) = 6(-3) + 12 = -6$ **(Concave down)**

$f''(-1) = 6(-1) + 12 = 6$ **(Concave up)**

Since the sign of $f''(x)$ changed, i.e., the concavity changed at $x = -2$, then this indicates an inflection point at $x = -2$.

Verification

$x = -3$	$\boxed{x = -2}$	$x = -1$
$f''(-3) = -6$		$f''(-1) = 6$
Concave down		*Concave up*
	$f''(-2) = 0$	
	Inflection point	

To find the other coordinate of the inflection point, we substitute the value of $\boldsymbol{x}$ by $(-\mathbf{2})$ into the original function as follows:

$$\boldsymbol{f(-2) = (-2)^3 + 6(-2)^2 - 18}$$

$$\boldsymbol{= -8 + 24 - 18 = -2}$$

***i.e.,* the inflection point is $(-\mathbf{2}, -\mathbf{2})$.**

2) Since, $\boldsymbol{f(x) = -10x^4 + 100}$

Then: $\boldsymbol{f'(x) = -40x^3}$,

And: $\boldsymbol{f''(x) = -120x^2}$

By setting $\boldsymbol{f''(x)}$ equal to zero in order to find the candidate locations for inflection points, then we have the following:

$$\boldsymbol{-120x^2 = 0}$$

$\therefore x = 0$ ***is considered a possible inflection point***

To verify that the point $x = 0$ is an inflection point, we find the value of $f''(x)$ at the point $x = -1$ which is located before at $x = 0$, and at the point $x = 1$ which is located after $x = 0$ as follows:

$f''(-1) = -120(-1)^2 = -120$ **(Concave down)**
$f''(1) = -120(1)^2 = -120$ **(Concave down)**

Since the sign of $f''(x)$ does not change, i.e., the concavity does not change at $x = 0$, therefore, the point $x = 0$ is not an inflection point. i.e., there is no inflection point for this function.

<u>(6-3): Maxima and Minima</u>

Many of the important applications of derivatives involve finding the maximum or minimum values for any specific function.

<u>For example</u>:

- **The profit, which a manufacturer takes on, depends on the price charged for the product, and the manufacturer is interested in knowing the price which makes his profit maximum. The optimum price (or best price) is**

determined by such process which is called *the maximization* or *optimization* of the profit function.

- **A railway company wants to know the average speed at which trains should run in order to *minimize* the cost per mile of operation.**
- **The economist may wish to know the level of tax in a country that will promote the *maximum* rate of growth of the economy.**

Before we look at such as the preceding applications, we will discuss the concept of the theory of maxima and minima, and how to find the maximum and minimum values for the functions.

(6-3-1): Relative (Local) Extreme Points :

We will define the relative (local) maximum and the relative (local) minimum.

- ***Relative Maximum:***

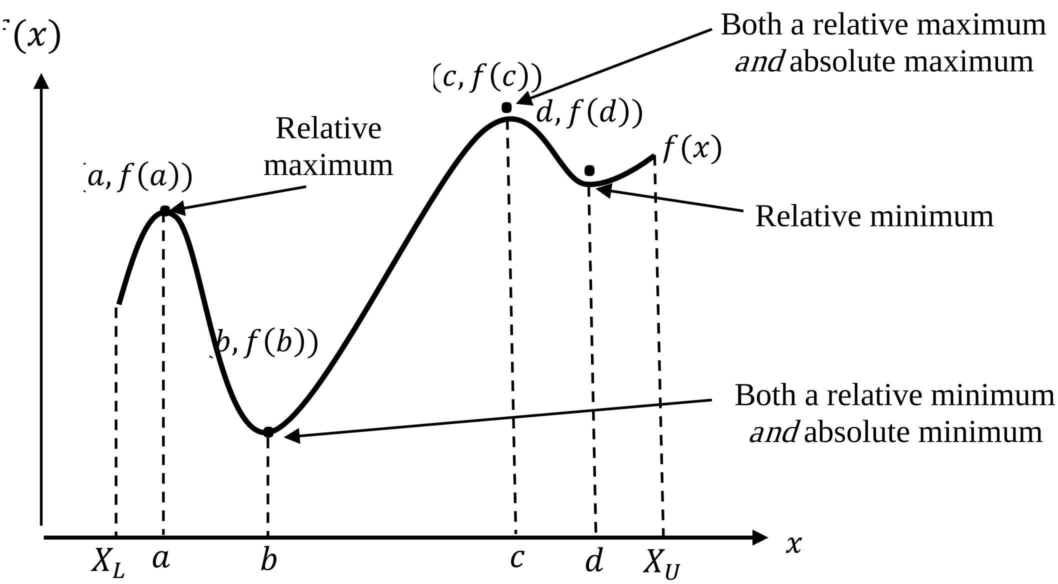
If $f(x)$ is defined on the interval $a \le x \le b$ which contains $x = c$, then $f(x)$ is said to reach a relative (local) maximum at $x = c$ if $f(c) > f(x)$ for all values of x which are closed enough to c.

- ***Relative Minimum:***

If $f(x)$ is defined on the interval $a \leq x \leq b$ which contains $x = c$, then $f(x)$ is said to reach a relative (local) minimum at $x = c$ if $f(c) < f(x)$ for all values of x which are closed enough to c.

Both the two definitions focus on the value of $f(x)$ within an interval. The relative maximum refers to the point at which the value of $f(x)$ is greater than the values for any points which are closed enough. Also, the relative minimum refers to a point where the value of $f(x)$ is lower than the values for any points which are closed enough. A one function may have more than one relative maximum or more than one relative minimum, as shown in the figure (6-5).

Figure (6-5): Relative extreme Points.



In figure (6-5), we find that:

- **$f(x)$ has a *relative maximum* at $x = a \;\; and \; x = c$.**
- **$f(x)$ has a *relative minimum* at $x = b \;\; and \; x = d$.**

The relative maximum or minimum are called *relative extrema*.

(6-3-2): Critical Points

We will have a particular interest in the relative maximum and minimum. It will be important to know how to identify and differs between them.

The value $x = c$ is called a *critical point* for a continuous functions $f(x)$ if:

i. **$f'(c) = 0$, or**

ii. **$f'(c)$ is undefined ($f'(x)$ fails to exist at $x = c$).**

In other words, given that the function$f(x)$, then the necessary conditions for the existence of a relative maximum or minimum that at $x = c$ are: $f'(c) = 0$, or $f'(c)$ is undefined.

The points which are satisfying either of these conditions in this definition are candidate for relative maximum or minimum. Such points are referred to as *critical points*.

The critical points are denoted with (x^*) in order to distinguish them from other values of x. Given a critical

value for$f(x)$, then the corresponding critical point is $[x^*, f(x^*)]$.

Graphically, the points which satisfy condition 1 are on the graph of $f(x)$ where the slope is equal to zero. *i.e.,* in order to determine the critical points, we should set $f'(x)$ equal to zero. The points satisfying condition 2 are the points at which the function is not continuous, or points where $f'(x)$ cannot be evaluated.

We emphasize the fact that for $x = c$ to be a critical point, then $f(x)$ must be well-defined. Consider, for example, $f(x) = x^{-1}$, whose 1st derivative is $f'(x) = -x^{-2}$. Clearly, $f'(x)$ becomes unbounded when$x \to 0$. However, $x = 0$ is not a critical point for this function since $f(0)$ does not exist?

Shortly, we will develop specific tests which will enable us to distinguish those critical points which are relative extreme from those which are not. Firstly, let us examine the critical points through some examples.

<u>Example (6-7):</u>

Determine the critical points for the following functions:

1) $f(x) = 2x^2 + 8x + 1$

2) $f(x) = x^3 - 9x^2 + 24x + 2$

3) $f(x) = x^2 \ln x$

Solution:

1) Since, $f(x) = 2x^2 + 8x + 1$

Then: $f'(x) = 4x + 8$

Substituting $f'(x)$ **equal to zero, then we get the following:**

$$4x + 8 = 0$$

$$x = -2$$

And then:

$$f(-2) = 2(-2)^2 + 8(-2) + 1$$

$$= 8 - 16 + 1 = -7$$

Therefore, the critical point is $(-2, -7)$**.**

2) Since, $f(x) = x^3 - 9x^2 + 24x + 2$

Then: $f'(x) = 3x^2 - 18x + 24$

Substituting $f'(x)$ **equal to zero, then we get the following:**

$$3x^2 - 18x + 24 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$i.e., \quad x = 2 \quad and \quad x = 4$$

And then:

$$f(2) = (2)^3 - 9(2)^2 + 24(2) + 2$$

$$= 8 - 36 + 48 + 2 = 22$$

$$f(4) = (4)^3 - 9(4)^2 + 24(4) + 2$$

$$= 64 - 144 + 96 + 2 = 18$$

Therefore, the critical points are $(2, 22)$ and $(4, 18)$.

3) Since, $f(x) = x^2 \ln x$

Then: $f'(x) = x^2\left(\frac{1}{x}\right) + \ln x \ (2x)$

$$= x + 2x\ln x$$

$$= x(1 + 2\ln x)$$

Substituting $f'(x)$ equal to zero, then we get the following:

$$x(1 + 2\ln x) = 0$$

$$i.e., \quad x = 0$$

$$or \quad 1 + 2\ln x = 0$$

$$2\ln x = -1$$

$$\ln x = \frac{-1}{2}$$

$$\therefore e^{\ln x} = e^{\frac{-1}{2}}$$

$$i.e., \quad x = e^{\frac{-1}{2}}$$

$\boldsymbol{Therefore,} \quad \boldsymbol{x = 0} \quad \boldsymbol{and} \quad \boldsymbol{x = e^{\frac{-1}{2}}}$

And then:

$$\boldsymbol{f(0) = (0)^2 \ln(0)}$$

Since, there is no logarithm to zero, then $\boldsymbol{f(0)}$ is undefined at $\boldsymbol{x = 0}$.

Therefore, at $\boldsymbol{x = 0}$ it is not a critical point (see the 2nd condition of the critical point).

Then:

$$\boldsymbol{f\left(e^{\frac{-1}{2}}\right) = (e^{\frac{-1}{2}})^2 \ln(e^{\frac{-1}{2}}) \qquad , (note\ that: e = 2.71828)}$$

$$\boldsymbol{= e^{-1} \ln(e^{\frac{-1}{2}}) = e^{-1}(\frac{-1}{2})}$$

$$\boldsymbol{= e^{-1} \ln(e^{\frac{-1}{2}}) = (\frac{-1}{2e})}$$

$\mathbf{Therefore, the\ critical\ point\ is}$ $\boldsymbol{\left(e^{\frac{-1}{2}}, \frac{-1}{2e}\right)}$.

In general, any critical point by which $\boldsymbol{f'(x) = 0}$ will be a *relative maximum*, or a *relative minimum*, or an *inflection point*. Now, we will develop specific tests which will enable us to distinguish those critical points which are relative extreme from those which are not.

(6-3-3): Tests for Relative Extrema:

In order to locate on the relative maximum or relative minimum points, the 1st step is to determine all critical points on the graph of the function. Given that a critical point may be either a relative maximum or minimum or an inflection point.

(6-3-3-1): The Second Derivative Test:

Consider the case when the relative extreme occurs at a critical point given by $f'(x) = 0$, *i.e.,* when the tangent line is horizontal at the point, on the graph of $f(x)$, representing to the relative maximum or minimum. Therefore, if the point is a relative maximum, then the graph is concave down, and if the point is a relative minimum, then the graph is concave up. But we know that whenever$f''(x) < 0$, then the graph of $f(x)$ is concave down, and whenever$f''(x) > 0$, then the graph of $f(x)$ is concave up. This leads to the following theorem:

Theorem (1): " *2nd Derivative Test* "

Let $f(x)$ be twice differentiable at the critical point $x = c$, then:

i. **$x = c$ is a *relative maximum* of $f(x)$ whenever:**

$$f'(c) = 0 \qquad and \qquad f''(c) < 0 \text{ ;}$$

ii. **$x = c$ is a *relative minimum* of $f(x)$ whenever:**

$$f'(c) = 0 \qquad and \qquad f''(c) > 0$$

The steps for performing the 2nd derivative test can be summarized as follows:

Summary of Determining the Relative Extrema by using the 2nd Derivative Test:

Step 1: Find $f'(x)$, then solve the equation: $f'(x) = 0$ in order to determine the critical points (x^*).

Step 2: Find $f''(x)$, then substitute x for every critical value you obtained in step(1). Let $x = c$ is one of those values, then:

- If $f''(c) < 0$, then $f(x)$ has a ***relative maximum*** at $[(c, f(c)]$.
- If $f''(c) > 0$, then $f(x)$ has a ***relative minimum*** at $[(c, f(c)]$.
- If $f''(c) = 0$ or $f''(c)$ is undefined, then the test ***fails***. and we can restore to the first derivative test that we discussed later.

Example (6-8):

Determine the relative maximum and minimum for the following function:

$$f(x) = 2x^3 + 3x^2 - 12x - 15$$

By using the 2nd derivative test.

Solution:

Since: $f(x) = 2x^3 + 3x^2 - 12x - 15$

Then: $f'(x) = 6x^2 + 6x - 12$

And: $f''(x) = 12x + 6$

In order to find the critical points, we set $f'(x)$ equal to zero, then we get:

$$6x^2 + 6x - 12 = 0$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$\therefore \quad x = -2 \quad and \quad x = 1$$

Therefore, there are critical points at $x = -2$ and $x = 1$.

The nature of these critical points can be determined by using the second derivative as follows:

Since, $f''(x) = 12x + 6$

Then: at $x = -2$:

$$f''(-2) = 12(-2) + 6$$

$$= -18 < 0$$

Since $f''(-2) < 0$, then $f(x)$ has a relative maximum at $x = -2$. The value of this relative maximum can be determined by substituting $x = -2$ into the original function as follows:

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) - 15$$

$$= -16 + 12 + 24 - 15 = 5$$

Therefore, the point $(-2, 5)$ represents a relative maximum for the function, and at this point the graph of the function is concave down.

And, at $x = 1$:

$$f''(1) = 12(1) + 6$$

$$= 18 > 0$$

Now, since $f''(1) > 0$, then $f(x)$ has a relative minimum at $x = 1$. The value of this relative minimum can be determined by substituting $x = 1$ into the original function as follows:

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1) - 15$$

$$= 2 + 3 - 12 - 15 = -22$$

Therefore, the point $(1, -22)$ represents a relative minimum for the function, and at this point the graph of the function is concave up.

Example (6-9):

If we have the following function:

$$f(x) = \frac{x^3}{3} - 2.5x^2 + 4x$$

Identify the critical points for this function and determine the nature of each by using the 2nd derivative test.

Solution:

Since: $f(x) = \frac{x^3}{3} - 2.5x^2 + 4x$

Then: $f'(x) = x^2 - 5x + 4$

And: $f''(x) = 2x - 5$

Now, in order to identify the critical points, we set $f'(x)$ equal to zero, then we get:

$$x^2 - 5x + 4 = 0$$

$$(x - 4)(x - 1) = 0$$

$$\therefore \quad x = 4 \quad and \quad x = 1$$

Therefore, there are critical points at $x = 4$ and $x = 1$.

$$f(4) = \frac{(4)^3}{3} - 2.5(4)^2 + 4(4) = \frac{-8}{3} = -2.67$$

$$f(1) = \frac{(1)^3}{3} - 2.5(1)^2 + 4(1) = \frac{11}{9} = 1.83$$

Therefore, the critical points are:

$$(4, -2.67) \qquad and \qquad (1, 1.83)$$

The nature of this critical points can be determined by using the 2nd derivative as follows:

Since, $f''(x) = 2x - 5$

Then, at $x = 4$ **:**

$$f''(4) = 2(4) - 5$$

$$= 3 > 0$$

Now, Since $f''(4) > 0$**, then the point** $(4, -2.67)$ **is a relative minimum for** $f(x)$ **and the value of this relative minimum is** (-2.67)**, and at this point the graph of the function is concave up.**

And, at $x = 1$ **:**

$$f''(1) = 2(1) - 5$$

$$= -3 < 0$$

Since $f''(4) < 0$, then the point $(1, 1.83)$ is a relative maximum for $f(x)$ and the value of this relative maximum is (1.83), and at this point the graph of the function is concave down.

It should be noted that the 2nd derivative test can be used for all relative extrema at which $f'(c) = 0$ and $f''(c) \neq 0$. When $f''(x) = 0$ at a critical point $= c$, or when $f''(c)$ fails to exist, then the 2nd derivative test cannot be used to determine whether $x = c$ is a relative maximum or minimum. In such cases, we must use another test, which is the 1st derivative test.

<u>Theorem (2):</u> ***"1st Derivative Test "***

If $x = c$ is a critical point for the function$f(x)$, i.e., $f'(c) = 0$ or $f'(c)$ is not defined, then:

i. **$x = c$ is a *relative maximum* of $f(x)$, if $f'(x) > 0$ for x just before c (less than and it is the left of c), it is denoted by x_l, and $f'(x) < 0$ for x just after c (greater than and it is the right of c), it is denoted by x_r. In other**

word, $x = c$ is a *relative maximum* of $f(x)$, if $f'(x_l) > 0$ and $f'(x_r) < 0$, where: $x_l < c < x_r$.

ii. **$x = c$ is a *relative minimum* of $f(x)$, if $f'(x) < 0$ for x_l (less than and it is the left of c), and $f'(x) > 0$ for x_r (greater than and it is the right of c). *i.e.*, $x = c$ is a *relative minimum* of $f(x)$, if $f'(x_l) < 0$ and $f'(x_r) > 0$, where: $x_l < c < x_r$.**

iii. **$x = c$ is *not a relative extrema* of $f(x)$, if $f'(x)$ has the same sign for x_l and x_r. In such cases, $x = c$ is either an inflection point or an angle on the graph of the function.**

Another way of describing this test is the following:

- **For a *relative maximum*, the value of the function is increasing to the left and decreasing to the right.**
- **For a *relative minimum*, the value of the function is decreasing to the left and increasing to the right.**
- **For an *inflection points*, the value of the function is either increasing both to the left and right or decreasing both to the left and right.**

The steps for performing the 1^{st} derivative test can be summarized as follows:

Summary of Determining the Relative Extrema by the 1st Derivative Test:

Step 1: Find $f'(x)$, then solve the equation: $f'(x) = 0$ in order to determine the critical points (x^*).

Step 2: For any critical value (c), determine the value of $f'(x)$ to the left (x_l) and right (x_r) of (c) where $x_l < c < x_r$, then:

- If $f'(x_l) > 0$ and $f'(x_r) < 0$, *i.e.,* $f'(x)$ changed its sign from positive to negative, then $f(x)$ has a ***relative maximum*** at $[(c, f(c)]$.
- If $f'(x_l) < 0$ and $f'(x_r) > 0$, *i.e.,* $f'(x)$ changed its sign from negative to positive, then $f(x)$ has a ***relative minimum*** at $[(c, f(c)]$.
- If $f'(x)$ has the same sign at both x_l and x_r, then there is an ***inflection point*** for $f(x)$ at $[(c, f(c)]$.

Example (6-10):

Determine the relative maximum and minimum for the function in example (6-8) by using the first-derivative test.

Solution:

Since: $\boldsymbol{f(x) = 2x^3 + 3x^2 - 12x - 15}$

Then: $\boldsymbol{f'(x) = 6x^2 + 6x - 12}$

Substituting $\boldsymbol{f'(x)}$ equal to zero, then the critical points are:

$$x = -2 \qquad and \qquad x = 1$$

The nature of these critical points can be determined by using the 1st derivative as follows:

at $x = -2$ **:**

We will choose two values for x, a value before -2, say $(x_l = -3)$, and a value after -2, say $(x_r = -1)$, then we find the value of $f'(x)$ at each point as follows:

$$f'(-3) = 6(-3)^2 + 6(-3) - 12$$

$$= 54 - 18 - 12 \quad = 24 > 0$$

$$f'(-1) = 6(-1)^2 + 6(-1) - 12$$

$$= 6 - 6 - 12 \quad = -12 < 0$$

Now, since $f'(x)$ changed its sign from positive to negative at $x = -2$, then $f(x)$ has a relative maximum at $x = -2$. And then the value of this relative maximum in Example (6-8) where we found that it is equal to 5. Therefore, $f(x)$ has a relative maximum at $(-2, 5)$.

And, at $x = 1$ **:**

Then, we have to choose two values for x, a value before 1, say $(x_l = 0)$, and a value after 1, say $(x_r = 2)$, then we find that the values of $f'(x)$ at each of them as follows:

$$f'(0) = 6(0)^2 + 6(0) - 12$$

$$= 0 - 0 - 12 \quad = -12 < 0$$

And, $f'(2) = 6(2)^2 + 6(2) - 12$

$$= 24 + 12 - 12 \quad = 24 > 0$$

Now, since $f'(x)$ changed its sign from negative to positive at $x = 1$, then $f(x)$ has a relative minimum at $x = 1$. And then the value of this relative minimum in Example (6-8) where we found that it is equal to -22. Therefore, $f(x)$ has a relative minimum at $(1, -22)$.

Therefore, we see that the 1st derivative test leads to the same results that we obtained by using the 2nd derivative in the example (6-8).

***Remark*:**

In the 1st derivative test, when selecting x_l and x_r, they must stay nearest close to the critical value x^*. If you tray too far left or right, you may reach for wrong results.

Example (6-11):

By using the 1st derivative test, find the relative maximum and minimum for the function in example (6-9).

Solution:

Since: $$f(x) = \frac{x^3}{3} - 2.5x^2 + 4x$$

Then: $$f'(x) = x^2 - 5x + 4$$

Substituting $f'(x)$ equal to zero, we found that the critical points are:

$$x = 4 \qquad and \qquad x = 1$$

The nature of these critical points can be determined by using the 1st derivative as follows:

at $x = 4$ **:**

We will select two values for x, a value before 4, say $(x_l = 3)$, and a value after 4, say $(x_r = 5)$, then we find that the value of $f'(x)$ at each of them as follows:

$$f'(3) = (3)^2 - 5(3) + 4$$

$$= 9 - 15 + 4 \quad = -2 < 0$$

$$f'(5) = (5)^2 - 5(5) + 4$$

$$= 25 - 25 + 4 \quad = 4 > 0$$

Now, since $f'(x)$ changed its sign from negative to positive through$x = 4$, then $f(x)$ has a relative minimum at $x = 4$. We found that the value of this relative minimum in Example (6-

9) where we found that it is equal to -2.67. Therefore, $f(x)$ has a relative minimum at $(4, -2.67)$.

And, at $x = 1$:

We will select two values for x, a value before 1, say $(x_l = 0)$, and a value after 1, say $(x_r = 2)$, then we find that the value of $f'(x)$ at each of them as follows:

$$f'(0) = (0)^2 - 5(0) + 4$$

$$= 0 - 0 + 4 \quad = 4 > 0$$

$$f'(2) = (2)^2 - 5(2) + 4$$

$$= 4 - 10 + 4 \quad = -2 < 0$$

Now, since $f'(x)$ changed its sign from positive to negative through$x = 1$, then $f(x)$ has a relative maximum at $x = 1$. We found that the value of this relative maximum in Example

(6-9) where we found that it is equal to 1.83. Therefore, $f(x)$ has a relative maximum at $(1, 1.83)$.

Also, we see that the 1st derivative test leads to the same results that we obtained by using the 2nd derivative in the example (6-9).

(6-4): Absolute Maxima and Minima:

In some problems, it happens that the independent variable x is restricted to some interval of values, say $a \leq x \leq b$, and we need to find the largest or smallest value of some function $f(x)$ over this set of values of x. For example, if x is the level of production by manufacturing firm, then x is restricted to the interval $x \geq 0$, where it is not logical for the level of production to take a negative quantity, and we are interested in the maximum value of the profit function in this interval.

This restriction on x does not affect any of the results we obtained, but there are some cases in which similar restrictions can affect the conclusions regarding the maxima and minima of the functions.

(6-4-1): Absolute Maximum:

A function $f(x)$ is said to reach an *absolute maximum* at $x = c$ if $f(c) > f(x)$ for any other x in the domain of $f(x)$.

In other words, the *absolute maximum* value of $f(x)$ over an interval $a \leq x \leq b$ of its domain is the largest value of $f(x)$ as x takes all the values from a to b.

(6-4-2): Absolute Minimum

A function $f(x)$ is said to reach an *absolute minimum* at $x = c$ if $f(c) < f(x)$ for any other x in the domain of $f(x)$.

In other words, the *absolute minimum* value of $f(x)$ over an interval $a \leq x \leq b$ of its domain is the smallest value of $f(x)$ as x increases from a to b.

It is intuitively obvious that $f(x)$ is continuous in $a \leq x \leq b$, then the point at which $f(x)$ reaches its absolute maximum must be either a relative maximum of $f(x)$ or one of the endpoints $a\ or\ b$. A similar statement holds for the absolute minimum. Therefore, in order to find the absolute maximum and absolute minimum values of $f(x)$ over $a \leq x \leq b$, we simply select the largest and the smallest values from among the values of $f(x)$ at the critical points lying in $a \leq x \leq b$ and the end points $a\ and\ b$. This is illustrated in the following example.

Example (6-12):

Determine the absolute maximum and minimum for the following function:

$$f(x) = \frac{2x^3}{3} + 3x^2 + 4x - 1 \qquad , where \ -3 \leq x \leq 5$$

Solution:

Since: $f'(x) = 2x^2 + 6x + 4$

Substituting $f'(x)$ equal to zero, hence, we get:

$$2x^2 + 6x + 4 = 0$$

$$x^2 + 3x + 2 = 0$$

$$(x + 1)(x + 2) = 0$$

$$\therefore \quad x = -1 \quad and \quad x = -2$$

Therefore, the critical points for $f(x)$ are $x = -1 \ and \ x = -1$.

Here, we notice that these two values lie in the defined interval $-3 \leq x \leq 5$.

Remark:

If one of the critical values does not fall in the domain of the interval by which the function is defined, then this value is excluded from the solution.

Now, to determine the nature of the critical points:

Since, $f''(x) = 4x + 6$

Then, at $x = -1$:

$$f''(-1) = 4(-1) + 6$$

$$= 2 > 0$$

Since $f''(-1) > 0$, then $f(x)$ has a relative minimum at $x = -1$. And the value of this relative minimum can be determined by substituting $x = -1$ into the original function as follows:

$$f(-1) = \frac{2(-1)^3}{3} + 3(-1)^2 + 4(-1) - 1$$

$$= \frac{-2}{3} + 3 - 4 - 1 = \frac{-8}{3}$$

Therefore, the point $(-1, \frac{-8}{3})$ represents a relative minimum for the function.

And, at $x = -2$:

$$f''(-2) = 4(-2) + 6$$

$$= -2 < 0$$

Since $f''(-2) < 0$, then $f(x)$ has a relative maximum at $x = -2$. The value of this relative maximum can be determined by substituting $x = -2$ into the original function as follows:

$$f(-2) = \frac{2(-2)^3}{3} + 3(-2)^2 + 4(-2) - 1$$

$$= \frac{-16}{3} + 12 - 8 - 1 = \frac{-7}{3}$$

Therefore, the point $(-2, \frac{-7}{3})$ represents a relative maximum for the function.

Then, we find that the value of the function at the extreme points of the interval, *i.e.,* at $x = -3 \quad and\, x = 5$:

$$f(-3) = \frac{2(-3)^3}{3} + 3(-3)^2 + 4(-3) - 1$$

$$= 18 + 27 - 12 - 1 = -4$$

$$f(5) = \frac{2(5)^3}{3} + 3(5)^2 + 4(5) - 1$$

$$= \frac{250}{3} + 75 + 20 - 1 = \frac{532}{3}$$

Therefore, we conclude that:

- $f(x)$ **has an absolute maximum equal to** $\frac{532}{3}$ **at** $x = 5$**.**
- $f(x)$ **has an absolute minimum equal to** -4 **at** $x = -3$**.**

<u>Example (6-13):</u>

Determine the absolute maximum and minimum for the following function:

$$f(x) = 3x^2 - 48x + 30 \qquad , where\; 0 \le x \le 10$$

<u>Solution:</u>

Since, $f'(x) = 6x - 48$

Substituting $f'(x)$ **equal to zero, we get:**

$$6x - 48 = 0$$

$$x = 8$$

$$f(8) = 3(8)^2 - 48(8) + 30 = -162$$

i.e., **the critical point is:** $(8, -162)$

To determine the nature of this critical point:

$$f''(x) = 6 > 0$$

Therefore, $f(x)$ **has a relative minimum equal to** -162 **at** $x = 8$**.**

Then we find the value of the function at the extreme points of the interval:

$f(0) = 3(0)^2 - 48(0) + 30 = 30$

$f(10) = 3(10)^2 - 48(10) + 30 = -150$

Therefore, we conclude that:

- $f(x)$ **has an absolute maximum equal to** 30 **at** $x = 0$**.**
- $f(x)$ **has an absolute minimum equal to** -162 **at** $x = 8$**.**

(6-5): Business Applications

In this section, we will present the applications of mathematical methods covered in this chapter in many business fields: economics, production, administration, marketing and accounting. These applications will include the following:

- **Analysis of cost, revenue and profit functions.**
- **Marginal cost analysis.**
- **Average cost analysis.**
- **Marginal revenue analysis.**
- **Cost optimization applications (minimum cost, minimum average cost).**
- **Revenue optimization applications (maximum revenue).**
- **Profit optimization applications (maximum profit).**
- **Marginal approach to profit maximization.**

(6-5-1): Analysis of Cost, Revenue and Profit Functions

Example (6-14):

If the cost function $C(q)$ for producing q units of a specific product is:

$$C(q) = 2000 + 10q$$

And the demand function is on the form:

$$p = 100 - 0.5q$$

Where:

$\boldsymbol{q\ is\ the\ quantity\ demanded,\ \ p\ is\ the\ unit\ price\ (L.E.)}$

Required:

Determine the production levels that make each of:

$\boldsymbol{i.}$ $\boldsymbol{The\ cost\ function.}$

$\boldsymbol{ii.}$ $\boldsymbol{The\ revenue\ function.}$

$\boldsymbol{iii.}$ $\boldsymbol{The\ profit\ function.}$

Increasing *or* Decreasing.

Solution:

***i*. Since** $\quad \boldsymbol{C(q) = 2000 + 10q}$

Then $\quad \boldsymbol{C'(q) = 10 > 0}$

Therefore, the cost function is always increasing. In other word, the cost increases with the increasing of the quantity produced.

***ii*.Since,** $\boldsymbol{The\ \ Total\ Revenue = Sales\ Quantity\ \times Unit\ Price}$

$$\boldsymbol{= q\ \times p = q(100 - 0.5q)}$$

$$\boldsymbol{= 100q - 0.5q^2}$$

***i.e.*, the total revenue function is:**

$$\boldsymbol{R(q) = 100q - 0.5q^2}$$

$$\boldsymbol{R'(q) = 100 - q}$$

Therefore, we find that: $R'(q) > 0$ **when:**

$$100 - q > 0$$

$$i.e., \quad q < 100$$

Also, we find that: $R'(q) < 0$ **when:**

$$100 - q < 0$$

$$i.e., \quad q > 100$$

Therefore, we conclude that the total revenue function will be increasing if the quantity of production is less than 100 units, and it is decreasing if the quantity of production is greater than 100 units.

$iii.$ **Since** $\quad The\ Total\ Profit = Total\ Revenue - Total\ Cost$

i.e., **the profit function is:**

$$P(q) = 100q - 0.5q^2 - (2000 + 10q)$$

$$= 90q - 0.5q^2 - 2000$$

$$P'(q) = 90 - q$$

Therefore, we find that: $P'(q) > 0$ **when:**

$$90 - q > 0$$

$$i.e., \quad q < 90$$

Also, we find that: $P'(q) < 0$ when:

$$90 - q < 0$$

$$i.e., \quad q > 90$$

Therefore, we find that the total profit function will be increasing if the quantity of production is less than 90 units, and it is decreasing if the quantity of production is greater than 90 units.

(6-5-2): Marginal Cost Analysis

The *marginal cost* is the extra cost of producing and selling an additional unit of the product.

Example (6-15):

If the cost of producing q thousand units of a specific product is determined by the following form:

$$C(q) = 2500 + 9q - 3q^2 + 2q^3$$

Required:

Determine the production levels at which the marginal cost is:

$i. Increasing$ $\qquad\qquad$ $ii. Decreasing$

Solution:

Since, the marginal cost is: $C'(q) = 9 - 6q + 6q^2$

Then, for the marginal cost function to be increasing, it must hold that:

$$C''(q) > 0$$

$$i.e., \qquad -6 + 12q > 0$$

$$\therefore \qquad q > 0.5$$

And, for it to be decreasing, it must hold that:

$$C''(q) < 0$$

$$i.e., \qquad -6 + 12q < 0$$

$$\therefore q < 0.5$$

Therefore, we find that the marginal cost is increasing at any level of production that exceeds from half thousand units (500 units), and it is decreasing at any level of production that is less than half thousand units (500 units).

(6-5-3): Average Cost Analysis:

Example (6-16):

If the cost of producing q units of a specific product is determined by the following function:

$$C(q) = 6 + \frac{2q(q+4)}{q+1}$$

Required:

Prove that the average cost is always decreasing at any level of production.

Solution:

$$\textbf{Since, the Average Cost} = \bar{C}(q) = \frac{Total\ Cost}{Number\ of\ Units} = \frac{C(q)}{q}$$

$$= \frac{6}{q} + \frac{2(q+4)}{q+1}$$

***i.e.,* the average cost function is:**

$$\bar{C}(q) = \frac{6}{q} + \frac{2(q+4)}{q+1}$$

To prove that the average cost is always decreasing for any level of production, we must find the 1st derivative of the average cost function. *i.e.,*

$$C'(q) = \frac{-6}{q^2} + \frac{(q+1)(2) - 2(q+4)(1)}{(q+1)^2}$$

by performing some algebra operations, we can reach to the following:

$$C'(q) = -6\left[\frac{2q^2 + 2q + 1}{q^2(q+1)^2}\right]$$

This expression is always negative for any value of $q > 0$.

Therefore, we find that the average cost is always decreasing for any level of production ($i.e., for\ q > 0$).

(6-5-4): Marginal Revenue Analysis:

The *marginal revenue* is the extra revenue of selling an additional unit of the product.

Example (6-17):

If the demand function for a commodity takes the following formula:

$$p = 600 - q^2$$

Where:

q *is the quantity demanded,* p *is the unit price* $(L.E.)$

Required:

Determine the production levels by which the marginal revenue is:

$i.$ *Increasing* $ii.$ *Decreasing*

Solution:

Since: $Total\ Revenue = Sales\ Quantity \times Unit\ Price$

$$i.e., \quad R(q) = q \times p = qp$$

$$= q(600 - q^2)$$

$$= 600q - q^3$$

i.e., **the total revenue function is:**

$$R(q) = 600q - q^3$$

Then, the marginal revenue function is:

$$R'(q) = 600 - 3q^2$$

In order to determine whether the marginal revenue increases or decreases at different levels of production, we must find the 1st derivative of the marginal revenue function, which represents, at the same time, the 2nd derivative of the original total revenue function is:

$$R''(q) = -6q$$

Since the quantity of production is greater than zero, then we can say that the marginal revenue is always decreasing as long as there are sales and it is never increasing.

(6-5-5): Cost Optimization Applications:

There is no doubt that any business firm aims to reduce its production costs to its minimum value in order to achieve the maximum of profits. In this section, we will discuss how we can apply the differential derivatives to determine the

production level which minimize the total cost and average cost.

- ***Minimum Cost:***

Example (6-18):

A manufacturer found that the annual cost of producing q units of his product is represented by the following function:

$$C(q) = \frac{20000}{q} + 0.5q + 8000$$

Required:

i. **What is the production level that achieves the minimum annual cost?**

ii. **What is the value of the minimum annual cost?**

Solution:

$i.$ $\quad Since, \quad C'(q) = \frac{-20000}{q^2} + 0.5 \quad , then:$

Substituting $C'(q)$ equal to zero, we get:

$$\frac{-20000}{q^2} + 0.5 = 0$$

$$i.e., \quad 0.5q^2 = 20000$$

$$q^2 = 40000$$

$$\therefore \quad q = 200$$

i.e., **there is a critical point at $q = 200$. And to determine the nature of this critical point, we use $C''(q)$ as follows:**

$$C''(q) = \frac{40000}{q^3} > 0 \;\; for\, all\, values\, of\; q > 0$$

Then, there is a relative minimum for the cost function. This means that the graph of the cost function is concave up. So, the cost reaches to its minimum level when$q = 200$, ***i.e.,*** **the production level by which the annual cost reaches to its minimum level is 200 units.**

$ii.$ **The value of the minimum annual cost is:**

$$C(200) = \frac{20000}{200} + 0.5(200) + 8000$$

$$= 8200 \quad pounds$$

- ***Minimum Average Cost :***

The ***average cost*** **is the cost per unit of production on average. It can be obtained by dividing the total cost by the number of units produced.**

Example (6-19):

If the total cost function for producing q units of a specific product is:

$$C(q) = 4000 + 3q + \frac{q^2}{1000}$$

Required:

i. **Determine the average cost as a function of the number of units produced(q).**

ii. **What is the number of units produced that makes the average cost of production as minimum as possible?**

iii. **By using your result in (*ii*), what is the value of the minimum average cost of production?**

Solution:

$i.$ $Since$: The Average Cost $\bar{C}(q) = \frac{Total\ Cost}{Number\ of\ Units}$

$$= \frac{C(q)}{q}$$

i.e., **the average cost function is:**

$$\bar{C}(q) = \frac{4000}{q} + 3 + \frac{q}{1000}$$

$ii.$ **To make the average cost of production in its minimum value as possible, we have to find the 1st derivative for the average cost function:**

$$i.e., \qquad \bar{C}'(q) = \frac{-4000}{q^2} + \frac{1}{1000}$$

Substituting $\bar{C}'(q)$ equal to zero, we get:

$$\frac{-4000}{q^2} + \frac{1}{1000} = 0$$

$$i.e., \qquad \frac{4000}{q^2} = \frac{1}{1000}$$

$$q^2 = 4000000$$

$$\therefore \quad q = 2000$$

i.e., **there is a critical point at $q = 2000$. And to determine the nature of this critical point we have to find the 2nd derivative for the average cost function:**

$$i.e., \qquad \bar{C}''(q) = \frac{8000}{q^3} > 0 \quad for\ all\ values\ of\ q > 0$$

i.e., **the average cost reaches to its minimum level when the number units produced is 2000 units.**

iii. **From the preceding result in (*ii*), then the value of the minimum average cost of production is:**

$$\bar{C}(2000) = \frac{4000}{2000} + 3 + \frac{2000}{1000}$$

$$= 7 \quad pounds$$

Therefore, the production level that makes the average cost as minimum as possible is 2000 units, and the average cost

is 7 pounds per unit at this level of production, which is the minimum average cost that can be achieved.

<u>Example (6-20):</u>

If the total cost function for a specific product is on the following form:

$$C(q) = 1000 + 5q + 0.1q^2$$

Where:

$q \text{ is the number of units produced per day}$.

If the maximum number of units produced per day is 80 units. Find the number of units produced at which the average cost is in its minimum as possible.

<u>Solution:</u> Since the maximum number of units produced per day is 80 units, then this means that: $q \leq 80$. Also, finding the average cost indicates that there is at least one unit produced, *i.e.*: $q \geq 1$, therefore, we find that: $1 \leq q \leq 80$. Then,

The average cost function is:

$$\bar{C}(q) = \frac{1000}{q} + 5 + 0.1q$$

To find the minimum average cost, we have to find the 1st derivative for $\bar{C}(q)$, *i.e.,*

$$\bar{C}'(q) = \frac{-1000}{q^2} + 0.1$$

Substituting $\bar{C}'(q)$ equal to zero, we get:

$$\frac{-1000}{q^2} + 0.1 = 0$$

$$i.e., \qquad \frac{1000}{q^2} = 0.1$$

$$\therefore \; q = 100$$

Since the value $q = 100$ lies out of the domain of q values in the interval $1 \leq q \leq 80$, then we reject this value.

We have already concluded that the cost function is defined in an interval of q values. To explore whether there is an absolute minimum for the average cost function, we find the value of the average cost function at both endpoints of the interval, *i.e.,* finding $\bar{C}(1)$ and $\bar{C}(80)$ as follows:

$$\bar{C}(1) = \frac{1000}{1} + 5 + 0.1(1) = 1005.1 \qquad pounds$$

$$\bar{C}(80) = \frac{1000}{80} + 5 + 0.1(80) = 25.5 \qquad pounds$$

Therefore, we can conclude that there is a minimum limit equal to $25.5\,pounds$ for the average cost function at the quantity of production $q = 80$ units per day. In other words, the average cost reaches to its minimum level, which is $25.5\,pounds$ per unit, at the production level $q = 80$ units per day.

(6-5-6):Revenue Optimization Applications(Maximum Revenue):

Any business firm aims to achieve the possible maximum revenue, just as it aims to reduce its production cost to the possible minimum level. Since this enables the firm to achieve the maximum amount of profits as a result of selling its products.

Example (6-21):

If the demand function for a commodity takes the following formula:

$$p = 15e^{\frac{-q}{3}} \qquad ; \qquad 0 \le q \le 8$$

Where:

$q\ is\ the\ quantity\ demanded,\ p\ is\ the\ unit\ price\ (L.E.)$

Required:

Determine the unit price of the commodity and the quantity of production which achieve the maximum revenue.

Solution:

Since:

$$\textit{The Total Revenue}$$

$$= \textit{Demand Quantity} \times \textit{Unit Selling Price}$$

$$= q \times p = qp$$

$$= q\left(15e^{\frac{-q}{3}}\right)$$

$$= 15qe^{\frac{-q}{3}}$$

i.e., **the total revenue function is:**

$$R(q) = 15qe^{\frac{-q}{3}}$$

Then, in order to find the maximum revenue, we first determine the critical points as follows:

$$R'(q) = 15qe^{\frac{-q}{3}}\left(\frac{-1}{3}\right) + 15e^{\frac{-q}{3}}$$

$$= e^{\frac{-q}{3}}(15 - 5q)$$

Substituting $R'(q)$ equal to zero, we get:

$$e^{\frac{-q}{3}}(15-5q)=0$$

$$(15-5q)=0 \qquad \therefore q=3$$

$$or \quad e^{\frac{-q}{3}}=0 \qquad \therefore q=\infty$$

Since $q=\infty$ lies out of the interval$0\le q\le 8$, we exclude this value.

Therefore, there is a critical point at $q=3$, and to determine the nature of this point, we have to find the 2nd derivative for $R(q)$:

$$R''(q)=e^{\frac{-q}{3}}(-5)+(15-5q)\,e^{\frac{-q}{3}}\left(\frac{-1}{3}\right)$$

$$=-5e^{\frac{-q}{3}}-5e^{\frac{-q}{3}}+\frac{5}{3}qe^{\frac{-q}{3}}$$

$$=\frac{5}{3}qe^{\frac{-q}{3}}-10e^{\frac{-q}{3}}$$

$$=e^{\frac{-q}{3}}\left(\frac{5}{3}q-10\right)$$

Then we find the value of $R''(q)$ at $q=3$ as follows:

$$R''(3)=e^{\frac{-3}{3}}\left(\frac{5}{3}(3)-10\right)$$

$$=-5e^{-1}<0 \qquad Maximum$$

Therefore, we conclude that there is a maximum value of the revenue function at the production level ($q = 3$). *i.e.,* the revenue reaches to the maximum possible level at the production level ($q = 3$), and when the unit price is:

$$p = 15e^{\frac{-3}{3}} = 15e^{-1}$$

$$= 5.52 \qquad\qquad note\ that: e = 2.71828$$

In order to determine the maximum possible revenue at the level of production and price referred to above, we substitute the value of q in the revenue function as follows:

$$R(3) = 15(3)e^{\frac{-3}{3}} = 45e^{-1} = 16.55$$

***The final result is*: the maximum possible revenue, which is equal to 16.55, is achieved at the unit price $p = 5.52$ and the production quantity $q = 3\ units$.**

<u>Example (6-22):</u>

If a library's owner for the school tools sells a specific type of pens at price 5 pounds per pen, then he can sell 200 pens. But if he sells the one pen at price 7 pounds, then his sales will drop to 100 pens per day.

<u>Required:</u>

i. **Find the demand function by assuming it is linear.**

ii. **Determine the pen price at which the library's owner should sell the pen in order to achieve the maximum possible revenue. Then find the value of this revenue.**

iii. **What is the value of the library owner's revenue if he sold 200 pens in one day?**

<u>Solution:</u>

i. **The linear demand function passes through the two points (5,200) and (7,100), where the $(x) - coordinate$ of the point indicates the pen price(p), while its $(y) - coordinate$ indicates the quantity of sales (q) (demand). So, the demand function can be determined by using the slope of the function and a point on it as follows:**

$$Slope = \frac{q_2 - q_1}{p_2 - p_1} = \frac{100 - 200}{7 - 5} = -50$$

By using the point slope equation by using the first point, we can determine the linear demand function as follows:

$$m = \frac{200 - q}{5 - p}$$

$$-50 = \frac{200 - q}{5 - p}$$

$$200 - q = -250 + 50p$$

***i.e.*, the linear demand function is:**

$$\boldsymbol{q = 450 - 50p}$$

Where:

$\boldsymbol{q \text{ is the demand quantity } (sales),\ p \text{ is the pen price } (L.E.)}$

$\boldsymbol{ii}$**. Since:**

T$\boldsymbol{he\ total\ Revenue = \ Pen\ Price \times Number\ of\ Sold\ pens}$

$$\boldsymbol{= p \ \times q = pq}$$

$$\boldsymbol{= p(450 - 50p)}$$

$$\boldsymbol{= 450p - 50p^2}$$

***i.e.*, the revenue function is:**

$$\boldsymbol{R(p) = 450p - 50p^2}$$

And to find the maximum revenue of the library's owner:

$$\boldsymbol{R'(p) = 450 - 100p}$$

Substituting $\boldsymbol{R'(p)}$ equal to zero, we get:

$$\boldsymbol{450 - 100p = 0}$$

$$\boldsymbol{p = 4.5}$$

This means that there is a critical point at $p = 4.5$, and to determine the nature of this point, we find the 2nd derivative for $R(p)$:

$$R''(p) = -100 < 0 \qquad Maximum$$

This indicates that there is a maximum value of the revenue function at $p = 4.5$. In other word, the library owner's revenue will reach its maximum if he sells the pen at a price equal to $4.5\ pounds$.

The maximum revenues of the library's owner will be:

$$R(4.5) = 450(4.5) - 50(4.5)^2$$

$$= 1012.5 \quad pounds$$

iii. **If the library's owner sold 200 pens, *i.e.* $q = 200$, in one day, then this means that he sold one pen at a price that is determined as follows:**

$$q = 450 - 50p = 200$$

$$i.e., \qquad 450 - 50p = 200$$

$$50p = 250 \quad , \quad \therefore p = 5$$

Therefore, we find that if the library's owner sold 200 pens in one day, then this means that he sold the pen at price 5 pounds. So, his revenue will be determined as follows:

$$R(5) = 450(5) - 50(5)^2$$

$$= 1000 \quad pounds$$

(6-5-7): Profit Optimization Applications (*Maximum Profits):*

Example (6-23):

An industrial firm can sell one of its products at a price equal to 2 pounds per unit. If the total cost of producing q units of this product is determined by the function:

$$C(q) = 1000 + \frac{1}{2}\left(\frac{q}{50}\right)^2$$

Required:

- *i.* **Finding the total profit function if q units are produced and sold.**
- *ii.* **What is the number of units which must be produced in order to the firm achieve the maximum possible profit? and what is the value of this profit?**
- *iii.* **Determine the profit of the firm if it produced and sold 6000 units.**

Solution:

$i.\ Since.$

$$Total\ Revenue = \ Unit\ Selling\ Price \times Number\ of\ Sold\ units$$

$$= p \times q = pq = 2q$$

And, since, $\boldsymbol{The\ Total\ profit = Total\ Revenue - Total\ Cost}$

Then, $\boldsymbol{P(q) = R(q) - C(q)}$

$$\boldsymbol{= 2q - \left[1000 + \frac{1}{2}\left(\frac{q}{50}\right)^2\right]}$$

***i.e.,* the total profit function is:**

$$\boldsymbol{P(q) = 2q - 1000 - \frac{q^2}{5000}}$$

$\boldsymbol{ii.}$ **Since:** $\boldsymbol{P(q) = 2q - 1000 - \frac{q^2}{5000}}$

Then: $\boldsymbol{P'(q) = 2 - \frac{q}{2500}}$

Substituting $\boldsymbol{P'(q)}$ equal to zero, then we have:

$$\boldsymbol{2 - \frac{q}{2500} = 0}$$

$$\boldsymbol{\therefore q = 5000}$$

***i.e.,* there is a critical point at $\boldsymbol{q = 5000}$, and to determine the nature of this point, we find the second derivative for $\boldsymbol{P(q)}$:**

$$\boldsymbol{P''(q) = \frac{-1}{2500} < 0 \qquad Maximum}$$

This indicates that the profit function reaches to its maximum level at $\boldsymbol{q = 5000}$. *i.e.,* the firm achieves the

maximum possible profit when the production level reaches 5000 units, and the value of this profit is determined as follows:

$$\boldsymbol{P(5000) = 2(5000) - 1000 - \frac{(5000)^2}{5000}}$$

$$\boldsymbol{= 4000 \quad pounds}$$

$\boldsymbol{iii}$. **the profit of the firm if it produced and sold $\boldsymbol{6000}$ units is:**

$$\boldsymbol{P(6000) = 2(6000) - 1000 - \frac{(6000)^2}{5000}}$$

$$\boldsymbol{= 3800 \quad pounds}$$

Note that the profit value in this case is less than the maximum profit in ($\boldsymbol{ii}$) by 200 pounds.

Example (6-24):

If the demand function for a specific product is:

$$\boldsymbol{p = 5 - 0.001q}$$

Where:

$\boldsymbol{q \text{ is the quantity demanded}, \ p \text{ is the unit price}}$.

And the cost function $\boldsymbol{C(q)}$ of producing $\boldsymbol{q}$ units is:

$$\boldsymbol{C(q) = 2800 + q}$$

Required:

What is the production level that will achieve the maximum profit and what is the amount of this profit?

Solution:

Since:

$$\boldsymbol{Total\ Revenue = Sold\ Quantity\ \times Unit\ selling\ Price}$$

$$\boldsymbol{= q\ \times p = qp}$$

$$\boldsymbol{= q(5 - 0.001q)}$$

$$\boldsymbol{= 5q - 0.001q^2}$$

i.e., **the total revenue function is:**

$$\boldsymbol{R(q) = 5q - 0.001q^2}$$

And, since: $\boldsymbol{Total\ profit = Total\ Revenue - Total\ Cost}$

Then, $\boldsymbol{P(q) = R(q) - C(q)}$

$$\boldsymbol{= 5q - 0.001q^2 - (2800 + q)}$$

i.e., **the total profit function is:**

$$\boldsymbol{P(q) = 4q - 0.001q^2 - 2800}$$

We can determine the maximum value of this function as follows:

$$\boldsymbol{P'(q) = 4 - 0.002q}$$

Substituting $P'(q)$ equal to zero, then we have the following:

$$4 - 0.002q = 0$$

$$\therefore q = 2000$$

***i.e.,* there is a critical point at $q = 2000$, and to determine the nature of this point, we have to find the 2nd derivative for $P(q)$:**

$$P''(q) = -0.002 < 0 \qquad Maximum$$

Therefore, the total profit is in its maximum possible level when the production level reaches 2000 units, and at this level, the profits are determined as follows:

$$P(2000) = 4(2000) - 0.001(2000)^2 - 2800$$

$$= 1200 \quad pounds$$

So, the profits reach their maximum level, which is 1200 pounds, at the production level equals to 2000 units.

(6-5-7): Marginal Approach to Profit Maximum*:*

We discussed before how we can determine the maximum profit using the 2nd derivative of the total profit function. An alternative approach for finding the profit maximization point involves *marginal analysis*. This

approach is based on marginal analysis for the cost and revenue.

Given that a firm is producing a specific number of units each year, then the marginal analysis would be concerned with the effect on profit if one additional unit is produced and sold.

As long as the additional revenue resulted from producing and selling an additional unit exceeds the additional cost of this unit, then there is a net profit from producing and selling this unit and the total profit increases. If, however, the additional revenue from selling the next unit is exceeded by the cost of producing and selling the additional unit, then there is a net loss from this next unit and the total profit decreases.

Assuming that the profit has a greatest importance, then we can follow the rule which is concerned whether or not to produce an additional unit:

Rule (1): Should an Additional Unit be produced?

i. If $Marginal\ Revenue(MR) > Marginal\ Cost(MC)$

Produce the next unit

ii. If $Marginal\ Revenue(MR) < Marginal\ Cost(MC)$

Do not produce the next unit

For many production situations, the marginal revenue exceeds the marginal cost at lower levels of production. As the production level increases, then the amount by which the marginal revenue exceeds the marginal cost becomes smaller. Eventually, the production level is reached at which $MR = MC$. For any different from this point we find that: $MR < MC$, and the total profit begins to decrease with any additional production. Therefore, if the point can be identified where $MR = MC$ for the last unit produced and sold, then the total profit will be maximized. This profit maximization level of production can be determined by the following rule:

Rule (2): Profit Maximization Criterion

Total profit is maximized at the production level by which:

$$Marginal\ Revenue(MR) = Marginal\ Cost(MC)$$

$$i.e., \qquad R'(q) = C'(q)$$

By using the derivatives, we can say that the production level at which the 1st derivative of the total revenue function is equal to the 1st derivative of the total cost function achieves the maximum total profit.

<u>Example (6-25):</u>

By using the marginal analysis, answer the required (*ii*) in example (6-23).

<u>Solution:</u>

Since: $R(q) = 2q$,

$$and \qquad C(q) = 1000 + \frac{1}{2}\left(\frac{q}{50}\right)^2$$

By finding the first derivative for each of these two functions, then we have:

$$R'(q) = 2$$

$$C'(q) = \frac{q}{2500}$$

In order to determine the maximum profit by using the marginal approach, then:

$$R'(q) = C'(q)$$

$$i.e., \qquad 2 = \frac{q}{2500}$$

$$\therefore q = 5000$$

And in example (6-23), we get:

$$P(q) = P(5000) = 4000 \quad pounds$$

Which is the same result that we obtained by using the second derivative in the example (6-23).

There is no doubt that the marginal analysis approach is easier to apply than the second derivative method.

<u>Example (6-25):</u>

By using the data in example (6-24), determine the production level that achieves the maximum profit, by using the marginal analysis method.

<u>Solution:</u>

In Example (6-24), we concluded:

$$R(q) = 5q - 0.001q^2 \qquad , \qquad C(q) = 2800 + q$$

By finding the first derivative for each of these preceding two functions, then we have:

$$R'(q) = 5 - 0.002q$$

$$C'(q) = 1$$

In order to determine the maximum profit by using the marginal approach, then we have:

$$R'(q) = C'(q)$$

$$i.e., \qquad 5 - 0.002q = 1$$

$$\mathbf{0.002}\boldsymbol{q} = \mathbf{5} - \mathbf{1} = \mathbf{4}$$

$$\therefore \boldsymbol{q} = \mathbf{2000} \qquad \boldsymbol{units}$$

Therefore, the quantity of production that achieves the maximum possible profit is 2000 units.

Exercises for Chapter (6)

1- Determine the values of x that make each of the following functions:

$-\ Increasing$ $\qquad -Decreasing$

$-Not\ increasing\ or\ decreasing$

$a)\quad f(x) = x^2 - 3x + 4$

$b)\quad f(x) = x + \frac{1}{x}$

$c)\quad f(x) = 2x^3 - 9x^2 - 24x + 10$

$d)\quad f(x) = \frac{x^3}{3} + \frac{x^2}{2} - 6$

$e)\quad f(x) = \frac{x+1}{x-1}$

$f)\quad f(x) = e^{-2x}$

2- By using $f''(x)$ for each of the following functions, determine the nature of concavity at $x = -2\ \ and\ \ x = 1$.

$a)\quad f(x) = x^3 + 12x + 1$

$b)\quad f(x) = 3x^2 + 2x - 3$

$c)\quad f(x) = \frac{5x^3}{3} + \frac{3x^2}{2} - 5x + 25$

$d)\quad f(x) = -\ln x$

$e)\quad f(x) = \dfrac{x^2}{1+x}$

3- Find the critical points for each of the following functions, and then determine the nature of each by using:

- ***The 1st derivative test.***
- ***The 2nd derivative test.***

$a)\quad f(x) = \dfrac{x^3}{3} + 3x^2 - 48x + 100$

$b)\quad f(x) = \ln x - 0.05x$

$c)\quad f(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 20x$

$d)\quad f(x) = 2x^3 - 3x^2 - 36x + 1$

$e)\quad f(x) = \dfrac{x}{\ln x}$

$f)\quad f(x) = 3x + \dfrac{1}{3x}$

$g)\quad f(x) = \dfrac{\ln x}{x}$

$h)\quad f(x) = 2x^3 - 9x^2 + 12x + 6$

$i)\quad f(x) = x^3 - 3x^2 - 9x + 7$

4- Prove that the following function has no relative maximum or minimum relative at $x = 1$.

$$f(x) = x^3 - 3x^2 + 3x + 7$$

5- Prove that the following function has relative maximum or minimum relative, but its relative maximum is less than its relative minimum.

$$f(x) = x + \frac{1}{x}$$

6- Find the absolute maximum and the absolute minimum for the following functions:

$a)$ $f(x) = x^2 - 6x + 7$ $\qquad 1 \le x \le 6$

$b)$ $f(x) = x^3 - 18x^2 + 60x$ $\qquad -1 \le x \le 5$

$c)$ $f(x) = \dfrac{(x+1)(x-6)}{x^2}$ $\qquad 0.5 \le x \le 2$

$d)$ $f(x) = x^3 - 12x^2$ $\qquad 2 \le x \le 10$

$e)$ $f(x) = -4x^2 + 6x - 10$ $\qquad 0 \le x \le 10$

$f)$ $f(x) = \ln(x^2 + 10)$ $\qquad -1 \le x \le 4$

$g)$ $f(x) = x - \ln x$ $\qquad e^{-1} \le x \le e$

$h)$ $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} + 2x - 10$ $\qquad -2 \le x \le 30$

7- By using the cost function and the demand function in each of the following two cases, determine the production levels q by which each of:

$-\, \textit{The cost function}$

$-\, \textit{The revenue function}$

$-\, \textit{The profit function}$

Increasing *or* Decreasing.

$a)\quad C(q) = 2000 + 10q \qquad , \qquad p = 100 - 0.5$

$b)\quad C(q) = \sqrt{100 + q^2} \qquad , \qquad p = a - \left(\frac{b}{q}\right) 100 + q^2$

$\textit{where}:\ \ b > a > 0$

8- If the cost function of producing q units of a given product is:

$$C(q) = 1500 + 25q - 0.1q^2 + 0.004q^3$$

Determine the production level at which the total cost reaches to its minimum.

9- If the cost of producing q units of a product is given by the function:

$$C(q) = 16000 + 3q + 10^{-6}q^3$$

Determine the production level at which the average cost reaches to its minimum, then find the minimum average cost.

10- One of the firms concluded that the total revenue can be determined by the following function:

$$R(q) = 4000000 - (q - 2000)^2$$

Where: $q\ is\ the\ number\ of\ units\ sold$

<u>Required:</u>

i. **The number of units sold that achieves the maximum possible total revenue.**

ii. **What is the maximum total revenue?**

iii. **Determine the total revenue value if the number of units sold is 2500 units.**

11- If the total cost of producing q units of a product is:

$$C(q) = 50 + 2q + 0.5q^2$$

And the total revenue function for selling q units of this product is:

$$R(q) = 20q - q^2$$

<u>Required:</u>

i. **Determine the production level by which the maximum profit is achieved (by using the 2nd derivative test).**

ii. **By using your result in (*i*), what is the maximum profit?**

12- One of the firms concluded that the total cost function and the total revenue function for a specific product are:

$$C(q) = 100 + 0.015q^2 \qquad \textit{the total cost function}$$

$$R(q) = 3q \qquad \textit{the total cost revenue}$$

Where: q *is the number of units produced and sold*

Required:

i. **Determine the production level that achieves the maximum possible profit, by using:**

$-$ *2nd Derivative Test*

$-$ *Marginal Analysis of Profit Maximization*

ii. **Find the amount of the maximum profit.**

iii. **Determine the amount of profit at the production level $q = 120$, then compare your results with what you obtained in (*ii*).**

Chapter (7)
Integration

Introduction

The integration is the inverse process for differentiation. Just like division as the inverse operation of multiplication or addition as the inverse operation of subtraction, using the roots as the inverse process of rising to powers. This is the case in many mathematical operations, where we find that a mathematical process cancels out the effect of another inverse mathematical process.

In our study of calculus, we have been concerned with the process of differentiation in chapter five, *i.e.,* the calculation and use the derivatives of functions which are used in our practical life as useful indicators of the rates of change and the slop of these functions. In many situations, we need to know the original function if we have the slope or the 1st derivative of this function. For example, we may be concerned with the cost model in which the marginal cost (the 1st derivative of the total cost function) is a known function of production level, and we may wish to determine the total cost of producing *(q)* units. The same applies to the marginal revenue and the marginal profit. In such cases, the

process that enables us to do this is the reverse process of differentiation which is called *integration*.

(7-1): The definition of Integration:

***Integration calculus* is the process of finding the function when its 1st derivative is given. Given a function $f(x)$, we are studied how to find the 1st derivative $f'(x)$. There are many situations in which we are given the 1st derivative $f'(x)$ and wish to determine the original function $f(x)$. Since the process of finding the original function is the reverse of differentiation, then $f(x)$ is said to be an *ant derivative* of $f'(x)$.**

Consider the 1st derivative:

$$f'(x) = 7 \qquad (1)$$

By using a trial-and-error approach, it is not difficult to conclude that the function:

$$f(x) = 7x \qquad (2)$$

is the original function whose 1st derivative on the form in equation (1). Another function having the same 1st derivative is:

$$f(x) = 7x + 3$$

In fact, any function having the form:

$$f(x) = 7x + C \qquad (3)$$

Where:(C) is any constant, will have the same derivative. Therefore, if we have the derivative in equation (1), then the original function is one of the families of equations characterized by equation (3). This family of equations is a set of linear functions whose members all have a slope is equal to *7* but different in *y*-intercepts(C). Whatever the value of the constant *(C*) ,then its derivative is always equal to zero.

(7-2): Rules of Integration:

Fortunately, we need not follow to a trial-and-error approach whenever we wish to find an ant derivative. As with differentiation, a set of rules has been developed for finding ant derivatives, i.e., performing the integration process. If a function has a specific form, then a rule may be available which allows us to determine its ant derivative very easy.

The notation of integration is:

$$\int f(x)\ \ dx \qquad (4)$$

Which is often used to indicates the *indefinite integral* of the function $f(x)$. Where the symbol $\int$ is the integration

symbol, $f(x)$ is called the *integrand equation*, or the function for which we want to find the indefinite integral, and dx indicates the variable with respect to which the integration process is performed. Two verbal descriptions of the process indicated in equation (4) are "integrate the function $f(x)$ with respect to the variable x" or "find the indefinite integral of $f(x)$ with respect to x".

The integration process can be expressed in general terms as follows:

$$\int \boldsymbol{f'(x)}\ \ \boldsymbol{dx = f(x) + C}$$

Where:

$$\frac{d}{dx}f(x) = f'(x) \quad ,$$

$\boldsymbol{and\ (C) is\ termed\ as\ the\ constant\ of\ integration}.$

The following rules are a set of rules for finding the indefinite integral of some common functions with business and economics applications:

Rule 1: Constant Function:

$$\int k\ \ dx = kx + C$$

Where k is any constant.

Example (7-1):

$$(a) \int 9 \, dx = 9x + C$$

$$(b) \int (-3) \, dx = -3x + C$$

$$(c) \int \frac{2}{5} \, dx = \frac{2}{5}x + C$$

$$(d) \int 0 \, dx = 0x + C = C$$

Rule 2: Power Rule:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad , \quad n \neq -1$$

This rule is analogous to the power rule of differentiation. It is not valid when $n = -1$. This rule states that when the integral is x raised to some real-valued power, then we increase the exponent of x by 1, and we divided by the new exponent, and finally add the constant of integration.

Example (7-2):

$$(a) \int x \, dx = \frac{x^2}{2} + C$$

$$(b) \int x^7 \ dx = \frac{x^8}{8} + C$$

$$(c) \int \sqrt{x} \ dx = \int x^{\frac{1}{2}} \ dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$(d) \int \frac{1}{x^3} \ dx = \int x^{-3} \ dx = \frac{x^{-2}}{-2} + C = \frac{-1}{2x^2} + C$$

Rule 3:Integerate a Constant multiplied by a function:

$$\int k \ f(x) \ dx = k \int f(x) \ dx$$

This rule states that the indefinite integral of a constant (k) times a function $f(x)$ is found by multiplying the constant by the indefinite integral of $f(x)$. In other words, whenever a constant can be factored from the integral function, then the constant may also be factored outside the integral.

<u>Example (7-3):</u>

$$(a) \int 10x \ dx = 10 \int x \ dx = 10\left(\frac{x^2}{2} + C_1\right)$$

$$= 5x^2 + 10C_1 = 5x^2 + C$$

Check:

$$If \quad f(x) = 5x^2 + C \quad , \quad then\ f'(x) = 10x$$

$$i.e., \quad the\ integration\ process\ is\ correct$$

$$(b) \int \frac{2x^3}{5}\ dx = \frac{2}{5}\int x^3\ dx = \frac{2}{5}\left(\frac{x^4}{4} + C_1\right)$$

$$= \frac{1}{10}x^4 + \frac{2}{5}C_1 = \frac{1}{10}x^4 + C$$

Check:

$$If \quad f(x) = \frac{1}{10}x^4 + C \quad , \quad then\ f'(x) = \frac{4}{10}x^3 = \frac{2x^3}{5}$$

$$i.e., \quad the\ integration\ process\ is\ correct$$

Note that :

With indefinite integrals, we always add the constant of integration. In using rule 3, the algebra suggests that any constant (*k*) factored out of the integral will be multiplied by the constant of integration (for example: $10C_1$ in the previous example). This multiplication is unnecessary. We simply need a constant of integration to indicate the *indefinite nature* of the integral. Thus, the convention is to add *C* and not a multiple of *C*. In the last step, the $10C_1$ term is rewritten

as just *C*, since *C* can represent any constant as well as $\mathbf{10C_1}$.

Rule 4:

$$If \quad \int f(x) \quad dx \quad and \int g(x) \quad dx \quad exist, \; then:$$

$$\int [f(x) \pm g(x)] \; dx = \int f(x) \quad dx \; \pm \int g(x) \quad dx$$

i.e., **the indefinite integral of the sum (or the difference) of at least two function is the sum (difference) of their respective indefinite integrals.**

<u>Example (7-4):</u>

$$(\boldsymbol{a}) \int (\mathbf{5}\boldsymbol{x} - \mathbf{4}) \quad \boldsymbol{dx} = \int \mathbf{5}\boldsymbol{x} \; \boldsymbol{dx} - \int \mathbf{4} \; \boldsymbol{dx}$$

$$= \mathbf{5}\frac{\boldsymbol{x}^{2}}{\mathbf{2}} + \boldsymbol{C}_{\mathbf{1}} - (\mathbf{4}\boldsymbol{x} + \boldsymbol{C}_{\mathbf{2}})$$

$$= \frac{\mathbf{5}\boldsymbol{x}^{2}}{\mathbf{2}} - \mathbf{4}\boldsymbol{x} + (\boldsymbol{C}_{\mathbf{1}} - \boldsymbol{C}_{\mathbf{2}})$$

$$= \frac{\mathbf{5}\boldsymbol{x}^{2}}{\mathbf{2}} - \mathbf{4}\boldsymbol{x} + \boldsymbol{C}$$

Note that we substituted the two constants $(C_1 - C_2)$ by one constant(C), and we can verify the correctness of our result as follows:

$$If:\ \ f(x) = \frac{5x^2}{2} - 4x + C\ \ ,\quad then: f'(x) = 5x - 4$$

$$i.e.,\qquad the\ integration\ process\ is\ \ correct$$

$$(b)\ \ \int (\ x^2 + 3x)\ \ dx = \int\ x^2\ \ dx + \int 3x\ \ dx$$

$$= \frac{x^3}{3} + C_1 + 3\left(\frac{x^2}{2} + C_2\right)$$

$$= \frac{x^3}{3} + C_1 + \frac{3x^2}{2} + 3C_2$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} + (C_1 + 3C_2)$$

$$= \frac{x^3}{3} + \frac{3x^2}{2} + C$$

We can verify this result as follows:

$$If:\ \ f(x) = \frac{x^3}{3} + \frac{3x^2}{2} + C\ \ ,$$

$$then: f'(x) = \frac{3x^2}{3} + \frac{6x}{2} = x^2 + 3x$$

$$i.e.,\qquad the\ integration\ process\ is\ \ correct$$

Rule 5: Power Rule Exception:

$$\int x^{-1}\,dx = \ln x + C$$

This rule is the exception associated with rule 2 (the power rule) where $n = -1$ for x^n. Remember our differentiation rules:

$$\boldsymbol{If}:\ f(x) = \ln x\ ,\ \ \boldsymbol{then}: f'(x) = x^{-1}$$

Rule 6: Integrate the Exponential Function:

$$\int e^x\,dx = e^x + C$$

Rule 7:

$$\int [f(x)]^n . f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + C\ ,\qquad n \neq -1$$

This rule is similar to the power rule (rule 2). In fact, the power rule is the special case of this rule when $f(x) = x$. This rule states that if the integral consists of the product of a function $f(x)$ raised to a power *n* and the derivative of $f(x)$, then the indefinite integral is found by increasing the exponent of $f(x)$ by 1 and dividing by the new exponent.

Example (7-5):

Find the following integration:

$$(a) \int (x^2 - 3x)^4 (2x - 3) \quad dx$$

$$(b) \int (3x^2 - 2x)(3x - 1) \quad dx$$

$$(c) \int (2x^3 - 3x)^3 (12x - 6) \quad dx$$

Solution:

(a) $We\ find\ f(x) = (x^2 - 3x)^4 \quad , \quad f'(x) = (2x - 3)$

$$\therefore \int (x^2 - 3x)^4 (2x - 3) \ dx = \frac{(x^2 - 3x)^5}{5} + C$$

$$(b) \int (3x^2 - 2x)(3x - 1) \ dx$$

In order to apply the 7th rule, we can multiply the integral function by 2, then we can rewrite as follows:

$$= \frac{1}{2} \int \overbrace{(3x^2 - 2x)}^{f(x)} \overbrace{(6x - 2)}^{f'(x)} \quad dx$$

$$= \frac{1}{2} \frac{(3x^2 - 2x)^2}{2} + C$$

$$= \frac{(3x^2 - 2x)^2}{4} + C$$

$(c)\quad \int (2x^3 - 3x)^3\,(12x - 6)\quad dx$

$$= 2\int \overbrace{(2x^3 - 3x)}^{f(x)}{}^3\,\overbrace{(6x - 3)}^{f'(x)}\quad dx$$

$$= 2 \times \frac{(2x^3 - 3x)^4}{4} + C$$

$$= \frac{(2x^3 - 3x)^4}{2} + C$$

It is so easy to verify the correctness of this result as follows:

$$If:\quad f(x) = \frac{(2x^3 - 3x)^4}{2} + C \quad ,$$

$$then: f'(x) = \frac{4(2x^3 - 3x)^3(6x^2 - 3)}{2}$$

$$= 2(2x^3 - 3x)^3(6x^2 - 3) = (2x^3 - 3x)^3(12x^2 - 6)$$

$$i.e., \qquad the\ integration\ process\ is\ correct$$

Rule 8:

$$\int f'(x).\,e^{f(x)}\quad dx = e^{f(x)} + C$$

Note that the 6th rule is considered the special case of the 8th rule when $f(x) = x$.

Example (7-6):

Find the following integration:

$$(a) \int 3x^2\, e^{x^3} \quad dx$$

$$(b) \int x\, e^{2x^2} \quad dx$$

$$(c) \int 12x^2\, e^{2x^3} \quad dx$$

$$(d) \int (x-1)\, e^{x^2-2x} \quad dx$$

$$(e) \int \frac{3x^2}{(x^3+4)^3} \quad dx$$

Solution:

(a) $Let \;\; f(x) = x^3 \;\; , then \quad f'(x) = 3x^2$

$$\therefore \int 3x^2\, e^{x^3} \quad dx = e^{x^3} + C$$

$$(b) \int x\, e^{2x^2} \quad dx = \frac{1}{4}\int 4x\, e^{2x^2} \quad dx$$

$$= \frac{1}{4}\, e^{2x^2} + C$$

$$(c)\ \int 12x^2\, e^{2x^3}\ dx = 2\int 6x^2\, e^{2x^3}\ dx$$

$$= 2\, e^{2x^3} + C$$

$$(d)\ \int (x-1)\, e^{x^2-2x}\ dx = \frac{1}{2}\int (2x-2)\, e^{x^2-2x}\ dx$$

$$= \frac{1}{2}\, e^{x^2-2x} + C$$

$$(e)\ \int \frac{3x^2}{(x^3+4)^3}\ dx = \int (x^3+4)^{-3}(3x^2)\ dx$$

$$= \frac{(x^3+4)^{-2}}{-2} + C$$

$$= \frac{-1}{2(x^3+4)^2} + C$$

We can verify this result as follows:

$$If:\ f(x) = \frac{-1}{2(x^3+4)^2} + C \quad ,$$

$$= \frac{-1}{2}(x^3+4)^{-2} + C$$

$$then:\ f'(x) = (x^3+4)^{-3}(3x^2)$$

$$= \frac{3x^2}{(x^3+4)^3}$$

$$i.e., \quad the\ integration\ process\ is\ correct$$

Rule 9:

$$\int \frac{f'(x)}{f(x)}\ dx = \ln f(x) + C$$

Example (7-7):

Find the following integration:

$$(a) \int \frac{4x-1}{2x^2-x}\ dx \qquad (b) \int \frac{x^2-x}{2x^3-3x^2}\ dx$$

$$(c) \int \frac{4x-2}{x^2-x+1}\ dx \qquad (d) \int \frac{-x}{x^2+5}\ dx$$

Solution:

$$(a) \int \frac{4x-1}{2x^2-x}\ dx$$

Let: $f(x) = 2x^2 - x$, $\quad then \quad f'(x) = 4x - 1$

$$\therefore \int \frac{4x-1}{2x^2-x}\ dx = \ln(2x^2-x) + C$$

$$(b) \int \frac{x^2-x}{2x^3-3x^2}\ dx = \frac{1}{6}\int \frac{6x^2-6x}{2x^3-3x^2}\ dx$$

$$= \frac{1}{6} \ln(2x^3 - 3x^2) + C$$

(c) $$\int \frac{4x-2}{x^2-x+1} \, dx = 2\int \frac{2x-1}{x^2-x+1} \, dx$$

$$= 2 \ln(x^2 - x + 1) + C$$

(d) $$\int \frac{-x}{x^2+5} \, dx = \frac{-1}{2}\int \frac{2x}{x^2+5} \, dx$$

$$= \frac{-1}{2} \ln(x^2 + 5) + C$$

(7-3): Economic Applications for Integration:

In this section, we will discuss some of the different applications of integration in the field of economics and business applications.

It is including the following topics:

- **Marginal cost**
- **Marginal Revenue**
- **Marginal Profit**
- **Consumer's and Producer's Surplus**

And we will deal with these topics in some detail and by using some examples as follows:

(7-3-1): Marginal Cost:

Example (7-8):

The following function describing the marginal cost (in pounds) for producing a specific product is:

$$C'(q) = 8q + 800$$

Where: *q* is the number of units produced.

And if the total cost of producing 40 units is 80000 pounds. Determine the total cost function.

Solution:

In order to determine the total cost function, we must find the anti-derivative of the marginal cost, *i.e.,* the integral of the marginal cost function, then we have the following:

$$C(q) = \int C'(q) \quad dq = \int (8q + 800) \quad dq$$

$$= \frac{8q^2}{2} + 800q + C$$

***i.e.,* the total cost function is:**

$$C(q) = 4q^2 + 800q + C$$

Now, in order to find the constant (C):

Since: $C(40) = 80000$

Then:

$$\boldsymbol{C(40) = 4(40)^2 + 800(40) + C = 80000}$$

$\boldsymbol{i.e,}$ $\quad \boldsymbol{6400 + 32000 + C = 80000}$

$\therefore$ $\quad \boldsymbol{C = 80000 - 38400 = 41600}$

Note that: the value of $\boldsymbol{C = 41600 \ \ pounds}$ represents the fixed costs of production.

Therefore, the total cost function is:

$$\boldsymbol{C(q) = 4q^2 + 800q + 41600}$$

<u>(7-3-2): Marginal Revenue:</u>

<u>Example (7-9):</u>

If the marginal revenue function for a specific product is:

$$\boldsymbol{R'(q) = 220000 - 18q}$$

<u>Required:</u>

c) **Determine the total revenue function.**
d) **What is the total revenue for producing and selling 100 units?**

<u>Solution :</u>

$\boldsymbol{a)}$ $\because$ **Total revenue function**

$$= \int \boldsymbol{Marginal} \textbf{ revenue function}$$

$$\therefore \quad R(q) \quad = \int (220000 - 18q) \quad dq$$

$$i.e., \; R(q) = 220000\text{q} - 9q^2 + C$$

Since, there is no revenue in the case that no units are sold (unless stipulated), *i.e.,* $R(0) = 0$, therefore, $C = 0$.

***i.e.,* the total revenue function is:**

$$R(q) = 220000\text{q} - 9q^2$$

$b)$ **The total revenue when producing and selling 100 units is:**

$$R(100) = 220000(100) - 9(100)^2$$

$$= 21910000 \quad pounds$$

This can be found in another way [see Example (7-10)].

<u>Example (7-10):</u>

If the marginal revenue function for a specific product is:

$$R'(q) = -0.04q + 10$$

Where: *q* is the number of sold units.

<u>Required:</u>

a) Find the total revenue of selling 200 units from this product.

b) What is the additional revenue as a result of the increase in sales from 100 units to 200 units?

<u>Solution:</u>

a) **In order to find the total revenue for selling 200 units, we can calculate it in a different way from that used in an example (6-9) as follows:**

$$\textbf{Since, the total revenue} = \int_{0}^{200} (-0.04q + 10) \; dq$$

$$= [-0.02q^2 + 10q]_0^{200}$$

$$= [-0.02(200)^2 + 10(200)] - [-0.02(0)^2 + 10(0)]$$

$$= 1200 \quad pounds.$$

$$b) \textbf{ The additional revenue} = \int_{100}^{200} (-0.04q + 10) \; dq$$

$$= [-0.02q^2 + 10q]_{100}^{200}$$

$$= [-0.02(200)^2 + 10(200)] - [-0.02(100)^2 + 10(100)]$$

$$= 1200 - 800 = 400 \quad pounds.$$

The required additional revenue can be expressed as follows:

$$\int_{100}^{200} R'(q) \; dq = \int_{0}^{200} R'(q) \; dq - \int_{0}^{100} R'(q) \; dq$$

$$= R(200) - R(100)$$

This leads to the same result.

(7-3-3): Marginal Profit:

Example (7-11):

If the marginal profit function for a firm is:

$$P'(q) = 5 - 0.002q$$

And the total profit resulted from selling 100 units is 310 pounds.

Required:

a) **Determine the total profit function.**

b) **Find the total profit when selling 200 units.**

Solution :

$a)$ Since,: $P(q) = \int P'(q) \quad dq$

$Then,$ $P(q) = \int (5 - 0.002q) \quad dq$

$$= 5q - 0.001q^2 + C$$

And Since: $P(100) = 310$

Then:

$$P(100) = 5(100) - 0.001(100)^2 + C = 310$$

$i.e,$ $500 - 10 + C = 310$

$\therefore \qquad C = 310 - 490 = -180$

Therefore, the total profit function is:

$$R(q) = 5q - 0.001q^2 - 180$$

$b)$ **The total profit when selling 200 units is:**

$$R(200) = 5(200) - 0.001(200)^2 - 180$$

$$= 780 \quad pounds.$$

<u>(7-3-4): Consumer's and Producer's Surplus:</u>

Let the demand curve be $p = f(q)$ for a specific commodity and let the supply curve for the same commodity be given by $p = g(q)$, where q denotes the quantity of commodity units by which can be sold or supplied at the price (p) per unit. In general, the demand function $f(q)$ is decreasing function indicating that the consumers will buy less if the price increases. On the other hand, the supply function $g(q)$ in general is increasing function because the producers are willing to supply more if they get higher prices. The market equilibrium is the point of intersection of the demand and supply curves. This means that at the equilibrium price p_o per unit, the consumers will buy the equilibrium quantity q_o units of the commodity, and the producers will sell the same number of units q_o.

Then, the *consumer's surplus* (CS) is given by the definite integral:

$$C.S = \int_0^{q_o} [f(q) - p_o] \ \ dq = \int_0^{q_o} f(q) \ \ dq \ - \ p_o q_o$$

Where: $\boldsymbol{p = f(q)}$ **is the demand function.**

And, the *producer's surplus* (PS) is given by the definite integral:

$$P.S = \int_0^{q_o} [p_o - g(q)] \ \ dq = \ p_o q_o - \int_0^{q_o} g(q) \ \ dq$$

Where: $\boldsymbol{p = g(q)}$ **is the supply function.**

<u>Example (7-12):</u>

The demand and supply functions for a specific product are given by:

$$\boldsymbol{demand\ function}: \quad \boldsymbol{p = 15 - 2q}$$

$$\boldsymbol{supply\ function}\ : \quad \boldsymbol{p = 3 + q}$$

Determine the consumer's surplus and the producer's surplus, by assuming that the market equilibrium has been established.

<u>Solution:</u>

The equilibrium point (q_o, p_o) is obtained by solving the demand and supply equations as follows:

$$15 - 2q = 3 + q$$

$$3q = 12 \qquad\qquad \therefore q = 4$$

By substituting the value of *q* into the demand function:

$$p = 15 - 2(4) = 7$$

Therefore, the equilibrium in the market is take on when the price of the commodity reached to 7 pounds, and at this price, the demanded quantity is equal to the supplied quantity = 4 units. i.e.,

$$q_o = 4 \qquad , \qquad p_o = 7$$

Now, since:

$$\text{The consumer's surplus} = \int_0^{q_o} f(q)\ dq - p_o q_o$$

$$f(q) = 15 - 2q \qquad\qquad \textit{demand function}$$

$$\therefore \text{The consumer's surplus} = \int_0^{4} (15 - 2q)\ dq - (7)(4)$$

$$= [15q - q^2]_0^4 \ - 28$$

$$= [15(4) - (4)^2] - 0 \ - 28$$

$$= 60 - 16 - 0 \ - 28 = 60 - 44 = 16$$

And, since:

$$\textbf{The producer's surplus} = p_o q_o - \int_0^{q_o} g(q) \ dq$$

$$g(q) = 3 + q \qquad \textit{supply function}$$

$$\therefore \textbf{The producer's surplus} = (7)(4) - \int_0^4 (3 + q) \ dq$$

$$= 28 - [3q + \frac{q^2}{2}]_0^4$$

$$= 28 - \left[3(4) + \frac{(4)^2}{2}\right] - 0 \ = 28 - 20 \ = 8$$

Exercises for Chapter (7)

1- Find the integral for each of the following functions:

$a)\quad f(x) = x^3 + x^2 + 6x$

$b)\quad f(x) = 20x^4 + 8x^3 - 4x$

$c)\quad f(x) = \sqrt{x^3 + 8x + 10}$

$d)\quad f(x) = x^3$

$e)\quad f(x) = -18x^5 + 9x^2 - 10x$

$f)\quad f(x) = (x + 2)^2$

2- Find the following integrals:

$(a)\ \int \frac{4x}{100 + x^2}\, dx$ $\qquad$ $(b)\ \int \frac{18}{6x + 5}\, dx$

$(c)\ \int \frac{x^3 - 1}{x^4 - 4x}\, dx$ $\qquad$ $(d)\ \int \sqrt{9x - 3x^2}\,(3 - 2x)\, dx$

$(e)\ \int (x + 2)\, dx$ $\qquad$ $(f)\ \int (4x^3 + 3x^2)\, dx$

$(g)\ \int (x + 2)^2\, dx$ $\qquad$ $(h)\ \int 12e^{6x}\, dx$

$(i)\ \int \frac{-12x^2}{2x^3 + 3}\, dx$

3- If the marginal cost for producing *(q)* units of a specific product in a firm is represented by the following function:

$$C'(q) = 24 - 0.03q + 0.006\, q^2$$

And the total cost of producing 200 units is 22700 pounds.

Required:

- **a) Find the variable total cost function.**
- **b) Determine the fixed cost, and then the total cost function for the firm.**
- **c) Find the cost for producing 500 units.**
- **d) If one unit of the product is sold at a price equal to 90 pounds, determine the production level that will achieve the maximum possible profit.**

4- If the marginal revenue function for a firm is:

$$R'(q) = 4 - 0.01q$$

Where: *q* is the number of units sold.

Required:

- **a) Find the total revenue obtained from selling *q* units of this product.**
- **b) What is the demand function for this product?**

5- If the marginal revenue for a specific product in a company is represented by the following function:

$$R'(q) = 12 - 0.2q - 0.03\, q^2$$

Where: *q* is the number of units sold.

Required:

- **a) Find the total revenue function.**
- **b) Find the total revenue obtained from selling 20 units of this product.**
- **c) What is the demand function for this product?**
- **d) What is the number of units that the company would be able to sell if it had paid 3 pounds for each unit sold?**

6- If the marginal profit function for producing and selling *q* units of a product is:

$$P'(q) = -6q + 450$$

And the total profits are equal to 5000 pounds, if 100 units of this product were sold.

Required:

A):Find the total profit function.

b):Determine the profit value when 1000 units are sold.

7- If the demand and supply functions for a specific product are given by:

$\boldsymbol{Demand\ function}$: $\boldsymbol{p = 100 - q^2}$

$\boldsymbol{Supply\ function}$: $\boldsymbol{p = 52 + 2q}$

Determine the consumer’s and producer’s surplus, assuming that the market equilibrium has been taken.

8- Find the consumer’s and producer’s surplus for a product whose demand and supply functions are given by:

$\boldsymbol{Demand\ function}$: $\boldsymbol{p = 17 - 0.5q}$

$\boldsymbol{Supply\ function}$: $\boldsymbol{p = 5 + 0.3q}$

Assuming that the market equilibrium has been achieved .