Mathematics for Computer Science Students

Math. I

2022/2023

Mathematics and mathematical modelling are of central importance in computer science. For this reason the teaching concepts of mathematics in computer science have to be constantly reconsidered, and the choice of material and the motivation have to be adapted. This applies in particular to mathematical analysis, whose significance has to be conveyed in an environment where thinking in discrete structures is predominant. On the one hand, an analysis course in computer science has to cover the essential basic knowledge. On the other hand, it has to convey the importance of mathematical analysis in applications, especially those which will be encountered by computer scientists in their professional life.

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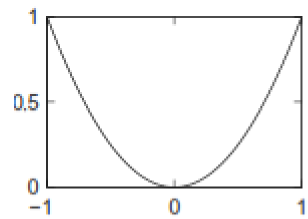
Real-Valued Functions

The notion of a function is the mathematical way of formalizing the idea that one or more independent quantities are assigned to one or more dependent quantities. Functions in general and their investigation are at the core of analysis. They help to model dependencies of variable quantities, from simple planar graphs, curves and surfaces in space to solutions of differential equations or the algorithmic construction of fractals. This chapter serves to introduce the basic concepts. The most important examples of real-valued, elementary functions are discussed here. These include the power functions, the exponential functions and their inverses.

Basic Notions

The simplest case of a real-valued function is a doula-row list of numbers, consisting of values from an independent quantity x and corresponding values of a dependent quantity.

Experiment. To study the mapping $y = x^2$. First choose the region D in which the x-values should vary, for instance $D = \{x \in \mathbb{R} : -1 \le x \le 1\}$. Using $y = x^2$ a row vector of the same length of corresponding y-value is generated. Finally plots the points $(x_1, y_1), \cdots, (x_n, y_n)$ in the coordinate plane and connects them with line segments. The result can be seen in the following figure.



Definition 1. A real-valued function f with domain D and range R is a rule which assigns to every $x \in D$ a real number $y \in R$.

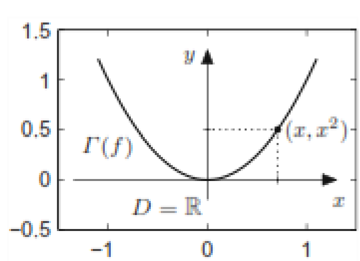
In general, D is an arbitrary set. In this section, however, it will be a subset of $\mathbb{R}$ (the set of all real numbers). For the expression function we also use the word mapping. A function is denoted by

$$f: D \to \mathbb{R}, \qquad x \longmapsto y = f(x)$$

The graph of a function f is the set of all points (x, y) with $x \in D, y = f(x)$.

Example 1. A part of the graph of the quadratic function $f: \mathbb{R} \to \mathbb{R}, f\ (x)\ =$ x^2

is shown as



If one chooses the domain to be $D = \mathbb{R}$, then the image is the interval $f\,(D) = [0, \infty)$.

Definition 2. (a) A function $f: D \to B$ is called injective or one-to-one, if different values of x always have different function values $f(x)$:

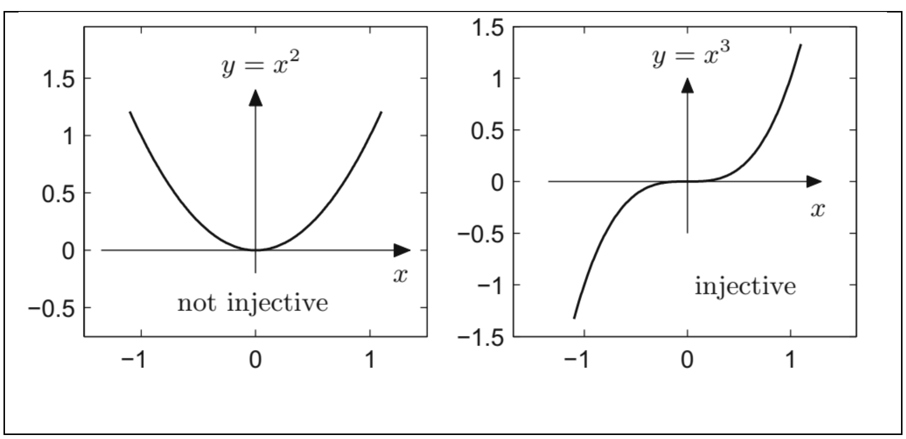
$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

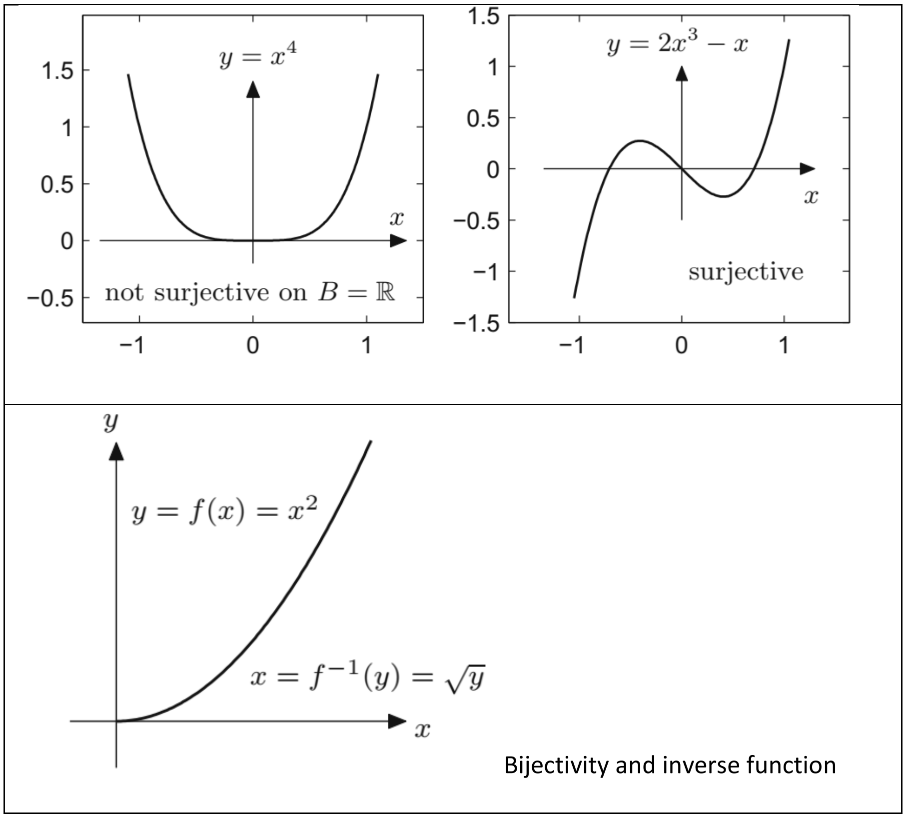
(b) A function $f: D \to B \subset \mathbb{R}$ is called surjective or onto from D to B, if each $y \in B$ appears as a function value:

$$\forall\, y \in B\; \exists\, x \in D\,; y = f(x).$$

(c) A function $f = D \to B$ is called bijective if it is injective and surjective.

Subjectivity can always be enforced to reducing the range B; for example, $f: D \to f(D)$ is always surjective. Likewise, injectivity can be obtained by restricting the domain to a subdomain. with $y = f(x)$. The mapping $x \longmapsto f(x)$ then defines the inverse of the mapping $x \mapsto y$.





Bijectivity and inverse function

Definition 3. If the function $f = D \to B$: $y = f(x)$ is bijective, then the function $f^{-1}: B \to D$: $x = f^{-1}(y)$ which maps each $y \in B$ to the unique $x \in D$ with $y = f(x)$ is called the inverse function of the function f.

Example 3. The quadratic function $f(x) = x^2$ is bijective from $D = [0, \infty)$ to $B = [0, \infty)$. In these intervals ($x \geq 0, y \geq 0$) one has

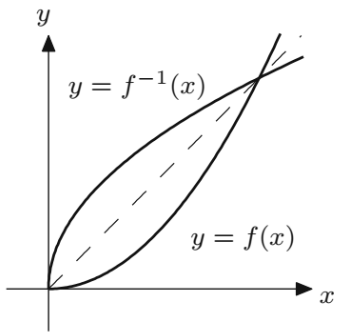
$$y = x^2 \Leftrightarrow x = \sqrt{y}.$$

Here $\sqrt{y}$ denotes the positive square root. Thus the inverse of the quadratic function on the above intervals is given by $f^{-1}(y) = \sqrt{y}$.

Once one has found the inverse function f^{-1}, it is usually written with variables

$y = f^{-1}(x)$. This corresponds to flipping the graph of y = f (x) about the diagonal

y = x, as is shown

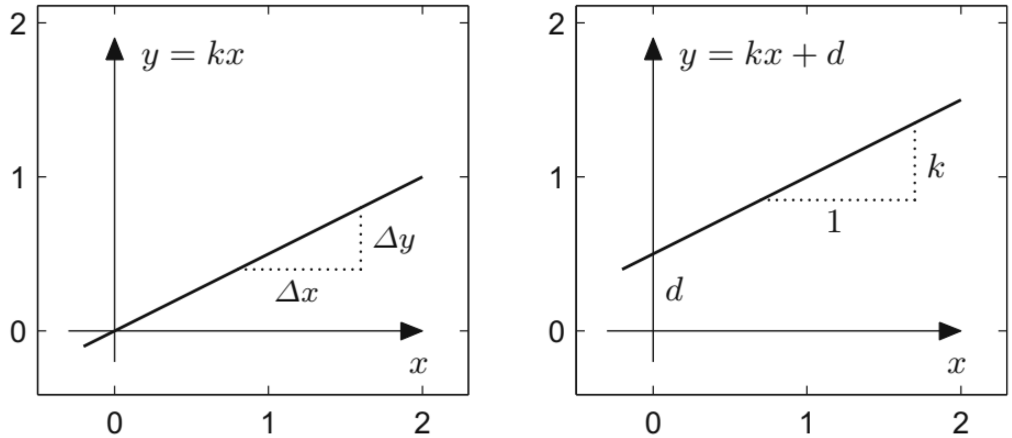


Some Elementary Functions

1. Linear functions (straight lines). A linear function $\mathbb{R} \to \mathbb{R}$ assigns each x-value a fixed multiple as y-value, i.e., $y = kx$. Here

$$k = \frac{\text{increase in height}}{\text{increase in length}} = \frac{\Delta y}{\Delta x}$$

Is the slop of the graph. This function presents a straight line through the origin. Adding an intercept $d \in \mathbb{R}$ translates the straight line d units in y-direction. The equation is then $\mathrm{y} = \mathrm{k\,x} + \mathrm{d}$. See the following figures

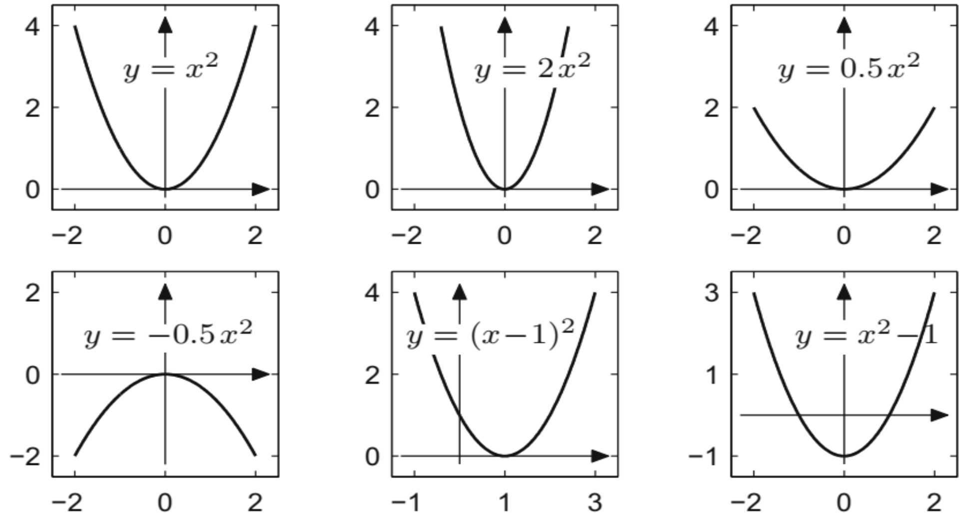


2. Quadratic parabolas. The quadratic function with demain $D = \mathbb{Z}$ (the set of all integers) in its basic form is given by $y = x^2$. From which Compression/stretching, horizontal and vertical translation are obtained via

$$y = a\,x^2, \quad y = (x - b)^2, \qquad y = x^2 + c$$

$a > 1$	compression in x-direction
$0 < a < 1$	stretching in x-direction
$a < 0$	reflection in the x-axis
$b > 0$	translation to the right
$b < 0$	translation to the left
$c > 0$	translation upwards
$c < 0$	translation downwards

The effect of these transformations on the graph can be seen in figure



quadratic function can be reduced to these cases by completing the square:

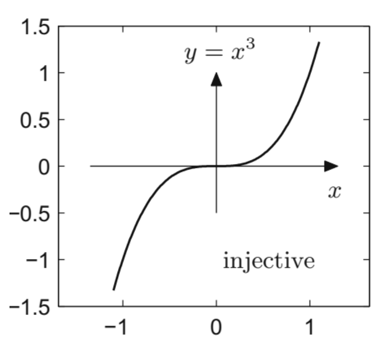
$$y = a\,x^2 + b\,x + c = \mathrm{a}\left(\mathrm{x} + \frac{\mathrm{b}}{2\mathrm{a}}\right)^2 + \mathrm{c} - \frac{\mathrm{b}^2}{4\mathrm{a}}$$

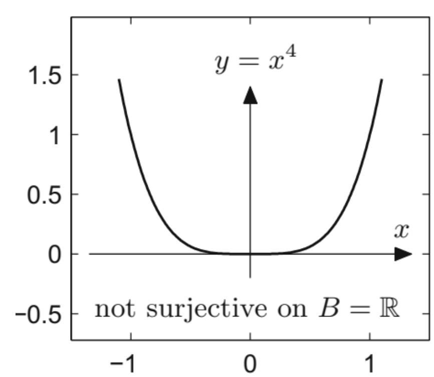
3. Power functions. If $n \in \mathbb{N}$ (n is an integer) the following rules apply

$$x^n = x \cdot x -\cdot\, x \cdots x \quad (n \text{ factors}), \quad x^1 = x$$

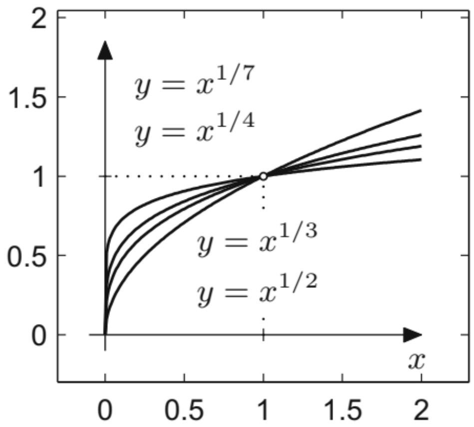
$$x^0 = 1, \quad x^{-n} = \frac{1}{x^n} \ \ (x \neq 0).$$

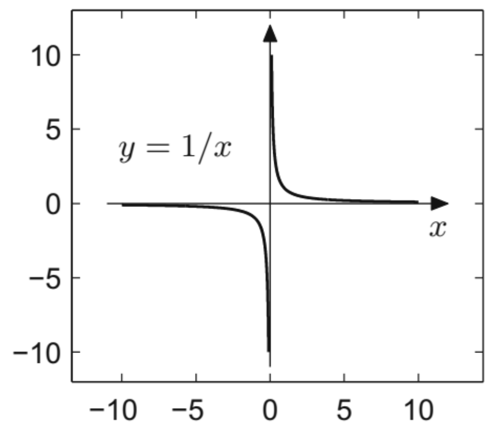
The behaviour of $y = x^3$ and $y = x^4$ can be seen as





Power functions with fractional and negative exponents

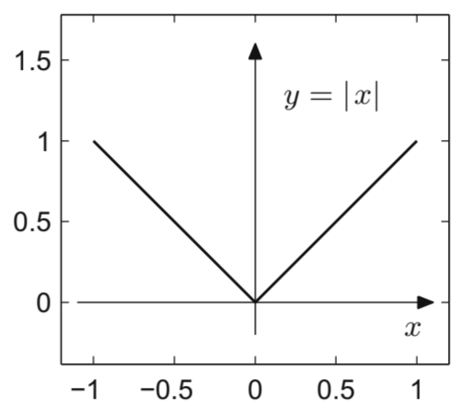




4. Absolute value, sign and indicator function. The absolute value function is

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

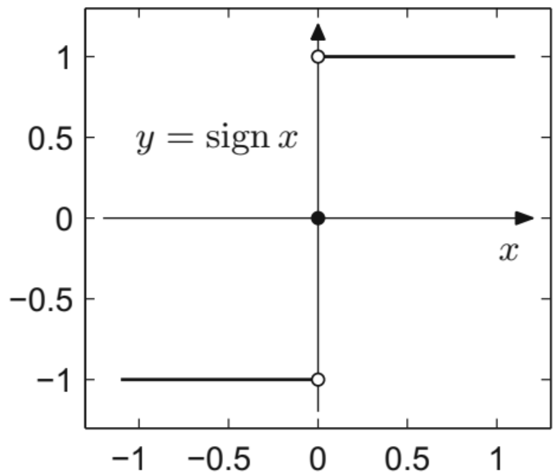
Its graph is



The sign function is

$$y = \text{sign}\, x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Its graph is

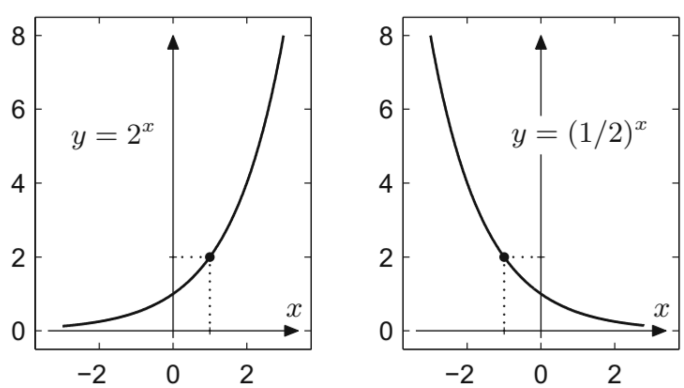


The indicator function of a subset $A \subset \mathbb{R}$ is defined as

$$I_A(x) = \begin{cases} x, & x \in A \\ 0, & x \notin A \end{cases}$$

5. Exponential functions and logarithms.

The ***exponential function*** with base a, the function $y = a^x$, increases for $a > 1$ and decreases for $a < 1$, see



the proper range is $B = (0, \infty)$; the exponential function is bijective from $\mathbb{R}$ to $(0, \infty)$. Integer powers of a number a > 0 have just been defined. Fractional (rational) powers give

$$a^{1/n} = \sqrt[n]{a}, \qquad a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

If r is an arbitrary real number then a^r is defined by its approximations $a \;\; m/n$, where $\frac{m}{n}$ is the rational approximation to r obtained by decimal expansion.

Example. 2^π is defined by the sequence

$$2^3, \qquad 2^{3.1}, \qquad 2^{3.14}, \qquad 2^{3.141}, \;\; 2^{3.1415}, \cdots$$

where

$$2^{3.1} = 2^{31/10} = \sqrt[10]{2^{31}}, \qquad 2^{3.14} = 2^{314/100} = \sqrt[100]{2^{314}}, \cdots$$

From the definition of the exponential function we obtain the following rules for rational exponents:

$$a^r\, a^x = a^{r+a}, \quad (a^r)^s = a^{rs} = (a^s)^r, \quad a^r\, b^r = (a\, b)^r.$$

The inverse function is the ***logarithm*** to the base a (with domain $(0, \infty)$ and range $\mathbb{R}$):

$$y \; = \; a^x \; \Leftrightarrow \; x \; = \log_a y\,.$$

For example, $\log_{10} 2$ is the power by which 10 needs to be raised to obtain 2:

$$2 \; = \; 10^{\log_{10} 2}.$$

Other examples are, for instance:

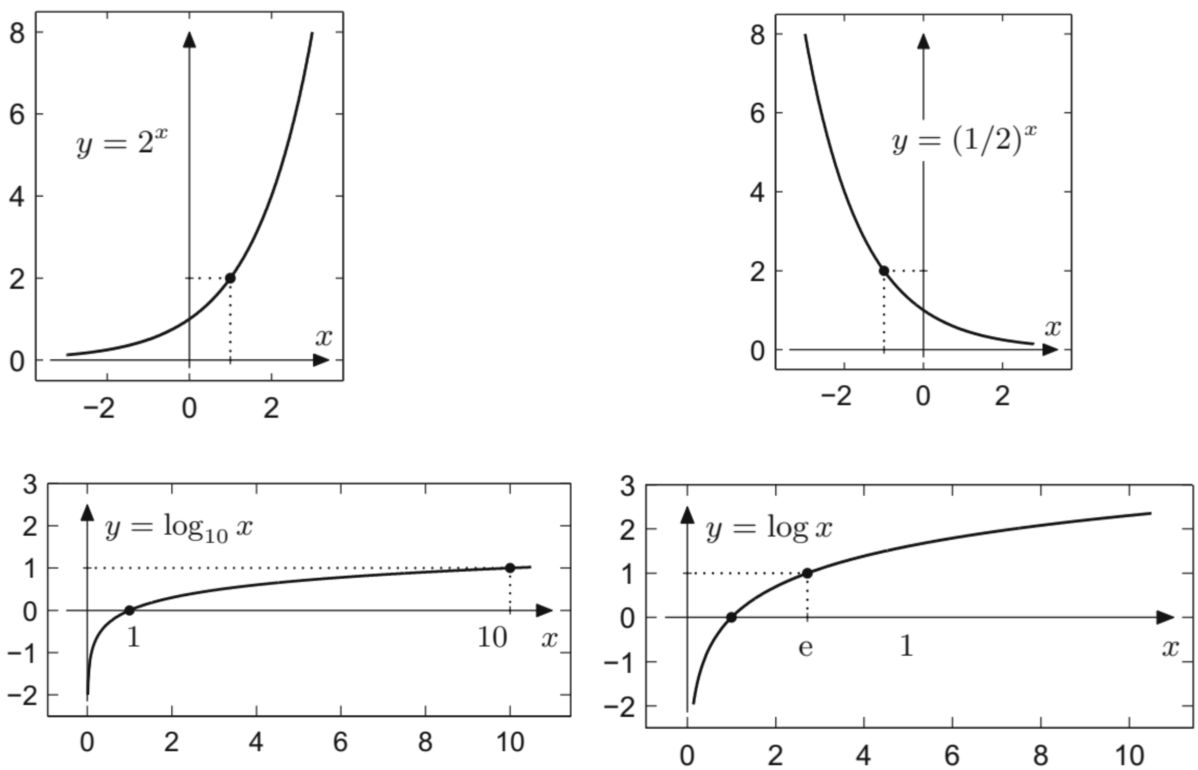
$$2 \; = \log_{10} 10^2, \;\; \log_{10} 10 = 1, \log_{10} 1 = 0, \qquad \log_{10} 0.001 = -3.$$

Euler's number e is defined by

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots = \sum_{j=0}^{\infty} \frac{1}{j!} \cong 2.718281828459 \cdots$$

The logarithm to the base e is called ***natural logarithm*** and is denoted by log (or ln) we write $\log x = \log_e x$ ($\ln x = \log_e x$).

The graphs of the exponential and logarithm functions are seen



We stick to the notation $\log x$ which is used, for example in MATLAB. The following rules are obtained directly from the rules for the exponential function:

$$u = e^{\log x}, \quad \log uv = \log u + \log v, \quad \log u^z = z \log u$$

$u, v > 0$ and arbitrary $z \in \mathbb{R}$. In addition, it holds that $u = \log(e^u)$ for all $u \in \mathbb{R}$ and $\log e = 1$. Moreover,

$$\log \frac{1}{u} = -\log u, \quad \log \frac{u}{v} = \log u - \log v.$$

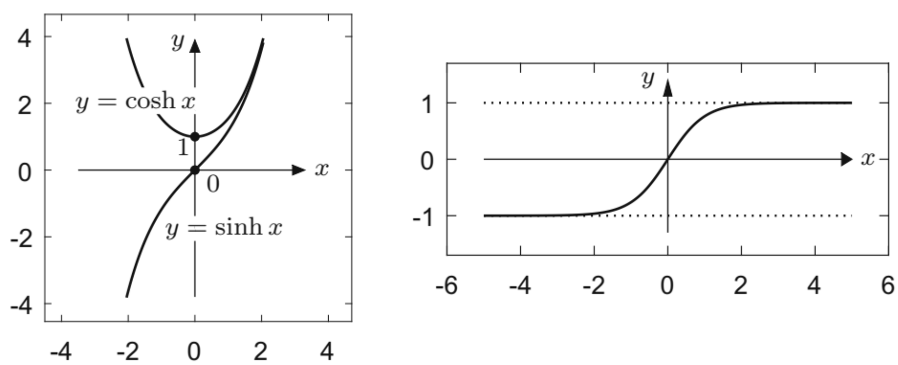
6. Hyperbolic functions and their inverses.

The hyperbolic sine, the hyperbolic cosine and the hyperbolic tangent are defined

by

$$\sinh x = \frac{1}{2}\,(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}\,(e^x + e^{-x}), \quad \tanh x = \frac{\sinh x}{\cosh x}$$

for $x \in \mathbb{R}$. Their graphs are displayed as the following, and important property is that identity $\cosh^2 x - \sinh^2 x = 1$ for all $x \in \mathbb{R}$.



These figures shows that the hyperbolic sine is bijective as a function from $\mathbb{R} \to \mathbb{R}$,

the hyperbolic cosine is bijective as a function from $[0, \infty) \to [1, \infty)$, and the hyperbolic tangent is bijective as a function from $\mathbb{R} \to (-1, 1)$. The inverse hyperbolic functions are referred to as inverse hyperbolic sine (cosine, tangent) or area hyperbolic sine (cosine, tangent). They can be expressed by means of logarithms as follows

$$\operatorname{arsinh} x = \log\left(x + \sqrt{x^2 + 1}\right) \text{ for } x \in \mathbb{R}$$

$$\operatorname{arcosh} x = \log\left(x + \sqrt{x^2 - 1}\right) \text{ for } x \geq 1$$

$$\operatorname{artanh} x = \frac{1}{2}\log\frac{1 + x}{1 - x} \text{ for } |x| < 1.$$

Exercises

1. How does the graph of an arbitrary function $y = f(x): \mathbb{R} \to \mathbb{R}$ change under the transformations

$$y = f(a\,x), \qquad y = f(x - b), \qquad y = c\,f(x), \qquad y = f(x) + d$$

with $a, b, c, d \in \mathbb{R}$? Distinguish the following different cases for a

$$a < -1 ., \qquad -1 \leq a < 0, \quad 0 < a \leq 1, \quad a \geq 1,$$

and for b, c, d the cases

$$b, c, d > 0, \quad b, c, d < 0$$

Sketch the resulting graphs.

2. Using the graph of the function $f: D \to \mathbb{R}: x \mapsto 3\,x^4 - 2\,x^3 - 3\,x^2 + 1$

$$D = [-1{,}1.5|, \quad D = [-0{,}5{,}0.5], \quad D = [0.5{,}1.5]$$

To explain the behaviour of the function for $d = \mathbb{R}$ and find

$$f([-1,1.5]),\ \ f\big((-.5,.5)\big),\ \ f((-\infty,1])\,.$$

3. Which of the following functions are injective/surjective/bijective?

$$\begin{aligned} f &: \mathbb{N} \to \mathbb{N}: & n &\mapsto n^2 - 6n + 10; \\ g &: \mathbb{R} \to \mathbb{R}: & x &\mapsto |x+1| - 3; \\ h &: \mathbb{R} \to \mathbb{R}: & x &\mapsto x^3. \end{aligned}$$

4. Sketch the graph of the function $y = x^2 - 4x$ and justify why it is bijective as a function from $D = (-\infty, 2]$ to $B = [-4, \infty)$. Compute its inverse function on the given domain.

5. Check that the following functions $D \to B$ are bijective in the given regions and compute the inverse function in each case:

$$\begin{aligned} y &= -2x + 3, & D &= \mathbb{R},\ B = \mathbb{R}; \\ y &= x^2 + 1, & D &= (-\infty, 0]\,,\ B = [1, \infty)\,; \\ y &= x^2 - 2x - 1, & D &= [1, \infty)\,,\ B = [-2, \infty)\,. \end{aligned}$$

6. Find the equation of the straight line through the points (1, 1) and (4, 3) as well as the equation of the quadratic parabola through the points (−1, 6), (0, 5) and (2, 21).

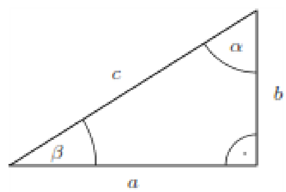
7.	Draw the graph of the function $f : \mathbb{R} \to \mathbb{R} : y = ax + \operatorname{sign} x$ for different values of a. Distinguish between the cases $a > 0$, $a = 0$, $a < 0$. For which values of a is the function f injective and surjective, respectively?
8.	Let $a > 0$, $b > 0$. Verify the *laws of exponents* $$a^r a^s = a^{r+s}, \qquad (a^r)^s = a^{rs}, \qquad a^r b^r = (ab)^r$$ for rational $r = k/l$, $s = m/n$.
9.	Using the arithmetics of exponentiation, verify the rules $\log(uv) = \log u + \log v$ and $\log u^z = z \log u$ for $u, v > 0$ and $z \in \mathbb{R}$.
10.	Verify the identity $\cosh^2 x - \sinh^2 x = 1$.

11.	Show that $\operatorname{arsinh} x = \log\left(x + \sqrt{x^2 + 1}\right)$ for $x \in \mathbb{R}$.

Trigonometry

Trigonometric Functions at the Triangle

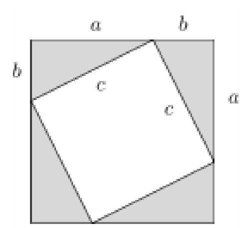
The definitions of the trigonometric functions are based on elementary properties of the right-angled triangle. The following figure shows a right-angled triangle. The sides adjacent to the right angle are called *legs*, the opposite side *hypotenuse*.



One of the basic properties of the right-angled triangle is expressed by Pythagoras' theorem.

Proposition (*Pythagoras*) In a right-angled triangle the sum of the squares of the legs equals the square of the hypotenuse $a^2 + b^2 = c^2$.

Proof. According to the figure



one can easily see that

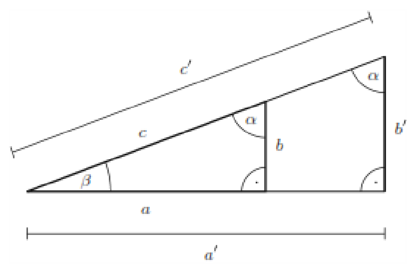
$$(a + b)^2 - c^2 = \text{area of the grey triangles} = 2ab$$

From this it follows that $a^2 + b^2 - c^2 = 0$.

A fundamental fact in Thales' intercept theorem which says that the ratios of the sides in a triangle are scale invariant: they do not depened on the size of the triangle.

Thales' theorem. the following ratios are valid:

$$\frac{a}{c} = \frac{a'}{c'}, \qquad \frac{b}{c} = \frac{b'}{c'}, \qquad \frac{a}{b} = \frac{a'}{b'}$$



The reason for this is that by changing the scale (enlargement or reduction) of the triangle all sides are changed by the same factor.

Definition. (*Trigonometric functions*) For $0^o \leq \alpha \leq 90^o$

$$\sin\alpha = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}} \qquad \text{(sine)},$$

$$\cos\alpha = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}} \qquad \text{(cosine)},$$

$$\tan\alpha = \frac{b}{c} = \frac{\text{opposite leg}}{\text{adjacent leg}} \qquad \text{(tangent)},$$

$$\cot\alpha = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{opposite leg}} \qquad \text{(cotangent)}.$$

Note that $\tan\alpha$ is not defined for $\alpha = 90^o$(since $= 0$) and that $\cot\alpha$ is not defined for $\alpha = 0^o$ (since $a = 0$). The following identities hold true

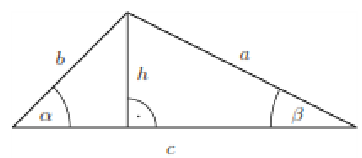
$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \cot\alpha = \frac{\cos\alpha}{\sin\alpha}, \quad \sin\alpha = \cos(90^o - \alpha)$$

Using Pythagoras' theorem the identity can be obtained

$$\sin^2\alpha + \cos^2\alpha = 1.$$

The trigonometric functions have many applications in mathematics. We mention some of these applications.

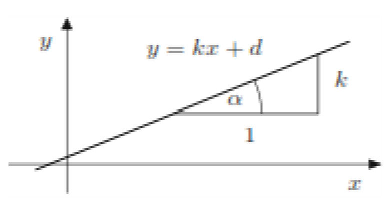
1. *The formula for the area of a general triangle*;



The area of the triangle in the figure is given by

$$A = \frac{1}{2}ch = \frac{1}{2}bc\sin\alpha = \frac{1}{2}ac\sin\beta = \frac{1}{2}ac\sin\gamma$$

2. *The slope of a straight line.*



For the straight line $y = kx + d$ the slope k is the change of the y-value per unit change in x. It is calculated from the triangle attached to the straight line in above figure as $k = \tan\alpha$.

Extension of the Trigonometric Functions to $\mathbb{R}$

The radian measure of the angle α (in degrees) is defined as the length ℓ of the corresponding arc of the unit circle with the sign of α.

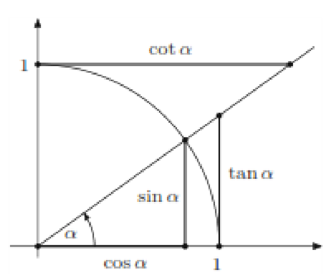
It is generally known the circumference of the unit circle is 2π with the constant

$$\pi = 3.141592653589793\cdots \cong \frac{22}{7}$$

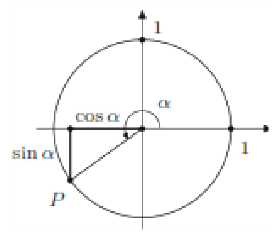
For the conversion between the two measures we use that 360^o corresponds to 2π in radian measure, for short, $360^o \leftrightarrow 2\pi[\text{rad}]$. So,

$$\alpha^o \leftrightarrow \frac{\pi}{180}\alpha[\text{rad}], \qquad \ell[\text{rad}] \leftrightarrow \left(\frac{180}{pi}\right)^o$$

• For $0 \le \alpha \le \frac{\pi}{2}$ the values $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ and $\cot\alpha$ have a simple interpretation on the unit circle; see the following figure



• One can extend the definition of the trigonometric functions for $0 \leq \alpha \leq 2\pi$ by continuation with the help of the unit circle. A general point P on the unit circle, which is defined by the angle α, is assigned the coordinates $P = (\cos\alpha, \sin\alpha)$



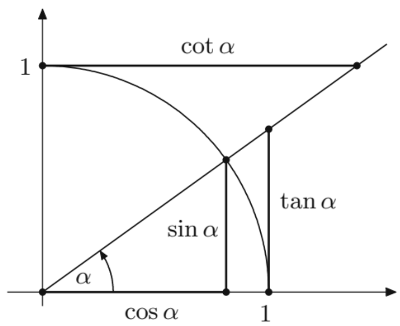
For $0 \leq \alpha \leq \frac{\pi}{2}$ this is compatible with the earlier definition. For larger angles the *sine* and *cosine* functions are extended to the interval $[0, 2\pi]$ by this convention. For example, it follows from the above that

$$\sin\alpha = \sin\left(\alpha - \frac{\pi}{2}\right), \qquad \cos\alpha = -\cos\left(\alpha - \frac{\pi}{2}\right) \text{ if } \frac{\pi}{2} \leq \alpha \leq \pi,$$

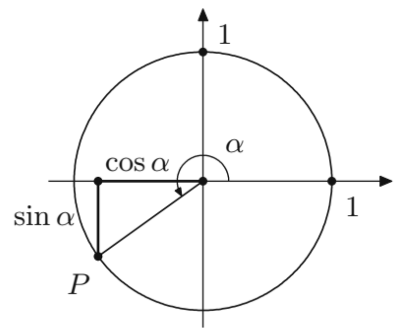
$$\sin\alpha = -\sin(\alpha - \pi), \qquad \cos\alpha = -\cos(\alpha - \pi) \text{ if } \pi \leq \alpha \leq 2\pi$$

• For arbitrary values $\alpha \in \mathbb{R}$ one finally defines $\sin\alpha$ and $\cos\alpha$ by periodic continuation with period 2π. For this purpose one first writes $\alpha = x + 2k\pi$ with a unique $x \in [0, 2\pi)$ and $k \in \mathbb{Z}$. Then one sets

$$\sin\alpha = \sin(x + 2k\pi) = \sin x, \cos\alpha = \cos(x + 2k\pi) = \cos x.$$

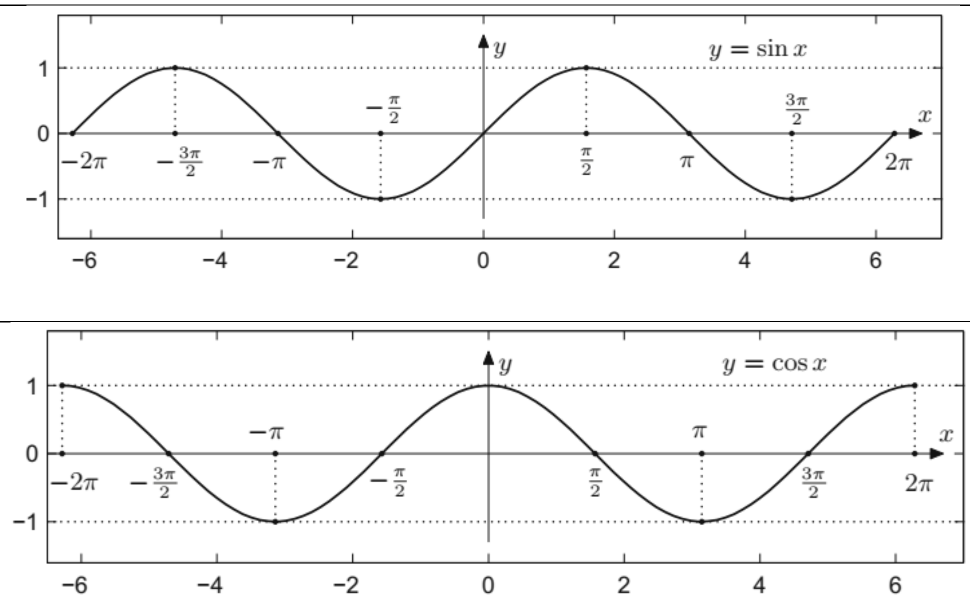


Definition of the trigonometric functions on the unit circle



Extension of the trigonometric functions on the unit circle

The domain of the functions $y = \sin x$, $y = \cos x$ is $D = \mathbb{R}$. Their graphs are



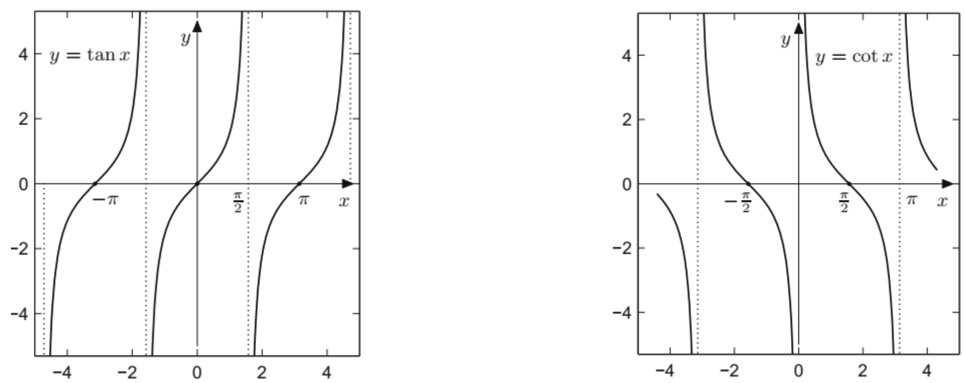
The graphs of the sine and cosine functions in the interval $[-2\pi, 2\pi]$

With the help of the formulas

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}$$

the tangent and cotangent functions are extended as well. Since the sine function equals zero for integer multiples of π the cotangent is not defined for such arguments. Likewise the tangent is not defined for odd multiples of $\frac{\pi}{2}$.

The domain of $y = \tan x$ is $D = \left\{x \in \mathbb{R}\,;\ x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$, the domain of $y = \cot x$ is $D = \{x \in \mathbb{R}\,;\ x \neq k\pi,\ k \in \mathbb{Z}\}$. The graphs of these functions are



The graphs of the tangent (left) and cotangent (right) functions

Many relations are valid between the trigonometric functions. For example, the following addition theorems, which can be proven by elementary geometrical considerations, are valid.

Proposition (Addition theorems) For $x, y \in \mathbb{R}$ it holds that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y\,,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

Cyclometric Functions

The cyclometric functions are inverse to the trigonometric functions in the appropriate bijectivity regions.

Sine and arcsine. The sine function is bijective from the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ to the range $[-1,1]$. This part of the graph is called principal branch of the sine. Its inverse function is called *arcsine* (or sometimes *inverse sine*)

$$\arcsin\ :[-1,1]\rightarrow\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

According to the definition of the inverse function it follows that

$$\sin(\arcsin y)=y\ \text{ for all }\ y\in[-1,1]\,.$$

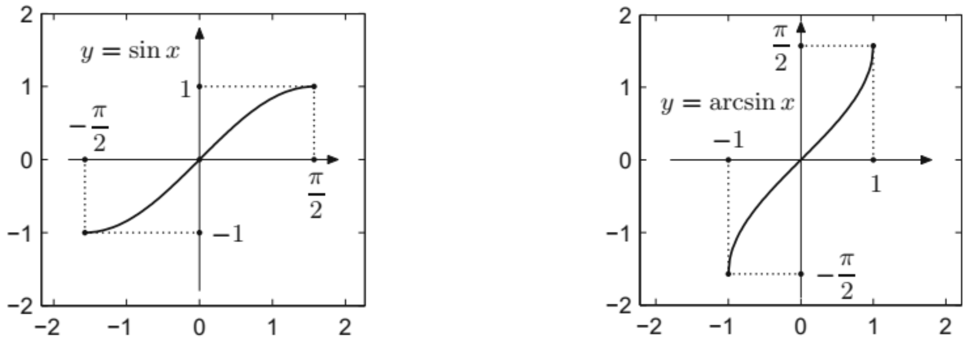
However, the converse formula is only valid for the principal branch; i.e.

$$\arcsin(\sin x)=x\ \text{ is only valid for }\ -\frac{\pi}{2}\leq\ x\leq\frac{\pi}{2}\,.$$

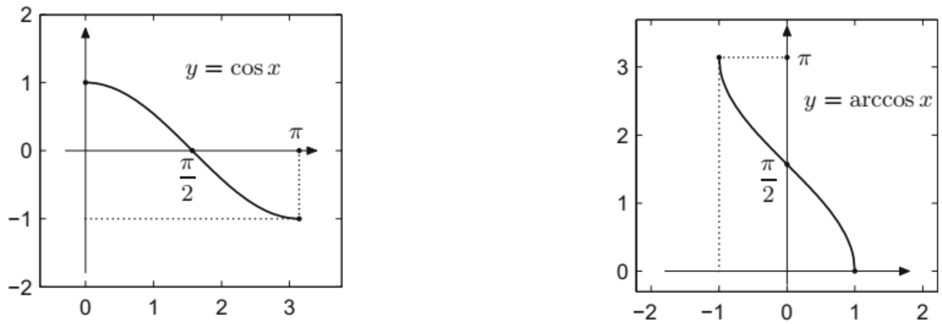
For example, $\arcsin(\sin 4)\ =-0.8584073\cdots\neq 4$.

Cosine and arccosine. Likewise, the cosine function is bijective from the interval $[0,\pi]$ to the range $[-1,1]$. Therefore, the principal branch of the cosine is defined as restriction of the cosine to the interval $[0,\pi]$ with range $[-1,1]$. The inverse function is called *arccosine* (or sometimes *inverse cosine*)

$$\arccos\ :[-1,1]\rightarrow[0,\pi]\,.$$

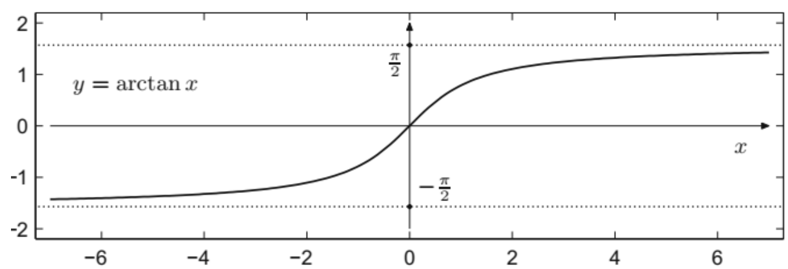


The principal branch of the sine (left); the arcsine function (right)



The principal branch of the cosine (left); the arccosine function (right)

Tangent and arctangent. As can be seen



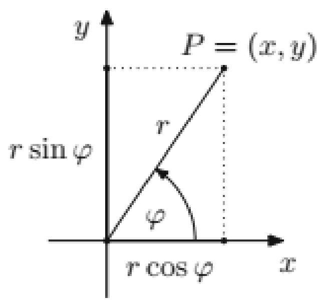
The principal branch of the arctangent

the restriction of the tangent to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is bijective. Its inverse function is called arctangent (or inverse tangent)

$$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) .$$

To be precise this is again the principal branch of the inverse tangent.

Application (Polar coordinates in the plane) The polar coordinates (r, φ) of a point $P = (x, y)$ in the plane are obtained by prescribing its distance r from the origin and the angle φ with the positive x-axis (in counterclockwise direction); see the following figure



The relation between Cartesian and polar coordinates is therefore described by

$$x = r\cos\varphi \ , \ \ y = r \ \sin\varphi,$$

where $0 \leq \varphi < 2\,\pi$ and $r \geq 0$. The range $-\pi < \varphi \leq \backslash pi$ is also often used.

The following conversion formulas are valid

$$r = \sqrt{x^2 + y^2}, \ \ \varphi = \arctan\frac{y}{x}, \text{if} \ \ x > 0\,; -\frac{\pi}{2} < \varphi < \frac{\pi}{2},$$

$$\varphi = \operatorname{sign} y \cdot \arccos\frac{x}{\sqrt{x^2 + y^2}}, \qquad \text{if} \ \ y \neq \ 0 \ \ \text{or} \ \ x > 0\,; -\pi < \varphi < \pi.$$

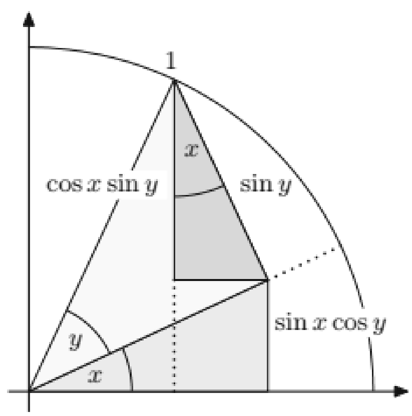
Exercises

1. Using geometric considerations at suitable right-angled triangles, determine the values of the sine, cosine and tangent of the angles $\alpha = 45°$, $\beta = 60°$, $\gamma = 30°$. Extend your result for $\alpha = 45°$ to the angles $135°$, $225°$, $-45°$ with the help of the unit circle. What are the values of the angles under consideration in radian measure?

2. Prove the addition theorem of the sine function

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

Hint. If the angles x, y and their sum $x + y$ are between 0 and $\frac{\pi}{2}$ you can directly argue with the help of figure

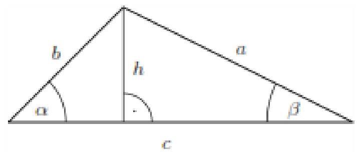


The remaining cases can be reduced to this case.

3. Prove the law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

Hint. Consider the triangle



The segment c is divided into two segments c_1 and c_2, to get two right triangles with common height h. Then applying the Pythagoras' theorem obtains the following identities

$$a^2 = h^2 + c_2^2, \quad b^2 = h^2 + c_1^2, \quad c = c_1 + c_2 \, .$$

Eliminating h gives $a^2 = b^2 + c^2 - 2\,c\,c_1$.

4. Compute the angles $\alpha,\ \beta,\ \gamma$ of the triangle with sides $a = 3, b = 4, c = 2$.

Hint. Use the law of cosines from Exercise 3.

5. Prove the law of sines

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

for the general triangle.

Hint. The first identity follows from $\sin\alpha = \frac{h}{b}, \sin\beta = \frac{h}{a}$.

6. Compute the missing sides and angles of the triangle with data $b = 5, \alpha = 43^\circ,\ \gamma = 62^\circ$.

Hint. Use the law of sines from Exercise 5.

7. The *secant* and *cosecant* functions are defined as the reciprocals of the cosine and the sine functions, respectively,

$$\sec\alpha = \frac{1}{\cos\alpha}\,, \quad \csc\alpha = \frac{1}{\sin\alpha}\,.$$

Due to the zeros of the cosine and the sine function, the secant is not defined for odd multiples of $\frac{\pi}{2}$, and the cosecant is not defined for integer multiples of $\backslash pi$.

(a) Prove the identities $1 + \tan^2\alpha = \sec^2\alpha$ and $1 + \cot^2\alpha = \csc^2\alpha$.

(b) Plot the graph of the functions $y = \sec x$ and $y = \csc x$ for x between $-2\,\pi$ and $2\,\pi$.

Limits and Continuity of Functions

In this section we introduce the notion of the limit of a function. Limits of functions form the basis of central themes in mathematical analysis, namely continuity and differentiation. It shows the behaviour of graphs of real functions

$$f:(a,b) \to \mathbb{R}$$

while approaching a point x in the open interval (a,b) or a boundary point of the closed interval $[a,b]$.

Definition 1. (Notion of the limit)

(a) The function f has a limit M at a point $x \in (a,b)$, if

$$\lim_{h\to 0} f(x+h) = M$$

or

$$f(x+h) \to M \text{ as } h \to 0.$$

(b) The function f has a right-hand limit R at the point $x \in [a,b)$, if

$$\lim_{h\to 0+} f(x+h) = R$$

(c) The function f has a left-hand limit L at the point $x \in (a,b]$, if

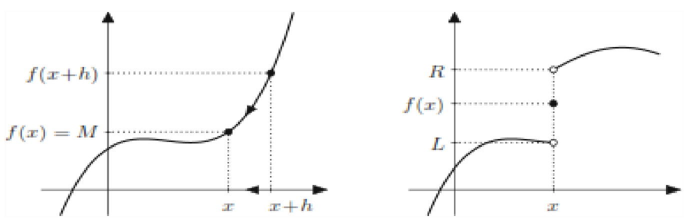
$$L = \lim_{h\to 0-} f(x+h) = \lim_{\xi\to x-} f(\xi)\,.$$

Definition 2. (the continuity)

(a) If f has a limit M at $x \in (a, b)$ which coincides with the value of the function, i.e. $f(x) = M$, then f is called continuous at the point x.

(b) If f is continuous at every $x \in (a,b)$, then f is said to be continuous on the open interval (a,b). If in addition f has right- and left-hand limits at the endpoints a and b, it is called continuous on the closed interval $[a, b]$.

The following figures illustrate the idea of approaching a point x for $h \to 0$ as well as possible differences between left-hand and right-hand limits and the value of the function.



If a function f is continuous at a point x, the function evaluation can be interchanged with the limit:

$$\lim_{\xi \to x} f(\xi) = f(x) = f\left(\lim_{\xi \to x} \xi\right).$$

Example 1. The quadratic function $f(x) = x^2$ is continuous at every $x \in \mathbb{R}$ since

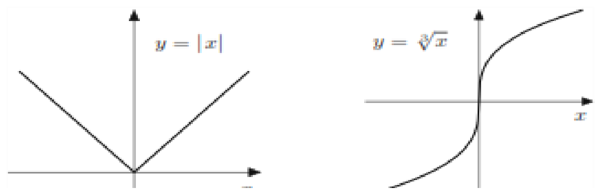
$$f(x+h) - f(x) = (x+h)^2 - x^2 = 2xh + h^2 \to 0$$

as $h \to \infty$ for any zero sequence $(h_n)_{n \geq 1}$. Therefore

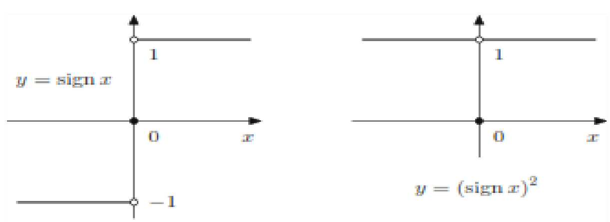
$$\lim_{h\to 0} f(x+h) = f(x).$$

Likewise the continuity of the power functions $x \mapsto x^m$ for $m \in \mathbb{N}$ can be shown.

Example 2. The absolute value function $f(x) = |x|$ and the third root $g(x) = \sqrt[3]{x}$ are everywhere continuous. The former has a kink at $x = 0$, the latter a vertical tangent; see the figure



Example 3. The sign function $f(x) = \text{sign}\, x$ has different left- and right-hand limits $L = -1, R = 1$ at $x = 0$. In particular, it is discontinuous at that point. At all other points $x \neq 0$ it is continuous; see



Example 4. The square of the sign function

$$g(x) = (\text{sign}\, x)^2 = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

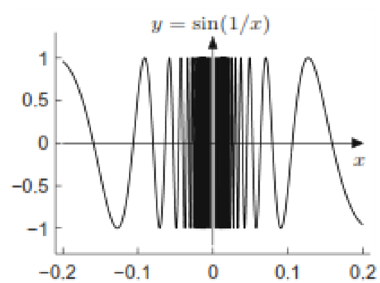
has equal left- and right-hand limits at $x = 0$. However, they are different from the value of the function (see the last figure):

$$\lim_{\xi \to 0} g(\xi) = 1 \neq 0 = g(0).$$

Therefore, g is discontinuous at $x = 0$.

Example 5. The functions $f(x) = \frac{1}{x}$ and $g(x) = \tan x$ have vertical asymptotes at $x = 0$ and $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$, respectively, and in particular no left- or right-hand limit at these points. At all other points, however, they are continuous.

Example 6. The function $f(x) = \sin\frac{1}{x}$ has no left- or right-hand limit at $x = 0$ but oscillates with non-vanishing amplitude; see the following figure



Indeed, one obtains different limits for different zero sequences. For example, for

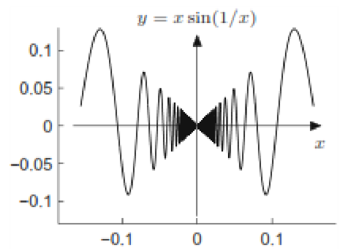
$$h_n = \frac{1}{n\pi}, \quad k_n = \frac{1}{\pi/2 + 2n\pi}, \quad l_n = \frac{1}{3\pi/2 + 2n\pi}$$

the respective limits are

$$\lim_{n \to \infty} f(h_n) = 0, \quad \lim_{n \to \infty} f(k_n) = 1, \quad \lim_{n \to \infty} f(l_n) = -1.$$

All other values in the interval [-1,1] can also be obtained as limits with the help of suitable zero sequences.

Example 7. The function $g(x) = x\sin\frac{1}{x}$ can be continuously extended by $g(0) = 0$ at $x = 0$; it oscillates with vanishing amplitude



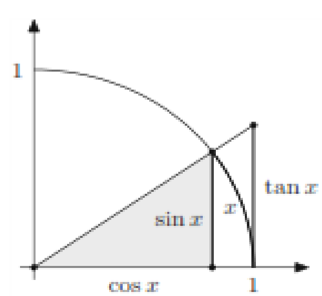
Indeed,

$$\left|g\left(h_n\right) - g(0)\right| = \left|h_n \sin\frac{1}{h_n} - 0\right| \le \left|h_n\right| \to 0$$

for all zero sequences $\left(h_n\right)_{n\ge1}$, thus $\lim_{h\to0} h\sin\frac{1}{h} = 0$.

Trigonometric Limits

Comparing the areas in the following figure shows that the area of the grey triangle



with sides $\cos x$ and $\sin x$ is smaller than the area of the sector which in tum is smaller or equal to the area of the big triangle with sides 1 and $\tan x$.

The area of a sector in the unit circle (with angle x in radian measure) equals $x/2$ as is well-known. In summary we obtain the inequalities

$$\frac{1}{2}\sin x\cos x \le \frac{x}{2} \le \frac{1}{2}\tan x$$

or after division by $\sin x$ and taking the reciprocal

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x},$$

valid for all x with $0<|x|<\pi/2$.

With the help of these inequalities we can compute several important limits. From an elementary geometric consideration, one obtains

$$|\cos x| \ge \frac{1}{2} \quad \text{for} \quad -\frac{\pi}{3} \le x \le \frac{\pi}{3},$$

and together with the previous inequalities

$$|\sin h| \le \frac{|h|}{|\cos h|} \le 2|h| \to 0 \text{ as } h \to 0.$$

This means that $\lim_{h\to 0}\sin h = 0.$

The sine function is therefore continuous at zero. From the continuity of the square function and the root function as well as the fact that $\cos h$ equals the positive square root of $1-\sin^2 h$ for small h it follows that

$$\lim_{h\to 0}\cos h = \lim_{h\to 0}\sqrt{1-\sin^2 h} = 1.$$

With this the continuity of the sine function at every point $x\in\mathbb{R}$ can be proven:

$$\lim_{h\to 0}\sin(x+h) = \lim_{h\to 0}(\sin x\cos h + \cos x\sin h) = \sin x.$$

The inequality illustrated at the beginning of the section allows one to deduce one of the most important trigonometric limits. It forms the basis of the differentiation rules for trigonometric functions.

Proposition. $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

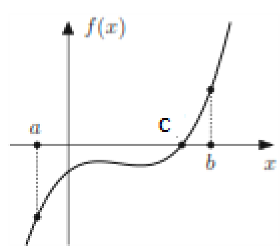
Proof. We combine the above result $\lim_{x\to 0}\cos x = 1$ with the inequality deduced earlier and obtain

$$1 = \lim_{x\to 0}\cos x \le \lim_{x\to 0}\frac{\sin x}{x} \le \lim_{x\to 0}\frac{1}{\cos x} = 1,$$

and therefore $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Zeros of Continuous Functions

The following figure



shows the graph of a function that is continuous on a closed interval $[a, b]$ and that is negative at the left endpoint and positive at the right endpoint. Geometrically the graph has to intersect the x-axis at least once since it has no jumps due to the continuity. This means that f has to have at least one zero in (a,b). This is a criterion that guarantees the existences of a solution to the equation $f(x)=0$. A first rigorous proof of this intuitively evident statement goes back to Bolzano.

Proposition.(*Intermediate value theorem*) $f:[a,b]\to\mathbb{R}$ be continuous and $f(a)<0, f(b)>0$. Then there exists a point $c\in(a,b)$ with $f(c)=0$.

Example 8. Calculation of $\sqrt{2}$ as the root of $f(x)=x^2-2=0$ in the interval [1,2] using the bisection method:

Start:	$f(1)=-1<0,\ f(2)=2>0;$	$a_1=1,\ b_1=2$
Step 1:	$f(1.5)=0.25>0;$	$a_2=1,\ b_2=1.5$
Step 2:	$f(1.25)=-0.4375<0;$	$a_3=1.25,\ b_3=1.5$
Step 3:	$f(1.375)=-0.109375<0;$	$a_4=1.375,\ b_4=1.5$
Step 4:	$f(1.4375)=0.066406\ldots>0;$	$a_5=1.375,\ b_5=1.4375$
Step 5:	$f(1.40625)=-0.022461\ldots<0;$	$a_6=1.40625,\ b_6=1.4375$
etc.		

After 5 steps the first decimal place is ascertained:

$$1.40625<\sqrt{2}<1.4375$$

Experiment. Sketch the graph of the function $y=x^3+3x^2-2$ on the interval $[-3,2]$, and try to first estimate graphically one of the roots by successive bisection. Execute the interval bisection with the help of the applet Bisection method. Assure yourself of the plausibility of the intermediate value theorem using the applet Animation of the intermediate value theorem.

As an important application of the intermediate value theorem we now show that images of intervals under continuous functions are again intervals.

Proposition. Let $I \subset \mathbb{R}$ be an interval (open, half-open or closed, bounded or improper) and $f : I \to \mathbb{R}$ a continuous function with proper range $J = f(I)$. Then J is also an interval.

Proposition. Let $I = [a, b]$ be a closed, bounded interval and $f : I \to \mathbb{R}$ a continuous function. Then the proper range $J = f(I)$ is also a closed, bounded

interval.

Corollary. Each continuous function defined on a closed interval $I = [a, b]$ attains its maximum and minimum there.

Exercises

1. (a) Investigate the behaviour of the functions

$$\frac{x + x^2}{|x|}, \quad \frac{\sqrt{1+x} - 1}{x}, \quad \frac{x^2 + \sin x}{\sqrt{1 - \cos^2 x}}$$

in a neighbourhood of $x=0$ by plotting their graphs for arguments in

$\left[-2,-\frac{1}{100}\right)\cup\left(\frac{1}{100},2\right]$.

(b) Find out by inspection of the graphs whether there are left- or right-hand limits at $x=0$. Which value do they have? Explain your results by rearranging the expressions in **(a)**.

2. Do the following functions have a limit at the given points? If so, what is its value?

(a) $y=x^3+5x+10, x=1$.

(b) $y=\frac{x^2-1}{x^2+x}, x=0, x=1, x=-1$.

(c) $y=\frac{1-\cos x}{x^2}, x=0$. Hint. Expand with $(1+\cos x)$.

(d) $y=\operatorname{sign} x\cdot\sin x, x=0$,

(e) $y=\operatorname{sign} x\cdot\cos x, x=0$

3. Argue with the help of the intermediate value theorem that $p(x)=x^3+5x+10$ has a zero in the interval $[-2,1]$. Compute this zero up to four decimal places using the applet Bisection method.

4. Compute all zeros of the following functions in the given interval with accuracy 10^{-3}, using the applet Bisection method.

$$f(x)=x^4-2, \qquad I=\mathbb{R}$$
$$g(x)=x-\cos x, \qquad I=\mathbb{R};$$
$$h(x)=\sin\frac{1}{x}, \qquad I=\left[\frac{1}{20},\frac{1}{10}\right].$$

The Derivative of a Function

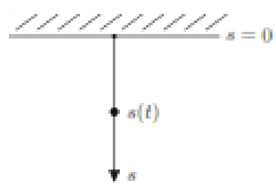
Starting from the problem to define the tangent to the graph of a function, we introduce the derivative of a function. Two points on the graph can always be joined by a secant, which is a good model for the tangent whenever these points are close to each other. In a limiting process, the secant (discrete model) is replaced by the tangent (continuous model). Differential calculus, which is based on this limiting process, has become one of the most important beilding blocks of mathematical modelling.

In this section we discuss the derivative of important elementary functions as well as general differentiation rules. Thanks to the meticulous implementation of these rules, expert systems such as maple have become helpful tools in mathematical analysis. Furthermore, we will discuss the interpretation of the derivative as linear approximation and as rate of change. These interpretations form the basis of numerous applications in science and engineering.

The concept of the numerical derivative follows the opposite direction. The continuous model is discretised, and the derivative is replaced by a difference quoticnt. We carry out a detailed error analysis which allows us to find an optimal approximation. Further, we will illustrate the relevance of symmetry in numerical procedures.

Motivation

Example 1. (*The free fall according to Galilei'*) Imagine an object, which released at time $t = 0$, falls down under the influence of gravity. We are interested in the position $s(t)$ of the object at time $t \geq 0$ as well as in its velocity $v(t)$, see the following figure.



Due to the definition of velocity as change in travelled distance divided by change

in time, the object has the average velocity

$$v_{\text{average}} = \frac{s(t+\Delta t) - s(t)}{\Delta t}$$

in the time interval $[t, t+\Delta t]$. In order to obtain the instantaneous velocity $v = v(t)$ we take the limit $\Delta t \to 0$ in the above formula and arrive at

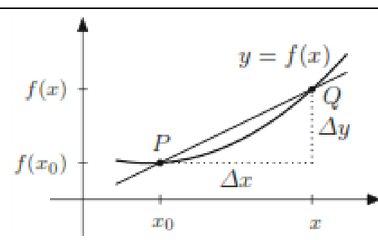
$$v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}.$$

Galilei discovered through his experiments that the travelled distance in free fall increases quadratically with the time passed, i.e. the law $s(t) = \frac{g}{2} t^2$ with $g \approx 9.81 \mathrm{m/s^2}$ holds. Thus we obtain the expression

$$v(t) = \lim_{\Delta t \to 0} \frac{\frac{g}{2}(t+\Delta t)^2 - \frac{g}{2}t^2}{\Delta t} = \frac{g}{2} \lim_{\Delta t \to 0} (2t + \Delta t) = gt$$

for the instantaneous velocity. The velocity is hence proportional to the time passed.

Example 2. (*The tangent problem***) Consider a real function** f and two different points $P = (x_0, f(x_0))$ and $Q = (x, f(x))$ on the graph of the function. The uniquely defined straight line through these two points is called secant of the function f through P and Q, see Fig. 7.2.

The slope of the secant is given by the difference quotient $$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}.$$	

As x tends to x_0 - the secant graphically turns into the tangent, provided the limit exists. Motivated by this idea we define the slope

$$k = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

of the function f at x_0. If this limit exists, we call the straight line

$$y = k - (x - x_0) + f(x_0)$$

the tangent to the graph of the function at the point $(x_0, f(x_0))$.

The Derivative

Motivated by the above applications we are going to define the derivative of a real valued function.

Definition 1. (*Derivative*) Let $I \subset \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a real valued function and $x_0 \in I$.

(a) The function f is called differentiable at x_0 if the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

has a (finite) limit for $x \to x_0$. In this case one writes

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and calls the limit derivative of f at the point x_0.

(b) The function f is called differentiable (in the interval I) if $f'(x)$ exists for all $x \in I$. In this case the function $f' : I \to \mathbb{R} : x \mapsto f'(x)$ is called the derivative of f. The process of computing f' from f is called differentiation.

In place of $f'(x)$ one often writes $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ or $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$, respectively. The following examples show how the derivative of a function is obtained by means of the limiting process above.

Example 3. (The constant function $f(x)=c$)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=0.$$

The derivative of a constant function is zero.

Example 4. (The affine function $g(x)=ax+b)$)

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{ax+ah+b-ax-b}{h}=\lim_{h\to 0}a=a.$$

The derivative is the slope a of the straight line $y=ax+b$.

Example 5. (The derivative of the quadratic function $y=x^2$)

$$y'=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{2hx+h^2}{h}=\lim_{h\to 0}(2x+h)=2x.$$

Similarly, one can show for the power function (with $n\in\mathbb{N}$)

$$f(x)=x^n \quad\Rightarrow\quad f'(x)=n\cdot x^{n-1}.$$

Example 6. (The derivative of the square root function $y=\sqrt{x}$ for $x>0$)

$$y' = \lim_{\zeta \to x} \frac{\sqrt{\xi} - \sqrt{x}}{\xi - x} = \lim_{\zeta \to x} \frac{\sqrt{\xi} - \sqrt{x}}{(\sqrt{\xi} - \sqrt{x})(\sqrt{\zeta} + \sqrt{x})} = \lim_{\zeta \to x} \frac{1}{\sqrt{\xi} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Example7. (Derivatives of the sine and cosine functions) We first recall that

$$\lim_{t \to 0} \frac{\sin t}{t} = 1$$

Due to

$$(\cos t - 1)(\cos t + 1) = -\sin^2 t$$

it also holds that

$$\frac{\cos t - 1}{t} = -\underbrace{\sin t}_{\to 0} \cdot \underbrace{\frac{\sin t}{t}}_{\to 1} \cdot \frac{1}{\underbrace{\cos t + 1}_{\to 1/2}} \to 0 \quad \text{for } t \to 0,$$

and thus

$$\lim_{t \to 0} \frac{\cos t - 1}{t} = 0.$$

Hence, we find

$$\begin{aligned} \sin' x &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \cdot \frac{\sin h}{h} \\ &= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{=0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{=0} = \cos x. \end{aligned}$$

This proves that $\sin' x = \cos x$. Likewise it can be shown that $\cos' x = -\sin x$.

Example 8. (The derivative of the exponential function with base e)

Rearranging terms in the series expansion of the exponential function (Proposition C.12) we obtain

$$\frac{\mathrm{e}^h - 1}{h} = \sum_{k=0}^{\infty} \frac{h^k}{(k+1)!} = 1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24} + \cdots$$

From that one infers

$$\left| \frac{\mathrm{e}^h - 1}{h} - 1 \right| \le |h| \left(\frac{1}{2} + \frac{|h|}{6} + \frac{|h|^3}{24} + \cdots \right) \le |h|\,\mathrm{e}^{|h|}.$$

Letting $h \to 0$ hence gives the important limit

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

The existence of the limit

$$\lim_{h \to 0} \frac{\mathrm{e}^{x+h} - \mathrm{e}^x}{h} = \mathrm{e}^x \cdot \lim_{h \to 0} \frac{\mathrm{e}^h - 1}{h} = \mathrm{e}^x$$

shows that the exponential function is differentiable and that $\left(\mathrm{c}^x\right)' = \mathrm{e}^x$.

Example 9. (New representation of Euler's number) By substituting $y = \mathrm{e}^h - 1$, $h = \log(y+1)$ in the above limit one obtains

$$\lim_{y \to 0} \frac{y}{\log(y+1)} = 1$$

and in this way

$$\lim_{y\to 0}\log(1+\alpha y)^{1/y} = \lim_{y\to 0}\frac{\log(1+\alpha y)}{y} = \alpha \lim_{y\to 0}\frac{\log(1+\alpha y)}{\alpha y} = \alpha.$$

Due to the continuity of the exponential function it further follows that

$$\lim_{y\to 0}(1+\alpha y)^{1/y} = \mathrm{e}^{\alpha}.$$

In particular, for $y = 1/n$, we obtain a new representation of the exponential function

$$\mathrm{c}^{a} = \lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^{n}.$$

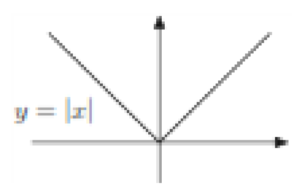
For $\alpha = 1$ the identity

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \sum_{k=0}^{\infty}\frac{1}{k!} = 2.718281828459\ldots$$

follows.

Example 10. Not every continuous function is differentiable. For instance,

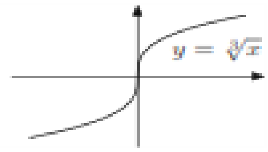
$$f(x) = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x \le 0 \end{cases}$$



is not differentiable at the vertex $x = 0$. However, it is differentiable for $x = 0$. With

$$(|x|)' = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0. \end{cases}$$

The function $g(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$ either. The reason for that is the vertical tangent.



Definition 1. If the function f' is again differentiable then

$$f''(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

is called the second derivative of f with respect to x. Likewise higher derivatives are defined recursively as

$$f'''(x) = \left(f''(x)\right)' \text{ or } \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)\right), \text{ etc.}$$

Interpretations of the Derivative

We introduced the derivative geometrically as *the slope of the tangent*, and we saw that the tangent to a graph of a differentiable function f at the point $\left(x_0, f(x_0)\right)$ is given by $y = f'(x_0)(x - x_0) + f(x_0)$.

Example 11. Let $f(x) = x^4 + 1$ with derivative $f'(x) = 4x^3$.

(i) The tangent to the graph of f at the point $(0,1)$ is

$$y = f'(0) \cdot (x - 0) + f(0) = 1$$

and thus horizontal.

(ii) The tangent to the graph of f at the point $(1,2)$ is

$$y = f'(1)(x-1)+2 = 4(x-1)+2 = 4x-2.$$

The derivative allows further interpretations.

Interpretation as linear approximation. We start off by emphasizing that every differentiable function f can be written in the form

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + R(x,x_0),$$

where the remainder $R(x,x_0)$ has the property

$$\lim_{x\to x_0} \frac{R(x,x_0)}{x-x_0} = 0.$$

This follows immediately from $R(x,x_0) = f(x) - f(x_0) - f'(x_0)(x-x_0)$ by dividing by $x-x_0$, since

$$\frac{f(x)-f(x_0)}{x-x_0} \to f'(x_0) \quad \text{as } x \to x_0.$$

Application. As we have just seen, a differentiable function f is characterised by the property that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + R(x,x_0),$$

where the remainder term $R(x,x_0)$ tends faster to zero than $x-x_0$. Taking the limit $x \to x_0$ in this equation shows in particular that every differentiable function is continuous.

Application. Let g be the function given by $g(x) = k\cdot(x-x_0) + f(x_0)$.

Its graph is the straight line with slope k passing through the point $\left(x_0, f\left(x_0\right)\right)$.

Since

$$\frac{f(x)-g(x)}{x-x_0}=\frac{f(x)-f\left(x_0\right)-k\cdot\left(x-x_0\right)}{x-x_0}=f'\left(x_0\right)-k+\underbrace{\frac{R\left(x,x_0\right)}{x-x_0}}_{\to 0}$$

as $x \to x0$, the tangent with $k = f'(x_0)$ is the straight line which approximates the graph best. One therefore calls

$g(x)=f\left(x_0\right)+f'\left(x_0\right)\cdot\left(x-x_0\right)$

the linear approximation to f at x. For x close to x_0 one can consider $g(x)$ as a good approximation to $f(x)$. In applications the (possibly complicated) function f is often replaced by its linear approximation g which is easier to handle.

Example 12. Let $f(x)=\sqrt{x}=x^{1/2}$. Consequently,

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}.$$

We want to find the linear approximation to the function f at $x_0 = a$.

According to the formula above it holds that

$$\sqrt{x}\approx g(x)=\sqrt{a}+\frac{1}{2\sqrt{a}}(x-a)$$

for x close to a, or, alternatively with $h=x-a$,

$$\sqrt{a+h} \approx \sqrt{a} + \frac{1}{2\sqrt{a}} h \quad \text{for small } h.$$

If we now substitute $a = 1$ and $h = 0.1$, we obtain the approximation

$$\sqrt{1.1} \approx 1 + \frac{0.1}{2} = 1.05.$$

The first digits of the actual value are $1.0488\ldots$

Physical interpretation as rate of change. In physical applications the derivative often plays the role of a rate of change. A well-known example from everyday life is the velocity. Consider a particle which is moving along a straight line. Let $s(t)$ be the position where the particle is at time t. The average velocity is given by the quotient

$$\frac{s(t) - s(t_0)}{t - t_0} \quad \text{(difference in displacement divided by difference in time).}$$

In the limit $t \to t_0$ the average velocity turns into the instantaneous velocity

$$v(t_0) = \frac{\mathrm{d}s}{\mathrm{d}t}(t_0) = \dot{s}(t_0) = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0}.$$

Note that one often writes $\dot{f}(t)$ instead of $f'(t)$ if the time t is the argument of the function f. In particular, in physics the dot notation is most commonly used.

Likewise one obtains the acceleration by differentiating the velocity

$$a(t) = \dot{v}(t) = \vec{s}(t).$$

The notion of velocity is also used in the modeling of other processes that vary over time, e.g. for growth or decay.

Differentiation Rules

In this section $I \subset \mathbb{R}$ denotes an open interval. We first note that differentiation is a linear process.

Proposition. (Linearity of the derivative) Let $f, g : I \to \mathbb{R}$ be two functions which are differentiable at $x \in I$ and take $c \in \mathbb{R}$. Then the functions $f+g$ and $c+f$ are differentiable at x as well and

$$(f(x)+g(x))' = f'(x)+g'(x), \quad (cf(x))' = cf'(x)$$

Proof. The result follows from the corresponding rules for limits. The first statement is true because

$$\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} = \underbrace{\frac{f(x+h)-f(x)}{h}}_{\to f'(x)} + \underbrace{\frac{g(x+h)-g(x)}{h}}_{\to g'(x)}$$

as $h \to 0$. The second statement follows similarly.

Linearity together with the differentiation rule $\left(x^m\right)' = mx^{m-1}$ for powers implies that every polynomial is differentiable. Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then its derivative has the form

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

For example, $\left(3x^7 - 4x^2 + 5x - 1\right)' = 21x^6 - 8x + 5$.

The following two rules allow one to determine the derivative of products and quotients of functions from their factors.

Proposition. (Product rule) Let $f, g : I \to \mathbb{R}$ be two functions which are differentiable at $x \in I$. Then the function $f - g$ is differentiable at x and

$$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x).$$

Proof. This fact follows again from the corresponding rules for limits

$$\begin{aligned}
&\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x+h)}{h} + \frac{f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h} \\
&= \underbrace{\frac{f(x+h) - f(x)}{h}}_{\to f'(x)} \cdot \underbrace{g(x+h)}_{\to g(x)} + f(x)\cdot \underbrace{\frac{g(x+h) - g(x)}{h}}_{\to g'(x)}
\end{aligned}$$

as $h \to 0$. The required continuity of g at x is a consequence of Application 7.15.

Proposition. (Quotient rule) Let $f, g : I \to \mathbb{R}$ be two functions differentiable at $x \in I$ and $g(x) \neq 0$. Then the quotient $\frac{f}{g}$ is differentiable at the point x and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}.$$

In particular,

$$\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{(g(x))^2}.$$

The proof is similar to the one for the product rule.

Example 13 An application of the quotient rule to $\tan x = \frac{\sin x}{\cos x}$ shows that

$$\tan' x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

Complicated functions can often be written as a composition of simpler functions. For example, the function

$$h:[2,\infty) \to \mathbb{R} : x \mapsto h(x) = \sqrt{\log(x-1)}$$

can be interpreted as $h(x) = f(g(x))$ with

$$f:[0,\infty) \to \mathbb{R} : y \mapsto \sqrt{y}, \quad g:[2,\infty) \to [0,\infty) : x \mapsto \log(x-1).$$

One denotes the composition of the functions f and g by $h = f \circ g$. The following proposition shows how such compound functions can be differentiated.

Proposition. (Chain rule) The composition of two differentiable functions

g : $I \to B$ and $f : B \to \mathbb{R}$ is also differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x)) \cdot g'(x).$$

In shorthand notation the rule is

$$(f \circ g)' = \left(f' \circ g\right) \cdot g'.$$

Proof. We write

$$\begin{aligned}\frac{1}{h}(f(g(x+h)) - f(g(x))) &= \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\ &= \frac{f(g(x)+k) - f(g(x))}{k} \cdot \frac{g(x+h) - g(x)}{h},\end{aligned}$$

where, due to the interpretation as a linear approximation, the expression

$$k = g(x+h) - g(x)$$

is of the form

$$k = g'(x)h + R(x+h, x)$$

and tends to zero itself as $h \to 0$. It follows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = \lim_{h\to 0} \frac{1}{h}(f(g(x+h)) - f(g(x))) \\ &= \lim_{h\to 0}\left(\frac{f(g(x)+k) - f(g(x))}{k} \cdot \frac{g(x+h) - g(x)}{h}\right) = f'(g(x)) \cdot g'(x)\end{aligned}$$

and hence the assertion of the proposition.

The differentiation of a composite function $h(x) = f(g(x))$ is consequently performed in three steps:

1. Identify the outer function f and the inner function g with

   $h(x) = f(g(x))$.

2. Differentiate the outer function f at the point $g(x)$, i.e. compute $f'(y)$ and then substitute $y = g(x)$. The result is $f'(g(x))$.

3. Inner derivative: Differentiate the inner function g and multiply it with the result of step 2. One obtains $h'(x) = f'(g(x)) \cdot g'(x)$.

In the case of three or more compositions, the above rules have to be applied recursively.

Example 14. (a) Let $h(x) = (\sin x)^3$. We identify the outer function $f(y) = y^3$ and the inner function $g(x) = \sin x$. Then, $h'(x) = 3(\sin x)^2 \cdot \cos x.$

(b) Let $h(x) = e^{-x^2}$. We identify $f(y) = e^y$ and $g(x) = -x^2$. Thus,

$h'(x) = e^{-x^2} \cdot (-2x).$

The last rule that we will discuss concerns the differentiation of the inverse of a differentiable function.

Proposition. (Inverse function rule) Let $f : I \to J$ be bijective, differentiable and $f'(y) \neq 0$ for all $y \in I$. Then $f^{-1} : J \to I$ is also differentiable and

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

In shorthand notation this rule is

$$\left(f^{-1}\right)' = \frac{1}{f' \circ f^{-1}}$$

Proof. We set $y = f^{-1}(x)$ and $\eta = f^{-1}(\xi)$. Due to the continuity of the inverse function we have that $\eta \to y$ as $\xi \to x$. It thus follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \lim_{\xi \to x} \frac{f^{-1}(\xi) - f^{-1}(x)}{\xi - x} = \lim_{\eta \to y} \frac{\eta - y}{f(\eta) - f(y)} = \lim_{\eta \to y} \left(\frac{f(\eta) - f(y)}{\eta - y} \right)^{-1}$$

$$= \frac{1}{f'(y)} = \frac{1}{f'\left(f^{-1}(x)\right)}$$

and hence the statement of the proposition.

Example 15. (Derivative of the logarithm) Since $y = \log x$ is the inverse function to $x = \mathrm{e}^y$, it follows from the inverse function rule that

$$(\log x)' = \frac{1}{\mathrm{e}^{\log x}} = \frac{1}{x}$$

for $x > 0$. Furthermore

$$\log |x| = \begin{cases} \log x, & x > 0, \\ \log(-x), & x < 0, \end{cases}$$

and thus

$$(\log |x|)' = \begin{cases} (\log x)' = \dfrac{1}{x}, & x > 0, \\ (\log(-x))' = \dfrac{1}{(-x)} \cdot (-1) = \dfrac{1}{x}, & x < 0. \end{cases}$$

Altogether one obtains the formula

$$(\log |x|)' = \frac{1}{x} \text{ for } x \neq 0.$$

For logarithms to the base a one has

$$\log_a x = \frac{\log x}{\log a}, \quad \text{thus } \left(\log_a x\right)' = \frac{1}{x \log a}.$$

Example 16. (Derivatives of general power functions) From $x^{\alpha} = \mathrm{e}^{\alpha \log x}$ we infer by the chain rule that

$$\left(x^{\alpha}\right)' = \mathrm{e}^{\alpha \log x} \cdot \frac{\alpha}{x} = x^{\alpha} \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1}$$

Example 17. (Derivative of the general exponential function) For $a > 0$ we have $a^x = \mathrm{e}^{x \log a}$. An application of the chain rule shows that

$$\left(a^x\right)' = \left(\mathrm{e}^{x \log a}\right)' = \mathrm{e}^{x \log a} \cdot \log a = a^x \log a.$$

Example 18. For $x > 0$ we have $x^x = \mathrm{e}^{x \log x}$ and thus

$$\left(x^x\right)' = \mathrm{e}^{x \log x}\left(\log x + \frac{x}{x}\right) = x^x(\log x + 1).$$

Example 19. (Derivatives of cyclometric functions) We recall the differentiation rules for the trigonometric functions on their principal branches:

$$(\sin x)' = \cos x = \sqrt{1-\sin^2 x}, -\frac{\pi}{2} \le x \le \frac{\pi}{2},$$

$$(\cos x)' = -\sin x = -\sqrt{1-\cos^2 x}, 0 \le x \le \pi,$$

$$(\tan x)' = 1+\tan^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The inverse function rule thus yields

$$(\arcsin x)' = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1,$$

$$(\arccos x)' = \frac{-1}{\sqrt{1-\cos^2(\arccos x)}} = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1,$$

$$(\arctan x)' = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}, -\infty < x < \infty.$$

Example 20. (Derivatives of hyperbolic and inverse hyperbolic functions)

The derivative of the hyperbolic sine is readily computed by invoking the defining formula:

$$(\sinh x)' = \left(\frac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right)\right)' = \frac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) = \cosh x.$$

Similarly, we find $(\cosh x)' = \sinh x, \quad (\tanh x)' = 1 - \tanh^2 x.$

The derivative of the inverse hyperbolic sine can be computed by means of the inverse function rule:

$$(\operatorname{arsinh} x)' = \frac{1}{\cosh(\operatorname{arsinh} x)} = \frac{1}{\sqrt{1+\sinh^2(\operatorname{arsinh} x)}} = \frac{1}{\sqrt{1+x^2}}$$

for $x \in \mathbb{R}$, where we have used the identity $\cosh^2 x - \sinh^2 x = 1$. In a similar way, the derivatives of the other inverse hyperbolic functions can be computed on their respective domains.

$$(\operatorname{arcosh} x)' = \frac{1}{\sqrt{x^2-1}}, x > 1, \ (\operatorname{artanh} x)' = \frac{1}{1-x^2}, -1 < x < 1.$$

Table 7.1 Derivatives of the elementary functions $(\alpha \in \mathbb{R}, a > 0)$

$f(x)$	$1 \quad x^{\alpha}$	e^{x}	a^{x}	$\log\lvert x\rvert$	$\log ax$	

$f'(x)$	0	$\alpha x^{\alpha-1}$	e^x	$a^x \log a$	$\frac{1}{x}$	$\frac{1}{x \log a}$
$f(x)$	$\sin x$	$\cos x$	$\tan x$	$\arcsin x$	$\arccos x$	$\arctan x$
$f'(x)$	$\cos x$	$-\sin x$	$1+\tan^2 x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$
$f(x)$	$\sinh x$	$\cosh x$	$\tanh x$	$\operatorname{arsinh} x$	$\operatorname{arcosh} x$	$\operatorname{artanh} x$
$f'(x)$	$\cosh x$	$\sinh x$	$1-\tanh^2 x$	$\frac{1}{\sqrt{1+x^2}}$	$\frac{1}{\sqrt{x^2-1}}$	$\frac{1}{1-x^2}$

The derivatives of the most important elementary functions are collected in Table 7.1. The formulas are valid on the respective domains.

Exercises

1. Compute the first derivative of the functions

$$f(x) = x^3, \quad g(t) = \frac{1}{t^2}, \quad h(x) = \cos x, \quad k(x) = \frac{1}{\sqrt{x}}, \quad \ell(t) = \tan t$$

using the definition of the derivative as a limit.

2. Compute the first derivative of the functions

$$a(x) = \frac{x^2-1}{x^2+2x+1}, \quad b(x) = \left(x^3-1\right)\sin^2 x, \quad c(t) = \sqrt{1+t^2}\arctan t\,,$$

$$d(t) = t^2 e^{\cos\left(t^2+1\right)}, \quad e(x) = x^{2\sin x}, \quad f(s) = \log\left(s+\sqrt{1+s^2}\right).$$

3. Derive the remaining formulas in Example 7.30. Start by computing the derivatives of the hyperbolic cosine and hyperbolic tangent. Use the inverse function rule to differentiate the inverse hyperbolic cosine and inverse hyperbolic tangent.

4. Compute an approximation of $\sqrt{34}$ by replacing the function $f(x)=\sqrt{x}$ at $x=36$ by its linear approximation. How accurate is your result?

5. Find the equation of the tangent line to the graph of the function $y=f(x)$ through the point $\left(x_0, f\left(x_0\right)\right)$, where $f(x)=\dfrac{x}{2}+\dfrac{x}{\log x}$ and (a) $x_0=\mathrm{e}$; (b) $x_0=\mathrm{e}^2$.

6. Sand runs from a conveyor belt onto a heap with a velocity of $2\,\mathrm{m}^3/\mathrm{min}$. The sand forms a cone-shaped pile whose height equals $\dfrac{4}{3}$ of the radius. With which velocity does the radius grow if the sand cone has a diameter of $6\,\mathrm{m}$?

Hint. Determine the volume V as a function of the radius r, consider V and r as functions of time t and differentiate the equation with respect to t. Compute $\dot{r}$.

7. Show that the n^{th} derivative of the power function $y=x^n$ equals $n!$for $n\geq 1$. Verify that the derivative of order $n+1$ of a polynomial $p(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x+a_0$ of degree n equals zero.

8. Compute the second derivative of the functions

$$f(x) = \mathrm{e}^{-x^2}, \quad g(x) = \log\left(x + \sqrt{1+x^2}\right), \quad h(x) = \log\frac{x+1}{x-1}$$

Applications of the Derivative

This sectin is devoted to some applications of the derivative which form part of the basic skills in modeling. We consider features of graphs.

Curve Sketching

In the following we investigate some geometric properties of graphs of functions using the derivative: maxima and minima, intervals of monotonicity and convexity. We further discuss the mean value theorem which is an important technical tool for proofs.

Definition 8.1 A function $f:[a,b]\to\mathbb{R}$ has

(a) a global maximum at $x_0\in[a,b]$ if $f(x)\le f(x_0)$ for all $x\in[a,b]$;

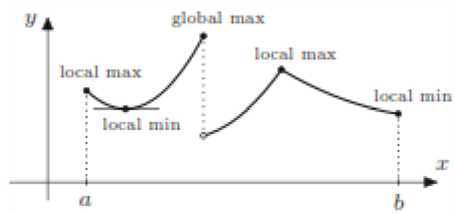
(b) a local maximum at $x_0\in[a,b]$, if there exists a neighbourhood $U_\varepsilon(x_0)$ so that

$$f(x)\le f(x_0) \text{ for all } x\in U_\varepsilon(x_0)\cap[a,b].$$

The maximum is called strict if the strict inequality $f(x)<f(x_0)$ holds in (a) or (b) for $x\ne x_0$.

The definition for minimum is analogous by inverting the inequalities.

Maxima and minima are subsumed under the term extrema. The following figure shows some possible situations. Note that the function there does not have a global minimum on the chosen interval.



For points x_0 in the open interval (a,b) one has a simple necessary condition for extrema of differentiable functions:

Proposition. Let $x_0 \in (a,b)$ and f be differentiable at x_0. If f has a local maximum or minimum at x_0 then $f'(x_0)=0$.

Proof. Due to the differentiability of f we have

$$f'(x_0)=\lim_{h\to 0+}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0-}\frac{f(x_0+h)-f(x_0)}{h}.$$

In the case of a maximum the slope of the secant satisfies the inequalities

$$\frac{f(x_0+h)-f(x_0)}{h}\le 0, \quad \text{if } h>0 \, \frac{f(x_0+h)-f(x_0)}{h}\ge 0, \quad \text{if } h<0$$

Consequently the limit $f'(x_0)$ has to be greater than or equal to zero as well as smaller than or equal to zero, thus necessarily $f'(x_0)=0$.

The function $f(x)=x^3$, whose derivative vanishes at $x=0$, shows that the condition of the proposition is not sufficient for the existence of a maximum or minimum.

The geometric content of the proposition is that in the case of differentiability the graph of the function has a horizontal tangent at a maximum or minimum. A point $x_0 \in (a,b)$ where $f'(x_0)=0$ is called a stationary point.

Remark. The proposition shows that the following point sets have to be checked in order to determine the maxima and minima of a function $f:[a,b]\to\mathbb{R}$:

(a) the boundary points $x_0=a, x_0=b$;

(b) points $x_0 \in (a,b)$ at which f is not differentiable;

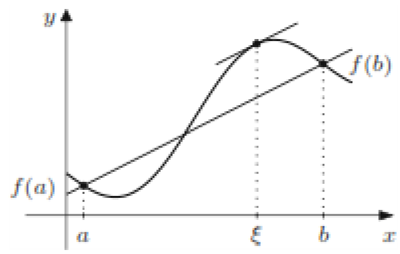
(c) points $x_0 \in (a,b)$ at which f is differentiable and $f'(x_0)=0$.

The following proposition is a useful technical tool for proofs. One of its applications lies in estimating the error of numerical methods. Similarly to the intermediate value theorem, the proof is based on the completeness of the real numbers.

Proposition (Mean value theorem) Let f be continuous on $[a, b]$ and differentiable on (a,b). Then there exists a point $\xi \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi).$$

Geometrically this means that the tangent at ξ has the same slope as the secant through $(a, f(a)),(b, f(b))$. The following figure illustrates this fact.



We now turn to the description of the behaviour of the slope of differentiable functions.

Definition. A function $f: I \to \mathbb{R}$ is called monotonically increasing, if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in I$. It is called strictly monotonically increasing, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

A function f is said to be (strictly) monotonically decreasing, if $-f$ is (strictly) monotonically increasing.

Examples of strictly monotonically increasing functions are the power functions $x \mapsto x^n$ with odd powers n; a monotonically, but not strictly monotonically increasing function is the sign function $x \mapsto \operatorname{sign} x$, for instance.

The behaviour of the slope of a differentiable function can be described by the sign of the first derivative.

Proposition. For differentiable functions $f:(a,b)\to\mathbb{R}$ the following implications hold:

(a) $f' \geq 0$ on $(a,b) \Leftrightarrow f$ is monotonically increasing; $f' > 0$ on (a,b) $\Rightarrow f$ is strictly monotonically increasing.

(b) $f' \leq 0$ on $(a,b) \Leftrightarrow f$ is monotonically decreasing; $f' < 0$ on $(a,b) \Rightarrow f$ is strictly monotonically decreasing.

Proof. **(a)** According to the mean value theorem we have $f(x_2) - f(x_1) = f'(\xi)\cdot(x_2 - x_1)$ for a certain $\xi \in (a,b)$. If $x_1 < x_2$ and $f'(\xi) \geq 0$ then $f(x_2) - f(x_1) \geq 0$. If $f'(\xi) > 0$ then $f(x_2) - f(x_1) > 0$. Conversely

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \geq 0,$$

if f is increasing. The proof for (b) is similar.

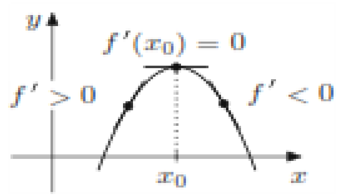
Remark. The example $f(x) = x^3$ shows that f can be strictly monotonically increasing even if $f' = 0$ at isolated points.

Proposition. (Criterion for local extrema) Let f be differentiable on (a,b), $x_0 \in (a,b)$ and $f'(x_0) = 0$. Then

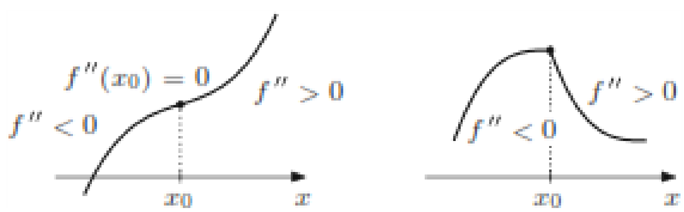
(a) $\left.\begin{matrix} f'(x) > 0 \text{ for } x < x_0 \\ f'(x) < 0 \text{ for } x > x_0 \end{matrix}\right\} \Rightarrow$ has a local maximum in x_0,

(b) $\left.\begin{matrix} f'(x) < 0 \text{ for } x < x_0 \\ f'(x) > 0 \text{ for } x > x_0 \end{matrix}\right\} \Rightarrow f$ has a local minimum in x_0.

Proof. The proof follows from the previous proposition which characterises the monotonic behaviour as shown in



Remark. (Convexity and concavity of a function graph) If $f'' > 0$ holds in an interval then f' is monotonically increasing there. Thus the graph of f is curved to the left or convex. On the other hand, if $f'' < 0$, then f' is monotonically decreasing and the graph of f is curved to the right or concave see the following figures.



Let x_0 be a point where $f'(x_0) = 0$. If f' does not change its sign at x_0, then x_0 is an inflection point. Here f changes from positive to negative curvature or vice versa.

Proposition. (Second derivative criterion for local extrema) Let f be twice continuously differentiable on $(a,b), x_0 \in (a,b)$ and $f'(x_0) = 0$.

(a) If $f''(x_0) > 0$ then f has a local minimum at x_0.

(b) If $f''(x_0) < 0$ then f has a local maximum at x_0.

Proof. **(a)** Since f'' is continuous, $f''(x) > 0$ for all x in a neighbourhood of x_0. According to Proposition 8.6, f' is strictly monotonically increasing in this neighbourhood. Because of $f'(x_0) = 0$ this means that $f'(x_0) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$; according to the criterion for local extrema, x_0 is a minimum. The assertion (b) can be shown similarly.

Remark. If $f''(x_0) = 0$ there can either be an inflection point or a minimum or maximum. The functions $f(x) = x^n, n = 3, 4, 5, \ldots$ supply a typical example. In fact, they have for n even a global minimum at $x = 0$, and an inflection point for n odd.

One of the applications of the previous propositions is curve sketching, which is the detailed investigation of the properties of the graph of a function using differential calculus. Even though graphs can easily be plotted in MATLAB or maple it is still often necessary to check the graphical output at certain points using analytic methods.

Experiment. Plot the function

$y = x(\text{sign}\, x - 1)(x+1)^3 + (\text{sign}(x-1)+1)\left((x-2)^4 - 1/2\right)$ on the interval $-2 \le x \le 3$ and try to read off the local and global extrema, the inflection points and the monotonic behaviour. Check your observations using the criteria discussed above.

A further application of the previous propositions consists in finding extrema, i.e. solving one-dimensional optimisation problems. We illustrate this topic using a standard example.

Example. Which rectangle with a given perimeter has the largest area? To answer this question we denote the lengths of the sides of the rectangle by x and y. Then the perimeter and the area are given by $U = 2x + 2y, \quad F = xy.$ Since U is fixed, we obtain $y = U/2 - x$, and from that $F = x(U/2 - x),$ where x can vary in the domain $0 \le x \le U/2$. We want to find the maximum of the function F on the interval $[0, U/2]$. Since F is differentiable, we only have to investigate the boundary points and the stationary points. At the boundary points $x = 0$ and $x = U/2$ we have $F(0) = 0$ and $F(U/2) = 0$. The stationary points are obtained by setting the derivative to zero $F'(x) = U/2 - 2x = 0,$ which brings us to $x = U/4$ with the function value $F(U/4) = U^2/16$.

As result we get that the maximum area is obtained at $x = U/4$, thus in the case of a square.

Exercises

1. Find out which of the following (continuous) functions are differentiable at

$$x = 0$$

$$y = x\,|\,x\,|; \quad y = |\,x\,|^{1/2}, \quad y = |\,x\,|^{3/2}, \quad y = x\sin(1/x).$$

2. Find all maxima and minima of the functions

$$f(x) = \frac{x}{x^2+1} \text{ and } g(x) = x^2 \mathrm{e}^{-x^2}.$$

3. Find the maxima of the functions

$$y = \frac{1}{x}\mathrm{e}^{-(\log x)^2/2}, x > 0 \quad \text{and} \quad y = \mathrm{e}^{-x}\mathrm{e}^{-\left(\mathrm{e}^{-x}\right)}, x \in \mathbb{R}.$$

These functions represent the densities of the standard lognormal distribution and of the Gumbel distribution, respectively.

4. Find all maxima and minima of the function $f(x) = \dfrac{x}{\sqrt{x^4+1}}$, determine on what intervals it is increasing or decreasing, analyse its behaviour as $x \to \pm\infty$, and sketch its graph.

5. Find the proportions of the cylinder which has the smallest surface area F for a given volume V.

Hint. $F = 2r\pi h + 2r^2\pi \to \min$. Calculate the height h as a function of the radius r from $V = r^2\pi h$, substitute and minimise $F(r)$.

6. (From mechanics of solids) The moment of inertia with respect to the central axis of a beam with rectangular cross section is $I = \frac{1}{12} bh^3$ (b the width, h the height). Find the proportions of the beam which can be cut from a log with circular cross section of given radius r such that its moment of inertia becomes maximal.

Hint. Write b as function of $h, I(h) \to \max$.

7. (From soil mechanics) The mobilized cohesion $c_{\mathrm{m}}(\theta)$ of a failure wedge with sliding surface, inclined by an angle θ, is

$$c_{\mathrm{m}}(\theta) = \frac{\gamma h \sin\left(\theta - \varphi_{\mathrm{m}}\right) \cos\theta}{2\cos\varphi_{\mathrm{m}}}.$$

Here h is the height of the failure wedge, φ_{m} the angle of internal friction, γ the specific weight of the soil (see Fig. 8.10). Show that the mobilised cohesion c_{m} with given $h, \varphi_{\mathrm{m}}, \gamma$ is a maximum for the angle of inclination $\theta = \varphi_{\mathrm{m}} / 2 + 45^{\circ}$.

Antiderivatives

The derivative of a function $y = F(x)$ describes its local rate of change, i.e. the change Δy of the y-value with respect to the change Δx of the x-value in the limit $\Delta x \to 0$; more precisely

$$f(x) = F'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}.$$

Conversely, the question about the reconstruction of a function F from its local rate of change f leads to the notion of indefinite integrals which comprises the totality of all functions that have f as their derivative, the antiderivatives of f.

By multiplying the rate of change $f(x)$ with the change Δx one obtains an approximation to the change of the values of the function of the antiderivative F in the segment of length Δx:

$$\Delta y = F(x+\Delta x) - F(x) \approx f(x)\Delta x.$$

Adding up these local changes in an interval, for instance between $x = a$ and $x = b$ in steps of length Δx, gives an approximation to the total change $F(b) - F(a)$. The limit $\Delta x \to 0$ (with an appropriate increase of the number of summands) leads to the notion of the definite integral of f in the interval $[a, b]$, which is the subject we discuss later in this chapter.

Indefinite Integrals

Already, it was shown that the derivative of a constant is zero. The following proposition shows that the converse is also true.

Proposition. If the function F is differentiable on (a,b) and $F'(x)=0$ for all $x \in (a,b)$ then F is constant. This means that $F(x)=c$ for a certain $c \in \mathbb{R}$ and all $x \in (a,b)$.

Proof . We choose an arbitrary $x_0 \in (a,b)$ and set $c = F(x_0)$. If now $x \in (a,b)$ then, according to the mean value theorem, $F(x) - F(x_0) = F'(c)(x - x_0)$ for a point c between x and x_0. Since $F'(c)=0$ it follows that $F(x) = F(x_0) = c$. This holds for all $x \in (a,b)$, consequently F has to be equal to the constant function with value c.

Definition. (Antiderivatives) Let f be a real-valued function on an interval (a,b). An antiderivative of f is a differentiable function $F:(a,b) \to \mathbb{R}$ whose derivative F' equals f.

Example. The function $F(x)=\frac{x^3}{3}$ is an antiderivative of $f(x)=x^2$, as is $G(x)=\frac{x^3}{3}+5$.

This proposition implies that antiderivatives are unique up to an additive constant. This means that if F and G are antiderivatives of f in (a,b). Then $F(x)= G(x)+c$ for a certain $c\in\mathbb{R}$ and all $x\in(a,b)$.

Proof. Since $F^{\prime}(x)-G^{\prime}(x)=f(x)-f(x)=0$ for all $x \in(a, b)$, an application of Proposition 10.1 gives the desired result.

Definition. (Indefinite integrals) The indefinite integral $\int f(x)\mathrm{d}x$ denotes the totality of all antiderivatives of f.

Once a particular antiderivative F has been found, one writes accordingly

$$\int f(x)\mathrm{d}x = F(x)+c.$$

Example. The indefinite integral of the quadratic function is $\int x^2\,\mathrm{d}x=\frac{x^3}{3}+c$.

Example. (a) An application of indefinite integration to the differential equation of the vertical throw: Let $w(t)$ denote the height (in meters $[m]$) at time t (in seconds $[s]$) of an object above ground level $(w=0)$. Then, $w'(t)=v(t)$ is the velocity of the object (positive in upward direction) and $v'(t)=a(t)$ the acceleration (positive in upward direction). In this coordinate system the gravitational acceleration $g = 9.81\left[\mathrm{m/s^2}\right]$ acts downwards, consequently $a(t) = -g$. Velocity and distance are obtained by inverting the differentiation process

$$v(t) = \int a(t)\mathrm{d}t + c_1 = -gt + c_1 w(t) = \int v(t)\mathrm{d}t + c_2 = \int(-gt + c_1)\mathrm{d}t + c_2 = -\frac{g}{2}t^2 + c_1 t + c_2$$

where the constants c_1, c_2 are determined by the initial conditions:

$$c_1 = v(0) \quad \ldots \quad \text{initial velocity,}$$

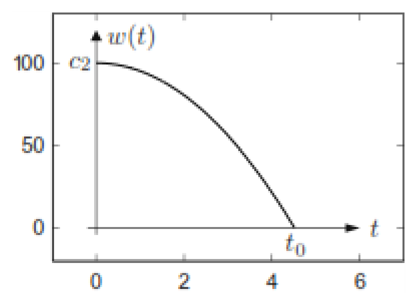
$$c_2 = w(0) \quad \ldots \quad \text{initial height.}$$

(b) A concrete example-the free fall from a height of $100\,\mathrm{m}$. Here

$$w(0) = 100, \quad v(0) = 0$$

and thus $w(t) = -\frac{1}{2}9.81t^2 + 100.$

The travelled distance as a function of time is given by a parabola.



The time of impact t_0 is obtained from the condition $w(t_0) = 0$, i.e.

$$0 = -\frac{1}{2}9.81t_0^2 + 100, \quad t_0 = \sqrt{200/9.81} \approx 4.5[\mathrm{s}],$$

the velocity at impact is $v(t_0) = -gt_0 \approx 44.3[\mathrm{m/s}] \approx 160[\mathrm{km/h}].$

Integration Formulas

It follows immediately from the definition that indefinite integration can be seen as the inversion of differentiation. It is, however, only unique up to a constant:

$$\left(\int f(x)\mathrm{d}x\right)' = f(x), \int g'(x)\mathrm{d}x = g(x) + c.$$

With this consideration and the differentiation Rules one easily obtains the basic integration formulas stated in the following table. The formulas are valid in the according domains.

The formulas in Table 10.1 are a direct consequence of those in Table 7.1.

Experiment 10.8 Antiderivatives can be calculated in maple using the command int. Explanations and further integration commands can be found in the maple

Table 10.1 Integrals of some elementary functions

$f(x)$	$x^\alpha, \alpha \neq -1$	$\frac{1}{x}$	e^x	a^x
$\int f(x)\mathrm{d}x$	$\frac{x^{\alpha+1}}{\alpha+1} + c$	$\log\|x\| + c$	$e^x + c$	$\frac{1}{\log a} a^x + c$
$f(x)$	$\sin x$	$\cos x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

$\int f(x)\mathrm{d}x$	$-\cos x + c$	$\sin x + c$	$\arcsin x + c$	$\arctan x + c$
$f(x)$	$\sinh x$	$\cosh x$	$\frac{1}{\sqrt{1+x^2}}$	$\frac{1}{\sqrt{x^2-1}}$
$\int f(x)\mathrm{d}x$	$\cosh x + c$	$\sinh x + c$	$\operatorname{arsinh} x + c$	$\operatorname{arcosh} x + c$

Functions that are obtained by combining power functions, exponential functions and trigonometric functions, as well as their inverses, are called elementary functions. The derivative of an elementary function is again an elementary function and can be obtained using the differentiation rules. In contrast to differentiation there is no general procedure for computing indefinite integrals. Not only does the calculation of an integral often turn out to be a difficult task, but there are also many elementary functions whose antiderivatives are not elementary. An algorithm to decide whether a functions has an elementary indefinite integral was first deduced by Liouville around 1835. This was the starting point for the field of symbolic integration.

Example. (Higher transcendental functions) Antiderivatives of functions that do not possess elementary integrals are frequently called higher transcendental functions. We give the following examples:

$$\frac{2}{\sqrt{\pi}}\int \mathrm{e}^{-x^2}\,\mathrm{d}x = \mathrm{Erf}(x) + c \quad \ldots \quad \text{Gaussian error function;}$$

$$\int \frac{e^x}{x}\mathrm{d}x = \mathcal{E}i(x) + c \quad \ldots \quad \text{exponential integral;}$$

$$\int \frac{1}{\log x}\mathrm{d}x = \ell i(x) + c \quad \ldots \quad \text{logarithmic integral;}$$

$$\int \frac{\sin x}{x}\mathrm{d}x = \mathcal{S}i(x) + c \quad \ldots \quad \text{sine integral;}$$

$$\int \sin\left(\frac{\pi}{2}x^2\right)\mathrm{d}x = \mathcal{S}(x) + c \quad \ldots \quad \text{Fresnel integral.}$$

Proposition. (Rules for indefinite integration) For indefinite integration the following rules hold:

(a) Sum: $\int (f(x)+g(x))\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$

(b) Constant factor: $\int \lambda f(x)\mathrm{d}x = \lambda \int f(x)\mathrm{d}x \quad (\lambda \in \mathbb{R})$

(c) Integration by parts: $\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$

(d) Substitution: $\int f(g(x))g'(x)\mathrm{d}x = \int f(y)\mathrm{d}y\Big|_{y=g(x)}$

Proof. **(a)** and **(b)** are clear; **(c)** follows from the product rule for the derivative

$$\int f(x)g'(x)\mathrm{d}x + \int f'(x)g(x)\mathrm{d}x = \int \big(f(x)g'(x) + f'(x)g(x)\big)\mathrm{d}x = \int (f(x)g(x))'\mathrm{d}x = f(x)g(x) + c$$

which can be rewritten as

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x.$$

In this formula we can drop the integration constant c since it is already contained in the notion of indefinite integrals, which appear on both sides.

Point (d) is an immediate consequence of the chain rule according to which an antiderivative of $f(g(x))g'(x)$ is given by the antiderivative of $f(y)$ evaluated at $y = g(x)$.

Example. The following five examples show how the rules of the last table can be applied.

(a) $\int \frac{dx}{\sqrt[3]{x}} = \int x^{-1/3}\,dx = \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c = \frac{3}{2}x^{2/3} + c$.

(b) $\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c$, which follows via integration by parts: we have $f(x) = x, g'(x) = \cos x$. Then, $f'(x) = 1, \quad g(x) = \sin x$.

(c) $\int \log x\,dx = \int 1\cdot\log x\,dx = x\log x - \int \frac{x}{x}dx = x\log x - x + c$,

via integration by parts: $f(x) = \log x, g(x) = x,$ and $g'(x) = 1, f'(x) = \frac{1}{x}$.

(d) $\int x\sin\left(x^2\right)dx = \int \frac{1}{2}\sin y\,dy\Big|_{y=x^2} = -\frac{1}{2}\cos y\Big|_{y=x^2} + c = -\frac{1}{2}\cos\left(x^2\right) + c$, which follows from the substitution rule with $y = g(x) = x^2, g'(x) = 2x$, $f(y) = \frac{1}{2}\sin y$.

(e) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\log|y|\|_{y=\cos x} + c = -\log|\cos x| + c$, again after substitution

with $y = g(x) = \cos x, g'(x) = -\sin x$ and $f(y) = -1/y$.

Example. (A simple expansion into partial fractions) In order to find the indefinite integral of $f(x) = 1/\left(x^2 - 1\right)$, we decompose the quadratic denominator in its linear factors $x^2 - 1 = (x-1)(x+1)$ and expand $f(x)$ into partial fractions of the form

$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$.Resolving the fractions leads to the equation

$1 = A(x+1) + B(x-1)$. Equating coefficients results in

$(A+B)x = 0, \quad A - B = 1$with the obvious solution $A = 1/2, B = -1/2$. Thus

$$\int \frac{1}{x^2-1} dx = \frac{1}{2}\left(\int \frac{dx}{x-1} - \int \frac{dx}{x+1}\right) = \frac{1}{2}(\log|x-1| - \log|x+1|) + C = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$$

Another antiderivative of $f(x) = 1/\left(x^2-1\right)$ is $F(x) = -\operatorname{artanh} x$. Thus,

$$\operatorname{artanh} x = -\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C = \frac{1}{2}\log\left|\frac{x+1}{x-1}\right| + C.$$

Inserting $x = 0$ on both sides shows that $C = 0$ and yields an expression of the inverse hyperbolic tangent in terms of the logarithm.

Exercises

1. An object is thrown vertically upwards from the ground with a velocity of $10[\mathrm{m/s}]$. Find its height $w(t)$ as a function of time t, the maximum height as well as the time of impact on the ground.

Hint. Integrate $w''(t) = -\mathrm{g} \approx 9.81\left[\mathrm{m/s^2}\right]$ twice indefinitely and determine the integration constants from the initial conditions $w(0)=0, w'(0)=10$.

2. Compute the following indefinite integrals by hand and with maple:

(a) $\int\left(x+3x^2+5x^4+7x^6\right)\mathrm{d}x$, (b) $\int\frac{\mathrm{d}x}{\sqrt{x}}$, (c) $\int x\mathrm{e}^{-x^2}\,\mathrm{d}x$ (substitution),

(d) $\int x\mathrm{e}^{x}\,\mathrm{d}x$ (integration by parts).

3. Compute the indefinite integrals **(a)** $\int\cos^2 x\,\mathrm{d}x$, **(b)** $\int\sqrt{1-x^2}\,\mathrm{d}x$.

Hints. For **(a)** use the identity $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

for **(b)** use the substitution $y = g(x) = \arcsin x, f(y) = 1-\sin^2 y$.

4. Compute the indefinite integrals **(a)** $\int\frac{\mathrm{d}x}{x^2+2x+5}\mathrm{d}x$, **(b)** $\int\frac{\mathrm{d}x}{x^2+2x-3}$.

Hints. Write the denominator in **(a)** in the form $(x+1)^2+4$ and reduce it to y^2+1 by means of a suitable substitution. Factorize the denominator in **(b)**.

5. Compute the indefinite integrals **(a)** $\int\frac{\mathrm{d}x}{x^2+2x}\mathrm{d}x$, **(b)** $\int\frac{\mathrm{d}x}{x^2+2x+1}$.

6. Compute the indefinite integrals **(a)** $\int x^2 \sin x \, dx$, **(b)** $\int x^2 e^{-3x} \, dx$.

Hint. Repeated integration by parts.

7. Compute the indefinite integrals **(a)** $\int \frac{e^x}{e^x+1} dx$, **(b)** $\int \sqrt{1+x^2} \, dx$

Hint. Substitution $y = e^x$ in case **(a)**, substitution $y = \sinh x$ in case **(b)**, invoking the formula $\cosh^2 y - \sinh^2 y = 1$ and repeated integration by parts or recourse to the definition of the hyperbolic functions.

8. Show that the functions $f(x) = \arctan x$ and $g(x) = \arctan \frac{1+x}{1-x}$ differ in the interval $(-\infty, 1)$ by a constant. Compute this constant. Answer the same question for the interval $(1, \infty)$.

9. Prove the identity $\operatorname{arsinh} x = \log\left(x + \sqrt{1+x^2}\right)$.

Hint. Note that the functions $f(x) = \operatorname{arsinh} x$ and $g(x) = \log\left(x + \sqrt{1+x^2}\right)$ have the same derivative.

Definite Integrals (Riemann's approach)

In the previous section the notion of the definite integral of a function f on an interval $[a, b]$ was already mentioned. It arises from taking limits of summing up expressions of the form $f(x)\Delta x$. Such sums appear in many applications including the calculation of areas, surface areas and volumes as well as the calculation of lengths of curves. This section introduces the notion of Riemann integrals as the basic concept of definite integration. Riemann's approach provides an intuitive concept in many applications, as will be elaborated in examples at the end of the section.

The main part of this section is dedicated to the properties of the integral. In particular, the two fundamental theorems of calculus are proven. The first theorem allows one to calculate a definite integral from the knowledge of an antiderivative. The second fundamental theorem states that the definite integral of a function f on an interval $[a, x]$ with variable upper bound provides an antiderivative of f. Since the definite integral can be approximated, for example by Riemann sums, the second fundamental theorem offers a possibility to approximate the antiderivative numerically. This is of importance, for example, for the calculation of distribution functions in statistics.

The Riemann Integral

Example. (From velocity to distance) How can one calculate the distance w which a vehicle travels between time a and time b if one only knows its velocity $v(t)$ for all times $a \le t \le b$? If $v(t) \equiv v$ is constant, one simply gets $w = v \cdot (b - a)$.

If the velocity $v(t)$ is time-dependent, one divides the time axis into smaller subintervals (see the following figure): $a = t_0 < t_1 < t_2 < \cdots < t_n = b$..

Choosing intermediate points $\tau_j \in [t_{j-1}, t_j]$ one obtains approximately $v(t) \approx v(\tau_j)$ for $t \in [t_{j-1}, t_j]$, if v is a continuous function of time. The approximation is the more precise, the shorter the intervals $[t_{j-1}, t_j]$ are chosen. The distance travelled in this interval is approximately equal to $w_j \approx v(\tau_j)(t_j - t_{j-1})$. The total distance covered between time a and time b is then $w = \sum_{j=1}^{n} w_j \approx \sum_{j=1}^{n} v(\tau_j)(t_j - t_{j-1})$. Letting the length of the subintervals $[t_{j-1}, t_j]$ tend to zero, one expects to obtain the actual value of the distance in the limit.

Example. (Area under the graph of a nonnegative function) In a similar way one can try to approximate the area under the graph of a function $y = f(x)$ by using rectangles which are successively refined (Fig. 11.2).

The sum of the areas of the rectangles

$$F \approx \sum_{j=1}^{n} f\left(\xi_j\right)\left(x_j - x_{j-1}\right)$$

form an approximation to the actual area under the graph.

The two examples are based on the same concept, the Riemann integral, [1] which we will now introduce. Let an interval $[a, b]$ and a function $f = [a,b] \to \mathbb{R}$ be given. Choosing points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$ the intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ form a partition Z of the interval $[a, b]$. We denote the length of the largest subinterval by $\Phi(Z)$, i.e. $\Phi(Z) = \max_{j=1,\ldots,n} \left|x_j - x_{j-1}\right|$.

For arbitrarily chosen intermediate points $\xi_j \in \left[x_{j-1}, x_j\right]$ one calls the expression

$S = \sum_{j=1}^{n} f\left(\xi_j\right)\left(x_j - x_{j-1}\right)$ a Riemann sum. In order to further specify the idea of the limiting process above, we take a sequence $Z_1, Z_2, Z_3, \ldots$ of partitions such that $\Phi\left(Z_N\right) \to 0$ as $N \to \infty$ and corresponding Riemann sums S_N.

Definition. A function f is called Riemann integrable in $[a,b]$ if, for arbitrary sequences of partitions $(Z_N)_{N\geq 1}$ with $\Phi(Z_N)\to 0$, the corresponding Riemann sums $(S_N)_{N\geq 1}$ tend to the same limit $I(f)$, independently of the choice of the intermediate points. This limit $I(f)=\int_a^b f(x)\mathrm{d}x$ is called the definite integral of f on $[a,b]$.

The intuitive approach in the above introductory Examples can now be made precise. If the respective functions f and v are Riemann integrable, then the integral

$$F=\int_a^b f(x)\mathrm{d}x$$

represents the area between the x-axis and the graph, and

$$w=\int_a^b v(t)\mathrm{d}t$$

gives the total distance covered.

The following examples illustrate the notion of Riemann inerrability.

Example . **(a)** Let $f(x)=c=$ constant. Then the area under the graph of the function is the area of the rectangle $c(b-a)$. On the other hand, any Riemann sum is of the form

$$\begin{aligned}&f(\xi_1)(x_1-x_0)+f(\xi_2)(x_2-x_1)+\cdots+f(\xi_n)(x_n-x_{n-1})\\&=c(x_1-x_0+x_2-x_1+\cdots+x_n-x_{n-1})=c(x_n-x_0)=c(b-a).\end{aligned}$$

All Riemann sums are equal and thus, as expected, $\int_a^b c\,\mathrm{d}x = c(b-a)$

(b) Let $f(x)=\frac{1}{x}$ for $x\in(0,1]$, $f(0)=0$. This function is not integrable in [0,1].

The corresponding Riemann sums are of the form

$$\frac{1}{\alpha_1}(x_1-0)+\frac{1}{\alpha_2}(x_2-x_1)+\cdots+\frac{1}{\alpha_n}(x_n-x_{n-1}).$$

By choosing α_1 close to 0 every such Riemann sum can be made arbitrarily large; thus the limit of the Riemann sums does not exist.

(c) Dirichlet's function

$$f(x)=\{1, x\in\mathbb{Q}0, x\notin\mathbb{Q}\}$$

is not integrable in [0,1]. The Riemann sums are of the form

$$S_N = f(\alpha_1)(x_1-x_0)+\cdots+f(\alpha_n)(x_n-x_{n-1}).$$

If all $\alpha_j\in\mathbb{Q}$ then $S_N=1$. If one takes all $\alpha_j\notin\mathbb{Q}$ then $S_N=0$; thus the limit depends on the choice of intermediate points α_j.

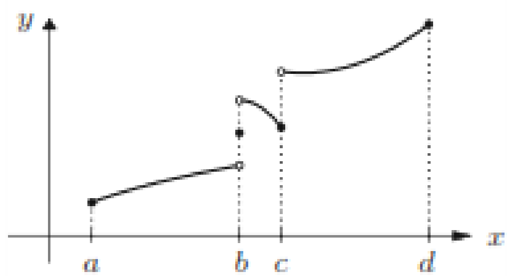
Remark. Riemann integrable functions $f:[a,b]\to\mathbb{R}$ are necessarily bounded. This fact can easily be shown by generalizing the argument in the previous example **(b)**.

The most important criteria for Riemann integrability are outlined in the following proposition. Its proof is simple, however, it requires a few technical considerations about refining partitions.

Proposition. (a) Every function which is bounded and monotonically increasing (monotonically decreasing) on an interval $[a,b]$ is Riemann integrable.

(b) Every piecewise continuous function on an interval $[a,b]$ is Riemann integrable.

A function is called piecewise continuous if it is continuous except for a finite number of points. At these points, the graph may have jumps but is required to have left- and right-hand limits, see the figure.



Remark. By taking equidistant grid points $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ for the partition, i.e. $x_j - x_{j-1} =: \Delta x = \frac{b-a}{n}$, the Riemann sums can be written as

$$S_N = \sum_{j=1}^{n} f\left(\xi_j\right) \Delta x.$$

The transition $\Delta x \to 0$ with simultaneous increase of the number of summands suggests the notation $\int_a^b f(x)\mathrm{d}x$. Originally it was introduced by Leibniz with the interpretation as an infinite sum of infinitely small rectangles of width $\mathrm{d}x$.

After centuries of dispute, this interpretation can be rigorously justified today within the framework of nonstandard analysis.

Note that the integration variable x in the definite integral is a bound variable and can be replaced by any other letter:

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(\xi)\mathrm{d}\xi = \cdots$$

This can be used with advantage in order to avoid possible confusion with other bound variables.

Proposition. (Properties of the definite integral) In the following let $a < b$ and f, g be Riemann integrable on $[a, b]$.

(a) Positivity: $f \geq 0 \text{ in } [a,b] \quad \Rightarrow \quad \int_a^b f(x)\mathrm{d}x \geq 0, \quad f \leq 0 \text{ in } [a,b] \quad \Rightarrow \quad \int_a^b f(x)\mathrm{d}x \leq 0.$

(b) Monotonicity: $f \leq g \text{ in } [a,b] \Rightarrow \int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$

In particular; with $m = \inf_{x\in[a,b]} f(x), \quad M = \sup_{x\in[a,b]} f(x)$ the following inequality holds

$$m(b-a) \leq \int_a^b f(x)\mathrm{d}x \leq M(b-a)$$

(c) Sum and constant factor (linearity):

$$\int_a^b (f(x)+g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x,$$
$$\int_a^b \lambda f(x)\mathrm{d}x = \lambda \int_a^b f(x)\mathrm{d}x \quad (\lambda \in \mathbb{R}).$$

(d) Partition of the integration domain: Let $a < b < c$ and f be integrable in $[a, c]$, then $\int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x$. If one defines

$$\int_a^a f(x)\mathrm{d}x = 0, \quad \int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x,$$

then one obtains the validity of the sum formula even for arbitrary $a,b,c \in \mathbb{R}$ if f is integrable on the respective intervals.

Proof. All justifications are easily obtained by considering the corresponding Riemann sums. **(a)** the interpretation of the integral as the area under the graph is only appropriate if $f \geq 0$. On the other hand, the interpretation of the integral of a velocity as travelled distance is also meaningful for negative velocities (change of direction). **(d)** is especially important for the integration of piecewise continuous functions. The integral is obtained as the sum of the single integrals.

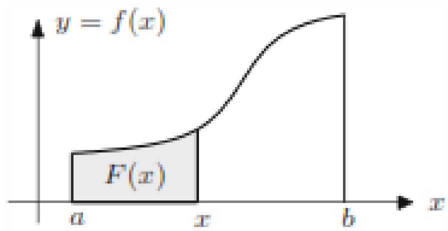
Fundamental Theorems of Calculus

For a Riemann integrable function \$f\$ we define a new function

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

It is obtained by considering the upper boundary of the integration domain as variable.

Remark. For positive f, the value $F(x)$ is the area under the graph of the function in the interval $[a, x]$; see



The interpretation of $F(x)$ as area

Proposition. (Fundamental theorems of calculus) Let f be continuous in $[a, b]$. Then the following assertions hold:

(a) *First fundamental theorem*: If G is an antiderivative of f then

$$\int_a^b f(x)\mathrm{d}x = G(b) - G(a).$$

(b) *Second fundamental theorem*: The function

$$F(x) = \int_a^x f(t)\mathrm{d}t$$

is an antiderivative of f, that is, F is differentiable and $F'(x) = f(x)$.

Proof. In the first step we prove the second fundamental theorem. For that let $x \in (a,b), h > 0$ and $x + h \in (a,b)$. According to the proposition the function f has a minimum and a maximum in the interval $[x, x+h]$:

$$m(h) = \min_{t\in[x,x+h]} f(t), \quad M(h) = \max_{t\in[x,x+h]} f(t).$$

The continuity of f implies the convergence $m(h) \to f(x)$ and $M(h) \to f(x)$ as $h \to 0$. According to item (b) in the proposition we have that

$$m(h) \cdot h \le F(x+h) - F(x) = \int_x^{x+h} f(t)\mathrm{d}t \le M(h) \cdot h.$$

This shows that F is differentiable at x and

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

The first fundamental theorem follows from the second fundamental theorem

$$\int_a^b f(t)\mathrm{d}t = F(b) = F(b) - F(a),$$

since $F(a) = 0$. If G is another antiderivative then $G = F + c$; hence

$$G(b) - G(a) = F(b) + c - (F(a) + c) = F(b) - F(a).$$

Thus $G(b) - G(a) = \int_a^b f(x)\mathrm{d}x$ as well.

Remark. For positive f, the second fundamental theorem of calculus has an intuitive interpretation. The value $F(x+h) - F(x)$ is the area under the graph of the function $y = f(x)$ in the interval $[x, x+h]$, while $hf(x)$ is the area of the approximating rectangle of height $f(x)$. The resulting approximation

$$\frac{F(x+h) - F(x)}{h} \approx f(x)$$

suggests that in the limit as $h \to 0$, $F'(x) = f(x)$.

Applications of the first fundamental theorem. The most important application consists in evaluating definite integrals $\int_a^b f(x)\mathrm{d}x$. For that, one determines an antiderivative $F(x)$, for instance as indefinite integral, and substitutes:

$$\int_a^b f(x)\mathrm{d}x = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a).$$

Example. As an application we compute the following integrals.

(a) $\int_1^3 x^2\,\mathrm{d}x = \left.\frac{x^3}{3}\right|_{x=1}^{x=3} = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$.

(b) $\int_0^{\pi/2} \cos x\,\mathrm{d}x = \sin x\Big|_{x=0}^{x=\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$.

(c) $\int_0^1 x\sin\left(x^2\right)\mathrm{d}x = -\frac{1}{2}\cos\left(x^2\right)\Big|_{x=0}^{x=1} = -\frac{1}{2}\cos 1 - \left(-\frac{1}{2}\cos 0\right) = -\frac{1}{2}\cos 1 + \frac{1}{2}.$

Applications of the second fundamental theorem. Usually, such applications are of theoretical nature, like the description of the relation between travelled distance and velocity,

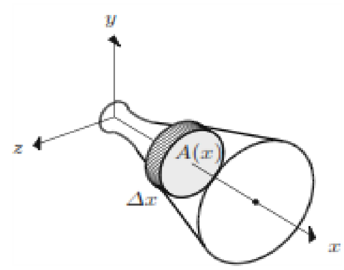
$$w(t) = w(0) + \int_0^t v(s)\mathrm{d}s, \quad w'(t) = v(t),$$

where $w(t)$ denotes the travelled distance from 0 to time t and $v(t)$ is the instantaneous velocity.

Applications of the Definite Integral

We now turn to further applications of the definite integral, which confirm the modeling power of the notion of the Riemann integral.

The volume of a solid of revolution. Assume first that for a three-dimensional solid (possibly after choosing an appropriate Cartesian coordinate system) the crosssectional area $A = A(x)$ is known for every $x \in [a,b]$; see



The volume of a thin slice of thickness Δx is approximately equal to $A(x)\Delta x$. Writing down the Riemann sums and taking limits one obtains for the volume V of the solid

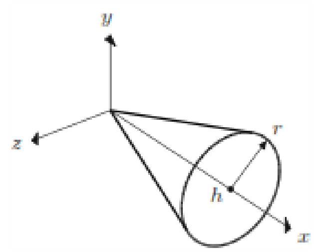
$$V = \int_a^b A(x)\mathrm{d}x.$$

A solid of revolution is obtained by rotating the plane curve $y = f(x), a \le x \le b$ around the x-axis. In this case, we have $A(x) = \pi f(x)^2$, and the volume is given by

$$V = \pi \int_a^b f(x)^2 \,\mathrm{d}x\,.$$

Example. (Volume of a cone) The rotation of the straight line $y = \frac{r}{h}x$ around the x-axis produces a cone of radius r and height h (see the following figure). Its volume is given by

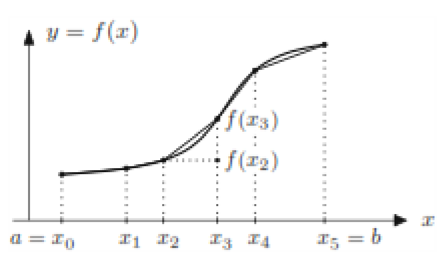
$$V = \pi \frac{r^2}{h^2} \int_0^h x^2 \, dx = \pi \frac{r^2}{h^2} \cdot \frac{x^3}{3} \bigg|_{x=0}^{x=h} = \pi r^2 \frac{h}{3}$$



Arc length of the graph of a function. To determine the arc length of the graph of a differentiable function with continuous derivative, we first partition the interval $[a,b]$, $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and replace the graph $y = f(x)$ on $[a,b]$ by line segments passing through the points $\left(x_0, f(x_0)\right), \left(x_1, f(x_1)\right), \ldots, \left(x_n, f(x_n)\right)$. The total length of the line segments is

$$s_n = \sum_{j=1}^{n} \sqrt{\left(x_j - x_{j-1}\right)^2 + \left(f\left(x_j\right) - f\left(x_{j-1}\right)\right)^2}.$$

It is simply given by the sum of the lengths of the individual segments



According to the mean value theorem we have

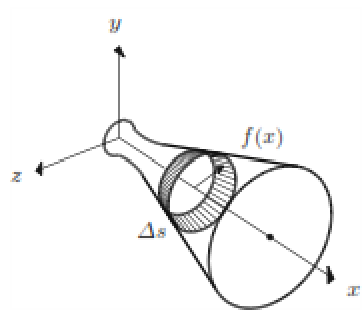
$$s_n = \sum_{j=1}^{n} \sqrt{\left(x_j - x_{j-1}\right)^2 + f'\left(c_j\right)^2 \left(x_j - x_{j-1}\right)^2} = \sum_{j=1}^{n} \sqrt{1 + f'\left(c_j\right)^2} \left(x_j - x_{j-1}\right)$$

with certain points $c_j \in \left[x_{j-1}, x_j\right]$. The sums s_n are easily identified as Riemann sums. Their limit is thus given by

$$s = \int_a^b \sqrt{1 + f'(x)^2}\, \mathrm{d}x.$$

Lateral surface area of a solid of revolution. The lateral surface of a solid of revolution is obtained by rotating the curve $y = f(x), a \le x \le b$ around the x-axis.

In order to determine its area, we split the solid into small slices of thickness Δx. Each of these slices is approximately a truncated cone with generator of length Δs and mean radius $f(x)$;



The lateral surface area of this truncated cone is equal to $2\pi f(x)\Delta s$. According to what has been said previously, $\Delta s \approx \sqrt{1 + f'(x)^2}\Delta x$ and thus the lateral surface area of a small slice is approximately equal to $2\pi f(x)\sqrt{1+f'(x)^2}\Delta x$. Writing down the Riemann sums and taking limits one obtains

$$M = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}x$$

for the lateral surface area.

Example. (Surface area of a sphere) The surface of a sphere of radius r is generated by rotation of the graph $f(x)=\sqrt{r^2-x^2}, -r\le x\le r$. One obtains

$$M = 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,\mathrm{d}x = 4\pi r^2.$$

Exercises

1. Prove that every function which is piecewise constant in an interval$[a, b]$is Riemann integrable (use Definition 11.3).

2. Compute the area between the graphs of $y=\sin x$ and $y=\sqrt{x}$ on the interval $[0,2\pi]$.

3. Rotation of the parabola $y=2\sqrt{x}, 0\le x\le 1$ around the x-axis produces a paraboloid. Sketch it and compute its volume and its lateral surface area.

4. Compute the arc length of the graph of the following functions:

(a) the parabola $f(x)=x^2/2$ for $0\le x\le 2$;

(b) the catenary $g(x)=\cosh x$ for $-1\le x\le 3$.

Hint. See Exercise 7 in Sect. 10.3.

5. The surface of a cooling tower can be described qualitatively by rotating the hyperbola $y=\sqrt{1+x^2}$ around the x-axis in the bounds $-1 \le x \le 2$.

(a) Compute the volume of the corresponding solid of revolution.

(b) Show that the lateral surface area is given by $M = 2\pi\int_{-1}^{2}\sqrt{1+2x^2}\,\mathrm{d}x$.

6. A lens-shaped body is obtained by rotating the graph of the sine function $y = \sin x$ around the x-axis in the bounds $0 \le x \le \pi$.

(a) Compute the volume of the body.

(b) Compute its lateral surface area.

Hint. For **(a)** use the identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$; for **(b)** use the substitution $g(x)=\cos x$.

7. (From probability theory) Let X be a random variable with values in an interval $[a,b]$ which possesses a probability density $f(x)$, that is, $f(x)\ge 0$ and $\int_a^b f(x)\mathrm{d}x = 1$. Its expectation value $\mu = \mathrm{E}(X)$, its second moment $\mathrm{E}\left(X^2\right)$ and its variance $\mathrm{V}(X)$ are defined by

$$\mathrm{E}(X)=\int_a^b x f(x)\mathrm{d}x, \quad \mathrm{E}\left(X^2\right)=\int_a^b x^2 f(x)\mathrm{d}x\,\mathrm{V}(X)=\int_a^b (x-\mu)^2 f(x)\mathrm{d}x$$

Show that $\mathrm{V}(X)=\mathrm{E}\left(X^2\right)-\mu^2$.

8. Compute the expectation value and the variance of a random variable which has (a) a uniform distribution on $[a, b]$, i.e. $f(x) = 1/(b-a)$ for $a \le x \le b$; (b) a (special) beta distribution on $[a, b]$ with density $f(x) = 6(x-a)(b-x)/(b-a)^3$

9. Compute the expectation value and the variance of a random variable which has a triangular distribution on $[a, b]$ with modal value m, i.e.

$$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(m-a)} \text{ for } a \le x \le m \frac{2(b-x)}{(b-a)(b-m)} \text{ for } m \le x \le b \end{cases}$$

Practice Problems

Review : Functions

For problems 1 – 4 the given functions perform the indicated function evaluations.

1. $f(x) = 3 - 5x - 2x^2$

   (a) $f(4)$ (b) $f(0)$ (c) $f(-3)$

   (d) $f(6-t)$ (e) $f(7-4x)$ (f) $f(x+h)$

2. $g(t) = \frac{t}{2t+6}$

   (a) $g(0)$ (b) $g(-3)$ (c) $g(10)$

   (d) $g(x^2)$ (e) $g(t+h)$ (f) $g(t^2 - 3t + 1)$

3. $h(z) = \sqrt{1 - z^2}$

   (a) $h(0)$ (b) $h(-\frac{1}{2})$ (c) $h(\frac{1}{2})$

   (d) $h(9z)$ (e) $h(z^2 - 2z)$ (f) $h(z+k)$

4. $R(x) = \sqrt{3+x} - \frac{4}{x+1}$

   (a) $R(0)$ (b) $R(6)$ (c) $R(-9)$

   (d) $R(x+1)$ (e) $R(x^4 - 3)$ (f) $R(\frac{1}{x} - 1)$

The **difference quotient** of a function $f(x)$ is defined to be,

$$\frac{f(x+h) - f(x)}{h}$$

For problems 5 – 9 compute the difference quotient of the given function.

5. $f(x) = 4x - 9$

6. $g(x) = 6 - x^2$

7. $f(t) = 2t^2 - 3t + 9$

8. $y(z) = \dfrac{1}{z+2}$

9. $A(t) = \dfrac{2t}{3-t}$

For problems 10 – 17 determine all the roots of the given function.

10. $f(x) = x^5 - 4x^4 - 32x^3$

11. $R(y) = 12y^2 + 11y - 5$

12. $h(t) = 18 - 3t - 2t^2$

13. $g(x) = x^3 + 7x^2 - x$

14. $W(x) = x^4 + 6x^2 - 27$

15. $f(t) = t^{\frac{5}{3}} - 7t^{\frac{4}{3}} - 8t$

16. $h(z) = \dfrac{z}{z-5} - \dfrac{4}{z-8}$

17. $g(w) = \dfrac{2w}{w+1} + \dfrac{w-4}{2w-3}$

For problems 18 – 22 find the domain and range of the given function.

18. $Y(t) = 3t^2 - 2t + 1$

19. $g(z) = -z^2 - 4z + 7$

20. $f(z) = 2 + \sqrt{z^2 + 1}$

21. $h(y) = -3\sqrt{14 + 3y}$

22. $M(x) = 5 - |x + 8|$

Review : Inverse Functions

For each of the following functions find the inverse of the function. Verify your inverse by computing one or both of the composition as discussed in this section.

1. $f(x) = 6x + 15$

2. $h(x) = 3 - 29x$

3. $R(x) = x^3 + 6$

4. $g(x) = 4(x-3)^5 + 21$

5. $W(x) = \sqrt[5]{9 - 11x}$

6. $f(x) = \sqrt[7]{5x + 8}$

7. $h(x) = \dfrac{1 + 9x}{4 - x}$

8. $f(x) = \dfrac{6 - 10x}{8x + 7}$

Review : Trig Functions

Determine the exact value of each of the following without using a calculator.

1. $\cos\left(\frac{5\pi}{6}\right)$

2. $\sin\left(-\frac{4\pi}{3}\right)$

3. $\sin\left(\frac{7\pi}{4}\right)$

4. $\cos\left(-\frac{2\pi}{3}\right)$

5. $\tan\left(\frac{3\pi}{4}\right)$

6. $\sec\left(-\frac{11\pi}{6}\right)$

7. $\cos\left(\frac{8\pi}{3}\right)$

8. $\tan\left(-\frac{\pi}{3}\right)$

9. $\tan\left(\frac{15\pi}{4}\right)$

10. $\sin\left(-\frac{11\pi}{3}\right)$

11. $\sec\left(\frac{29\pi}{4}\right)$

Review : Solving Trig Equations

Without using a calculator find the solution(s) to the following equations. If an interval is given then find only those solutions that are in the interval. If no interval is given then find all solutions to the equation.

1. $4\sin(3t) = 2$

2. $4\sin(3t) = 2$ in $\left[0, \frac{4\pi}{3}\right]$

3. $2\cos\left(\frac{x}{3}\right) + \sqrt{2} = 0$

4. $2\cos\left(\frac{x}{3}\right) + \sqrt{2} = 0$ in $[-7\pi, 7\pi]$

5. $4\cos(6z) = \sqrt{12}$ in $\left[0, \frac{\pi}{2}\right]$

6. $2\sin\left(\frac{3y}{2}\right) + \sqrt{3} = 0$ in $\left[-\frac{7\pi}{3}, 0\right]$

7. $8\tan(2x) - 5 = 3$ in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$

Review : Exponential Functions

Sketch the graphs of each of the following functions.

1. $f(x) = 3^{1+2x}$
2. $h(x) = 2^{3-\frac{x}{4}} - 7$
3. $h(t) = 8 + 3\mathbf{e}^{2t-4}$
4. $g(z) = 10 - \frac{1}{4}\mathbf{e}^{-2-3z}$

Review : Logarithm Functions

Without using a calculator determine the exact value of each of the following.

1. $\log_3 81$
2. $\log_5 125$
3. $\log_2 \frac{1}{8}$
4. $\log_{\frac{1}{4}} 16$
5. $\ln \mathbf{e}^4$
6. $\log \frac{1}{100}$

Write each of the following in terms of simpler logarithms

7. $\log\left(3x^4y^{-7}\right)$

8. $\ln\left(x\sqrt{y^2+z^2}\right)$

9. $\log_4\left(\dfrac{x-4}{y^2\sqrt[5]{z}}\right)$

Combine each of the following into a single logarithm with a coefficient of one.

10. $2\log_4 x + 5\log_4 y - \frac{1}{2}\log_4 z$

11. $3\ln(t+5) - 4\ln t - 2\ln(s-1)$

12. $\frac{1}{3}\log a - 6\log b + 2$

Review : Common Graphs

Without using a graphing calculator sketch the graph of each of the following.

1. $y = \frac{4}{3}x - 2$
2. $f(x) = |x - 3|$
3. $g(x) = \sin(x) + 6$
4. $f(x) = \ln(x) - 5$
5. $h(x) = \cos\left(x + \frac{\pi}{2}\right)$
6. $h(x) = (x - 3)^2 + 4$
7. $W(x) = \mathrm{e}^{x+2} - 3$
8. $f(y) = (y - 1)^2 + 2$
9. $R(x) = -\sqrt{x}$
10. $g(x) = \sqrt{-x}$

1. For the function $f(x) = 3(x+2)^2$ and the point P given by $x = -3$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) -3.5	**(ii)** -3.1	**(iii)** -3.01	**(iv)** -3.001	**(v)** -3.0001
(vi) -2.5	**(vii)** -2.9	**(viii)** -2.99	**(ix)** -2.999	**(x)** -2.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $f(x)$ at $x = -3$ and write down the equation of the tangent line.

2. For the function $g(x) = \sqrt{4x+8}$ and the point P given by $x = 2$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $g(x)$ at $x = 2$ and write down the equation of the tangent line.

3. For the function $W(x) = \ln(1+x^4)$ and the point P given by $x = 1$ answer each of the following questions.

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 1.5	**(ii)** 1.1	**(iii)** 1.01	**(iv)** 1.001	**(v)** 1.0001
(vi) 0.5	**(vii)** 0.9	**(viii)** 0.99	**(ix)** 0.999	**(x)** 0.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $W(x)$ at $x = 1$ and write down the equation of the tangent line.

4. The volume of air in a balloon is given by $V(t)=\frac{6}{4t+1}$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t=0.25$ and the following values of *t*.

(i) 1	**(ii)** 0.5	**(iii)** 0.251	**(iv)** 0.2501	**(v)** 0.25001
(vi) 0	**(vii)** 0.1	**(viii)** 0.249	**(ix)** 0.2499	**(x)** 0.24999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the volume of air in the balloon at $t=0.25$.

5. The population (in hundreds) of fish in a pond is given by $P(t)=2t+\sin(2t-10)$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t=5$ and the following values of *t*. Make sure your calculator is set to radians for the computations.

(i) 5.5	**(ii)** 5.1	**(iii)** 5.01	**(iv)** 5.001	**(v)** 5.0001
(vi) 4.5	**(vii)** 4.9	**(viii)** 4.99	**(ix)** 4.999	**(x)** 4.9999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the population of the fish at $t=5$.

6. The position of an object is given by $s(t)=\cos^2\left(\frac{3t-6}{2}\right)$ answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t=2$ and the following values of *t*. Make sure your calculator is set to radians for the computations.

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t=2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

7. The position of an object is given by $s(t)=(8-t)(t+6)^{\frac{3}{2}}$. Note that a negative position here simply means that the position is to the left of the "zero position" and is perfectly acceptable.

Answer each of the following questions.

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of t.

(i) 10.5	**(ii)** 10.1	**(iii)** 10.01	**(iv)** 10.001	**(v)** 10.0001
(vi) 9.5	**(vii)** 9.9	**(viii)** 9.99	**(ix)** 9.999	**(x)** 9.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero).

The Limit

1. For the function $f(x) = \dfrac{8 - x^3}{x^2 - 4}$ answer each of the following questions.

(a) Evaluate the function the following values of x compute (accurate to at least 8 decimal places).

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from **(a)** to estimate the value of $\lim\limits_{x \to 2} \dfrac{8 - x^3}{x^2 - 4}$.

2. For the function $R(t) = \dfrac{2 - \sqrt{t^2 + 3}}{t + 1}$ answer each of the following questions.

(a) Evaluate the function the following values of t compute (accurate to at least 8 decimal places).

(i) -0.5	**(ii)** -0.9	**(iii)** -0.99	**(iv)** -0.999	**(v)** -0.9999
(vi) -1.5	**(vii)** -1.1	**(viii)** -1.01	**(ix)** -1.001	**(x)** -1.0001

(b) Use the information from **(a)** to estimate the value of $\lim\limits_{t \to -1} \dfrac{2 - \sqrt{t^2 + 3}}{t + 1}$.

3. For the function $g(\theta) = \dfrac{\sin(7\theta)}{\theta}$ answer each of the following questions.

(a) Evaluate the function the following values of θ compute (accurate to at least 8 decimal places). Make sure your calculator is set to radians for the computations.

(i) 0.5	**(ii)** 0.1	**(iii)** 0.01	**(iv)** 0.001	**(v)** 0.0001
(vi) -0.5	**(vii)** -0.1	**(viii)** -0.01	**(ix)** -0.001	**(x)** -0.0001

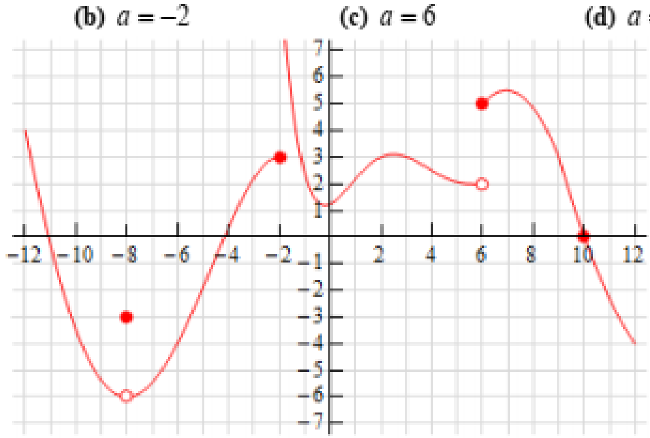
(b) Use the information from **(a)** to estimate the value of $\lim_{\theta \to 0} \frac{\sin(7\theta)}{\theta}$.

4. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -3$ **(b)** $a = -1$ **(c)** $a = 2$ **(d)** $a = 4$

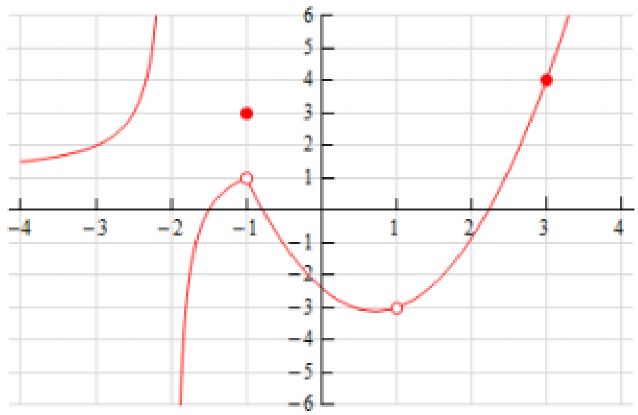
5. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -8$ **(b)** $a = -2$ **(c)** $a = 6$ **(d)** $a = 10$



6. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

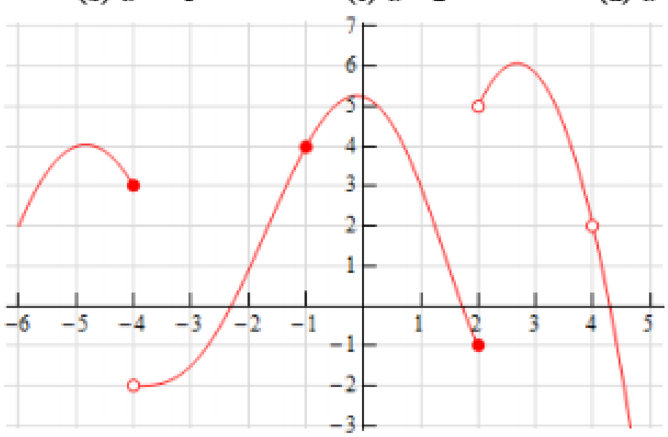
(a) $a = -2$ **(b)** $a = -1$ **(c)** $a = 1$ **(d)** $a = 3$



One-Sided Limits

1. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a^+} f(x)$, and $\lim_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -4$ **(b)** $a = -1$ **(c)** $a = 2$ **(d)** $a = 4$



2. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim_{x\to a^-} f(x)$, $\lim_{x\to a^+} f(x)$, and $\lim_{x\to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a=-2$ **(b)** $a=1$ **(c)** $a=3$ **(d)** $a=5$

3. Sketch a graph of a function that satisfies each of the following conditions.

$\lim_{x\to 2^-} f(x)=1$ $\lim_{x\to 2^+} f(x)=-4$ $f(2)=1$

4. Sketch a graph of a function that satisfies each of the following conditions.

$\lim_{x\to 3^-} f(x)=0$ $\lim_{x\to 3^+} f(x)=4$ $f(3)$ does not exist

$\lim_{x\to -1} f(x)=-3$ $f(-1)=2$

Limit Properties

1. Given $\lim_{x\to 8} f(x)=-9$, $\lim_{x\to 8} g(x)=2$ and $\lim_{x\to 8} h(x)=4$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim_{x\to 8}\left[2f(x)-12h(x)\right]$ **(b)** $\lim_{x\to 8}\left[3h(x)-6\right]$

(c) $\lim_{x\to 8}\left[g(x)h(x)-f(x)\right]$ **(d)** $\lim_{x\to 8}\left[f(x)-g(x)+h(x)\right]$

2. Given $\lim_{x\to -4} f(x)=1$, $\lim_{x\to -4} g(x)=10$ and $\lim_{x\to -4} h(x)=-7$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim_{x\to -4}\left[\frac{f(x)}{g(x)}-\frac{h(x)}{f(x)}\right]$ **(b)** $\lim_{x\to -4}\left[f(x)g(x)h(x)\right]$

(c) $\lim_{x\to -4}\left[\frac{1}{h(x)}+\frac{3-f(x)}{g(x)+h(x)}\right]$ **(d)** $\lim_{x\to -4}\left[2h(x)-\frac{1}{h(x)+7f(x)}\right]$

3. Given $\lim_{x \to 0} f(x) = 6$, $\lim_{x \to 0} g(x) = -4$ and $\lim_{x \to 0} h(x) = -1$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim_{x \to 0} \left[f(x) + h(x) \right]^3$

(b) $\lim_{x \to 0} \sqrt{g(x) h(x)}$

(c) $\lim_{x \to 0} \sqrt[3]{11 + \left[g(x) \right]^2}$

(d) $\lim_{x \to 0} \sqrt{\dfrac{f(x)}{h(x) - g(x)}}$

For each of the following limits use the limit properties given in this section to compute the limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

4. $\lim_{t \to -2} \left(14 - 6t + t^3 \right)$

5. $\lim_{x \to 6} \left(3x^2 + 7x - 16 \right)$

6. $\lim_{w \to 3} \dfrac{w^2 - 8w}{4 - 7w}$

7. $\lim_{x \to -5} \dfrac{x + 7}{x^2 + 3x - 10}$

8. $\lim_{z \to 0} \sqrt{z^2 + 6}$

Computing Limits

For problems 1 – 9 evaluate the limit, if it exists.

1. $\lim\limits_{x \to 2} \left(8 - 3x + 12x^2\right)$

2. $\lim\limits_{t \to -3} \dfrac{6 + 4t}{t^2 + 1}$

3. $\lim\limits_{x \to -5} \dfrac{x^2 - 25}{x^2 + 2x - 15}$

4. $\lim\limits_{z \to 8} \dfrac{2z^2 - 17z + 8}{8 - z}$

5. $\lim\limits_{y \to 7} \dfrac{y^2 - 4y - 21}{3y^2 - 17y - 28}$

6. $\lim\limits_{h \to 0} \dfrac{(6 + h)^2 - 36}{h}$

7. $\lim\limits_{z \to 4} \dfrac{\sqrt{z} - 2}{z - 4}$

8. $\lim\limits_{x \to -3} \dfrac{\sqrt{2x + 22} - 4}{x + 3}$

9. $\lim\limits_{x \to 0} \dfrac{x}{3 - \sqrt{x + 9}}$

10. Given the function

$$f(x) = \begin{cases} 7 - 4x & x < 1 \\ x^2 + 2 & x \geq 1 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim_{x \to -6} f(x)$ (b) $\lim_{x \to 1} f(x)$

11. Given

$$h(z) = \begin{cases} 6z & z \le -4 \\ 1 - 9z & z > -4 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim_{z \to 7} h(z)$ (b) $\lim_{z \to -4} h(z)$

Infinite Limits

For problems 1 – 6 evaluate the indicated limits, if they exist.

1. For $f(x) = \dfrac{9}{(x-3)^5}$ evaluate,

(a) $\lim_{x \to 3^-} f(x)$ **(b)** $\lim_{x \to 3^+} f(x)$ **(c)** $\lim_{x \to 3} f(x)$

2. For $h(t) = \dfrac{2t}{6+t}$ evaluate,

(a) $\lim_{t \to -6^-} h(t)$ **(b)** $\lim_{t \to -6^+} h(t)$ **(c)** $\lim_{t \to -6} h(t)$

3. For $g(z) = \dfrac{z+3}{(z+1)^2}$ evaluate,

(a) $\lim_{z \to -1^-} g(z)$ **(b)** $\lim_{z \to -1^+} g(z)$ **(c)** $\lim_{z \to -1} g(z)$

4. For $g(x) = \dfrac{x+7}{x^2-4}$ evaluate,

(a) $\lim_{x \to 2^-} g(x)$ **(b)** $\lim_{x \to 2^+} g(x)$ **(c)** $\lim_{x \to 2} g(x)$

5. For $h(x) = \ln(-x)$ evaluate,

(a) $\lim_{x \to 0^-} h(x)$ **(b)** $\lim_{x \to 0^+} h(x)$ **(c)** $\lim_{x \to 0} h(x)$

6. For $R(y) = \tan(y)$ evaluate,

(a) $\lim_{y \to \frac{3\pi}{2}^-} R(y)$ **(b)** $\lim_{y \to \frac{3\pi}{2}^+} R(y)$ **(c)** $\lim_{y \to \frac{3\pi}{2}} R(y)$

Limits At Infinity, Part I

1. For $f(x) = 4x^7 - 18x^3 + 9$ evaluate each of the following limits.

(a) $\lim\limits_{x \to -\infty} f(x)$ **(b)** $\lim\limits_{x \to \infty} f(x)$

2. For $h(t) = \sqrt[3]{t} + 12t - 2t^2$ evaluate each of the following limits.

(a) $\lim\limits_{t \to -\infty} h(t)$ **(b)** $\lim\limits_{t \to \infty} h(t)$

For problems 3 – 10 answer each of the following questions.

(a) Evaluate $\lim\limits_{x \to -\infty} f(x)$.

(b) Evaluate $\lim\limits_{x \to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

3. $f(x) = \dfrac{8 - 4x^2}{9x^2 + 5x}$

4. $f(x) = \dfrac{3x^7 - 4x^2 + 1}{5 - 10x^2}$

5. $f(x) = \dfrac{20x^4 - 7x^3}{2x + 9x^2 + 5x^4}$

6. $f(x) = \dfrac{x^3 - 2x + 11}{3 - 6x^5}$

7. $f(x) = \dfrac{x^6 - x^4 + x^2 - 1}{7x^6 + 4x^3 + 10}$

8. $f(x) = \dfrac{\sqrt{7 + 9x^2}}{1 - 2x}$

9. $f(x) = \dfrac{x + 8}{\sqrt{2x^2 + 3}}$

10. $f(x) = \dfrac{8 + x - 4x^2}{\sqrt{6 + x^2 + 7x^4}}$

Limits At Infinity, Part II

For problems 1 – 6 evaluate **(a)** $\lim\limits_{x \to -\infty} f(x)$ and **(b)** $\lim\limits_{x \to \infty} f(x)$.

1. $f(x) = \mathbf{e}^{8 + 2x - x^3}$

2. $f(x) = \mathbf{e}^{\frac{6x^2 + x}{5 + 3x}}$

3. $f(x) = 2\mathbf{e}^{6x} - \mathbf{e}^{-7x} - 10\mathbf{e}^{4x}$

4. $f(x) = 3\mathbf{e}^{-x} - 8\mathbf{e}^{-5x} - \mathbf{e}^{10x}$

5. $f(x) = \dfrac{\mathbf{e}^{-3x} - 2\mathbf{e}^{8x}}{9\mathbf{e}^{8x} - 7\mathbf{e}^{-3x}}$

6. $f(x) = \dfrac{\mathbf{e}^{-7x} - 2\mathbf{e}^{3x} - \mathbf{e}^{x}}{\mathbf{e}^{-x} + 16\mathbf{e}^{10x} + 2\mathbf{e}^{-4x}}$

For problems 7 – 12 evaluate the given limit.

7. $\lim\limits_{t \to -\infty} \ln\left(4 - 9t - t^3\right)$

8. $\lim\limits_{z \to -\infty} \ln\left(\dfrac{3z^4 - 8}{2 + z^2}\right)$

9. $\lim\limits_{x \to \infty} \ln\left(\dfrac{11 + 8x}{x^3 + 7x}\right)$

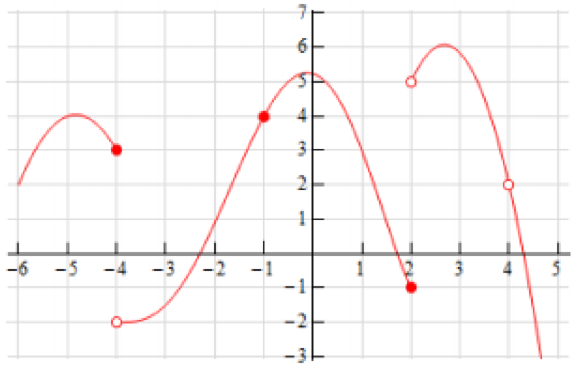
10. $\lim\limits_{x \to -\infty} \tan^{-1}\left(7 - x + 3x^5\right)$

11. $\lim\limits_{t \to \infty} \tan^{-1}\left(\dfrac{4 + 7t}{2 - t}\right)$

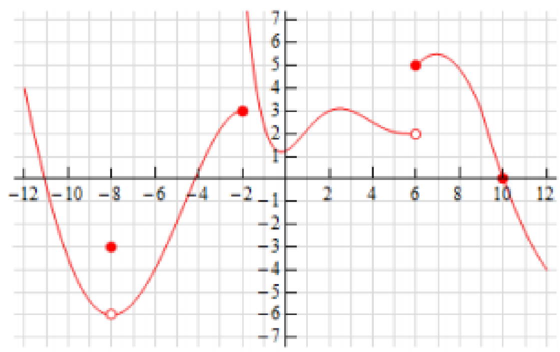
12. $\lim\limits_{w \to -\infty} \tan^{-1}\left(\dfrac{3w^2 - 9w^4}{4w - w^3}\right)$

Continuity

1. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



2. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



For problems 3 – 7 determine if the given function is continuous or discontinuous at the indicated points.

3. $f(x) = \dfrac{4x+5}{9-3x}$
   (a) $x=-1$, **(b)** $x=0$, **(c)** $x=3$?

4. $g(z) = \dfrac{6}{z^2-3z-10}$
   (a) $z=-2$, **(b)** $z=0$, **(c)** $z=5$?

5. $g(x) = \begin{cases} 2x & x<6 \\ x-1 & x \geq 6 \end{cases}$
   (a) $x=4$, **(b)** $x=6$?

6. $h(t) = \begin{cases} t^2 & t<-2 \\ t+6 & t \geq -2 \end{cases}$
   (a) $t=-2$, **(b)** $t=10$?

7. $g(x) = \begin{cases} 1-3x & x<-6 \\ 7 & x=-6 \\ x^3 & -6<x<1 \\ 1 & x=1 \\ 2-x & x>1 \end{cases}$
   (a) $x=-6$, **(b)** $x=1$?

The Definition of the Derivative

Use the definition of the derivative to find the derivative of the following functions.

1. $f(x) = 6$

2. $V(t) = 3 - 14t$

3. $g(x) = x^2$

4. $Q(t) = 10 + 5t - t^2$

Differentiation Formulas

For problems 1 – 12 find the derivative of the given function.

1. $f(x) = 6x^3 - 9x + 4$

2. $y = 2t^4 - 10t^2 + 13t$

3. $g(z) = 4z^7 - 3z^{-7} + 9z$

4. $h(y) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$

5. $y = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$

6. $f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$

7. $f(t) = \frac{4}{t} - \frac{1}{6t^3} + \frac{8}{t^5}$

8. $R(z) = \frac{6}{\sqrt{z^3}} + \frac{1}{8z^4} - \frac{1}{3z^{10}}$

9. $z = x(3x^2 - 9)$

10. $g(y) = (y-4)(2y + y^2)$

11. $h(x) = \frac{4x^3 - 7x + 8}{x}$

12. $f(y) = \frac{y^5 - 5y^3 + 2y}{y^3}$

Product and Quotient Rule

For problems 1 – 6 use the Product Rule or the Quotient Rule to find the derivative of the given function.

1. $f(t) = (4t^2 - t)(t^3 - 8t^2 + 12)$

2. $y = \left(1 + \sqrt{x^3}\right)\left(x^{-3} - 2\sqrt[3]{x}\right)$

3. $h(z) = (1 + 2z + 3z^2)(5z + 8z^2 - z^3)$

4. $g(x) = \frac{6x^2}{2 - x}$

5. $R(w) = \frac{3w + w^4}{2w^2 + 1}$

6. $f(x) = \frac{\sqrt{x} + 2x}{7x - 4x^2}$

7. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$ determine the value of $(f\,g)'(2)$

8. If $f(x) = x^3 g(x)$, $g(-7) = 2$, $g'(-7) = -9$ determine the value of $f'(-7)$.

9. Find the equation of the tangent line to $f(x) = \left(1 + 12\sqrt{x}\right)(4 - x^2)$ at $x = 9$.

10. Determine where $f(x) = \frac{x - x^2}{1 + 8x^2}$ is increasing and decreasing.

11. Determine where $V(t) = (4 - t^2)(1 + 5t^2)$ is increasing and decreasing.

Derivatives of Trig Functions

For problems 1 – 3 evaluate the given limit.

1. $\lim\limits_{z\to 0}\dfrac{\sin(10z)}{z}$

2. $\lim\limits_{\alpha\to 0}\dfrac{\sin(12\alpha)}{\sin(5\alpha)}$

3. $\lim\limits_{x\to 0}\dfrac{\cos(4x)-1}{x}$

For problems 4 – 10 differentiate the given function.

4. $f(x)=2\cos(x)-6\sec(x)+3$

5. $g(z)=10\tan(z)-2\cot(z)$

6. $f(w)=\tan(w)\sec(w)$

7. $h(t)=t^3-t^2\sin(t)$

8. $y=6+4\sqrt{x}\csc(x)$

9. $R(t)=\dfrac{1}{2\sin(t)-4\cos(t)}$

10. $Z(v)=\dfrac{v+\tan(v)}{1+\csc(v)}$

Derivatives of Exponential and Logarithm Functions

For problems 1 – 6 differentiate the given function.

1. $f(x)=2\mathbf{e}^x-8^x$

2. $g(t)=4\log_3(t)-\ln(t)$

3. $R(w)=3^w\log(w)$

4. $y=z^5-\mathbf{e}^z\ln(z)$

5. $h(y)=\dfrac{y}{1-\mathbf{e}^y}$

6. $f(t)=\dfrac{1+5t}{\ln(t)}$

7. Find the tangent line to $f(x)=7^x+4\mathbf{e}^x$ at $x=0$.

8. Find the tangent line to $f(x)=\ln(x)\log_2(x)$ at $x=2$.

9. Determine if $V(t)=\dfrac{t}{\mathbf{e}^t}$ is increasing or decreasing at the following points.

(a) $t=-4$ **(b)** $t=0$ **(c)** $t=10$

Derivatives of Inverse Trig Functions

For each of the following problems differentiate the given function.

1. $T(z) = 2\cos(z) + 6\cos^{-1}(z)$

2. $g(t) = \csc^{-1}(t) - 4\cot^{-1}(t)$

3. $y = 5x^6 - \sec^{-1}(x)$

4. $f(w) = \sin(w) + w^2 \tan^{-1}(w)$

5. $h(x) = \dfrac{\sin^{-1}(x)}{1+x}$

Derivatives of Hyperbolic Functions

For each of the following problems differentiate the given function.

1. $f(x) = \sinh(x) + 2\cosh(x) - \operatorname{sech}(x)$

2. $R(t) = \tan(t) + t^2 \operatorname{csch}(t)$

3. $g(z) = \dfrac{z+1}{\tanh(z)}$

Chain Rule

For problems 1 – 26 differentiate the given function.

1. $f(x)=\left(6x^2+7x\right)^4$

2. $g(t)=\left(4t^2-3t+2\right)^{-2}$

3. $y=\sqrt[3]{1-8z}$

4. $R(w)=\csc(7w)$

5. $G(x)=2\sin\left(3x+\tan(x)\right)$

6. $h(u)=\tan(4+10u)$

7. $f(t)=5+\mathbf{e}^{4t+t^7}$

8. $g(x)=\mathbf{e}^{1-\cos(x)}$

9. $H(z)=2^{1-6z}$

10. $u(t)=\tan^{-1}(3t-1)$

11. $F(y)=\ln\left(1-5y^2+y^3\right)$

12. $V(x)=\ln\left(\sin(x)-\cot(x)\right)$

13. $h(z)=\sin\left(z^6\right)+\sin^6(z)$

14. $S(w)=\sqrt{7w}+\mathbf{e}^{-w}$

15. $g(z)=3z^7-\sin\left(z^2+6\right)$

16. $f(x)=\ln\left(\sin(x)\right)-\left(x^4-3x\right)^{10}$

17. $h(t)=t^6\sqrt{5t^2-t}$

18. $q(t)=t^2\ln\left(t^5\right)$

19. $g(w)=\cos(3w)\sec(1-w)$

20. $y=\dfrac{\sin(3t)}{1+t^2}$

21. $K(x)=\dfrac{1+\mathbf{e}^{-2x}}{x+\tan(12x)}$

22. $f(x)=\cos\left(x^2\mathbf{e}^x\right)$

23. $z=\sqrt{5x+\tan(4x)}$

24. $f(t)=\left(\mathbf{e}^{-6t}+\sin(2-t)\right)^3$

25. $g(x)=\left(\ln\left(x^2+1\right)-\tan^{-1}(6x)\right)^{10}$

26. $h(z)=\tan^4\left(z^2+1\right)$

Implicit Differentiation

For problems 1 – 3 do each of the following.

(a) Find y' by solving the equation for y and differentiating directly.
(b) Find y' by implicit differentiation.
(c) Check that the derivatives in **(a)** and **(b)** are the same.

1. $\dfrac{x}{y^3}=1$

2. $x^2+y^3=4$

3. $x^2+y^2=2$

For problems 4 – 9 find y' by implicit differentiation.

4. $2y^3+4x^2-y=x^6$

5. $7y^2+\sin(3x)=12-y^4$

6. $\mathbf{e}^x-\sin(y)=x$

7. $4x^2y^7-2x=x^5+4y^3$

8. $\cos\left(x^2+2y\right)+x\,\mathbf{e}^{y^2}=1$

9. $\tan\left(x^2y^4\right)=3x+y^2$

Higher Order Derivatives

For problems 1 – 5 determine the fourth derivative of the given function.

1. $h(t)=3t^7-6t^4+8t^3-12t+18$

2. $V(x)=x^3-x^2+x-1$

3. $f(x)=4\sqrt[5]{x^3}-\dfrac{1}{8x^2}-\sqrt{x}$

4. $f(w)=7\sin\left(\frac{w}{3}\right)+\cos(1-2w)$

5. $y=\mathbf{e}^{-5z}+8\ln\left(2z^4\right)$

For problems 6 – 9 determine the second derivative of the given function.

6. $g(x) = \sin(2x^3 - 9x)$

7. $z = \ln(7 - x^3)$

8. $Q(v) = \dfrac{2}{(6 + 2v - v^2)^4}$

9. $H(t) = \cos^2(7t)$

For problems 10 & 11 determine the second derivative of the given function.

10. $2x^3 + y^2 = 1 - 4y$

11. $6y - xy^2 = 1$

Logarithmic Differentiation

For problems 1 – 3 use logarithmic differentiation to find the first derivative of the given function.

1. $f(x) = (5 - 3x^2)^7 \sqrt{6x^2 + 8x - 12}$

2. $y = \dfrac{\sin(3z + z^2)}{(6 - z^4)^3}$

3. $h(t) = \dfrac{\sqrt{5t + 8}\ \sqrt[3]{1 - 9\cos(4t)}}{\sqrt[4]{t^2 + 10t}}$

For problems 4 & 5 find the first derivative of the given function.

4. $g(w) = (3w - 7)^{4w}$

5. $f(x) = (2x - \mathbf{e}^{8x})^{\sin(2x)}$

Indefinite Integrals

1. Evaluate each of the following indefinite integrals.

(a) $\int 6x^5 - 18x^2 + 7\,dx$

(b) $\int 6x^5\,dx - 18x^2 + 7$

2. Evaluate each of the following indefinite integrals.

(a) $\int 40x^3 + 12x^2 - 9x + 14\,dx$

(b) $\int 40x^3 + 12x^2 - 9x\,dx + 14$

(c) $\int 40x^3 + 12x^2\,dx - 9x + 14$

For problems 3 – 5 evaluate the indefinite integral.

3. $\int 12t^7 - t^2 - t + 3\,dt$

4. $\int 10w^4 + 9w^3 + 7w\,dw$

5. $\int z^6 + 4z^4 - z^2\,dz$

6. Determine $f(x)$ given that $f'(x) = 6x^8 - 20x^4 + x^2 + 9$.

7. Determine $h(t)$ given that $h'(t) = t^4 - t^3 + t^2 + t - 1$.

Computing Indefinite Integrals

For problems 1 – 21 evaluate the given integral.

1. $\int 4x^6 - 2x^3 + 7x - 4\,dx$

2. $\int z^7 - 48z^{11} - 5z^{16}\,dz$

3. $\int 10t^{-3} + 12t^{-9} + 4t^3\,dt$

4. $\int w^{-2} + 10w^{-5} - 8\,dw$

5. $\int 12\,dy$

6. $\int \sqrt[3]{w} + 10\sqrt[5]{w^3}\,dw$

7. $\int \sqrt{x^7} - 7\sqrt[6]{x^5} + 17\sqrt[3]{x^{10}}\,dx$

8. $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}\,dx$

9. $\int \frac{7}{3y^6} + \frac{1}{y^{10}} - \frac{2}{\sqrt[3]{y^4}}\,dy$

10. $\int (t^2 - 1)(4 + 3t)\,dt$

11. $\int \sqrt{z}\left(z^2 - \frac{1}{4z}\right)dz$

12. $\int \frac{z^8 - 6z^5 + 4z^3 - 2}{z^4}\,dz$

13. $\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}\,dx$

14. $\int \sin(x) + 10\csc^2(x)\,dx$

15. $\int 2\cos(w) - \sec(w)\tan(w)\,dw$

16. $\int 12 + \csc(\theta)\left[\sin(\theta) + \csc(\theta)\right]d\theta$

17. $\int 4\mathbf{e}^z + 15 - \frac{1}{6z}\,dz$

18. $\int t^3 - \frac{\mathbf{e}^{-t} - 4}{\mathbf{e}^{-t}}\,dt$

19. $\int \frac{6}{w^3} - \frac{2}{w}\,dw$

20. $\int \frac{1}{1 + x^2} + \frac{12}{\sqrt{1 - x^2}}\,dx$

21. $\int 6\cos(z) + \frac{4}{\sqrt{1 - z^2}}\,dz$

22. Determine $f(x)$ given that $f'(x) = 12x^2 - 4x$ and $f(-3) = 17$.

23. Determine $g(z)$ given that $g'(z) = 3z^3 + \frac{7}{2\sqrt{z}} - \mathbf{e}^z$ and $g(1) = 15 - \mathbf{e}$.

24. Determine $h(t)$ given that $h''(t) = 24t^2 - 48t + 2$, $h(1) = -9$ and $h(-2) = -4$.

Substitution Rule for Indefinite Integrals

For problems 1 – 16 evaluate the given integral.

1. $\int (8x - 12)(4x^2 - 12x)^4\,dx$

2. $\int 3t^{-4}(2 + 4t^{-3})^{-7}\,dt$

3. $\int (3 - 4w)(4w^2 - 6w + 7)^{10}\,dw$

4. $\int 5(z - 4)\sqrt[3]{z^2 - 8z}\,dz$

5. $\int 90x^2\sin(2 + 6x^3)\,dx$

6. $\int \sec(1 - z)\tan(1 - z)\,dz$

7. $\int (15t^{-2} - 5t)\cos(6t^{-1} + t^2)\,dt$

8. $\int (7y - 2y^3)\mathbf{e}^{y^4 - 7y^2}\,dy$

9. $\int \frac{4w + 3}{4w^2 + 6w - 1}\,dw$

10. $\int (\cos(3t) - t^2)(\sin(3t) - t^3)^5\,dt$

11. $\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right)\cos(\mathbf{e}^{-z} + \ln z)\,dz$

12. $\int \sec^2(v)\mathbf{e}^{1 + \tan(v)}\,dv$

13. $\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x) - 5}\,dx$

14. $\int \frac{\csc(x)\cot(x)}{2 - \csc(x)}\,dx$

15. $\int \frac{6}{7 + y^2}\,dy$

16. $\int \frac{1}{\sqrt{4 - 9w^2}}\,dw$

17. Evaluate each of the following integrals.

(a) $\int \frac{3x}{1+9x^2}\,dx$ (b) $\int \frac{3x}{\left(1+9x^2\right)^4}\,dx$ (c) $\int \frac{3}{1+9x^2}\,dx$

For problems 1 & 2 use the definition of the definite integral to evaluate the integral. Use the right end point of each interval for x_i^*.

1. $\int_1^4 2x+3\,dx$

2. $\int_0^1 6x\left(x-1\right)dx$

3. Evaluate : $\int_4^4 \frac{\cos\left(\mathbf{e}^{3x}+x^2\right)}{x^4+1}\,dx$

For problems 4 & 5 determine the value of the given integral given that $\int_6^{11} f\left(x\right)dx=-7$ and $\int_6^{11} g\left(x\right)dx=24$.

4. $\int_{11}^{6} 9f\left(x\right)dx$

5. $\int_6^{11} 6g\left(x\right)-10f\left(x\right)dx$

6. Determine the value of $\int_{2}^{9} f(x)\,dx$ given that $\int_{5}^{2} f(x)\,dx = 3$ and $\int_{5}^{9} f(x)\,dx = 8$.

7. Determine the value of $\int_{-4}^{20} f(x)\,dx$ given that $\int_{-4}^{0} f(x)\,dx = -2$, $\int_{31}^{0} f(x)\,dx = 19$ and $\int_{20}^{31} f(x)\,dx = -21$.

For problems 8 & 9 sketch the graph of the integrand and use the area interpretation of the definite integral to determine the value of the integral.

8. $\int_{1}^{4} 3x - 2\,dx$

9. $\int_{0}^{5} -4x\,dx$

For problems 10 – 12 differentiate each of the following integrals with respect to x.

10. $\int_{4}^{x} 9\cos^{2}\left(t^{2} - 6t + 1\right)dt$

11. $\int_{7}^{\sin(6x)} \sqrt{t^{2} + 4}\,dt$

12. $\displaystyle\int_{3x^{2}}^{-1} \frac{e^{t} - 1}{t}\,dt$

Substitution Rule for Definite Integrals

Evaluate each of the following integrals, if possible. If it is not possible clearly explain why it is not possible to evaluate the integral.

1. $\int_{0}^{1} 3\left(4x + x^{4}\right)\left(10x^{2} + x^{5} - 2\right)^{6} dx$

2. $\displaystyle\int_{0}^{\frac{\pi}{4}} \frac{8\cos(2t)}{\sqrt{9 - 5\sin(2t)}}\,dt$

3. $\int_{\pi}^{0} \sin(z)\cos^{3}(z)\,dz$

4. $\int_{1}^{4} \sqrt{w}\, e^{1-\sqrt{w^{3}}}\,dw$

Area Between Curves

1. Determine the area below $f(x) = 3 + 2x - x^2$ and above the x-axis.

2. Determine the area to the left of $g(y) = 3 - y^2$ and to the right of $x = -1$.

For problems 3 – 4 determine the area of the region bounded by the given set of curves

3. $y = x^2 + 2$, $y = \sin(x)$, $x = -1$ and $x = 2$

4. $y = \frac{8}{x}$, $y = 2x$ and $x = 4$

Volumes of Solids of Revolution / Method of Rings

For problems 1 – 4 use the method of disks/rings to determine the volume of the solid obtained by rotating the region bounded by the given curves about the given axis

1. Rotate the region bounded by $y = \sqrt{x}$, $y = 3$ and the y-axis about the y-axis.

2. Rotate the region bounded by $y = 7 - x^2$, $x = -2$, $x = 2$ and the x-axis about the x-axis.

3. Rotate the region bounded by $x = y^2 - 6y + 10$ and $x = 5$ about the y-axis.

4. Rotate the region bounded by $y = 2x^2$ and $y = x^3$ about the x-axis.